Physica A 391 (2012) 3816–3827

Contents lists available at SciVerse ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

A study on the action of non-Gaussian noise on a Brownian particle

Welles A.M. Morgado [a,*], Thiago Guerreiro [b]

[a] *Departamento de Física, Pontifícia, Universidade Católica and National Institute of Science and Technology for Complex Systems, 22452-970, Rio de Janeiro, Brazil*

[b] *GAP-Optique, Université de Genève, Rue de l'Ecole de Medecine 20, CH-1211 Geneve 4, Switzerland*

ARTICLE INFO

Article history:
Received 15 August 2011
Received in revised form 6 February 2012
Available online 9 March 2012

Keywords:
Non Gaussian noise
Exact results
Time Averaging

ABSTRACT

We analytically address the non-equilibrium problem of a Brownian particle in contact with a thermal reservoir by means of a non-Gaussian Langevin noise term $\eta(t)$. The presence of noise kurtosis is akin to a second temperature reservoir acting on the system, and we exploit its consequences by means of studying a converging exact form for the stationary probability distribution.

1. Introduction

The Langevin equation is a useful tool for studying large classical systems where rapidly fluctuating forces are present [1]. These fluctuating forces are a consequence of the fast microscopic degrees of freedom of the system as initially explained by Einstein [2]. For most usual thermodynamic systems at equilibrium, these forces are quite well represented by white Gaussian noise, since they are the result of the coarse-graining of the actual quantum degrees of freedom [3]. Then, the Boltzmann–Gibbs (BG) distribution can be derived for general Hamiltonians [1,4–6].

However, for non usual noise (colored or not) the form of the stationary distribution is not BG and depends on the noise details [7–9], highlighting the essential role played by the properties of the Langevin force in the determination of the stationary state. In consequence, true equilibrium does not hold for general types of noise, but non-equilibrium stationary states might arise instead. In this case, characteristic functional approaches can be developed for non-Gaussian systems [10]. Many important systems fall into the category, such as is the case with Poisson noise, which possesses an infinite number of non-zero cumulants [3].

From a technological point of view, recent experiments show evidence of the presence of skewness in the distributions of currents for coherent quantum conductors induced by external magnetic fields [11–13]. This is a consequence of their far from equilibrium characteristics: electrical conductors stray away from the symmetries present at equilibrium and the associated equilibrium fluctuation relations are no longer valid [12]. These are hierarchies of fluctuation relations connecting the equilibrium value of the cumulants to linear and non-linear conductances and susceptibilities. They can be verified experimentally [13], confirming the non-Gaussian properties of these far from equilibrium settings. In fact, this is an example of a mechanism for building up cumulants of higher order than two. Similarly, for a thermal system far from equilibrium, the presence of kurtosis in the noise distribution shall induce the presence of kurtosis on the distribution of positions and velocities. Hence, we shall exploit some of the consequences of the presence of higher order cumulants on the noise distribution.

* Corresponding author. Tel.: +55 21 3114 1263; fax: +55 21 3114 1271.
E-mail address: welles@fis.puc-rio.br (W.A.M. Morgado).

0378-4371/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.physa.2012.03.012

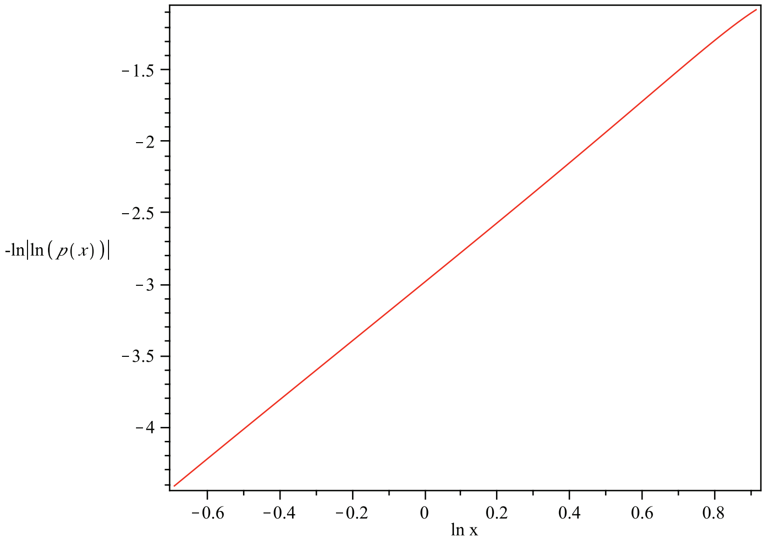

Fig. 1. In this plot, the position distribution is almost indistinguishable from a Gaussian for the low T_κ case. The parameter values are $k = 1.0$, $\gamma = 1.0$, $T = 0.5$, $T_\kappa = 1.5$, $m = 1.0$. This form of plotting the curve allows us to obtain the power for stretched exponentials as the angular coefficient of the curve. In this case it is 2.1, very close to a Gaussian, even for the case $T_\kappa/T = 3.0$.

We assume that there exists a non-equilibrium mechanism, breaking up the equilibrium symmetries and favoring the build up of higher order cumulant correlations in the bath, and that these non-Gaussian fluctuations will spill over the system (for instance, in Ref. [14] higher order cumulants arise for transferred particles, which corresponds to the heat for the present model). For simplicity sake, we choose a rather artificial form of symmetric noise which presents non-zero kurtosis, with small higher order cumulants since. According to Marcinkiewcz [15], the cumulant generating function cannot be a finite order polynomial, thus the kurtosis cannot be the highest non-zero cumulant and an infinite number of non-zero cumulants arise, such as for Poisson noise [16,17]. This type of distribution has fatter tails and smaller peaks than the normal distribution. We shall assume that the higher order cumulants are small and do not contribute significantly to the shape of the stationary distribution.

As we shall see, the presence of higher order noise cumulants (skewness and higher order ones) is equivalent to the coexistence of distinct temperatures, namely T and T_κ in the present case, respectively associated with the variance and the kurtosis of the noise. The equipartition theorem will not be affected, since T_κ will not arise from averages of quadratic energy terms. However, the presence of eventual non-quadratic energy terms would imply a feeding of energy into the system dependent on T and T_κ, as will be more clear later.

We will follow a method that is well suited to the obtaining of the exact stationary distribution for a linear Langevin or Langevin-like (non-Markovian) system [9,18], which has also been used to study, analytically, simple models for thermal conductance [19,20] and fluctuation theorems [21]. A recent article presents a treatment for Poisson noise, which itself presents infinite non-zero cumulants, and goes along similar lines to the current work [17]. However, in the present work we are able to obtain closed expressions for the probabilities, in terms of series expansions, instead of getting to know the cumulant generating function only [17]. The present model consists of a Brownian particle coupled to a harmonic potential, and a noise term with non-zero variance and kurtosis. The corresponding Langevin equation is then exactly solved via a time-averaging procedure. The main advantage of this method over Fokker–Planck approaches (FPE) is that, given the exact noise form (all of its cumulants), all the cumulants for the dynamical variables are calculated correctly, which can only be guaranteed for the first two cumulants in the case of the FPE. One of the interesting aspects of the present method is that we can deal quite easily with the presence of the mass of the Brownian particle, so all dynamical effects are taken into account.

