MONOGRAFIAS

XXXIII

STRONG INTERACTION PHYSICS

(SELECTED TOPICS)

por

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C O N T E N T S

The present monograph is based on the post-graduate lectures given by the author during the first semester of 1970. The motivation was to introduce some basic aspects of strong interaction physics and symmetry principles in sufficiently comprehensive fashion which may facilitate clear understanding of advanced topics of research in particle physics.

1

LORENTZ AND POINCARÉ GROUPS

The linear space of real 4-vectors x^μ: (x^0, x^1, x^2, x^3) with scalar product *

$$x \cdot y = x^\mu y^\nu g_{\mu\nu} = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3 \qquad (1)$$

is called Minskowski Space. Here $g_{\mu\nu}$ is the (μ,ν) component of the metric matrix:

$$G = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & & 0 & \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \end{array} \qquad (2)$$

Real linear transformations in this space defined by

$$x^\mu \to x^{\mu\prime} = \Lambda^\mu{}_\nu \; x^\nu + a^\mu \qquad (3)$$

with *

$$g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta} \qquad (4)$$

are called Inhomogeneous Lorentz Transformations (IHL) and will be denoted by (Λ, a). The (4 x 4) matrix acting in four dimensional Minskowski Space

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* Summation convention on repeated indices is used.

in the sense of ordinary matrix multiplication are of the type

$$\Lambda \equiv (\Lambda^{\mu}{}_{\nu}) = \begin{pmatrix} \Lambda^{0}{}_{0} & \Lambda^{0}{}_{1} & \Lambda^{0}{}_{2} & \Lambda^{0}{}_{3} \\ \Lambda^{1}{}_{0} & \Lambda^{1}{}_{1} & \Lambda^{1}{}_{2} & \Lambda^{1}{}_{3} \\ \Lambda^{2}{}_{0} & \Lambda^{2}{}_{1} & \Lambda^{2}{}_{2} & \Lambda^{2}{}_{3} \\ \Lambda^{3}{}_{0} & \Lambda^{3}{}_{1} & \Lambda^{3}{}_{2} & \Lambda^{3}{}_{3} \end{pmatrix} \tag{5}$$

and differ from the matrices defined by using the elements $\Lambda_{\mu}{}^{\nu}$, $\Lambda_{\mu\nu}$ etc. *
Transpose matrix Λ^{T} is defined by $(\Lambda^{T})^{\mu}{}_{\nu} = (\Lambda)^{\nu}{}_{\mu}$.

Equation (4) can be rewritten as:

$$\Lambda^{T}\, G\, \Lambda = G \tag{6}$$

since

$$\Lambda_{\mu\beta} = g_{\mu\nu}\, \Lambda^{\nu}{}_{\beta} = (G\, \Lambda)_{\mu\beta}$$

and

$$\Lambda^{\mu}{}_{\alpha}\, (G\, \Lambda)_{\mu\beta} = (\Lambda^{T})^{\alpha}{}_{\mu}\, (G\, \Lambda)_{\mu\beta}$$

The condition (4) or (6) ensures that the scalar product is preserved under Homogeneous Lorentz Transformations (HL) $(\Lambda, 0)$:

$$x^{\mu\prime}\, y^{\nu\prime}\, g_{\mu\nu} = \Lambda^{\mu}{}_{\rho}\, \Lambda^{\nu}{}_{\sigma}\, g_{\mu\nu}\, x^{\rho}\, y^{\sigma} = g_{\rho\sigma}\, x^{\rho}\, y^{\sigma} \tag{7}$$

The invariance of scalar product may be used as alternative definition of Homogeneous Lorentz Transformations (HL).

In addition to the contravariant vectors x^{μ} we can also

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* The first index represent row and the second the column.

introduce covariant vectors x_μ in our four-space defined by

$$x_\mu = g_{\mu\nu} x^\nu \tag{8}$$

and correspondingly a metric $g^{\mu\nu}$ to raise the indices defined by

$$g_{\mu\alpha} g^{\alpha\nu} = g^\mu{}_\nu = \delta^\mu{}_\nu \tag{9}$$

we find that the matrix formed by elements $g^{\mu\nu}$ coincides with G. The scalar product can thus be written as

$$x^\mu y^\nu g_{\mu\nu} = x_\mu y_\nu g^{\mu\nu} = x^\mu y_\mu = x^0 y^0 - \vec{x}\,\vec{y}$$

Here $\vec{x} = (x^1, x^2, x^3) = (-x_1, -x_2, -x_3)$.

Inhomogenous Lorentz Transformations (Λ, a) form group called Poincaré Group.

We readily verify that

$$(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1 \Lambda_2, a_1 + \Lambda_1 a_2) \tag{8}$$

The inverse and identity are *:

$$(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1} a) \tag{9}$$

$$\mathbf{1} = (\mathbf{1}, 0)$$

From equation (8) we show

$$(\Lambda, a) = (\mathbf{1}, a)(\Lambda, 0) \tag{10}$$

Translations $(\mathbf{1}, a)$ themselves form a group called translation Group

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* **From equation (4)**

$$\Lambda_{\nu\alpha} \Lambda^\nu{}_\beta = g_{\alpha\beta}$$

or

$$\Lambda_\nu{}^\alpha \Lambda^\nu{}_\beta = \delta^\alpha{}_\beta \quad \text{so that} \quad (\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu = g_{\nu\rho} \Lambda^\rho{}_\sigma g^{\mu\sigma}$$

T, and so do the homogeneous transformations $(\Lambda, 0)$ which constitute Homogeneous Lorentz Group $\mathcal{L}$. In fact T is invariant subgroup of $\mathcal{P}$; $(\Lambda, a)(\mathbf{1}, a')(\Lambda, a)^{-1} = (1, \Lambda a')$, and the Factor or Quotient group of $\mathcal{P}$ by the translation group T is just the group $\mathcal{L} = (\frac{\mathcal{P}}{T})$. The groups $\mathcal{P}$, T, $\mathcal{L}$ are Lie groups and thus can be generated from successive infinitesimal transformations.

From equations (4) or (6) we find $\det \Lambda = \pm 1$ and $g_{00} = 1 = (\Lambda^0{}_0)^2 - \sum_{j=1}^{3} (\Lambda^j{}_0)^2$ leading to $(\Lambda^0_0)^2 \geqslant 1$ which implies $\Lambda^0{}_0 \geqslant 1$ or $\Lambda^0{}_0 \leqslant -1$. The Homogeneous Lorentz Group $\mathcal{L}$ has four disconnected pieces, each of which is connected in the sense that two Lorentz Transformations in the same piece can be connected to one another by a continuous curve of Lorentz transformations, but no Lorentz Transformation in one piece can be connected to another piece. The four pieces are characterized by $\det \Lambda$ and $\text{sgn}\, \Lambda^0{}_0$:

$\mathcal{L}_+^\uparrow$:	$\det \Lambda = +1$	$\text{sgn}\, \Lambda^0{}_0 = +1$	contains $\mathbf{1}$
$\mathcal{L}_-^\uparrow$:	$= -1$	$= +1$	contains I_s
$\mathcal{L}_+^\downarrow$:	$= +1$	$= -1$	contains I_{st}
$\mathcal{L}_-^\downarrow$:	$= -1$	$= -1$	contains I_t

Here, the Lorentz Transformations I_s (Space inversion), I_t (time inversion) and I_{st} (space-time inversion) are defined by:

$$I_s = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}, \qquad I_t = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

and

$$I_{st} = \begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} = I_s \, I_t$$

Clearly $\mathcal{L}_-^{\uparrow}$ is mapped one to one into $\mathcal{L}_+^{\uparrow}$ by I_s, $\mathcal{L}_-^{\downarrow}$ onto $\mathcal{L}_+^{\uparrow}$ by I_t and $\mathcal{L}_+^{\downarrow}$ onto $\mathcal{L}_+$ by I_{st}. All Λ for which $\Lambda^0{}_0 \geqslant 1$ are called orthochronous, Λ for which det $\Lambda = +1$, proper and Λ for which (sgn Λ^0_0) (det Λ) = + 1, orthochorous. It is sufficient to show that $\mathcal{L}_+^{\uparrow}$ is connected and a proof may be found in ref. (1).

The following (homogeneous) sub-groups can be built from the above pieces:

$\mathcal{L}_+^{\uparrow}$	Proper Orthochronous Lorentz Group or Restricted Homogeneous Lorentz Group (RHL)
$\mathcal{L}^{\uparrow} = \mathcal{L}_+^{\uparrow} \cup \mathcal{L}_-^{\uparrow}$	Orthochronous Lorentz Group
$\mathcal{L}_+ = \mathcal{L}_+^{\uparrow} \cup \mathcal{L}_+^{\downarrow}$	Proper Lorentz Group
$\mathcal{L}_0 = \mathcal{L}_+^{\uparrow} \cup \mathcal{L}_-^{\downarrow}$	Orthochorous Lorentz Group

Associated with $\mathcal{L}_+^{\uparrow}$ is the group of (2 x 2) complex matrices of determinant one denoted by SL(2, C). To any 4-vector x^{μ} we can associate (2 x 2) matrix

$$\mathbb{X} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} = \sum_{\alpha=0}^{3} x^\alpha \quad \sigma^\alpha$$

where

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$x^\mu = \frac{1}{2} \operatorname{tr}(\mathbb{X}\, \sigma^\mu) \tag{15}$$

For x^μ real $\mathbb{X}$ is hermitian; if $\mathbb{X}$ is hermitian equation (   ) yields a real four-vector. Also $\det \mathbb{X} = x^\mu x_\mu$ and $\frac{1}{2}\left[\det(\mathbb{X} + \mathbb{Y}) - \det \mathbb{X} - \det \mathbb{Y}\right] =$ $= x^\mu y_\mu$. If A is any (2 x 2) matrix of determinant 1, then:

$$\mathbb{X}' = A\, \mathbb{X}\, A^+; \quad \det A = 1 \tag{16}$$

defines a real linear mapping of fourvectors x^μ onto four-vectors $x^{\mu\prime}$ which satisfies

$$x^{\mu\prime} \; y'_\mu = x^\mu \, y_\mu$$

This mapping is thus a homogeneous Lorentz Transformation, say, denoted by $\Lambda(A)$. Actually, $\Lambda(A)$ is restricted Lorentz transformation for we can vary A continously until it is identity while the corresponding $\Lambda(A)$ varies continously to reach the identity. We can easily show

$$\begin{aligned} \Lambda(A)\, \Lambda(B) &= \Lambda(AB) \\ \Lambda(1) &= \mathbb{1} \end{aligned} \tag{17}$$

and note also $\Lambda(-A) = \Lambda(A)$ and that $\Lambda(A) = \Lambda(B)$ implies $A = \pm B$. Thus $A \to \Lambda(A)$ is a homomorphism of SL(2, C) onto $\mathcal{L}_+^\uparrow$. It is a non-unitary representation of the homogeneous proper orthochronous Lorentz group.

The matrices A due to condition det A = 1 depend upon 6 parameters, just as the Lorentz transformations (see latter). The generators of the group SL(2, C) are represented by six matrices:

$$\vec{J} = \frac{1}{2}\vec{\sigma} \qquad\qquad \vec{K} = \frac{1}{2} i\vec{\sigma} \tag{18}$$

where $\vec{J}$ are hermitian while $\vec{K}$ are anti-hermitian. There satisfy the commutation relations:

$$\left[J_i, J_j\right] = i\, \varepsilon_{ijk}\, J_k , \qquad \left[K_i, K_j\right] = - i\, \varepsilon_{ijk}\, J_k \text{ and } \left[J_i, K_j\right] = $$
$$= i\, \varepsilon_{ijk}\, K_k \tag{19}$$

Since $\vec{J}$ are generators of group SU(2) (the 2 x 2 unimodular unitary group), SL(2, C) embeds SU(2). In fact, from (16) we see that, if, A is also unitary: tr $\mathbf{X}'$ = tr $\mathbf{X}$ leading to $x^{0\prime} = x^0$ and $\vec{x}\cdot\vec{y} = \vec{x}\cdot\vec{y}$ which implies that $A \to \Lambda(A)$, when $A^+ = A^{-1}$, is a homomorphic mapping of SU(2) onto the rotation (sub-group of $_+^\uparrow$ in 3-dimensional space).

We also note from equation (16) that $\Lambda(A^*) = \Lambda(-A^*) = \Lambda(A) = \Lambda(-A)$ so that matrices $\pm A$ and $\pm A^*$ correspond to the same Lorentz transformation. In the case, A belongs to SU(2) A and A* are equivalent representations from the quantum mechanical point of view. In fact, any matrix of SU(2) can be written as $A = e^{i\vec{a}\cdot\vec{\sigma}}$ with same real vector $\vec{a}$. Consequently, $A^* = \overline{e}^{\,ia\cdot\sigma^*} = C^{-1} A C$ where $C = i\sigma_2$, $C^+ = C^{-1}$ and we have used $C^{-1} \vec{\sigma} C =$ $= -\vec{\sigma}^*$. Thus so far as space rotations are concerned the matrices A and A* are equivalent 2 x 2 matrix representations. This is no more true for the group SL(2, C) representing a general Lorentz transformation. We can write any matrix of SL(2, C) as $e^{i\vec{a}\cdot\vec{\sigma}} = e^{i(\vec{a}_1 + i\vec{a}_2)\cdot\vec{\sigma}}$, $\vec{a}_1$ and $\vec{a}_2$ being real vectors equation it is obtained by making the parameters '$\vec{a}$' of the Lie algebra of SU(2) complex. A* and A cannot be equivalent any more since $\vec{a}$ is no more real and the proof given above does not go through. They constitute two different representations of the Lorentz group acting upon two distinct two-dimensional vector spaces. We may take these two independent representations to be

$$D^{(\frac{1}{2})}(\Lambda) = A = e^{i(\vec{a}_1 + i\vec{a}_2)\cdot\vec{\sigma}}$$

and

$$\overline{D}^{(\frac{1}{2})}(\Lambda) = CA^*C^{-1} = e^{i(\vec{a}_1 - i\vec{a}_2)\cdot\vec{\sigma}} \tag{20}$$

The vectors ξ_α and $\eta_{\overline{\alpha}}$ ($\alpha, \overline{\alpha} = \pm 1/2$) called spinors of the two (distinct) vector spaces on which $D^{(\frac{1}{2})}(\Lambda)$ and $\overline{D}^{\frac{1}{2}}(\Lambda)$ act respectively transform under a Lorentz transformation as:

$$\xi'_\alpha = D^{(\frac{1}{2})}_{\alpha\beta}\,\xi_\beta \quad \text{and} \quad \eta'_{\overline{\alpha}} = D^{(\frac{1}{2})}_{\overline{\alpha}\,\overline{\beta}}\,\eta_{\overline{\beta}} \tag{21}$$

It is easily shown that $\eta_{\overline{\alpha}}$ transforms like ($C\xi^*$) and that for rotations $D^{(\frac{1}{2})}(R) = D^{-(\frac{1}{2})}(R)$. The index $\frac{1}{2}$ is to remind that these representations are complex extensions of spin 1/2 SU(2) representation of the rotation group to a representation of the Lorentz group.

It is possible to use only one of these two representations (e.g.use 2-spinors) for particles without parity. However if we want to represent parity operation (in addition to the transformations $\mathcal{L}_+^\uparrow$) we need both the representations and the basic vector space is now four dimensional.

This is seen as follows: the parity operation commutes with rotation group so that if a (2 x 2) matrix representations of parity operation be possible it will commute with all the (2 x 2) matrices of SU(2) representing rotations. By Schur's Lemma, then, the (2 x 2) matrix representing parity operation must be proportional to identity matrix and thus commutes with any (2 x 2) matrix of SL(2, C) representing Lorentz transformation. This implies that parity operation commutes with Lorentz Transformation which is clearly not true.

2

INVARIANCE PRINCIPLE. TRANSFORMATION OF STATE VECTORS AND OPERATORS UNDER POINCARÉ GROUP

1. *INVARIANCE PRINCIPLE*

We will now study the action of Lorentz transformation on physical state vectors *. We recall that a physical system can be represented by any vector $e^{i\lambda}|\alpha>$ where $|\alpha>$ is a state vector of a Hilbert space and λ is an arbitrary phase. Transition probability from a state $|\alpha>$ to state $|\beta>$ is given by $|<\beta|\alpha>|^2$. An invariance principle or symmetry operation is a one-to-one correspondence which assigns to each physical state $|\alpha>$ another state $|\alpha'>$, in another Hilbert space, such that all transition probabilities are preserved i.e. $|<\beta'|\alpha'>|^2 = |<\beta|\alpha>|^2$. Consequently, there is a correspondence between the observables in the two alternative descriptions. Wigner has shown that the invariance of the probabilities is possible if the symmetry operation is realized by means of a unitary or an anti-unitary operations. The essential features of the two cases are:

(i) *Unitary symmetry operation:*

$$|\alpha> \to |\alpha'> = U|\alpha> = |U\alpha> \tag{1}$$

$$U(\lambda|\alpha>) = \lambda . U|\alpha>$$

$$U(|\alpha_1> + |\alpha_2>) = U|\alpha_1> + U|\alpha_2>$$

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* Heisenberg state vectors to be precise. A Heisenberg state vector describes a system throughout all time.

and

$$< \beta'|\alpha' > = < \beta|\alpha > \tag{2}$$

implying

$$U^{+} U = UU^{+} = 1 \tag{3}$$

For the definition of conjugate operator for linear operators we have

$$< \beta|U^{+}|\alpha > \equiv <\beta|U^{+}\alpha > = < U\beta|\alpha > = < \alpha|U|\beta >^{*} \tag{4}$$

For any (linear) operator $\mathcal{O}$

$$< \beta|\mathcal{O}|\alpha > = < \beta|U^{-1} U \mathcal{O} U^{-1} U|\alpha> = < \beta'|\mathcal{O}'|\alpha' > \tag{5}$$

where

$$\mathcal{O}' = U \mathcal{O} U^{-1} \tag{6}$$

If the operator $\mathcal{O}$ is invariant under the symmetry operation U, then

$$\mathcal{O}' = \mathcal{O} \qquad \text{or} \qquad [U, \mathcal{O}] = 0 \tag{7}$$

$$< \beta|\mathcal{O}|\alpha > = < \beta'|\mathcal{O}|\alpha' > \tag{8}$$

ii) *Antiunitary symmetry operation:*

$$|\alpha > \rightarrow |\alpha'> = |A\alpha > = A|\alpha >$$

$$AA^{+} = A^{+}A = 1$$

$$A(\lambda|\alpha>) = \lambda^{*} A|\alpha > \quad \text{(antilinear operator)} \tag{10}$$

$$A(|\alpha_1> + |\alpha_2 >) = A|\alpha_1 > + A|\alpha_2 >$$

and

$$< \beta'|\alpha' > = < \beta|\alpha>^{*} = < \alpha|\beta > \tag{11}$$

For the definition of conjugate operator for antilinear operators * we have

* For antilinear operators the parenthesis notation is more convinient. See the chapter on Time Reversal operation.

$$< \beta|A^{+}|\alpha > = < \beta|A^{+}\alpha > < A\beta|\alpha >^{*} = < \alpha|A|\beta > \qquad (12)$$

For any (linear) operator $\mathcal{O}$ we have

$$< \beta|\mathcal{O}|\alpha > = <\beta|A^{-1} A\mathcal{O}A^{-1} A|\alpha > = <\beta'|(A\mathcal{O}A^{-1})|\alpha'>^{*}$$

$$= < \alpha'|(A\mathcal{O}A^{-1})^{+}|\beta'>= <\alpha'|\mathcal{O}'|\beta' > \qquad (13)$$

where $\mathcal{O}'$ is the operator linear

$$\mathcal{O}' \;,= (A\,\mathcal{O}A^{-1})^{+} \qquad (14)$$

giving the transformation of operators under antisymmetry transformation. If $\mathcal{O}$ is invariant under the transformation that is $\mathcal{O}' = \mathcal{O}$, then

$$\mathcal{O}^{+} = A\mathcal{O}A^{-1} \qquad (15)$$

and

$$< \beta|\mathcal{O}|\alpha > = < \alpha'|\mathcal{O}|\beta' > \qquad (16)$$

The case of antiunitary transformation is applicable for transformations that can be reached continuously from the identity transformation. The identity transformation is unitary which cannot be linked continuously to an antiunitary operator. In case the transformation is not continuous with identity, for example, space inversion, time reversal etc., it will be necessary to investigate if case (i) or (ii) has to be used taking into considerations the physical requirements. For the case of Poincaré transformations $\mathcal{P}_{+}^{\uparrow}$ unitary transformation has to be selected. In what follows, unless indicated otherwise, we will understand by Lorentz Transformation, an operation belonging to the group $\mathcal{P}_{+}^{\uparrow}$.

The principle of Lorentz invariance, that is the invariance under proper, orthochronous Lorentz transformation, asserts that the laws of physics are invariant under these transformations which relate the space time coordinate of the same physical event as it is observed from two different inertial frames. This implies then that there exists a unitary operator relating the state vectors and the observables which represent the physical system in the two descriptions. This formulation of relativity principle is called the "passive" formulation. It relates the description of the same physical system as seen from two coordinate frames. An alternative point of view, called "active" formulation, changing the object, is, however, more convinients and equivalent to the "passive" interpretation. Instead of considering two different coordinate frames (or observers) related by a Lorentz transformation we may consider the coordinate frame (observer) fixed and consider two physical events which are related by the same Lorentz transformation as the two reference frames in the passive formulation. The active interpretation implies that if $|\alpha\rangle$ is a possible state of the system, then $|\alpha'\rangle = U|\alpha\rangle$, where U represents a Lorentz transformation is also a possible state of the system as seen in a fixed reference frame. The observations are now made on two physical systems which are related by a Lorentz transformation.

For transformations of geometrical nature such as the one discussed above it is feasible to test the corresponding invariance principle via both of the interpretations and establish their equivalence. For transformations like space inversion, time reversal changed conjugation etc., we can in effect only test the invariance principle in its active formulation, since we cannot realize the "inverted" observer. This, however, does not impede in giving in some cases a passive interpretation through coordinate transformation. The mathematical operation will not be physical ly meaningful unless it represents a real invariance principle. A non-invariance operation in the active formulation will correspondingly lead to states outside the (physical) Hilbert space under consideration and may be sometimes a non-physical state.

For our discussions we will mostly adopt the active point of view and always use the operators in active formulation.

We note also that the requirement of the invariance of the transition probabilities under a symmetry operation can be expressed as the equality between the squared modulus of certain S-matrix elements. In the active formulation symmetry operation yields relations between the S-matrix elements corresponding to different physical (experimental) processes. This in turn combined with other general properties of S-matrix leads to relations between different measurable quantities which can be tested experimentally. We do not need to know the explicit form of S-matrix to derive such relations.

2 - *POINCARÉ INVARIANCE:* We will now discuss specifically the invariance under Poincaré transformation $\mathcal{P}_+^{\uparrow}$. A transformation (Λ, a) induces a unitary transformation $U(\Lambda, a)$ on the normalized vectors

$$|\alpha\rangle \longrightarrow |\alpha'\rangle = e^{i\omega}\, U(\Lambda, a)|\alpha\rangle \tag{17}$$

$$U^{+}(\Lambda, a) = U(\Lambda, a)^{-1}$$

where ω is an arbitrary phase *. The product of two Lorentz transforma-

* Note that a physical state is represented by a ray.

tions (Λ_1, a_1) and (Λ_2, a_2) is again a Lorentz transformation:

$$(\Lambda, a) = (\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1 \Lambda_2, a_1 + \Lambda_1 a_2) \tag{18}$$

It follows then

$$U(\Lambda, a)|\alpha > = e^{i(\omega_1 + \omega_2 - \omega)} U(\Lambda_1, a_1) U(\Lambda_2, a_2)|\alpha > \tag{19}$$

Wigner * has also shown that, in the case of Poincaré group, we can choose the transformation $U(\Lambda, a)$ such that we can get rid of the phase to ensure

$$U(\Lambda, a) = U(\Lambda_1, a_1) U(\Lambda_2, a_2) \tag{20}$$

or

$$U(\Lambda_1, a_1) U(\Lambda_2, a_2) = U(\Lambda_1 \Lambda_2, a_1 + \Lambda_1 a_2) \tag{21}$$

It follows:

$$\begin{aligned} U(\Lambda, a) &= U(\mathbb{1}, a) U(\Lambda, 0) \\ &= U(a) U(\Lambda) \end{aligned} \tag{22}$$

where we write

$$U(a) = U(\mathbb{1}, a) \tag{23}$$

$$U(\Lambda) = U(\Lambda, 0) \tag{24}$$

We also verify

$$\begin{aligned} &U(\Lambda_1) U(\Lambda_2) = U(\Lambda_1 \Lambda_2) \\ &U(a) U(a') = U(a + a') = U(a') U(a) \\ &U^{-1}(\Lambda) = U(\Lambda^{-1}) \end{aligned} \tag{25}$$

The operators $U(\Lambda, a)$ having the same algebraic properties as the Poincaré

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* Wigner, E. P., Ann. Math. 40, 149 (1939).

Halpern, F. R. Special Relativity and Quantum Mechanics Prentice Hall (1968).

transformations constitute a unitary representation of the Poincaré group.

The unitarity of $U(a,\Lambda)$ together with the fact that it is continuous with identity allows us to consider infinitesimal Lorentz transformation from which a finite transformation is generated by interaction of successive infinitesimal transformations. The number of independent parameters characterizing a transformation (Λ, a) due to restriction in equ. (1.4) is ten;six corresponding to homogeneous transformations and four for translations. An infinitesimal (Λ, a) can be expressed as:

$$x^{\mu\prime} = x^{\mu} + \alpha^{\mu}{}_{\nu}\, x^{\nu} + \varepsilon^{\mu} + O(\alpha^2,\varepsilon^2) \qquad (26)$$

thus

$$\Lambda^{\mu} = g^{\mu}{}_{\nu} + \alpha^{\mu}{}_{\nu} + O(\alpha^2) \qquad (27)$$

Equation (1.4) leads to

$$\alpha_{\mu\nu} = -\,\alpha_{\nu\mu} \qquad (28)$$

The corresponding infinitesimal transformation $U(\Lambda, a)$ can likewise be expressed as

$$U(\Lambda, a) = \mathbb{I} + \frac{i}{2}\,\alpha_{\mu\nu}\, M^{\mu\nu} + i\, P^{\mu}\,\varepsilon_{\mu} + \ldots \qquad (29)$$

Here $M^{\mu\nu} = -\,M^{\nu\mu}$ are the six generators of homogenous Lorentz transformations and four P^{μ} are generators of translations. From unitary of $U(\Lambda,a)$ it follows that the ten generators are hermitian operators. The finite transformations are then expressed by

$$U(a) = e^{iP^{\mu} a_{\mu}}$$

$$U(\Lambda) = e^{i/2\, M^{\mu\nu}\,\alpha_{\mu\nu}} \qquad (30)$$

$$U(\Lambda,a) = U(a)\,U(\Lambda) = e^{iP^{\mu} a_{\mu}}\, e^{i/2\, M^{\mu\nu}\,\alpha_{\mu\nu}}$$

The commutation relations satisfied by the generators will now be derived.

From U(a) U(a') = U(a') U(a) we find, say, considering infinitesimal transformations

$$[P^{\mu}, P^{\nu}] = 0 \tag{31}$$

The [M, M] commutator can be found by considering

$$U(\Lambda)\, U(\Lambda')\, U^{-1}(\Lambda) = U(\Lambda\Lambda'\,\Lambda^{-1}) \tag{32}$$

for infinitesimal transformations and comparing the coeficient of cross term ($\alpha\alpha'$) on the two sides. The result is

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\,(M^{\mu\rho}\, g^{\nu\sigma} + M^{\nu\sigma}\, g^{\mu\rho} - M^{\nu\rho}\, g^{\mu\sigma} - M^{\mu\sigma}\, g^{\nu\rho}) \tag{33}$$

Finally from the identity

$$(\Lambda, a)(\mathbb{I}, a')(\Lambda, a)^{-1} = (\mathbb{I}, \Lambda\, a') \tag{34}$$

which implies the relation

$$U(\Lambda)\, U(a)\, U^{-1}(\Lambda) = U(\Lambda a) \tag{35}$$

or

$$U(\Lambda) e^{iP^{\mu} a_{\mu}}\, U^{-1}(\Lambda) = e^{iP^{\nu} (\Lambda a)_{\nu}}$$

it follows *

$$U(\Lambda)\, P^{\mu}\, U^{-1}(\Lambda) = (\Lambda^{-1})^{\mu}{}_{\nu}\, P^{\nu} \tag{36}$$

Similarly, we can show

$$U(\Lambda)\, M^{\mu\nu}\, U^{-1}(\Lambda) = (\Lambda^{-1})^{\mu}{}_{\lambda}\, (\Lambda^{-1})^{\nu}{}_{\rho}\, M^{\lambda\rho} \tag{37}$$

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* Note that $(\Lambda a)_{\nu} = g_{\nu\mu}(\Lambda a)^{\mu} = g_{\nu\mu}\, \Lambda^{\mu}{}_{\rho}\, a^{\rho} = g_{\nu\mu}\, \Lambda^{\mu}{}_{\rho'}\, g^{\rho'\rho}\, a_{\rho}$

$$= \Lambda_{\nu}{}^{\rho}\, a_{\rho} = (\Lambda^{-1})^{\rho}{}_{\nu}\, a_{\rho}$$

Equation (36) implies that P^μ transforms as a four-vector. Considering infinitesimal transformations it leads to

$$\left[M^{\mu\nu}, P^{\sigma}\right] = -i(P^\mu g^{\nu\sigma} - P^\nu g^{\mu\sigma}) \tag{38}$$

Define

$$J_i = -\frac{1}{2} \varepsilon_{oijk} M^{jk} \tag{39}$$

$$K_i = M_{io} = M^{oi}$$

where $\varepsilon_{\mu\nu\rho\lambda}$ is the usual completely antisymmetric tensor with $\varepsilon_{0123} = - \varepsilon^{0123} = +1$ and $\varepsilon_{ijk} = \varepsilon_{oijk}$. Then the commutation relations for J_i, K_i and P^μ are deduced to be:

$$\begin{aligned} \left[P^\mu, P^\nu\right] &= 0 \\ \left[J_i, P_k\right] &= i\, \varepsilon_{ik\ell} P_\ell \\ \left[J_i, P_0\right] &= 0 \\ \left[K_i, P_k\right] &= i\, P_0\, g_{ik} = -i\, P_0\, \delta_{ik} \\ \left[K_i, P_0\right] &= -i\, P_i \end{aligned} \tag{40}$$

and

$$\begin{aligned} \left[J_i, J_j\right] &= i\, \varepsilon_{ijk} J_k \\ \left[J_i, K_j\right] &= i\, \varepsilon_{ijk} K_k \\ \left[K_i, K_j\right] &= -i\, \varepsilon_{ijk} J_k \end{aligned} \tag{41}$$

we will also adopt the notation:

$$\vec{J}: (J_1, J_2, J_3)$$
$$\vec{K}: (K_1, K_2, K_3) \qquad (42)$$

but

$$\vec{P}: (P^1, P^2, P^3)$$

and

$$\vec{X}: (X^1, X^2, X^3)$$

Space rotations are generated by $\vec{J}$. The above commutation relations show that $\vec{P}$, $\vec{K}$ and $\vec{J}$ transform as vector or pseudo-vector operators while P_0 transforms as scalar operator under space rotations. In fact $\vec{J}$ is a pseudo-vector, while $\vec{P}$ and $\vec{K}$ behave as vector operators.

3 - *MATRIX REPRESENTATION OF POINCARÉ GROUP GENERATORS* A (2 x 2) matrix representations of $\vec{J}$ and $\vec{K}$ satisfying the commutation relation in equations (40) and (41) is given by equations (16) and (18) and the corresponding group is the SL(2, C) as discussed before. A (4 x 4) matrix representation is easily found by considering transformation properties of the four-vector X^μ. For infinitesimal transformation $(\Lambda, 0)$:

$$\begin{aligned} X'^\mu &= X^\mu + \alpha^\mu{}_\nu X^\nu + \ldots \\ &= (\delta^\mu{}_\nu + \alpha^\mu{}_\nu) X^\nu + \ldots \\ &= (\mathbb{I} + \frac{i}{2}\alpha_{\lambda\rho} M^{\lambda\rho})^\mu{}_\nu X^\nu \end{aligned} \qquad (43)$$

Thus

$$\alpha^\mu{}_\nu = \frac{i}{2}\alpha_{\lambda\rho} (M^{\lambda\rho})^\mu{}_\nu$$

or

$$\alpha_{\mu\nu} = \frac{i}{2}\alpha_{\lambda\rho} (M^{\lambda\rho})_{\mu\nu} \qquad (44)$$

leading to:

$$(M^{\mu\nu})_{\lambda\rho} = - i(\delta^{\mu}{}_{\lambda}\, \delta^{\nu}{}_{\rho} - \delta^{\mu}{}_{\rho}\delta^{\nu}{}_{\lambda})$$

or (45)

$$(M^{\mu\nu})^{\lambda}{}_{\rho} = - i(g^{\lambda\mu}\, \delta^{\nu}{}_{\rho} - g^{\nu\lambda}\, \delta^{\mu}{}_{\rho})$$

Noting that $g^{\alpha 0} = g^{0\alpha} = \delta^{\alpha}{}_{0}$ and $g^{i\alpha} = g^{\alpha i} = - \delta^{\alpha}{}_{i}$ the (4 x 4) matrices for $\vec{J}$ and $\vec{K}$ are

$$(J_i)^{\lambda} = i\, \varepsilon_{i\rho\lambda} \tag{46}$$

$$(K_i)^{\lambda}{}_{\rho} = - i(\delta^{\lambda}{}_{0}\delta^{i}{}_{\rho} + \delta^{0}{}_{\rho}\ \delta^{\lambda}_{i})\ .$$

The matrix K is antihermitian:

$$(K_i^{+})^{\lambda}{}_{\rho} = (K_i^{*T})^{\lambda}{}_{\rho} = (K_i{}^{*})^{\rho}{}_{\lambda} = + (K_i^{*})^{\lambda}{}_{\rho}$$

$$= - (K_i)^{\lambda}{}_{\rho}$$

while $(J_i{}^{+})^{\lambda}{}_{\rho} = + (J_i)^{\lambda}{}_{\rho}$ is hermitian. We can further derive:

$$J_i\, J_j\, J_k + J_k\, J_j\, J_i = (J_i\, \delta_{jk} + J_k\, \delta_{ij})$$

(47)

$$K_i\, K_j\, K_k + K_k\, K_j\, K_i = -(K_i\, \delta_{kj} + K_k\, \delta_{ij})$$

and $(\vec{n}\cdot\vec{J})^3 = (\vec{n}\cdot\vec{J})$ while $(\vec{n}\cdot\vec{K})^3 = - (\vec{n}\cdot\vec{K})$ if $\vec{n}$ is a unit vector.

These relations allow us to write

$$e^{i(\vec{J}\cdot\vec{n})\theta} = \left[1-(\vec{J}\cdot\vec{n})^2\right] + (\vec{J}\cdot\vec{n})^2 \cos\theta + i(\vec{J}\cdot\vec{n}) \sin\theta$$

(48)

$$e^{i(\vec{K}\cdot\vec{n})\phi} = \left[1 + (\vec{K}\cdot\vec{n})^2\right] - (\vec{K}\cdot\vec{n})^2 \cos\phi + i(\vec{K}\cdot\vec{n}) \sinh\phi$$

To obtain a matrix representation of the ten generators of the inhomogeneous Lorentz group we introduce five coordinates X^0, X^1, X^2, X^3 and X^4 so that:

$$X^{\mu'} = \Lambda^{\mu}{}_{\nu} X^{\nu} + a^{\mu} = \Lambda^{\mu}{}_{\nu} X^{\nu} + a^{\mu} X^4$$
$$= (\Lambda, a)^A{}_B X^B \qquad \text{where A, B} = 0, 1, 2, 3, 4. \tag{49}$$

Here $X^A = (X^0, X^1, X^2, X^3, X^4)$ and $(\Lambda, a)^A{}_B$ is (5 x 5) matrix such that

$$(\Lambda, a)_{\nu} = \Lambda^{\mu}{}_{\nu} , \; (\Lambda, a)^{\mu}{}_{4} = a^{\mu} ,$$
$$(\Lambda, a)^4{}_{\mu} = 0 \quad (\Lambda, a)^4{}_4 = 1 \tag{50}$$

$$(\Lambda, a)^A{}_B = \left(\begin{array}{c|c} \Lambda^{(4x4)} & \overset{(4x1)}{a} \\ \hline 0^{(1x4)} & 1^{(1x1)} \end{array}\right) \tag{51}$$

clearly,

$$X^{4'} = X^4 \tag{52}$$

From infinitesimal (Λ, a) we find that (5 x 5) matrices $M^{\mu\nu}$ can be chosen to be extended by zeros, viz,

$$(M^{\mu\nu})^4{}_A = (M^{\mu\nu})^A{}_4 = 0 \tag{53}$$

The $(P^{\mu})^A$ are (5 x 5) matrices with the non-vanishing elements given by - $-(P^0)^0{}_4 = (P^1)^1{}_4 = (P^2)^2{}_4 = (P^3)^3{}_4 = + i$. The commutation relation of the matrices may be verified to be the same as given in equations (40) and (41). If should, however, be noted that these finite dimensional matrix representations of $\vec{J}$, $\vec{K}$, P^{μ}, since $\vec{K}$ is anti-hermitian, lead to a non-unitary finite dimensional matrix representation of the generators of Poincaré group. In fact, the unitary representations of Lorentz group are infinite dimensional as we will see below. Thus these finite dimensional representations do not lead to a unitary representation we are looking for. The matrices corresponding to I_s, I_t, I_{st} can also be easily constructed:

$$I_s = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix} ; \qquad I_t = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \tag{54}$$

4 - *MOMENTUM EIGENSTATES* - From equation (31) it follows that the hermitian operators P^μ can be diagonalized simultanously. We can label the states $|\alpha >$ by the eigenvalues p^μ of P^μ which form a four-vector p and other quantum numbers to be discussed below.

$$P^\mu |p,.. > = p^\mu |p,.. > \tag{55}$$

A translation ($\mathbb{1}$, a) will thus be represented by

$$U(a)|p,.. > = e^{-ip\cdot a} |p,.. > \tag{56}$$

Thus the eigenvalue of P^μ is left unchanged on translation on the other hand a Lorentz transformation (Λ, 0) acting on the above state will generate a new state with momenta (Λp). It follows easily from equation (36):

$$P^{\mu}(U(\Lambda)|p,..>) = U(\Lambda)U^{-1}(\Lambda)\ P^{\mu}\ U(\Lambda)|p,..>$$
$$= U(\Lambda)\ \Lambda^{\mu}{}_{\nu}\ P^{\nu}|p,..> = (\Lambda^{\mu}{}_{\nu}\ p^{\nu})(U(\Lambda)|p,..>) \tag{57}$$
$$= (\Lambda p)^{\mu}\ (U(\Lambda)|p,..>)$$

while

$$U(\Lambda)P^{\mu}\ U^{-1}(\Lambda)(U(\Lambda)|p,..>) = p^{\mu}(U(\Lambda)|p,..>)$$

Also, it shows that, under Lorentz transformation $(\Lambda, 0)$ the value of $\mathcal{M}^2 = (P^{\mu}\ P_{\mu})$ is not changed. We may thus identify the operator P^{μ} with the energy momentum operator and $\mathcal{M}^2$ with the square of the total centre of mass energy i.e. $(\text{mass})^2$ of the physical system. In fact, we can show

$$\left[\mathcal{M}^2, P^{\mu}\right] = 0 \qquad \left[\mathcal{M}^2\ , M^{\mu\nu}\right] = 0 \tag{58}$$

implying that $\mathcal{M}^2$ is a Casimir operator of Poincaré group (and is consequently proportional to identity operator). Its eigenvalues can be used to characterize irreducible representations of the Poincaré group. In the case of the state of a single particle $\mathcal{M}^2 = m^2\ I$ where m is the mass of the particle.

5 - *INVARIANT SPIN OPERATOR. LITTLE GROUP* - To find other quantum members which together with p^{μ} describe the state vector we consider the sub-space of states of a given mass 'm' which is characterized by a fixed value of fourmomenta p^{μ} e.g.

$$\begin{aligned} P^{\mu}|m, p, ..> &= p^{\mu}|m, p, ..> \\ \mathcal{M}^2|m, p, ..> &= m^2|m, p, ..> \end{aligned} \tag{59}$$

where $p^2 = m^2 > 0$ and $p^0 > 0$.

The sub-set of Lorentz transformations $(\Lambda, 0)$ that leaves the eigen-value p^μ of P^μ invariant constitute a sub-group of $\mathcal{L}_+^\uparrow$ called the "Little Group" associated with the four-momenta p^μ. These transformations are thus restricted by

$$\Lambda^\mu{}_\nu\, p^\nu = p^\mu \tag{60}$$

For infinitesimal transformation

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \alpha^\mu{}_\nu + \ldots \tag{61}$$

with $\alpha_{\mu\nu} = -\alpha_{\nu\mu}$ leads to

$$\alpha^\mu{}_\nu{}^\nu = 0 \tag{62}$$

The most general form of $\alpha_{\mu\nu}$ satisfying equation (62) can be written as:

$$\alpha_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma}\, p^\rho\, n^\sigma \tag{63}$$

where n^σ is an arbitrary (infitesimal) four-vector. The corresponding unitary transformation of the "Little Group" is thus written as

$$U(\Lambda) = \mathbb{1} + \frac{i}{2}\,\varepsilon_{\mu\nu\rho\sigma}\;p^\rho\, n^\sigma\, M^{\mu\nu} + \ldots \tag{64}$$

$$= \mathbb{1} - i\, n^\sigma\, W_\sigma(p) + \ldots$$

where

$$W_\sigma(p) = -\frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\, M^{\mu\nu}\, p^\rho \tag{65}$$

are the four hermitian generators of the Little Group associated with momenta p^μ. Since the transformations under consideration act on the momentum eigenstates we may as well write the generators of the Little Group by

$$W_\sigma = +\frac{1}{2}\,\epsilon_{\sigma\mu\nu\rho}\, M^{\mu\nu}\, P^\rho \qquad (66)$$

This is the four-vector of Pauli-Lubanski. It is readily verified that

$$\left[W_\sigma,\ P^\mu\right] = 0 \qquad (67)$$

A finite transformation belonging to the Little Group is given by $(e^{-in^\mu W_\mu})$ and all the states $e^{-in^\mu W_\mu}|m, p^\mu,.. >$ are eigenstates of P^μ with the same eigenvalue p^μ. A commutation relation similar to equation (38) can be derived e.g.

$$\left[M^{\mu\nu},\ W^\sigma\right] = -\,i(W^\mu g^{\nu\sigma} - W^\nu g^{\mu\sigma}) \qquad (68)$$

showing that W_σ is a four-vector like P^μ. This relation implies (cf. P^μ).

$$U(\Lambda)\, W^\mu\, U^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu\, W^\nu \qquad (69)$$

There is an important difference in that the components of W_σ do not commute. In fact, we can show

$$\left[W_\lambda,\ W_\sigma\right] = i\ \epsilon_{\lambda\sigma\alpha\beta}\, W^\alpha\, P^\beta \qquad (70)$$

Also, there are only three of the components independent in W_σ due to the relation

$$W_\sigma\, P^\sigma = 0\ . \qquad (71)$$

We also verify that $(W_\sigma W^\sigma)$ commutes with all the ten generators of Poincaré group and thus constitutes along with $\mathcal{M}^2$, the two Casimir operators of this group. It can be shown that these are the only possible invariant operators of Poincaré group.

