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XXVII

# HOLOMORPHIC MAPPINGS AND DOMAINS OF HOLOMORPHY

by

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ABSTRACT

Three aspects of the theory of holomorphic functions in the infinite dimensional case are considered. The first aspect is Zorn's characterization theorem for holomorphic mappings between Banach spaces. This theorem is established for a mapping from an open subset of a Hausdorff Baire complex topological vector space into a Banach space. This fact is used to prove other theorems, including a generalization of a classical Hartogs theorem concerning separately holomorphic functions. Domains and envelopes of holomorphy form the second subject of this study. The results of Alexander's thesis are generalized for Riemann domains over a complex locally convex space for which the closed convex hull of every compact subset is compact. The existence and characterization of the envelope of holomorphy of a Riemann domain over $\mathbb{C}^{\mathbb{N}}$ are established. A Cartan-Thullen theorem is proved for the so-called domains of $\tau$-holomorphy. The last aspect considered is the Malgrange-Gupta approximation theorem. If E is a complex locally convex space having a Fréchet space as its strong dual, the space $\mathbb{H}_{Nb}(E)$ is introduced and the Malgrange-Gupta theorem is proved.

INTRODUCTION

This work is concerned with the study of some aspects of the theory of holomorphic mappings in the non-finite dimensional case.

The first aspect considered is Zorn's characterization of a holomorphic mapping between Banach spaces. See ref. 3 and 4. In Chapter I, Zorn's theorem is established for a mapping f from an open subset U of a complex Hausdorff Baire topological vector space into a Banach space F: f is holomorphic in U if and only if f is B-continuous and G-holomorphic in U. This fact is used to prove other theorems, including a generalization of a classical Hartogs theorem concerning separately holomorphic functions. 2, 16 and 17 are references to topics related to these matters.

Domains and envelopes of holomorphy form the second subject of this study. In Chapter II, Alexander's results (see ref. 5) are generalized for Riemann domains over complex locally convex spaces for which the closed convex hull of every compact subset is compact. In this case, the topology of the uniform convergence over compactsets is replaced by the topology generated by the seminorms ported by compact sets. In Chapter III, the existence and characterization of the envelope of holomorphy of a Riemann domain over $\mathbb{C}^{\mathbb{N}}$ are established. In Chapter IV, a Cartan Thullen theorem is proved for the so-called domains of $\tau$- holomorphy. 12, 18, 11, 19, 20 and 21 are works related to the preceding matters.

The last aspect considered is the Malgrange-Gupta approximation theorem (see ref. 14). If E is a complex locally convex space having a Fréchet space as its strong dual, the space $\mathcal{H}_{Nb}(E)$ is introduced and the Malgrange-Gupta theorem is proved in Chapter V. Other works related to this topic are 22, 23, 24, 25 and 26.

# CHAPTER I

# HOLOMORPHIC MAPPINGS

## 1. Holomorphic Mappings Between Topological Vector Spaces

All topological vector spaces we consider are complex and Hausdorff unless the contrary is stated explicitly.

The notation is the same as the one used by Leopoldo Nachbin in reference 1.

Let f be a mapping from an open subset U of a topological vector space E into another topological vector space F. f is holomorphic at a point x of U if there is a sequence of elements $P_n \in \mathcal{P}(^nE; F)$, $n = 0, 1, 2, \ldots$ such that

$$f(x+h) = \sum_{n=0}^{\infty} P_n(h)$$

the series converging uniformly for h in a neighbourhood of zero in E. The sequence $(P_n)_{n=0}^{\infty}$ is unique and each $P_n(h)$ will be denoted by $\frac{1}{n!}\hat{d}^n f(x)(h)$. If $T_n \in \mathcal{L}(^nE;F)$ defines $P_n$, it will be denoted by $\frac{1}{n!} d^n f(x)$. f is holomorphic in U if it is holomorphic at each one of the points of U. f is G-holomorphic in U if, for each x in U and y in E, the mapping $\lambda \in \{\lambda \in \mathbb{C};\ x+\lambda y \in U\} \longmapsto f(x+\lambda y) \in F$ is holomorphic in its domain of definition.

Proposition 1.1 - Let F be locally convex and such that the closed convex hull of every compact subset is compact. If f is G-holomorphic in U, then, for each x in U, there is a sequence of elements $P_n \in \mathcal{P}_a(^nE; F)$, $n = 0, 1, 2, \ldots$, such that

$$f(x+h) = \sum_{n=0}^{\infty} P_n(h)$$

for every x+h in the largest x-equilibrated open neighbourhood of x in E contained in U. The sequence $(P_n)_{n=0}^{\infty}$ is unique and

$$P_n(h) = \frac{1}{2\pi i} \int_{|\xi|=1} f(x+\xi h)\xi^{-n-1}\, d\xi$$

for every h in E as above and n = 0, 1,... If we take the mapping $x \in U \mapsto \delta_x^h f = \delta f(x;h) = \lim_{\lambda \to 0} \lambda^{-1} [f(x+\lambda h) - f(x)] \in F$ we see that it is well defined and G-holomorphic in U for each h in E. With the notations $\delta^0 f(x;h) = f(x)$, $\delta^n f(x; h_1, h_2, h_3, \ldots h_n) = \delta_x^{h_n} \delta_x^{h_{n-1}} \ldots \delta_x^{h_1} f$ and $\delta^n f(x;h) = \delta^n f(x;h, h, \ldots, h)$ for all h, $h_1$, $h_2$, ...$h_n$ in E, n = 1, 2, ..., x in U, it follows that $P_n(h)$ is equal to $(n!)^{-1}\, \delta^n f(x;h)$ for all h in E and n = 0, 1, 2, ...

f is <u>locally bounded in</u> U if each x in U has a neighbourhood V such that f(V) is a bounded subset of F.

<u>Proposition 1.2</u> - If f is a Banach space, the following conditions are equivalent:

(a) f is holomorphic in U;

(b) f is G-holomorphic and continuous in U;

(c) f is G-holomorphic and locally bounded in U.

Proofs of the above results may be found in ref. 2.

## 2. <u>Holomorphic Mappings Defined in Baire Topological Vector Spaces</u>

A mapping f from a topological space X into another topological space Y is <u>B-continuous</u> if it is continuous everywhere except at the points of a first category subset of X.

Proposition 2.1 - If $(f_n)_{n=0}^{\infty}$ is a sequence of B-continuous mappings from a topological space X into a Banach space F such that $\lim_{n\to\infty} f_n(x) = f(x)$ exists for each $x \in X$, then f is B-continuous in X.

Proof - It is enough to show the preposition for a sequence $(f_n)_{n=0}^{\infty}$ of continuous mappings from X into F. For $\varepsilon > 0$ and $m = 1, 2, \ldots$ let $B_m(\varepsilon) = \{t \in X; \|f_m(t) - f(t)\| \leq \varepsilon\}$ and $G(\varepsilon) = \bigcup_{m=1}^{\infty} \text{Int } B_m(\varepsilon)$. We show that $C = \bigcap_{n=1}^{\infty} G(1/n)$ is the set of all points of X where f is continuous. Let f be continuous at $t_o$. There is m such that $\|f(t_o) - f_m(t_o)\| < \varepsilon/3$. From the continuity of f and $f_m$ at $t_o$ we get an open neighbourhood $U(t_o)$ of $t_o$ such that $\|f(t)-f(t_o)\| < \varepsilon/3$ and $\|f_m(t) - f_m(t_o)\| < \varepsilon/3$ for every t in $U(t_o)$. Thus $U(t_o)$ is contained in the interior of $B_m(\varepsilon)$ and $t_o \in G(\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, $t_o$ is in C. On the other hand, if $t_o \in C$, $t_o$ is in $G(\varepsilon/3)$ for every $\varepsilon > 0$. Hence there is m such that $t_o \in \text{Int } B_m(\varepsilon/3)$ and, therefore, $\|f(t) - f_m(t)\| < \varepsilon/3$ for each t in some neighbourhood $U(t_o)$ of $t_o$. By the continuity of $f_m$ and the arbitrariety of $\varepsilon > 0$ it follows that f is continuous at $t_o$. Now we take the following set $F_m(\varepsilon) = \{t \in X; \|f_m(t) - f_{m+k}(t)\| \leq \varepsilon, k = 1, 2, \ldots\}$ which is closed in X. Since $\lim_{m\to\infty} f_m(t) = f(t)$ exists for all t in X, we have $X = \bigcup_{m=1}^{\infty} F_m(\varepsilon)$. Furthermore $F_m(\varepsilon) \subset B_m(\varepsilon)$ and $\text{Int } F_m(\varepsilon) \subset \text{Int } B_m(\varepsilon)$. Thus $\bigcup_{m=1}^{\infty} \text{Int } F_m(\varepsilon) \subset G(\varepsilon)$. Hence we have $X - \bigcup_{m=1}^{\infty} \text{Int } F_m(\varepsilon) \subset \bigcup_{m=1}^{\infty} [F_m(\varepsilon) - \text{Int } F_m(\varepsilon)]$ which is a first category set. It follows that $X - C = X - \bigcap_{n=1}^{\infty} G(1/n) = \bigcup_{n=1}^{\infty} [X - G(1/n)]$ is a first category set and coincides with the collection of all points of X where f is not continuous.