Our stationary results can be expressed in terms of converging series, as can be seen in Fig. 1. For a range of values of the parameters the distributions can be numerically close to Gaussian ones. Another way of approaching this problem is by looking at the thermostat and the heat it injects into the system. This can also be obtained analytically and the results are in complete agreement with the stationary state ones, such as the equipartition theorem, which is still valid for the present model.

This paper is organized as follows. In Section 2, we present the model used. In Section 3, we develop the methodology for solving it via time-average. In Section 4, we discuss the results for the exact series for the distributions. In Section 5, we discuss the injection of energy into the system, and, in Section 6, we elaborate our conclusions.

2. Model

The kurtosis, defined commonly as $\gamma_2 \equiv \frac{\kappa_4}{\kappa_2^2}$, where κ_4 and κ_2 are the fourth and second order cumulants respectively, is a measure of the "bulging" of a distribution. The more peaked distributions will have higher kurtosis values (leptokurtic) whereas the broader ones will have smaller, or even negative, values for kurtosis (platykurtic). The kurtosis can also be an indicator of processes happening in biological systems: for some plants, solar illumination does not seem to affect the average speed of the gravitropic reaction but only its kurtosis [22]. Finance is another field where the presence of non-Gaussian fluctuations has been observed and studied carefully [23]. Simple models can illustrate the kind of distribution we shall use in the following.[1] For symmetric distributions we can write

$$\gamma_2 = \frac{\kappa_4}{\kappa_2^2} \equiv \frac{\mu_4}{\sigma^4} - 3, \tag{1}$$

where μ_4 is the fourth moment of the distribution, and σ^2 its variance. In our model, the only non-zero cumulants for the noise will be variance and kurtosis (see discussion below).

2.1. Langevin equation for non-Gaussian noise

We shall model a Markovian type of noise, with interactions between the Brownian particle and the bath, so that there is no memory function [24], that can be expressed by

$$\dot{x}(t) = v(t), \tag{2}$$

$$m\,\dot{v}(t) = -kx(t) - \gamma\, v(t) + \eta(t). \tag{3}$$

The main cumulants for the noise distribution are only variance and kurtosis. This is a rather strong approximation but it is one that is quite straightforward to implement [17]. However, it is known that a distribution possessing only kurtosis and variance for cumulants cannot exist [15], since only Gaussians can have characteristic functions which are exponentials of finite order polynomials. What we propose is to construct a rather artificial form of a distribution that has small cumulants of order higher than the kurtosis.[2] Then, the ensuing characteristic function will be controlled by the kurtosis.

The form for the noise correlations we shall use is given by:

$$\langle \eta(t_1)\rangle_c = 0, \tag{4}$$

$$\langle \eta(t_1)\eta(t_2)\rangle_c = 2\,\gamma\, T\,\delta(t_1 - t_2), \tag{5}$$

$$\langle \eta(t_1)\eta(t_2)\eta(t_3)\rangle_c = 0, \tag{6}$$

$$\langle \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4)\rangle_c = A_{(c4)}\,\delta(t_1 - t_2)\delta(t_2 - t_3)\delta(t_3 - t_4), \tag{7}$$

where we ignore the higher order cumulants. For pure Gaussian distributions, we have $A_{(c4)} = 0$ (mesokurtic). Leptokurtic distributions have $A_{(c4)} > 0$, while for platykurtic distributions $A_{(c4)} < 0$. In the following, we will assume a platykurtic distribution defining

$$A_{(c4)} = -\frac{4\,m\,\left(3\,\gamma^2 + 4\,km\right)}{\gamma}\,T_\kappa^2, \tag{8}$$

since for a leptokurtic distribution, the higher order cumulants, the time-averaged distribution series diverge: a positive cumulant of order $2n > 4$ will be necessary to make the integration of the cumulant generating functions (see Eq. (23)) convergent.

The main reason for the divergence comes from the fat tails, associated with the leptokurtic case. In order to obtain those, the kurtosis has to be positive but needs non-zero higher order cumulants to prevent the divergence of the integrand, due to $\exp\left\{\gamma_2\, q^4/4!\right\}$ when $q \to \infty$. Two leptokurtic cases are illustrative: the (well behaved) Poisson distribution, with an infinite number of positive cumulants (including kurtosis) and the (not so well behaved) Cauchy distribution, which presents fat tails. For the Poisson distribution, the non-trivial higher order cumulants assure the convergence of the probability distribution. For the Cauchy case, the characteristic function is non-analytic and the cumulants cannot be defined. So, the present model will be appropriate for platykurtic distributions only.

In what follows, the results obtained are to be interpreted in the sense that they are approximations to the behavior of a distribution that possesses small but non-zero higher order cumulants.

---

1 As an example, let us discretize time into intervals of size ε, and define a on–off noise in those intervals. The on probability is given by θ while the off case is given by $1-\theta$. When the noise ϕ is on, it can assume two values: $+\varphi_0$ and $-\varphi_0$ with probabilities p and $1-p$, respectively. It is easy to show that the interval averages are given by $\langle \phi^{2n+1}\rangle = \theta\,\varphi_0^{2n+1}\,(2p-1)$, $\langle \phi^{2n}\rangle = \theta\,\varphi_0^{2n}$, where $n = 0, 1, 2, \ldots$. The kurtosis γ_2 can be obtained as

$$\gamma_2 = -\frac{16\,p\,(p-1)\left(-6\,\theta^2 p\,(1-p) + 1 - 3\,\theta + 3\,\theta^2\right)}{\left(-1 - 4\,\theta\,p\,(1-p) + \theta\right)^2} - \frac{(\theta-1)\left(6\,\theta^2 - 6\,\theta + 1\right)}{\theta\,\left(-1 - 4\,\theta\,p\,(1-p) + \theta\right)^2},$$

and can assume positive or negative values according to the values of θ and p. In fact, for the symmetrical case, $p = 1/2$, we can have $-2 \le \gamma_2 < 0$.

2 A specific example is given by cumulants such as $\kappa_n < \varepsilon\, a^n$ yielding cumulant generating functions which are smaller than $\sum_{n=3}^{\infty} \frac{(-ik)}{n!}\,\varepsilon\, a^n = \varepsilon\cos(ak) - \varepsilon\left(1 - \frac{a^2k^2}{2}\right) < 2\varepsilon$, which is convergent and small as $\varepsilon \to 0$.