6 - *EIGENVALUES OF* $(W_\sigma W^\sigma)$ -

(I) *Non-Vanishing Mass* $(p^2 > 0)$ *Eigenstates* $|P^\mu, J, M,.. >$

For non-vanishing mass the momentum four-vector in the rest frame (or C.M. frame) is $\tilde{p}^\mu = (m, \vec{0})$. A state with four-momentum p^μ is obtained by a Lorentz transformation L(p) such that any four-vector $a^\mu = (a^0, \vec{a})$ is related to the corresponding four-vector $\tilde{a}$ in the rest frame through $a^\mu = L(p)^\mu{}_\nu \tilde{a}^\nu$ where $(\sqrt{p^2} = m)$

$$\vec{a} = \vec{\tilde{a}} + \frac{\vec{p}}{\sqrt{p^2}} \left(\frac{\vec{p}\cdot\vec{\tilde{a}}}{p^0 + \sqrt{p^2}} + \tilde{a}^0 \right) \tag{72}$$

$$a^0 = \frac{1}{\sqrt{p^2}} (\tilde{a}^0 p^0 + \vec{p}\cdot\vec{\tilde{a}})$$

determine the Lorentz matrix L(p) - the "boost" of 'p'. The inverse $L^{-1}(p)$ is given by

$$\vec{\tilde{a}} = \vec{a} + \frac{\vec{p}}{\sqrt{p^2}} \left(\frac{\vec{p}\cdot\vec{a}}{p^0 + \sqrt{p^2}} - a^0 \right) \tag{73}$$

$$\tilde{a}^0 = \frac{1}{\sqrt{p^2}} (a^0 p^0 - \vec{p}\cdot\vec{a}) = \frac{(a\cdot p)}{\sqrt{p^2}}$$

L(p) corresponds to a Lorentz transformation along the direction of momentum $\vec{p}$ with velocity $\vec{\beta} = \frac{\vec{p}}{E}$ (and $\gamma = E/m$). We note also the convention $L(\tilde{p}) = 1$. The eigenstate of momentum p^μ is then given by

$$|m, p, .. > = U(L(p))|m, \tilde{p}, .. > \tag{74}$$

since from equation (57)

$$P^{\mu}|m, p.. > = p^{\mu}|m, p >$$

and

$$(W_{\mu} W^{\mu})|m,p,.. > = (W_{\mu} W^{\mu}) U(L(p))|m, \bar{p} >$$

$$= U(L(p)) (W_{\mu} W^{\mu})|m, \bar{p} > \qquad (75)$$

since $(W_{\mu} W^{\mu})$ commutes with all the generators of Poincaré group.

Now, from the definition W_o, we find:

$$W_i = \frac{1}{2} \varepsilon_{i\mu\nu\rho} M^{\mu\nu} P^{\rho} = \frac{1}{2} \varepsilon_{ijko} M^{jk} P^{o} + \frac{1}{2} \varepsilon_{i\mu\nu k} M^{\mu\nu} P^{k}$$

$$= - \frac{1}{2} \varepsilon_{oijk} M^{jk} P^{o} + \varepsilon_{iojk} M^{oj} P^{k} \qquad (76)$$

$$= P^{o} J_i + \varepsilon_{oijk} K_j P_k$$

$$W_o = \frac{1}{2} \varepsilon_{oijk} M^{ij} P^{k} = - J_k P^{k}$$

or

$$W = (- \vec{J} \cdot \vec{P}, P^{o} \vec{J} - \vec{K} \times \vec{P}) \qquad (77)$$

where $\vec{J} = (J_1, J_2, J_3)$, $\vec{P} = (P^1, P^2, P^3)$ and $(\vec{J} \cdot \vec{P}) = (J_k P^k) = - J_k P_k$.
For the state in the rest frame $(\vec{J} \equiv \vec{S})$

$$W_{\sigma}|m, \bar{p}, .. > = W_{\sigma}(\bar{p}) |m, \bar{p}, .. >$$

where

$$W_{\sigma}(\bar{p}) = (0, m \vec{J}) \equiv (0, m \vec{S}) \qquad (78)$$

For single particle state S is the intrinsic spin angular momentum. W^{σ} may thus be thought of as relativistic generalization of spin operator.

From equations (75) and (78) it follows from the theory of angular momentum operator:

$$W_{\sigma} W^{\sigma}|m,p, \ldots > = - m^2 U(L(p)) S^2|m, \tilde{p}, \ldots > \quad (79)$$
$$= - m^2 s(s+1)|m, p, \ldots >$$

where $s = 0, \frac{1}{2}, 1, \frac{3}{2}$

The result in equation (78), for $m \neq 0$ case, indicates that the Little Group associated with $\tilde{p}$ is the space rotation group SO(3) whose covering group is SU(2). The result is otherwise obvious since any space rotation will leave $\tilde{p}$ invariant. We can thus specify an irreducible unitary representation of the rotation group on the rest states. The states at rest, hence, can thus be characterized by, in addition to $\tilde{p}$(and m), s and a projection quantum number m_s on a z-axis defined in the rest frame. The rest states will thus indicated as $|[m,s], \tilde{p}, m_s,.. >$. The states corresponding to four momenta 'p' are defined using equation (74), ($L(p)\tilde{p} = p$ and $L^{-1}(p)p = \tilde{p}$)

$$|[m,s], p, m_s,.. > = U(L(p))|[m,s], \tilde{p}, m_s,\ldots > \quad (80)$$

Here it should be remembered that

$$J_z |\tilde{p}, m_s,.. > = m_s|\tilde{p}, m_s,.. >$$

and (81)

$$U(L(p))J_z U(L^{-1}(p))|[m,s],p,m_s,..> = m_s|[m,s], p, m_s, .. >$$

that is the projection 'm_s' for a moving particle is measured along the axis derived from the z-axis in the rest frame by the Lorentz transformation L(p). This convention is quite inconvinient when many particles are involved, say, in scattering process. An alternative more convinient way of labelling the spin projection in the form of helicity states will be discussed latter.

Before leaving the present discussion, we remark that the operators $U(L(p))\, W_\sigma\, U(L^{-1}(p)) = W_\sigma'$ likewise behave, while acting on state with momentum p, like W_σ acting on the rest state. From equations (69) and (73) we obtain $W_0' = \frac{1}{m} W_\sigma p^\sigma$ so that $W_0'(p) = 0$ against equation (78) giving $W_0(\bar{p}) = 0$. One can also verify that $\left[W_i'(p), W_j'(p)\right] = i\, \varepsilon_{ijk}\, m\, W_k'(p)$ allowing us to identify $(W_i'(p)|m)$ with the angular momentum operator $\vec{S}$ (since $\vec{W}'^2(p) = \vec{W}^2(\bar{p})$) in the rest frame. It also shows that the Little group associated with p^μ is isomorphic to SO(3). In fact little groups at different points of the orbit ($p^2 = m^2$) are seen to be conjugate to each other. We also show latter that the manifold of state vectors representing the possible states which particle can occupy form a representation space (infinite dimensions) of the unitary irreducible representation $[m,s]$ of the Poincaré group. In other words given the state vector representing a possible state of the particle, all other physical state vectors representing the same particle are obtained by means of a unitary Lorentz transformation acting on the original state.

From equations (57) and (81) we note that while the state $|\bar{p}, m_s>$ is an eigenstate of the "comuting" operators P^μ and J_z (e.g. $\left[J_z, P^\mu\right]|\bar{p}, m_s> = 0$) the state $|p, m_s>$ is eigenstate of the "commuting" operators P^μ and $J_z' = U(L(p))J_z\, U^{-1}(L(p))$. In fact from equation (81),

$$P^\mu J_z'|p,m_s> = m_s p^\mu |p,m_s> \text{ and } J_z' P^\mu|p, m_s> = p^\mu m_s |p, m_s> \qquad (82)$$

showing that $\left[P^{\mu}, J_z'\right] = 0$ when applied on the states $|p, m_s>$. These states are not simultaneous eigenstates P^{μ} and J_z. We have, by definition, chosen to specify the state with momentum p^{μ} with the same additional labels as the rest state corresponding to momentum $\tilde{p}$. That is to say we choose different sets of commuting operators to specify the rest state corresponding to momentum p ; the operators P^{μ}, P^2, W^2 being common to the two sets. The latter operators are obtained from the former operators through a Lorentz transformation U(L(p)).

(II) *Massless Case, ($p^2 = 0$, $p^{\mu} \neq 0$):*

In the case $P_{\mu} P^{\mu} = 0$ ($p^{\mu} \neq 0$), no rest frame exists. Because of $p^{0\,2} = \vec{p}^{\,2}$ it is clear that by a space rotation we can reach the standard state with $p_R^{\mu} = (p, 0, 0, p > 0)$. Equation (71) implies

$$W_{\sigma} P^{\sigma} |p_R, \ldots > = p(W_0 - W_3)|p_R, \ldots > = 0 \tag{83}$$

or

$$W_0|p_R, \ldots > = W_3|p_R, \ldots >$$

Also

$$W_{\sigma} W^{\sigma}|p_R, \ldots > = -(W_1{}^2 + W_2{}^2)|p_R, \ldots > = -\rho^2|p_R, \ldots > \tag{84}$$

where ρ is a Poincaré invariant. From the commutation relation of W_{σ} we also derive:

$$\begin{aligned} \left[W_1, W_2\right]|p_R, \ldots > &= i\,\epsilon_{1230}(W^3\ P^0 - W^0\ P^3)|p_R, \ldots > = 0 \\ \left[W_3, W_1\right]|p_R, \ldots > &= ipW_2|p_R, \ldots > \\ \left[W_3, W_2\right]|p_R, \ldots > &= -i\,p\,W_1|p_R, \ldots > \end{aligned} \tag{85}$$

e.g. $[W_1, W_2] = 0$; $[W_3, W_1] = i\,p\,W_2$; $[W_3, W_2] = -\,i\,p\,W_1$, in the state vector space under consideration. Also

$$W_0 = W_3 = p\,M^{12};\; W_1 = (M^{20} - M^{23})p;\; W_2 = (-M^{01} + M^{13})p. \qquad (86)$$

Thus the Little Group corresponding to p_R is the two dimensional Euclidean group E(2) in two dimension $E(2) \equiv SO(2) \wedge T(2)$ (semi-direct product) * - W_1 and W_2 representing two translation generators along axes 1 and 2 while (W_3/p) the generator of rotation in (12) plane.

It follows that $(W_1^2 + W_2^2)$ that is ρ^2 can take any value. In fact, in the case of $SO(2) \wedge T(2)$ the irreducible unitary representations are either infinite dimensional or one-dimensional. In the latter case the two "translations" e.g. W_1 and W_2 are mapped on zero so that $\rho = 0$ and the little group is effectively only the rotation group of the plane. Its irreducible unitary representation are one-dimensional characterized by a projection quantum number which takes values o, ± 1/2, ± 1. For integral values the representation is single-valued while for half integers it is double valued. Now in nature there are no physical states corresponding to continuous spin, so that we restrict the physical massless states to be such that

$$W_1|p_R, .. > = W_2|p_R, .. > = 0 \qquad (87)$$

$$W_\sigma W^\sigma|p_R, .. > = 0 \qquad \text{e.g. } \rho = 0$$

We may thus write

$$W_\mu|p_R, \lambda.. > = -\lambda\, P_\mu|p_R, \lambda.. > \qquad (88)$$

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* See for example: J. D. Talman, Special Functions, W. A. Benjamin (1968).

Utilizing the fact that W_μ, like P_μ, is a four-vector we can write using equations (36) and (69)

$$W_\mu|p,\lambda \,..\, \rangle = -\lambda\, P_\mu|p,\lambda ..\rangle \qquad (89)$$
$$W_\sigma W^\sigma|p,\lambda, \,..\, \rangle = 0$$

Here λ is an invariant quantity with the dimensions of angular momentum and could be used to characterize the state of a massless particle in addition to the four-vector 'p'. For $\mu = 0$ equation (89) reads

$$W_0|p,\lambda ..\rangle = -(\vec{J}\cdot\vec{P})|p, ..\rangle = -\lambda p_0|p,\lambda \,..\,\rangle \qquad (90)$$

or

$$\left(\frac{\vec{J}\cdot\vec{P}}{|\vec{p}|}\right)|p,\lambda,..\rangle = |p,\lambda ..\rangle$$

It shows that for massless states the "Helicity" operator $\frac{\vec{J}\cdot\vec{P}}{|\vec{p}|} = h(\vec{P})$ representing the projection of angular momentum operator $\vec{J}$ along the direction of motion is an invariant operator and its eigenvalue λ can be used to characterize the spin state just as (J,M) was used in the non-vanishing mass case. In the present case the intrinsic angular momentum state is, however, described by a single number $\lambda(=0, \pm 1/2, \pm 1, ..)$ e.g. there is only one spin state unlike in the nonvanishing mass case where for each J (or S) we had a set of (2J+1) states. $|\lambda|$ is often called the spin of the massless particle. We must remark, however, that for $\lambda \neq 0$ if the particle has definite parity both helicity states $\lambda = \pm|\lambda|$ are possible e.g. there are <u>two</u> states possible for a given spin $|\lambda|$. This is seen from the fact that under parity operation the helicity operator is pseudo-scalar ($\vec{p} \to -\vec{p}$, $\vec{J} \to \vec{J}$), thus changing λ to $-\lambda$. This is not in contradiction to invariant character of λ, since parity operation does not belong to proper Lorentz group discussion above. If the particle does not have definite parity, only one helicity state is allowed. The examples for the two cases are γ having $\lambda_{\gamma} = \pm 1$ and ν with $\lambda_{\nu} = -1/2$ while $\bar{\nu}$ with $\lambda_{\bar{\nu}} = +1/2$.

(III) *Other cases:*

In the case of space like four-vector p^{μ} the corresponding little group is the non-compact rotation group SO(2, 1) while in the case of well vector ($p^{\mu} \equiv 0$) the little group is the homogeneous Lorentz Group SO(3, 1). The representations in these cases and in the case I and II have been reviewed and generalized by Salam et al. (Partial wave analysis, IC/67/9).

HELICITY FORMALISM

3.1 - *HELICITY STATES FOR MASSIVE PARTICLE*

We saw in the last chapter that any massive particle is characterized by its mass 'm' and spin 'S'. The latter is identical to a proper angular momentum in its rest frame. Spin can also be concieved as a four-vector W_μ. For massless particles this four-vector must be parallel to the four-momentum. The constant of proportionality allows us to define a 'spin' for massless particles. However, massless particle has only two or one spin states according to its being or not an eigenstate of parity. For massive particles we could use the states $|[m,s], p^\mu, m_s, .. >$, with the known transformation properties under Poincaré group to describe experiments and the S-matrix. An alternative representation called Helicity representation, however, is more convenient in that it allows us to put the treatment of massive and massless particles on the same footing.

The single particle states of a massive particle can be described by the commuting set of observables $P^2 = P^\mu P_\mu$, $\frac{W^2}{m^2} = \frac{W_\mu W^\mu}{m^2}$, P^μ and one of the (non-vanishing) components of $1/m\ W_\mu$ (or a linear combination $a_0 \frac{W_0}{m} + a_1 \frac{W_1}{m^1} + a_2 \frac{W_2}{m^2} + a_3 \frac{W_3}{m^3}$). The eigenvalues of the last operator are, however, not known except in the rest frame where it is simply a projection of the spin operator. We may thus start from the rest frame and obtain the states for arbitrary momentum by an appropriate Lorentz transformation. The classification of the rest states is equivalent to the well known classification of the irreducible representations of S^2 and, say, the component S_3. The (2s+1) linearly independent states denoted by $\{|[m,s], \tilde{p}, m_s>; m_s = -s, -(s-1) \ldots +s\}$ form a complete set in the rest frame of a particle with spin s. The relative phases between these states are defined as usual by

$$S_{\pm}|\tilde{p};m_s> = \sqrt{(s\mp m_s)(s\pm m_s+1)}\,|\tilde{p}; m_s \pm 1> \tag{1}$$

where $S_{\pm} = (S_1 \pm i\, S_2)$ and

$$S_3|\cdot\tilde{p}; m_s> = m_s|\cdot\tilde{p}; m_s> \tag{2}$$

Under a finite rotation $R(\alpha, \beta, \gamma)$ these rest states transform according to a (2s + 1) dimensional unitary irreducible representation of the rotation group SO(3):

$$U(R(\alpha,\beta,\gamma))|[m,s], \tilde{p}, m_s> = \sum_{m_s'=-s}^{m_s'=+s} D^{(s)}_{m_s' m_s}(\alpha,\beta,\gamma)|[m,s],\tilde{p},m_s'> \tag{3}$$

where

$$U(R(\alpha,\beta,\gamma)) = e^{-iJ_3\alpha}\, e^{-iJ_2\beta}\, e^{-iJ_3\gamma} \tag{4}$$

and *

$$D^{(j)}_{m'm}(R(\alpha,\beta,\gamma)) \equiv D^{(j)}_{m'm}(\alpha,\beta,\gamma) = <j\, m'|U(R(\alpha,\beta,\gamma))|jm>$$

$$= e^{-i(m'\alpha+m\gamma)}\, d^{(j)}_{m'm}(\beta) \tag{5}$$

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* See for example: K. Gottfried, Quantum Mechanics, Vol. I.

with the real rotation matrices

$$d^{(j)}_{m'm}(\beta) = \langle jm'|e^{-i J_2 \beta}|j m\rangle \tag{6}$$

Starting from a state at rest we may obtain the states in motion. We define * (see also equations (17) and (19)):

$$\begin{aligned} |[m,s], \vec{p}, m_s\rangle &= U(R(\phi,\theta, -\phi))\; U(L_3(v))\; U^{-1}(R\,(\phi,\theta,-\phi))|[m,s], \tilde{p}, m_s\rangle \\ &\equiv U(L(p))|[m,s], \tilde{p}, m_s\rangle \end{aligned} \tag{7}$$

(defined already in equation (2.81)) and

$$|[m,s], \vec{p}, \lambda\rangle \equiv U(R(\phi, \theta,-\phi))\; U(L_3(v))|[m,s], \tilde{p}, \lambda\rangle \tag{8}$$

Here (θ,ϕ) are polar angles of $\vec{p}$ and $\vec{v} = \frac{\vec{p}}{p^0}$. The Lorentz transformation $L_3(v)$ along the axis can be expressed in terms of infinitesimal generator K_3 as $(c = 1)$

$$L_3(v) = e^{i K_3 \psi}\;; \quad 0 < \psi < \infty \tag{9}$$

where $v = \tanh\phi$, $p^0 = m\cosh\phi$ and $|\vec{p}| \equiv p = m\sinh\phi$. The Lorentz transformation along $\vec{v}$ (or $\vec{p}$) is obtained to be

$$R(\phi, \theta,-\phi)e^{i K_3 \psi}R^{-1}(\phi, \theta,-\phi) = e^{i \frac{v\cdot K}{v}\psi} = L(\vec{v}) \equiv L(\vec{p}) \tag{10}$$

the rotation operator is taken to be * $R\,(\phi, \theta, -\phi)$ so that $R(\phi,\theta,-\phi)=\mathbb{1}$. The states in equation (8) are called Helicity States since they are simultaneous eigenstates of helicity operator $\frac{\vec{J}\cdot\vec{P}}{|\vec{P}|} = h(\vec{P})$, P^2, W^2 and P^μ. That they are eigenstates of P^μ with momenta p^μ follows from the vector character of P^μ; it may also be verified to be so by direct

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* Note that both the operators $R\,L_3\,R^{-1}$ and RL_3 take $\tilde{p}$ to p.

calculation using commutation relations of the generators. That they are eigenstates of helicity operator follows from the observations: (1) $\left[h(\vec{P}),\vec{J}\right] = 0$ eg. $h(\vec{p})$ is invariant under space rotations and (2) $h(\vec{p})$ remains invariant under pure Lorentz transformations along the direction $(\frac{\vec{p}}{p})$ (unless such a Lorentz transformation transforms beyond the rest system and thereby changes the sign of $\vec{p}$ and consequently that of $h(\vec{p})$). This is a special case of the commutation relation $\left[\vec{J}\cdot\vec{n}, \vec{K}\cdot\vec{n}\right] = 0$ e.g. the space rotation around an axis $\vec{n}$ commutes with the Lorentz transformation along the same axis. In fact, we have

$$J_3|\bar{p}, \lambda> = \lambda|\bar{p}, \lambda > \tag{11}$$

Applying the "boost" $U(L_3(v)) = e^{iK_3\psi}$ and using $\left[J_3, K_3\right] = 0$ we find, writing $p_R^\mu = (p^0, 0, 0, p > 0)$

$$\left(\frac{\vec{J}\cdot\vec{p}_R}{|\vec{p}_R|}\right)|p_R, \lambda > \equiv J_3|p_R, \lambda > = \lambda|p_R, \lambda > \tag{12}$$

where

$$|p_R, \lambda > = e^{iK_3\psi}|\bar{p}, \lambda > \tag{13}$$

Now applying a rotation $U(R(\phi, \Theta, -\phi))$ so that p_R^μ goes over to p^μ with $\vec{p}$ pointing along (Θ, ϕ) we have

$$U(R(\phi,\Theta, -\phi))\,\frac{\vec{J}\cdot\vec{p}_R}{|\vec{p}_R|}\,U^{-1}(R(\phi,\Theta, -\phi))|\vec{p},\lambda> = \lambda|\vec{p},\lambda > \tag{14}$$

But

$$U(R(\phi,\Theta, -\phi))\,\vec{J}\cdot\vec{n}\,U^{-1}(R(\phi,\Theta, -\phi)) = \vec{J}\cdot(R\vec{n}) \tag{15}$$

so that

$$\frac{\vec{J}\cdot\vec{p}}{|\vec{p}|}\,|\vec{p},\lambda> = \lambda|\vec{p}, \lambda > \tag{16}$$

with $\lambda = -s, -(s-1)\ldots(s+1), s$. Alternatively,

$$\frac{\vec{J}\cdot\vec{p}}{|\vec{p}|}\,|\vec{p},\lambda\rangle = U(R(\phi,\theta,-\phi))\;\frac{\vec{J}\cdot\vec{P}}{|\vec{P}|}\,e^{iK_3\psi}|\tilde{p},\lambda\rangle$$

$$= U(R(\phi,\theta,-\phi))\,J_3\,e^{iK_3\psi}|\tilde{p},\lambda\rangle \tag{17}$$

$$= \lambda|\vec{p},\lambda\rangle$$

Still another way of arriving at the helicity operator is to simply (cf. equ. (2.81)) note that if Λ_p is a Lorentz transformation such that $\Lambda_p\,\tilde{p} = p$ then from equ. (2.57),

$$P^\mu(U(\Lambda_p)|[m,s];\tilde{p},\alpha\rangle) = p^\mu(U(\Lambda_p)|[m,s];\tilde{p},\alpha\rangle) \tag{18}$$

Defining

$$|[m,s];p,\alpha\rangle = U(\Lambda_p)|[m,s];\tilde{p},\alpha\rangle$$

$$P^\mu|[m,s];p,\alpha\rangle = p^\mu|[m,s];p,\alpha\rangle \tag{19}$$

If the rest state is also an eigenstate of J_3 with eigen-value α e.g.

$$J_3|[m,s];\tilde{p},\alpha\rangle = \alpha|[m,s];\tilde{p},\alpha\rangle \tag{20}$$

it is clear that

$$U(\Lambda_p)\,J_3\,U^{-1}(\Lambda_p)|[m,s];p,\alpha\rangle = \alpha|[m,s];p,\alpha\rangle \tag{21}$$

For the choice $\Lambda_p \equiv R(\phi,\theta,-\phi)\,L_3(v)$ (in place of $\Lambda_p = L(p) \equiv R(\phi,\theta,-\phi)\,L_3(v)\,R^{-1}(\phi,\theta,-\phi)$ we obtain, using equ. (15),

$$U(\Lambda_p)\,J_3\,U^{-1}(\Lambda_p) = U(R)\,U(L_3)\,J_3\,U^{-1}(L_3)\,U^{-1}(R) = U(R)\,J_3\,U^{-1}(R) = \frac{\vec{J}\cdot\vec{p}}{|\vec{p}|} \tag{22}$$

The difference between the helicity states and the states specified by m_s lies in the choice of Λ_p operator.

The relation between the two sets of basis vectors is seen to be

$$|[m,s];\vec{p},\lambda> = U(R)\ U(L_3)\ U^{-1}(R)\ U(R)|[m,s];\bar{p},\lambda>$$

$$= \sum_{m_s=-s}^{+s} D^{(s)}_{\lambda m_s}\ (R(\phi,\Theta,-\phi))|[m,s];\vec{p},m_s> \qquad (23)$$

$$|[m,s];\vec{p},m_s> = \sum_{\lambda} D^{(s)}_{\lambda m_s}\ (R^{-1}(\phi,\Theta,-\phi))|[m,s];\vec{p},\lambda> \qquad (24)$$

and we remind that (see equations (2) and (7)):

$$U(L(p))\ J_3\ U^{-1}(L(p))\ [m,s];\vec{p},m_s> = m_s|[m,s],\vec{p},m_s> \qquad (25)$$

Unlike the case of massless particles where helicity λ is an invariant parameter characterizing the state, in the case of massive states it is not so. The transformation of massive states under a Lorentz transformation will be studied latter in this chapter.

We next define helicity states with momenta pointing along negative Z axis that is for states with momenta $p_R' = (p^0, 0, 0, -p)$ where $p > 0$. It will be taken to be ($e^{-i\pi J_2}\ \vec{p}_R = -\vec{p}_R$)

$$|[m,s];p_R',\lambda> \equiv |[m,s];-\vec{p}_R,\lambda> = (-1)^{s-\lambda} e^{-i\pi J_2}|[m,s];\vec{p}_R,\lambda>$$

$$= (-1)^{s-\lambda}\ e^{-i\pi J_2}\ e^{iK_3\psi}|[m,s];\bar{p},\lambda> \qquad (26)$$

$$= e^{-i K_3\psi}\ |[m,s],\bar{p},-\lambda>$$

The last equality follows from *

$$S^{-1} e^{A} S = e^{S^{-1}AS}$$

$$e^{-i\pi J_2} K_3 e^{i\pi J_2} = - K_3 \qquad (27)$$

$$e^{-i\pi J_2} | [m,s]; \tilde{p}, \lambda > = (-1)^{s-\lambda} | [m,s], \tilde{p}, - \lambda >$$

The last result follows from the well known results in angular momentum theory,

$$e^{-i\pi J_2} |jm > = \sum_{m'} d^{j}_{m'm}(\pi) |j,m' > = \sum_{m'} (-1)^{j-m} \delta_{m',-m} |jm'>$$

$$= (-1)^{j-m} |j, -m > \qquad (28)$$

and

$$e^{-i(2\pi)\vec{J}\cdot\vec{n}} |jm > = (-1)^{2j} |jm > \qquad (29)$$

We also verify for helicity state that ($\vec{p}_R' = - \vec{p}_R$)

$$\frac{\vec{J}\cdot\vec{p}_R'}{|\vec{p}_R|} | [m,s]; - \vec{p}_R, \lambda > = - J_3 | [m,s]; -\vec{p}_R, \lambda >$$

$$= - J_3 e^{-iK_3\psi} | [m,s]; \tilde{p}, -\lambda> = \lambda | [m,s], - \vec{p}_R, \lambda> \qquad (30)$$

The result in equation (30) is not surprizing since the helicity operator for state with momentum ($-\vec{p}_R$) is ($-J_3$) while it is ($+ J_3$) for state with momenta $\vec{p}_R$). Thus we start with a rest state which has eigenvalue ($-\lambda$) of J_3 and then impart momentum p along -ve Z axis through the boost

* Note also $e^{-i\pi J_2} e^{-i\pi J_2} K_3 e^{i\pi J_2} e^{i\pi J_2} = K_3$. and that the states $|\pm\vec{p}_R,\lambda>$ are eigen-states of J_3 with eigenvalues $\pm\lambda$.

$\bar{e}^{iK_3\psi}$. We also note with our phase conventions

$$|[m,s], -\vec{p}_R \to \vec{0}^-, \lambda > = |[m,s], \vec{p}_R \to \vec{0}^+, -\lambda > \tag{31}$$

For $\lambda = s > 0$, the maximum value, we obtain

$$|[m,s], -\vec{p}_R, s| = \bar{e}^{i\pi J_2} |[m,s], \vec{p}_R, s > \tag{32}$$

By analogy we define for the massless states *

$$|m = 0, -\vec{p}_R, |\lambda| > = \bar{e}^{i\pi J_2}|m = 0, \vec{p}_R, |\lambda| > \tag{33}$$

$$|m = 0, -\vec{p}_R, -|\lambda|> = (-1)^{2|\lambda|} \bar{e}^{i\pi J_2}|m = 0, \vec{p}_R, -|\lambda| >$$

In case it has a well defined parity, state with $-|\lambda|$ is also possible and can be reached by parity operation to be discussed latter. The states with momentum $(-\vec{p})$ are then obtained by applying the rotation operator $U(R(\phi,\Theta, -\phi))$.

3.2 - *TRANSFORMATION OF MASSIVE STATES UNDER POINCARÉ GROUP*

We consider the action of Poincaré transformation on States of massive particles. Since $U(\Lambda, a) = U(a)\, U(\Lambda)$ it leads to

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* Then

$$|m = 0, \vec{p}_R, |\lambda| > = (-1)^{2|\lambda|} \bar{e}^{i\pi J_2} \ |m = 0, -|\vec{p}_R|, |\lambda| >$$

$$= e^{i\pi J_2} \ |m = 0, -\vec{p}_R, |\lambda| >$$

$$U(\Lambda,a)|[m,s];p,..> = e^{iP\cdot a}\, U(\Lambda)|[m,s];\, p,..>$$

$$= U(\Lambda)\, U^{-1}(\Lambda)\, e^{iP\cdot a}\, U(\Lambda)|[m,s];\, p,..> \quad (34)$$

$$= U(\Lambda)\, e^{ip\cdot(\Lambda a)}|[m,s];\, p,\, ..>$$

We can thus dispense of translation and consider only homogeneous Lorentz Transformations. Now

$$U(\Lambda)|[m,s],\vec{p},\, m_s > = U(\Lambda)\, U(L(p))|[m,s],\, \tilde{p},\, m_s >$$
$$= U(L(\Lambda p))\, U(L^{-1}(\Lambda p)\, \Lambda L(p))|[m,s],\, \tilde{p},\, m_s > \quad (35)$$

Clearly,

$$R(\Lambda p,p) = (L^{-1}(\Lambda p)\, \Lambda L(p)) \quad (36)$$

is a rotation since, for example, $L^{-1}(\Lambda p)\, \Lambda L(p)\tilde{p} = L^{-1}(\Lambda p)(\Lambda p)\, \tilde{p} = \tilde{p}$ or $R(\Lambda p,\, p)\, \tilde{p} = \tilde{p}$. It follows from equation (3)

$$U(R(\Lambda p,\, p))|[m,s],\, \tilde{p},\, m_s >$$

$$= \sum_{m_s'} D^{(s)}_{m_s' m_s}\, (R(\Lambda p,\, p)|[m,s],\, \tilde{p},\, m_s' > \quad (37)$$

whence

$$U(\Lambda)|[m,s]\vec{p},\, m_s > = \sum_{m_s'=-s}^{+s} D^{(s)}_{m_s' m_s}(R(\Lambda p,p))|[m,s],\Lambda p,\, m_s' > \quad (38)$$

The rotation $R(\Lambda p,\, p)$ is called Wigner's rotation.

The transformation of helicity states follows from equation (18):

$U(\Lambda)||m,s|, \vec{p}, \lambda>$

$$= \sum_{m_s} \sum_{m_s'} D^{(s)}_{m_s\lambda}(\phi,\Theta,-\phi)\, D^{(s)}_{m_s'm_s}(R(\Lambda p,p))|[m,s], \Lambda p, m_s'>$$

$$= \sum_{m_s'} D^{(s)}_{m_s'\lambda}(L^{-1}(\Lambda p)\,\Lambda L(p)\, R(\phi,\Theta,-\phi)|[m,s], \Lambda p, m_s'>$$

$$= \sum_{m_s'} \sum_{\mu} D^{(s)}_{m_s'\lambda}(L^{-1}(\Lambda p)\Lambda L(p)R(\phi,\Theta,-\phi))\, D^{(s)}_{\mu m_s'}(R^{-1}(\phi,\Theta,-\phi)|[m,s], \Lambda p, \mu>$$

$$= \sum_{\mu} D^{(s)}_{\mu\lambda}(R^{-1}(\phi,\Theta,-\phi)L^{-1}(\Lambda p)\,\Lambda L(p)\, R(\phi,\Theta,-\phi))|[m,s]\ \Lambda p, \mu>$$

$$= \sum_{\mu} D^{(s)}_{\mu\lambda}(R^{-1}(\phi,\Theta,-\phi))\, R(\Lambda p, p)\, R(\phi,\Theta,-\phi))|[m,s], \Lambda p, \mu> \qquad (39)$$

We also note from equation (10)

$$L(\vec{p})\, R(\phi,\Theta,-\phi) = R(\phi,\Theta,-\phi)\, L_3(v) \qquad (40)$$

Combining with equation (28) we obtain essentially a unitary operator representation of $U(a,\Lambda)$. In fact if we define the norm * of our state vectors in invariant fashion

$$<[m,s]; \vec{p}', m_s' |[m,s]; \vec{p}, m_s> = (2\pi)^3 2p^0\, \delta^3(\vec{p}-\vec{p}')\, \delta_{m_s m_s'} \qquad (41)$$

and a similar definition for helicity states, we can show that

$$UU^{\dagger} = U^{\dagger}U = I \qquad (42)$$

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* See next section.

For example,

$$< [m,s]\,;\ \vec{p}\,',\ m_s' | U^{\dagger}(\Lambda,\ a)\ U(\Lambda,\ a) | [m,s]\,;\ \vec{p},\ m_s >$$

$$= e^{-i(p'-p)\cdot\Lambda a} < [m,s]\,;\ \vec{p}\,',\ m_s'\ | U^{\dagger}(\Lambda)\ U(\Lambda) | [m,s]\,;\ \vec{p},\ m_s >$$

$$= e^{-i(p'-p)\cdot\Lambda a} \sum_{m_1,\ m_2} D^{(s)\,*}_{m_2 m_s'}(R(\Lambda p',\ p')\ D^{(s)}_{m_1 m_s}(R(\Lambda p,\ p))\ \cdot$$

$$\cdot < [m,s]\,;\ \Lambda\vec{p}\,',\ m_2 | [m,s]\,;\ \Lambda\vec{p},\ m_1 >$$

The Lorentz invariance of $p^0\delta^3(\vec{p}-\vec{p}\,')$ and the relation

$D^{(j)}_{m'm}(R) = D^{(j)}_{mm'}(R^{\dagger}) = D^{(j)}_{mm'}(R^{-1})$ leads to

$$= (2\pi)^3 2p^0\delta^3(\vec{p}-\vec{p}\,') \sum_{m'} D^{(s)}_{m_s' m_1}(R^{-1}(\Lambda p,\ p))\ D^{(s)}_{m_1 m_s}(R(\Lambda p,\ p)) \tag{43}$$

$$= (2\pi)^3\ 2p^0\delta^3(\vec{p}-\vec{p}\,')\ \delta_{m_s\ m_s'}$$

implying the relation in equation (42). Thus these states constitute a basis (infinite dimensional) for an irreducible unitary representation of the Poincaré group.

An alternative way of checking the unitary is as follows. From the normalization condition we have the resolution of the identity viz:

$$\sum_{\lambda} \int d^4p\ \frac{\eta}{(2\pi)^3}\ \ \Theta(p^0)\ \delta(p^2 - m^2)\ |\vec{p},\lambda > < \vec{p},\lambda| = I \tag{44}$$

Then

$$U(\Lambda)\ U^{\dagger}(\Lambda) = U(\Lambda)\ I\ U^{\dagger}(\Lambda)$$

$$= \int d^4p \frac{\eta}{(2\pi)^3} \Theta(p^0)\ \delta(p^2 - m^2) \sum_{\lambda} U(\Lambda)|\vec{p}\lambda\rangle\langle\vec{p}\lambda|U^{\dagger}(\Lambda)$$

From equation (32)

$$= \int d^4p \frac{}{(2\pi)^3} \Theta(p^0)\ \delta(p^2 - m^2) \sum_{\lambda} \Big\{ \sum_{\mu} D^{(s)}_{\mu\lambda} \quad (R^{-1}R_W R)|\vec{p}\,',\mu\rangle$$

$$\sum_{\sigma} D^{(s)}_{\sigma\lambda}{}^{*} \quad (R^{-1}R_W\ R)\langle\ \vec{p}\,',\ \sigma| \Big\}$$

where $p' = \Lambda p$, $R \equiv R(\phi,\Theta,-\phi)$ and $R_W \equiv R(\Lambda p,\ p)$

$$= \int d^4p' \frac{\eta}{(2\pi)^3} \Theta(p'^0)\ \delta(p'^2-m^2) \sum_{\lambda\mu\sigma} \Big\{ D^{(s)}_{\mu\lambda}(R^{-}R_W R) D^{(s)}_{\sigma\lambda}(R^{-1}R_W R) \; |\vec{p}\,'\mu\rangle\langle\ \vec{p}\,'\sigma| \Big\}$$

$$= \int d^4p' \frac{\eta}{(2\pi)^3} \Theta(p'^0)\ \delta(p'^2 - m^2) \sum_{\mu\sigma} \delta_{\mu\sigma}\ |\vec{p}\,'\mu\rangle\langle\ \vec{p}\,'\sigma\ |$$

Since D(R) form a unitary representation

$$= I$$

Thus the unitary of $U(\Lambda)$ is tied to that of the representation of the little group of the four-vector $\bar{p}$.

3.3 - *HELICITY STATES IN ANGULAR MOMENTUM REPRESENTATION*

From the helicity states constructed above as simultaneous eigen-states of the operators P^2, W^2, P^{μ} and $\frac{\vec{J}\cdot\vec{P}}{|\vec{p}|}$. We can construct helicity states which are simultaneous eigenstates of the commuting operators P^2,

W^2, $|\vec{P}|$, $\frac{\vec{J}\cdot\vec{P}}{|\vec{P}|}$, J^2 and J_3. To attain this we make use of the "projection theorem" of angular momentum theory.

Projection theorem: Suppose $\{|jm\,\gamma\rangle\}$ be the "standard basis" angular momentum representation for the set of states $\{|\psi\rangle\}$; then the (j, m) projection of the state $|\psi\rangle$ is obtained by

$$|jm\,\gamma\rangle \equiv \mathbb{N}_j \int D^{(j)^*}_{mm'}(\mathcal{R})\; U_a(\mathcal{R})|\psi\rangle\, dR \tag{45}$$

where $\mathcal{R} \equiv R(\alpha, \beta, \gamma)$. By "standard basis" we mean that the operators J^2, J_Z and the set $\{\Gamma\}$ of operators which together form a complete set also constitute a mutually commutating set. Here $\mathbb{N}_j$ is an appropriate normalization factor. The invariante volume element of the rotation group (or rather of the covering group SU(2)) ** dR is given by $dR = d\alpha\; d\beta\; d\gamma\; \sin\beta$ with $0 < \alpha < 2\pi$, $0 < \beta < \pi$, $0 < \gamma < 4\pi$, and the integration is taken over the whole group space of SU(2) (Hurwitz integral).

We will show now that the constructions in equation (45) of angular momentum states is adequate by demonstrating that the states so defined, transform according to (2j+1) dimensional unitary irreducible representation of the rotation group SO(3) (or rather of the group SU(2)). In fact, we have

$$U_a(\mathcal{R}')|jm\,\gamma\rangle = \mathbb{N}_j \int D^{(j)^*}_{mm'}(\mathcal{R})\; U_a(\mathcal{R}'\mathcal{R})|\psi\rangle\, dR$$

$$= \mathbb{N}_j \int \sum_{m''} D^{(j)}_{mm''}(\mathcal{R}'^{-1}) D^{(j)^*}_{m''m'}(\mathcal{R}'\mathcal{R})\; U_a(\mathcal{R}'\mathcal{R})|\psi\rangle\, dR$$

* The situation is analogous to the expansion of free particle wave function $e^{i\vec{k}\cdot\vec{r}}$ which is eigenstate of $\vec{P}$ in terms of wave functions $j_\ell(kr)Y^m_\ell(\Theta,\phi)$ which are simultaneous eigenfunctions of $|\vec{P}|$, L^2 and L_3.

** See for example: J. D. Talman, Special Functions, W. A. Benjamin Inc. (68) Hammermesh:

Under the rotation $\mathcal{R}'(\alpha',\beta',\gamma')$ the Euler angles (α,β,γ) transform to $(\alpha'', \beta'', \gamma'')$ such that

$$d\alpha\, d\beta\, d\gamma \sin\beta = d\alpha''\, d\beta''\, d\gamma'' \frac{\partial(\alpha\beta\gamma)}{\partial(\alpha''\,\beta''\,\gamma'')} \sin\beta$$

The Jacobian can be shown to be

$$\frac{\partial(\alpha\beta\gamma)}{\partial(\alpha''\,\beta''\,\gamma'')} = \frac{\sin\beta''}{\sin\beta}$$

so that

$$dR = d(R'R) = d(RR') \tag{46}$$

e.g. the volume element remains invariant under a rotation. Thus

$$U_a(\mathcal{R}')|jm\gamma\rangle = \mathbb{N}_j \sum_{m''} D^{(j)*}_{mm''}(\mathcal{R}')^{-1} \int D^{(j)*}_{m''m'}(\mathcal{R}'\mathcal{R})\, U_a(\mathcal{R}'\mathcal{R})|\psi\rangle\, d(R'\,R)$$

or in view of equation (35)

$$U_a(\mathcal{R}')|jm\,\gamma\rangle = \sum_{m''} |jm''\,\gamma\rangle D^{(j)}_{m''m}(\mathcal{R}') \tag{47}$$

In case the states $|\psi\rangle$ are eigenstates of J_3, say $J_3|\psi\rangle = M|\psi\rangle$ so that $U(R(\alpha,\beta,\gamma))|\psi\rangle = U(R(\alpha,\beta,0))|\psi\rangle e^{-iM\gamma}$ we find $|jm\gamma\rangle =$

$$= \mathbb{N}_j \int D^{(j)*}_{mm'}(\alpha,\beta,0)\, U(R(\alpha,\beta,0))|\psi\rangle_M\, e^{i(m'-M)\gamma}\, d\alpha\, d\beta\, d\gamma \sin\beta$$

$$= N_j\, \delta_{m',M} \int D^{(j)*}_{mM}(\alpha,\beta,0)\, U(R(\alpha,\beta,0))|\psi\rangle_M\, d\alpha\, d\beta\, \sin\beta$$

Writing ϕ in place of α and Θ in place of β

$$\begin{aligned} |jm\,\gamma\rangle &= N_j \int D^{(j)*}_{mM}(\phi,\Theta,0)\, U(R(\phi,\Theta,0))|\psi\rangle_M\, d\Omega \\ &= N_j \int D^{(j)*}_{mM}(\phi,\Theta,-\phi)\, U(R(\phi,\Theta,-\phi))|\psi\rangle_M\, d\Omega \end{aligned} \tag{48}$$

Here (Θ, ϕ) are the spherical polar coordinates and the solid angle $d\Omega =$ $= \sin\Theta \, d\Theta \, d\phi$.