Proposition 2.2 - Let f be a B-continuous and G-holomorphic mapping from an open subset U of a topological vector space E into a Banach space F. For each x in U and each $n = 0, 1, 2, 3, \ldots$ the mapping $h \in E \longmapsto \delta^n f(x;h) \in F$ is B-continuous. If E is a Baire space it is continuous.

Proof - Note that

$$\delta^n f(x;h) = \lim_{\lambda\to 0} \lambda^{-n} \left[\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(x+j\lambda h)\right]$$

for all n = 0, 1, 2, ..., h $\varepsilon$ E and x $\varepsilon$ U. To prove this it is enough to write the Taylor series of $g(\xi) = f(x + \xi h)$ at 0 for each term of the sum, to reorder the sum taking together the homogeneous polynomials of the same degree, and to pass to the limit as $\lambda$ goes to zero.

For all x in U, h in E and n = 0, 1, 2, ..., we can write

$$\delta^n f(x;h) = \lim_{m\to\infty} m^n \left[\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \tilde{f}(x+jh/m)\right]$$

where $\tilde{f}$ is equal to f in U and equal to 0 in E - U. Since for all x in U, n = = 0, 1, 2, ..., m = 1, 2, ..., the mapping $h \in E \longmapsto m^n\left[\sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} \tilde{f}(x+jh/m)\right] \in F$ is B-continuous, Proposition 2.1 implies that $h \in E \longmapsto \delta^n f(x;h) \in F$ is B-continuous for each x in U and n = 0, 1, 2, ...

Theorem 2.1 - Let f be a mapping from an open subset U of a Baire topological vector space E into a Banach space F. f is holomorphic in U if and only if f is G-holomorphic and B-continuous in U.

We need the following two lemmas whose proofs are in ref. 2.

Lemma 2.1 - Let $(f_n)_{n=0}^{\infty}$ be a sequence of continuous mappings from an open subset U of a Baire space E into a normed space F having a limit f(x) for all x in U. The set of all points of U where $(f_n)_{n=0}^{\infty}$ is locally bounded is open and dense in U.

Lemma 2.2 - Let E be a topological vector space and F a locally convex space. If $P \in \mathcal{P}_a({}^nE;F)$ is bounded in $x_o+A$, where $x_o$ is a point of E and A is a balanced subset of E, then P is bounded in $A_o = \bigcup_{|\lambda|=1} \{\lambda x_o+A\}$. More precisely, if q is a continuous seminorm in F and $\sup\{q(P(x_o+a));\ a \in A\} \leq K_q$, then $\sup\{q(P(a));\ a \in A_o\} \leq K_q$.

Proof of Theorem 2.1 - Let f be G-holomorphic and B-continuous in U. By Proposition 1.1, for x in U and h in the largest open balanced neighbourhood V of 0 in E such that $x + V \subset U$,

$$f(x+h) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n f(x;h) .$$

If $P_n(h) = \frac{1}{n!} \delta^n f(x;h)$ for all h in E, $P_n$ is in $\mathcal{P}(^nE;F)$ by Proposition 2.2. Since $\lim_{n\to\infty} P_n(h) = 0$ for each $h \in V$, Lemma 2.1 applies and the set of all points of V where $(P_n)_{n=0}^{\infty}$ is locally bounded is open and dense in V. It follows that there are $x_o$ in V and an open balanced neighbourhood A of zero in E such that $x_o + A \subset V$ and $\sup\{||P_n(x_o+a)||;\ a \in A,\ n \in \mathbb{N}\} \leq K < +\infty$. Thus, by Lemma 2.2, $\sup\{||P_n(a)||;\ a \in A_o,\ n \in \mathbb{N}\} \leq K$. For $0 < \gamma < 1$ and h in $\gamma A_o$

$$\left|\left|f(x+h) - \sum_{j=0}^{N} \frac{1}{j!} \delta^j f(x;h)\right|\right| \leq \left|\left| \sum_{j>N} \frac{1}{j!} \delta^j f(x;h)\right|\right| \leq \sum_{j>N}\left|\left| P_j(h)\right|\right| \leq K \sum_{j>N} \gamma^j$$

Hence f is holomorphic at x since the above inequality implies that $\sum_{n=0}^{\infty} P_n(h)$ converges uniformly to f(x+h) in $\gamma A_o$.

Theorem 2.2 - Let x be a point of a Baire topological vector space E and A a balanced open neighbourhood of 0 in E. If f is a G-holomorphic mapping from x+A into a Banach space F and f is locally bounded at a point $x_o$ of x+A, then f is holomorphic in x+A.

Proof - There is no loss of generality in supposing x=0. If $y \in A$, $A_y$ denotes the largest balanced neighbourhood of zero in E such that $y+A_y \subset A$. The set L' of all points of A where f is locally bounded is non-empty. If y is in L'

$$f(y+h) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n f(y;h)$$

for all $h \in A_y$. Since f is bounded in a neighbourhood of the point y, the polynomial $P_n(h) = \frac{1}{n!} \delta^n f(y;h)$ is bounded in a neighbourhood of 0 for all n = 0,1,... Hence $P_n \in \mathcal{P}(^nE;F)$ for n = 0, 1, ... Proposition 2.1 implies that f is B-continuous in $y + A_y$. Thus, by Theorem 2.1, f is holomorphic in $y + A_y$. Hence L' is open in A. If L" = A-L', the above argument gives $\{y \in A;\ z - y \in A_y\} \cap L' = \emptyset$ for all z in L". Since $\{y \in A;\ z - y \in A_y\}$ is open in A and z is in it, L" is open in A. Therefore, as A is connected, A=L' and f is holomorphic in A.

<u>Corollary 2.1</u> - Let f be a G-holomorphic function from an open connected subset U of a Baire topological vector space E into a Banach space F. If f is locally bounded at one point x of U, it is holomorphic in U.

<u>Corollary 2.2</u> - Let f be a G-holomorphic function from an open subset U of a Baire topological vector space E into a Banach space E. The set of all points where f is locally bounded is open and closed in U.

The above results may be used to prove the following theorem:

<u>Theorem 2.3</u> - Let $E_1$ and $E_2$ be two Baire topological vector spaces with $E_1$ metrizable and $E_1 \times E_2$ a Baire space. Let f be a mapping from an open subset U of $E_1 \times E_2$ into a Banach space F. f is holomorphic in U if and only if it is separately holomorphic in each variable.

We need the following result proved in ref. 2.

<u>Lemma 2.3</u> - Under the conditions of Theorem 2.3 if f is separately continuous in U, then every open subset W of U contains an open set where f is bounded.

<u>Proof of Theorem 2.3</u> - If f is separately holomorphic in U, then the mapping $(\lambda, \xi) \in \{(\lambda,\xi) \in \mathbb{C}^2 ; (x_1 + \lambda x_1', x_2 + \xi x_2') \in U\} \longmapsto f(x_1+\lambda x_1', x_2+\xi x_2') \in F$ is separately holomorphic for each $(x_1, x_2)$ in U and each $(x_1', x_2)$ in $E_1 \times E_2$. By the classical Hartogs theorem it is holomorphic in its domain of definition. Thus the mapping $\lambda \in \{\lambda \in \mathbb{C} ; (x_1+\lambda x_1', x_2+\lambda x_2') \in U\} \longmapsto f(x_1+\lambda x_1', x_2+\lambda x_2') \in F$ is holomorphic for each $(x_1, x_2) \in U$ and $(x_1', x_2') \in E_1 \times E_2$ and f is G-holomorphic in U. By Lemma 2.3 and Corollary 2.1 f is holomorphic in U.

<u>Remarks</u>

(1) Let us recall the following result proved in ref. 2: "A mapping f from an open subset U of a metrizable topological vector space E into a locally convex space F is G-holomorphic and continuous in U if and only if $T \circ f$ is holomorphic in U for every T in E'." Hence, if we understand holomorphy as G-holomorphy and continuity and if, in Theorem 2.1, Theorem 2.2, Corollary 2.1, Corollary 2.2, Theorem 2.3, we consider E, $E_1$, $E_2$ metrizable, then we may take F locally convex and all the results will still be true when we replace the words "locally bounded" by the word "continuous" in Corollary 2.2.

(2) The results of this section were proved by Zorn in ref. 3 and 4 for functions defined in open subsets of a Banach space.

(3) Corollary 2.2 and Theorem 2.3 were proved by Noverraz in ref. 2 for E, $E_1$ and $E_2$ metrizable complete.

## 3. Holomorphic Complex Mappings in a Riemann Domain

In this section it is considered the concept and some properties of a holomorphic function defined in a Riemann domain over a locally convex space with complex values. They will be used in Chapter II.

Let E be a locally convex space. A pair $(\mathcal{U}, \phi)$ is a Riemann domain over E if $\mathcal{U}$ is a connected Hausdorff topological space and $\phi$ is a local homeomorphism from $\mathcal{U}$ into E. A mapping f from $\mathcal{U}$ into $\mathbb{C}$ is holomorphic in $\mathcal{U}$ if $f \circ [\phi|\omega]^{-1}$ is a holomorphic in $\phi(\omega)$ for every open subset $\omega$ of $\mathcal{U}$ where $\phi$ is a homeomorphism.

Let $(\mathcal{U}, \phi)$ be a Riemann domain over E.

Notations

(i) $\mathcal{H}(\mathcal{U})$ denotes the algebra of all holomorphic complex mappings in $\mathcal{U}$.