2.2. Laplace transformations

Since the effect of the initial conditions rapidly vanish [9], without loss of generality, we may assume that $x(t=0)=v(t=0)=0$. By taking the Laplace transformations of Eqs. (2) and (3) (with $\mathrm{Re}(s)>0$), we obtain

$$s\,\tilde{x}(s)=\tilde{v}(s) \tag{9}$$

and

$$\tilde{x}(s)=\frac{\tilde{\eta}(s)}{G(s)}, \tag{10}$$

where $G(s)\equiv m\,s^2+\gamma s+k=m(s-\kappa_+)(s-\kappa_-)$. The zeros of $G(s)$ are

$$\kappa_\pm=-\frac{\gamma}{2m}\pm \mathrm{i}\sqrt{\frac{k}{m}-\frac{\gamma^2}{4m^2}}.$$

The Laplace transforms for the non-zero noise cumulants are

$$\langle\tilde{\eta}(s_1)\tilde{\eta}(s_2)\rangle_c=\frac{2\,\gamma\,T}{s_1+s_2}, \tag{11}$$

$$\langle\tilde{\eta}(s_1)\tilde{\eta}(s_2)\tilde{\eta}(s_3)\tilde{\eta}(s_4)\rangle_c=-\frac{4\,m\,\left(3\,\gamma^2+4\,k_0 m\right)}{\gamma}\,\frac{T_\kappa^2}{s_1+s_2+s_3+s_4}. \tag{12}$$

2.3. Second temperature

The main effect of kurtosis is to introduce a new temperature-like variable T_κ into play. This gives rise to an energy feeding mechanism, a bit distinct from the usual equilibrium thermal injection, a mechanism that operates through higher order correlations. For the specific linear model under analysis, the T_κ factor will not affect the energy averages, but will make the probability distribution non-Gaussian. However, any nonlinearities of the potential energy would lead to a steady-state average energy dependent on T_κ. It is easily seen that for a quartic potential such as $\phi=\frac{1}{2}k_3x^4$ we have,

$$\langle\phi\rangle=\frac{1}{2}k_3\left\langle x^4\right\rangle_c+\frac{3}{2}k_3\left\langle x^2\right\rangle_c^2=\frac{1}{2}\,k_3\,\sigma^4\,(\gamma_2+3),$$

where the non-equilibrium contribution ($\gamma_2\equiv\gamma_2(T_\kappa)$) is evident. In the following we study the stationary properties of the probability distributions.

Distinctively from the equilibrium case, a single non-equilibrium reservoir might keep a system at a non-equilibrium stationary state (NESS) without any net energy flux or entropy production. Indeed, to keep a system at a NESS usually involves at least two reservoirs at distinct temperatures with the consequence of non-zero energy fluxes and entropy production. On the other hand, for the non-equilibrium reservoir, the entropy production processes must be internal, and not being due to the interaction with the system when the last is at a NESS.

3. Time averaging

The stationary distribution can be written as a function of products of the Laplace transforms of the variables x and v [9,18,19,21]. Due to the linearity of the model, the solution is obtained by simple complex plane integrations over the correct poles.

The time-averaged expression for the steady-state probability distribution is thus (see Appendix A and Refs. [9,18,19,21]):

$$p^{ss}(x,v)=\lim_{z\to0}\lim_{\epsilon\to0}\sum_{l,m=0}^{\infty}\int_{-\infty}^{+\infty}\frac{\mathrm{d}Q}{2\pi}\frac{\mathrm{d}P}{2\pi}\mathrm{e}^{\mathrm{i}Qx+\mathrm{i}Pv}\frac{(-\mathrm{i}Q)^l}{l!}\frac{(-\mathrm{i}P)^m}{m!}$$

$$\times\int_{-\infty}^{+\infty}\prod_{h=1}^{m+l}\frac{\mathrm{d}p_h}{2\pi}\,\frac{z}{z-\left[\sum_{h=1}^{m+l}\mathrm{i}p_h+(l+m)\epsilon\right]}\,\frac{\prod_{h=1}^{m}(\mathrm{i}p_h+\epsilon)}{\prod_{k=1}^{m+l}G(\mathrm{i}p_h+\epsilon)}\left\langle\prod_{h=1}^{m+l}\tilde{\eta}(\mathrm{i}p_h+\epsilon)\right\rangle. \tag{13}$$

Some typical derivations, necessary to transform Eq. (13) into a final stationary form, are done in the following.

Let us define

$$I_{l+m}^{(l,m)}=\lim_{z\to0}\lim_{\epsilon\to0}\int_{-\infty}^{+\infty}\prod_{h=1}^{m+l}\frac{\mathrm{d}p_h}{2\pi}\,\frac{z}{z-\left[\sum_{h=1}^{m+l}\mathrm{i}p_h+(l+m)\epsilon\right]}\,\frac{\prod_{h=1}^{m}(\mathrm{i}p_h+\epsilon)}{\prod_{k=1}^{m+l}G(\mathrm{i}p_h+\epsilon)}\left\langle\prod_{h=1}^{m+l}\tilde{\eta}(\mathrm{i}p_h+\epsilon)\right\rangle_c, \tag{14}$$

where only the terms that make the p's in the denominator of $I(z) = z/\left(z - \left[\sum_{h=1}^{m+l} \mathrm{i}p_h + (l+m)\epsilon\right]\right)$ vanish shall contribute to the final result, since a non-zero residue will make $I(z)$ vanish as $z \to 0$ [9].

We can, by manipulating the products of averages of the noise in Eq. (13), rewrite Eq. (13) as a product of $I_{l+m}^{(l,m)}$ above. We see that the typical contributions are two-fold, since they come from the two non-zero cumulants: due to grouping the η's in sets of two (related to the variance and giving rise to integrals of type I_2) or groups of four (related to the kurtosis and giving rise to integrals of type I_4). The non-zero contributions are:

$$I_2^{(2,0)} = \frac{T}{k}, \tag{15}$$

$$I_2^{(0,2)} = \frac{T}{m}. \tag{16}$$

Similarly, we have:

$$I_4^{(4,0)} = -\frac{3\,{T_\kappa}^2 m}{k\,\gamma^2}, \tag{17}$$

$$I_4^{(2,2)} = -\frac{{T_\kappa}^2}{\gamma^2}, \tag{18}$$

$$I_4^{(1,3)} = -\frac{{T_\kappa}^2}{\gamma\, m}, \tag{19}$$

$$I_4^{(0,4)} = -\frac{3\,{T_\kappa}^2\left(\gamma^2 + k_0 m\right)}{m^2\,\gamma^2}. \tag{20}$$

4. Probability distribution expressions

4.1. Position distribution

4.1.1. Pure Gaussian case: $T_\kappa = 0$

For the pure Gaussian case, by residue integrating the pairs of p's, we get rid of all terms in $I(z) = z/(z - [\sum_{h=1}^{m+l} \mathrm{i}p_h + (l+m)\epsilon])$ and then $\lim_{z\to 0} z/z = 1$. There are $\frac{(2n)!}{2^n\, n!}$ ways of arranging $2n$ indices 2 by 2, which is the case when kurtosis is zero. By doing all the integrations and applying the correct multiplicity factors we obtain

$$\begin{aligned} p^{ss}(x,v) &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}Q}{2\pi}\frac{\mathrm{d}P}{2\pi} \mathrm{e}^{\mathrm{i}Qx+\mathrm{i}Pv} \sum_{l=0}^{\infty}\left[\frac{1}{l!}\left(-\frac{Q^2T}{2k}\right)^l\right] \sum_{m=0}^{\infty}\left[\frac{1}{m!}\left(-\frac{P^2T}{2m}\right)^m\right] \\ &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}Q}{2\pi}\frac{\mathrm{d}P}{2\pi} \exp\left(\mathrm{i}Qx - \frac{Q^2T}{2k}\right)\exp\left(\mathrm{i}Pv - \frac{P^2T}{2m}\right). \end{aligned} \tag{21}$$

Completing the squares and integrating we obtain

$$p^{ss}(x,v) = \sqrt{\frac{k}{2\,\pi\,T}}\sqrt{\frac{m}{2\,\pi\,T}} \exp\left(-\frac{k\,x^2}{2T}\right)\exp\left(-\frac{m\,v^2}{2T}\right). \tag{22}$$

This is the Boltzmann expression. The presence of kurtosis will alter this simple picture.