Applying this result to the helicity states $|\vec{p}_R, \lambda >$ the (J,M) projection is given by

$$
\begin{aligned}
|[m,s], |\vec{p}|; J\,M\,\lambda > &= N_J \int D^{(J)\,*}_{M\lambda}(\phi,\Theta,-\phi)\; U(R(\phi,\Theta,-\phi))|\vec{p}_R, \lambda > d\Omega \\
&= N_J \int D^{(J)\,*}_{M\lambda}(\phi,\Theta,-\phi)\,|[m,s], \vec{p}\;;\lambda > d\Omega \qquad (49)\\
&= N_J \int D^{(J)\,*}_{M\lambda}(\phi,\Theta,-\phi)\,|[m,s], |\vec{p}|\Theta\phi;\lambda > \; d\Omega
\end{aligned}
$$

The normalization factor N_J will be calculated below.

3.4 - *NORMALIZATION COMPLETENESS RELATION:*

Since we have used a set of mutually commuting hermitian operators to specify the state of a single particle they are orthogonal with respect to the corresponding eigenvalues. For momentum eigenstates under consideration $\mathcal{M}^2 = m^2 = p^2$ takes discrete (fixed) values so that only the spatial part $\vec{p}$ of p^μ is needed to specify p^μ e.g. the point on the orbit $p^2 = m^2$ The inner product of states belonging to p^μ and p'^μ (with the same value of m) must then contain a factor $\gamma(\vec{p}, m)\,\delta^3(\vec{p}-\vec{p}')$ where $\gamma(\vec{p}, m)$ is so chosen that $\gamma\,\delta^3(\vec{p}-\vec{p}')$ is Lorentz invariant. From $(E_p =$ $= +\sqrt{\vec{p}^2 + m^2})$ the relations,

$$
\begin{aligned}
1 &= \int d^3p\,\delta^3(\vec{p}-\vec{p}') = \int \frac{d^3p}{2E_p}\cdot 2\,E_p\,\delta^3(\vec{p}-\vec{p}') \\
&= \int d^4p\,\Theta(p^0)\,\delta(p^2-m^2)\cdot 2\,p^0\,\delta^3(\vec{p}-\vec{p}')
\end{aligned}
\qquad (50)
$$

it follows that $2p^0\ \delta^3(\vec{p}-\vec{p}\,')$ is invariant under proper orthochronous Lorentz transformations. We will adopt thus the following invariant normalization

$$< [m,s],\ \vec{p},\ m_s,\ \sigma \,|\, [m,s],\ \vec{p}\,',\ m_s',\ \sigma'>$$
$$= \frac{(2\pi)^3}{\eta_|}\ 2\,E_p\ \delta^3(\vec{p}-\vec{p}\,')\ \delta_{m_s m_s'}\ \delta_{\sigma\sigma'} \qquad (51)$$

Here σ refers to the additional set of quantum numbers to specify the particle e.g. iso-spin, hypercharge etc. For, helicity states m_s, m_s' are replaced by λ, λ' and $\delta_{\lambda\lambda'}$ appears in place of $\delta_{m_s m_s'}$. The factor $\eta_|$ is defined to be

$$\begin{aligned} \eta_| &= 1 \quad \text{for bosons} \\ &= 2\,m \quad \text{for fermions}^{*} \end{aligned} \qquad (52)$$

Explicit covariance of the normalization adopted can be also seen from writing it as follows

$$< \vec{p},\ m_s | \vec{p}\,',\ m_s' > \Delta^+(p) = \frac{(2\pi)^4}{\eta}\ \delta^4(p^\mu - p'^\mu)\ \delta_{m_s m_s'} \qquad (53)$$

where

$$\begin{aligned} \Delta^+(p) &= 2\pi\Theta(p^0)\ \delta(p^2 - m^2) \\ &= 2\pi\ \frac{\Theta(p^0)}{2\,E_p}\ \delta(p^0 - E_p) \end{aligned} \qquad (54)$$

For two particle states

$$|[m_1 s_1][m_2 s_2];\ \vec{p}_1,\ \vec{p}_2;\ \lambda_1\ \lambda_2 > \equiv |[m_1 s_1],\ \vec{p}_1, \lambda_1> \otimes\ |[m_2 s_2],\ \vec{p}_2,\ \lambda_2> \quad (55)$$

the normalization condition is

---

* For m=0 fermions one may give a finite mass. The physical quantities do not depend on it and the limit $m \to 0$ is the same as putting $m \equiv 1$.

$$< [m_1 s_1][m_2 s_2];\ \vec{p}_1 \vec{p}_2;\ \lambda_1 \lambda_2 \,|\, [m_1 s_1][m_2 s_2];\ \vec{p}_1{}' \ \vec{p}_2{}' \ ;\ \lambda_1' \ \lambda_2' >$$

$$= \frac{(2\pi)^6}{\eta|_1 \ \eta|_2} \delta^3(\vec{p}_1 - \vec{p}_1{}') \ \delta^3(\vec{p}_2 - \vec{p}_2{}') \ \delta_{\lambda_1 \lambda_1'} \ \delta_{\lambda_2 \lambda_2'} \tag{56}$$

The completeness relation then reads

$$1\!\!| = |vac> <vac| + \sum_{m,s} \sum_{\lambda} \frac{\eta|}{(2\pi)^3} \int \frac{d^3p}{2E_p} \,|\, [m,s] \ \vec{p}\lambda >< [m,s] \ \vec{p}\lambda \,|$$

$$+ \sum \frac{\eta|_1 \ \eta|_2}{(2\pi)^6} \int \frac{d^3 \ p_1 \ d^3 \ p_2}{2E_{p_1} \ \ 2E_{p_2}} \,|\vec{p}_1 \ \vec{p}_2 \ \lambda_1 \lambda_2 >< \vec{p}_1 \ \vec{p}_2 \ \lambda_1 \lambda_2| + \ldots \tag{57}$$

$$= |vac> <vac| + \sum \frac{\eta|}{(2\pi)^3} \int |\ \vec{p}\lambda> \Theta(p^0) \ \delta(p^2 - m^2) \ d^4 \ p < \vec{p} \ \lambda|$$

$$+ \sum \frac{\eta|_1 \ \eta|_2}{(2\pi)^6} \int |p_1 \ \vec{p}_2 ; \lambda_1 \lambda_2 > \ \Theta(p_1^0) \ \Theta(p_2^0) \ \delta(p_1^2 - m_1^2) \ \delta(p_2^2 - m_2^2)$$

$$d^4p_1 \ \ d^4p_2 < \vec{p}_1 \ \vec{p}_2 ; \ \lambda_1 \ \lambda_2| + \ldots \tag{58}$$

We remark also that, with the invariant normalization of states a matrix element of a covariant operator $\mathcal{O}$, viz, $< a|\mathcal{O}|b >$ is also a covariant expression.

The density of momentum states with the above normalization is given by *

---

* This is clear, for example, from the completeness relation.

$$\frac{\eta_|}{(2\pi)^3}\ \frac{d^3p}{2E_p} = \frac{\eta_|}{(2\pi)^3}\int \Theta(p^0)\ \delta(p^2 - m^2)\ d^4p \tag{59}$$

which is clearly an invariant expression.

3.5 - *NORMALIZATION OF ANGULAR MOMENTUM HELICITY STATES*

We will now determine the normalization factor N_J. The invariant normalization of momentum eigen-states reads

$$<[m,s],\vec{p};\lambda|[m,s],\ \vec{p}\,';\ \lambda'\ > \equiv <[m,s],\ |\vec{p}|,\ \Theta,\ \phi;\lambda|\ [M,s],\ |\vec{p}\,'|,\ \Theta',\ \phi';\ \lambda'\ >$$

$$= \frac{(2\pi)^3}{\eta_|}\ 2\ E_p\ \delta^3(\vec{p}-\vec{p}\,')\ \delta_{\lambda\lambda'}$$

$$= N_p\ \delta(|\vec{p}| - |\vec{p}\,'|)\ \delta(\Omega-\Omega')\delta_{\lambda\lambda'} \tag{60}$$

where

$$\delta(\Omega-\Omega') = \delta(\cos\Theta - \cos\Theta')\ \delta(\phi-\phi')$$

and $p^2 = p'^2 = m^2$ with $p^0,\ p^{0\prime} > 0$ are understood and

$$N_p = \frac{(2\pi)^3}{\eta_|}\ \ \frac{2E_p}{|\vec{p}|^2} \tag{61}$$

we <u>define</u> by analogy

$$<[m,s],|\vec{p}|,\ J'\ M'\ \lambda'\ |\ [m,s],\ |\vec{p}|,\ J\ M\ \lambda\ >$$

$$= N_p\ \delta(|\vec{p}| - |\vec{p}\,'|)\delta_{\lambda\lambda'}\ \delta_{JJ'}\ \delta_{MM'} \tag{62}$$

Now

$$N_p\, \delta(|\vec{p}| - |\vec{p}|)\, \delta_{\lambda\lambda'}\, \delta_{JJ'}\, \delta_{MM'}$$

$$= N_J\, N_{J'}^{*}\, N_p\, \delta(|\vec{p}| - |\vec{p}'|) \iint d\Omega d\Omega'\, D^{(J')}_{M'\lambda}(\phi',\theta',-\phi')\, D^{(J)}_{M\lambda}{}^{*}(\phi,\theta,-\phi)\delta_{\lambda\lambda'}$$

$$= N_J N_{J'}^{*}\, N_p\, \delta(|\vec{p}| - |\vec{p}|\,)\, \delta_{\lambda\lambda'} \int d\Omega\, D^{(J')}_{M'\lambda}\, (\phi,\theta,\, -\phi)\, D^{(J)}_{M\lambda}{}^{*}\, (\phi,\theta,\, -\phi)$$

$$= |N_J|^2\, N_p\, \delta(|\vec{p}| - |\vec{p}'|)\, \delta_{\lambda\lambda'} \cdot \frac{4\pi}{(2J+1)}\, \delta_{MM'}\, \delta_{JJ'}$$

on using

$$\int d\Omega\, D^{*(J)}_{M\lambda}\, (\phi,\theta,\, -\phi)\, D^{(J)}_{M'\lambda}\, (\phi,\theta,\, -\phi) = \frac{4\pi}{(2J+1)}\, \delta_{JJ'}\, \delta_{MM'} \tag{63}$$

We obtain

$$|N_J|^2 = \frac{2J+1}{4\pi}$$

We choose the convenient arbitrary phase convertion to write

$$N_J = +\sqrt{\frac{2J+1}{4\pi}} \tag{64}$$

3.6 - *CLEBSCH-GORDON COEFFICIENTS: INVERSE RELATION:*

The matrix elements $<\vec{p}',\lambda'||\vec{p}|J\ M\ \lambda>$ can be easily calculated:

$$< |\vec{p}'|\theta'\phi';\, \lambda'||\vec{p}|\ J\ M\ \lambda >$$

$$= N_J \int d\Omega\, D^{(J)}_{M\lambda}{}^{*}(\phi,\theta,\, -\phi) < |\vec{p}'|\ \theta'\phi';\lambda'||\vec{p}|\ \theta\phi;\lambda >$$

$$= N_J\, N_p\, \delta(|\vec{p}| - |\vec{p}'|)\, D^{(J)}_{M\lambda}{}^{*}(\phi',\theta',\, -\phi')\delta_{\lambda\lambda'} \tag{65}$$

The normalization condition for the states $||\vec{p}|JM;\lambda>$ implies a completeness relation of the type

$$\sum_J \sum_M \sum_\lambda \int \frac{d|\vec{p}|}{N_p} \quad ||\vec{p}|\ J\ M\ ;\ \lambda > <|\vec{p}|\ J\ M;\ \lambda\ | = I \tag{66}$$

where I on the righthard side is the formal identity in the Hilbert space of one-particle states under consideration. The relation can be checked for example, by multiplying both sides by $||\vec{p}|\ J\ M;\ \lambda >$.

We can use this relation to obtain the inverse relation expressing states $|\vec{p},\ \lambda >$ in terms of the angular momentum states $||\vec{p}|\ J\ M;\ \lambda >$.

$$|\vec{p}\,',\lambda' > \equiv\ ||\vec{p}\,'|\Theta'\phi'\lambda' >$$

$$= \sum \int d\ |\vec{p}|\ ||\vec{p}|\ J\ M\ \lambda > \frac{1}{N_p}\ < |\vec{p}|\ J\ M\ \lambda||\vec{p}\,'|\ \Theta'\phi'\lambda' >$$

$$= \sum_J \sum_M\ N_J^*\ D_{M\lambda}^{(J)}\ (\phi',\Theta',\ -\phi')\ ||\vec{p}|\ J\ M\ \lambda' > \tag{67}$$

With the normalization conditions adapted above we can verify easily

$$\delta(\cos\Theta-\cos\Theta')\delta(\phi-\phi') = \sum_J \left(\frac{2J+1}{4\pi} \right) D_{M\lambda}^{(J)^*}(\phi,\Theta,\ -\phi)\ D_{M\lambda}^{(J)}(\phi,\Theta,\ -\phi) \tag{68}$$

and note the formal relations:

$$< \Theta'\ \phi'\ \lambda'|\ \Theta\ \phi\ \lambda > = \delta(\Omega - \Omega')\delta_{\lambda\lambda'}$$

$$< J'M'\lambda'|\ J\ M\ \lambda > = \delta_{JJ'}\ \delta_{MM'}\ \delta_{\lambda\lambda'} \tag{69}$$

For spinless particles $\lambda = 0$, $J \equiv \ell$, $M \equiv m$, $|\vec{p}| \equiv p$,

$$|\vec{p}> = \sum_{\ell}\sum_{m} \sqrt{\frac{2\ell+1}{4\pi}}\ D^{(\ell)}_{mo}\ (\phi_p,\ \Theta_p,\ -\ \phi_p)|p,\ell,m>$$

$$= \sum\sum_{m} |p\,\ell\, m > Y^m_\ell\ (\Theta_p,\ \phi_p)^* \qquad (70)$$

on using

$$D^{(\ell)}_{mo}\ (\alpha,\beta) = \sqrt{\frac{4\pi}{2\ell+1}}\ Y^{m\ *}_{\ell}\ (\beta,\alpha)\quad .$$

Also, since *

$$Y^m_\ell(\Theta_p,\ \phi_p) = <\Theta_p\ \phi_p\ |\ \ell m>$$

we have

$$|\vec{p}> = \sum_{\ell}\sum_{m} |p\ \ell\ m> <\ell m|\Theta_p\ \phi_p> \qquad (71)$$

In coordinate space representation $(<\vec{r}|\vec{r}'> = \delta^3(\vec{r}-\vec{r}')$

$$<\vec{r}|\vec{p}> \equiv \phi_{\vec{p}}(\vec{r}) = \sum_{\ell}\sum_{m} <\vec{r}|\ p\ \ell\ m> Y^{m\ *}_{\ell}(\Theta_p,\ \phi_p)$$

Our normalization $<\vec{p}'|\vec{p}> = \frac{(2\pi)^3}{\eta}\ 2\ E_p\ \delta^3(\vec{p}-\vec{p}')$

implies

$$\phi_{\vec{p}}(\vec{r}) = \sqrt{\frac{2E_p}{\eta}}\ e^{i\vec{p}\cdot\vec{r}} \qquad (73)$$

From the expansion

$$e^{i\vec{p}\cdot\vec{r}} = 4\pi\ \sum_{\ell}\sum_{m} i^\ell\ j_\ell(pr)\ Y^{m\ *}_{\ell}(\Theta_p,\ \phi_p)\ Y^m_\ell(\Theta,\phi) \qquad (74)$$

---

* Note $<\Theta_{\vec{r}}\ \phi_{\vec{r}}|\ell m> = i^\ell\ Y^m_\ell(\Theta_{\vec{r}}\ \ \phi_{\vec{r}})$ in order that under time reversal the state transforms in the same way under the coordinate and momentum space representations.

it follows:

$$< \vec{r}|p\ \ell\ m > = 4\pi\ \ j_\ell(pr) i^\ell\ Y_\ell^m(\Theta,\phi) \sqrt{\frac{2E_p}{\eta}} \tag{75}$$

or

$$< r\ \Theta\ \phi|p\ \ell\ m > = < r|p\ell > < \Theta\phi|\ell m > \tag{76}$$

where

$$< \Theta\phi|\ell m > = i^\ell\ Y_\ell^m\ (\Theta,\phi)$$

and *

$$< r|p\ell > = \sqrt{\frac{2E_p}{\eta}}\ 4\ \pi\ j_\ell(pr) \tag{77}$$

$$< r|r'\ > = \frac{\delta(r-r')}{rr'} \tag{78}$$

so that

$$< p'\ell|p\ell > = \int_0^\infty r^2 dr < p'\ell|r > < r|p\ell >$$

$$= \int_0^\infty\ r^2\ dr\ j_\ell(p'r)\ j_\ell(pr)(4\pi)^2\ (2E_{p/\eta})$$

$$= \frac{(2\pi)^3}{\eta}\ 2\ E_p\ \frac{\delta(p\ -\ p')}{p^2} \tag{79}$$

$$= N_p\ \delta(p-p')$$

Finally, the inverse relation is easily obtained

---

* $< r\Theta\phi|r'\Theta'\phi' > = \delta^3(\vec{r}-\vec{r}') = \frac{\delta(r-r')}{rr'}\ \delta(\Omega-\Omega')$ suggest us to write $< \Theta\phi|\Theta'\phi' > = \delta(\Omega-\Omega')$, $< r|r' > = \delta(r-r')/rr'$ and $\int|\Theta\phi><\Theta\phi|\ d\Omega = \mathbb{1}$ while $\int|r> r^2dr<r| = \mathbb{1}$ as completeness conditions.

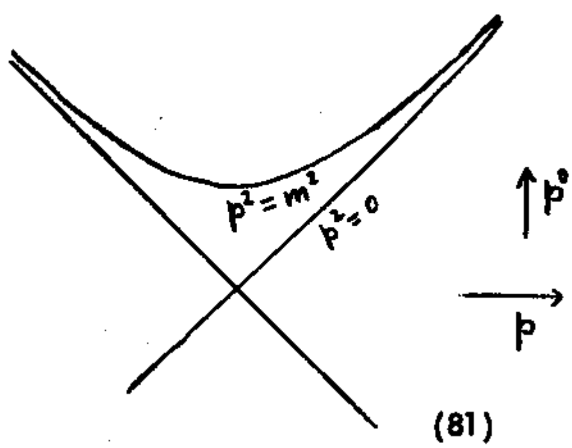
$$|p\ell m> = \int |\vec{p}> Y_\ell^m(\theta_p, \phi_p)\, d\Omega_p \tag{80}$$

3.7 - *STRUCTURE OF HILBERT SPACE OF STATE VECTORS:*

From the discussion above we see that the eigenstate of P_μ is not, in general, fully specified by the eigenvalue p_μ since P_μ's may not form a complete set by themselves. The set of momentum eigenstates belonging to a given p^μ constitute a Hilbert space which is a sub-space of the total Hilbert space H.

It is clear that the spaces H_p, on a given orbit ($p^2 = m^2$ fixed), corresponding to different p^μ are isomorphic to each other. This can be seen as follows. Select one particular 4-vector $\bar{p}^\mu$ on the orbit.

We can reach a 4-momentum p^μ by a Lorentz transformation Λ_p e.g. $\Lambda_p \bar{p} = p$. Since Λ_p belongs to a group there exists an unique inverse transformation Λ_p^{-1} such that $\Lambda_p^{-1} p = \bar{p}$. The operator $U(\Lambda p)$ maps $H_{\bar{p}}$ into H_p since



$$P^\mu(U(\Lambda p)|m; \bar{p}>) = p^\mu(U(\Lambda p)|m; \bar{p}>) \tag{81}$$

that is $|m;p> = U(\Lambda_p)|m;p>$; $|m; \bar{p}> = U(\Lambda_p^{-1})|m; p>$

and

$$P^\mu|m; p> = p^\mu|m; p> \tag{82}$$

The inverse $U(\Lambda_p^{-1}) = U^{-1}(\Lambda_p)$ is defined so that the mapping is one-to-one implying that all the spaces H_p are isomorphie to each other. The Hilbert space to which all the spaces H_p, on a given orbit, are isomorphic will be denoted by h.

There exists thus a one-one correspondence between the vectors $|\alpha_p>$ of each space H_p and the vectors $|\alpha>$ of h. The state vectors $|\alpha_p>$ of H belonging to a definite orbit may thus be written explicitly as direct product

$$|m; f> \otimes |m; \alpha> \tag{83}$$

where $\{|m;f>\}$ is the space of vectors fully labelled by p^μ's on the orbit in question. Such a state is, for example, the wave-packet state

$$|m;f> = \int d^4p\, \delta(p^2-m^2)\, \Theta(p^0)\, f(p^\mu)\, |m; p^\mu> \tag{84}$$

Here $f(p^\mu)$ is weight factor corresponding to p^μ. Since $\{|m; p^\mu>\}$ constitute a basis for the space $\{|m; f>\}$ the space H is spanned by the vectors of type *

$$|m; p^\mu> \otimes |m; \alpha> \equiv |m; \vec{p}> \otimes |m;\alpha> \equiv |m,p^\mu;\alpha> \tag{85}$$

Here $\{|m;\alpha>\}$ 'a set of vectors faming a basis in h.

From

$$P^\mu|m,p^\mu; \alpha> = p^\mu|m, p^\mu; \alpha> \tag{86}$$

or

$$P^\mu(|m,p^\mu> \otimes |m;\alpha>) = p^\mu(|m; p^\mu> \otimes |m;\alpha>)$$

It follows, that $|m;\alpha>$ is translation invariant. We define the relation between the reference state $|m, \bar{p}; \alpha>$ and the state $|m, p; \alpha>$ by

* Note $|m,p^\mu> \equiv |m,\vec{p}>$ for obvious reasons.

$$|m,p;\alpha\rangle \equiv U(\Lambda_p)|m,\hat{p};\alpha\rangle \qquad (87)$$

where $\Lambda_p\,\hat{p} = p$ and $\Lambda_p^{-1}p = \hat{p}$. Written in direct product form

$$|m;p\rangle \otimes |m;\alpha\rangle \equiv U(\Lambda_p)(|m;\hat{p}\rangle \otimes |m;\alpha\rangle) \qquad (88)$$

That it is possible to choose the some additional labels on both sides can be seen from the following. Suppose α be the eigenvalue of operators Γ which must then commute with P^μ, say, when acting on the reference state e.g.

$$P^\mu(|m;\hat{p}\rangle \otimes |m;\alpha\rangle) = \hat{p}^\mu(|m;\hat{p}\rangle \otimes |m;\alpha\rangle) \qquad (89)$$

$$\Gamma(|m;\hat{p}\rangle \otimes |m;\alpha\rangle) = \alpha(|m;\hat{p}\rangle \otimes |m;\alpha\rangle)$$

It follows then (see above)

$$P^\mu(|m;p\rangle \otimes |m;\alpha\rangle) = p^\mu(|m;p\rangle \otimes |m;\alpha\rangle) \qquad (90)$$

but

$$U(\Lambda_p)\,\Gamma\,U(\Lambda_p^{-})(|m;p\rangle \otimes |m;\alpha\rangle) = \alpha(|m;p\rangle \otimes |m;\alpha\rangle) \qquad (91)$$

Thus label α on the state $|m,p;\alpha\rangle$ refers to the eigen-state of $\Gamma' = U(\Lambda_p)\,\Gamma\,U(\Lambda^{-1}_p)$ and not of Γ, which clearly does not commute with P^μ when acting on the states $|m,p;\alpha\rangle$. To describe simultaneously p^μ and α on the state $|m,p;\alpha\rangle$ we use the commuting operators P^μ and Γ' while to specify $\hat{p}^\mu$ and α on the reference state $|m,\hat{p};\alpha\rangle$ we use the commuting operators P^μ and Γ. It is clear that the algebraic structure of the operators Γ' (similarity transformed) is the same as that of the

* We consider here $m \neq 0$ case. For $m = 0$ case $\hat{p} \equiv (p, 0, 0, p)$. The corresponding little group has already been discussed.

operator Γ.

Now $U(\Lambda_p)|m;\hat{p}> = |m;p>$ so that $U(\Lambda_p)$ leaves $|m;\alpha>$ invariant, viz, $U(\Lambda_p)|m;\alpha> = |m;\alpha>$. However, in general, (writing $|\alpha> \equiv m;\alpha>$ for conciseness),

$$U(\Lambda)(|m;p> \otimes |\alpha>) = |m;p'> \otimes |\alpha'> \tag{92}$$

Where $p' = \Lambda p$. The relation between $|\alpha'>$ and $|\alpha>$ can be easily deduced

$$U(\Lambda)\, U(\Lambda_p)(|m;\hat{p}> \otimes |\alpha>) = U(\Lambda_{p'})(|m;\hat{p}> \otimes |\alpha'>)$$

or

$$U^{-1}(\Lambda_{p'})\, U(\Lambda)\, U(\Lambda_p)(|m;\hat{p}> \otimes |\alpha>) = (|m;\hat{p}> \otimes |\alpha'>) \tag{93}$$

Thus

$$U(\Lambda_{p'}^{-}, \Lambda\, \Lambda_p) \equiv U^{-1}(\Lambda_{p'})\, U(\Lambda)\, U(\Lambda_p) \quad \text{leaves} \quad |m;\hat{p}>$$

invariant while transforms $|\alpha>$ to $|\alpha'>$. For time like vectors * under consideration $p^2 = m^2 > 0$ we may choose $\hat{p}$ to be the rest four-vector $\hat{p} =$ $= (m, \vec{0})$ whose little group is rotation group. The transformation is then a space rotation in 3-dimensional space:

$$\Lambda_{p'}^{-1}\ \Lambda\ \Lambda_p \equiv R(\Lambda p, p) \equiv \mathcal{R}_w \tag{94}$$

is often called Wigner rotation. It follows that once we specify a unitary representation of the rotation group (a subgroup of Lorentz group) on h we know the transformation of a general vector of H under RIHL. Thus

$$U(\Lambda,a)(|m;p> \otimes |\alpha>) = e^{iP\cdot a}\, U(\Lambda)(|m;p> \otimes |\alpha>)$$
$$= e^{iP\cdot a}\,(|m;p'> \otimes |\alpha'>) = e^{ip'\cdot a}(|m;p'> \otimes |\alpha'>) \tag{95}$$

---

* We consider here $m \neq 0$ case. For $m=0$ case $\hat{p} \equiv (p, 0, 0, p)$. The corresponding little group has already been discussed.

and

$$|\alpha' > = U(R(\Lambda p,p))|\alpha > \equiv D(R_W)\ |\alpha > \qquad (96)$$

where $p' = \Lambda p$ and $D(R)$ is a unitary representation of the rotation group on h.

The normalization of the basis vectors will be chosen to be invariant so that

$$\begin{aligned} < m,p;\alpha|m,p;\alpha > &= <m,p;\alpha|U^{\dagger}(\Lambda)\ U(\Lambda)|m,p;\alpha > \\ &= <m,p';\alpha'|m,p';\alpha'> \end{aligned} \qquad (97)$$

Noting that

$$< \alpha'|\alpha' > = < \alpha|D^{\dagger}(R)\ D(R)|\alpha > = <\alpha|\alpha> \qquad (98)$$

we have

$$< m;p|m;p > = < m;p'|m;\ p' > \qquad (99)$$

A convinient invariant normalization, as discussed before, is

$$< m;p|m;p' > = (2\pi)^3\ 2p^0\ \delta^3(\vec{p} - \vec{p}')$$

The scalar product of wave-packet states will then be

$$\begin{aligned} < m;g|m;f > &= \int d^4p\ d^4p'\ \delta(p'^2 - m^2)\ \delta(p^2 - m^2)\ \Theta(p^0)\ \Theta(p'^0) \\ &\quad g^*(p')\ \ f(p) < m;p'|m;p > \\ &= (2\pi)^3 \int d^4p \int d^3p' \int dp'_0\ \delta(p^2 - m^2)\ \Theta(p^0)\ \Theta(p'^0) \\ &\quad \delta(p'^2 - m^2)\ 2p'^0\ \delta^3(\vec{p}-\vec{p}')\ g^*(p')\ f(p) \\ &= (2\pi)^3 \int d^4p\ \ \delta(p^2 - m^2)\ \Theta(p^0)\ g^*(p)\ \ f(p) \end{aligned} \qquad (100)$$

3.8 - *TWO PARTICLE HELICITY STATES:*

To describe two-particle collision we need to construct helicity states of two (or more) particles *. Since we will be using S-matrix formalism the only multi-particle states we shall need are those describing non-interacting particles.

A non-interacting 2-particle state can be written as direct product of two single-particle, momentum eigen-states:

$$|[m_1 s_1][m_2 s_2]; \vec{p}_1\ \lambda_1; \vec{p}_2\ \lambda_2 >$$

$$\equiv |[m_1 s_1]; \vec{p}_1\ \lambda_1> \otimes |[m_2 s_2]; \vec{p}_2\ \lambda_2 > \qquad (101)$$

We shall, for simplicity, label it by $|\vec{p}_1\ \vec{p}_2;\ \lambda_1\lambda_2;\ \gamma>$ where γ includes the spins and additional quantum numbers needed to describe separate particles.

Under a pure translation

$$U(a)|\vec{p}_1\vec{p}_2;\lambda_1\ \lambda_2;\ \gamma> = e^{i\ P_1\cdot a} \otimes e^{i\ P_2\cdot a}\ |\vec{p}_1\ \vec{p}_2;\ \lambda_1\ \lambda_2;\gamma> =$$

$$= e^{i(p_1+p_2)\cdot a}\ |\vec{p}_1\ \vec{p}_2;\ \lambda_1\ \lambda_2;\ \gamma > \qquad (102)$$

following from the definition of 2-particle state. It follows that

$$(P^\mu \equiv P_1^\mu + P_2^\mu \equiv P_1^\mu \otimes I + I \otimes P_2^\mu)$$

$$P^\mu|\vec{p}_1\ \vec{p}_2;\ \lambda_1\ \lambda_2;\gamma > = (p_1+ p_2)^\mu\ |\vec{p}_1\ \vec{p}_2;\ \lambda_1\ \lambda_2;\ \gamma > \qquad (103)$$

---

* For helicity states of more than two particles see, for example, G. Tindle, Phys. Rev. D3, 1468 (1971) and the references contained therein.

and that $(p_1+p_2)^2 = \varepsilon^2 > 0$ is Lorentz invariant. ε defined to be positive, can be regarded as the "mass of the 2-particle system" and specifies an orbit. However, in the present case, a continuum of values are possible $\infty > \varepsilon > (m_1+m_2)$ in contrast to the single particle case where it takes discrete values. The direct product states even for fixed ε do not define an irreducible representation * of the restricted inhomogeneous Lorentz group (RIHL). They do define a representation which can be decomposed into a direct sum of irreducible representation.

For one-particle states an irreducible representation was constructed by starting from the vectors corresponding to a particle in its rest frame, and defining on these vectors an irreducible representation of the rotation group. The rest of the irreducible representations can be obtained via the appropriate Lorentz boost. We, likewise, in the case of two particles appeal to centre of mass frame of the two particles.

For two non-interacting particles we could, as well describe the state by using the operators $P^\mu = P_1^\mu + P_2^\mu$ and $K^\mu = \frac{1}{2}(P_1^\mu - P_2^\mu)$ in place of the operators P_1^μ, and P_2^μ. The state is then labelled as

$$|\vec{p}\ \vec{k};\ \lambda_1\lambda_2;\ \gamma> \equiv |\varepsilon, p^\mu:\ \theta\phi;\ \lambda_1\lambda_2;\ \gamma>$$
$$\equiv |\vec{p}_1\ \vec{p}_2;\ \lambda_1\lambda_2\ \ \gamma> \equiv |\vec{p}_1\lambda_1> \otimes |\vec{p}_2\lambda_2> \otimes |\gamma> \qquad (104)$$

where $p^\mu = (p_1+p_2)^\mu$, $k^\mu = \frac{1}{2}(p_1-p_2)^\mu$, θ and ϕ are the polar angles of $\vec{k}$ with respect to $\vec{p}$. We note

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* This is obvious, say, by considering the case $\vec{p}_1 = \vec{p}_2 = 0$ and using the angular momentum theory.

$$P^{\mu}|\vec{p}\ \vec{k};\ \lambda_1\lambda_2;\ \gamma> = p^{\mu}|\vec{p}\ \vec{k};\ \lambda_1\lambda_2;\ \gamma> \tag{105}$$

$$K^{\mu}|\vec{p}\ \vec{k};\ \lambda_1\lambda_2;\ \gamma> = k^{\mu}|\vec{p}\ \vec{k};\ \lambda_1\lambda_2;\ \gamma> \tag{106}$$

and $(E_p(m) = +\sqrt{\vec{p}^2 + m^2} > 0)$

$$p^0 = E_{p_1}(m_1) + E_{p_2}(m_2) > 0\ , \qquad k^0 = \frac{1}{2}\left(E_{p_1}(m_1) - E_{p_2}(m_2)\right) \tag{107}$$

From the relations $p_1^{\mu} = k^{\mu} + \frac{1}{2}p^{\mu}$ and $p_2^{\mu} = -k^{\mu} + \frac{1}{2}p^{\mu}$ we can write:

$$\delta^3(\vec{p}_1 - \vec{p}_1')\delta^3(\vec{p}_2 - \vec{p}_2') = \delta^3(\vec{k} - \vec{k}' + \frac{1}{2}[\vec{p}-\vec{p}'])\ \delta^3(-\vec{k}+\vec{k}' + \frac{1}{2}[\vec{p}-\vec{p}'])$$

$$= \delta^3(\vec{p}-\vec{p}')\delta^3(-\vec{k}+\vec{k}' + \frac{1}{2}[\vec{p}-\vec{p}']) = \delta^3(\vec{p}-\vec{p}')\ \delta^3(\vec{k}-\vec{k}') \tag{108}$$

For the volume element in momentum space we note

$$d^3p_1 d^3p_2 = \left|\frac{\partial(p_1^1,\ p_1^2,\ p_1^3;\ p_2^1,\ p_2^2;\ p_2^3)}{(k^1,\ k^2,\ k^3;\ p^1,\ p^2,\ p^3)}\right|\ d^3k\ d^3p = d^3k\ \ d^3p \tag{109}$$

the Jacobian being unity. The right hand side can be expressed as

$$d^3p\,|\vec{k}|^2\ d|\vec{k}|\ d\Omega\ (\theta,\phi) \tag{110}$$

where (Θ,ϕ) denote the polar angles of $\vec{k}$ with respect to $\vec{p}$. Now

$$(p_1^0)^2 = |\vec{k}| + \frac{1}{4}\ |\vec{p}|^2 + |\vec{k}||\vec{p}|\ \cos\theta + m_1^2$$

$$(p_2^0)^2 = |\vec{k}|^2 + \frac{1}{4}\ |\vec{p}|^2 - |\vec{k}||\vec{p}|\ \cos\theta + m_2^2$$

For $\vec{p}$ and $\Omega(\theta,\ \phi)$ fixed

$$p_1^0\ dp_1^0 = d|\vec{k}|\cdot(|\vec{k}| + \frac{1}{2}\ |\vec{p}|\ \cos\theta\ )$$

$$p_2^0\ dp_2^0 = d|\vec{k}|\cdot(|\vec{k}| - \frac{1}{2}\ |\vec{p}|\ \cos\theta\ )$$

$$dp^0 = dp_1^0 + dp_2^0 = \frac{\left[|\vec{k}|p^0 - k^0|\vec{p}| \cos\theta\right]}{p_1^0\ p_2^0}\ d|\vec{k}|$$

Thus

$$d^3k\ \ d^3p = \frac{p_1^0\ p_2^0\ |\vec{k}|^2}{(|\vec{k}|p^0 - k^0|\vec{p}|\ \cos\theta)}\ d^4p\ \ d\Omega$$

From the fact that $\frac{d^3p_1}{2p_1^0}\ \frac{d^3p_2}{2p_2^0}$ is Lorentz invariant we conclude that

$$\frac{d^3k\ d^3p}{p_1^0\ \ p_2^0} = \frac{|\vec{k}|^2\ d\Omega}{(|\vec{k}|p^0 - k^0|\vec{p}|\cos\theta)} \cdot d^4p \qquad (111)$$

is invariant and so is the factor, on the right hand side, multiplying the invariant volume element d^4p. In the centre of mass frame ($\vec{p} = 0$) the r.h.s takes simple form:

$$\frac{d^3p_1\ d^3p_2}{p_1^0\ p_2^0} = \frac{d^3k\ d^3p}{p_1^0\ p_2^0} = \left(\frac{|\vec{k}|}{p^0}\ d\Omega\right)_{c.m}\ d^4p \qquad (112)$$

Furthermore, in the expression

$$\delta^3(\vec{p}-\vec{p}\,')\ \delta^3(\vec{k}-\vec{k}\,') = \delta^3(\vec{p}-\vec{p}\,')\ \frac{\delta(|\vec{k}|-|\vec{k}\,'|)}{|\vec{k}|^2}\ \delta(\Omega-\Omega')$$

where

$$\delta(\Omega-\Omega') = \delta(\cos\theta - \cos\theta')\ \delta(\phi - \phi') = \frac{\delta(\theta-\theta')\ \delta(\phi-\phi')}{\sin\theta}$$

we can use the relation (note $\vec{p} = \vec{p}\,'$, $\Omega = \Omega'$ are fixed)

$$\delta(|\vec{k}|-|\vec{k}|') = \frac{\delta(p^0 - p'^0)}{\left(\frac{\partial|\vec{k}|}{\partial p^0}\right)} = \frac{(|\vec{k}|p^0 - k^0|\vec{p}|\cos\theta)}{p_1^0\ p_2^0}\ \delta(p^0 - p'^0)$$

to obtain *

$$\delta^3(\vec{p}-\vec{p}')\delta^3(\vec{k}-\vec{k}') = \frac{(|\vec{k}|p^0 - k^0|\vec{p}|\cos\theta)}{p_1^0\ p_2^0\ \ |\vec{k}|^2}\ \delta^4(p^\mu - p'^\mu)\ \delta(\Omega-\Omega') \qquad (113)$$

Finally, for the normalization condition we obtain:

$$< \varepsilon, p'; \theta'\phi';\ \lambda_1'\lambda_2' | \varepsilon, p; \theta\phi;\ \lambda_1\lambda_2 > = < \vec{p}'\vec{k}';\ \lambda_1'\lambda_2' | \vec{p}\ \vec{k};\ \lambda_1\lambda_2 >$$

$$= < \vec{p}_1'\ \vec{p}_2';\ \lambda_1'\ \lambda_2'\ |\vec{p}_1\ \vec{p}_2;\ \lambda_1\lambda_2 >$$

$$= \frac{(2\pi)^6}{\eta_1\ \eta_2}\ 2\ p_1^0\ \ 2\ p_2^0\ \ \delta^3(\vec{p}_1 - \vec{p}_1')\delta^3(\vec{p}_2 - \vec{p}_2')\ \delta_{\lambda_1\lambda_1'}\ \delta_{\lambda_2\lambda_2'}$$

$$= \frac{(2\pi)^6}{\eta_1\ \eta_2}\left(\frac{4(|\vec{k}|p^0 - k^0|\vec{p}|\cos\theta)}{|\vec{k}|^2}\right)\delta^4(p^\mu - p'^\mu)\ \delta(\Omega-\Omega')\ \delta_{\lambda_1\lambda_1'}\ \delta_{\lambda_2\lambda_2'} \qquad (114)$$

The normalization is Lorentz invariant. Calculated in the c.m. frame the r.h.s. takes the form ($\delta(\Omega-\Omega')_{cm} \equiv \delta(\cos\Theta - \cos\Theta')\ \delta(\phi-\phi')$) =

$$= \frac{(2\pi)^6}{\eta_1\eta_2}\left(\frac{4p^0}{|\vec{k}|}\ \delta(\Omega-\Omega')\right)_{c.m.}\ \delta^4(p^\mu - p'^\mu)\ \delta_{\lambda_1\lambda_1'}\ \delta_{\lambda_2\lambda_2'} \qquad (115)$$

Here (Θ,ϕ) are measured w.r.t. a z-axis defined in c.m. frame.

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* This result is rather expected. From the relations $p^2 = \varepsilon^2 = p^{0\,2} - |\vec{p}|^2$, $m_1^2 + m_2^2 = p_1^2 + p_2^2 = 2(k^{0\,2} - |\vec{k}|^2) + \frac{1}{2}|\vec{p}|^2$ and $m_1^2 - m_2^2 = 2\ k\cdot p = 2(k^0\ p^0 - \vec{k}\cdot\vec{p})$ we conclude that p^μ, Θ and ϕ are sufficient to specify $\vec{p}$ and $\vec{k}$.

Unlike in the case of single particle case a $\delta^4(p_\mu - p'_\mu)$ appears in the present case. This is expected since $p^2 = (p_1 + p_2)^2$ is not any more fixed and takes on values continuously; in fact, $\infty > p^2 \geqslant (m_1 + m_2)$. The separating out of this factor simply expresses the overall total four momentum conservation between the initial and the final states. It is clearly suggested that we may write

$$|\varepsilon, p^\mu; \ \theta\phi; \lambda_1\lambda_2 > \equiv c_1|\varepsilon, p^\mu> \otimes \ |\theta\phi; \lambda_1\lambda_2 > \qquad (116)$$

where c is a convinient factor. We will adopt the normalization

$$< \varepsilon, p'^\mu|\varepsilon, p^\mu > = (2\pi)^4\delta^4(p^\mu - p'^\mu) \qquad (117)$$

Like in the single particle case the spaces H_p, on a given orbit ($p^2 = \varepsilon^2$ fixed), corresponding to different p^μ are isomorphic to each other. Denoting the Hilbert space to which all the spaces are isomorphic by h the state vectors of the total Hilbert space of two particles (direct product of two vector spaces associated with the irreducible representations $U^{(1)}$ and $U^{(2)}$ of two distinct Poincaré groups) are spanned by the vectors of type (ε indicating the orbit):

$$|\varepsilon; p^\mu> \otimes \ |\varepsilon;\alpha > \equiv |\varepsilon \ , p^\mu; \alpha > \qquad (118)$$

where

$$P^\mu(|\varepsilon; p^\mu> \otimes \ |\varepsilon;\alpha >) = p^\mu(|\varepsilon; p^\mu> \ \otimes \ |\varepsilon;\alpha >) \qquad (119)$$

$$P^\mu|\varepsilon; p^\mu > = p^\mu|\varepsilon; p^\mu > \qquad (120)$$

and

$$|\varepsilon; p^\mu > \ = U(\Lambda_p)|\varepsilon; \tilde{p}^\mu > \qquad (121)$$

where $\Lambda_p\, \hat{p} = p$. It follows ($|\varepsilon;\alpha\rangle \equiv |\alpha\rangle$ for brevity)

$$P^{\mu}|\alpha\rangle = |\alpha\rangle \tag{122}$$

The relation between $|\varepsilon, p; \alpha\rangle$ and $|\varepsilon, \hat{p}; \alpha\rangle$ is <u>defined</u> * as

$$U(\Lambda_p)(|\varepsilon;\hat{p}\rangle \otimes |\alpha\rangle) \equiv |\varepsilon;p\rangle \otimes |\alpha\rangle \tag{123}$$

like in the single particle case implying

$$U(\Lambda_p)|\alpha\rangle = |\alpha\rangle \tag{124}$$

For general Lorentz transformation Λ

$$U(\Lambda)(|\varepsilon;p\rangle \otimes |\alpha\rangle) = |\varepsilon;p'\rangle \otimes |\alpha'\rangle \tag{125}$$

which leads to ($p' = \Lambda p$)

$$|\alpha'\rangle = U(\Lambda_{p'}^{-1}\,\Lambda\,\Lambda_p)|\alpha\rangle \tag{126}$$

where $(\Lambda_{p'}^{-1}\,\Lambda\,\Lambda_p)$ leaves $\hat{p}$ invariant. For the present case under discussion $p^2 > 0$ e.g. time like and we may choose $\hat{p}$ to be the C.M. frame 4-vector $\hat{p} \equiv (p^0 = \varepsilon > 0, \vec{0}) = \hat{p}$. The transformation $(\Lambda_{p'}^{-1}\,\Lambda\,\Lambda_p) \equiv R(\Lambda p,p)$ is then a, pure 3-dimensional rotation (Wigner rotation) belonging to SO(3) whose covering group is SU(2). Thus we need to study simply the C.M. state $|\varepsilon, \hat{p}\rangle \otimes |\alpha\rangle$ and set up the appropriate SU(2) group labels or quantum numbers (so that we know the transformation of $|\alpha\rangle$ under the transformation $U(\Lambda)$ on a general vector $|\varepsilon;p\rangle \otimes |\alpha\rangle$.