(ii) If f is a holomorphic mapping from $\mathcal{U}$ into $\mathbb{C}$ and $u \in \omega \subset \mathcal{U}$, $\omega$ being an open set where $\phi$ is a homeomorphism, there is a sequence $(P_n)_{n=0}^{\infty}$ of elements $P_n \in \mathcal{P}(^nE)$ and an open convex balanced neighbourhood U of zero in E such that

$$f \circ [\phi|\omega]^{-1} (\phi(u) + h) = \sum_{n=0}^{\infty} P_n(h) \qquad (1)$$

the series converging uniformly for h in U. The sequence $(P_n)_{n=0}$ is unique and each $P_n$ is denoted by $\frac{1}{n!} \hat{d}^n f(u)$. If $T \in \mathcal{L}(^nE)$ defines $P_n$ it is denoted by $\frac{1}{n!} d^n f(u)$.

(iii) If $a \in \mathcal{U}$ and $A \subset E$, a + A means that

$$a + A = [\phi|\omega]^{-1} (\phi(a) + A)$$

$\omega$ being an open subset of $\mathcal{U}$ where $\phi$ is a homeomorphism and $a \in \mathcal{W}$ and $\phi(a) + A \subset \phi(\omega)$. If $A = \{h\}$, $a + \{h\}$ is denoted by a+h. If $\mathcal{B} \subset \mathcal{U}$ then $\mathcal{B} + A = \bigcup_{b \in \mathcal{B}} (b+A)$ where b+A have the meaning just stated.

(iv) From (ii) and (iii) it follows that (1) may be written

$$f(u+h) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{d}^n f(u)(h) \tag{2}$$

Proposition 3.1 - If f is a holomorphic mapping from $\mathcal{U}$ into $\mathbb{C}$ and $h \in E$, then the mapping $\hat{d}^n f(\ )(h): u \in \mathcal{U} \longmapsto \hat{d}^n f(u)(h) \in \mathbb{C}$ is holomorphic in $\mathcal{U}$, n = = 0, 1, ...

Proposition 3.2 - If f is a holomorphic mapping from $\mathcal{U}$ into C, then:

(1) $\hat{d}[\hat{d}f(\ )(v)](\ )(u) = \hat{d}\left[\hat{d}f(\ )(u)\right](\ )(v)$

(2) $\hat{d}^n f(\ )(u+v) = \sum_{s+t=n} \frac{n!}{s!t!} \hat{d}^s\left[\hat{d}^t f(\ )(v)\right](\ )(u)$

for all u, v in E and n = 0, 1, 2, ...

Proposition 3.3 - Let f be a holomorphic mapping from $\mathcal{U}$ into $\mathbb{C}$, $x \in \mathcal{U}$ and V a balanced convex open neighbourhood of zero in E such that $x + V \subset \mathcal{U}$. Then

$$f(x+v) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{d}^n f(x)(v)$$

for all  in V.

Proposition 3.4 - If f and g are in $\mathcal{H}(\mathcal{U})$, then f.g is in $\mathcal{H}(\mathcal{U})$ and it is true that

$$\hat{d}^n(f.g)(u)(h) = \sum_{s+t=n} \frac{n!}{s!t!} \left[\hat{d}^s f(u)(h)\right]\left[\hat{d}^t g(u)(h)\right]$$

for all u in $\mathcal{U}$ and h in E.

Proposition 3.5 - If f is a holomorphic mapping from $\mathcal{U}$ into $\mathbb{C}$ and g is an entire complex function in $\mathbb{C}$, the mapping $g \circ f$ is holomorphic in $\mathcal{U}$.

# CHAPTER II

## EXTENSIONS OF HOLOMORPHIC FUNCTIONS

### 1. The Spectrum

Let E be a locally convex space such that the closed convex hull of every compact subset is compact. If $(\mathcal{U}, \phi)$ is a Riemann domain over E, we consider in $\mathcal{H}(\mathcal{U})$ the topology $\tau_{\omega_a}$ generated by all seminorms p such that

(1) p is ported by some compact subset $\mathcal{K}$ of $\mathcal{U}$, that is: for every open subset $\mathcal{V}$ of $\mathcal{U}$ containing $\mathcal{K}$ there is $c(\mathcal{V}) > 0$ such that

$$p(f) \leqslant c(\mathcal{V}) \sup_{v \in \mathcal{V}} |f(v)|$$

for all f in $\mathcal{H}(\mathcal{U})$;

(2) $p(fg) \leqslant p(f)p(g)$, for all f and g in $\mathcal{H}(\mathcal{U})$.

With this topology $\mathcal{H}(\mathcal{U})$ is an m-convex topological algebra. The collection $S(\mathcal{U})$ of all continuous homomorphisms from $\mathcal{H}(\mathcal{U})$ onto $\mathbb{C}$ is called the spectrum of $\mathcal{H}(\mathcal{U})$. For every $h \in S(\mathcal{U})$ there is a compact subset $\mathcal{K}$ of $\mathcal{U}$ such that

$$\left|h(f)\right| \leqslant \sup_{k \in \mathcal{K}} \left|f(k)\right|$$

for all f in $\mathcal{H}(\mathcal{U})$. This fact is denoted $h \,\dashv\, \mathcal{K}$.

Proposition 1.1 - For every $h \in S(\mathcal{U})$ there is a unique $a_h$ in E such that $T(a_h) = h(T \circ \phi)$ for all T in the topological dual E' of E.

We need the following lemma:

Lemma 1.1 - If $T = (T_1, T_2, \ldots, T_n)$ is a continuous linear mapping from E into $\mathbb{C}^n$ and G is a linear mapping from $\mathbb{C}^n$ into $\mathbb{C}$, then the mapping $f\colon u \in \mathcal{U} \longmapsto f(u) = \exp G(T \circ \phi(u)) \in \mathbb{C}$ is such that h(f) is equal to $\exp G(h(T_1 \circ \phi), h(T_2 \circ \phi), \ldots, h(T_n \circ \phi))$ for all h in $S(\mathcal{U})$.

Proof - By Proposition 3.5 from Chapter I f is in $\mathcal{H}(\mathcal{U})$. It is known that

$$\exp G(T(x)) = \sum_{n=0}^{\infty} \frac{1}{n!} \Big[G(T_1(x), T_2(x), \ldots, T_n(x))\Big]^n$$

uniformly for x in some neighbourhood of every compact subset of E. If a $\tau_{\omega_a}$-continuous seminorm p in $\mathcal{H}(\mathcal{U})$ is ported by a compact subset $\mathcal{K}$ of $\mathcal{U}$, there is an open subset V of $\phi(\mathcal{U})$ containing $\phi(\mathcal{K})$ such that

$$\exp G(T \circ \phi(u)) = \sum_{n=0}^{\infty} \frac{1}{n!} \Big[G(T_1 \circ \phi(u), \ldots, T_n \circ \phi(u))\Big]^n$$

uniformly in $\phi^{-1}(V) \supset \mathcal{K}$. It follows that the sequence of the partial sums of the above series converges to f for the $\tau_{\omega_a}$-topology. Thus

$$h\left[\sum_{n=0}^{N} \frac{1}{n!} \Big[G(T_1 \circ \phi, \ldots, T_n \circ \phi)\Big]^n\right] = \sum_{n=0}^{N} \frac{1}{n!} \Big[G(h(T_1 \circ \phi), \ldots, h(T_n \circ \phi))\Big]^n$$

converges to h(f) as N goes to $\infty$ and the lemma follows.

Proof of Proposition 1.1 - Consider: $h \rangle \mathcal{K}$, $K = \phi(\mathcal{K})$, $\hat{K}$ the closed convex hull of K in E. If $\mathcal{G} = \{T_1, T_2, \ldots, T_n\} \subset E'$ and $\alpha_{\mathcal{G}} = \{x \in \hat{K}; T_i(x) = h(T_i \circ \phi)$, $i = 1, 2, \ldots, n\}$, then $\alpha_{\mathcal{G}} \neq \phi$. In fact, if $\alpha_{\mathcal{G}} = \phi$, $T = (T_1, T_2, \ldots, T_n) \in \mathcal{L}(E;\mathbb{C}^n)$ is such that $(h(T_1 \circ \phi), h(T_2 \circ \phi), \ldots, h(T_n \circ \phi)) \notin T(\hat{K})$. Hence there is G in $\mathcal{L}(\mathbb{C}^n)$ such that

$$\operatorname{Re} G(h(T_1 \circ \phi), \ldots, h(T_n \circ \phi)) > \sup_{t \in \hat{K}} \operatorname{Re} G(T(t)) \tag{1}$$

From Lemma 1.1 and (1) it follows that $|h(f)| > \sup_{u \in K} |f(u)|$ for the holomorphic mapping f defined as in the preceeding lemma. This is a contradiction to $h \rangle \mathcal{K}$ and we must have $\alpha_{\mathcal{G}} \neq \phi$. Since $\hat{K}$ is compact and the collection $\{\alpha_{\mathcal{G}} | \mathcal{G} \subset E', \mathcal{G}$ finite $\}$ of closed subsets of $\hat{K}$ has the finite intersection property, the inter section of the whole collection is non-empty and any $a_h$ of it has the property this proposition requires. The uniqueness of $a_h$ follows from the fact that E' separates the points of E.