4.1.2. Position distribution: The kurtosis effect

The kurtosis can be expressed directly by[3]

$$p^{ss}(x) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}Q}{2\pi} \exp\left\{\mathrm{i}Qx - Q^2\frac{I_2^{(2,0)}}{2} + Q^4\frac{I_4^{(4,0)}}{4!}\right\}.$$

---

[3] The integral expression can also be obtained by direct summation as follows. The integrations for the kurtosis derived contributions will lead to products of $I_2^{(2,0)}$ and $I_4^{(4,0)}$. For the cases of $4n$, and $4n+2$, terms, we can accommodate them into up to n 4-uples, the remaining in couples. For instance, the number of ways of doing it, for a choice of l $(0 \le l \le n)$ are either

$$\frac{(4n+2)!}{(4l)!(4n+2-4l)!}\,\frac{(4l)!}{(4!)^l l!}\,\frac{(4n+2-4l)!}{2^{2n+1-2l}(2n+1-2l)!} = \frac{(4n+2)!}{(4!)^l\, l!\, 2^{2n+1-2l}(2n+1-2l)!},$$

or

$$\frac{(4n)!}{(4l)!(4n-4l)!}\,\frac{(4l)!}{(4!)^l l!}\,\frac{(4n-4l)!}{2^{2n-2l}(2n-2l)!} = \frac{(4n)!}{(4!)^l\, l!\, 2^{2n-2l}(2n-2l)!}.$$

The integral above diverges if $I_4^{(4,0)} > 0$ (leptokurtic). The only solutions are to be found for sub-Gaussian (platykurtic) distributions $I_4^{(4,0)} \leq 0 \Rightarrow A_{(c4)} \leq 0$. In consequence:

$$p^{ss}(x) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}Q}{2\pi} \exp\left\{ \mathrm{i}Qx - Q^2 \frac{T}{k} - Q^4 \frac{{T_\kappa}^2 m}{8\,k\,\gamma^2} \right\}. \tag{23}$$

The generating function $G(k) = \left\langle \mathrm{e}^{-\mathrm{i}kx} \right\rangle$ is given by

$$G(k) = \exp\left\{ -k^2 \frac{T}{k} - k^4 \frac{{T_\kappa}^2 m}{8\,k\,\gamma^2} \right\} \Rightarrow \langle x^2 \rangle_c = \frac{2T}{k}; \qquad \langle x^4 \rangle_c = -\frac{3\,{T_\kappa}^2 m}{8\,k\,\gamma^2}.$$

The presence of non-zero kurtosis on the noise induces kurtosis on the position distribution.

4.2. Series expansion for $p^{ss}(x)$

4.2.1. Small T

A series expansion for $p^{ss}(x)$ can be obtained by expanding Eq. (23) as follows (for $T_\kappa \neq 0$)

$$\begin{aligned} p^{ss}(x) &= \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} \frac{\mathrm{d}Q}{2\pi} \frac{(-Q^2 x^2)^n}{(2n)!} \exp\left\{ -Q^2 \frac{T}{k} - Q^4 \frac{{T_\kappa}^2 m}{8\,k\,\gamma^2} \right\} \\ &= \sum_{n=0}^{\infty} \sum_{o=0}^{\infty} \frac{(-x^2)^n}{(2n)!} \frac{\left(-\frac{T}{k}\right)^o}{o!} \int_{-\infty}^{+\infty} \frac{\mathrm{d}Q}{2\pi} Q^{2n+2o} \exp\left\{ -Q^4 \frac{{T_\kappa}^2 m}{8\,k\,\gamma^2} \right\}. \end{aligned}$$

For $T_\kappa = 0$ the Boltzmann–Gibbs case is recovered by direct integration of the generating function.

On the other hand, by integrating on Q when $T_\kappa \neq 0$, the convergent series for $p^{ss}(x)$ reads

$$p^{ss}(x) = \frac{1}{4\pi} \sqrt{\frac{k}{T_\kappa} \sqrt{\frac{8\,\gamma^2}{k\,m}}} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{i+j}{2} + \frac{1}{4}\right)}{\Gamma(j+1)\,\Gamma(2i+1)} \left(-\frac{k\,x^2}{T_\kappa} \sqrt{\frac{8\,\gamma^2}{k\,m}} \right)^i \left(-\frac{T}{T_\kappa} \sqrt{\frac{8\,\gamma^2}{k\,m}} \right)^j. \tag{24}$$

An interesting limit is $T/T_\kappa \to 0$: Eq. (24) then reads[4]

$$\begin{aligned} p^{ss}_{T_\kappa}(x) &= \frac{1}{4\pi} \sqrt{\frac{k}{T_\kappa} \sqrt{\frac{8\,\gamma^2}{k\,m}}} \sum_{i=0}^{\infty} \frac{\Gamma\left(\frac{2i+1}{4}\right)}{\Gamma(2i+1)} \left(-\frac{k\,x^2}{T_\kappa} \sqrt{\frac{8\,\gamma^2}{k\,m}} \right)^i \\ &= -1/4\,k\sqrt{2}\gamma \sqrt{\frac{1}{km}} x^2 {}_0F_2\left([\,,\,]; [5/4, 3/2]; \frac{1}{128}\, x^4 k\sqrt{2}\gamma \sqrt{\frac{1}{km}} {T_\kappa}^{-1} \right) \Gamma\,(3/4)\, {T_\kappa}^{-1} \pi^{-1} \\ &\quad + 1/2 \sqrt{k\sqrt{2}\gamma \sqrt{\frac{1}{km}} {T_\kappa}^{-1}}\, {}_0F_2\left([\,,\,]; [1/2, 3/4]; \frac{1}{128}\, x^4 k\sqrt{2}\gamma \sqrt{\frac{1}{km}} {T_\kappa}^{-1} \right) (\Gamma\,(3/4))^{-1}, \end{aligned} \tag{25}$$

where ${}_0F_2(n; d; z)$ is a generalized hypergeometric function.[5] More information can be obtained in Ref. [25].

The limit above is clearly not Gaussian[6] but for small x there is some similarity between them. Physically, we notice that, in this pure non-equilibrium limit, the second temperature T_κ behaves as a usual temperature, renormalized by factors of $\sqrt{\frac{m\,k}{\gamma^2}}$. Such distribution is, as we have seen, similar to a Gaussian for small x.