It is clear that $P_\mu = P_\mu^{(1)} + P_\mu^{(2)}$ and $\vec{J} = (\vec{J}^{(1)} + \vec{J}^{(2)})$ commute while acting on the C.M. states ($\vec{p} = \vec{p}^{(1)} + p^{(2)} = \vec{0}$) of two particles.

* e.g. the labels α are maintained under $U(\Lambda_p)$.

We can thus include among the labels α the labels (J,M) corresponding to the operators J^2 and J_z. The C.M. states thus read

$$C_2(|\varepsilon;\ \bar{p}> \otimes\ |JM\sigma>) \equiv |\varepsilon\ ,\ \bar{p};\ JM\sigma > \qquad (127)$$

where σ are other labels to be determined and C_2 a convinient factor to be discussed below. These states define an irreducible representation for two particles in their C.M. frame.

To connect these states with the $(|\vec{p}_1\ \lambda_1> \otimes\ |\vec{p}_2\ \lambda_2>)$ we make use of the projection theorem of angular momentum theory. We define

$$|\varepsilon,\ \bar{p};\ J\ M\ \sigma > \propto \int_{SU(2)} D^{(J)^*}_{MM'}(\alpha,\beta,\gamma)\ U(R(\alpha,\beta,\gamma))|\vec{p}_1\ \vec{p}_2;\ \lambda_1\ \lambda_2 > dR \qquad (128)$$

where $0 \leq \alpha \leq 2\pi$, $0 \leq \gamma \leq 4\pi$, $0 \leq \beta \leq \pi$ and $dR = \sin\beta\ d\alpha\ d\beta\ d\gamma$, the invariant volume element and the integration is over the whole space of SU(2) group*. Also $\vec{P}|\varepsilon,\ \bar{p};\ J\ M\ \sigma > = (\vec{p}^{(1)} + \vec{p}^{(2)})|\varepsilon,\ \bar{p};\ J\ M\ \sigma > = 0$ then implies (since M' is arbitrary)

$$\vec{P}\ U(R)|\vec{p}_1\ \vec{p}_2;\ \lambda_1\ \lambda_2 > = 0 \qquad (129)$$

for every U(R). This is only possible if

$$\vec{P}|\vec{p}_1\ \vec{p}_2;\ \lambda_1\ \lambda_2 > = 0 \qquad (130)$$

e.g. $|\vec{p}_1\ \vec{p}_2;\ \lambda_1\ \lambda_2 >$ be also a C.M. state. To be precise we define:

$$|\varepsilon,\bar{p};\ J\ M\ \sigma > = N \int_{SU(2)} D^{(J)^*}_{MM'}(\alpha,\beta,\gamma)\ U(R(\alpha,\beta,\gamma))|\vec{p}_1 = \vec{k}_R,\ \vec{p}_2 = -\vec{k}_R;\lambda_1\lambda_2> \qquad (131)$$

where $\vec{k}_R = (0,\ 0,\ k > 0)$ and N is a normalization factor. From the definitions given earlier of one particle helicity states

* The whole SU(2) group is covered by taking either α or γ vary from 0 to 4π and $0 \leq \beta \leq \pi$.

$$|\vec{k}_R\ \lambda_1 > = e^{i\ K_3^{(1)}\psi}\ |\vec{0},\ \lambda_1 >$$

$$|-\vec{k}_R\ \lambda_2 > = (-1)^{S_2-\lambda_2}\ e^{-i\pi J_2^{(2)}}\ |\vec{k}_R\ \lambda_2 > = e^{-iK_3^{(2)}\psi}|\vec{0},\ -\lambda_2 >$$

and $J_3^{(1)}|\vec{0},\ \lambda_1 > = \lambda_1|\vec{0},\ \lambda_1>$ while $J_3^{(2)}|\vec{0},\ -\lambda_2 > = -\lambda_2|\vec{0},\ -\lambda_2 >$

From the commutator $\left[J_3^{(1)},\ K_3^{(1)}\right] = \left[J_3^{(2)},\ K_3^{(2)}\right] = 0$ it follows:

$$J_3^{(1)}|\vec{k}_R\ \lambda_1 > = \lambda_1\ |\vec{k}_R\ \lambda_1 > \tag{132}$$

and

$$J_3^{(2)}|-\vec{k}_R\ \lambda_2 > = -\lambda_2|-\vec{k}_R\ \lambda_2 >$$

The results are otherwise obvious. For (helicity) state with momentum $\vec{k}_R$ the helicity operator is J_3 while for the state with momentum $-\vec{k}_R$ it is $(-J_3)$. Thus

$$\begin{aligned} J_3(|\vec{k}_R\ \lambda_1>\otimes\ |-\vec{k}_R\lambda_2 >) &= (J_3^{(1)}+J_3^{(2)})(|\vec{k}_R\lambda_1>\ \otimes\ \ |-\vec{k}_R\lambda_2 >) \\ &= (\lambda_1-\lambda_2)(|\vec{k}_R\lambda_1>\ \otimes\ \ |-\vec{k}_R\lambda_2>) \end{aligned} \tag{133}$$

Now

$$\begin{aligned} &D^{(J)^*}_{MM'}(\alpha,\beta,\gamma)\ U(R(\alpha,\beta,\gamma)|\vec{k}_R,\ -\vec{k}_R;\ \lambda_1\ \lambda_2 > \\ &= D^{(J)^*}_{MM'}(\alpha,\beta,0)\ U(R(\alpha,\beta,0))|\vec{k}_R,-\vec{k}_R;\ \lambda_1\lambda_2>e^{i(M'-\lambda_1+\lambda_2\ )\gamma} \end{aligned} \tag{134}$$

Performing the integration over γ gives $\delta_{M';\lambda_1-\lambda_2}$.

We can thus write (substituting α by ϕ and β by Θ)

$$|\varepsilon,\tilde{p};J\ M\ \sigma > \delta_{\mu,(\lambda_1-\lambda_2)} \equiv N_J \int_{\phi=0}^{2\pi} \int_{\Theta=0}^{\pi} D_{M,\mu}^{(J)^*}(\phi,\Theta,0)\ U(R(\phi,\Theta,0))\ |\vec{k}_R,-\vec{k}_R;\lambda_1\lambda_2>\sin\Theta\ d\Theta\ d\phi$$

$$= N_J \int D_{M,\mu}^{(J)^*}(\phi,\Theta,-\phi)\ U(R(\phi,\Theta,-\phi))|\vec{k}_R,-\vec{k}_R;\ \lambda_1\lambda_2>\ d\Omega \quad (135)$$

It also follows that

$$J = |\lambda_1-\lambda_2|,\ |\lambda_1-\lambda_2| + 1,\ldots \quad (136)$$

for otherwise $D_{M,\mu}^{(J)}$ vanishes.

From the fact that helicity remains invariant under a rotation we verify that the state $|\varepsilon,\ \tilde{p};\ J\ M\ \sigma>$ is also eigenstate of helicity operators $h^{(1)}$ and $h^{(2)}$ allowing us to write

$$|\varepsilon,\tilde{p};JM\lambda_1\lambda_2>\delta_{\mu,(\lambda_1-\lambda_2)}=N_J \int D_{M,\mu}^{(J)^*}(\phi,\Theta,-\phi)\ U(R(\phi,\Theta,-\phi))|\vec{k}_R,-\vec{k}_R;\ \lambda_1\lambda_2>.\ d\Omega \quad (137)$$

We defined for the single particle states

$$|\vec{k}\ \lambda > = U(R(\phi,\Theta,\ -\phi))|\vec{k}_R\ \lambda >$$

$$|-\vec{k}\ \lambda > = U(R(\phi,\Theta,\ -\phi))|-\vec{p}_R\ \lambda >$$

so that

$$|\vec{k}\ \lambda_1 > \otimes\ |-\vec{k}\ \lambda_2 > = U^{(1)}(R(\phi,\Theta,-\phi))|\vec{k}_R\lambda_1>\otimes\ U^{(2)}(R(\phi,\Theta,-\phi))|-\vec{k}_R\lambda_2>$$

$$\equiv U(R(\phi,\Theta,-\phi))(|\vec{k}_R\lambda_1>\otimes\ |-\vec{k}_R\ \lambda_2\ >) \quad (138)$$

where θ and ϕ are polar angles of $\vec{k}$. That is *

$$|\epsilon,\ \tilde{p};\ \theta\phi;\ \lambda_1\ \lambda_2\ > = U(R(\phi,\theta,-\phi))|\epsilon,\ \tilde{p};\ 0\ 0;\ \lambda_1\lambda_2\ > \qquad (139)$$

Thus we have

$$|\epsilon,\ \tilde{p};\ J\ M\ \lambda_1\ \lambda_2\ > = N_J \int d\Omega\ D^{(J)^*}_{M,(\lambda_1-\lambda_2)}(\phi,\theta,-\phi)|\vec{k},-\vec{k};\ \lambda_1\ \lambda_2\ >$$

$$= N_J \int d\Omega\ D^{(J)*}_{M,(\lambda_1-\lambda_2)}(\phi,\theta,-\phi)|\epsilon,\tilde{p};\theta\phi;\ \lambda_1\lambda_2\ > \qquad (140)$$

Defining the states $|\theta\phi;\ \lambda_1\lambda_2>$ in c.m. frame by

$$|\epsilon,\tilde{p};\theta\phi;\ \lambda_1\lambda_2> = |\epsilon;\bar{p}> \otimes |\theta\phi\ ;\ \lambda_1\lambda_2\ > \frac{2\pi}{\sqrt{\eta_1\ \eta_2}}\sqrt{\frac{4\epsilon}{|\vec{k}|_{c.m.}}} \qquad (141)$$

we obtain from the normalization condition

$$<\theta'\phi';\lambda_1'\lambda_2'|\theta\phi\ ;\ \lambda_1\lambda_2\ > = \delta(\Omega-\Omega')\ \delta_{\lambda_1\ \lambda_1'}\ \delta_{\lambda_2\lambda_2'} \qquad (142)$$

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* Here (θ,ϕ) are polar angles of $\vec{k}$ in C.M. frame w.r.t. a z-axis in the C.M. frame. The general helicity state $|\epsilon,\ p^\mu;\ \theta\Phi;\ \lambda_1\ \lambda_2\ >$ is obtained from the C.M. state $|\epsilon,\ \tilde{p};\ \theta\phi;\ \lambda_1\ \lambda_2>$ by ($\lambda_1\lambda_2$ labels are maintained).

$$|\epsilon,\ p^\mu;\ \theta\ \Phi;\ \lambda_1\ \lambda_2\ > = (U(\Lambda_p)|\epsilon,\ \tilde{p};\ \theta\phi;\ \lambda_1\lambda_2\ >$$

where

$$U(\Lambda_p) = U(R(\alpha\beta\gamma))e^{-iK_3\psi}$$

The boost gives a momentum along z axis so that $\tilde{p} \to p_R$ at the same time the polar angles of $\vec{k}$(w.r.t. K_{p_R}) change to some value (θ',Φ'). Additional rotation takes $\vec{p}_R$ to $\vec{p}$ while changes $(\theta',\ \Phi')$ to (θ,Φ).

on using $< \varepsilon; p'^{\mu} | \varepsilon; p^{\mu} > = (2\pi)^4 \delta^4(p'^{\mu} - p^{\mu})$.

We likewise define the states $|JM; \lambda_1\lambda_2>$ by

$$|\varepsilon, \tilde{p}; J\ M\ \lambda_1\lambda_2 > = |\varepsilon; \tilde{p}> \otimes |JM; \lambda_1\lambda_2> \frac{2\pi}{\sqrt{\eta_1\ \eta_2}} \sqrt{\frac{4\varepsilon}{|\vec{k}|_{c.m.}}} \quad (143)$$

We obtain, for these states $|\Theta\phi; \lambda_1\lambda_2 >$ and $|JM;\lambda_1\lambda_2>$ (in the c.m. frame):

$$|JM;\lambda_1\lambda_2 > = N_J \int d\Omega\ D^{(J)^*}_{M,(\lambda_1-\lambda_2)}(\phi,\Theta,-\phi)|\Theta\phi;\lambda_1\lambda_2 > \quad (144)$$

The normalization factor N_J is determined by requiring:

$$<J'M';\lambda_1'\lambda_2'\ |JM;\lambda_1\lambda_2> = \delta_{JJ'}\ \delta_{MM'}\ \delta_{\lambda_1\lambda_1'}\ \delta_{\lambda_2\lambda_2'} \quad (145)$$

Hence:

$$\delta_{JJ'}\ \delta_{MM'} = N^*_{J'}\ N_J \iint d\Omega\ d\Omega'\ D^{(J')}_{M',\mu}(\phi',\Theta';-\phi')\ D^{(J)*}_{M,\mu}(\phi,\Theta,-\phi)\ \delta(\Omega-\Omega')$$

$$= N^*_{J'}\ N_J \int d\Omega\ D^{(J')}_{M'\mu}(\phi,\Theta,-\phi)\ D^{(J)*}_{M,\mu}(\phi,\Theta,-\phi)$$

$$= |N_J|^2 \frac{4\pi}{(2J+1)}\ \delta_{JJ'}\ \delta_{MM'}$$

or

$$N_J = +\sqrt{\frac{2J+1}{4}} \quad (146)$$

defining the convinient phase convention.

The inverse relation can be obtained on using the completeness relation (with the normalization adopted above):

$$\sum_J \sum_M \sum_{\lambda_1} \sum_{\lambda_2} |J\ M;\lambda_1\lambda_2 > \ < J\ M;\lambda_1\lambda_2| = I \tag{147}$$

Therefore

$$|\Theta\phi;\lambda_1\lambda_2 > = \sum |J\ M;\ \lambda_1\lambda_2> < J\ M;\ \lambda_1\lambda_2|\Theta\phi;\lambda_1\lambda_2 > \tag{148}$$

The Clebsch-Gordon coefficients are given by

$$\begin{aligned} &< \Theta'\phi';\lambda_1'\lambda_2'\ |J\ M;\ \lambda_1\lambda_2 > \\ &\quad = N_J \int d\Omega\ D^{(J)*}_{M,(\lambda_1-\lambda_2)}(\phi,\Theta,-\phi)\ \delta(\Omega-\Omega')\ \delta_{\lambda_1\lambda_1'}\ \delta_{\lambda_2\lambda_2'} \\ &\quad = N_J\ D^{(J)*}_{M,(\lambda_1-\lambda_2)}(\phi',\ \Theta',\ -\phi')\ \delta_{\lambda_1\lambda_1'}\ \delta_{\lambda_2\lambda_2'} \end{aligned} \tag{149}$$

Thus

$$|\theta\phi;\ \lambda_1\lambda_2> = \sum_J \sum_{M=-J}^{+J} N_J^*\ D^{(J)}_{M'(\lambda_1-\lambda_2)}\ (\phi,\Theta,-\phi)|J\ M;\ \lambda_1\lambda_2 > \tag{150}$$

and

$$< \Theta\phi;\ \lambda_1\lambda_2| = \sum_J \sum_M N_J\ D^{(J)^*}_{M,(\lambda_1-\lambda_2)}\ (\phi,\Theta,-\phi) < J\ M;\ \lambda_1\lambda_2\ | \tag{151}$$

since $D^{(J)}_{M,(\lambda_1-\lambda_2)}$ are merely expansion coefficients. Clearly,

$$|\varepsilon,\ \bar{p};\ \Theta\phi;\ \lambda_1\lambda_2 > = \sum_J \sum_M\ N_J^*\ D^{(J)}_{M,(\lambda_1-\lambda_2)}(\phi,\Theta,-\phi)|\varepsilon,\ \bar{p};\ J\ M\ \lambda_1\lambda_2> \tag{152}$$

Thus expressing the states $|\varepsilon,\ \bar{p};\ \Theta\phi;\ \lambda_1\lambda_2 >$ in terms of the set of basis vectors given by all the irreducible unitary representations of the rotation group (or SU(2) group). Therefore, a representation U(R) is defined on the set of vectors $|\varepsilon,\ \tilde{p};\ \Theta\phi;\ \lambda_1\lambda_2 >$ with $\tilde{p}$, λ_1 and λ_2 fixed (since rotation leaves helicities and $\vec{p} = 0$ unchanged). In fact $U(R(\alpha,\beta,\gamma))\,|\varepsilon, \tilde{p};\, \Theta\phi;\, \lambda_1\lambda_2 > = e^{\pm i\omega}\,|\varepsilon, \tilde{p};\Theta'\phi';\, \lambda_1\lambda_2 >$ where $k' = R(\alpha,\beta,\gamma)\,\vec{k}$ and (Θ', ϕ') are angles of $\vec{k}'$ (in C.M. frame).

The irreducible representation of the RIHL defined on two particle state vectors with total four momentum p^μ, on a fixed orbit $p^2 = \varepsilon^2$, is given by the states

$$|\varepsilon, p^\mu\,;\, J\, M\, \lambda_1\lambda_2 > \equiv |\varepsilon, p^\mu > \otimes\, |J\, M;\, \lambda_1\lambda_2 > = U(\Lambda_p)|\varepsilon, \tilde{p};\, J\, M\, \lambda_1\lambda_2 > \tag{153}$$

where $\Lambda_p\,\tilde{p} = p$.

Under a general transformation $U(\Lambda, a)$ of RIHL

$$U(\Lambda,a)|\varepsilon,p^\mu;J\,M\,\lambda_1\lambda_2 > = e^{i\,P\cdot a}U(\Lambda)|\varepsilon,p^\mu;\, J\, M\,\lambda_1\lambda_2 > = e^{ip'\cdot a}(|\varepsilon,p'^\mu>\otimes\, U(\Lambda_{p'}^{-1}\,\Lambda\Lambda_p)|JM;\lambda_1\lambda_2>) \tag{154}$$

where $p' = \Lambda p$ and $\Lambda_{p'}^{-1}\,\Lambda\,\Lambda_p = R(\Lambda p,p)$ is the Wigner rotation. Thus

$$(\Lambda,a)|\varepsilon,p^\mu;\, J\, M\,\lambda_1\lambda_2 > = e^{ip'\cdot a}(|\varepsilon,p'> \otimes \sum_{M'} D^{(J)}_{M'M}\,(R(\Lambda p,p))|J\, M';\, \lambda_1\lambda_2 >) \tag{155}$$

$$= e^{ip'\cdot a}\sum_{M'}|\varepsilon,\, p';\, J\, M'\,\lambda_1\lambda_2 > D^{(J)}_{M'M}\,(R(\Lambda p,p))$$

The unitary can be verified, say, by using the identity:

$$I = \sum_J\sum_M\sum_{\lambda_1}\sum_{\lambda_2}\int\frac{d^4p}{(2\pi)^4}\left(\frac{\eta|_1\;\eta|_2}{(2\pi)^2}\quad\frac{|\vec{k}|_{C.M.}}{4\varepsilon}\right)|\varepsilon,p;JM\lambda_1\lambda_2 >< \varepsilon,p;JM\lambda_1\lambda_2| \tag{156}$$

The normalization of states $|\varepsilon,p;\, J\, M\,\lambda_1\lambda_2 >$ is given by

$$\langle s, |P' m; J' M' \lambda_1' \lambda_2' J | m; P; J M \lambda_1 \lambda_2 \rangle$$

$$= \frac{(2\pi)^4}{\eta|_1 \, \eta|_2} \, \frac{4s}{|\vec{k}|_{C.M.}} \, \delta^4(p'^{\mu} - p^{\mu}) \, \delta_{JJ'} \, \delta_{MM'} \, \delta_{\lambda_1 \lambda_1'} \, \delta_{\lambda_2 \lambda_2'} \tag{157}$$

The discussion for one massless and other massive particle or both massless goes on similar lines. The discussion above starts essentially from the reference state $|\vec{k}_R, -\vec{k}_R; \lambda_1 \lambda_2 \rangle$

4

SPACE REFLECTION

We will next investigate the transformation properties of the helicity states under the discrete transformations of space reflection (parity operation) and time reversal in order to be able to apply these invariance principles to the particle reactions. The four operations $\{I, I_s, I_t, I_{st}\}$ constitute an Abelian group called Four Group. Each of these four transformations characterize one of the four disconnected pieces of the HLG.

4.1 - *PARITY TRANSFORMATION: PASSIVE AND ACTIVE FORMULATION*

Parity is an important property of many functions. If $f(-\vec{r}) = f(\vec{r})$ we say f has even parity and if $f(-\vec{r}) = - f(\vec{r})$ it has negative parity. Functions like $e^{-i\vec{k}\cdot\vec{r}}$ do not have any well defined parity. We can define the parity operations as the passive coordinate transformation

$$I_s: \quad \begin{array}{l} \vec{x}' = -\vec{x} \\ x'^0 = x^0 \end{array} \quad ; \qquad I_s = I_s^{-1} = I_s^T \tag{1}$$

Equivalently, we may describe it as the transformation *

$$\vec{e}_1' = -\vec{e}_1 \qquad \vec{e}_2' = -\vec{e}_2 \qquad \vec{e}_3' = -\vec{e}_3$$

where $\vec{e}_i$ are the unit vectors along space axes. The vectors $\vec{e}'$ clearly form a right handed system of coordinates if $\vec{e}_i$'s form a left-handed system. In fact $\vec{e}_1' \times \vec{e}_2' = -\vec{e}_3'$. It can be easily verified that any space rotation leaves invariant the relation $\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$. Thus the space reflection or parity operations is not a rotation. This is otherwise obvious since the determinant of the transformation is (-1). The parity operation thus implies looking at the physical system from the coordinate frame with opposite handedness. Parity transformation takes any 3-vector $\vec{V}$, like $\vec{x},\vec{p}$ etc. to $-\vec{V}$ while 3-axial vector** $\vec{A}$, like $(\vec{r} \times \vec{p})$, to $+\vec{A}$.

Parity operator $\mathcal{P}$ describing the effects of a space (coordinates) reflection on a point function $f(\vec{x})$ is defined by $(\vec{x}' = -\vec{x})$:

$$f'(\vec{x}') = f(\vec{x}) = \mathcal{P} f(\vec{x}')$$

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* Note $\vec{e}_i' = \sum_{j=1}^{3} (R^T)^{-1}_{ij}\, \vec{e}_j$, so that $\vec{e}_i' = \sum_{j=1}^{3} R_{ij}\, e_j$ while for parity $\vec{e}_i' = -\vec{e}_i$. The transformation of $\vec{e}_i'$ follows from $\vec{x} \equiv \sum x_i \cdot \vec{e}_i = \sum x_i'\, \vec{e}_i'$ and $x_i' = \sum R_{ij}\, x_j$.

** In case of 4-pseudo vector the parity changes the sign of the time component while leaving it unaltered for a 4-vector. For an antisymmetric tensor of second rank the space-space components behave as 3-axial vector while space-time components as 3-vector. See for example: Landau and Lifshitz, Classical Theory of Fields (3rd Edition).

or

$$\mathcal{P} f(\vec{x}) = f(-\vec{x})$$

It follows $\mathcal{P}^2 f(\vec{x}) = f(\vec{x})$ (choosing the phases in the definition of $\mathcal{P}$ such that $\mathcal{P}^2 = 1$) so that the eigen-value of parity operator $\mathcal{P}$ are ± 1 corresponding to even or odd parity functions $f_e(-\vec{x}) = f_e(\vec{x})$ $f_0(-\vec{x}) = - f_0(\vec{x})$. That $\mathcal{P}$ must be a (linear) unitary operator is showed below using active formulation; the results of any analysis are independent of the two view points of th formulation of parity.

In the active sense formulation of space reflection the space axes fixed and the "body" (or body axes) is acted on. The active transformation reflects the physical system * through the origin so that what was at point $\vec{x}$ is now at point $(-\vec{x})$. The procedure changes $\vec{p}$ to $-\vec{p}$ but axial vectors like $\vec{L} = \vec{x} \times \vec{p}$ are left unchanged. Parity transformation may also be realized in equivalent way by reversing one coordinate and rotating through π around the axis of that coordinate, for example,

$$\begin{aligned} \mathcal{R}_{13}\, \vec{x} &= (x^1, -x^2, x^3) \\ e^{\pm i\pi J_2}\, \mathcal{R}_{13}\, \vec{x} &= (-x^1, -x^2, -x^3) = -\vec{r} \end{aligned} \tag{2}$$

The reversing of one coordinate is essentially equivalent to taking the mirror image of the system in the plane perpendicular to the corresponding coordinate axis. There is no difficulty in taking the mirror

* Note that one particular vector $\vec{x}$ can be through to $-\vec{x}$ by a rotation of 180° in any of the infinite families of planes containing $\vec{x}$ and $-\vec{x}$. However, it is impossible to achieve this for all the vectors $\vec{x}$ attached to the body at the same time using a single rotation. The situation is clearly different in 2-dimensional case.

image of vectors. Axial vectors (or rather antisymmetric tensors of second rank) corresponding to magnetic field or angular momentum are difficult to handle and the easiest way is to consider the corresponding vector e.g. a loop of electric curvent in place of magnetic field. One verifies that the axial vector components parallel to the mirror surface reverse their direction in going from object to the image while the axial-vector component perpendicular to the mirror are unchanged. The additional rotation by π of the image in the plane of the mirror turns an axial vector image into its object e.g. the parity leaves axial vectors unchanged.

To decide if $\mathcal{P}$ is unitary or antiunitary operator consider a single spinless particle and the position and momentum eigenkets *:

$$\hat{\vec{x}}|\vec{x}'\rangle = \vec{x}'|\vec{x}'\rangle \qquad \hat{\vec{p}}|\vec{p}'\rangle = \vec{p}'|\vec{p}'\rangle$$

We choose the phases so that

$$\mathcal{P}|\vec{x}'\rangle = |-\vec{x}'\rangle \qquad \mathcal{P}|\vec{p}'\rangle = |-\vec{p}'\rangle$$

$$\mathcal{P}\hat{\vec{x}}\mathcal{P}^{-1}\mathcal{P}|\vec{x}'\rangle = \mathcal{P}\hat{\vec{x}}\mathcal{P}^{-1}|-\vec{x}'\rangle = \vec{x}\mathcal{P}|\vec{x}'\rangle = \vec{x}'|-\vec{x}'\rangle = -\hat{\vec{x}}|-\vec{x}'\rangle$$

or

$$\mathcal{P}\hat{\vec{x}}\mathcal{P}^{-1} = -\hat{\vec{x}}$$

Similarly

$$\mathcal{P}\hat{\vec{p}}\mathcal{P}^{-1} = -\hat{\vec{p}} \qquad (3)$$

Clearly

$$\mathcal{P}\hat{\vec{L}}\mathcal{P}^{-1} = \hat{\vec{L}}$$

Let us use the geometrical fact that

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* ^ indicates an operator acting or kets in discussion.

$$\mathcal{P}\, T(\vec{a}) = T(-\vec{a})\, \mathcal{P} \tag{4}$$

where $T(\vec{a}) = e^{-i\hat{\vec{p}}\cdot\vec{a}}$ is translation operator. Thus

$$\mathcal{P}\, e^{-i\vec{p}\cdot\vec{a}}\, \mathcal{P}^{-1} = e^{i\vec{p}\cdot\vec{a}}$$

or $\mathcal{P}(i\vec{p})\,\mathcal{P}^{-1} = -\, i\vec{p}$ implying $\mathcal{P} i \mathcal{P} = i$ whence $\mathcal{P}$ must be a unitary operator,

$$\mathcal{P}^{\dagger}\mathcal{P} = \mathcal{P}\mathcal{P}^{\dagger} = \mathbb{1}\ . \tag{5}$$

That $\mathcal{P}$ must be unitary follows also from the requirement that the energy spectrum is restricted to +ve values. We discuss it below.

We note that $\mathcal{P}^2$ can be looked upon as a complete rotation and as such, for example, for j = 1/2 states we have $\mathcal{P}^2 = -\,\mathbb{1}$. This stems from the double valuedness of spin 1/2 states as a function of the co-ordinates. However, even in this case we may redefine the parity operator $\mathcal{P}$ - since a phase factor is always at ones disposal in any unitary operator so as to secure $\mathcal{P}^2 = \mathbb{1}$. No physical restriction, however, is obtained on the state vectors by this convention. *

Parity of the angular momentum state $|\ell m>$ can be easily obtained. For any state vector $|\alpha>$ we can write

$$|\alpha> = \int d^3\, r' \;\; |\vec{r}'><\vec{r}'|\alpha> \tag{6}$$

$<\vec{r}'|\alpha> \equiv \psi_\alpha(\vec{r}')$ is the wave function in the coordinate space. Like-wise for the transformed state.

* Contrast this with the situation in time reversal operation case.

$$|\alpha;\ \text{ref.}> \equiv \mathcal{P}|\alpha> = \int d^3\vec{r}\,'|\vec{r}\,'><\vec{r}\,'|\mathcal{P}|\alpha> = \int d^3\vec{r}\,'|\vec{r}\,'>< -\vec{r}\,'|\alpha > \qquad (7)$$

since

$$< \vec{r}\,'|\ \ |\alpha > = <\alpha|\ ^{\dagger}|\vec{r}\,'>^{*} \ = <-\vec{r}\,'|\alpha >$$

Indicating parity operator in coordinate representation by the same symbol $\mathcal{P}$ we may write

$$\mathcal{P} < \vec{r}\,'|\alpha > = < -\vec{r}\,'|\alpha > \equiv < \vec{r}\,'|\alpha;\ \text{refl.}> \qquad (8)$$

For the states $|\ell m >$ we have *

$$|\ell m > = \int d\Omega\ |\Theta\phi><\Theta\phi|\ \ell m >$$

$$= \int d\Omega\ |\Theta\phi >\left[i^{\ell}\ Y_{\ell}^{m}(\Theta,\phi)\right] \qquad (9)$$

Since $-\vec{r}$ has polar angles $(\pi-\Theta,\ \pi+\phi)$

$$\mathcal{P}|\ell m > = \int d\Omega\ |\Theta\phi > \left[i^{\ell}\ Y_{\ell}^{m}\ (\pi-\Theta,\ \pi+\phi)\right]$$

but

$$Y_{\ell}^{m}(\pi-\Theta,\ \pi+\phi) \equiv \mathcal{P}\,Y_{\ell}^{m}(\Theta,\phi) = (-1)^{\ell}\ Y_{\ell}^{m}(\Theta,\phi) \qquad (10)$$

Thus

$$|\ell m;\ \text{refl.} > = \mathcal{P}|\ell m > = (-1)^{\ell}|\ell m\ \ > \qquad (11)$$

e.g. the $|\ell m >$ states are eigenstates of parity with parity $(-1)^{\ell}$ like

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* To obtain the conventional phase on time reversal operation we write $< \Theta\phi|\ell m > = i^{\ell}\ Y_{\ell}^{m}(\Theta,\phi)$ in coordinate representation while $<\Theta_p\phi_p|\ell m> = Y_{\ell}^{m}(\Theta_p,\ \phi_p)$ in momentum space representation.

the spherical harmonics Y_ℓ^m. The same result could also be obtained using momentum space representation.

The parity of a single object, described by a unique state vector $|\alpha\rangle$, is a dichotomic variable; the eigenvalues of parity being ± 1. We may introduce a 2-dimensional parity space and represent in it the two parity states by $\binom{1}{0}$ and $\binom{0}{1}$ corresponding to eigenvalues (+) and (-) respectively of the parity operator represented by 2 x 2 matrix $\mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. A state vector $|\alpha\rangle$ can be then decomposed as

$$|\alpha\rangle = |\alpha; +\rangle \otimes \binom{1}{0} + |\alpha; -\rangle \otimes \binom{0}{1} \tag{12}$$

where $|\alpha; +\rangle$ behave as functions (number) only w.r.t. the parity space; with respect to all other attributes they are still state vectors, with n/2 components if n is the number of components * of $|\alpha\rangle$. A state with both $|\alpha; \pm\rangle \neq 0$ is called asstate of mixed parity. We note

$$\mathcal{P}|\alpha\rangle = |\alpha;+\rangle \otimes \binom{1}{0} - |\alpha; -\rangle \otimes \binom{0}{1} \tag{13}$$

so that

$$\begin{aligned} |\alpha;+\rangle \otimes \binom{1}{0} &= \frac{1}{2}(|\alpha\rangle + |\alpha; \text{ref.}\rangle) = \frac{1}{2}(\mathbb{1} + \mathcal{P})|\alpha\rangle \\ |\alpha;+\rangle \otimes \binom{0}{1} &= \frac{1}{2}(|\alpha\rangle - |\alpha; \text{ref.}\rangle) = \frac{1}{2}(\mathbb{1} - \mathcal{P})|\alpha\rangle \end{aligned} \tag{14}$$

are states of definite parity constructed from $|\alpha\rangle$ and $|\alpha; \text{ref.}\rangle$. Under parity operation $|\alpha; +\rangle \longrightarrow |\alpha; +\rangle$ while $|\alpha; -\rangle \longrightarrow -|\alpha; -\rangle$.

The transformation of spin operator under parity is defined in analogy to the case of $\vec{L}$

$$\mathcal{P}\vec{S}\mathcal{P}^{-1} = \vec{S} \tag{15}$$

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* The Dirac wave function constructed out of two 2-spinors.

so that $\mathcal{P}\vec{J}\mathcal{P}^{-1} = \vec{J}$ e.g. parity commutes with rotation operator $\vec{J}$. This could also be checked from the matrix representation of $\mathcal{P}$ and $\vec{J}$ already discussed. The operators $\vec{K}$, however, behave as vectors (see also next section)

$$\mathcal{P}\vec{K}\mathcal{P}^{-1} = -\vec{K} \tag{16}$$

as can be verified by using say the matrix representations of poincare generators. The helicity operator is hence a pseudo-scalar operator and helicity changes sign under parity operation, $\mathcal{P}\frac{\vec{J}\cdot\vec{P}}{|\vec{P}|}\mathcal{P}^{-1} = -\frac{\vec{J}\cdot\vec{P}}{|\vec{P}|}$.

4.2 - *PARITY OPERATOR ON HILBERT SPACE: SINGLE PARTICLE STATES*

We now consider the parity operation acting in the Hilbert space H of state vectors. We continue to denote the operator in H (correspond̲ing to I_s) by $\mathcal{P}$. From the group multiplication law for operators

$$\mathcal{P}U(L)\mathcal{P}^{-1} = e^{i\omega(L)}\, U(I_s\, L\, I_s^{-1}) \tag{17}$$

where $L \equiv (\Lambda, a)$ is any element of RIHL and $U(L)$ the corresponding unitary operator in H. Then the identity

$$\mathcal{P}\, U(L_1\, L_2)\mathcal{P}^{-1} = \mathcal{P}U(L_1)\mathcal{P}^{-1}\,\mathcal{P}\, U(L_2)\mathcal{P}^{-1}$$

leads to

$$e^{i\omega(L_1L_2)} = e^{i\omega(L_1)}\, e^{i\omega(L_2)} \tag{18}$$

e.g. $e^{i\omega(L)}$ form a representation of RIHL by complex numbers. It may be shown * that there is only one such representation viz. $e^{i\omega(L)} = 1$. Then

$$\mathcal{P} U(L) \mathcal{P}^{-} = U(I_s \; L \; I_s^{-})$$ (19)

The unitary of $\mathcal{P}$ can also be easily demonstrated. Consider a time translation i.e. $L = (\mathbb{1}, a)$ with $a^\mu = (t, \vec{0})$. The corresponding operator is $U(L) = e^{iP^0 t}$. Thus $\mathcal{P} e^{iP^0 t} \mathcal{P}^{-1} = U(\mathbb{1}, a_s) = e^{\pm iP^0 t}$. Here $a_s =$ $= (a^0, -\vec{a})$ and we used $I_s(\mathbb{1}, a^\mu) I_s^{-1} = (\mathbb{1}, a_s^\mu)$. Let $|\chi>$ be a state vector which is an energy eigenstate with positive energy E. Then

$$e^{iP^0 t} \mathcal{P}|\chi> = \mathcal{P} e^{iP^0 t}|\chi> = \mathcal{P} e^{iEt}|\chi> = e^{\pm iEt}\mathcal{P}|\chi>$$ (20)

According as $\mathcal{P}$ is unitary or anti-unitary operator. If we require $\mathcal{P}|\chi>$ to be a positive energy state, say, by restricting the energy spectrum to positive values, $\mathcal{P}$ must be a unitary operator. It is also possible then to choose the phase in the operator so as to secure $\mathcal{P}^2 = I$.

Commutation relation of $\mathcal{P}$ with P^μ easily follow **:

$$\mathcal{P} e^{ia\cdot P} \mathcal{P}^{-1} = e^{ia\cdot \mathcal{P} P \mathcal{P}^{-1}} = U(I_s \; (\mathbb{1}, a) \; I_s^{-1}) = U(\mathbb{1}, a_s) = e^{ia_s \cdot P}$$

which for infinitesimal translations gives

$$\mathcal{P} P^k \mathcal{P}^{-1} = - P^k \qquad \mathcal{P} P^0 \mathcal{P}^{-1} = \mathcal{P}^0$$ (21)

We discussed earlier that the Hilbert space H of state vectors can always be decomposed in terms of spaces on which irreducible unitary representations of RIHL are defined. We thus need consider only the problem of defining parity on such an irreducible representation space.

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* See Wigner:
Halpern:

** For IHL (Λ, a), I_s, I_t etc. are represented by 5 x 5 matrices.

Consider vectors corresponding to the particle rest frame (or the c.m. frame in many particle case). From the commutation relations just obtained it is allowed to write:

$$\mathcal{P}(|m;\tilde{p}> \otimes |\alpha>) = |m;\tilde{p}> \otimes D(\mathcal{P})|\alpha> \tag{22}$$

where $D(\mathcal{P})$ is a unitary operator defined on h. Now from the relation $\mathcal{P}U(L)\mathcal{P}^{-1} = U(I_s\, L\, I_s^{-1})$ applied for pure rotations to the above state vectors, we obtain

$$D(\mathcal{P})\ D(R)\ D(\mathcal{P})^{-1} = D(I_s R\, I_s^{-1}) = D(R) \tag{23}$$

for all the matrices D(R) belonging to the irreducible representation of the rotation group. From Schur's lemma it follows $D(\mathcal{P}) = \lambda I$ and the convention $\mathcal{P}^2 = I$ then gives ($\eta = \pm 1$) for single particle states

$$\mathcal{P}(|m;\tilde{p}> \otimes |\alpha>) = \eta(|m;\tilde{p}> \otimes |\alpha>)$$

and (24)

$$\mathcal{P}|m,\tilde{p};\alpha> = \eta|m,\tilde{p};\alpha>$$

η is called the intrinsic parity.

Now from

$$|m,p;\alpha> \equiv U(\Lambda_p)|m,\tilde{p};\alpha> \tag{25}$$

we have

$$\mathcal{P}|m,p;\alpha> = \mathcal{P}U(\Lambda_p)\mathcal{P}^{-1}\mathcal{P}|m,\tilde{p};\alpha> = \eta\mathcal{P}U(\Lambda_p)\mathcal{P}^{-1}|m,\tilde{p};\alpha> \tag{26}$$

where

$$\mathcal{P}U(\Lambda_p)\mathcal{P}^{-1} = U(I_s\, \Lambda_p\, I_s^{-1})$$

It is clear that $I_s\, \Lambda_p\, I_s^{-1}\, \tilde{p} = p_s$ where $p_s = (p^0, -\vec{p})$.

We consider, for example, to be explicit, the helicity states. For the rest state we have ($\eta^2 = 1$)

$$\mathcal{P}|\vec{0}\ \lambda > = \eta|\vec{0}\ \lambda > \tag{27}$$

where η is the intrinsic parity; the rest state being an eigenstate of parity operator. For helicity states

$$U(\Lambda_p) \equiv U(R(\phi,\ \theta,\ -\phi))\ e^{iK_3\psi}$$

giving

$$\mathcal{P}\,U(\Lambda_p)\,\mathcal{P}^{-1} = \mathcal{P}\,U(R)\mathcal{P}^{-1}\mathcal{P}\ e^{iK_3\psi}\mathcal{P}^{-1} = U(R)\ e^{-iK_3\psi}$$

Thus

$$\begin{aligned}\mathcal{P}|p^0,\ \vec{p},\ \lambda > &= \eta\ U(R(\phi,\theta,-\phi))e^{-iK_3\psi}|\vec{0}\ \lambda > \\ &= \eta\ U(R(\phi,\theta,-\phi))|p^0,\ -\vec{p}_R,-\lambda > \\ &= \eta|p^0,\ -\vec{p},\ -\lambda >\end{aligned} \tag{28}$$

That the parity changes the sign of helicity is expected since helicity operator defines the spin direction in conjunction with the momentum direction; in fact $\vec{J}\cdot\vec{P}/|\vec{P}|$ changes sign under reflection operation eg. it is pseudo-scalar. Likewise we show

$$\mathcal{P}|p^0,\ -\vec{p},\ \lambda > = \eta\ U(R)e^{iK_3\psi}|\vec{0},\ -\lambda > = \eta|p^0,\ \vec{p},\ -\lambda > \tag{29}$$

Combining the two results

$$\mathcal{P}^2|p^0,\ \vec{p},\ \lambda > = \eta^2|p^0,\ \vec{p},\ \lambda > = |p^0,\ \vec{p},\ \lambda > \tag{30}$$

For the case of massless particles no rest frame exists and only two helicity states $|m = 0,\ \vec{p}_R,\ |\lambda| >$ and $|m = 0,\ \vec{p}_R,\ -|\lambda| >$, $\lambda = 0, \pm\frac{1}{2}, \pm 1, \ldots$ are possible for each $p_R = (p,\ 0,\ 0,\ p > 0)$. The relative phases of these states can be given if parity is defined on these states. Unlike in the case of massive particles no ladder operator is defined to relate the phases of these states of massless particles. We define in analogy with massive case

$$\mathcal{P}|m = 0, \vec{p}_R, |\lambda| > = \eta|m = 0, -\vec{p}_R, -|\lambda| > \tag{31}$$

since parity commutes with rotation

$$\mathcal{P}|m = 0, \vec{p}, |\lambda| > = \eta|m = 0, -\vec{p}, -|\lambda| > \tag{32}$$

Note also the definition give earlier:

$$|m = 0, -\vec{p}_R, |\lambda| > = e^{-i\pi J_2}|m = 0, \vec{p}_R, |\lambda| >$$

so that

$$|m = 0, -\vec{p}, |\lambda| > = U(R(\phi,\theta, -\phi))\, e^{-i\pi J_2}\, |m = 0, \vec{p}_R, |\lambda| > \tag{33}$$

The parity operation on angular momentum states can be now defined,

$$||\vec{p}|J\ M\ \lambda > = N_J \int D^{(J)^*}_{M\lambda}(\phi,\theta,-\phi)|\vec{p}, \lambda > d\Omega_{\vec{p}}(\theta,\phi)$$

$$\mathcal{P}||\vec{p}|J\ M\ \lambda > = \eta\ N_J \int D^{(J)^*}_{M\lambda}(\phi,\theta,-\phi)|-\vec{p}, -\lambda > d\Omega_{\vec{p}}$$

$$= \eta\ N_J \int D^{(J)^*}_{M\lambda}(\pi+\phi', \pi-\theta', -\pi-\phi')|\vec{p}', -\lambda > d\Omega_{\vec{p}'} \tag{34}$$

In the case of spinless particle $J \equiv \ell \quad M \equiv m$

$$||\vec{p}|\ell m > = \int |\vec{p} > Y^m_\ell(\theta,\phi)\ d\Omega_{\vec{p}} \tag{35}$$

and

$$\mathcal{P}\,||\vec{p}|\ell m > = \eta(-1)^\ell||\vec{p}|\ell m >$$

on using

$$Y_{\ell}^{m}\,(\pi-\Theta,\ \phi+\pi) = (-1)^{\ell}\ Y_{\ell}^{m}(\Theta,\phi)$$

Y Operator (Mirror Reflections):

It is often more convinient to consider the operation of reflection, say, on the (13) plane than consider parity operation. We define this operator by

$$Y \equiv \mathcal{P}\, e^{-i\pi J_2} = e^{-i\pi J_2}\ \mathcal{P} \tag{37}$$

Under this operation

$$x^1 \to x^1 \qquad\qquad x^2 \to -x^2 \qquad\qquad x^3 \to x^3$$

and so do the components of any 3-vector while for axial vector $A^1 \to -A^1$, $A^2 \to A^2$, $A^3 \to -A^3$. We also note

$$\left[Y,\ e^{-i\pi J_2}\right] = 0 \tag{38}$$

and

$$\left[Y, P^1\right] = \left[Y,\ K^1\right] = \left[Y, P^3\right] = \left[Y,\ K^3\right] = 0 \tag{39}$$

For one particle helicity states

$$\begin{aligned} Y\left|\left[m,s\right],\ \vec{p}_R,\ \lambda\right> &= e^{-i\pi J_2}\left|\left[m,s\right],\ \vec{p}_R,\ \lambda\right> \\ &= (-1)^{s-\lambda}\ \left|\left|m,s,\ -\vec{p}_R,\ \lambda\right>\right. \\ &= \eta(-1)^{s-\lambda}\ \left|\left[m,s\right],\ \vec{p}_R,\ -\lambda\right> \end{aligned} \tag{40}$$

This is physically expected since under reflection in (13) plane $\vec{p}_R$ clearly remains unaltered but the helicity, which represents some kind of rotation around $\vec{p}_R$ axis, gets the opposite sign.