Remark - It is possible to give a topology to $S(\mathcal{U})$ in such a way that the mapping $\pi: h \in S(\mathcal{U}) \longmapsto \pi(h) = a_h \in E$ is a local homeomorphism and $(S(\mathcal{U}), \pi)$ has nice properties. The remaining part of this section is devoted to the construction of this topology.

Lemma 1.2 - If $\mathcal{K}$ is a compact subset of $\mathcal{U}$, there is a convex balanced open neighbourhood U of zero in E such that $\mathcal{K}+U \subset \mathcal{U}$ and $\mathcal{K}+L$ is compact for every compact subset L of U.

Proof - For each $k \in \mathcal{K}$ let $\mathcal{W}_k$ be an open subset of $\mathcal{U}$ containing k and such that $\phi|\mathcal{W}_k$ is a homeomorphism from $\mathcal{W}_k$ onto $\phi(\mathcal{W}_k)$. If $V_k$ is a convex balanced open neighbourhood of zero in E such that $\phi(\mathcal{W}_k) \supset \phi(k) + 4V_k$ and $2\overline{V}_k \subset 4V_k$, we have

$$\bigcup_{k\in\mathcal{K}} [\phi|\mathcal{W}_k]^{-1}(\phi(k) + V_k) \supset \mathcal{K}\,.$$

There are $k_1, k_2, \ldots, k_n \in \mathcal{K}$ such that

$$\bigcup_{i=1}^{n} \left[\phi|\mathcal{W}_{k_i}\right]^{-1} (\phi(k_i) + V_{k_i}) \supset \mathcal{K}\,.$$

Let $V = \bigcap_{i=1}^{n} V_{k_i}$. We have:

$$\mathcal{K} + V \subset \bigcup_{i=1}^{n} \left[\phi|\mathcal{W}_{k_i}\right]^{-1}(\phi(k_i) + 2\overline{V}_{k_i})$$

and, if L is a compact subset of V

$$\mathcal{K} \hat{+} L = \underbrace{\bigcup_{i=1}^{n} \left[\phi|\mathcal{W}_{k_i}\right]^{-1} \underbrace{\left[(\phi(k_i) + 2\overline{V}_{k_i}) \cap (\phi(\mathcal{K}) + L)\right]}_{\text{compact}}}_{\text{compact}}$$

<u>Remark</u> - Throughout this chapter the results given without a proof are to be considered as having the same proofs (formally) as the corresponding results for the Banach space case in Alexander thesis (see ref. 5).

<u>Proposition 1.2</u> - Let h in $S(\mathcal{U})$ be such that $h \}\mathcal{K}$. Let U be an open convex balanced neighbourhood of zero in E such that $\mathcal{K} + U \subset \mathcal{U}$ and $\mathcal{K} + L$ is compact for every compact subset L of U. If, for every $u \in U$, we set formally

$$h_u(f) = \sum_{n=0}^{\infty} \frac{1}{n!}\, h(\hat{d}^n f(\ )(u)) \tag{2}$$

for all f in $\mathcal{H}(\mathcal{U})$, then

(i) The series in (2) converges absolutely, $h_u$ is in $S(\mathcal{U})$ and $h_u \{ \mathcal{K}_u$, where $\mathcal{K}_u$ =

$= \mathcal{K} + \{\lambda u \mid |\lambda| \leq \alpha\}$ for a $\alpha > 1$ such that $\{\lambda u \mid |\lambda| \leq \alpha\} \subset U$;

ii) $\pi(h_u) = \pi(h) + u$.

If h is in $S(\mathcal{U})$, $h \nmid \mathcal{K}$, and if U is an open balanced convex neighbourhood of zero in E such that $\mathcal{K} + U \subset \mathcal{U}$ and $\mathcal{K} + L$ is compact for every compact subset L of U, then we set $N_{h,U}$ as the collection of all $h_u$ for u in U. We consider in $S(\mathcal{U})$ the topology having the family of all such $N_{h,U}$ as a basis.

Proposition 1.3 - The mapping $\pi$ is a local homeomorphism from $S(\mathcal{U})$ into E and $\pi|N_{h,U}$ is a homeomorphism between $N_{h,U}$ and $\pi(h) + U$.

## 2. Extensions Pairs

In the preceeding section $(S(\mathcal{U}), \pi)$ was defined. If $S(\mathcal{U})$ is connected, then $(S(\mathcal{U}),\pi)$ is a Riemann domain. We do not know if $S(\mathcal{U})$ is connected, but we can find a connected open subset $E(\mathcal{U})$ of $S(\mathcal{U})$ such that $(E(\mathcal{U}),\pi)$ is a Riemann domain over E with very interesting properties.

We define a holomorphic mapping from $(S(\mathcal{U}),\pi)$ into $\mathbb{C}$ in the same way a holomorphic mapping from a Riemann domain into $\mathbb{C}$ is defined.

Proposition 2.1 - If f is in $\mathcal{H}(\mathcal{U})$, the mapping $\tilde{f}: h \in S(\mathcal{U}) \longmapsto \tilde{f}(h) = h(f) \in \mathbb{C}$ is holomorphic in $S(\mathcal{U})$.

The mapping $i: u \in \mathcal{U} \longmapsto i(u) \in S(\mathcal{U})$, where i(u) is the evaluation homomorphism at u, is such that $i(u)(T \circ \phi) = T \circ \phi(u)$ for all T in E' and u in $\mathcal{U}$. If follows that $\pi \circ i = \phi$. If $\mathcal{H}(\mathcal{U})$ separates the points of $\mathcal{U}$, then i is 1-1 and it is possible to show that i is a local homeomorphism from $\mathcal{U}$ onto an open subset $\mathcal{U}_S$ of $S(\mathcal{U})$ with the following properties: (i) if f is in $\mathcal{H}(\mathcal{U})$, $f \circ i^{-1}$ is in $\mathcal{H}(\mathcal{U}_S)$;

(ii) $g \circ i$ is in $\mathcal{H}(\mathcal{U})$ for each g in $\mathcal{H}(\mathcal{U}_S)$. In this case it is usual to call i a biholomorphism from $\mathcal{U}$ onto $\mathcal{U}_S$. Since $\mathcal{U}$ is connected there is a connected component $E(\mathcal{U})$ of $S(\mathcal{U})$ containing $\mathcal{U}_S$. If we also denote $f|E(\mathcal{U})$ by $\bar{f}$, we have $\bar{f}(i(u)) = i(u)(f) = f(u)$ for all u in $\mathcal{U}$. Hence, identifying $\mathcal{U}$ to $\mathcal{U}_S$, we may consider $\bar{f}$ as a holomorphic extension of f to $E(\mathcal{U})$.

Let $(\mathcal{E}, \pi)$ and $(\mathcal{D}, \phi)$ be Riemann domains over E with $\mathcal{D}$ canonically identified to an open subset of $\mathcal{E}$ (by means of a biholomorphism from $\mathcal{U}$ onto an open subset of $\mathcal{E}$). $(\mathcal{E}, \mathcal{D})$ is called an extension pair if for each f in $\mathcal{H}(\mathcal{D})$ there is an $\bar{f}$ in $\mathcal{H}(\mathcal{E})$ such that $\bar{f}|\mathcal{D} = f$. In this case $f \in \mathcal{H}(\mathcal{D}) \longmapsto \bar{f} \in \mathcal{H}(\mathcal{E})$ is an algebraic isomorphism. If it is also a homeomorphism, when we consider the topology $\tau_{\omega_a}$ in both spaces, we say that $(\mathcal{E}, \mathcal{D})$ is a normal extension pair.

If $\mathcal{H}(\mathcal{U})$ separates the points of $\mathcal{U}$, then $(E(\mathcal{U}), \mathcal{U})$ is an extension pair as we saw above. The following theorem implies that it is also a normal extension pair.

<u>Theorem 2.1</u> - Let $(\mathcal{E}, \mathcal{D})$ be an extension pair. If, for every x in $\mathcal{E}$, the mapping $f \in \mathcal{H}(\mathcal{D}) \longmapsto \bar{f}(x) \in \mathbb{C}$ is linear and continuous for the topology $\tau_{\omega_a}$ in $\mathcal{H}(\mathcal{D})$; then $(\mathcal{E}, \mathcal{D})$ is normal.