---

4 Calculations with the help of Maple.

5 The generalized hypergeometric function is defined upon the Pochhammer symbol

$${}_pF_q(n; d; z) = \sum_{k=0}^{\infty} z^k/k! * \prod_{i=1}^{p \equiv \mathrm{nops}(n)} \mathrm{pochhammer}(n[i], k) \Bigg/ \prod_{j=1}^{q \equiv \mathrm{nops}(d)} \mathrm{pochhammer}(d[j], k),$$

where nops is the number of terms in the brackets, and

$$\mathrm{pochhammer}(d[j], k) = \frac{\Gamma(k + d[j])}{\Gamma(d[j])}.$$

6 Gaussian

$$\mathrm{e}^{-Ax^2} = \sum_{i=0}^{\infty} \frac{1}{\Gamma(i+1)} \left(-A\,x^2\right)^i.$$

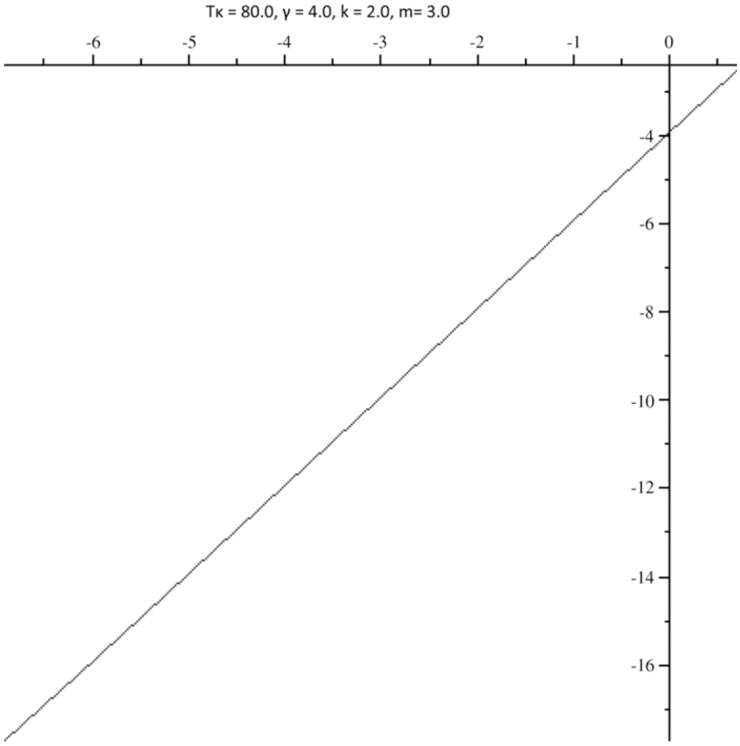

Fig. 2. In this plot, the parameter values are $k = 2.0$, $\gamma = 4.0$, $m = 3.0$, $T = 0.0$, $T_\kappa = 80.0$, $m = 1.0$. The form is nearly Gaussian.

By plotting the expression above for a typical range of parameters corresponding to the case of Eq. (25), see Fig. 1, we observe that the function is very close to a Gaussian despite the large relative value of T_κ/T. Even for the extreme case ($T = 0$), for small values of x, the distribution also does no differ much from a Gaussian (see Fig. 2).

4.2.2. Small T_κ

Following similar lines to the result above, we may obtain a small T_κ expansion series for $p^{ss}(x)$, which reads

$$p^{ss}(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2n+4m)!}{m!(2n)!(n+2m)!}\left(-\frac{kx^2}{2T}\right)^n \left(\frac{k}{2\pi T}\right)^{1/2} \left(-\left(\frac{T_\kappa}{T}\right)^2 \frac{m\,k}{768\,\gamma^2}\right)^m. \tag{26}$$

For $T_\kappa = 0$ the Boltzmann–Gibbs case is recovered.

4.3. Expected energy values

The average for the energy is given by

$$E = \left\langle \frac{mv^2}{2} \right\rangle + \left\langle \frac{kx^2}{2} \right\rangle = \frac{m}{2} I_2^{(0,2)} + \frac{k}{2} I_2^{(2,0)} = T.$$

The presence of kurtosis does not affect the relationship between the temperature (second cumulant of the noise) and the average energy. The equipartition theorem is still valid in this case. However, the fluctuations of the energy, and the specific heat, would nevertheless depend on the kurtosis.

4.4. Expression for the joint distribution

The expression for the exact probability distribution function series has no closed form. However, we can write it as a convergent series. The calculations are shown in Appendix B. The complete expression reads after replacing the integrals type I_4:

$$p^{ss}(x, v) = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{e=0}^{\infty}\sum_{f=0}^{\infty} \frac{1}{16\pi^2} \frac{\Gamma\left(\frac{a+c+d+f}{2}+\frac{1}{4}\right)\Gamma\left(\frac{b+d+e+3f}{2}+\frac{1}{4}\right)}{(2a)!\,(2b)!\,c!\,d!\,e!\,(2f)!}$$
$$\times \frac{\left(-x^2\right)^a \left(-v^2\right)^b \left(-\frac{T}{2k}\right)^c \left(-\frac{1}{4\gamma^2}\right)^d \left(-\frac{T}{2m}\right)^e \left(\frac{1}{6\gamma m}\right)^{2f}}{\left(T_k^2\right)^{\frac{a+b+c+e}{2}+\frac{1}{2}} \left(\frac{m}{8k\gamma^2}\right)^{\frac{a+c+d+f}{2}+\frac{1}{4}} \left(\frac{(\gamma^2+k_0 m)}{8m^2\gamma^2}\right)^{\frac{b+d+e+3f}{2}+\frac{1}{4}}}$$

$$+\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{e=0}^{\infty}\sum_{f=0}^{\infty}\frac{1}{16\pi^2}\frac{\Gamma\left(\frac{a+c+d+f}{2}+\frac{3}{4}\right)\Gamma\left(\frac{b+d+e+3f}{2}+\frac{5}{4}\right)}{(2a+1)!\,(2b+1)!\,c!\,d!\,e!\,(2f+1)!}$$

$$\times\frac{(-x^2)^a\,(xv)\,(-v^2)^b\left(-\frac{T}{2k}\right)^c\left(-\frac{T_\kappa{}^2}{4\gamma^2}\right)^d\left(-\frac{T}{2m}\right)^e\left(\frac{T_\kappa{}^2}{6\,\gamma\,m}\right)^{2f+1}}{\left(\frac{T_\kappa{}^2 m}{8\,k\,\gamma^2}\right)^{\frac{a+c+d+f+1}{2}+\frac{1}{4}}\left(\frac{T_\kappa{}^2(\gamma^2+k_0 m)}{8\,m^2\,\gamma^2}\right)^{\frac{b+d+e+3f+2}{2}+\frac{1}{4}}}. \quad (27)$$

5. Injection and dissipation of energy

Let us study the heat injected into the system by means of the kurtosis producing non-equilibrium thermal bath of the previous section.

Heat is composed of two contributions: an injected term J_{IT} and a dissipative term J_{DT}. Let us consider the case $x_0 = v_0 = 0$. The general form of heat injection reads

$$J_T = \int_0^\tau \mathrm{d}t\, v(t)\,\left(\eta(t) - \gamma\, v(t)\right), \quad (28)$$

where we integrate the instantaneous power $P(t)$, i.e.,

$$P(t) = F(t)\,v(t),$$

where the resultant force over the particle is $F(t) = \eta(t) - \gamma\, v(t)$. Thus, two terms arise: an injected (i.e., stochastically taken from the reservoir) power term $\eta(t)\,v(t)$ and a dissipative power (i.e., leaked back to the reservoir) term $-\gamma\, v(t)$.

Let us observe that (we assume without loss of generality $v(0) = 0$):

$$\begin{aligned} J_T(\tau) &= \int_0^\tau \mathrm{d}t\, v(t)\,\left(\eta(t) - \gamma\, v(t)\right) \\ &= \int_0^\tau \mathrm{d}t\, v(t)\left(m\,\dot{v}(t) + \frac{\partial V(x)}{\partial x}\right) \\ &= \left[\frac{1}{2} m\, v^2(\tau) + V(x(\tau))\right] \\ &= E(\tau). \end{aligned} \quad (29)$$

The interpretation of the above integrations should be done in the Stratonovich way [1]. This way the integrals are consistent and calculable [17].