For the massless case

$$Y|m=0,\ \vec{p}_R,\ |\lambda|> = \mathcal{P}\, e^{-i\pi J_2}|m=0,\ \vec{p}_R, |\lambda|> = \mathcal{P}|m=0,\ -\vec{p}_R,\ |\lambda|\ >$$

$$= \eta|m=0,\ \vec{p}_R, -\ |\lambda|\ > \qquad (41)$$

Same relations are obtained if we replace $\vec{p}_R$ by $-\vec{p}_R$. Also we note

$$Y|[m,s], \vec{p}_R,\ \lambda=s\ > = \eta|[m,s],\ \vec{p}_R,\ -s\ >$$

$$Y|[m,s], \vec{p}_R,\ \lambda = -s> = \eta(-1)^{2S}|[m,s],\ \vec{p}_R,\ s\ > \qquad (42)$$

$$Y\ |m=0,\ \vec{p}_R,\ -\ |\lambda|> = \eta(-1)^{2|\lambda|}| \text{[illegible]}\ \vec{p}_R, |\lambda|\ >$$

4.3 - *PARITY OPERATION ON TWO-PARTICLE STATES:*

For the case of time like total momentum states the discussion is analogous to the case of single particle states *.

To be precise we consider the two particle helicity states in the c.m. frame. We recall the notation

$$|\varepsilon,p^\mu;\Theta,\phi;\lambda_1\lambda_2> \equiv |\vec{p}_1\lambda_1> \otimes |\vec{p}_2\lambda_2\ >$$

$$|\varepsilon,\tilde{p};00;\ \lambda_1\lambda_2> \equiv |k_R\lambda_1> \otimes |-\vec{k}_R\ \lambda_2\ >$$

$$|\varepsilon,\tilde{p};\Theta\phi;\ \lambda_1\lambda_2> \equiv |\vec{k}\ \lambda_1> \otimes |-\vec{k}\ \lambda_2> \equiv U(R(\phi,\ \Theta,-\phi))|\varepsilon,\tilde{p};00;\ \lambda_1\lambda_2>$$

where $\vec{k}_R \equiv (0,0,k > 0)$ and (Θ,ϕ) are polar angles of $\vec{k}$ w.r.t. z axis taken along the incidence direction of the first particle. The operator U(R) above stands for the direct product $U^{(1)}(R) \otimes U^{(2)}(R)$.

Since space reflection and rotation commute it is enough to consider parity on the state $|\varepsilon,\tilde{p};00;\lambda_1\lambda_2\ >$. We have

* We, however, do not have $|\varepsilon,\tilde{p}> \otimes\ |\alpha\ > = \eta(|\varepsilon,\tilde{p}> \otimes\ |\alpha>)$.

$$\mathcal{P}(|\vec{k}_R\lambda_1> \otimes |-\vec{k}_R\lambda_2>) = \mathcal{P}^{(1)}|\vec{k}_R\lambda_1> \otimes \mathcal{P}^{(2)}|-\vec{k}_R\ \lambda_2>$$

$$= \eta_1\eta_2(|-\vec{k}_R\lambda_1> \otimes |\vec{k}_R\lambda_2>)$$

$$= \eta_1\eta_2(-1)^{s_1+\lambda_1+s_2-\lambda_2}(e^{-i\pi J_2^{(1)}}|\vec{k}_R,-\lambda_1> \otimes e^{-i\pi J_2^{(2)}} |-\vec{k}_R, -\lambda_2>)$$

Since $|\pm\vec{k}_R, \lambda>$ are eigenstates of J_3 and J^2 (e.g. states $|jm>$) we may use *

$$e^{-i\pi J_2} e^{-i\pi J_2} |jm> = e^{-i2\pi J_2}|jm> = (-1)^{2j} |jm> \tag{44}$$

to write the r.h.s. in the alternate form

$$= \eta_1\eta_2(-1)^{s_1+s_2-\lambda_1+\lambda_2}(e^{i\pi J_2^{(1)}} | \vec{k}_R,-\lambda_1> \otimes e^{i\pi J_2^{(2)}} |-\vec{k}_R,-\lambda_2>)$$

Thus

$$\mathcal{P}|\varepsilon,\tilde{p};00;\lambda_1\lambda_2> = \eta_1\eta_2(-1)^{s_1+s_2+\lambda_1-\lambda_2}e^{-i\pi J_2}|\varepsilon,\tilde{p};00;-\lambda_1,-\lambda_2>$$

$$= \eta_1\eta_2(-1)^{s_1+s_2-\lambda_1+\lambda_2}e^{i\pi J_2}|\varepsilon,\tilde{p};00;-\lambda_1,-\lambda_2> \tag{45}$$

and

$$\mathcal{P}|\varepsilon,\tilde{p};\theta\phi;\lambda_1\lambda_2> = U(R(\phi,\theta,-\phi))\mathcal{P}|\varepsilon,\tilde{p};00;\lambda_1\lambda_2> \tag{46}$$

The space reflection operation on the angular momentum states of

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* In fact $e^{-i\pi(\vec{J}\cdot\vec{n})} e^{-i\pi(\vec{J}\cdot\vec{n})}|jm> = e^{-i2\pi(\vec{J}\cdot\vec{n})} |jm> = (-1)^{2j}|jm>$.

two particles in c.m. frame can be obtained from the partial wave expansion

$$|\varepsilon,\bar{p};00;\lambda_1\lambda_2 > = \sum_J \sum_M N_J^* D_{M\lambda}^{(J)^*}(0,0,0)|\varepsilon,\bar{p};\, J\, M\, \lambda_1\lambda_2 >$$

$$= \sum_J N_J^* |\varepsilon,\bar{p};\, J\, \lambda\;\; \lambda_1\, \lambda_2 > \tag{47}$$

where $\lambda = (\lambda_1-\lambda_2)$. Clearly $|\lambda_1 - \lambda_2| < J$ or the possible J values are given by

$$J = |\lambda_1 - \lambda_2|,\ |\lambda_1 - \lambda_2| + 1,\ |\lambda_1 - \lambda_2| + 2, \ldots \tag{48}$$

The discussion for two massless particles or one massless and another massive goes along similar lines. Thus

$$\sum_J N_J^* \;\; |\varepsilon,\bar{p};J\;\lambda\;\lambda_1\;\;\lambda_2> = \eta_1\eta_2(-1)^{s_1+s_2-\lambda_1+\lambda_2} \sum_J \sum_M N_J^*\, \delta_{M,-\lambda}$$

$$e^{i\pi J_2}\;|\varepsilon,\bar{p};\, J\, M,-\lambda_1,\, -\,\lambda_2 >$$

But

$$e^{i\pi J_2}|\varepsilon,\bar{p};JM\lambda_1\lambda_2> = \sum_{M'} d^J_{M'M}(-\pi)|\varepsilon,\bar{p};JM'\lambda_1\lambda_2> = (-1)^{-J-M}|\varepsilon,\bar{p};J,-M\lambda_1\lambda_2> \tag{49}$$

Thus

$$\sum_J N_J^* \mathcal{P}|\varepsilon,\bar{p};J\lambda\;\lambda_1\lambda_2> = \eta_1\eta_2(-1)^{s_1+s_2-\lambda_1+\lambda_2}\sum_J N_J^*(-1)^{\lambda-J}|\varepsilon,\bar{p};J,\lambda,-\lambda_1,-\lambda_2>$$

Since $\mathcal{P}$ commutes with $\vec{J}$ the transformed state on parity operation has the same value of J and M = λ. The states in the expansion above with distinct J values are orthogonal we can equate the terms on each side for same J, obtaining

$$\mathcal{P}|\varepsilon,\tilde{p};\, J\,\lambda\ \ \lambda_1\ \lambda_2> = (-1)^{s_1+s_2-J}\eta_1\eta_2|\varepsilon,\tilde{p};\, J\lambda,\ -\lambda_1,-\lambda_2\ > \qquad (50)$$

We may apply the ladder operators $J\pm$ which commute with $\mathcal{P}$ to obtain

$$\mathcal{P}|\varepsilon,\tilde{p};JM\lambda_1\lambda_2\ > = \eta_1\eta_2(-1)^{s_1+s_2-J}|\varepsilon,\tilde{p};\ J\ M,\ -\lambda_1,\ -\lambda_2\ >$$

$$= \eta_1\eta_2(-1)^{J-s_1-s_2}|\varepsilon,\tilde{p};J\ M,\ -\lambda_1,\ -\lambda_2\ > \qquad (51)$$

since $(J-s_1-s_2)$ is always an integer. We can also construct parity eigenstates by noting that $(|\alpha\ > \pm\mathcal{P}|\alpha>)$ are eigenstates of parity with parity +1 and -1.

For the states $|JM\lambda_1\lambda_2>$ defined earlier by

$$|\varepsilon,\tilde{p};JM\lambda_1\lambda_2> = \frac{2\pi}{\eta|_1\ \eta|_2}\sqrt{\frac{4\varepsilon}{|\vec{k}|_{c.m.}}}\ |\varepsilon,\tilde{p}> \otimes\ |JM;\lambda_1\lambda_2\ > \qquad (52)$$

it is clear that $(\tilde{p} \equiv (\varepsilon,\vec{0})$:

$$\mathcal{P}|\ M;\lambda_1\lambda_2> = \eta_1\eta_2\ (-1)^{J-s_1-s_2}|JM;-\lambda_1,\ -\lambda_2> \qquad (53)$$

The parity eigenstates then are

$$\left[|JM;\lambda_1\lambda_2\ > \pm\ \eta_1\eta_2(-1)^{J-s_1-s_2}|JM;-\lambda_1,\ -\ \lambda_2\ >\right] \qquad (54)$$

with parity $\pm$ 1.

For particle 1 massless and 2 massive or both massless the discussion goes along similar lines.

Parity operation on two particle state with total four momentum p^μ, on a fixed orbit $p^2 = \epsilon^2$ is given by

$$\mathcal{P}|\epsilon,p^\mu;JM\lambda_1\lambda_2> = \mathcal{P}\,U(\Lambda_p)\mathcal{P}^{-1}\mathcal{P}|\epsilon,\tilde{p};JM\lambda_1\lambda_2 > \tag{54}$$

TIME REVERSAL OPERATION

1. *INTRODUCTION. TIME REVERSED STATES.*

The time inversion operation is defined as the passive coordinate transformation

$$I_t: \begin{cases} x'^i = x^i \\ x' = -x \end{cases} \tag{1}$$

In classical dynamics the trajectory of the physical system is given by $\{q(t), p(t)\}$. If the Lagrangian does not depend on t explicitly and involves only even powers of p(t) then the "time rerversed" orbit described by $\{q_T(t), p_T(t)\}$ where

$$q_T(t) = q(-t) \qquad p_T(t) = -p(-t) \tag{2}$$

is also a physically possible solution. Since initial conditions completely describe the evolution of the trajectory of the system for future time it is clear from $q_T(0) = q(0)$ and $p_T(0) = -p(0)$ that the "time reversed" orbit (for example, a Kepler orbit) will be traced in the reversed sense. In case the reversed orbit is physically realizable, the system is said to be time reversal invariant.

The concept of time reversal is property through of in the active sense as "reversal of motion", it is accompanied by reversing the direction of the momenta and spins, but permiting the time to continue to run forward.. Such a reversal of motion is connected with the passive transformation $t \rightarrow -t$ through the fact that observables that are odd in t change their sign, while functions that are even do not. Let $q(t_0)$, $p(t_0)$ be the coordinate and momentum of a particle at $t = t_0$. After time τ its coordinates will be $q(t_0 + \tau)$, $p(t_0 + \tau)$. Start another identical particle off, now, (at $t = t_0 + \tau$) at $q(t_0 + \tau)$ and with momentum $-p(t_0 + \tau)$. Then at a latter time $(t_0 + 2\tau)$ if we find the positions and momentum to be $q(t_0)$ and $p(t_0)$ we say that the system is time reversal invariant otherwise it is not so.

In quantum mechanics the wave function of the system is described by the Schrödinger equation

$$i \frac{\partial}{\partial t} \psi(\vec{x}, t) = H(t) \psi(\vec{x},t) \tag{3}$$

The wave function $\psi(\vec{x},0) \equiv \psi(0)$ evolves to wave function $\psi(\vec{x}, \tau) \equiv \psi(\tau)$ at a latter time $t = \tau$. Consider now a time reversed state $\psi'(\tau)$ corresponding to $\psi(\tau)$; then the system is invariant under reversal of motion or time reversal if the state described by $\psi'(\tau)$ develops after additional interval of time τ to the state $\psi'(0)$. The state $\psi'(\tau)$ thus develops backward with time. It is then suggested to consider the functions (t_0 fixed),

$$\phi(t) \equiv \psi(t_0 - t) \tag{4}$$

for which $\psi(0) = \phi(t_0)$ and $\psi(t_0) = \phi(0)$, that is, while $\psi(0)$ goes to $\psi(t_0)$, when t varies from 0 to t_0, $\phi(t_0)$ varies to $\phi(0)$. Also $t \rightarrow (t_0-t)$ leads to

$$- i \frac{\partial\phi(t)}{\partial t} = H(t_0 - t)\ \phi(t) \tag{5}$$

This is not the Schrodinger equation and thus $\phi(t)$ cannot describe a physical time reversal state. The complex conjugate of this equations, however, gives

$$i \frac{\partial\phi^*(t)}{\partial t} = H^*(t_0 - t)\ \phi^*(t) \tag{6}$$

If $H^*(t_0-t) = H(t)$ then it becomes Schrodinger equation; in this case $\phi^*(t) = \psi^*(\vec{x}, t_0 - t)$ may be used to describe the time reversed state.Thus we define it by

$$\psi'(\vec{x},t) = \psi^*(\vec{x}, t_0 - t) \tag{7}$$

In more general case $H^* \neq H$ but there may exist a unitary operator such that $U H^*(t_0-t)U^{-1} = H(t)$ then the time reversed wave function satisfying the Schrodinger equations is

$$\psi'(\vec{x},t) = U\ \psi^*(\vec{x}, t_0 - t) \tag{8}$$

It is clear that the mapping $\psi \rightarrow \psi'$ is antilinear due to the antilinear operation of complex conjugation. In classical mechanics like wise the transformation reverses the sign in the Poisson bracket relationships e.g. it is not a canonical (but rather anti-canonical) transformation. The Hamiltons equations under $q(t) \rightarrow q'(t)$; $p(t) \rightarrow -p(t) = p'(t)$ change sign $\dot{q}'(t) = - \frac{\partial H'}{\partial p'(t)} \cdot \dot{p}'(t) = \frac{\partial H'}{\partial q'(t)}$ where $H'(q,p) = H(q',-p')$. The sign may be corrected if the transformation is $q(t) \rightarrow q_T(t) = q'(-t) =$ $= q(-t)$ and $p(t) \rightarrow p_T(t) = p'(-t) = -p(-t)$ since

$$\frac{dq_T(t)}{dt} = \frac{\partial H_T}{\partial p_T(t)} \qquad \frac{d\,p_T(t)}{dt} = -\frac{\partial H_T}{\partial q_T(t)} \tag{9}$$

Here $H_T(q_T(t), p_T(t)) = H(q_T(-t), -p_T(-t))$. Hence if H_T is the same function of (q_T, p_T) as H is of (q,p). Then if $(q(t), p(t))$ is a solution of the equation of motion so also is $(q_T(t), p_T(t)) = (q(-t),-p(-t))$.

We thus know that the mapping $|\Psi> \to |\Psi_T>$ is not unitary. For physical reasons we must require

$$|<\Psi_T(2)|\Psi_T(1)>|^2 = |<\Psi(2)|\Psi(1)>|^2 \tag{10}$$

Since $<\Psi_T(2)|\Psi_T(1)> = <\Psi(2)|\Psi(1)>$ holds only for unitary transformation we must have

$$<\Psi_T(2)|\Psi_T(1)> = <\Psi(1)|\Psi(2)> \tag{11}$$

and the corresponding mapping (according to Wigner theorem) is antiunitary. Such a mapping can be realized in two ways:

1. map a complete set of states $\{|n>\}$ on to *any other* complete set $\{|\bar{n}>\}$ and map a general state

$$|\Psi> = \Sigma\, C_n|n> = \Sigma|n><n|\Psi> \tag{12}$$

to

$$|\bar{\Psi}> = \Sigma\, C_n^*|\bar{n}> \tag{13}$$

We will call $|\bar{n}>$ as reversed states. For example, $\{|\bar{n}>\}$ may be time reversed states and $|\bar{\Psi}>$ is then the time reversal state of $|\Psi>$.

2. map the set $\{|n\rangle\}$ on to set of bras $\{\langle\bar{n}|\}$ and $|\Psi\rangle$ on the bra

$$\langle\bar{\Psi}| = \Sigma\, C_n \langle\bar{n}| = (|\bar{\Psi}\rangle)^{\dagger} \qquad (14)$$

The end results are related in the two cases by hermitian conjugation. It is worth remarking that the definition of the mapping above depends on a definite choice of basis vectors $|n\rangle$. It is convenient to introduce a unitary operator R such that $R|n\rangle = |\bar{n}\rangle$, $R^{\dagger}R = RR^{\dagger} = I$ e.g. $R = \sum_n |\bar{n}\rangle\langle n|$ and an antilinear operator K which takes complex conjugation of all the expansion coefficients of the arbitrary ket in terms of the particular basis $\{|n\rangle\}$ with which one is working e.g.,

$$K|\Psi\rangle = K\,\Sigma\, C_n|n\rangle = \Sigma\, C_n^{*}|n\rangle \qquad (15)$$

Since $K^2 = I$ it follows $K = K^{-1}$. Thus

$$|\bar{\Psi}\rangle = R\,K\,|\Psi\rangle = Ⓗ\,|\Psi\rangle \qquad (16)$$

where $Ⓗ = R\,K$ is antiunitary operator with the inverse $Ⓗ^{-1} = K\,R^{-1}$.
A few properties of antilinear operators are worth reminding.

5.2 - *ANTILINEAR OPERATORS*

An operator is antilinear if it satisfies

$$A(\lambda\psi) = \lambda^{*}(A\psi) \qquad \text{anti-} \qquad (17)$$

$$A(\psi_1 + \psi_2)\; A\,\psi_1 + A\,\psi_2 \qquad \text{- linear}$$

The inverse A^{-1} is also antilinear. The bra-ket notation is inconvinient and we will use the parenthesis notation. A hermitian conjugate or adjoint operator A^{+} can be introduced by the relation:

$$(\psi,\ A\,\varphi) = (A^{\dagger}\,\psi,\ \varphi)^{*} = (\varphi,\ A^{\dagger}\,\psi) \qquad (18)$$

The demonstration goes as follows: $(\psi, A\varphi)^{*} \equiv a_{\psi}(\varphi)$ for a fixed ψ, is a linear function of φ e.q. $a_{\psi}(\varphi_1 + \varphi_2) = a_{\psi}(\varphi_1) + a_{\psi}(\varphi_1)$ and $a_{\psi}(\lambda\varphi) = \lambda a_{\psi}(\varphi)$. Thus there exists a (uniquely) defined vector α_{ψ} such that $a_{\psi}(\varphi) = (\alpha_{\psi}, \varphi)$. We note also $a_{\lambda\psi}(\varphi) = (\alpha_{\lambda\psi}, \varphi) = (\lambda\psi, A\varphi)^{*} = \lambda a_{\psi}(\varphi) = (\lambda^{*}\alpha_{\psi}, \varphi)$ giving $\alpha_{\lambda\psi} = \lambda^{*}\alpha_{\psi}$. Varying now ψ one sees that there exists a uniquely defined antilinear operator $A^{\dagger}$ such that $\alpha_{\psi} = A^{\dagger}\psi$, obtaining $(\psi, A\varphi)^{*} = (A'\psi, \varphi)$. Similar considerations applied to $(\psi, \mathbb{A}\varphi)$ where $\mathbb{A}$ is linear operator leads to introduction of adjoint $\mathbb{A}^{\dagger}$ but with the corresponding relation $(\psi, \mathbb{A}\,\varphi) = (\mathbb{A}^{\dagger}\psi, \varphi)$.

We can easily demonstrate that $(A^{\dagger})^{\dagger} = A$, $(AB)^{\dagger} = B^{\dagger}\,A^{\dagger}$ where each of the operators is either linear or antilinear and we note that the product of two antilinear operators is linear operator.

An operator is antihermitian if it is anti-linear and satisfies $A^{\dagger} = A$, is antiunitary if $A^{\dagger} = A^{-1}$. The adjoint $K^{\dagger}$ of complex conjugation operator K can be seen to be antihermitian and antiunitary. In fact from the definition of K it is clear that it leaves the basis vectors $\{|n\rangle \equiv u_n\}$ invariant,

$$K\,u_n = u_n \qquad \text{or} \qquad K|n\rangle = |n\rangle \qquad (19)$$

Thus if $\varphi = \Sigma\, d_n\, u_n$ and $\psi = \Sigma\, C_n\, u_n$ we have

$$(\psi,\ K\varphi)^{*} = (\sum_m C_m\, u_m,\ \Sigma\, d_n^{*}\, u_n)^{*} = \Sigma\,\Sigma\, C_m\, d_n(u_m,\ u_n)^{*}$$

while $(K^{\dagger}\psi, \varphi) = (\Sigma\, C_m^{*}\, K^{\dagger}\, u_m,\ \Sigma\, d_n\, u_n) = \Sigma\,\Sigma\, C_m\, d_n(u_m,\ u_n)$

From $(\psi, K\varphi)^{*} = (K^{\dagger}\psi, \varphi)$ and the fact that the basis vectors are orthogonal it follows that $K' = K$. The operator $\textcircled{H} = R\,K$ where R is unitary linear operator is antiunitary, since $\textcircled{H}^{\dagger} = K^{\dagger}R^{\dagger} = K\,R^{-1} = \textcircled{H}^{-1}$. It is instructive to verify the definition of $\textcircled{H}^{\dagger}$ explicitly (R any linear operator):

$$(\psi,\textcircled{H}\,\varphi)^{*} = (\Sigma\, C_m\, u_m,\ R \sum_n d_n^{*}\, u_n)^{*} = \sum_m \sum_n C_m d_n\ (u_m,\ R\, u_n)^{*}$$

$$(\textcircled{H}^{\dagger}\ \psi,\varphi) = (K^{\dagger}R^{\dagger}\Sigma C_m\ u_m,\ \Sigma d_n\ u_n) = (K \sum_m \sum_s C_m\ u_s(u_s, R^{\dagger}\ u_m),\ \Sigma d_n\ u_n\ )$$

$$= (\sum_m \sum_s C_m^{*}\ (u_s, R^{\dagger} u_m)^{*} u_s,\ \sum_n d_n u_n) = \sum_m \sum_s \sum_n C_m d_n (u_s, R^{\dagger} u_m)(u_s, u_n)$$

$$= \sum_m \sum_n C_m d_n (u_n, R^{\dagger}\ u_m) = \sum_m \sum_n C_m d_n (u_m,\ R\ u_n)^{*}$$

Further properties of K operator are

$$(u_m,\ K\,\psi) = (K^{\dagger}\ u_n,\ \psi)^{*} = (u_n,\ \psi)^{*} \tag{20}$$

$$(K\psi,\ u_n) = (\psi,\ u_n)^{*} \tag{21}$$

$$(u_m,\ K\,B\,K\ u_n) = (u_m,\ K\,B\ u_n) = (u_m,\ B\ u_n)^{*}, \tag{22}$$

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* In bra-ket notation the definitions of adjoint read $<\psi|(A|\varphi> = <\varphi|(A^{\dagger}|\psi>$ and $<\psi|A|\varphi>^{*} = <\varphi|A^{\dagger}|\psi>$. For antilinear operators, however, one has to exercise some came since one has $|\chi\ |A)|u> = [<\chi|(A|u>]^{*}$ etc. See for example, A. Messiah, Quantum Mechanics Chap. VII and XV. See also E. Wigner, J. Math. Phys. 1, 409, 414 (1960).

where B is a linear operator. Thus in the representation we are working the operation of K amounts merely to complex conjugation. Any arbitrary antilinear operator A can always be written as

$$A = (AK)K = K(KA) \tag{23}$$

Clearly (AK) and (KA) are linear operators and satisfy

$$(KA) = K(AK)K \tag{24}$$

Complex Conjugate of linear operator B is defined by

$$B^* = K\ B\ K \tag{25}$$

Care must be exercised since the definition of K depends on the basis in which we are working. We note

$$(BC)^* = B^*\ C^* \tag{26}$$

$$(B^*)^* = B \tag{27}$$

$$(\psi,\ B^*\ \varphi) = (\psi,\ KBK\varphi) = (K\psi,\ BK\varphi)^* \tag{28}$$

$$(K\psi,\ B^*\ K\varphi) = (\psi,\ B\ \varphi)^* \tag{29}$$

Transpose of linear operator B is defined as:

$$B^T = (B^\dagger)^* = (B^*)^\dagger = (K\ B\ K)^\dagger = K\ B^\dagger\ K \tag{30}$$

We note

$$(BC)^T = \left[(BC)^\dagger\right]^* = (C^\dagger)^*\ (B^\dagger)^* = C^T\ B^T \tag{31}$$

$$(K\psi,\ B^T\ K\varphi) = (\psi,\ B^\dagger\varphi)^* = (\varphi,\ B\ \psi) \tag{32}$$

5.3 - *CHANGE OF BASIS*

Consider a change of basis from $|n>$ to $|\nu>$ and let $|\bar{\nu}>$ denote the reversed of $|\nu>$. Then $\textcircled{H} = \mathbb{R}\,\mathbb{K}$ where $\mathbb{K}$ is complex conjugation operator w.r.t. the new basis, clearly $\mathbb{R} = \Sigma|\bar{\nu}><\nu|$. We have

$$\psi = \Sigma\, u_n(u_n, \psi) = \Sigma\, \xi_\nu(\xi_\nu, \psi) \tag{33}$$

where we write $u_n = |n>$ and $\xi_\nu \equiv |\nu>$ etc. for parenthesis notation. We compute $\textcircled{H}\psi$ w.r.t. the two basis:

$$\textcircled{H}\psi = R\,K\,\psi = \Sigma\, R\, u_{n'}(u_{n'},\psi)^* = \Sigma\, u_n(u_n, R\, u_{n'})(u_{n'}, \psi)^*$$

$$\textcircled{H}\psi = \mathbb{R}\,\mathbb{K}\,\psi = \Sigma\, \xi_{\nu'}(\xi_{\nu'}, \mathbb{R}\,\xi_\nu)(\xi_\nu, \psi)^*$$

$$= \Sigma\, u_n(u_n, \xi_{\nu'})(\xi_{\nu'}, \mathbb{R}\,\xi_\nu)(\xi_\nu\, , u_{n'})^*\,(u_{n'}, \psi)^*$$

Here the summation over repeated indices is understood. Comparing the two expressions we obtain

$$(u_n, R\, u_{n'}) = \Sigma(u_n, \xi_{\nu'})(\xi_{\nu'}, \mathbb{R}\,\xi_\nu)(\xi_\nu, u_{n'})^*$$

or

$$R_{nn'} = W^*_{\nu' n}\, \mathbb{R}_{\nu'\nu}\, W^*_{\nu n'} = (W^\dagger\, \mathbb{R}\, W^*)_{nn'}$$

where $W_{\nu n} = (\xi_\nu, u_n)$. The transformation $\{u_n\} \rightarrow \{\xi_\nu\}$ is unitary since

$$u_n = \Sigma\, \xi_\nu\, W_{\nu n}$$

and $\Sigma\, W_{\nu n}\, W^*_{\nu n'} = \Sigma(u_n, \xi_\nu)^*\,(u_{n'}, \xi_\nu) = \delta_{nn'}$ etc. Thus

$$\mathbb{R} = W\, R\, W^T \tag{34}$$

instead of the usual transformation $W\, R\, W^\dagger = W\, R\, W^{-1}$. Only if the transformation coefficients $W_{n\nu}$ are real (i.e. W is orthogonal) do we get the similarity transformation theory in this problem. Note that RK takes ψ to the reversed state w.r.t. the basis $\{|n>\}$ while $\mathbb{R}\,\mathbb{K}$ takes to the reversed state w.r.t. the new basis $\{|\nu>\}$.

5.4 - *TIME REVERSED OPERATORS*

To determine the operator R in order to define the time reversal operation we require, in analogy with classical case

$$< \psi_T|\vec{r}|\psi_T > = < \psi|\vec{r}|\psi >$$
$$< \psi_T|\vec{p}|\psi_T > = - < \psi|\vec{p}|\psi > \qquad (35)$$

where $|\psi_T >$ is the time reversal state $|\psi_T > = \Theta|\psi > = R\,K|\psi >$. We require thus

$$< \psi|\vec{r}|\psi > = (R\,K\,\psi,\ \vec{r}\,R\,K\,\psi) = (K\psi,\ R^{\dagger}\vec{r}\,R\,K\,\psi) \qquad (36)$$

A suitable choise for R will then be such that

$$R^{\dagger}\vec{r}\,R = \vec{r}^{\,*} \qquad \text{or} \qquad \vec{r} = R\,\vec{r}^{\,*}\,R^{-1} = \Theta\,\vec{r}\,\Theta^{-1} \qquad (37)$$

for then the right hand side becomes $(\psi,\ \vec{r}^{\,*}\,K\,\psi) = (\psi,\ \vec{r}\,\psi)^{*} = (\psi,\ \vec{r}\,\psi)$ since $\vec{r}$ is hermitian operator. Likewise we require that

$$-\vec{p} = R\,\vec{p}^{\,*}\,R^{-1} = \Theta\,\vec{p}\,\Theta^{-1} \qquad (38)$$

Clearly $\Theta\,[p_i,\ x_j]\,\Theta^{-1} = -\,[p_i,\ x_j] = +\,i\,\delta_{ij}$ (39)

and

$$\Theta \vec{L} \Theta^{-1} = -\vec{L} \tag{40}$$

The time reversed operator corresponding to any general operator $\mathcal{O}$ is then defined by

$$\mathcal{O}_T = R\mathcal{O}^* R^{-1} = R K \mathcal{O} K R^{-1} = \Theta \mathcal{O} \Theta^{-1} \tag{41}$$

where

$$\Theta = R K \quad \text{and} \quad \Theta^{\dagger} = \Theta^{-1} = K R^{-1} \tag{42}$$

In particular $\vec{r}_T = \vec{r}$ and $\vec{p}_T = -\vec{p}$. Note that R is defined up to an arbitrary phase. The definition implies for any linear operator (hermitian or not)

$$(\mathcal{O}\psi)_T = \mathcal{O}_T \psi_T \tag{43}$$

and

$$(\psi_T(2), \mathcal{O}_T \psi_T(1)) = (\psi(2), \mathcal{O}\psi(1))^* = (\psi(1), \mathcal{O}^{\dagger} \psi(2) \tag{44}$$

for

$$(\mathcal{O}\psi)_T = R K \mathcal{O} \quad \psi = R K \mathcal{O} K R^{\dagger} R K \psi = R\mathcal{O}^* R \psi_T = \mathcal{O}_T \psi_T$$

$$(\psi_T(2), \mathcal{O}_T \psi_T(1)) = (K\psi(2), R^{\dagger} R\mathcal{O}^* K \psi(1)) = (K\psi(2), \mathcal{O}^* k \psi(1)) =$$

$$= (\psi(2), \mathcal{O}\psi(1))^*$$

For $\mathcal{O} = \lambda$ a complex number, $\mathcal{O}_T = \lambda^*$ and

$$(\psi_T(2), \lambda^* \psi_T(1)) = (\psi(2), \lambda\psi(1))^*$$

or

$$(\psi_T(2), \psi_T(1)) = (\psi(1), \psi(2))$$

In Schrodinger picture the time dependence of states is given by

$$i \frac{\partial}{\partial t} |\psi (t) > = H(t)|\psi(t) >$$

we deduce

$$- i \frac{\partial}{\partial t} |\psi_T(t) > = Ⓗ H(t) Ⓗ^{-1}|\psi_T(t) > \qquad (45)$$

changing $t \rightarrow (t_0 - t)$

$$i \frac{\partial}{\partial t} |\psi_T(t_0-t) > = Ⓗ H(t_0-t) Ⓗ^{-1}|\psi_T(t_0-t) > \qquad (46)$$

If $H_T = Ⓗ H(t_0-t) Ⓗ^{-1} = R H^* R^{-1} = H$ e.g. H is invariant under time reversal, and if $|\psi (t) >$ is a possible state of the system so is $|\psi_{rev}(t) > \equiv |\psi_T(t_0-t)>$. The dynamical state represented by $|\psi_{rev} (t) >$ at a given time t is the time-reversal transform of the state represented by $|\psi (t) >$ at time (t_0-t). In the Heisenberg picture one can show that if $\vec{r}(t)$ and $\vec{p}(t)$ satisfy the canonical equations of motion, so also do $\vec{r}_{rev}(t) = \vec{r}_T(-t) = \vec{r}(-t)$ and $\vec{p}_{rev}(t) = \vec{p}_T(-t) = -\vec{p}(-t)$.

It is clear that Ⓗ commutes with all the spatial transformations. From $Ⓗ \; p_i \; Ⓗ^{-1} = - p_i$ it follows $Ⓗ \; e^{-i\vec{p}\cdot\vec{a}} = e^{-i\vec{p}\cdot\vec{a}} \; Ⓗ$. From $Ⓗ (\vec{r} \times \vec{p}) Ⓗ^{-1} = - (\vec{r} \times \vec{p})$ it follows $Ⓗ \; e^{-i\vec{L}\cdot\vec{n}\omega} = e^{-i\vec{L}\cdot\vec{n}\omega} \; Ⓗ$. Reflections too commute with time reversal.

Since spin is a particular angular momentum, we assume

$$Ⓗ \vec{S} Ⓗ^{-1} = - \vec{S} \qquad \text{or} \qquad R \vec{S}^* R^{-1} = - \vec{S} \qquad (47)$$

The definition preserves the property of commutation of time reversal with spatial transformations. It follows

$$\text{Ⓗ} \vec{J}\, \text{Ⓗ}^{-1} = -\vec{J} \quad \text{or} \quad R\, \vec{J}^* R^{-1} = -\vec{J} \tag{48}$$

and consequently $[\text{Ⓗ}, \mathcal{R}] = 0$, $\mathcal{R}$ being rotation operator ($\mathcal{R} = e^{-i\vec{J}\cdot\vec{n}\omega}$).

The invariance of an operator B under time reversal is expressed by $B_T = B$, that is, $\text{Ⓗ} B = B\, \text{Ⓗ}$ or $RB^* = BR$ or $R\, B^T R^{-1} = B^\dagger$ or $RB^* R^{-1} = B$. The invariance condition for the operator $S = e^{-iB\alpha}$ (α real) is then given by $\text{Ⓗ}\, S\, \text{Ⓗ}^{-1} = e^{i\alpha \text{Ⓗ} B \text{Ⓗ}^{-1}} = e^{i\alpha B}$, that is, $\text{Ⓗ}\, S\, \text{Ⓗ}^{-1} = S^{-1}$. If B is hermitian $\text{Ⓗ}\, S\, \text{Ⓗ}^{-1} = S^\dagger$. This is to be contrasted with the case of invariance under (linear) unitary transformation where the invariance condition is expressed by $U\, B\, U^{-1} = B$ and $U\, S\, U^{-1} = S$. The relation $\text{Ⓗ}\, S\, \text{Ⓗ}^{-1} = S$ is obtained for time odd operators satisfying $\text{Ⓗ}\, B\, \text{Ⓗ}^{-1} = -B$. We note an important relation for the matrix elements of S when B is hermitian and invariant under time reversal:

$$(\psi_T(2), \text{Ⓗ}\, S\, \text{Ⓗ}^{-1} \psi_T(1)) = (\psi(2), S\, \psi(1))^* = (\psi_T(2), S^\dagger \psi_T(1))$$

$$= (S\, \psi_T(2), \psi_T(1)),$$

S being linear operator, thus

$$(\psi_T(1), S\, \psi_T(2)) = (\psi(2), S\, \psi(1)) \tag{49}$$

Note the interchange of initial and final states.

We will now determine the operator R for some special cases.

In coordinate space representation spanned by the kets in the wave function $\psi(\vec{r},t) = \langle\vec{r}|\Psi(t)\rangle$ is defined by

$$|\Psi(t)\rangle = \int d^3\vec{r}\, |\vec{r}\rangle \langle\vec{r}|\Psi(t)\rangle = \int d^3\vec{r}\, |\vec{r}\rangle\, \psi(\vec{r},t) \tag{50}$$

so that

$$K|\Psi(t)> = \int d^3\vec{r}|\vec{r}> < \vec{r}|\Psi(t)>^*$$

$$\textcircled{H}\Psi(t) > = \int d^3\vec{r} \int d^3 \vec{r}'|\vec{r}' > < \vec{r}'|R|\vec{r} > < \vec{r}|\Psi(t) >^*$$

$$= \int d^3\vec{r} \int d^3\vec{r}' \ |\vec{r} > < \vec{r}'|\textcircled{H}|\vec{r}> < \vec{r}|\Psi(t) >^* \tag{51}$$

Since $\vec{r}_{op}(\textcircled{H}|\vec{r}>) = \vec{r}(\textcircled{H}|\vec{r}>)$ we can write $\vec{r}'|\textcircled{H}|\vec{r}> = \delta^3(\vec{r}'-\vec{r})R$ where R is the coordinate space representation of the operator R, and acts on the spin components of $\psi^*(\vec{r},t)$.

$$\textcircled{H} \ |\Psi(t) > = \int d^3\vec{r}|\vec{r} > R\ \psi^*(\vec{r},t) \tag{52}$$

e.g. in the coordinate space the time reversal operation is realized by the complex conjugation of the wave function followed by a unitary transformation R which acts on the spin components of the wave function.

In momentum space representation we have

$$|\Psi(t) > = \int d^3\vec{p}|\vec{p}><\vec{p}|\Psi(t) > = \int d^3\vec{p}|\vec{p} > \Phi(\vec{p},t) \tag{53}$$

and

$$\textcircled{H}|\Psi(t)> = \int d^3p \int d^3\vec{p}'|\vec{p}'><\vec{p}'|\textcircled{H}|\vec{p}> \Phi^*(\vec{p},t) \tag{54}$$

Now

$$\vec{p}_{op}(\textcircled{H}|\vec{p}>) = -\textcircled{H}\ \vec{p}_{op}|\vec{p}> = -\vec{p}\ (\textcircled{H}|\vec{p}>)$$

thus

$$<\vec{p}'|\textcircled{H}|\vec{p}> = <\vec{p}'|R|\vec{p}> = \delta^3(\vec{p}+\vec{p})\mathbb{R}$$

where $\mathbb{R}$ is the operator acting on the spin components of the momentum space wave function. Then

$$Ⓗ\,|\Psi(t)\rangle = \int d^3\vec{p}\,'\;|\vec{p}\,'\rangle\;\mathbb{R}\;\phi^*(-\vec{p}\,',\,t) \tag{55}$$

e.g. in the momentum space the time reversal operation is realized by complex conjugation of the momentum wave function, changing $\vec{p}$ to $-\vec{p}$ and following with a unitary transformation $\mathbb{R}$ acting on the spin components of the wave function.

For example consider the wave packet in momentum space for a spinless particle:

$$|\Psi(t)\rangle = \int \frac{d^3k}{(2\pi)^3}\,a(\vec{k})e^{-iE_kt}|\vec{k}\rangle = \int d^3k\;|\vec{k}\rangle\,\Phi(\vec{k},t) \tag{56}$$

$$|\Psi_T(t)\rangle = Ⓗ\;|\Psi(t)\rangle = \int \frac{d^3k}{(2\pi)^3}\,a^*(-k)e^{iE_kt}\;|\vec{k}\rangle$$

$$|\Psi_{rev}(t)\rangle = |\Psi_T(-t)\rangle = \int \frac{d^3k}{(2\pi)^3}\;a^*(-k)\;e^{-iE_kt}\;|\vec{k}\rangle \tag{57}$$

Clearly,

$$\Phi_{rev}(\vec{k},t) = \Phi^*(-\vec{k},\,-t) \tag{58}$$

The coordinate space wave function is easily obtained:

$$\psi(\vec{r},t) = \langle\vec{r}|\Psi(t)\rangle = \int \frac{d^3k}{(2\pi)^3}\,a(k)\;e^{i(\vec{k}\cdot\vec{r}-E_kt)} = \int d^3k\;\Phi^*(\vec{k},t)e^{i\vec{k}\cdot\vec{r}}$$

$$\psi_{rev}(\vec{r},t) = \langle\vec{r}|\Psi_T(-t)\rangle = \int \frac{d^3k}{(2\pi)^3}\,a^*(-\vec{k})e^{i(\vec{k}\cdot\vec{r}-E_kt)} = \int d^3k\;\Phi^*(-\vec{k},-t)e^{i\vec{k}\cdot\vec{r}}$$

$$= \int \frac{d^3k}{(2\pi)}\,a^*(\vec{k})e^{-i(\vec{k}\cdot\vec{r}+E_kt)} = \int d^3k\;\Phi^*(\vec{k},-t)e^{-i\vec{k}\cdot\vec{r}}$$

$$= \psi(\vec{r},\,-t)^* \tag{59}$$

We can calculate the average momentum in reversed state

$$\int d^3\vec{r} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3\vec{k}'}{(2\pi)^3} a(\vec{k})e^{i(\vec{k}\cdot\vec{r}+E_k t)} (\tfrac{1}{i}\vec{\nabla})\ a^*(\vec{k}')e^{-i(\vec{k}'\cdot\vec{r}+E_{k'} t)}$$

$$= \iint \frac{d^3k\ d^3\vec{k}'}{(2\pi)^3} (-\vec{k}')\ a(\vec{k})a^*(\vec{k})\ \delta^3(\vec{k}-\vec{k}') = -\int \frac{d^3k}{(2\pi)^3} \vec{k}|a(\vec{k})|^2$$

which is opposite in sign to the average momentum in the original state.