<u>Proof</u> - Let $\pi$ and $\phi$ be the local homeomorphisms defining the Riemann domains $\mathcal{E}$ and $\mathcal{D}$ respectively. To prove this theorem it is enough to show that for every algebra seminorm p in $\mathcal{H}(\mathcal{E})$, ported by a compact set $\mathcal{L}$ contained in $\mathcal{E}$, there is an algebra seminorm q in $\mathcal{H}(\mathcal{D})$, ported by a compact set $\mathcal{K}$ contained in $\mathcal{D}$, such that $p(\bar{f}) \leq q(f)$ for all f in $\mathcal{H}(\mathcal{D})$. By the assumptions of the theorem, for each x in $\mathcal{L}$ there is a compact subset $\mathcal{K}_x$ of $\mathcal{D}$ such that $|\bar{f}(x)| \leq \sup_{t \in \mathcal{K}_x} |f(t)|$ for every f in $\mathcal{H}(\mathcal{D})$. Let $V_x$ be a closed balanced neighbourhood of zero in E such that $V_x$ is contained in the interior $\overset{\circ}{\widehat{2V_x}}$ of $2V_x$ and (i) $x + 2V_x \subset \mathcal{E}$, (ii) $\mathcal{K}_x + 4V_x \subset \mathcal{D}$, (iii) $\mathcal{K}_x + L$ is compact for all compact subsets L of $4V_x$. Since $\mathcal{L}$ is

compact, there is a finite cover $\{x_i + Vx_i = x_i + V_i \mid i = 1, 2, \dots, n\}$ of $\mathcal{L}$. For $i = 1, 2, \dots, n$ we set

$$\mathcal{L}_i = \mathcal{K}_{x_i} + \bigcup_{|\lambda|<2} \lambda \{ [\pi(\mathcal{L} \cap (x_i + V_i)) - \pi(x_i)] \cap V_i \}$$

Hence $\mathcal{L}_i \subset \mathcal{K}_{x_i} + 2V_i \subset \mathcal{D}$ and $\mathcal{L}_i$ is compact. Let $\mathcal{K}$ be the union of the $\mathcal{L}_i$, $i = 1, 2, \dots, n$, which is a compact subset of $\mathcal{D}$. For every open subset $\mathcal{W}$ of $\mathcal{D}$ containing $\mathcal{K}$, there is an open balanced neighbourhood A of zero in E such that

$$\mathcal{W}_i = \mathcal{K}_{x_i} + \bigcup_{|\lambda|\leq 2} \lambda \{ [\pi((\mathcal{L} + A) \cap (x_i + \mathring{\widehat{2V}}_i)) - \pi(x_i)] \cap \mathring{\widehat{2V}}_i \}$$

is an open subset of $\mathcal{D}$ containing $\mathcal{L}_i$ for all $i = 1, 2, \dots n$. The union of all such $\mathcal{W}_i$ contains $\mathcal{K}$. We may take A so that $\bigcup_{i=1}^{n} \mathcal{W}_i \subset \mathcal{W}$.

$$\mathcal{B}(\mathcal{W}) = \bigcup_{i=1}^{n} [(\mathcal{L} + A) \cap (x \ \ + \mathring{\widehat{2V}}_i)]$$

is an open subset of $\mathcal{E}$ containing $\mathcal{L}$. If $x \in \mathcal{B}(\mathcal{W})$, then x is in $x_i + \mathring{\widehat{2V}}_i$ for some i, $\pi(x) - \pi(x_i)$ belongs to $\mathring{\widehat{2V}}_i$ and, for $|\lambda| \leq 1$, $x_i + \lambda(\pi(x) - \pi(x_i))$ is in $\mathcal{E}$ by (i). The function $g(\lambda) = \bar{f}(x_i + \lambda(\pi(x) - \pi(x_i))$ defined in a neighbourhood of the set $\{\lambda \in \mathbf{C};\ |\lambda| \leq 1\}$ is such that

$$g(\lambda) = \sum_{m=0}^{\infty} \lambda^m \frac{1}{m!} g^{(m)}(0) \text{ and } \bar{f}(x) = g(1) = \sum_{m=0}^{\infty} \frac{1}{m!} g^{(m)}(0)$$

Therefore:

$$\bar{f}(x) \leq \sum_{m=0}^{\infty} \frac{1}{m!} |\hat{d}^m \bar{f}(x_i)\ (\pi(x) - \pi(x_i)| \leq$$

$$\leq \sum_{m=0}^{\infty} \frac{1}{m!} \sup_{k \in \mathcal{K}_{x_i}} |\hat{d}^m f(k)(\pi(x) - \pi(x_i))| \tag{4}$$

If $k \in \mathcal{K}_{x_i}$, then, by (ii), $k + \lambda(\pi(x) - \pi(x_i))$ is in $\mathcal{D}$ for all $|\lambda| \leq 2$. The function $g_1(\lambda) = f(k + \lambda(\pi(x) - \pi(x_i)))$ is defined for all $|\lambda| \leq 2$. The Cauchy inequalities

$$|g_1^{(m)}(0)| \leq m!\ 2^{-m} \sup_{|\lambda|\leq 2} |g_1(\lambda)|$$

for $m = 0, 1, 2, \ldots$, imply

$$|\hat{d}^m f(k)(\pi(x) - \pi(x_i))| \leq m!\ 2^{-m} \sup_{t\in\mathcal{W}_i} |f(t)|$$

for $m = 0, 1, 2, \ldots$ These inequalities and (3) imply

$$|\overline{f}(x)| \leq \sup_{t\in\mathcal{W}_i} |f(t)| \sum_{m=0}^{\infty} 2^{-m} \leq 2 \sup_{t\in\mathcal{W}} |f(t)|$$

Therefore

$$\sup_{x\in\mathcal{B}(\mathcal{W})} |\overline{f}(x)| \leq 2 \sup_{t\in\mathcal{W}} |f(t)|$$

for every open subset $\mathcal{W}$ of $\mathcal{D}$ such that $\mathcal{K}$ is contained in $\mathcal{W}$. It follows that for every open subset $\mathcal{W}$ of $\mathcal{D}$ containing $\mathcal{K}$ there is $c(\mathcal{B}(\mathcal{W})) > 0$ such that

$$p(\overline{f}) \leq 2\ c(\mathcal{B}(\mathcal{W})) \sup_{t\in\mathcal{W}} |f(t)| \tag{4}$$

for all $f$ in $\mathcal{H}(\mathcal{D})$.

We need the following lemma which has a simple proof.

<u>Lemma 2.1</u> - Let $\mathcal{P}$ be the collection of all seminorms $s$ in $\mathcal{H}(\mathcal{D})$ ported by $\mathcal{K}$ and such that (a) $s(f.g) \leq s(f).s(g)$, (b) $s(f) \leq 2\, c(\mathcal{B}(\mathcal{W})). \sup_{t\in\mathcal{W}} |f(t)|$, for all $f$, and $g$ in $\mathcal{H}(\mathcal{D})$ and for all open subsets $\mathcal{W}$ of $\mathcal{D}$ containing $\mathcal{K}$. The supremum of all these seminorms is a seminorm belonging to $\mathcal{P}$.

Returning to the proof of the theorem, we see that the seminorm $s$ in $\mathcal{H}(\mathcal{D})$, defined by $s(f) = p(\overline{f})$, for all $f$ in $\mathcal{H}(\mathcal{D})$, is a member of the collection $\mathcal{P}$ of Lemma 2.1. This lemma implies that there is an algebra seminorm $q$ in $\mathcal{H}(\mathcal{D})$ ported by $\mathcal{K}$, such that $p(\overline{f}) \leq q(f)$ for all $f$ in $\mathcal{H}(\mathcal{D})$.

Theorem 2.2 - If $(\mathcal{U}, \phi)$ is a Riemann domain over E such that $\mathcal{H}(\mathcal{U})$ separates the points of $\mathcal{U}$, then $(E(\mathcal{U}), \pi)$ and the biholomorphism i from $\mathcal{U}$ onto the open subset $\mathcal{U}_S$ of $E(\mathcal{U})$ are such that:

(a) $\mathcal{H}(E(\mathcal{U}))$ separates the points of $E(\mathcal{U})$;

(b) $(E(\mathcal{U}), \mathcal{U})$ is a normal extension pair.

Moreover, $(E(\mathcal{U}), \pi)$ is maximum relatively to (a) and (b) in the following sense: if $(\mathcal{M}, \psi)$ is a Riemann domain over E and j is a biholomorphism from $\mathcal{U}$ onto an open subset $\mathcal{U}_{\mathcal{M}}$ of $\mathcal{M}$ and (a) and (b) are satisfied when $E(\mathcal{U})$ is replaced by $\mathcal{M}$, then $\mathcal{M}$ may be identified to an open subset of $E(\mathcal{U})$ by a biholomorphism preserving the points of $\mathcal{U}$.

Remark - Other results in Alexander's thesis may be generalized to the case we are considering. Since they are not going to be needed here we do not enunciate them.

# CHAPTER III

# RIEMANN DOMAINS OF HOLOMORPHY OVER $\mathbb{C}^N$

Throughout this chapter the points and the subsets of $\mathbb{C}^n$ are identified respectively to the points and subsets of $\mathbb{C}^n x(0,0,\ldots) \subset \mathbb{C}^N$.

A Riemann domain $(\mathcal{U}, \phi)$ over $\mathbb{C}^N$ is <u>of order n at a point</u> u of $\mathcal{U}$ if n is the smallest positive integer such that there is an open polydisc B in $\pi_n(\mathbb{C}^N)$ with center 0 for which $u+v \in \mathcal{U}$ for every v in $\pi_n^{-1}(B)$. Here $\pi_n$ is the projection mapping from $\mathbb{C}^N$ onto the space of the first n variables. $(\mathcal{U}, \phi)$ is <u>of order (at most) n in a subset</u> $\mathcal{A}$ of $\mathcal{U}$ if $(\mathcal{U}, \phi)$ is of order (at most) n at each point of $\mathcal{A}$. $(\mathcal{U}, \phi)$ is <u>locally pseudo-convex</u> if $(\mathcal{U}_V, \phi_V)$ is pseudo-convex for each affine subspace V of $\mathbb{C}^N$ of dimension two. $\mathcal{U}_V$ denotes the topological subspace $\phi^{-1}[\phi(\mathcal{U}) \cap V]$ of $\mathcal{U}$ and $\phi_V$ denotes the restriction of $\phi$ to $\mathcal{U}_V$. In this case, see ref. 6 and 7, for each v in V, the mapping $z \in \mathcal{U}_V \longmapsto -\log \delta_{\mathcal{U}_V}(z,v) \in \mathbb{C}$ is plurisubharmonic. Recall that $\delta_{\mathcal{U}_V}(z,v) = \inf \{|\lambda|;\ u + \lambda v \notin \mathcal{U}_V\}$.