Thus, taking the thermal average of J_T leads to the average energy E in the limit $\tau \to \infty$:

$$E = \lim_{\tau\to\infty}\left(\frac{1}{2} m\,\left\langle v^2(\tau)\right\rangle + \langle V(x(\tau))\rangle\right). \quad (30)$$

In Eq. (29) we have assumed a general form for the potential the Brownian particle is bounded to. In the remaining, let us come back to the previous sections potential $V(x) = 1/2kx^2$:

$$E = \lim_{\tau\to\infty}\left(\frac{1}{2} m\,\left\langle v^2\right\rangle + \frac{1}{2} k\left\langle x^2\right\rangle\right), \quad (31)$$

which according to the equipartition theorem will be equal to T.

5.1. Injection contribution

The injection of energy can be written ($R(s) = s^2 + \theta\, s + \omega^2 = G(s)/m$)

$$J_{IT0} = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_2}{2\pi}\,\frac{\mathrm{e}^{(\mathrm{i}\,q_1+\mathrm{i}\,q_2+2\epsilon)\tau}-1}{(\mathrm{i}\,q_1+\mathrm{i}\,q_2+2\epsilon)}\,\frac{(\mathrm{i}\,q_1+\epsilon)\,\tilde{\eta}(\mathrm{i}\,q_1+\epsilon)\tilde{\eta}(\mathrm{i}\,q_2+\epsilon)}{m\,R(\mathrm{i}\,q_1+\epsilon)}, \quad (32)$$

where after taking the thermal average, we get

$$\langle J_{IT0}\rangle = \left(\frac{2\,\gamma\, T}{m}\right)\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_2}{2\pi}\,\frac{\mathrm{e}^{(\mathrm{i}\,q_1+\mathrm{i}\,q_2+2\epsilon)\tau}-1}{(\mathrm{i}\,q_1+\mathrm{i}\,q_2+2\epsilon)^2}\,\frac{(\mathrm{i}\,q_1+\epsilon)}{R(\mathrm{i}\,q_1+\epsilon)} = \frac{\gamma\, T\tau}{m}, \quad (33)$$

where a factor $1/2$ comes into play due to the non-vanishing integration of q_1 on the semi-circle.

5.2. Dissipation contribution

The dissipative term is:

$$J_{DT0} = -\gamma \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_2}{2\pi}\,\frac{\mathrm{e}^{(\mathrm{i}\,q_1+\mathrm{i}\,q_2+2\epsilon)\tau}-1}{(\mathrm{i}\,q_1+\mathrm{i}\,q_2+2\epsilon)}\,\frac{(\mathrm{i}\,q_1+\epsilon)\,\tilde{\eta}(\mathrm{i}\,q_1+\epsilon)}{m\,R(\mathrm{i}\,q_1+\epsilon)}\,\frac{(\mathrm{i}\,q_2+\epsilon)\,\tilde{\eta}(\mathrm{i}\,q_2+\epsilon)}{m\,R(\mathrm{i}\,q_2+\epsilon)}. \tag{34}$$

It becomes, after taking the thermal average,

$$\begin{aligned}\langle J_{DT0}\rangle &= \left(-\frac{\gamma}{m^2}\right)\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_2}{2\pi}\,\frac{\mathrm{e}^{(\mathrm{i}\,q_1+\mathrm{i}\,q_2+2\epsilon)\tau}-1}{(\mathrm{i}\,q_1+\mathrm{i}\,q_2+2\epsilon)}\,\frac{(\mathrm{i}\,q_1+\epsilon)}{R(\mathrm{i}\,q_1+\epsilon)}\,\frac{(\mathrm{i}\,q_2+\epsilon)}{R(\mathrm{i}\,q_2+\epsilon)}\left[\frac{2\gamma\,T}{\mathrm{i}\,q_1+\mathrm{i}\,q_2+2\,\epsilon}\right]\\ &= -\frac{\gamma\,T\tau}{m}-\frac{T\left(\theta^2\mathrm{e}^{-2\theta\,\tau}\cos\left(2\sqrt{\omega^2-\theta^2}\tau\right)-\theta^2+\omega^2-\mathrm{e}^{-2\theta\,\tau}\omega^2\right)}{-\omega^2+\theta^2}.\end{aligned} \tag{35}$$

Asymptotically, since the second right hand term reduces to T as the exponentials vanish as $\tau\to\infty$, we have:

$$\langle J_{DT0}\rangle\,(\tau) = -\frac{\gamma\,T\tau}{m}+T. \tag{36}$$

5.3. Heat and entropy production

Thus, the asymptotic value for the total heat injected, $Q = \lim_{\tau\to\infty}(\langle J_{IT0}\rangle+\langle J_{DT0}\rangle)$, that takes the system from $x_0 = v_0 = 0 = E_0$ to the final energy is T, consistent with $\lim_{\tau\to\infty}E(\tau) = T$ above.

The rate above is nothing but the entropy production rate J_s for the model [26]:

$$\begin{aligned}J_s &= \lim_{\tau\to\infty}\frac{1}{\tau}\frac{\langle J_{IT0}\rangle+\langle J_{DT0}\rangle}{T}\\ &= \lim_{\tau\to\infty}\frac{\left(\theta^2\mathrm{e}^{-2\theta\,\tau}\cos\left(2\sqrt{\omega^2-\theta^2}\tau\right)-\theta^2+\omega^2-\mathrm{e}^{-2\theta\,\tau}\omega^2\right)}{\tau\,(\omega^2+\theta^2)} = 0,\end{aligned} \tag{37}$$

which tends to zero as the stationary state is approached and the total entropy produced is finite. This a non-usual non-equilibrium situation, since entropy is continuously produced only inside the thermal reservoir which interacts with the particle via the non-Gaussian noise. That production is the cost of maintaining the reservoir in a non-equilibrium state.

6. Conclusions

We have studied a simple case of a linear system under the influence of a clearly non-equilibrium noise with significant kurtosis. The stationary behavior of the system can be obtained, yielding non-zero kurtosis for the dynamical variables of the system. It is important to stress out that each noise cumulant gives rise only to dynamical variables' cumulants of the same order, as can be seen in Eq. (B.2). In this work, due to Marcinkiewcz [15] we know that the presence of nonzero kurtosis implies that an infinite number of higher order cumulants must also exist, such as the case of Poisson distributions [17].

One of the goals of the present work is to investigate the role played by such non usual cumulants into the energy processes of the system. We characterize their role in the distributions and energy exchanges between system and reservoir.

Within our approximations we were able to express the stationary distribution by means of convergent approximative series. At some simple limiting cases, such as $p^{ss}(x)$ for $T_\kappa/T\to 0$, the series are shown to be expressible as closed expressions involving the hypergeometric function. Unfortunately, this result does not hold for the general case.