The time reversal, it is clear leaves the total charge invariant while reverses the direction of the current. This is sometimes called *Wigner time reversal*. It is also possible to define time reversal often called *Schwinger time reversal* which leaves the current invariant and reverses the charge, and indeed is Wigner time reversal either preceded by or followed by charge conjugation.

We now determine the unitary transformation R for particles with spin. Working in coordinate space representation, R acts solely on the spin components of state vector. We will adopt also the usual "standard basis" in spin (angular momentum) space in order to define K. In this representation space J_3 and $J_\pm = (J_1 \pm i\ J_2)$ have real matrix elements so that J_1 and J_3 have real while J_2 has pure imaginary matrix elements. R must then satisfy ($R^+R = RR^+ = I$).

$$R\ J_1\ R^{-1} = -J_1 \qquad R\ J_2\ R^{-1} = J_2 \qquad R\ J_3\ R^{-1} = -J_3$$

since

$$K\ J_1\ K^{-1} = J_1 \qquad K\ J_2\ K^{-1} = -J_2 \qquad K\ J_3\ K^{-1} = J_3$$

Then

$$R = e^{-i\pi J_2}$$

corresponding to rotation by π around 2-axis, as can be easily verified *. In coordinate representation, thus, time reversal operator is ($\vec{S}$ being spin operator)

$$Ⓗ \equiv e^{-i\pi S_2} K = K e^{-i\pi S_2} \tag{60}$$

since in our representation $(-iJ_2)$ has real matrix elements. From $[Ⓗ, J^2] = 0$ it follows that time reversal does not change the value of j. Also of interest is the unitary operator $Ⓗ^2$. It commutes with $\vec{J}$, $\vec{p}$, $\vec{r}$. Time reversal applied twice in succession will bring the physical system to its original state. Hence

$$Ⓗ^2 = \varepsilon I \qquad \text{where } |\varepsilon|^2 = 1 \tag{61}$$

Since Ⓗ is antiunitary the introduction of a phase factor does not alter $Ⓗ^2$ e.g.

$$(e^{i\delta}Ⓗ)^2 = e^{i\delta}Ⓗ \; e^{i\delta}Ⓗ = e^{i\delta} \; e^{-i\delta}Ⓗ^2 = Ⓗ^2 \tag{62}$$

Also we have

$$Ⓗ^3 = \varepsilon Ⓗ = Ⓗ \varepsilon \tag{63}$$

implying $\varepsilon = \varepsilon^*$. Thus

$$Ⓗ^2 = \pm I \tag{64}$$

In other words all states must be eigenstates of $Ⓗ^2$ with eigenvalues +1 or -1. It may be remarked that in the present case of antiunitary

* Note then $R\mathcal{R}(\alpha,\beta,\gamma)R^{-1} = \mathcal{R}^*(\alpha,\beta,\gamma)$ in "Standard basis representation".

operator we do not have eigenstates of Ⓗ analogous to the case of space reflection. In the standard basis representation of angular momentum states we have:

$$Ⓗ^2 = e^{-i\pi S_2} K e^{-i\pi S_2} K = e^{-i\pi S_2} e^{-i\pi S_2} = e^{-i2\pi S_2} \tag{65}$$

From the result

$$e^{-i(2\pi)\vec{J}\cdot\vec{n}} |jm\rangle = (-1)^{2j}|jm\rangle$$

we see that $Ⓗ^2$ has eigenvalues +1 for integral spin S and -1 for half-odd -integer S, we ignore all other internal attributes.

As an example we consider the case of S = 1/2 particle in coordinate representation:

$$R = e^{-i\frac{\pi}{2}\sigma_2} = -i\,\sigma_2$$

$$Ⓗ = -i\,\sigma_2\,K = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} K = K\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \tag{66}$$

$$Ⓗ^2 = -I$$

In coordinate representation the representative of Ⓗ is $\langle\vec{r}|Ⓗ|\vec{r}'\rangle$ = $= R\,K\,\delta^3(\vec{r}-\vec{r}')$. Time reversed state then is

$$\psi_{rev}(\vec{r},t) = \int d^3\vec{r}'\,\langle\vec{r}|Ⓗ|\vec{r}'\rangle\;\psi(\vec{r}',-t) = R\,K\,\psi(\vec{r},-t) \tag{67}$$

For free particle with m = 1/2

$$\psi(\vec{r},t) = e^{i(\vec{k}\cdot\vec{r}-Et)}\begin{pmatrix}1\\0\end{pmatrix} \quad e^{i(\vec{k}\cdot\vec{r}-Et)}\,|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

Then

$$\psi_{rev}(\vec{r},t) = e^{+i(-\vec{k}\cdot\vec{r}-Et)}\begin{pmatrix}0\\1\end{pmatrix} \equiv e^{-i(\vec{k}\cdot\vec{r}+Et)}|\tfrac{1}{2},-\tfrac{1}{2}\rangle \tag{68}$$

ψ_{rev} has spin and momentum components opposite to those of ψ. The same goes for the orbital angular momentum.

Time reversal invariance requires that any spin independent Hamiltonian must be real. For spin-orbit term $V_{so}(r)\ \vec{\sigma}\cdot\vec{L}$ the invariance requires that $V_{so}(r)$ be real. However for a term $V_p(r)\vec{\sigma}\cdot\vec{r}$ the invariance condition is

$$R\ V_p^*(r)(\vec{\sigma}^*\cdot\vec{r})\ R^{-1} = -\ V_p^*(r)\ \vec{\sigma}\cdot\vec{r} = V_p(r)(\vec{\sigma}\cdot\vec{r})$$

that is, $V_p(r)$ is imaginary. This term, however, is not reflection invariant.

5.5 - *TRANSFORMATION OF ANGULAR MOMENTUM STATES*

In latter discussions we will need to know the transformation properties of angular momentum states under time reversal. From $[\textcircled{H}, J^2] = 0$ it follows that the values of j is unaltered and only the eigenvalue of J_3 will get altered. From

$$\textcircled{H}\ J_3 \textcircled{H}^{-1} \textcircled{H} |jm> = -J_3 \textcircled{H} |jm> = \textcircled{H} m|jm$$

$$J_3(\textcircled{H} |jm>) = -\ m\ (\textcircled{H} |jm>)$$

that is,

$$\textcircled{H} |jm> = \mu(j, -m)|j, -m> \tag{69}$$

where $\mu(j, -m)$ is a phase factor. Also

$$J_{\pm}|jm> = \sqrt{j(j+1) - m(m \mp 1)}\,|jm \pm 1>$$

and

$$Ⓗ\; J_{\pm}Ⓗ^{-1} = - J_{\mp}$$

give

$$Ⓗ\; J_{-}|jm> = - J_{+}Ⓗ|jm> = -\mu(j,-m)\; J_{+}\, |j, -m>$$

$$= -\mu(j,-m)\, \sqrt{j(j+1)-m(m-1)}\,|j,-m+1>$$

But

$$Ⓗ\; J_{-}\,|jm> = \sqrt{j(j+1)-m(m-1)}\,Ⓗ|j, m-1>$$

$$= \mu(j,-m+1)\, \sqrt{j(j+1)-m(m-1)}\,|j,-m+1>$$

We obtain

$$\mu(j, -m) = - \mu(j, -m+1)$$

Hence

$$\mu(j, -m) = (-1)^{m}\, \mu_j \qquad \text{where} \quad |\mu_j| = 1 \tag{70}$$

It is desirable to choose μ_j so that the form of $\mu(j, -m)$ remains invariant under addition of angular momentum. We have

$$|jm\; j_1\; j_2> = \Sigma < j_1\; j_2\; m_1\; m_2|jm > |j_1\; m_1> |j_2\; m_2 > \tag{71}$$

where the Clebsch-Gordon coefficients are real in the standard convention of phases. Thus we require

$$Ⓗ|jm\; j_1 j_2> = (-1)^{m}\, \mu_j |j_1 j_2\; j, -m >$$

$$= \Sigma < j_1 j_2\; m_1 m_2|jm >(-1)^{m_1+m_2}\, \mu_{j_1}\mu_{j_2}\, |j_1 -m_1>|j_2 -m_2 >$$

$$= (-1)^{m}\, \mu_{j_1}\mu_{j_2}(-1)^{j_1+j_2-j}\; \Sigma<j_1 j_2 -m_1 -m_2|j-m>|j_1 -m_1>|j_2 -m_2$$

or

$$\mu_j(-1)^j = \mu_{j_1}(-1)^{j_1}\ \mu_{j_2}(-1)^{j_2} \tag{72}$$

we could satisfy thus by choosing

$$\mu_j = (-1)^{-j} = i^{-2j} \tag{73}$$

Thus

$$\text{Ⓗ}\,|jm> = (-1)^{j-m}|j, -m> \tag{74}$$

This choice of phase is also consistent with the choice of phase in the representation of time reversal operator by $\text{Ⓗ} = e^{-i\pi S_2} K$ in the representation space of standard basis. For

$$K|jm> = |jm> \tag{75}$$

and

$$\text{Ⓗ}\,|jm> = e^{-i\pi J_2}|jm> = \sum_{m'} d^j_{m'm}(\pi)|jm> = (-1)^{j-m}|j,-m>$$

In the special case of $\vec{J} = \vec{L}$

$$\text{Ⓗ}\,|\ell, m> = (-1)^{\ell-m}|\ell,-m \tag{76}$$

implying that in the coordinate representation

$$<\Theta\phi|\text{Ⓗ}|\ell m> = \text{Ⓗ}<\Theta\phi|\ell m> = (-1)^{\ell-m} <\Theta\phi|\ell m> \tag{77}$$

In the coordinate representation for spinless case time reversal can by realized simply by complex conjugation, therefore

$$\text{Ⓗ}<\Theta\phi|\ell m> = <\Theta\phi|\ell m>^* = (-1)^{\ell-m} <\Theta\phi|\ell, -m>$$

Since standard spherical harmonics satisfy $Y^m_\ell(\Theta,\phi)^* = (-1)^m\, Y^m_\ell(\Theta,\phi)$ it follows that

$$< \Theta\phi|\ell m > = i^{\ell}\, Y_{\ell}^{m}\,(\Theta,\phi) \tag{78}$$

However, in the momentum space representation $\vec{p} \to -\vec{p}$ in addition to the complex conjugation of the (spinless) momentum space wave function e.g.

$$Ⓗ< \Theta_p\phi_p|\ell m > = (-1)^{\ell-m} < \Theta_p\phi_p|\ell,-m > = <\pi-\Theta_p,\pi+\phi_p|\ell,m >^{*}$$

Now

$$Y_{\ell}^{m}(\pi-\Theta,\ \pi+\phi)^{*} = (-1)^{\ell-m}\, Y_{\ell}^{m}\,(\Theta,\phi)$$

thus

$$<\Theta_p,\ \phi_p|\ell m > = Y_{\ell}^{m}(\Theta_p,\ \phi_p) \tag{79}$$

We note that $Ⓗ^2 = 1$ for orbital angular momentum.

We also note an important *superselection rule*: there can exist no observables quantity which have non-zero matrix elements connecting states of integral and half-odd-integral spin *. This follows from $Ⓗ^2 = \pm I$ and $Ⓗ^2|j\ldots > = (-1)^{2j}|j\ \ldots >$ so that

$$\begin{aligned} <j'\ldots|B|j\ldots > &= < j'\ldots|\,Ⓗ^2\ B\ Ⓗ^2|j\ \ldots > \\ &= (-1)^{2(j-j')} < j'\ldots|B|j\ldots> \end{aligned} \tag{80}$$

Hence the theorem.

5.6 - *KRAMERS DEGENERACY*

Another important consequence of time reversal invariance is the *Kramers degeneracy* in the case $Ⓗ^2 = -I$ when the states $|\Psi >$ and $Ⓗ|\Psi >$ are necessarily orthogonal:

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* We ignore here other internal attribute.

$$(\psi, \Theta\psi) = -(\Theta^2\psi, \Theta\psi) = -(\Theta\psi, \Theta^{\dagger}\Theta\psi)^* = -(\psi, \Theta\psi) \qquad (81)$$

and thus linearly independent. Thus eigenvectors of Θ^2 in this case are two fold degenerate. If time reversal operation is a symmetry operation the system, in the case of $\Theta^2 = -I$, one requires for its complete description an additional two dimensions in the abstract state vector space. For example, if the Hamiltonian of a particle with spin half-odd integer be invariant under time reversal, the energy eigenvalues are at least doubly degenerate (or degenerate an even number of time). In each subspace corresponding to a fixed energy value we can choose an orthogonal basis made up of the pairs formed by a vector and its time reversed vector . If time reversal is not a symmetry operator Kramers degeneracy does not apply e.g. in the presence of external magnetic field ($\ell = 0$ $J = \frac{1}{2}$) states are split lifting the Kramers degeneracy.

In the case of $\Theta^2 = I$ it can be easily shown that we can choose in this case an orthogonal basis all of whose vectors satisfy

$$\Theta|\alpha> = |\alpha> \qquad (82)$$

Such a *real* representation in which basis vectors are invariant under Θ is sometimes quite useful. One must note, however, $\Theta(i|\alpha>) = -(i|\alpha>)$. Since $i|\alpha>$ and $|\alpha>$ describe the same physical state it is not possible to introduce any physically meaningful quantum number here like in case of parity. Θ^2, of course, is invariant under multiplication of states by a phase factor and is a meaningful label for the state. We note that in a state invariant under Θ the expectation value of any time odd hermitian operator, viz, $\Theta A \Theta^{-1} = -A$, vanishes:

$$(\psi, A\psi) = -(\psi, \text{Ⓗ} A \text{Ⓗ}^{-1}\psi) = -(\text{Ⓗ}^{-1}, A\text{Ⓗ}^{-1}\psi)^{*} = -(\psi, A\psi)^{*}$$

where $\text{Ⓗ}\psi = e^{i\lambda}\psi$ or $\text{Ⓗ}^{-1}\psi = e^{i\cdot\lambda}\psi$ giving $Re(\psi, A\,\psi)^{*} = 0$. If in addition A hermitian $(\psi, A\,\psi) = 0$.

5.7 - *TIME REVERSAL OPERATOR ON HILBERT SPACE: SINCE PARTICLE STATES:*

We now consider the time reversal operation acting in the Hilbert space of state vectors. Like in the case of parity operation we have the group multiplication law

$$\text{Ⓗ}\, U(L)\, \text{Ⓗ}^{-1} = U(I_t\, L\, I_t^{-1}) \tag{83}$$

Considering a time translation, that is, $L \equiv (\mathbb{1}, a)$ with $a^{\mu} = (t, 0, 0, 0)$ we obtain $\text{Ⓗ} e^{iP^0 t}\text{Ⓗ}^{-1} = U(L(\mathbb{1}, a)) = e^{-iP^0 t}$. Let $|\chi>$ be the energy eigenstate with energy $E > 0$, then

$$e^{iP^0 t}\ \text{Ⓗ}\,|\chi> = \text{Ⓗ}\, e^{-iP^0 t}|\chi> = \text{Ⓗ}\,\bar{e}^{iEt}|\chi> = e^{\mp iEt}\ \text{Ⓗ}\,|\chi> \tag{84}$$

If we require $\text{Ⓗ}|\chi>$ to be +ve energy state vectors, Ⓗ must be an antiunitary operator. Commutation relations of Ⓗ with P^{μ} easily follow:

$$\text{Ⓗ}\ e^{iP^{\mu}a_{\mu}}\text{Ⓗ}^{-1} = U(I_t(\mathbb{1}, a)\ I_t^{-1}) = U(\mathbb{1}; -a^0, \vec{a}) \tag{85}$$

or

$$\bar{e}^{ia\cdot}\text{Ⓗ}\, P\, \text{Ⓗ}^{-1} = \bar{e}^{iP_0 a_0} + I\vec{P}\cdot\vec{a}$$

Thus

$$\text{Ⓗ}\ P^0\,\text{Ⓗ}^{-1} = P^0 \quad \text{and} \quad \text{Ⓗ}\, P^k\, \text{Ⓗ}^{-1} = -\,P^k \tag{86}$$

same as in the case of parity operation.

By adding space-inversion and time-reversal elements to the restricted group we obtain the so called *extended* or full inhomogeneous group (IHL) and by adding the representatives of P and T in   to a representation of RIHL we get a representation of the IHL. The procedure adapted for the case of space reflection.

Consider vectors corresponding to the particle rest frame. From the commutation relations it follows that we can write:

$$\textcircled{H}(|m;\tilde{p}> \otimes |\alpha>) = |m;\tilde{p}> \otimes T|\alpha> \tag{87}$$

where T is an antiunitary operator defined on h. From

$$\begin{aligned} \textcircled{H}\, U(R)\textcircled{H}^{-1}(|m;\tilde{p}> \otimes |\alpha>) &= |m;\tilde{p}> \otimes T\, D(R)\, T^{-1}|\alpha> \\ &= U(I_t R\, I_t^{-1})(|m;\tilde{p}> \otimes |\alpha>) \\ &= U(R)(|m;\tilde{p}> \otimes |\alpha>) = |m;\tilde{p}> \otimes D(R)|\alpha> \end{aligned} \tag{88}$$

We have thus,

$$T\, D(R) T^{-1} = D(R) \tag{89}$$

Since T is antilinear Schur's Lemma is not applicable. However, we showed that we can express, T as

$$T = D(\textcircled{H})K, \qquad D^{+}D = DD^{+} = I \tag{90}$$

where K is the antiunitary operator, w.r.t. a spacific basis, which replaces the components of a vector by their complex conjugates and $D(\textcircled{H})$ is a unitary linear operator. We must then specify a basis in h/w.r.t. which K and consequently T are defined. It follows

$$D(\textcircled{H})\, K\, D(R)\, K\, D^{-1}(\textcircled{H}) = D(R)$$

or

$$D^{-1}(Ⓗ)\ \ D(R)\ D(Ⓗ) = D^{*}(R) \tag{91}$$

Showing that D(R) and $D^{*}(R)$ are equivalent representations and D(Ⓗ) is the unitary matrix corresponding to time reversal operator - which transforms them into one another. The matrix D(Ⓗ), we recall, was found to be

$$D(Ⓗ) = e^{-i\pi S_2}$$

the arbitrary phase chosen so as to ensure $D(Ⓗ)\ K|jm> = (-1)^{j-m}|j,-m>$. From the commutation relations of J_k operators one in fact has

$$\mathcal{R}_2(\pi)J_{1,3}\mathcal{R}^{-1}_2(\pi) = -J_{1,3};\ \mathcal{R}_2(\pi)J_2\mathcal{R}^{-1}_2(\pi) = J_2 \tag{92}$$

In the *standard basis* representation J_1, J_2 have real matrix elements while J_2 has pure imaginary elements. If follows

$$\mathcal{R}_2(\pi)\mathcal{R}(\alpha,\beta,\gamma)\mathcal{R}^{-1}_2(\pi) = \mathcal{R}^{*}(\alpha,\beta,\gamma) \tag{93}$$

We recall also

$$d^{j}_{mm'}(\pi) = (-1)^{j-m}\,\delta_{m,-m'} \tag{94}$$

and

$$d^{j}(\pi)\ d^{j*}(\pi) = (-1)^{2j}\ \Pi$$

We also note that Ⓗ commutes with rotation operator. The relation may be derived from

$$Ⓗ\ U(R)\ Ⓗ^{-1} = U(I_t\ R\ I_t^{-1}) = U(R) \tag{95}$$

where $U(R) = e^{-i(\vec{J}\cdot\vec{n})}$. Likewise considering pure Lorentz transformation, say along 3-axis we show Ⓗ K_3Ⓗ$^{-1} = K_3$ which implies (Ⓗ being antiunitary)

$$Ⓗ\, e^{-iK_3} = e^{iK_3\varphi}Ⓗ$$

Helicity operator clearly commutes with time reversal operator $\left[Ⓗ, \frac{\vec{J}\cdot\vec{P}}{|\vec{p}|}\right] = 0$. From

$$|m,p;\alpha > \equiv U(\Lambda_p)(|m;\tilde{p} > \otimes\ |\alpha >\ ) = |m;\tilde{p}> \otimes\ |\alpha >$$

$$Ⓗ|m,p;\alpha > = Ⓗ U(\Lambda_p) Ⓗ^{-1}(|m;\tilde{p} > \otimes\ T|\alpha >) = Ⓗ\ U(\Lambda_p) Ⓗ^{-1} Ⓗ(|m,\tilde{p};\alpha>)$$

$$= |m;p^0, -\vec{p} > \otimes\ T|\alpha >$$

since $I_t\ \Lambda_p\ I_t^{-1}\ \tilde{p} = p_s$, where $p_s = (p^0, -\vec{p})$ and $Ⓗ\ U(\Lambda_p) Ⓗ^{-1} = U(\Lambda_{p_s})$ is linear operator, as expected from the commutation relations of Ⓗ with p^μ.

To be explicit we study time reversal operation on the helicity states. From the construction of the rest states and the phase convention adapted

$$Ⓗ|[m,s];\vec{0}\ \lambda > = (-1)^{s-\lambda}|[m,s];\vec{0},-\lambda> = e^{-i\pi J_2}|[m,s];\vec{0},\lambda > \tag{96}$$

For helicity states

$$U(\Lambda_p) = U(R(\phi,\Theta,-\phi))e^{iK_3\varphi}$$

and

$$Ⓗ\ U(\Lambda_p) Ⓗ^{-1} = U(R(\phi,\Theta,-\phi))e^{-i\varphi Ⓗ K_3 Ⓗ^{-1}} = U(R(\phi,\Theta,-\phi))e^{-i\varphi K_3} \tag{97}$$

therefore

$$Ⓗ|[m,s];p^0\vec{p};\lambda>= (-1)^{s-\lambda}\ U(R(\phi,\Theta,-\phi))|[m,s];p^0,-\vec{p}_R;\lambda >$$

$$= (-1)^{s-\lambda}|[m,s];\ p^0,\ -\vec{p},\ \lambda > \tag{98}$$

where we remember that the last label is the helicity label which for the rest states it is an eigenvalue of J_3 operator. We also note

$$Ⓗ\,|[m,s];\, p^0\vec{p}_R;\lambda > = (-1)^{s-\lambda}\,|[m,s];\, p^0,\, -\vec{p}_R;\lambda >$$

$$= e^{-i\pi J_2}\,|[m,s];\, p^0,\, \vec{p}_R;\lambda > \qquad (99)$$

To be precise time reversal is defined in the standard basis representation of angular momentum. The operation of complex conjugation K leaves basis states invariant and is thus suppressed. Care must be exercised when applying time reversal on a linear combination of these states. With our choice of phase conventions no extra phase factor arises for time reversal acting on $|\vec{p}_R, \lambda >$ states neither depending on p nor on λ. For the case of massless particles we may define

$$Ⓗ\,|m{=}0\ \vec{p}_R,\ \pm\,|\lambda|\ > = e^{-i\pi J_2}|m{=}0\ \ \vec{p}_R,\ \pm\,|\lambda|\ > \qquad (100)$$

Not there arises no extra phase dependence in going from $|\lambda|$ state to $-\,|\lambda|$ state can be seen as follows. The two states are connected by Y operator

$$Y|m{=}0,\, \vec{p}_R,\, |\lambda|> = \eta|m{=}0,\, \vec{p}_R,\, -\,|\lambda|\, > \qquad (101)$$

where $Y = \mathcal{P}e^{-i\pi J_2} = e^{-i\pi J_2}\mathcal{P}$, while $Ⓗ = e^{-i\pi J_2}\,K = K\,e^{-i\pi J_2}$

$$ⒽY = YⒽ \qquad \text{and} \qquad Y\,e^{-i\pi J_2} = e^{-i\pi J_2}\,Y \qquad (102)$$

Therefore

$$Ⓗ Y\,|\vec{p}_R,\,|\lambda|> = \eta\; Ⓗ\;|\vec{p}_R,-\,|\lambda|> = Y\;Ⓗ|\vec{p}_R,\,|\lambda|> = Y\,e^{-i\pi J_2}|\vec{p}_R,\,|\lambda|\,>$$

$$= e^{-i\pi J_2}Y|\vec{p}_R,\,-\,|\lambda|> = \eta\,e^{-i\pi J_2}|\vec{p}_R,\,-|\lambda|\,> \tag{103}$$

Analogous result when $\vec{p}_R \to -\vec{p}_R$ are obtained $Ⓗ|-\vec{p}_R,\lambda> =$ $= e^{-i\pi J_2}|-\vec{p}_R,\,\lambda>$ for both massive and massless case. Since space rotation commutes with time reversal we have also

$$Ⓗ|\pm\vec{p},\,\lambda> = U(R(\phi,\Theta,\,-\phi))e^{-i\pi J_2}|\pm\vec{p}_R,\lambda> \tag{104}$$

The time reversal operation on angular momentum states can be derived from

$$||\vec{p}|JM\lambda> = N_J\int D^{(J)^*}_{M\lambda}(\phi,\Theta,-\phi)|\vec{p},\lambda>d\Omega$$

$$Ⓗ\;||\vec{p}|\;JM\lambda> = N^*_J\int D^{(J)}_{M\lambda}(\phi,\Theta,\,-\phi)\;U(R(\phi,\Theta,\,-\phi))\;e^{-i\pi J_2}|\vec{p}_R,\lambda>\,d\Omega$$

$$= N^*_J(-1)^{s-\lambda}\int D^{(J)}_{M\lambda}(\phi,\Theta,-\phi)|-\vec{p},\lambda>\,d\Omega = (-1)^{s-\lambda}N^*_J\int D^{(J)}_{M\lambda}(\pi+\phi,\pi-\Theta,-\pi-\phi)|\vec{p},\lambda>d\Omega \tag{105}$$

For spinless particles $J \equiv \ell$ $M = m$ $N_J = N^*_J = \sqrt{(2\ell+1)/4\pi}$

$$||\vec{p}|\ell m> = \int|\vec{p}>Y^m_\ell(\Theta,\phi)d\Omega$$

and

$$Ⓗ||\vec{p}|\ell m> = \int Y^{m*}_\ell(\pi-\Theta,\,\pi+\phi)|\vec{p}>\,d\Omega = (-1)^{\ell-m}\int Y^{-m}_\ell(\Theta,\phi)|\vec{p}\,d\Omega$$

$$= (-1)^{\ell-m}\,||\vec{p}|\ell,\,-m> \tag{106}$$

5.8 - *TIME REVERSAL ON TWO - PARTICLE STATES*

We consider the states in c.m. frame with the particles moving along 3-axis

$$|\varepsilon,\tilde{p};00;\lambda_1\lambda_2\rangle = |\vec{k}_R\ \lambda_1\rangle \otimes |-\vec{k}_R\ \lambda_2\rangle$$

then

$$Ⓗ|\varepsilon,\tilde{p};00;\lambda_1\lambda_2\rangle = e^{-i\pi J_2^{(1)}}|\vec{k}_R\ \lambda_1\rangle \otimes e^{-i\pi J_2^{(2)}}|-\vec{k}_R\lambda_2\rangle \tag{107}$$

This is by definition same as

$$= e^{-i\pi J_2}(|\vec{k}_R\ \lambda_1\rangle \otimes |-\vec{k}_R\ \lambda_2\rangle) \tag{108}$$

where $\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$ is total angular momentum operator. Thus

$$Ⓗ|\varepsilon,\tilde{p};00;\ \lambda_1\lambda_2\rangle = K\ e^{-i\pi J_2}\ |\varepsilon,\tilde{p};00;\lambda_1\lambda_2\rangle \tag{109}$$

where we insert K to remind us of antilinear nature of Ⓗ. From the partial wave expansion

$$|\varepsilon,\tilde{p};00;\ \lambda_1\lambda_2\rangle = \sum_J N_J^*|\varepsilon,\ \tilde{p};\ J\ \lambda\ \lambda_1\lambda_2\rangle$$

we obtain

$$(K|\varepsilon,\tilde{p};JM\ \lambda_1\lambda_2\rangle = |\varepsilon,\tilde{p};\ JM;\ \lambda_1\lambda_2\rangle)$$

$$\sum_J N_J Ⓗ|\varepsilon,\tilde{p};J\ \lambda\ \lambda_1\lambda_2\rangle = \sum_J\sum_{M'} N_J(-1)^{J-\lambda}\delta_{M',-\lambda}|\varepsilon,\tilde{p};JM'\ \lambda_1\lambda_2\rangle$$

$$= \sum_J N_J(-1)^{J-\lambda}|\varepsilon,\tilde{p};\ J,-\lambda,\ \lambda_1\lambda_2\rangle$$

Since Ⓗ commutes with rotation operator (or from $[Ⓗ, J^2] = 0$) it follows therefore

$$\text{Ⓗ}|\epsilon,\tilde{p};J\ \lambda\ \lambda_1\lambda_2> = (-1)^{J-\lambda}|\epsilon,\tilde{p};J,-\lambda,\ \lambda_1\lambda_2\ > \tag{110}$$

From

$$\text{Ⓗ}J_{\pm}\text{Ⓗ}^{-1} = -\ J_{\mp}$$

$$\text{Ⓗ}J_{\pm}|\epsilon,\tilde{p};\ J\ \lambda\ \lambda_1\lambda_2> = -(J_{\mp})|\epsilon,\tilde{p};J,-\lambda,\ \lambda_1\lambda_2>(-1)^{J-\lambda}$$

or

$$\text{Ⓗ}|\epsilon,\tilde{p};J\lambda \pm 1,\ \lambda_1\lambda_2> = (-1)^{J-\lambda\mp 1}|\epsilon,\tilde{p};\ J,-\lambda \mp 1,\ \lambda_1\lambda_2\ >$$

Hence

$$\text{Ⓗ}|\epsilon,\tilde{p};\ JM;\ \lambda_1\lambda_2> = (-1)^{J-M}|\epsilon,\tilde{p};\ J,\ -M;\ \lambda_1\lambda_2\ > \tag{111}$$

as was expected.

For the vectors $|JM;\ \lambda_1\lambda_2>$ defined earlier, we find, since the coefficient is real and $|\epsilon,\tilde{p}\ >$ is unaltered ($\text{Ⓗ}\ P^0\,\text{Ⓗ}^{-1} = P^0$)

$$\text{Ⓗ}\ |JM;\ \lambda_1\lambda_2> = (-1)^{J-M}\ |J,-M;\lambda_1\lambda_2\ > \tag{112}$$

Time reversal operator can be calculated for the c.m. state with relative momentum of the particles oriented along $\vec{k}(\Theta,\phi)$ by observing that

$$|\epsilon,\tilde{p};\Theta\phi;\ \lambda_1\lambda_2> = U(R(\phi,\Theta,-\phi))\ |\epsilon,\tilde{p};00;\ \lambda_1\lambda_2\ > \tag{113}$$

and that Ⓗ commutes with rotation operator.

OTHER SYMMETRY PRINCIPLES

6.1 - *IDENTICAL PARTICLES:* * *SYMMETRIZED STATES:*

When the two particles are identical the two-particle state has to be symmetrized or antisymmetrized. We introduce the symmetry operator $\mathcal{P}_{12}$ which interchanges particles 1 and 2. Since $\mathcal{P}_{12}$ commutes with rotation operator we may only consider the state

$$|\varepsilon,\bar{p};00;\lambda_1\lambda_2> \equiv |[m,s];\vec{k}_R,\lambda_1;1> \otimes |[m,s];-\vec{k}_R,\lambda_2;2> \qquad (1)$$

where we introduce extra labels to distinguish the particles. Thus

$$\begin{aligned}\mathcal{P}_{12}|\varepsilon,\bar{p};00;\lambda_1\lambda_2> &= |\vec{k}_R,\lambda_1;2> \otimes |-\vec{k}_R,\lambda_2;1> \\ &= \{(-1)^{s-\lambda_1}\, e^{i\pi J_2^{(2)}}|-\vec{k}_R,\lambda_1;2> \otimes \\ &\quad (-1)^{s-\lambda_2}\, e^{-i\pi J_2^{(1)}}|\vec{k}_R,\lambda_2;1>\} \\ &= (-1)^{-(\lambda_1+\lambda_2)}\, e^{-i\pi J_2}\{|-\vec{k}_R,\lambda_1;2> \otimes |\vec{k}_R,\lambda_2;1>\} \qquad (2)\end{aligned}$$

where $\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$, or alternatively $((-1)^{4s} = 1)$:

$$= (-1)^{-(\lambda_1+\lambda_2)}\, e^{+i\pi J_2}\{|-\vec{k}_R,\lambda_1;2> \otimes |\vec{k}_R,\lambda_2;1>\} \qquad (3)$$

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* See for example: Quantum Mechanics vol. II, A. Messiah, Chapter XIV; Quantum Mechanics, L. Landau.

Hence ($[\lambda_1 - \lambda_2]$ is integer)

$$\mathcal{P}_{12}|\varepsilon,\tilde{p};00;\lambda_1\lambda_2 > = (-1)^{2s-\lambda_1+\lambda_2} e^{\mp i\pi J_2}|\varepsilon,\tilde{p};00;\lambda_2\lambda_1 > \quad (4)$$

which is analogous to the parity operation on two particle state. The permutation operator is represented by a unitary operator. Since $\mathcal{P}_{12}$ commutes with rotations we can derive, using partial wave expansion of the states $|\varepsilon,\tilde{p};00;\lambda_1\lambda_2 >$ the transformation on angular momentum states $\mathcal{P}_{12}|JM;\lambda_1\lambda_2 > = (-1)^{J-2s}|JM;\lambda_2\lambda_1 >$.

According to spin-statistics theorem bosons have integral spin and fermions have half integral spin. Also two particle state must be symmetric if the particles are bosons and antisymmetric if they are fermions. Thus the appropriate states are obtained by applying the operator

$$\left[I + (-1)^{2s}\,\mathcal{P}_{12}\right] \quad (5)$$

to the unsymmetrized 2-particle state.

The appropriate states for identical particles are

$$\left[|\varepsilon,\tilde{p};00;\lambda_1\lambda_2 > + (-1)^{\lambda_1-\lambda_2} e^{\pm i\pi J_2}|\varepsilon,\tilde{p};00;\lambda_2\lambda_1 >\right] \quad (6)$$

For angular momentum states they are

$$N\left[|JM;\lambda_1\lambda_2 > + (-1)^J\,|JM;\lambda_2\lambda_1 >\right] \quad (7)$$

where J is necessarily integer in the present case. We also note that *

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* $\lambda = (\lambda_1 - \lambda_2)$ is simply the J_3 value e.g. the total spin component in the C.M. frame along the 3-axis and thus the possible J values are $J = |\lambda_1 - \lambda_2|, |\lambda_1 - \lambda_2| + 1 \ldots$

$|\lambda_1 - \lambda_2| \leqslant J$ and for J odd only $\lambda_1 \neq \lambda_2$ states are possible. For $\lambda_1 \neq \lambda_2$ the normalization factor N is $\frac{1}{\sqrt{2}}$ while for $\lambda_1 = \lambda_2$ it is 1/2.

This expression should be compared with parity eigenstates for identical particles

$$\left[|JM\,\lambda_1\lambda_2> \pm (-1)^{2S}\;(-1)^J\,|JM, -\lambda_1, -\lambda_2>\right] \qquad (8)$$

For example for two photon states ($\lambda_\gamma = \pm 1$) the states $|11>$ and $|-1, -1>$ must have J = 0, 2, 4 ... while the states $|1, -1>$ and $|-1, 1>$ must carry $J \geqslant 2$. Thus a spin one particle cannot decay into two photons. The symmetrized eigenstates are given by $|11>$, $|-1, -1>$ and $\frac{1}{\sqrt{2}}\left[|1,-1> \pm |-1,1>\right]$ according as J is even or odd ($J \geqslant 2$). The parity eigenstates are (S=1), $\frac{1}{\sqrt{2}}\left[|11> \pm |-1, -1>\right]$ and $\frac{1}{\sqrt{2}}\left[|1,-1> \pm (-1)^J\,|-1,1>\right]$. Hence only states possible for two photons are $0^{\pm}$, $2^{\pm}$, $4^{\pm}$ ... of the type $\frac{1}{\sqrt{2}}\left[|11> \pm |-1, -1>\right]$ and 2^+, 3^+, 4^+... of the type $\frac{1}{\sqrt{2}}\left[1, -1> \pm |-1, 1>\right]$, upper sign corresponds to J($\geqslant 2$) even while the lower to J odd. Another example is $\pi^0\,\pi^0$ state. The symmetrized states for J odd does not exist e.g. ω^0, ρ^0 are forbidden to decay into $2\pi^0$. The J = 0 state in this case carries even parity implying that a 0^- particle cannot decay into $2\pi^0$ e.g. $\eta \not\rightarrow 2\pi^0$.

The principle of definite symmetry properties for two particle state can be generalized to include all particles belonging to the same isospin multiplet if the electromagnetic interaction is neglected. In such a case all the (2I+1) particles belonging to the same isospin I multiplet are treated identical (e.g. different charge states). The complete wave function of the particle is now the direct product of the wave function in charge space with the wave function constructed above and appropriate symmetriz state of two identical particles is obtained by applying the operator

$$\left[I + (-1)^{2s} \mathcal{P}_{12} \mathcal{P}_{12}^{I}\right] \tag{9}$$

where $\mathcal{P}_{12}^{I}$ is the permutation operator in isospin space.

From angular momentum theory we know

$$\begin{aligned}
\mathcal{P}_{12}|j_1 j_2 jm> &= \sum_{m_1} \sum_{m_2} \mathcal{P}_{12}(|j_1 m_1>|j_2 m_2>) < j_1 j_2 m_1 m_2|jm> \delta_{m,m_1+m_2} \\
&= \sum \sum j_2 m_2>|j_1 m_1> <j_1 j_2 m_1 m_2|jm> \delta_{m,m_1+m_2} \\
&= \sum \sum (|j_2 m_2>|j_1 m_1> (-1)^{j-j_1-j_2} <j_2 j_1 m_2 m_1|jm> \delta_{m,m_1+m_2} \\
&= (-1)^{j-j_1-j_2} |j_2 j_1 jm>
\end{aligned} \tag{10}$$

Hence for identical particles belonging to the same isospin multiplet I,and carrying the total isospin (I, I_3)

$$\mathcal{P}_{12}^{I} = (-1)^{I-2I_1} \tag{11}$$

and appropriatly symmetrized total wave function incorporating the spin statistics connection is

$$N\left[|JM;\lambda_1\lambda_2; I \;\; I_3 > + (-1)^{J+I-2I_1}|JM;\lambda_2\lambda_1; I \; I_3 >\right] \tag{12}$$

If $\lambda_1 = \lambda_2$ then $(J + I - 2I_1)$ must be even.

In the LS-coupling scheme

$$|JM;LS> = \sum |LM_L> |S_1 S_2\ SM_S> < LSM_L\ M_S\ |JM>$$

$$= \sum |LM_L>< LSM_L M_S|JM> (|S_1\ m_1><S_2 m_2><S_1 S_2 m_1 m_2|SM_S>$$

$$\mathcal{P}_{12}|JM;LS>_{S_1 S_2} = \sum(\mathcal{P}_{12}|LM_L>)<LSM_L M_S|JM>< S_1 S_2 m_1 m_2|SM_S> \mathcal{P}_{12}(|S_1 m_1>|S_2 m_2>)$$

$$= (-1)^L(-1)^{S-s_1-s_2}\sum|LM_L>< LSM_L M_S\ JM> |S_2 S_1\ SM_S>$$

$$= (-1)^{L+S-s_1-s_2}|JM;LS>_{S_2 S_1} \qquad (13)$$

where we used the fact that for relative orbital angular momentum the $\mathcal{P}_{12}$ operation is the same as the parity operation.

Thus for identical particles the symmetrized states in this case are

$$N\left[I + (-1)^{2s_1}\ (-1)^{L+S-2s_1}\ (-1)^{\ I-2I_1}\right]\ |JM;\ LS;\ I\ I_3>$$

$$= N\left[I + (-1)^{L+S+I-2I_1}\right]|JM;LS;I\ I_3> \qquad (14)$$

that is, $(L+S+I-2I_1)$ must be even.

Note that we couple in this scheme the spins of particles which are in movement in c.m. frame. However, the nonrelativistic treatment used can be given relativistic justification. *

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* See, for example, Mc Kerrell, Il Nuovo Cimento, 34, 1289 (1964).

6.2 - *PARTICLE-ANTIPARTICLE CONJUGATION: (CHARGE CONJUGATION)*

Charge conjugation operation is defined as the mapping of the physical system into another physical system in which each particle has been replaced by one with opposite value of internal quantum numbers like of charge, baryon number strangeness and lepton number. Thus C transforms a particle into its anti-particle leaving unaltered, however, the space time description of the system.

$$C|[m,s]; \vec{p}, \lambda; \alpha> = \eta_\alpha |[m,s]; \vec{p}, \lambda; \bar{\alpha}>$$

where α refer to the quantum number Q, B, S etc. The anti-particle label $\bar{\alpha}$ contains these labels with their signs reversed. Clearly

$$C^2 = \varepsilon I \quad .$$

Requiring that $C|\chi>$ be also a positive energy state if $|\chi>$ is so, that is,

$$e^{i P^0 t} (C|\chi>) = e^{iEt}(C|\chi>) = C e^{i P^0 t} |\chi>$$

We conclude from $C^{-1}(i P^0)C = i P^0$ and $C^{-1} P^0 C = P^0$ that C must be a unitary operator. Than we can redefine h the phase convention to ensure

$$C^2 = I$$

and $C^\dagger = C^{-1} = C$. Also $\eta_\alpha = \eta^*_{\bar{\alpha}}$ with $|\eta_\alpha| = 1$.