<u>Proposition 1</u> - Let $(\mathcal{U}, \phi)$ be a locally pseudo-convex Riemann domain over $\mathbb{C}^N$. There is a positive integer n such that $(\mathcal{U}, \phi)$ is of order n in $\mathcal{U}$ and $\phi(\mathcal{U}) = \pi_n \circ \phi(\mathcal{U}) \times \mathbb{C}^{N - [0,n-1]}$.

<u>Lemma 1</u> - Let $(\mathcal{U}, \phi)$ be a locally pseudo-convex Riemann domain over $\mathbb{C}^N$ of order n at a point u of $\mathcal{U}$. Let r be the largest positive real number such that $u+b \in \mathcal{U}$ for each b in the open polydisc $B_r$ in $\pi_n(\mathbb{C}^N)$ with center 0 and radius r. Then $u+v \in \mathcal{U}$

for each $v$ in $\pi_n^{-1}(B_r)$. $(\mathcal{U}, \phi)$ is of order at most n at each one of these $u+v$.

Proof - $\phi(u)$ may be considered equal to zero without any loss of generality. Let $w = (w_j)_{j\in N} \in \pi_n^{-1}(B_r)$. If $\pi_n(w) = 0$, then $u+w \in \mathcal{U}$ because $(\mathcal{U},\phi)$ is of order n at u. If $\pi_n(w) \neq 0$ and $w_j = 0$ for each $j \geqslant n$, $u + w \in \mathcal{U}$ because $w \in B_r$. If $\pi_n(w) \neq 0$ and $w_j \neq 0$ for some $j \geqslant n$, consider $z = (z_j)_{j\in N} = (0,\ldots,0, w_n, w_{n+1},\ldots) \in \mathbb{C}^N$. The subspace V of $\mathbb{C}^N$ generated by z and w has dimension 2. Since $(\mathcal{U}_V, \phi|\mathcal{U}_V)$ is pseudoconvex, $-\log \delta_{\mathcal{U}_V}(t, w-z)$ is a plurisubharmonic function of t in $\mathcal{U}_V$. $(\mathcal{U}, \phi)$ of order n at u implies that there are positive real numbers $\varepsilon_0,\ldots, \varepsilon_{n-1}$ such that $u + v \in \mathcal{U}$ for each v in $\mathbb{C}^N$ such that $|v_i| < \varepsilon_i$, $i = 0, 1,\ldots, n-1$. Hence $u+ \lambda z \in \mathcal{U}_V$ for each $\lambda\in\mathbb{C}$ because $\lambda z_i = 0$, $i = 0, 1,\ldots, n-1$, and $\phi(u+\lambda z) = \lambda z \in V$. Consequently $-\log \delta_{\mathcal{U}_V}(u+ \lambda z, w-z)$ is a subharmonic function of $\lambda$ in $\mathbb{C}$. If $\varepsilon = \min\{ \varepsilon_i;\ i = 0, 1,\ldots, n-1\}$ and $\delta$ is the product of $\varepsilon$ by the inverse of $\sup\{|w_i|;\ i = 0, 1,\ldots, n-1\}$, $|\alpha(w_i - z_i)| < \varepsilon$ for each $|\alpha|<\delta$ and $i = 0, 1, \ldots, n-1$. It follows that $u+ \lambda z+\alpha(w-z)$ is in $\mathcal{U}_V$ for all $\lambda$ in $\mathbb{C}$ and $|\alpha|<\delta$. Thus $-\log \delta_{\mathcal{U}_V}(u+\lambda z, w-z)$ is a bounded above subharmonic function of $\lambda$ in $\mathbb{C}$, hence constant. Since $\delta_{\mathcal{U}_V}(u+0z, w-z)>1$, $\delta_{\mathcal{U}_V}(u+z, w-z)>1$ and $u+w \in \mathcal{U}$.

Lemma 2 - Let $(\mathcal{U}, \phi)$ and u be as in Lemma 1. Let $\mathcal{W}$ be an open connected neighborhood of u such that $\phi|\mathcal{W}$ is a homeomorphism from $\mathcal{W}$ onto $\phi(u) + A_0\times\ldots\times A_s \times \mathbb{C}^{N-[0,s]}$, where each $A_i$ is an open ball in $\mathbb{C}$ with center 0 and $s \geqslant n-1$. Then $(\mathcal{U}, \phi)$ is of order at most n in $\mathcal{W}$ and $\phi(\mathcal{U}) \supset \phi(u) + A_0\times\ldots\times A_{n-1} \times \mathbb{C}^{N-[0,n-1]}$.

Proof - $\phi(u)$ may be considered equal to 0 without any loss of generality. Let $W_n = A_0\times\ldots\times A_{n-1}$ and $W_n'$ the set of all points w of $W_n$ such that $(\mathcal{U}, \phi)$ is of order at most n at $[\phi|\mathcal{W}]^{-1}(w)$. By Lemma 1, if $w\in W_n'$, $W_n'$ contains the largest polydisc of radius $r > 0$ with center $w$ which is contained in $W_n$. Since $0 \in W'_n$, it follows that $W_n = W'_n$. If $\omega \in \mathcal{U}$, $\phi(\omega) \in W'_n \times \mathbb{C}^{N-[0,n-1]}$. Hence, applying Lemma 1 for $u = \phi^{-1}[\pi_n \phi(\omega)]$, it is easy to see that $(\mathcal{U}, \phi)$ is of order at most n at $\omega = u + (\phi(\omega) - \pi_n \phi(\omega))$.

<u>Proof of Proposition 1</u> - Let $\mathcal{V}$ be the set of all points of $\mathcal{U}$ where $(\mathcal{U}, \phi)$ is of order at most n. By Lemma 2, $\mathcal{V}$ is open. Let $(x_k)_{k=0}^{\infty}$ be a sequence of points of $\mathcal{V}$ converging to x in $\mathcal{U}$. Let $\mathcal{W}$ be an open connected neighbourhood of x in $\mathcal{U}$ such that $\phi|\mathcal{W}$ is a homeomorphism from $\mathcal{W}$ onto $\phi(x) + A_0 x \ldots x A_s \times \mathbb{C}^{N-[0,s]}$, where each $A_i$ is an open ball $\mathbb{C}$ with center 0. Thus $x_k \in \mathcal{W}$ for k large enough and, by Lemma 2, $(\mathcal{U}, \phi)$ is of order at most n in $\mathcal{W}$. Hence $x \in \mathcal{V}$ and $\mathcal{V}$ is closed in $\mathcal{U}$. Since $\mathcal{U}$ is connected, $\mathcal{V}$ is equal to $\mathcal{U}$. Now the remaining part of the proof follows easily.

<u>Proposition 2</u> - Let $(\mathcal{U}, \phi)$ be a locally pseudo-convex Riemann domain over $\mathbb{C}^N$. There is $n > 0$ in $\mathbb{N}$ such that $(\mathcal{U}, \phi)$ is of order n in $\mathcal{U}$ and $(\mathcal{U}_n, \phi_n)$ is a manifold of holomorphy spread over $\mathbb{C}^n$ if $\mathcal{U}_n = \phi^{-1}[\pi_n \circ \phi(\mathcal{U})]$ and $\phi_n = \phi|\mathcal{U}_n$. (See ref. 7 for the concept of manifold spread over $\mathbb{C}^n$).

<u>Proof</u> - Proposition 1 implies that there is a positive n in $\mathbb{N}$ such that $(\mathcal{U}, \phi)$ is of order n in $\mathcal{U}$. Thus $\phi(\mathcal{U}) = \pi_n \circ \phi(\mathcal{U}) \times \mathbb{C}^{N-[0,n-1]}$. If $(\mathcal{U}_n, \phi_n)$ is defined as above, it is a manifold spread over $\mathbb{C}^n$. Since $(\mathcal{U}, \phi)$ is locally pseudo-convex, $(\mathcal{U}_n, \phi_n)$ is locally pseudo-convex, hence a manifold of holomorphy spread over $\mathbb{C}^n$.

<u>Proposition 3</u> - Let $(\mathcal{U}, \phi)$ be a Riemann domain over $\mathbb{C}^N$ of order n in $\mathcal{U}$ and let v be a point of $\mathbb{C}^N$. Consider the manifolds spread over $\mathbb{C}^n$, $(\mathcal{U}_n, \phi_n)$ and $(\mathcal{V}_n, \psi_n)$

given by $\mathcal{U}_n = \phi^{-1}[\pi_n \circ \phi(\mathcal{U})]$, $\mathcal{V}_n = \phi^{-1}[v - \pi_n(v) + \pi_n \circ \phi(\mathcal{U})]$, $\phi_n = \phi|\mathcal{U}_n$, $\psi_n =$ $= \phi|\ \mathcal{V}_n$. Then there is a biholomorphism between them. In particular, if $(\mathcal{U}_n, \phi_n)$ is of holomorphy, $(\mathcal{V}_n, \psi_n)$ is also of holomorphy.