Finally, the energy injection mechanism is then analyzed and shown to be consistent with the stationary properties in the present treatment for the noise. The kurtosis term does not influence the injected energy due to the linear character of the present model. An interesting consequence appears for models with non-linear (harmonic) force, namely the kurtosis present in the noise acts as a new thermal reservoir! However, even for quadratic potentials the kurtosis will affect the shape of the final probability distribution, without changing the average energy of the system.

In fact, the average of quartic terms in the potential energy has non-Gaussian components whenever the noise presents kurtosis, i.e., components dependent on T_κ. Indeed, the energy injection term, Eq. (28), via a non-linear coupling between the Laplace transforms of the dynamical variables and those of the noise, will present a term that depends on T_κ.

The main difference between this type of non-equilibrium settings and more traditional ones, such as subjecting a system to contacts with thermal reservoirs with distinct temperatures is that the reservoir generating kurtosis (or any higher order noise cumulant for that matter) is itself an out of equilibrium reservoir. Besides, it only acts upon the non-quadratic energy degrees of freedom for the system, whereas for the quadratic ones only the usual temperature is relevant.

Acknowledgment

The first author was partially supported by FAPERJ and CNPq, Brazil.

Appendix A. Stationary probability distribution function

The definition for the instantaneous probability distribution [9,18,19]:

$$
\begin{aligned}
p(x,v,t) &= \langle \delta(x-x(t))\delta(v-v(t))\rangle \\
&= \int_{-\infty}^{+\infty}\frac{dQ}{2\pi}e^{iQx}\int_{-\infty}^{+\infty}\frac{dP}{2\pi}e^{iPv}\sum_{l=0}^{\infty}\frac{(-iQ)^l}{l!}\sum_{m=0}^{\infty}\frac{(-iP)^m}{m!} \\
&\quad\times\int_0^{\infty}\prod_{f=1}^{l}dt_{lf}\int_0^{\infty}\prod_{h=1}^{m}dt_{mh}\delta(t-t_{la})\delta(t-t_{mb})\left\langle\prod_{f=1}^{l}x(t_{lf})\prod_{h=1}^{m}v(t_{mh})\right\rangle \\
&= \lim_{\epsilon\to0}\int_{-\infty}^{+\infty}\frac{dQ}{2\pi}e^{iQx}\int_{-\infty}^{+\infty}\frac{dP}{2\pi}e^{iPv}\sum_{l=0}^{\infty}\frac{(-iQ)^l}{l!}\sum_{m=0}^{\infty}\frac{(-iP)^m}{m!}\int_{-\infty}^{+\infty}\prod_{f=1}^{l}\frac{dq_f}{2\pi}\prod_{h=1}^{m}\frac{dp_h}{2\pi} \\
&\quad\times\int_0^{\infty}\prod_{f=1}^{l}dt_{lf}\int_0^{\infty}\prod_{h=1}^{m}dt_{mh}e^{\sum_{a=1}^{l}(t-t_{la})(iq_a+\epsilon)+\sum_{b=1}^{m}(t-t_{mb})(ip_b+\epsilon)}\left\langle\prod_{f=1}^{l}x(t_{lf})\prod_{h=1}^{m}v(t_{mh})\right\rangle \\
p^{ss}(x,v) &= \lim_{z\to0}z\int_0^{\infty}dt\,e^{-zt}\,p(x,v,t) \\
&= \lim_{z\to0}\lim_{\epsilon\to0}\sum_{l,m=0}^{\infty}\int_{-\infty}^{+\infty}\frac{dQ}{2\pi}\frac{dP}{2\pi}e^{iQx+iPv}\frac{(-iQ)^l}{l!}\frac{(-iP)^m}{m!} \\
&\quad\times\int_{-\infty}^{+\infty}\prod_{h=1}^{m+l}\frac{dp_h}{2\pi}\frac{z}{z-\left[\sum_{h=1}^{m+l}ip_h+(l+m)\epsilon\right]}\left\langle\prod_{h=1}^{m}(ip_h+\epsilon)\,\tilde{x}(ip_h+\epsilon)\prod_{h=m+1}^{m+l}\tilde{x}(ip_h+\epsilon)\right\rangle \\
&= \lim_{z\to0}\lim_{\epsilon\to0}\sum_{l,m=0}^{\infty}\int_{-\infty}^{+\infty}\frac{dQ}{2\pi}\frac{dP}{2\pi}e^{iQx+iPv}\frac{(-iQ)^l}{l!}\frac{(-iP)^m}{m!} \\
&\quad\times\int_{-\infty}^{+\infty}\prod_{h=1}^{m+l}\frac{dp_h}{2\pi}\frac{z}{z-\left[\sum_{h=1}^{m+l}ip_h+(l+m)\epsilon\right]}\left\langle\prod_{h=1}^{m}\frac{(ip_h+\epsilon)\tilde{\eta}(ip_h+\epsilon)}{G(ip_h+\epsilon)}\prod_{k=m+1}^{m+l}\frac{\tilde{\eta}(ip_k+\epsilon)}{G(ip_k+\epsilon)}\right\rangle,
\end{aligned}
$$

where the t-integration defines the integration path, an infinite semi-circle on the top complex plane, and the expression of Eq. (13) is readily obtained.

Appendix B. Complete series

It is quite straightforward to see that the integral $I_{l+m}^{(l,m)}$ in Eq. (14) is exactly the cumulant $\langle x^l v^m\rangle_c$ [17], since in Eq. (13) the similar integral (corresponding to the momenta of the noise Laplace transforms, and not their cumulants) corresponds to the average $\langle x^l v^m\rangle$. The form for the stationary distribution reads ($I_4^{(4,1)}=0$):

$$
\begin{aligned}
p^{ss}(x,v) &= \int_{-\infty}^{+\infty}\frac{dQ}{2\pi}\frac{dP}{2\pi}e^{iQx+iPv}\exp\left\{-Q^2\frac{I_2^{(2,0)}}{2}-P^2\frac{I_2^{(2,2)}}{2}\right\} \\
&\quad\times\exp\left\{-Q^4\frac{\left|I_4^{(4,0)}\right|}{24}-Q^2P^2\frac{\left|I_4^{(2,2)}\right|}{4}-QP^3\frac{\left|I_4^{(1,3)}\right|}{6}-P^4\frac{\left|I_4^{(0,4)}\right|}{24}\right\}
\end{aligned}
\tag{B.1}
$$

where the generating function for the SPDF is shown below:

$$
\begin{aligned}
G(Q,P) &\equiv \left\langle e^{iQx+iPv}\right\rangle \\
&= \exp\left\{-Q^2\frac{I_2^{(2,0)}}{2}-P^2\frac{I_2^{(0,2)}}{2}-Q^4\frac{\left|I_4^{(4,0)}\right|}{24}-Q^2P^2\frac{\left|I_4^{(2,2)}\right|}{4}-QP^3\frac{\left|I_4^{(1,3)}\right|}{6}-P^4\frac{\left|I_4^{(0,4)}\right|}{24}\right\},
\end{aligned}
\tag{B.2}
$$

with $\ln G(Q,P)$ yielding the non-zero cumulants for the dynamical variables x and v.