The quantum numbers Q and B are always conserved e.g. satisfy super-selection rules meaning a state can never be a superposition of states with different values of Q and B. If Q = B = 0 we can construct eigenstates of C,

$$C\{|\ldots\alpha> \pm \eta_\alpha|\ldots\bar{\alpha}>\} = \pm\{|\ldots\alpha> \pm \eta_\alpha|\ldots\bar{\alpha}>\}$$

For the case in which particle state is identical with its antiparticle state (e.g. Q=0 B=0 S=0, etc.) we can define charge parity $\eta = \pm 1$ for the particle. The one-particle states of this type are η^0, ρ^0, π^0, ω^0, ϕ^0, γ etc. The action of C on many particle states is obtained easily if we regard the state as direct product of single particle states and C is applied separately to each single particle state. For a particle-antiparticle pair specially we note for angular-momentum states, for example,

$$C\,|JM;\,\lambda_1\lambda_2\,;\,\gamma> = |JM;\,\lambda_1\lambda_2;\,\bar{\gamma}>$$

the phase factor on the R.H.S. is one since the product of single particle and its antiparticle phase factors was seen above to be one. Regarding the particle and antiparticle as identical particles differing only in their internal charge labels the connection between spin and statistics may be applied e.g. the appropriately symmetrized states for particle-antiparticle pair are given by applying the operator $[I + (-1)^{2S}\,C\,\mathcal{P}_{12}]$ to unsymmetrized state where $\mathcal{P}_{12}$ is the permutation operator which interchanges the space and spin labels of the particle. Thus the appropriate states are

$$\left[|JM;\lambda_1\lambda_2;\gamma> + (-1)^J\;C|JM;\lambda_2\lambda_1;\gamma>\right] \equiv |JM;\lambda_1\lambda_2\;;\gamma>>$$

Since

$$C\,\mathcal{P}_{12}|JM;\lambda_1\lambda_2;\gamma>> = (-1)^{2S}\;|JM;\lambda_1\lambda_2;\gamma>>$$

and

$$\mathcal{P}_{12}|JM;\lambda_1\lambda_2;\gamma>> = (-1)^{J-2S}\;|JM;\lambda_2\lambda_1;\gamma>>$$

we have

$$C|JM;\lambda_1\lambda_2;\gamma>> = (-1)^J\;|JM;\lambda_2\lambda_1;\gamma>>$$

The symmetrized angular momentum states of particle-antiparticle are thus eigenstates of C only if $\lambda_1 = \lambda_2$. When $\lambda_1 \neq \lambda_2$ the states

$$\left[|JM;\lambda_1\lambda_2;\gamma >> \pm |JM;\lambda_2\lambda_1;\gamma >>\right]$$

are eigenstates of C with eigenvalue $\pm (-1)^J$.

For states in LS coupling scheme the symmetrized states are also eigen states of C operator,

$$\left[|JM;LS;\gamma > + (-1)^{L+S} C|JM;LS;\gamma >\right] \equiv |JM;LS;\gamma >>$$

$$C|JM;LS;\gamma >> = (-1)^{L+S} |JM;LS;\gamma >>$$

for particle-antiparticle pair.

Eigenstates of C are possible only when total internal charges in the system are zero, that is, B=0 S=0 and Q=0. Charge conjugation symmetry (C-invariance) invariance thus gives selection rules for transitions between such states. For strong interactions, however C-invariance can be combined with isospin invariance to extend the selection rules for Q≠0 cases if charge independence is assumed to hold *.

* See for example: Carruthurus, Unitary Symmetry, (Benjamin N.Y.).

7

S-OPERATOR AND S-MATRIX OF "IN" AND "OUT" STATES:

7.1 - *S-MATRIX FOR STRONG INTERACTIONS:*

Any scattering process due to strong interactions may be described in terms of initial and final states of non-interacting particles. This is possible due to short range of these interactions.

We assume the existence of a Hilbert space of physical states (Heisenberg states) which is spanned by both a complete set of: orthonormal Heisenberg "in" states $\{|\psi_a; \text{in}>\}$ and by a complete set of orthonormal "out" states $\{|\psi_a; \text{out}>\}$. Both "in" and "out" states contain vacuum, one particle states, two particle states (non-interacting) etc. We assume also $|0; \text{in}> = |0; \text{out}>$ and $|1, \text{in}> = |1; \text{out}>$ for the vacuum and one particle states. Since we are in Heisenberg picture the states are fixed and the operators carry the time dependence. The "in" and "out" states are not states which go over one into another as time changes from $t = -\infty$ to $+\infty$, rather the system is described once and for ever either by the "in" states or by the "out" states. These are the eigenstates of the assymptotic limits of changing operators. Since these operators are different for $t = = +\infty$ and $t = -\infty$ the corresponding "out" and "in" states differ also.

The S-matrix describes de entity of all possible results of measurements at $t = +\infty$ when the state at $t = -\infty$ has been given. According to rules of quantum mechanics we expand the given state $|\psi_a; in>$ w.r.t. the eigenstates of the operators which belong to the next measurement viz, the "out" operators. Then

$$|\psi_a; in> = \sum_b S_{ba}|\psi_b; out> \tag{1}$$

Then $|S_{ba}|^2$ gives the probability to find the system in $|\psi_b$, out > state when a measurement "A_{out}" is performed and the system was in state $|\psi_a$, in > before the measurement. S_{ba} depends on the interaction that took place. Clearly

$$S_{ba} = <\psi_b; out|\psi_a; in> \tag{2}$$

Also $\sum_b |S_{ba}|^2 = <\psi_a; in|\psi_a; in> = 1$ and $\sum_b S_{ba} S^*_{bc} = \sum_b S_{ab} S^*_{cb} = \delta_{ac}$

We may thus define a S-matrix operator S as a linear unitary operator which connects the complete sets of "in" and "out" states

$$|\psi_a; in> = S|\psi_a; out>, \qquad |\psi_a; out> = S^+|\psi_a; in> \tag{3}$$

and

$$A_{in} = S\, A_{out}\, S^{-1}, \qquad SS^\dagger = S^\dagger S = I \tag{4}$$

It follows:

$$\begin{aligned} S_{ba} &= <\psi_b; out\ |S|\psi_a; out> \\ &= <\psi_b; in\ |S|\ \psi_a; in> \end{aligned} \tag{5}$$

S is clearly given by $S = \sum_a |\psi_a; in><\psi_a; out|$. If there is no interaction $|\psi_a; out> = |\psi_a; in>$ and

$$S_{ba} = < \psi_b; in|\psi_a; in > \tag{6}$$

In the presence of interaction transition to final states can take place in two distinct ways: without having taken place any interaction or after having taken place actual interaction. It is thus suggested to define a non-unitary transition matrix T_{ba} by

$$S_{ba} = <\psi_b; in|\psi_a; in > + i\ T_{ba} \tag{7}$$

and introduce a T-operator

$$S = I + iT \tag{8}$$

$$T_{ba} = <\psi_b;out|T|\psi_a;out > = <\psi_b;in|T|\psi_a; in > \tag{9}$$

Unitary of S implies

$$TT^{\dagger} = T^{\dagger}T = i(T^{\dagger}-T) \tag{10}$$

Transition probability is then given by

$$|S_{ba} - <\psi_b; in|\psi_a; in >|^2 = |T_{ba}|^2 \tag{11}$$

We note also

$$|\psi_a; out > = |\psi_a; in> - i\ T^{\dagger}|\psi_a; in > \tag{12}$$

The invariance of S under a invariance transformation affected by an operator O is, as we discussed earlier, is stated by

$$O\ S\ O^{-1} = S \qquad \text{for unitary transformation}$$

and by

$$O\ S\ O^{-1} = S^{\dagger} \qquad \text{for anti-unitary transformation}$$

This corresponds to the commutativity with O of the corresponding hermitian generator of the unitarity operator S, viz, if $S = e^{i B\alpha}$, $B^+ = B$ and α real then

$$[O,B] = 0 \tag{13}$$

We will assume in the spirit of the principle of relativity that the S-matrix is Lorentz invariant (or rather invariant under RIHL). The matrix elements S_{ba} regarded as functions of the states $|\psi_a>$ and $|\psi_b>$ is thus a scalar under Lorentz transformations *. We showed that a unitary (infinite dimensional) representation of the Lorentz group is defined on the Hilbert space $\mathcal{H}$ to which the states belong; the unitary operator associated with transformation (Λ,a) is $U(\Lambda,a)$ defined on $\mathcal{H}$. Then (suppressing the in ("out") labels)

$$<\psi_b|S|\psi_a> = <\psi_b|U^\dagger S\, U|\psi_a> = (U\psi_b,\, S\, U\ \psi_a) \tag{14}$$

Lorentz invariance allows us to calculate the S matrix elements in any convinient frame of reference connected to the laboratory frame by a Lorentz transformation. We will frequently use the c.m. frame. Translation and rotation invariance in particular imply that the energy-momentum 4-vector and the total angular momentum is conserved during the transition.

From ** $U^\dagger S\, U = S$ it follows

$$[P^\mu, S] = 0 \quad \text{and} \quad [\vec{J}, S] = 0 \tag{15}$$

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* This clearly implies that the states, say $|\psi_a;\,\text{in}>$ are normalized in invariant fashion since S_{ba} reduces to $<\psi_b;\,\text{in}|\psi_a;\text{in}>$ in the absence of interaction.

** $U(\Pi, a) = e^{iP\cdot a}$ and $U(R,0) = e^{-i\vec{J}\cdot\vec{n}\Phi}$.

Thus if the initial and final states are eigenstates of 4-momentum we have

$$<\psi_b|P^\mu S - S P^\mu|\psi_a> = 0 \tag{16}$$

or

$$(p_a^\mu - p_b^\mu) < \psi_b|S|\psi_a> = 0$$

leading to

$$<\psi_b|S|\psi_a> \propto \delta^4(p_b^\mu - p_a^\mu) < \psi_b|S(p_a)|\psi_a> \tag{17}$$

Thus the invariance of S-matrix w.r.t. space-time translations allows us to write

$$S_{ba} = (2\pi)^4 \, \delta^4(p_b^\mu - p_a^\mu) \, S_{ba}(p_a) \tag{18}$$

where $S_{ba}(p_a)$ are the matrix elements of S operator on the energy-momentum surface. We may write

$$S_{ba}(p_a) \equiv <\alpha_b|S(p_a)|\alpha_a> \tag{19}$$

Also

$$<\psi_b|\psi_a> = (2\pi)^4 \, \delta^4(p_b - p_a) < \alpha_b|\alpha_a>$$

Here α are labels other than total four momentum. In fact we showed earlier that the states in $\mathcal{H}$ are spanned by the states of type $|p> \otimes |\alpha>$ where $<p'|p> = (2\pi)^4 \delta^4(p-p')$. Thus

$$i\, T_{ba} = (2\pi)^4 \delta^4(p_b^\mu - p_a^\mu) \left[S_{ba}(p_a) - <\alpha_b|\alpha_a>\right] \tag{20}$$

We may define transition matrix elements on energy-momentum surface by

$$T_{ba} = (2\pi)^4 \, \delta^4(p_b^\mu - p_a^\mu) \, T_{ba}(p_a) \tag{21}$$

where

$$i\, T_{ba}(p_a) = S_{ba}(p_a) - \langle \alpha_b | \alpha_a \rangle \tag{22}$$

In more formal way we may write

$$S = I \otimes S(P) \tag{23}$$

where

$$S_{ba}(p_a) = \langle \psi_b | S(p) | \psi_a \rangle \equiv \langle \alpha_b | S(p_a) | \alpha_a \rangle \tag{24}$$

and $|\psi\rangle \equiv |p\rangle \otimes |\alpha\rangle$. Likewise we may write

$$T = I \otimes T(P) \tag{25}$$

where

$$S(P) = I + i\, T(P) \tag{26}$$

giving

$$\langle \alpha_b | S(p_a) | \alpha_a \rangle = \langle \alpha_b | \alpha_a \rangle + i \langle \alpha_b | T(p_a) | \alpha_a \rangle \tag{27}$$

We note that we have adopted covariant normalization for the states so that $\langle \alpha_b | T(p_a) | \alpha_a \rangle$ is also Lorentz invariant.

The completeness relation corresponding to the covariant normalization adopted (suppressing the factors η_I):

$$\langle p_1', p_2', \ldots p_n'; \lambda | p_1, p_2, p_3 \ldots p_n; \lambda \rangle = (2\pi)^{3n} (2p_1^0 \ldots 2p_n^0) \delta^3(\vec{p}_1 - \vec{p}_1') \ldots \delta^3(\vec{p}_n - \vec{p}_n') \tag{28}$$

is given by (say for out -states)

$$I = |0;\text{out}\rangle\langle 0;\text{out}| + \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3\, 2\, p^0} |\vec{p};\lambda\ \text{out}\rangle \langle \vec{p};\lambda\ \text{out}| +$$

$$\sum_{\lambda_1} \sum_{\lambda_2} \iint \frac{d^3p_1}{(2\pi)^3\, 2p_1^0} \frac{d^3p_2}{(2\pi)^3\, 2p_2^0} |\vec{p}_1\lambda_1 ; \vec{p}_2\lambda_2; \text{out}\rangle \langle \vec{p}_1\lambda_1; \vec{p}_2\lambda_2; \text{out}| + \ldots \tag{29}$$

Therefore

$|\psi_a;in>= S|\psi_a;out>$

$$= |0;out><0;out|S|\psi_a;out>+ \sum_\lambda \int \frac{d^3p}{(2\pi)^3\ 2p^0}\ |\vec{p},\lambda;out><\vec{p},\lambda;out|S|\psi_a;out>\ldots \quad (30)$$

It is clear that we can interpret, (in our normalization), $\frac{d^3\vec{p}}{(2\pi)^3\ 2p^0}$, since the completeness relation involves summation over all the states, as the number of momentum states $\{|\vec{p}>\}$ available with momentum $\vec{p}$ centered around $\vec{p}$, in the range $\vec{p} \leqslant \vec{p} \leqslant \vec{p} + d^3p$. In other wards the density of states in momentum space is $\frac{1}{(2\pi)^3\ 2p^0}$. Thus, for example, the probability for finding , in the final state, after interaction has taken place, two particles 1 and 2 with helicities λ_1 and λ_2 and with final particle momentum in the range $\vec{p}_i \leqslant \vec{p}_i \leqslant \vec{p}_i + d^3p_i$. is proportional to *

$$|<\vec{p}_1\lambda_1;\ \vec{p}_2\ \lambda_2;\ out\ |S|\ \psi_a;\ out>|^2\ \frac{d^3p}{(2\pi)^3\ 2p_1^0}\ \ \frac{d^3p}{(2\pi)^3\ 2p_2^0}$$

$$= |<\vec{p}_1\lambda_1;\ \vec{p}_2\lambda_2;\ in\ |S|\ \ \psi_a;\ in>|^2 \frac{d^3p_1}{(2\pi)^3\ 2p_1^0}\ \ \frac{d^3p_2}{(2\pi)^3\ 2p_2^0} \quad (31)$$

The probability amplitude ,w.r.t. measure d^3p in momentum space, is proportional to:

$$\frac{1}{(2\pi)^{3/2}\ \sqrt{2p_1^0}}\ \ \frac{1}{(2\pi)^{3/2}\ \sqrt{2p_2^0}} < \vec{p}_1\lambda_1\ ;\ \vec{p}_2\lambda_2;\ in|S|\psi_a;\ in> \quad (32)$$

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* If $\{|i>\}$ is a complete set of states normalized as $<i|j> = N_i\ \delta_{ij}$ the resolution of identity reads $\sum|i> \frac{1}{N_i} <i| = I$ as can be cheked by applying it to a state $|j>$. The expansion of a state reads $|\psi> = \sum|i> \frac{1}{N_i} <i|\psi>$ and $<\psi|\psi> = \sum \frac{1}{N_i}\ |<i|\psi>|^2$. Thus the probability amplitude corresponding to state $|i>$ is $<i|\psi>\ /\ \sqrt{N_i\ <\psi|\psi>}$.

which is clearly not Lorentz invariant since it has been defined w.r.t. to the non-covariant measure d^3p for the integral in momentum space. This non-invariant probability amplitude multiplied by the factor $\sqrt{2p_1^0 \cdot 2p_2^0}$ gives the invariant probability amplitude $< \vec{p}_1\lambda_1 ; \vec{p}_2\lambda_2 | S | \psi_a >$ defined w.r.t. the covariant measure $\frac{d^3p}{(2\pi)^3\, 2p^0}$ in the momentum space. This observation was first made by Møller in 1949.

We note also that for the cases $|\psi_a> \neq |\psi_b>$ we have $S_{ba} = i\, T_{ba}$ and probability $|S_{ba}|^2 = |T_{ba}|^2$, the transition probability. In the case the initial and final state may be coincident it can be shown * say, by constructing wave packets for the initial and final states that the transition probability is still given by the amplitude $|T_{ba}|^2$. This is otherwise clear since it is T_{ba} which determines the transition amplitude.

7.2 - *CROSS-SECTIONS AND DECAY RATES:*

The transition probability can be connected to the experimentally measured quantities, for example, the decay rate of a particle or scattering cross section of two incident particles.

It is convenient to go over to box normalization **. The spatial part of the wave function of a plane wave state of momentum $\vec{p}$ is $e^{i\vec{p}\cdot\vec{x}}$ satisfying

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* See for example: Goldberger and Watson, Collision Theory or K. Gottfried Quantum Mechanics, Vol. I or S. Davydov, Quantum Mechanics, Chap. XI.

** See for example, Martin and Spearman, Elementary Particle Theory, North Holland.

$$\int d^3x (e^{i\vec{p}\cdot\vec{x}})^* \ (e^{i\vec{p}'\cdot\vec{x}}) = (2\pi)^3\delta^3(\vec{p}-\vec{p}') \tag{33}$$

as the scalar product of two plane wave states. The delta function indicates that there is a continuum of values of $\vec{p}$ due to integration over an infinite range. If, however, we restrict ourselves to a finite box in space such that $-L \leqslant x_i \leqslant L$ and impose further on the momentum eigenstates ($\vec{p}_{op}\, e^{i\vec{p}\cdot\vec{x}} = \vec{p}\, e^{i\vec{p}\cdot\vec{x}}$) the periodic boundary conditions ($i = 1, 2, 3$), $\phi_{\vec{p}}(\vec{x}) \sim e^{i\vec{p}\cdot\vec{x}}$,

$$\phi_{\vec{p}}(x_i = -L) = \phi_{\vec{p}}(x_i = L) \tag{34}$$

the momentum spectrum becomes discrete e.g.

$$p_i = n_i\left(\frac{\pi}{L}\right) \qquad n_i = 0, \pm 1, \pm 2, \ldots \tag{35}$$

The allowed momentum eigenvalues form a three-dimensional lattice of points $\vec{p}^{(n)}$ with spacing $\frac{\pi}{L}$. The corresponding wave functions $\phi_{\vec{p}(n)}(\vec{x})$ from a complete set of periodic functions in the box with the finite normalization

$$\int_V (e^{i\,\vec{p}^{(m)}\cdot x})^* \ (e^{i\,\vec{p}^{(n)}\cdot x})\ d^3x = V\,\delta_{\vec{p}^{(m)},\,\vec{p}^{(n)}} \tag{36}$$

where V is the volume ($V = L^3$) of the box and $\delta_{\vec{p}^{(m)},\,\vec{p}^{(n)}}$ is Kronecker delta. Thus $\frac{1}{\sqrt{V}}\ e^{i\vec{p}^{(n)}\cdot\vec{x}}$ are orthonormalized wave functions. The rule of going from box normalization to infinite volume normalization is clearly

$$V\,\delta_{\vec{p}^{(n)},\,\vec{p}^{(n')}} \longrightarrow (2\pi)^3\,\delta^3(\vec{p}-\vec{p}\,') \tag{37}$$

Since

$$\sum_{\vec{p}^{(n')}} \delta_{\vec{p}^{(n)},\,\vec{p}^{(n')}}\, f(\vec{p}^{(n')}) = f(\vec{p}^{(n)}) \tag{38}$$

and

$$\int \delta(\vec{p}-\vec{p}\,')\, f(\vec{p}\,')\, d^3p' = f(\vec{p}) \tag{39}$$

we conclude the rule ($\vec{p}^{(n)} \longrightarrow \vec{p}$):

$$\frac{1}{V} \sum_{\vec{p}^{(n)}} \longrightarrow \frac{1}{(2\pi)^3} \int d^3p \tag{40}$$

The normalized momentum eigenstates

$$\phi_{\vec{p}^{(n)}}(\vec{x}) \equiv \langle\vec{x}|\vec{p}^{(n)}\rangle = \frac{1}{\sqrt{V}}\, e^{i\,\vec{p}^{(n)}\cdot\vec{x}} \tag{41}$$

correspond to probability density $\frac{1}{V}$ at any point in the box, while the total probability density integrated over, the box volume V is one. Since

$$\langle\vec{p}^{(n)}|p^{(n')}\rangle = \delta_{\vec{p}^{(n)},\,\vec{p}^{(n')}} \tag{42}$$

and

$$\langle\vec{p}|\vec{p}\,'\rangle = (2\pi)^3\, 2\, p^0\, \delta^3(\vec{p}-\vec{p}\,') \tag{43}$$

we have the rule

$$\sqrt{V}\,|p^{(n)}\rangle \longrightarrow \frac{1}{\sqrt{2p^0}}\,|\vec{p}\rangle \tag{44}$$

For example, it implies:

$$\langle\vec{x}|\vec{p}\rangle = \sqrt{2\,p^0}\; e^{i\vec{p}\cdot\vec{x}} \tag{45}$$

which is clearly true. Also

$$\int \frac{d^3p}{(2\pi)^3\ 2p^0}\,|\vec{p}\rangle\langle\vec{p}| \longrightarrow \sum_{\vec{p}^{(n)}} |\vec{p}^{(n)}\rangle\langle\vec{p}^{(n)}| \tag{46}$$

so that the completeness relation reads

$$I = |0,\text{out}\rangle\langle 0,\text{out}| + \sum_{\lambda}\sum_{\vec{p}(n)} |\vec{p}^{(n)},\lambda;\text{out}\rangle\langle\vec{p}^{(n)},\lambda;\text{out}| + \sum_{\lambda_1}\sum_{\lambda_2}\sum_{\vec{p}^{(n)}}\sum_{\vec{p}^{(m)}} |\vec{p}_1^{(n)}\ \lambda_1;\vec{p}_2^{(m)}\ \lambda_2\ \text{out}\rangle\langle\vec{p}_1^{(n)}\lambda_1\ ;\vec{p}_2^{(m)}\ \lambda_2\text{out}| + \ldots \tag{47}$$

and

$$|\psi_a,\text{in}\rangle = S|\psi_a;\text{out}\rangle = |0,\text{out}\rangle\langle 0,\text{out}|S|\psi_a;\text{out}\rangle + \sum|\vec{p}^{(n)},\lambda;\text{out}\rangle\langle\vec{p}^{(n)},\lambda;\text{out}|+|S|\psi_a;\text{out}\rangle+\sum\sum|\vec{p}_1^{(n)}\lambda_1;\vec{p}_2^{(m)}\lambda_2\text{out}\rangle \langle\vec{p}_1^{(n)}\lambda_1;\vec{p}_2^{(m)}\ \lambda_2\ \text{out}|S|\psi_a;\text{out}\rangle + \ldots \tag{48}$$

Thus the transition probability amplitude from a state of two particles to a state of n particles is

$$i\,\langle\vec{p}^{(n_1)},\vec{p}^{(n_2)},\ldots\vec{p}^{(n_n)};\lambda\,|T|\,\vec{p}^{(n_p)},\vec{p}^{(n_q)};\overline{\lambda}\rangle \tag{49}$$

where we used $\langle\vec{p}^{(n_p)},\vec{p}^{(n_q)};\overline{\lambda}|\vec{p}^{(n_p)},\vec{p}^{(n_q)};\overline{\lambda}\rangle = 1$ and where λ, $\overline{\lambda}$ denote collectively quantum numbers other than momentum in the final and the initial state respectively. We may use the four momentum conservation to take out the Kronecker delta and write equation (7.49) as

$$i\,\langle\alpha_f^{\text{box}}|T(p_i)|\alpha_i^{\text{box}}\rangle\ \ \delta_{\vec{p}_i,\vec{p}_f}\ \delta_{p_i^0,p_f^0} \tag{50}$$

where p_f^μ and p_i^μ are total 4-momentum vectors in the final and the initial state.

We may translate it into continuum formulation of ordinary momentum space (with infinite or continuum normalization) by the rules given above. It is, say, in the example above,

$$i \frac{1}{\sqrt{2p_1^0 V}} \cdots \frac{1}{\sqrt{2\ p_n^0\ V}} < \vec{p}_1\ \vec{p}_2 \cdots \vec{p}_n;\ \lambda|T|p,q;\bar{\lambda} > \frac{1}{\sqrt{2p^0 V}\sqrt{2q^0 V}} \qquad (51)$$

We may remove the 4-momentum delta function to obtain

$$i(2\pi)^4\ \delta^4(p_f - p_i) < \alpha_f|T(p_i)|\ \alpha_i > \frac{1}{\sqrt{2p^0 V \cdot 2q^0 V}} \prod_{k=1}^{n} \frac{1}{\sqrt{2p_k^0\ V}} \qquad (52)$$

Thus the rule of translation for transition matrix elements is given by

$$< \alpha_f^{\text{box}}\ |T(p_i)|\alpha_i^{\text{box}} > \longrightarrow V\ t_0\ < \alpha_f|T(p_i)|\ \alpha_i > \frac{1}{\sqrt{2p^0 V\ \ 2q^0 V}} \prod_{k=1}^{n} \frac{1}{\sqrt{2p_k^0 V}}$$

(53)

where we have used the fact that * when we use finite box in space we should take a finite time interval t_0 and

$$t_0\ \delta_{p^0(n),\ p^0(n')} \longrightarrow (2\pi)\ \delta(p^0 - p'^0) \qquad (54)$$

so that

$$V\ t_0\ \delta_{\vec{p}(n),\ \vec{p}(n')}\ \delta_{p^0(n),\ p^0(n')} \longrightarrow (2\pi)^4 \delta^4(p^\mu - p'^\mu) \qquad (55)$$

The transition probability is

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* Note that with box normalization the energy spectrum is also discrete.

$$\sum_{\vec{p}(n_1)} \cdots \sum_{\vec{p}(n_n)} |<\alpha_f^{box}|T(p_i)|\alpha_i^{box}>|^2 \, \delta_{\vec{p}_f,\vec{p}_i} \, \delta_{p_f^0,\, p_i^0} \tag{56}$$

where we used $(\delta_{mn})^2 = \delta_{mn}$. Translated into continum language (when V is very large) the transition probability density reads:

$$\sum_{\vec{p}^{(n_1)}} \cdots \sum_{\vec{p}^{(n_n)}} (V\ t_0)^2 |<\alpha_f|T(p_i)|\alpha_i\ >|^2 \left(\prod_{k=1}^{n} \frac{1}{2p_k^0 V}\right) \frac{1}{2p^0 V \cdot 2q^0 V} \cdot \frac{(2\pi)^4 \delta^4(p_f^\mu - p_i^\mu)}{V t_0}$$

$$= \sum \ldots \sum (2\pi)^4 V t_0 \delta^4(p_f - p_i) |<\alpha_f|T(p_i)|\alpha_i>|^2 \left(\prod_{k=1}^{n} \frac{1}{2p_k^0 V}\right) \frac{1}{2p^0 V \cdot 2q^0 V} \tag{57}$$

Consider, for example, the *decay probability density* of a one particle initial state into n final particles. It is then given by

$$\sum \ldots \sum (2\pi)^4 \, \delta^4(p_f - p_i) V\ t_0 |<\alpha_f|T(p_i)|\alpha_i>|^2 \left(\prod_{k=1}^{n} \frac{1}{2p_k^0 V}\right) \frac{1}{2p^0 V} \tag{58}$$

From the rule, for V very large, we can replace

$$\sum_{\vec{p}(n)} \longrightarrow \frac{V}{(2\pi)^3} \int d^3p \tag{59}$$

We conclude that the transition probability for a one particle state with momentum p^μ going over to n particles state with 3-momenta lying inside the range $\vec{p}_i < \vec{p}_i < \vec{p}_i + d^3p_i$, i = 1, 2, .. n is given by

$$\frac{1}{2p^0} (2\pi)^4 \, \delta^4(p_f - p_i) \cdot t_0 \cdot |<\alpha_f|T(p_i)|\alpha_i>|^2 \left(\prod_{k=1}^{n} \frac{d^3p_k}{(2\pi)^3 \, 2\, p_k^0}\right) \tag{60}$$

and is proportional to time interval 't_0'. The transition probability per unit time called decay rate is then

$$d\lambda = \frac{(2\pi)^4\delta^4(p_f^\mu - p_i^\mu)}{2\ p^0} \cdot |\ \langle\alpha_f|T(p_i)|\alpha_i\rangle|^2 \cdot \prod_{k=1}^{n} \frac{d^3p_k}{2p_k^0\ (2\pi)^3} \qquad (61)$$

The total decay rate or inverse mean life time is obtained by integrating and taking appropriate summation over other variables (like, spin, isospin, etc.) over <u>all</u> final states *. The total decay rate is clearly the sum of partial decay rates.

We remaind ourselves that the probability density $|\psi(\vec{x},t)|^2$ corresponding to a wave function $\psi(\vec{x},t)$ can also be given the interpretation of the (average) particle density at position $\vec{x}$ and at time t. In fact if we have $\int_V |\psi(\vec{x},t)|^2 d^3x = 1$ we may imagine an ensemble of n identical non-interacting particles inside the volume V described by the common wave function $\psi(\vec{x},t)$. The relation $\int_V n|\psi|^2\ d^3x = n$ allows us to interpret $|\psi|^2$ as the average particle density at position $\vec{x}$ at time t. Interpreting one particle in the box the particle density corresponding to wave function $\frac{1}{\sqrt{V}}\ e^{i\vec{p}^{(n)}\cdot\vec{x}}$ is $\frac{1}{V}$. The decay probability density per unit time above thus also can be interpreted as the decay probability per unit time of one initial particle into n final particles. The expression is clearly independent of the volume of the box.

We next discuss the case of *two particle initial state*. (With the (box) normalization condition we can interpret it as a state representing two non

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* $d\lambda$ may also be interpreted, as is clear from the expression, as the probability density per unit space-time volume divided by the density of initial particle ($\frac{1}{V}$).

interacting particles in volume V). The transition probability density is

$$(2\pi)^4\ \delta^4(p_f-p_i)\ V\ t_0 |<\alpha_f|T(p_i)|\alpha_i>|^2 \left(\prod_{k=1}^{n} \frac{d^3p_k}{(2\pi)^3\ 2p_k^0} \right) \frac{1}{2p^0V\ 2q^0V} \quad (62)$$

The probability density per unit time is then

$$(2\pi)^4\ \delta^4(p_f-p_i) |<\alpha_f|T(p_i)|\alpha_i>|^2 \left(\prod_{k=1}^{n} \frac{d^3p_k}{(2\pi)^3\ 2p_k^0} \right) \frac{1}{2p^0\cdot 2q^0V} \quad (63)$$

Experimentally *, the transition of two particles (states) to n particles (state) is characterized by a"scattering cross section" with the dimensions of an area and which is independent of the details of the source of incident particles (as well as that of the aperture of the detector). We define the theoretical scattering cross section (which coincides with experimental definition) as the transition probability density per unit time and per unit incident flux density, say, in the laboratory frame where the other particle is at rest. For example if $\vec{q} = 0$ the current probability density e.g. the flux density is $v \frac{1}{V}$ where $v = \frac{|\vec{p}|}{p^0}$ is the relative velocity and $\frac{1}{V}$ is the probability density of incident particle. The theoretical cross section for the scattering of two particles is thus (in the lab. frame)

$$\sigma \equiv \sum \int d\sigma \equiv \sum \int \frac{1}{(4\ p^0q^0v)_{lab}} \left\{ (2\pi)^4\ \delta^4(p_f-p_i) |<\alpha_f|T(p_i)|\alpha_i>|^2 \cdot \prod_{k=1}^{n} \frac{d^3\ p_k}{(2\pi)^3\ 2\ p_k^0} \right\} \quad (64)$$

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* See for example, The Quantum Theory of Scattering, L. S. Rodberg and R. H. Thaler, Academic Press (1967), p. 20.

It is easy to check that $d\sigma$ has the dimensions of area (perpendicular to the direction of the collision axis). In fact with the covariant normalization adopted by us the states $|\vec{p}\lambda>$ carry a dimension $L(\hbar = c = 1)$. The dimension of $< p_1 .. p_n;\lambda| \, p,q;\bar{\lambda} >$ is then $L^{(n+2)}$ implying a dimension L^{n-2} for $< \alpha_f|T(p_i)|\alpha_i>$. The quantity inside curly brackets is thus dimensionless (as well as Lorentz invariant) and $d\sigma$ is seen to have dimensions of an area. The expression for $d\sigma$ is clearly independent of the box volume. Interpreting the (box) normalization as representing one particle in volume V, as discussed above, we recognize the flux density as the incident particle flux and the transition probability density per unit time as the probability per unit time of one incident particle interacting with one target particle producing n final particles.

From the expression above we may also interpret differential cross section as the probability per unit space-time volume divided by the flux of incident particle and the density of target particles. A similar interpretation for the decay rate may also be given.

The differential cross section for the scattering of two particles is thus given by (including $\eta|$'s)

$$d\sigma = \frac{(2\pi)^4 |<\alpha_f|T(p_i)|\alpha_i>|^2}{4F} \; (\eta|_p \, \eta|_q \, \eta|_{p_1} \cdots \eta|_{p_n}) \; dQ \qquad (65)$$

where $\eta| = 1$ for bosons and $= 2m$ for fermions corresponding to the normalization $<\vec{p}\lambda|p'\lambda'> = (2\pi)^3 \frac{2\,p^0}{\eta|} \, \delta^3(\vec{p}-\vec{p}\,')\delta_{\lambda\lambda'}$, and dQ is the covariant phase space factor *

$$dQ = \delta^4(p_f - p_i) \prod_{k=1}^{n} \frac{d^3 p_k}{(2\pi)^3 2 p_k^0}$$

$$= \delta^4(p_f - p_i) \prod_{k=1}^{n} \int_{(p_k^0)} \frac{d^4 p_k}{(2\pi)^3} \Theta(p_k^0)\, \delta(p_k^2 - m_k^2) \qquad (66)$$

The factor $F = |\vec{v}_p - \vec{v}_q| p^0 q^0$ can be expressed as

$$F = p^0 q^0 \left| \frac{\vec{p}}{p^0} - \frac{\vec{q}}{q^0} \right| = \{(q^0 \vec{p} - p^0 \vec{q})^2\}^{1/2} \qquad (67)$$

$$= \{(q.p)^2 - m^2 m'^2 + (\vec{q} \times \vec{p})^2\}^{1/2}$$

Here $p^2 = m^2$ and $p'^2 = m'^2$. In the frames of reference normally of interest (e.g. c.m. frame or a convenient lab frame) the beam and target particles are parallel or antiparallel and F takes explicitly covariant form (and hence $d\sigma$)

$$F = \left[(q \cdot p)^2 - m^2 m'^2\right]^{1/2} \qquad (68)$$

Thus $F = (v\, p^0 q^0)_{lab} = m(q_0^2 - m'^2)^{1/2} = m\, |\vec{q}|_{lab}$ in the lab. frame with $\vec{p} = 0$. In the c.m. frame $\vec{p} + \vec{q} = 0$ we get $F = |k\varepsilon$ where $|k$ is the magnitude of 3-momentum of initial particles and ε the total energy

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* A useful recurrence formulae for calculating this factor was first given by P. Srivastava and E. Sudarshan, Phys. Rev. 110, 765 (1958), in connection with a covariant formulation of Fermi's Statistical Model for collisions at high energies.

in the c.m. frame. It is clear that the value of $d\sigma$ remains invariant with respect a Lorentz transformation along the collision axis, in the c.m. frame.

7.3 - *DENSITY MATRIX; MIXTURE STATES:*

We may write, from the discussion in section (7.1) the transition matrix for a process A + B $\longrightarrow$ C + D + E + ...

$$< p_C\, p_D\, p_E \cdots ;\ \lambda_C\, \lambda_D\, \lambda_E;\ \begin{smallmatrix}\text{in}\\ \text{out}\end{smallmatrix}|\ T(p_i)\ | p_A\, p_B;\ \lambda_A\, \lambda_B\ ;\ \begin{smallmatrix}\text{in}\\ \text{out}\end{smallmatrix} >$$

$$= < \lambda_C\, \lambda_D\, \lambda_E \cdots ;\ \begin{smallmatrix}\text{in}\\ \text{out}\end{smallmatrix} |\ \mathbb{T}_{p_i}(p_A p_B;\ p_C p_D p_E \cdots) | \lambda_A \lambda_B;\ \begin{smallmatrix}\text{in}\\ \text{out}\end{smallmatrix} > \tag{69}$$

where $\mathbb{T}_{p_i}(p_A p_B;\ p_C p_D p_E \cdots)$ is an operator in spin space. This is clear since a state $|\vec{p}\lambda>$ is connected to the rest state $|\vec{0}\lambda>$ by a Lorentz trans formation for massive particles and for massless case the "spin" quantum number is an invariant quantity.

$\mathbb{T}_{p_i}$ is thus a matrix $(n_f \times n_i)$ in the "spin space".

$$\mathbb{T}^{\lambda_C\, \lambda_D\, \lambda_E\, \cdots}_{\qquad\qquad\qquad \lambda_A\, \lambda_B} \tag{70}$$

where

$$n_f = (2\, s_C + 1)(2\, s_D + 1)(2 s_E + 1) \ldots$$
$$n_i = (2\, s_A + 1)(2\, s_B + 1)$$

and for ν_e, ν_μ or $\bar{\nu}_e$, $\bar{\nu}_\mu$, $(2s + 1)$ is replaced by 1 white for photons by 2.

When the initial states is a mixture state use of density matrix must be made to describe the state of the system.

For p_A and p_B fixed the spin density matrix in the initial mixture state is

$$\rho_i \equiv \rho_i(p_A,p_B) = \sum_{\lambda_A} \sum_{\lambda_B} (|\lambda_A \lambda_B \text{ in}\rangle \langle \lambda_A \lambda_B \text{ in}|)\, \omega(\lambda_A)\ \ \omega(\lambda_B) \tag{71}$$

where $\sum_{\lambda_A} \omega(\lambda_A) = 1, \ \sum_{\lambda_B} \omega(\lambda_B) = 1, \ \sum_{\lambda_A}\sum_{\lambda_B} \omega(\lambda_A)\ \ \omega(\lambda_B) = 1$

e.g.* $\rho_i = \rho_i^{\ A} \otimes \rho_i^{\ B}$. Here $\omega(\lambda_A)$ is the probability that the spin state of particle A is given by λ_A etc.

Now for a reaction initiated by "pure" state $|\lambda_A \ \lambda_B\rangle$ the probability amplitude for finding the final "<u>out</u>" state $|\ \lambda_C\ \lambda_D\ \lambda_E \ \cdots ;\ \text{out}\ \rangle$ is given by ($p_C, p_D, p_E \cdots$ fixed)

$$\begin{aligned} &\langle \lambda_C\ \lambda_D\ \lambda_E \ldots \text{out}|\ \Pi_{p_i}|\ \lambda_A\ \lambda_B;\ \text{out}\ \rangle \\ &= \langle \lambda_C\ \lambda_D\ \lambda_E \ \cdots ;\ \text{in}|\ \Pi_{p_i}\ |\ \lambda_A\ \lambda_B;\ \text{in}\ \rangle \end{aligned} \tag{72}$$

Thus the "out" state of particles C, D, E, ... corresponding to a fixed initial state $|\ \lambda_A\ \lambda_B;\ \text{in}\ \rangle$ is given by

$$\begin{aligned} &\sum_{\lambda_C\ \lambda_D\ \lambda_E\cdots} |\lambda_C\ \lambda_D\ \lambda_E\ ;\ \text{out}\rangle\ \ \langle \lambda_C\ \lambda_D\ \lambda_E\ \ldots \text{out}\ |\Pi|\ \ \lambda_A\ \lambda_B;\ \text{out}\ \rangle \\ &\qquad = \Pi\ \ |\lambda_A\ \lambda_B;\ \text{out}\ \rangle \end{aligned} \tag{73}$$

where we made use of the resolution of the identity for p_C, p_D, p_E ...

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* Note taking partial trace w.r.t. particle A we obtain

$$\text{Tr}_A(\rho_i) = (\text{Tr}\ \rho^A_{\ i}) \otimes \ \rho_i^{\ B} = \rho_i^{\ B}$$

fixed. Thus the density matrix for the final state is

$$\rho_f \equiv \rho_f(p_C, p_D, p_E, \ldots) = C \sum_{\lambda_A} \sum_{\lambda_B} (\mathbb{T}|\lambda_A \lambda_B; out\rangle\langle\lambda_A \lambda_B; out|\mathbb{T}^\dagger)\omega(\lambda_A)\omega(\lambda_B) \quad (74)$$

where C is a constant to take care of the lack of normalization of states $\mathbb{T}| \lambda_A \lambda_B; out \rangle$. Let us calculate mean value of an operator $A_{out} = S^{-1} A_{in} S$ in the 'out' state.

$$\langle A\rangle = C\, Tr(\rho_f A_{out}) = Tr(A_{out}\, \rho_f)$$

$$= C\, Tr\, (A_{out} \sum_{\lambda_A} \sum_{\lambda_B} (\mathbb{T}|\lambda_A \lambda_B; out\rangle\langle\lambda_A \lambda_B; out|\mathbb{T}^\dagger)\, \omega(\lambda_A)\, \omega(\lambda_B))$$

$$= C \sum_{\lambda_A} \sum_{\lambda_B} \sum_{\lambda_C \lambda_D \lambda_E \cdots} \langle \lambda_C\, \lambda_D\, \lambda_E \,.. out|A_{out}\, \mathbb{T}\, |\lambda_A\, \lambda_B; out \rangle$$

$$\cdot\langle \lambda_A\, \lambda_B; out|\mathbb{T}^\dagger|\, \lambda_C\, \lambda_D\, \lambda_E \,..; out \rangle\, \omega(\lambda_A)\, \omega(\lambda_B)$$

$$= C \sum_{\lambda_A} \sum_{\lambda_B} \sum_{\lambda_C \lambda_D ..} \langle\, \lambda_C\, \lambda_D\, \lambda_E ..; in|\, A_{in}\, \mathbb{T}|\, \lambda_A\, \lambda_B; in \rangle$$

$$\langle \lambda_A\, \lambda_B; in\, |\mathbb{T}^\dagger\, |\, \lambda_C\, \lambda_D\, \lambda_E ..; in\rangle\, \omega(\lambda_A)\, \omega(\lambda_B)$$

$$= C\, Tr(A_{in}\, \mathbb{T}\, \rho_i\, \mathbb{T}^\dagger) \quad (75)$$

Thus if we work with only "in" states and "in" operators we may write the final state density operator as

$$\rho_f = C \ \mathbb{T} \ \rho_i \ \mathbb{T}^{\dagger} \tag{76}$$

or

$$\rho_f(p_C, p_D, p_E, \ldots) = C \ \mathbb{T}(p_A p_B, p_C p_D p_E \cdots) \ \rho_i(p_A, p_B) \ \mathbb{T}^{\dagger}(p_A p_B ; p_C p_D p_E \cdots)$$

C is given by the condition

$$\mathrm{Tr} \ \rho_f = 1 \tag{77}$$

Thus

$$\rho_f = \frac{\mathbb{T} \ \rho_i \ \mathbb{T}^{\dagger}}{\mathrm{Tr} \ (\mathbb{T} \ \rho_i \ \mathbb{T}^{\dagger})} \tag{78}$$

We can calculate the denominator

$$\mathrm{Tr}(\mathbb{T} \ \rho_i \ \mathbb{T}^{\dagger}) = \sum_{\lambda_A \lambda_B} \omega(\lambda_A) \ \omega(\lambda_B) \ \mathrm{Tr} \ (\mathbb{T} | \lambda_A \lambda_B ; \mathrm{in}\rangle \ \langle \lambda_A \lambda_B ; \mathrm{in} | \mathbb{T}^{\dagger})$$

$$= \sum_{\lambda_C} \sum_{\lambda_D} \cdots \sum_{\lambda_A} \sum_{\lambda_B} \omega(\lambda_A) \omega(\lambda_B) \langle \lambda_C \lambda_D \ldots ; \mathrm{in} | \ \mathbb{T} \ | \lambda_A \lambda_B ; \mathrm{in} \rangle$$

$$\cdot \langle \lambda_A \ \lambda_B ; \mathrm{in} | \ \mathbb{T}^{\dagger} | \ \lambda_C \lambda_D \ \ldots ; \mathrm{in} \rangle$$

$$= \sum_{\lambda_A} \sum_{\lambda_B} \sum_{\lambda_C \lambda_D \lambda_E \ldots} \omega(\lambda_A) \ \omega(\lambda_B) | \langle \lambda_C \lambda_D \lambda_E \ \ldots | \mathbb{T} | \lambda_A \lambda_B \rangle |^2 \tag{79}$$

is clearly the differential cross section,* upto a factor, due to the incoherent initial mixed states described by the density matrix ρ_i, summed over the final polarizations and averaged over the initial polarizations.