<u>Proof</u> - Consider the mappings

$$b_1: \ x \in \mathcal{V}_n \longmapsto x + [\pi_n\phi(x) - \phi(x)] \in \mathcal{U}_n$$

and

$$b_2: \ z \in \mathcal{U}_n \longmapsto z + [v - \pi_n(v)] \in \mathcal{V}_n$$

It is easy to see that $b_1 \circ b_2$ and $b_2 \circ b_1$ are the identity mappings in $\mathcal{U}_n$ and in $\mathcal{V}_n$ respectively. They are also local homeomorphisms. In fact: consider x in $\mathcal{V}_n$ and an open neighbourhood $\mathcal{W}'$ of x in $\mathcal{U}$ such that $\phi|\mathcal{W}'$ is a homeomorphism and $\phi(\mathcal{W}') =$ $= \phi(x) + A_o x \ldots x A_t x\, \mathbb{C}^{N-[0,t]}$, where each $A_i$ is an open ball in $\mathbb{C}$ with center 0. Let $\mathcal{W}$ be an open neighbourhood of $x + [\pi_n\phi(x) - \phi(x)]$ in $\mathcal{U}$ such that $\phi|\mathcal{W}$ is a homeomorphism and $\phi(\mathcal{W}) = \pi_n\phi(x) + B_o x \ldots x B_{s-1} \times B_s \times \mathbb{C}^{N-[0,s]}$ is contained in $\pi_n\phi(x) +$ $+ A_o x \ldots x A_t \times \mathbb{C}^{N-[0,t]}$, each $B_i$ being an open ball in $\mathbb{C}$ with center 0. Let $\mathcal{W}''$ be the open neighbourhood of $x \in \mathcal{U}$ given by $[\phi|\mathcal{W}']^{-1}\ [\phi(x) + B_o \times\ B_1 x \ldots x B_s \times \mathbb{C}^{N-[0,s]}]$ Now it is easy to see that $b_1|\mathcal{W}'' \cap \mathcal{V}_n$ is a homeomorphism from $\mathcal{W}'' \cap \mathcal{V}_n$ onto $\mathcal{W} \cap \mathcal{U}_n$. It is also easy to verify that $f \circ b_1 \in \mathcal{H}(\mathcal{V}_n)$ and $g \circ b_2 \in \mathcal{H}(\mathcal{U}_n)$ for every $f \in \mathcal{H}(\mathcal{U}_n)$ and $g \in \mathcal{H}(\mathcal{V}_n)$.

A Riemann domain $(\mathcal{U},\phi)$ over $\mathbb{C}^N$ is a <u>domain of holomorphy</u> if there is $f \in \mathcal{H}(\mathcal{U})$ with no extension $\bar{f} \in \mathcal{H}(\bar{\mathcal{U}})$ for every Riemann domain $(\bar{\mathcal{U}}, \bar{\phi})$ over $\mathbb{C}^N$ extending $(\mathcal{U},\phi)$ <u>proper</u> ly. $(\bar{\mathcal{U}}, \bar{\phi})$ <u>extends</u> $(\mathcal{U}, \phi)$ <u>properly</u> if there is a biholomorphism j (see Page 22) from $\mathcal{U}$ onto a proper open subset $\mathcal{U}_o$ of $\bar{\mathcal{U}}$. In this case, f has an extension $\bar{f} \in \mathcal{H}(\bar{\mathcal{U}})$ if $\bar{f} \circ j = f$.

<u>Proposition 4</u> - Let $(\mathcal{U}, \phi)$ be a Riemann domain over $\mathbb{C}^N$ of order n in $\mathcal{U}$ and such that $(\mathcal{U}_n, \phi_n)$, defined as above, is a manifold of holomorphy spread over $\mathbb{C}^n$. Then $(\mathcal{U}, \phi)$ is a domain of holomorphy.

<u>Proof</u> - There is $f_n \in \mathcal{H}(\mathcal{U}_n)$ with no extension $\overline{f}_n \in \mathcal{H}(\overline{\mathcal{U}}_n)$ for each manifold $(\overline{\mathcal{U}}_n, \overline{\phi}_n)$ spread over $\mathbb{C}^n$ extending $(\mathcal{U}_n, \phi_n)$ properly.

$$f: x \in \mathcal{U} \longmapsto f(x) = f_n\left[x+(\pi_n\phi(x) - \phi(x))\right] \in \mathbb{C}$$

holomorphic in $\mathcal{U}$. In fact: if x is in $\mathcal{U}$, let $\mathcal{W}''$ and $\mathcal{W}$ be considered as in the proof of Proposition 3. It is quite clear that

$$f \circ \left[\phi|\mathcal{W}''\right]^{-1}(\phi(x)+b) = f \circ \left[\phi(\mathcal{W})\right]^{-1} \left[\pi_n \; \phi(x)+b\right] \qquad (*)$$

for each b in $B_o \times B_1 \times \ldots \times B_s \times \mathbb{C}^{N-[0,s]}$. But, in $\phi(\mathcal{W})$, $f \circ \left[\phi|\mathcal{W}\right]^{-1}$ depends only on the first n variables and it is holomorphic in $\pi_n\phi(\mathcal{W})$ since it is equal to $f_n \circ \left[\phi|\mathcal{W}\cap\mathcal{U}_n\right]^{-1}$ there. Hence it is holomorphic in $\phi(\mathcal{W})$ (see ref. 8, 9 and 10). Since $\phi(\mathcal{W})$ is a translation of $\phi(\mathcal{W}'')$ and (*) holds $f \circ \left[\phi|\mathcal{W}''\right]$ is holomorphic in $\phi(\mathcal{W}'')$. Thus f is an element of $\mathcal{H}(\mathcal{U})$ and the restriction of it to $\mathcal{U}_n$ is equal to $f_n$. If f has an extension $\overline{f}$ in $\mathcal{H}(\overline{\mathcal{U}})$ for some Riemann domain $(\overline{\mathcal{U}}, \overline{\phi})$ over $\mathbb{C}^N$ extending $(\mathcal{U}, \phi)$ properly, there is some manifold $(\mathcal{V}_n, \psi_n)$ spread over $\mathbb{C}^n$ (of the type used in the proof of Proposition 3) which is not of holomorphy. Proposition 3 implies that $(\mathcal{U}_n, \phi_n)$ is not a manifold of holomorphy spread over $\mathbb{C}^n$, a contradiction to the hypothesis of this proposition. Therefore $(\mathcal{U}, \phi)$ is a domain of holomorphy.

<u>Proposition 5</u> - Let $(\mathcal{U}, \phi)$ be a Riemann domain of holomorphy over $\mathbb{C}^N$ such that $\mathcal{H}(\mathcal{U})$ separates the points of $\mathcal{U}$. Then $(E(\mathcal{U}), \pi)$ is canonically identified to $(\mathcal{U}, \phi)$.

The proof of this proposition is an immediate consequence of Theorem 2.2, Chapter II.

A Riemann domain $(\mathcal{U}, \phi)$ over $\mathbb{C}^N$ is <u>pseudo-convex</u> if, for each compact subset $\mathcal{K}$ of $\mathcal{U}$ and each balanced convex open neighbourhood U of 0 in $\mathbb{C}^N$ such that $\mathcal{K}+U\subset\mathcal{U}$ and $\mathcal{K}+L$ is compact for every compact subset L of U,

$$\hat{\mathcal{K}}_{\mathcal{U}} + U \subset \mathcal{U},$$

where

$$\hat{\mathcal{K}}_{\mathcal{U}} = \{u\in\mathcal{U};\ |f(u)| \leqslant \sup_{t\in\mathcal{K}} |f(t)|,\ \forall\ f \in \mathcal{H}(\mathcal{U})\}$$

<u>Proposition 6</u> - Let $(\mathcal{U}, \phi)$ be a Riemann domain over $\mathbb{C}^N$ such that $\mathcal{H}(\mathcal{U})$ separates the points of $\mathcal{U}$ and $(E(\mathcal{U}), \pi)$ is canonically identified to $(\mathcal{U}, \phi)$. Then $(\mathcal{U}, \phi)$ is pseudo-convex.

<u>Proof</u> - Let $\mathcal{K}$ be a compact subset of $\mathcal{U}$ and let U be an open balanced convex neighbourhood of 0 in $\mathbb{C}^N$ such that $\mathcal{K} + U\subset\mathcal{U}$ and $\mathcal{K}+L$ is compact for every compact subset L of U. Then, by Proposition 1.2, Chapter II, $\phi(\hat{\mathcal{K}}_u) + U\subset \phi(\mathcal{U})$. Proposition 1.3, Chapter II, implies that $N_{i(x),U} \subset S(\mathcal{U})$ for each x in $\hat{\mathcal{K}}_u$, where i(x) is the evaluation homomorphism at the point x. Since $N_{i(x),U}$ is open connected and x is in $N_{i(x),U} \cap E(\mathcal{U})$, it follows that $N_{i(x),U}$ is contained in $E(\mathcal{U})$ for each x in $\hat{\mathcal{K}}_u$. But $\mathcal{U}$ is identified to $E(\mathcal{U})$ and $N_{i(x),U}$ is the same set as x + U for each x in $\hat{\mathcal{K}}_{\mathcal{U}}$. Hence $\hat{\mathcal{K}}_{\mathcal{U}} + U\subset \mathcal{U}$.