B.1. An expression for the joint distribution

The expression of Eq. (B.1) has no closed form. However, we can write it as a convergent series. The calculations below show the way to do it. We start from Eq. (B.1) and expand it. By using the identity

$$I_\theta^\alpha = \int_{-\infty}^{+\infty} \frac{dQ}{2\pi}\, Q^{2\theta}\, e^{-\alpha\, Q^4} = \frac{1}{4\pi}\, \frac{\Gamma\left(\frac{\theta}{2}+\frac{1}{4}\right)}{\alpha^{\frac{\theta}{2}+\frac{1}{4}}}, \tag{B.3}$$

we obtain

$$\begin{aligned} p^{ss}(x,v) =& \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{e=0}^{\infty}\sum_{f=0}^{\infty} \frac{1}{16\pi^2}\, \frac{\Gamma\left(\frac{a+c+d+f}{2}+\frac{1}{4}\right)\Gamma\left(\frac{b+d+e+3f}{2}+\frac{1}{4}\right)}{(2a)!\,(2b)!\,c!\,d!\,e!\,(2f)!} \\ &\times \frac{\left(-x^2\right)^a\left(-v^2\right)^b\left(-\frac{T}{2k}\right)^c\left(-\frac{\left|I_4^{(4,2)}\right|}{4}\right)^d\left(-\frac{T}{2m}\right)^e\left(\frac{\left|I_4^{(4,3)}\right|}{6}\right)^{2f}}{\left(\frac{\left|I_4^{(4,0)}\right|}{24}\right)^{\frac{a+c+d+f}{2}+\frac{1}{4}}\left(\frac{\left|I_4^{(4,4)}\right|}{24}\right)^{\frac{b+d+e+3f}{2}+\frac{1}{4}}} \\ &+ \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{e=0}^{\infty}\sum_{f=0}^{\infty} \frac{1}{16\pi^2}\, \frac{\Gamma\left(\frac{a+c+d+f+1}{2}+\frac{1}{4}\right)\Gamma\left(\frac{b+d+e+3f+2}{2}+\frac{1}{4}\right)}{(2a+1)!\,(2b+1)!\,c!\,d!\,e!\,(2f+1)!} \\ &\times \frac{\left(-x^2\right)^a (xv)\left(-v^2\right)^b\left(-\frac{T}{2k}\right)^c\left(-\frac{\left|I_4^{(4,2)}\right|}{4}\right)^d\left(-\frac{T}{2m}\right)^e\left(\frac{\left|I_4^{(4,3)}\right|}{6}\right)^{2f+1}}{\left(\frac{\left|I_4^{(4,0)}\right|}{24}\right)^{\frac{a+c+d+f+1}{2}+\frac{1}{4}}\left(\frac{\left|I_4^{(4,4)}\right|}{24}\right)^{\frac{b+d+e+3f+2}{2}+\frac{1}{4}}}. \end{aligned} \tag{B.4}$$

B.2. Convergence of the distribution

The convergence of Eq. (B.4) will be determined by the behavior of the Gamma function at high values of its argument. We shall use Stirling's approximation $\lim_{n\to\infty}\Gamma(n) \sim n^{n+1/2}\,e^{-n}$ and try and check the convergence of the two series in Eq. (B.4) by freezing all but a chosen index and taking the limit of that index to infinity. The only problematic index is f, all others being well behaved (converge uniformly). The case of d corresponds to a converging alternating series.

So, let us analyze these two cases. We freeze (a, b, c, e, d) and take the limit $f \to \infty$. Keeping the essential summation yields

$$\sum_{f=0}^{\infty} \frac{\Gamma\left(\frac{f}{2}\right)\Gamma\left(\frac{3f}{2}\right)}{(2f)!} \times \frac{\left(\frac{\left|I_4^{(4,3)}\right|}{6}\right)^{2f}}{\left(\frac{\left|I_4^{(4,0)}\right|}{24}\right)^{\frac{f}{2}}\left(\frac{\left|I_4^{(4,4)}\right|}{24}\right)^{\frac{3f}{2}}} \sim \sum_{f=0}^{\infty} \frac{\sqrt{3f}\,(3)^{-\frac{f}{2}}}{2\sqrt{2}} \times \frac{\left(\frac{k\,m}{\gamma^2}\right)^{\frac{f}{2}}}{\left(1+\frac{k_0 m}{\gamma^2}\right)^{\frac{3f}{2}}},$$

which converges uniformly since for $x > 0$, $x\,(1+x)^{-3} < 1$. Then, the summations in Eq. (B.4) form a convergent distribution.

References

[1] N.G. van Kampen, Stochastic Processes in Physics and Chemistry, North-Holland, Amsterdam, 1992.
[2] A. Einstein, Annalen der Physik 322 (1905) 549.
[3] N.G. van Kampen, The Journal of Physical Chemistry B 109 (2005) 21293.
[4] H.S. Wio, P. Colet, M. San Miguel, L. Pesquera, M.A. Rodríguez, Physical Review A 40 (1989) 7312.
[5] T. Taniguchi, E.G.D. Cohen, Journal of Statistical Physics 126 (2007) 1.
[6] T. Taniguchi, E.G.D. Cohen, Journal of Statistical Physics 130 (2008) 1.
[7] M.O. Cáceres, Physical Review E 60 (1999) 5208.
[8] M.O. Cáceres, Physical Review E 67 (2003) 016102.
[9] D.O. Soares-Pinto, W.A.M. Morgado, Physica A 365 (2006) 289.
[10] A. Budini, M. Cáceres, Physical Review E 70 (2004) 046104.
[11] D. Sánchez, M. Büttiker, Physical Review Letters 93 (2004) 106802.
[12] H. Förster, M. Büttiker, Physical Review Letters 101 (2008) 136805.
[13] S. Nakamura, Y. Yamauchi, M. Hashisaka, K. Chida, K. Kobayashi, T. Ono, R. Leturcq, K. Ensslin, K. Saito, Y. Utsumi, A.C. Gossard, Physical Review Letters 104 (2010) 080602.
[14] B. Derrida, Journal of Statistical Physics (2007) P07023.
[15] J. Marcinkiewicz, Mathematische Zeitschrift 44 (1939) 612.
[16] P. Hanggi, K.E. Shuler, I. Oppenheim, Physica A 107 (1981) 143.
[17] W.A.M. Morgado, S.M. Duarte Queirós, D.O. Soares-Pinto, Journal of Statistical Physics 365 (2011) P06010.

[18] D.O. Soares-Pinto, W.A.M. Morgado, Physical Review E 77 (2008) 011103.
[19] W.A.M. Morgado, D.O. Soares-Pinto, Physical Review E 79 (2009) 051116.
[20] W.A.M. Morgado, S.M. Duarte Queirós, 2012 (submitted for publication).
[21] W.A.M. Morgado, D.O. Soares-Pinto, Physical Review E 82 (2010) 021112.
[22] A. Meskauskas, Gravitorpic reaction: the role of calcium and phytochrome, Vilnius State University, Vilnius, Lithuania, 1997.
[23] S.M. Duarte Queirós, Quantitative Finance 05 (2005) 475.
[24] H. Mori, Physical Review 112 (1958) 1829.
[25] W.N. Bailey, Generalized Hypergeometric Series, in: Cambridge Tracts in Mathematics and Mathematical Physics, Cambridge University Press, London, 1935, Also: http://en.wikipedia.org/wiki/Generalized_hypergeometric_function.
[26] S. de Groot, P. Mazur, Non-Equilibrium Thermodynamics, Dover, New York, 1984.