If the final state represented by the density matrix ρ_f is observed

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* e.g. ($\frac{d\sigma}{d\Omega}$).

through an apparatus, the setting of which is represented by a hermitian operator A acting on $|\lambda_C\ \lambda_D\ \lambda_E \cdots >$ the transitian rate is

$$\omega(i,a) = Tr(A\ \rho_f) \tag{80}$$

We may also obtain the polarization density matrix of one or a subset of the final particles when one does not observe the polarization of the other particles. This can be obtained by taking a partial trace with respect the latter particles. For example, if the polarization of the particle 1 is not observed, the polarization density matrix of the other final particles is $Tr_1\ (\rho_f)$. The discussion above clearly applies to a decay of one particle state. *

7.4 - *SCATTERING PROCESS: $a + b \rightarrow c + d$.*

For a process of type a + b → c + d we have explicity, in the c.m. frame,

$$|\alpha_i> = \frac{2\pi}{\sqrt{\eta_{|a}\ \eta_{|b}}} \sqrt{\frac{4\varepsilon}{|k}}\ |\Theta_0\ \phi_0\ ;\ \lambda_a\ \lambda_b >$$

$$|\alpha_f> = \frac{2\ \pi}{\sqrt{\eta_{|c}\ \eta_{|d}}} \sqrt{\frac{4\varepsilon}{|k'}}\ |\Theta\phi;\ \lambda_c\ \lambda_d > \tag{81}$$

where |k and |k' are magnitude and $(\Theta_0\ \phi_0)$ and (Θ,ϕ) are the polar angles

* For some applications to specific processes see for example, Jacob and Wick, Annals of Physics 7, 404 (1959); Jackson and

of the 3-momenta of the initial particle "a" and final particle "c" respectively. The invariant phase space factor dQ takes the simple form (eq. 3.112)

$$(2\pi)^6\ dQ = \frac{|k'|}{4\varepsilon}\, d\Omega(\Theta,\phi)\cdot d^4p_f\ \delta^4(p_f-p_i) \tag{82}$$

in the c.m. frame. The differential cross-section ($\frac{d\sigma}{d\Omega}$) is obtained on integrating $d\sigma$ over p_f to remove the delta function.

$$\frac{d\sigma}{d\Omega} = \frac{(2\pi)^4 |<\alpha_f|T(p_i)|\alpha_i>|^2\ \eta|_a\ \eta|_b\ \eta|_c\ \eta|_d}{4F}\ \frac{1}{(2\pi)^6}\ \frac{|k'|}{4\varepsilon}$$

$$= \frac{1}{8\pi\varepsilon^2}\ \left(\frac{|k'|}{|k|}\right) |<\alpha_f|T(p_i)|\alpha_i>|^2\ \eta|_a\ \eta|_b\ \eta|_c\ \eta|_d$$

$$= \left(\frac{2\pi}{|k|}\right)^2 |<\Theta\phi;\ \lambda_c\ \lambda_d\ |T(p_i)|\ \Theta_o\ \phi_o;\ \lambda_a\ \lambda_b>|^2 \tag{83}$$

We may choose the Z-axis in the c.m. frame so that it coincides with $\vec{k}$, that is, $\Theta_o = \phi_o = 0$. Then

$$\frac{d\sigma}{d\Omega} = \left|\ f^{(\Theta,\phi)}_{\lambda_c\ \lambda_d;\ \lambda_a\ \lambda_b}\ \right|^2 \tag{84}$$

where $$f^{(\Theta,\phi)}_{\lambda_c\lambda_d;\ \lambda_a\lambda_b} = \left(\frac{2\pi}{|k|}\right) < \Theta\phi;\ \lambda_c\ \lambda_d\ |T(p_i)|0\vec{0};\ \lambda_a\ \lambda_b > \tag{85}$$

We note also for the invariant matrix element:

$$<\alpha_f|T(p_i)|\alpha_i> = \frac{(2\pi)^2}{\sqrt{\eta_{|a}\,\eta_{|b}\,\eta_{|c}\,\eta_{|d}}} \quad \frac{4\varepsilon}{\sqrt{|k|k'}} < \Theta\phi;\ \lambda_c\ \lambda_d|T(p_i)|\Theta_o\phi_o;\lambda_a\lambda_b >$$

$$= \frac{(2\pi)^2}{\sqrt{\eta_{|a}\,\eta_{|b}\,\eta_{|c}\,\eta_{|d}}}\ \Upsilon(s,t) \tag{86}$$

where $\Upsilon(s,t) = \frac{2\varepsilon}{\pi}\sqrt{\frac{|k|}{|k'|}}\ f_{\lambda_c\lambda_d;\lambda_a\lambda_b}$ is the invariant Υ matrix element as func tion of the invariants $s = (p_a+p_b)^2 = \varepsilon^2$ and $t = (p_a-p_c)^2$. We remark that through $d\sigma$ is "invariant", the factor $d\Omega$ and hence $(d\sigma/d\Omega)$ is not invarian quantity.

We now will make the expansion of the invariant matrix elements in terms of angular momentum states.

$<\Theta\phi;\ \lambda_c\ \lambda_d|T(p_i)|00;\ \lambda_a\ \lambda_b>$

$$= \sum_J \sum_M N_J\, D^{*}_{M,\ \lambda_c-\lambda_d}(\phi,\Theta,-\phi) <JM;\ \lambda_c\lambda_d|T(p_i) \sum_{J'} N^{*}_{J'}|J',\ \lambda_a-\lambda_b;\ \lambda_a,\lambda_b> \tag{87}$$

Since the total angular momentum is conserved, that is, S-matrix operator is rotation invariant, $[S,\vec{J}] = 0$, we have

$< JM;\ \lambda_c\ \lambda_d\ |T(p_i)|J'M';\ \lambda_a\ \lambda_b >$

$$= \delta_{JJ'}\ \delta_{MM'} < \lambda_c\ \lambda_d|T^J(\epsilon)|\ \lambda_a\ \lambda_b > \tag{88}$$

Thus

$<\Theta\phi;\ \lambda_c\ \lambda_d\ |T(p_i)|00;\ \lambda_a\ \lambda_b>$

$$= \sum_J \left(\frac{2J+1}{4\pi}\right) D^{(J)*}_{\lambda,\mu}\ (\phi\ \Theta\ -\phi) <\ \lambda_c\ \lambda_d|T^J(\varepsilon)|\lambda_a\ \lambda_b > \tag{89}$$

Where $\lambda = (\lambda_a - \lambda_b)$ and $\mu = (\lambda_c - \lambda_d)$. The angular momentum expansion of $f(\Theta,\phi)$ (or $\tau(s,t)$) is thus

$$f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\Theta,\phi) = \frac{1}{2k}\sum_J (2J+1)\, D^{(J)^*}_{\lambda\mu}(\phi,\Theta,-\phi)<\lambda_c\, \lambda_d|T^J(\varepsilon)|\lambda_a\, \lambda_b> \tag{90}$$
$$= e^{i(\lambda-\mu)}\ \frac{1}{2k}\Sigma(2J+1)\, d^{(J)}_{\lambda\mu}(\Theta) < \lambda_c\, \lambda_d|T^J(\varepsilon)|\lambda_a\, \lambda_b>$$

This may be referred to as (generalized) partial wave expansion. In the special case of spinless particles it reduces to

$$f(\Theta) \doteq f(\Theta,0) = \frac{1}{2\,ik}\sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\Theta)\, T^\ell(\varepsilon)$$

$$= \sum_\ell (2\ell+1)\, P_\ell(\cos\Theta)\, f_\ell(\varepsilon) \tag{91}$$

where the partial wave amplitude is defined by $f_\ell(\varepsilon) = T^\ell(\varepsilon)/2\,k$. We derive easily

$$f_\ell(\varepsilon) = \frac{1}{2}\int_{-1}^{1} P_\ell(\cos\Theta)\, f(\Theta)\, d(\cos\Theta) \tag{92}$$

7.5 - *UNITARITY CONDITION:*

We now discuss the condition of unitary of S matrix on the matrix elements. From

$$T - T^\dagger = i\, T^\dagger T \tag{93}$$

it follows

$$< p_f;\, \alpha_f|(T-T^\dagger)|p_i;\, \alpha_i > = i < p_f;\alpha_f\, |T^\dagger\, T|p_i;\, \alpha_i > \tag{94}$$

where $|p;\alpha>$ represents a state with total four momentum p^{μ} and α indicates the rest of the labels and $|p;\alpha> \equiv |p> \otimes |\alpha>$. We now insert on the right hand side the resolution of the identity * in terms of the complete set of states of the Hilbert space, viz,

$$I = \oint d^4p \; dQ \; |p;\alpha><p;\alpha| \tag{95}$$

where $dQ = \delta^4(p_1 + p_2 + \ldots + p_n - p)\left(\prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3 \, 2p_i^0}\right)$ and $\oint$ refers to summation over all internal variables and integration over momentum space of the n particles. Note that $|p;\alpha> \equiv |p_1, p_2 .. p_n;\lambda>$.

For each T matrix elements we can use

$$<p'';\alpha''|T|p';\alpha'> = (2\pi)^4 \delta^4(p''-p') < \alpha''|T(p')|\alpha'> \tag{96}$$

to obtain

$$(2\pi)^4\delta^4(p_f-p_i)\left[<\alpha_f|T(p_i)|\alpha_i> - <\alpha_f|T^{\dagger}(p_i)|\alpha_i>\right]$$

$$= i\left[(2\pi)^4\right]^2 \delta^4(p_f-p_i) \oint \{dQ <\alpha_f|T^{\dagger}(p_i)|\alpha><\alpha|T(p_i)|\alpha_i>\}_{p=p_i}$$

Thus

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* Note with our normalization $\oint dQ|\alpha><\alpha| = I$ (with p fixed) corresponds to normalization $<\alpha|\alpha> = \delta(Q-Q')\,\delta_{\lambda\lambda'}$, in obvious notation. The discussion here follows closely Martin and Spearman.

$$[< \alpha_f|T(p_i)|\alpha_i > - <\alpha_f|T^\dagger(p_i)|\alpha_i>]$$

$$= i(2\pi)^4 \oint \{dQ < \alpha_f|T^\dagger(p_i)|\alpha><\alpha|T(p_i)|\alpha_i >\}_{p=p_i} \tag{98}$$

where it is clear that only intermediate states $|\alpha >$ with a total four momentum $p = p_i = p_f$ contribute.

For a hermitian K matrix defined by

$$S = \frac{I + iK}{I - iK} \tag{99}$$

so that T = 2K + i K T, the K-matrix elements can be defined similarly, viz,

$$< p'; \alpha'|K|p;\alpha > = < p'|p > <\alpha'|K(p)|\alpha >$$

$$= (2\pi)^4 \delta^4(p-p') < \alpha'|K(p) |\alpha > \tag{100}$$

Then we have

$$< \alpha_f|K(p_i)|\alpha_i > = < \alpha_i|K(p_i)| \alpha_f>^* \tag{101}$$

and

$$< \alpha_f|T(p_i)| \alpha_i> = 2 < \alpha_f|K(p_i)|\alpha_i >$$

$$+ i(2\pi)^4 \oint \{dQ<\alpha_f|K(p_i)|\alpha><\alpha|T(p_i)|\alpha_i>\}_{p=p_i} \tag{102}$$

For example, for two particle scattering a + b → c + d, the unitary relation becomes (choosing $\Theta_o = \phi_o = 0$)

$$<\Theta\phi;\lambda_c\lambda_d;\gamma'|T(p_i)|00;\lambda_a\lambda_b;\gamma>-<\Theta\phi;\lambda_c\lambda_d;\gamma'|T^\dagger(p_i)|00;\lambda_a\lambda_b;\gamma >$$

$$= i(2\pi)^4\oint dQ < \Theta\phi; \lambda_c\lambda_d; \gamma'|T^\dagger(p_i)|\alpha><\alpha|T(p_i)|00; \lambda_a\lambda_b;\gamma > \tag{103}$$

where $\oint dQ$ extends over states $|\alpha>$ which have the same 4-momentum as the initial (final) momentum. If total momentum is below the threshold of any three particle intermediate state, only two particle intermediate states are possible and the (2-particle) unitarity conditions is

$$<\Theta\phi;\lambda_c\lambda_d;\gamma'|T(p_i)|00;\lambda_a\lambda_b;\gamma>-<\Theta\phi;\lambda_c\lambda_d;\gamma'|T^{\dagger}(p_i)|00;\lambda_a\lambda_b;\gamma>$$

$$= i\oint d\Omega'' <\Theta\phi;\lambda_c\lambda_d;\gamma'|T^{\dagger}(p_i)|\Theta''\phi'';\lambda_1\lambda_2;\gamma''><\Theta''\phi'';\lambda_1\lambda_2;\gamma''|T(p_i)|00;\lambda_a\lambda_b;\gamma>$$

$$= i\oint d\Omega''<\Theta''\phi'';\lambda_1\lambda_2;\gamma''|T(p_i)|\Theta\phi;\lambda_c\lambda_d;\gamma''>^{*}<\Theta''\phi'',\lambda_1\lambda_2;\gamma''|T(p_i)|00;\lambda_a\lambda_b;\gamma> \tag{104}$$

We may now use expression in partial waves to obtain the partial wave unitary condition

$$<\lambda_c\lambda_d;\gamma'|T^J(p_i)|\lambda_a\lambda_b;\gamma>-<\lambda_c\lambda_d;\gamma\;|T^{\dagger J}(p_i)|\lambda_a\lambda_b;\gamma>$$

$$= i\sum_{\lambda_1}\sum_{\lambda_2}\sum_{\gamma''} <\lambda_c\lambda_d;\gamma'|T^{\dagger J}(p_i)|\lambda_1\lambda_2;\gamma''><\lambda_1\lambda_2;\gamma''|T^J(p_i)|\lambda_a\lambda_b;\gamma> \tag{105}$$

Symbolically

$$T^J_{\lambda'\lambda}-T^{J^*}_{\lambda\lambda'} = i\sum_{\lambda''} T^{J^*}_{\lambda''\lambda'}\,T^J_{\lambda''\lambda} \tag{106}$$

or regarding T^J as matrix whose components are labelled by (the set) λ, λ'. We may write

$$T^J - T^{\dagger J} = i\,T^{\dagger J}\;T^J \tag{107}$$

For the matrix S^J defined by

$$S^J_{\lambda\lambda'} = \delta_{\lambda\lambda'} + i\,T^J_{\lambda\lambda'} \tag{108}$$

we obtain

$$S^{J^\dagger} S^J = I \tag{109}$$

For matrix K^J the unitarity condition is

$$K^{J^\dagger} = K^J \qquad \text{or} \qquad K^J_{\lambda'\lambda} = K^{J^*}_{\lambda\lambda'} \tag{110}$$

Also

$$T^J = 2\, K^J + i\, K^J\, T^J$$

or

$$2(T^J)^{-1} = (K^J)^{-1} - i\, I \tag{111}$$

7.6 - *OPTICAL THEOREM*

From (suppressing indices $\eta_|$)

$$<00;\lambda_a\lambda_b;\gamma|T(p_i)|00;\lambda_a\lambda_b;\gamma> - <00;\lambda_a\lambda_b;\gamma|T^\dagger(p_i)|00;\lambda_a\lambda_b;\gamma>$$

$$= i(2\pi)^4 \oint dQ <00;\lambda_a\lambda_b;\gamma|T^\dagger(p_i)|\alpha><\alpha|T(p_i)|00;\lambda_a\lambda_b;\gamma>$$

$$= i(2\pi)^4 \oint dQ\ |\alpha|T(p_i)|00;\lambda_a\lambda_b;\gamma>|^2$$

$$= i\, \frac{\mathbb{k}}{4\varepsilon}\ \frac{(2\pi)^4}{(2\pi)^2} \oint dQ\ <\alpha|T(p_i)|\alpha_i>|^2$$

$$= i\, \frac{\mathbb{k}}{4\varepsilon}\cdot(2\pi)^2\cdot\frac{4\,\mathbb{k}\,\varepsilon}{(2\pi)^4} \oint dQ\, \frac{d\sigma}{dQ} = i\left(\frac{\mathbb{k}}{2\pi}\right)^2 \sigma_{tot.} \tag{112}$$

L.H.S. can be written as

$$2i\ \mathrm{Im} < 00;\ \lambda_a\lambda_b;\gamma\ |T(p_i)|00;\ \lambda_a\lambda_b;\gamma >$$

$$= 2i\ \frac{|k}{2\pi}\ \mathrm{Im}\ f^{(0,0)}_{\lambda_a\lambda_b\ \gamma;\lambda_a\lambda_b\gamma} \tag{113}$$

Thus

$$\mathrm{Im}\ f^{(0,0)}_{\lambda_a\lambda_b\ \gamma;\ \lambda_a\lambda_b\gamma} = \frac{|k}{4\pi}\ \sigma_{tot} \tag{114}$$

or

$$\mathrm{Im}\ \tau(s,\ t = 0) = \frac{|k\varepsilon}{2\pi^2}\ \sigma_{tot} \tag{115}$$

Here σ_{tot} is the total cross sections for scattering from an initial state with quantum numbers $\lambda_a, \lambda_b, \gamma$ and $f_{\lambda',\lambda}(0,0)$ is the forward elastic scattering amplitude.

7.7 - *INVARIANCE CONDITIONS ON PARTIAL WAVE AMPLITUDES:*

1. Parity Invariance:

$$< \lambda_c\lambda_d|T^J(\varepsilon)|\lambda_a\lambda_b > \ <JM\ \lambda_c\ \lambda_d\ |T(\varepsilon)|JM;\ \lambda_a\ \lambda_b >$$

$$= <JM;\lambda_c\lambda_d|\mathcal{P}^{\dagger}\,T(\varepsilon)\,\mathcal{P}|JM;\lambda_a\lambda_b >$$

$$= \eta_a\ \eta_b\ \eta_c\ \eta_d\ (-1)^{2J-s_a-s_b-s_c-s_d}$$

$$\cdot < JM;\ -\lambda_c,\ -\ \lambda_d\ |T(\varepsilon)|JM;\ -\ \lambda_a,\ -\ \lambda_b >$$

$$= \eta < -\lambda_c,-\lambda_d\ |T^J(\varepsilon)|\ -\lambda_a,\ -\ \lambda_b > \tag{116}$$

where

$$\eta = \eta_a \, \eta_b \, \eta_c \, \eta_d \, (-1)^{s_c + s_d - s_a - s_b} \tag{117}$$

since $(J-s_c-s_d)$ is certainly an integer. For the scattering amplitude

$$f_{\lambda_c \lambda_d;\lambda_a \lambda_b}(\Theta,\phi)$$

$$= \frac{1}{2\,ik} \sum_J (2J+1)\, D^{(J)^*}_{\lambda\mu}(\phi,\Theta,-\phi) \;\langle \lambda_c\, \lambda_d | T^J(\varepsilon) | \lambda_a\, \lambda_b \rangle$$

$$= \frac{1}{2\,ik} \sum_J (2J+1)\, D^{(J)^*}_{\lambda\mu}(\phi,\Theta,-\phi) \langle -\lambda_c, -\lambda_d | T^J(\varepsilon) | -\lambda_a, -\lambda_b \rangle \eta$$

$$= \frac{\eta}{2\,ik} \sum_J (2J+1)\, D^{(J)^*}_{-\lambda,-\mu}(\pi-\phi,\Theta,-\pi+\phi) \langle -\lambda_c, -\lambda_d | T^J(\varepsilon) | -\lambda_a, -\lambda_b \rangle$$

$$= \eta\, f_{-\lambda_c, -\lambda_d; -\lambda_a, -\lambda_b}(\Theta, \pi-\phi) \tag{118}$$

If we choose $\phi = 0$ to be the plane of scattering we find

$$f_{\lambda_c \lambda_d;\lambda_a \lambda_b}(\Theta) = \eta(-1)^{\mu-\lambda} f_{-\lambda_c, -\lambda_d; -\lambda_a, -\lambda_b}(\Theta) \tag{119}$$

2. *Time Reversal Invariance*

As discussed earlier by time reversal invariance of unitary S operator we mean the invariance under time reversal of its generator. This leads to the time reversal condition

$$Ⓗ\, S\, Ⓗ^{-1} = S^{\dagger}$$

$$< \lambda_c\ \lambda_d|S^J(\varepsilon)|\lambda_a\ \lambda_b > \equiv < JM;\ \lambda_c\ \lambda_d\ |S(\varepsilon)|JM;\ \lambda_a\ \lambda_b >$$

$$= (\psi_{JM;\ \lambda_c\ \lambda_d},\ Ⓗ^{\dagger}\ S^{\dagger}(\varepsilon)\ Ⓗ\ \psi_{JM;\ \lambda_a\ \lambda_b})$$

$$= (Ⓗ\,\psi_{JM;\lambda_c\lambda_d},\ S^{\dagger}(\varepsilon)\ \ Ⓗ\ \ \psi_{JM;\ \lambda_a\ \lambda_b})^{*}$$

$$= (-1)^{2(J-M)}(\psi_{J,-M;\lambda_c\lambda_d},\ S^{\dagger}(\varepsilon)\ \psi_{J,-M;\lambda_a\lambda_b})^{*} \tag{121}$$

$$= (\psi_{J,-M;\lambda_a\lambda_b},\ S\ \psi_{J,-M;\lambda_c\lambda_d})$$

$$= < J,-M;\lambda_a\lambda_b|S(\varepsilon)|J,-M;\lambda_c\lambda_d> = <\lambda_a\lambda_b|S^J(\varepsilon)|\lambda_c\lambda_d>$$

and similar relations holds if we replace S^J by T^J or K^J. Since K^J is also hermitian it follows that K^J matrix is a real symmetric matrix. From

$$\Sigma(2J+1)\ D^{(J)*}_{\lambda\mu}\ \ (\phi,\Theta,\ -\phi) < \lambda_a\ \lambda_b\ |T^J(\varepsilon)|\lambda_c\ \lambda_d >$$

$$= \Sigma(2J+1)\ D^{(J)^{*}}_{\mu\lambda}\ \ (\pi-\phi,\Theta,\ \phi-\pi) < \lambda_a\ \lambda_b\ |T^J(\varepsilon)|\lambda_c\ \lambda_d > \tag{122}$$

it follows, if time reversal holds

$$\mathbb{k}\ f_{\lambda_c\lambda_d;\lambda_a\lambda_b}^{(\Theta,\phi)} = \mathbb{k}'\ f_{\lambda_a\lambda_b;\ \lambda_c\lambda_d}^{(\Theta,\pi\ -\phi)} \tag{123}$$

or

$$\mathbb{k}\ f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\Theta\ \phi) = (-1)^{\mu-\lambda}\ \mathbb{k}'\ f_{\lambda_a\lambda_b;\lambda_c\lambda_d}^{(\Theta\ \phi)} \tag{124}$$

where (Θ,ϕ) on the right hand side are polar angles of the particle "a" in the inverse reaction $c + d \to a + b$ with c bearing the angles (0,0).

3 - Detailed Balancing

Time reversal invariance leads to a relation between unpolarized cross sections of reactions $a + b \to c + d$ and $c + d \to a + b$.

From the relation just obtained

$$|k^2|f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\Theta,\phi)|^2 = |k'^2\ |f_{\lambda_a\lambda_b;\lambda_c\lambda_d}(\Theta,\phi)|^2 \tag{125}$$

The differential cross section summed over all the final spins (or helicities) and averaged over all the initial spins for reaction $a + b \to c + d$ is

$$\left(\frac{d\sigma}{d\Omega}\right)_{ab\to cd} = \frac{1}{(2s_a+1)(2s_b+1)} \sum_{\substack{\lambda_a,\ \lambda_b\\ \lambda_c,\ \lambda_d}} |f_{\lambda_c\lambda_d;\lambda_a\lambda_b}^{(\Theta,\phi)}|^2 \tag{126}$$

Like wise for reaction $c + d \to a + b$ (with "a" emitted along (Θ,ϕ)).

$$\left(\frac{d\sigma}{d\Omega}\right)_{cd\to ab} = \frac{1}{(2s_c+1)(2s_d+1)} \sum_{\substack{\lambda_a,\lambda_b\\ \lambda_c,\lambda_d}} |f_{\lambda_a\lambda_b;\lambda_c\lambda_d}^{(\Theta,\phi)}|^2 \tag{127}$$

We find thus

$$\frac{(d\sigma/d\Omega)_{a+b\to c+d}}{(d\sigma/d\Omega)_{c+d\to a+b}} = \frac{|k'^2}{|k^2}\ \frac{(2s_c+1)(2s_d+1)}{(2s_a+1)(2s_b+1)} \tag{128}$$

Both cross sections are evaluated at the same total energy and angle in the c.m. frame. This equation is called the *principle of detailed balancing*. It was used to determine the spin of π^+ in the reaction $\pi^+ + d \rightleftarrows p + p$.

8

PION - NUCLEAR SCATTERING:

$$N + \pi \rightarrow N + \pi$$
$$a \quad b \quad c \quad d$$

We have $\lambda_\pi = 0$ $\lambda_N = \pm\frac{1}{2}$ and may use $\mu(=2(\lambda_N-\lambda_\pi)) = \pm$ to indicate the helicity state in place of pair (λ_π, λ_N). T^J is thus a (2 x 2) matrix. The parity operation on $|JM;\mu>$ state is given by

$$\begin{aligned} \mathcal{P}|JM;\mu> &= \eta_N\, \eta_\pi(-1)^{J-\frac{1}{2}}\,|JM;-\mu> \\ &= (-1)^{J+\frac{1}{2}}\,|JM;-\mu> \end{aligned} \tag{1}$$

using $\eta_N = +1$, $\eta_\pi = -1$. Eigenstates of parity are given by

$$\begin{aligned} |JM;\mu>_{\pm} &= \frac{1}{\sqrt{2}}\left[|JM;\mu> \pm |JM;-\mu>\right] \\ \mathcal{P}|JM;\mu>_{\pm} &= (-1)(-1)^{J\mp\frac{1}{2}}\,|JM;\mu>_{\pm} = \pm(-1)^{J-\frac{3}{2}}|JM;\mu>_{\pm} \end{aligned} \tag{2}$$

We may call $\ell = (J \mp \frac{1}{2})$, which is an integer, the "orbital" parity of πN system. Note, however, that "ℓ" has no place in the relativistic theory we have been discussing *. The parity conserving transition amplitudes (in the c.m. frame) may be defined by

---

* See however, the footnote on page 132.

$$F^{J\pm}_{\lambda\mu}(\varepsilon) = \langle \underset{\pm}{JM};\lambda|T(\varepsilon)|JM;\underset{\pm}{\mu}\rangle$$

$$= \frac{1}{2}\left\{ T^J_{\lambda\mu} + T^J_{-\lambda,-\mu} \pm T^J_{\lambda,-\mu} \pm T^J_{-\lambda,\mu} \right\} \quad (3)$$

where

$$T^J_{\lambda\mu}(\varepsilon) = \langle\lambda|T^J(\varepsilon)|\mu\rangle = \langle JM;\lambda|T(\varepsilon)|JM;\mu\rangle \quad (4)$$

and $\pm$ in $F^{J\pm}$ labels the parity states with parity $(-)(-1)^{J\mp\frac{1}{2}}$. If parity is conserved (we discussed above)

$$T^J_{\lambda\mu} = T^J_{-\lambda,-\mu} \text{ (or } T^J_{++} = T^J_{--} \text{ and } T^J_{+-} = T^J_{-+}) \quad (5)$$

since $\eta = +1$; it follows

$$F^{J\pm}_{\lambda\mu} = T^J_{\lambda\mu} \pm T^J_{\lambda,-\mu} = F^{J\pm}_{-\lambda,-\mu} \quad (6)$$

Also $T(\varepsilon)$ or $T^J(\varepsilon)$ is diagonal in the parity eigenstate basis if space reflection invariance holds. Time invariance requires

$$T^J_{\lambda\mu} = T^J_{\mu\lambda} \qquad \text{(or } T_{+-} = T_{-+}) \quad (7)$$

and is already satisfied if parity is conserved. We also see that only two amplitudes are independent if parity conservation holds. We take them to be

$$T_{J-} \equiv T^J_{\ell+} = F^{J+}_{++} = (T^J_{++} + T^J_{+-})$$

$$T_{J+} \equiv T^J_{\ell-} = F^{J-}_{++} = (T^J_{++} - T^J_{+-}) \quad (8)$$

where $T^J_{\ell+}$ corresponds to parity conserving amplitude with the states having $J = \ell + \frac{1}{2}$ so that the parity of the states is $-(-1)^{\ell} = (-1)(-1)^{J-\frac{1}{2}}$.

Similar interpretations apply to other notations for partial wave amplitudes and the corresponding phase shifts. One often uses the amplitude f^J,

$$f^J(\varepsilon) = \frac{T^J(\varepsilon)}{2\,k}$$

For helicity amplitudes we have the decomposition $(\frac{1}{2} + 0 \to \frac{1}{2} + 0)$

$$f_{2\mu,2\lambda}(\Theta,\phi) = e^{i(\lambda-\mu)\phi} \, \Sigma(2J+1) \, d^J_{\lambda\mu}(\Theta) \, \frac{\langle 2\mu|T^J(\varepsilon)|2\lambda\rangle}{2\,k} \tag{10}$$

where $\lambda = (\lambda_a - \lambda_b)$, $\mu = (\lambda_c - \lambda_d)$ and (Θ,ϕ) represent polar angles of nucleon momentum.

Thus

$$f_{++}(\Theta,\phi) = \sum_J (2J+1) \, d^J_{\frac{1}{2}\frac{1}{2}} (\Theta) \, f^J_{++}(\varepsilon)$$

$$f_{+-}(\Theta,\phi) = e^{-i\phi} \sum_J (2J+1) \, d^J_{-\frac{1}{2}\frac{1}{2}} (\Theta) \, f^J_{+-}(\varepsilon) \tag{11}$$

or

$$f_{++}(\Theta,\phi) = \sum_J \cos\frac{\Theta}{2} \left[P'_{J+\frac{1}{2}} (\cos\Theta) - P'_{J-\frac{1}{2}} (\cos\Theta)\right] (f_{J+} + f_{J-})$$

$$f_{+-}(\Theta,\phi) = \sum_J e^{-i\phi} \sin\frac{\Theta}{2} \left[P'_{J+\frac{1}{2}}(\cos\Theta) + P'_{J-\frac{1}{2}} (\cos\frac{\Theta}{2})\right] (f_{J-} - f_{J+}) \tag{12}$$

on using the explicity expressions for $d^J(\Theta)$.

Parity conservation condition implies

$$f_{--}(\Theta,\phi) = f_{++}(\Theta,\phi) \tag{13}$$

$$f_{-+}(\Theta,\phi) = e^{i\phi} \, \Sigma(2J+1) \, d^J_{\frac{1}{2},-\frac{1}{2}}(\Theta) \, f^J_{+-}(\varepsilon) = -\,e^{2i\phi} \, f_{+-}(\Theta,\phi)$$

$$= e^{-i\phi} \, \Sigma(2J+1) \, d^J_{-\frac{1}{2},\frac{1}{2}} (\Theta) \, f^J_{+-}(\varepsilon) \tag{14}$$

on using $d_{m'm}(\Theta) = (-1)^{m'-m}\, d_{mm'}(\Theta)$, and $f^J_{+-} = f^J_{-+}$.

f_{++} and f_{+-} are referred to as helicity non-flip and helicity-flip amplitudes. Clearly $f_{++}(\Theta = \pi, \phi) = 0$ and $f_{+-}(\Theta = 0, \phi) = 0$ which also follows from angular momentum conservation considerations.

From orthogonality of $d(\Theta)$ matrices we may derive

$$
\begin{aligned}
f^J_{++}(\varepsilon) &= \frac{1}{4\pi}\int f_{++}(\Theta)\, d^J_{\frac{1}{2}\frac{1}{2}}(\Theta)\, d\Omega \\
f^J_{+-}(\varepsilon) &= \frac{1}{4\pi}\int f_{+-}(\Theta,\phi)\, e^{i\phi}\, d^J_{-\frac{1}{2}\frac{1}{2}}(\Theta)\, d\Omega
\end{aligned}
\tag{15}
$$

The formulae for the unpolarized differential cross section and the polarization are given by

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= |f_{++}|^2 + |f_{+-}|^2 \\
\vec{P}(\Theta) &= \frac{2\,\mathrm{Im}\,(f_{++}\, f^*_{+-})}{|f_{++}|^2 + |f_{+-}|^2}\,\vec{n}
\end{aligned}
\tag{16}
$$

where $\vec{n} = \dfrac{\vec{q} \times \vec{q}'}{|\vec{q} \times \vec{q}'|}$ and $\vec{q}$ and $\vec{q}'$ refer to the momentum of incoming and outgoing nucleons.

In order to connect f_{++}, f_{+-} with other amplitudes in use in literature we have to introduce explicit wave function for particles involved. For the spinless pion the wave function in coordinate representation is simply

$$
\langle \vec{r}|\vec{k}\rangle = \sqrt{2k^0}\, e^{i\vec{k}\cdot\vec{r}}
\tag{17}
$$

k^μ being the energy and momentum of pion. For spin S = 1/2 nucleon state with momentum p and helicity λ the wave function is

$$< \vec{r} \left[m, \frac{1}{2} \right] \vec{p}\ \lambda > = e^{i\vec{p}\cdot\vec{r}}\ \mathcal{U}_\lambda(\vec{p}) \tag{18}$$

where $\mathcal{U}_\lambda(\vec{p})$ is the Dirac 4-spinor corresponding to four momentum p^μ and helicity λ. The helicity operator can be shown to be given by *

$$h(\vec{p}) = \frac{i}{2}\gamma_5(\gamma\cdot e^L(p)) \tag{19}$$

where $e^L_\mu(p) = \left(\frac{E_p}{m}\ \frac{\vec{p}}{p}\ ,\ i\ \frac{|\vec{p}|}{m} \right)$ and γ's are the 4 x 4 Dirac matrices and $\mu = 1, 2, 3, 4$.

The helicity operator reduces to $h(p) = \frac{\vec{\Sigma}\cdot\vec{p}}{2|\vec{p}|}$ for $E > 0$ and $h(p) = -\frac{\vec{\Sigma}\cdot\vec{p}}{2|\vec{p}|}$ for $E < 0$ states; where $\frac{1}{2}\vec{\Sigma}$ is the spin operator. It also commutes with $(\gamma\cdot p)$. In the Dirac - Pauli representation of the gamma matrices two linearly independent plane wave spinors satisfying Dirac equation $(\not{p} - m) = 0$ are given by $(E > 0)$:

$$\mathcal{U}_i(\vec{p}) = \sqrt{\frac{2m}{m+E_p}} \left(\frac{m + \not{p}}{2m} \right) \mathcal{U}_i(0) \qquad i = 1, 2 \tag{20}$$

where $\mathcal{U}_i(0) = \mathcal{U}_i(\vec{p}=0)$ satisfy $\gamma_0\,\mathcal{U}_i(0) = \mathcal{U}_i(0)$ has the form $\mathcal{U}_i(0) = \begin{pmatrix} \chi_i \\ 0 \end{pmatrix}$. The helicity states with the phase convention adopted by us can be constructed as follows. For the two linearly independent two spinors χ_i, we choose them as eigenstates of $\frac{1}{2}\sigma_3$ viz $\chi_{\frac{1}{2}} = \binom{1}{0}$ and $\chi_{-\frac{1}{2}} = \binom{0}{1}$ so

* See for example, J. Sakurai, Advanced Quantum Mechanics or Lurie, Particles and Fields.

that the corresponding rest frame states $\mathcal{U}_\lambda(0) = \binom{\chi_\lambda}{0}$, $\lambda = \pm\frac{1}{2}$, are eigenstates of the $\frac{1}{2}\Sigma_3$, the third component of the spin oeprator. The states

$$\mathcal{U}_\lambda(\vec{p}_k \equiv (p^0, 0, 0, p)) = \sqrt{\frac{2m}{m+E_p}}\left(\frac{m+\not{p}_R}{2m}\right)\mathcal{U}_\lambda(0) \qquad (21)$$

are clearly helicity states for a particle moving along 3-axis. The helicity states for particle moving along $\vec{p}(\Theta,\phi)$ is then obtained by applying a rotation

$$\mathcal{U}_\lambda(\vec{p}) = \mathcal{R}(\phi,\Theta,-\phi)\,\mathcal{U}_\lambda(\vec{p}_R)$$

$$= \sqrt{\frac{2m}{m+E_p}}\left(\frac{m+\not{p}}{2m}\right)\mathcal{R}(\phi,\Theta,-\phi)\,\mathcal{U}_\lambda(0) \qquad (22)$$

where

$$\mathcal{R}(\phi,\Theta,-\phi)\,\mathcal{U}_\lambda(0) = \begin{pmatrix}\mathcal{R}(\phi,\Theta,-\phi)\chi_\lambda \\ 0\end{pmatrix} \qquad (23)$$

and $\mathcal{R}(\phi,\Theta,-\phi)\,\chi_\lambda = e^{-i\sigma_3\phi/2}\,e^{-i\sigma_2\Theta/2}\,e^{i\sigma_3\phi/2}\,\chi_\lambda$. Also $\mathcal{U}_\lambda(|\vec{p}| \to 0) = \mathcal{R}(\phi,\Theta,-\phi)\,\mathcal{U}_\lambda(0)$ are eigenstates of $\frac{1}{2}(\vec{\Sigma}\cdot\vec{n})$ where $\vec{n} = \vec{n}(\Theta,\phi)$ while $\mathcal{R}\chi_\lambda$ are eigenstates of $\frac{1}{2}\vec{\sigma}\cdot\vec{n}$. We easily show

$$\begin{aligned}\mathcal{R}(\phi,\Theta,-\phi)\chi_{\frac{1}{2}} &= \cos\frac{\Theta}{2}\,\chi_{1/2} + e^{i\phi}\sin\frac{\Theta}{2}\,\chi_{-\frac{1}{2}} \\ \mathcal{R}(\phi,\Theta,-\phi)\,\chi_{-\frac{1}{2}} &= -\left(\sin\frac{\Theta}{2}\right)e^{-i\phi}\,\chi_{1/2} + \cos\frac{\Theta}{2}\,\chi_{-\frac{1}{2}}\end{aligned} \qquad (24)$$

These may be termed rest (helicity) states corresponding to particle moving along (Θ,ϕ).

For the process $N + \pi \to N + \pi$ the scattering amplitude clearly takes the form

$$\bar{\mathcal{U}}_f(q')\,\mathbb{M}\,\mathcal{U}_i(q)$$

In the Dirac-Pauli representation we can reduce it to the form

$$\chi_f^{\dagger} M \chi_i$$

where $\chi_{f,i}$ are the Pauli 2-spinors representing spin states and $M(q,q')$ is 2 x 2 matrix in spin space *. The most general form of M consistent with rotation invariance and parity conservation in our case is

$$M = f_1 I + f_2 (\vec{\sigma}\cdot\vec{q}')(\vec{\sigma}\cdot\vec{q})/|\vec{q}||\vec{q}'| \quad (26)$$

where $\vec{q}$ is the initial nucleon momentum and $\vec{q}'$ that of the final nucleon. To relate f_1 and f_2 to helicity amplitudes f_{++} and f_{+-} we calculate the scattering amplitude between helicity states. Taking $\vec{q}$ along 3-axis and $\vec{q}'$ along (Θ,ϕ) we have

$$f_{++}(\Theta,\phi) = (\mathcal{R}\chi_{1/2})^{\dagger} M \chi_{1/2} = (f_1 + f_2) \cos\frac{\Theta}{2}$$

$$f_{+-}(\Theta,\phi) = (\mathcal{R}\chi_{1/2})^{\dagger} M \chi_{-\frac{1}{2}} = (f_1 - f_2) e^{-i\phi} \sin\frac{\Theta}{2} \quad (27)$$

where we used $(\vec{\sigma}\cdot\vec{q}')(\mathcal{R}\chi_{\pm\frac{1}{2}}) = \pm |\vec{q}'|(\mathcal{R}\chi_{\pm\frac{1}{2}})$ and $(\vec{\sigma}\cdot\vec{q})\chi_\lambda = \lambda|\vec{q}|\chi_\lambda$.
Frequently, M is written as

$$M = f I + i g \vec{\sigma}\cdot\vec{n} \quad (28)$$

where $\vec{n} = \dfrac{\vec{q} \times \vec{q}'}{|\vec{q} \times \vec{q}'|}$. We find easily

$$f = f_1 + f_2 \cos\Theta \quad (29)$$

$$g = - f_2 \sin\Theta$$

---

* Compare with discussion in section 7.3 on Density matrix.

The expressions for cross section and polarization become

$$\frac{d\sigma}{d\Omega} = |f|^2 + |g|^2$$

$$\vec{P} = - \frac{2 \text{ Im } (f^* g)}{|f|^2 + |g|^2} \, n$$

and the partial wave expansions,

$$f = \sum_{\ell=0}^{\infty} \left[(\ell+1) f_{\ell+} + \ell f_{\ell-}\right] P_\ell(\cos\Theta)$$

$$g = \sum_{\ell=1}^{\infty} \left[f_{\ell+} - f_{\ell-}\right] P_\ell'(\cos\Theta) \sin\Theta \tag{30}$$

Here we denote partial wave amplitude by $f_{\ell\pm} \equiv f_{J\pm}$ corresponding to $J = (\ell \pm \frac{1}{2})$. The inverse relations are

$$f_{\ell\pm}(\varepsilon) = \frac{1}{2} \int_{-1}^{1} \left[f_1 \, P_\ell(\cos\Theta) + f_2 \, P_{\ell\pm 1} (\cos\Theta)\right] d(\cos\Theta) \tag{31}$$

Another common notation for the partial wave amplitudes is $L_{2I,2J}$ where I is isospin of (πN) system e.g. S_{11}, P_{11}, S_{31}, $P_{3.}$, P_{33}... partial waves.

Invariant scattering amplitudes A and B are also in use. They are defined by *

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^4 (p_f - p_i) 2m \, \bar{u}_f(q') \, T \, u_i(q)$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{16 \pi^2 W^2} \sum_{\text{spins}} |\bar{u}_f \, T \, u_i|^2 \tag{32}$$

where T can be expressed, in its most general form in terms of two invariant amplitudes A and B

* See for example: Eden, J. High Energy Collisions

$$T = A(s,t) - \frac{i}{2} B(s,t) \gamma_\mu (q + q')^\mu \qquad (33)$$

In c.m. frame we can reduce $\bar{\mathcal{U}}_f \, T \, \mathcal{U}_i$ to the form

$$\chi_f^\dagger M \chi_i \qquad (34)$$

We can then show

$$f_1 = \frac{E + m}{8\pi W} \left[A + (W - m)B \right]$$

$$f_2 = \frac{E - m}{8\pi W} \left[-A + (W + m)B \right] \qquad (35)$$

where E is the energy of nucleon, W the c.m. energy and m the nuclear mass.

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