<u>Proposition 7</u> - If $(\mathcal{U}, \phi)$ is a pseudo-convex Riemann domain over $\mathbb{C}^N$, it is locally pseudo-convex.

<u>Proof</u> - If V is an affine subspace in $\mathbb{C}^N$ of dimension two, $(\mathcal{U}_V, \phi_V)$ is a manifold

spread over $\mathbb{C}^2$, where $\mathcal{U}_V = \phi^{-1}[\phi(\mathcal{U}) \cap V]$ and $\phi_V = \phi|\mathcal{U}_V$. To show that $\mathcal{U}_V$ is pseudo-convex, it is enough to prove that $d(\hat{\mathcal{K}}_{\mathcal{U}_V}) > 0$ (see ref. 7 for this notation) for every compact subset $\mathcal{K}$ of $\mathcal{U}_V$. Let $\mathcal{K}$ be a compact subset of $\mathcal{U}_V$ and let U be an open balanced convex neighbourhood of 0 in $\mathbb{C}^N$ such that $\mathcal{K} + U \subset \mathcal{U}$ and $\mathcal{K} + L$ is compact for each compact subset L of U. Then $\hat{\mathcal{K}}_{\mathcal{U}} + U \subset \mathcal{U}$ and $\phi(\hat{\mathcal{K}}_{\mathcal{U}}) + U \subset \phi(\mathcal{U})$. Since $\phi(\hat{\mathcal{K}}_{\mathcal{U}})$ is contained in the closed convex hull of $\phi(\mathcal{K})$ and $\phi(\mathcal{K}) = \phi_V(\mathcal{K}) \subset V$ (see Chapter II), it follows that

$$\phi(\hat{\mathcal{K}}_{\mathcal{U}}) + U \cap V \subset \phi(\mathcal{U}) \cap V = \phi_V(\mathcal{U}_V) .$$

Now

$$\hat{\mathcal{K}}_{\mathcal{U}_V} + U \cap V \subset \hat{\mathcal{K}}_{\mathcal{U}} \cap \mathcal{U}_V + U \cap V \subset$$

$$\subset (\hat{\mathcal{K}}_{\mathcal{U}} + U) \cap \mathcal{U}_V \subset \mathcal{U} \cap \mathcal{U}_V \subset \mathcal{U}_V .$$

It follows that there is a polydisc B in V with center 0 and radius $r > 0$ such that $\mathcal{K}_{\mathcal{U}_V} + B \subset \mathcal{U}_V$. This means that $d(\hat{\mathcal{K}}_{\mathcal{U}_V}) > 0$.

Now it is possible to enunciate the following theorem whose proof we have just finished.

<u>Theorem 1</u> - Let $(\mathcal{U}, \phi)$ be a Riemann domain over $\mathbb{C}^N$ such that $\mathcal{H}(\mathcal{U})$ separates the points of $\mathcal{U}$. The following conditions are equivalent:

(1) $(\mathcal{U}, \phi)$ is a domain of holomorphy.

(2) $(E(\mathcal{U}), \pi)$ is canonically identified to $(\mathcal{U}, \phi)$.

(3) $(\mathcal{U}, \phi)$ is pseudo-convex.

(4) $(\mathcal{U}, \phi)$ is locally pseudo-convex.

(5) There is $n > 0$ in N such that $(\mathcal{U}, \phi)$ is of order n in $\mathcal{U}$ and $(\mathcal{U}_n, \phi_n)$ is a manifold of holomorphy spread over $\mathbb{C}^n$, if $\phi_n = \phi|\mathcal{U}_n$, $\mathcal{U}_n = \phi^{-1}\left[\pi_n \circ \phi(\mathcal{U})\right]$.

Remarks - (1) Theorem 1 was proved by Hirschowitz in ref. 10 for the case in which $(\mathcal{U}, \phi)$ is an open subset of $\mathbb{C}^N$.

(2) The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are true for a Riemann domain $(\mathcal{U}, \phi)$ over a locally convex space E such that the closed convex hull of each compact subset is compact. The proofs are exactly the same as above.

Let $(\mathcal{U}, \phi)$ be a Riemann domain over $\mathbb{C}^N$ such that $\mathcal{H}(\mathcal{U})$ separates the points of $\mathcal{U}$. The envelope of holomorphy of $(\mathcal{U}, \phi)$ is a Riemann domain $(\mathcal{U}_o, \phi_o)$ over $\mathbb{C}^N$ which is maximum in the sense stated in Theorem 2.2, Chapter II, with the word "normal" erased in condition (b).

The equivalent conditions (1) and (2) in Theorem 1 and Theorem 2.2, Chapter II, imply that the following results are true.

Theorem 2 - If $(\mathcal{U}, \phi)$ is a Riemann domain over $\mathbb{C}^N$ and $\mathcal{H}(\mathcal{U})$ separates the points of $\mathcal{U}$, then $(E(\mathcal{U}), \pi)$ is the envelope of holomorphy of $(\mathcal{U}, \phi)$.

Corollary 1 - Every extension pair of Riemann domains over $\mathbb{C}^N$ is normal.

# CHAPTER IV

## DOMAINS OF $\tau$-HOLOMORPHY IN A SEPARABLE BANACH SPACE

Let $U \neq \phi$ be an open subset of a complex separable Banach space E. Let $\tau$ be a positive lower semi-continuous function in U such that $\tau(x) \leqslant d(x, \partial U)$, the distance of x to the boundary $\partial U$ of U, for every x in U. $\mathcal{H}_\tau(U)$ denotes the algebra of all the complex valued holomorphic functions in U which are bounded on each closed ball with center x in U and radius strictly smaller than $\tau(x)$ (see ref. 11). The collection of all finite unions of balls of the above type is denoted by $\mathcal{B}_\tau(U)$. In $\mathcal{H}_\tau(U)$ it is considered the Fréchet topology of the uniform convergence over the elements of $\mathcal{B}_\tau(U)$. Observe that the union of the $\mathcal{H}_\tau(U)$, for all $\tau$, is the algebra $\mathcal{H}(U)$ of the complex valued holomorphic functions in U. U is an <u>open set of $\tau$-holomorphy</u> if it is impossible to find two open connected sub sets $U_1$ and $U_2$ of E such that (a) $U \cap U_1$ contains $U_2 \neq \phi$, (b) $U_1$ is not contained in U, (c) for each f in $\mathcal{H}_\tau(U)$, there is $f_1$ in $\mathcal{H}(U_1)$ having the same values as f in $U_2$. If X is a subset of U, $\hat{X}_U^\tau$ denotes the set of all x in U such that

$$|f(x)| \leqslant \sup \{|f(t)|;\ t \in X\}$$

for each f in $\mathcal{H}_\tau(U)$. It is easy to see that the following properties are satisfied:

(1) $\hat{X}_U^\tau$ is closed in U;

(2) If $X \subset U \subset V$, where U and V are open subsets of E, then $\hat{X}_U^\tau \subset \hat{X}_V^\tau$;

(3) If $X \subset Y \subset U$, then $\hat{X}_U^\tau \subset \hat{Y}_U^\tau$;

(4) $\sup \{|f(t)|;\ t \in X\} \leqslant \sup \{|f(t)|;\ t \in \hat{X}_U^\tau\}$ for each f in $\mathcal{H}_\tau(U)$.

Proposition 1 - If X is a subset of E, then $\hat{X}_E^\tau$ is contained in the closed convex hull $\hat{X}$ of X.

Proof - By a separation theorem it follows that $\zeta \in \hat{X}$ if, and only if, $\phi(\zeta) \leq \sup \{\phi(x);\ x \in X\}$ for every real continuous linear mapping $\phi$ on E, considered as a real Banach space. The mapping $\psi : x \in E \longmapsto \psi(x) = \phi(x) - i\phi(i.x) \in \mathbb{C}$ is a continuous linear form on E and the function $f : x \in E \longmapsto f(x) = \exp \psi(x)$ is in $\mathcal{H}_\tau(E)$. If $\xi \in \hat{X}_U^\tau$, $|f(\xi)| \leq \sup \{|f(x)|;\ x \in X\}$ and $\phi(\xi) \leq \sup \{\phi(x);\ x \in X\}$. Hence $\xi \in \hat{X}$.

Corollary 1 - If X is a bounded set contained in the open subset U of E, $\hat{X}_U^\tau$ is bounded.

Proof - Apply Proposition 1 and property (2).

Theorem 1 - Let U be an open connected subset of E. The following conditions are equivalent:

(a) U is a domain of $\tau$-holomorphy.

(b) If A is in $\mathcal{B}_\tau(U)$, $\hat{A}_U^\tau$ is a bounded closed set in E and $d(\hat{A}_U^\tau, \partial U) > 0$.

(c) There is f in $\mathcal{H}_\tau(U)$ such that it is impossible to find open connected subsets $U_1$ anf $U_2$ of E such that

(i) $U \cap U_1 \supset U_2 \neq \phi$ and $U_1 \not\subset U$,

(ii) there is $f_1$ in $\mathcal{H}(U_1)$ such that f is equal to $f_1$ in $U_2$.

Proof - It is clear that (c) $\Longrightarrow$ (a)

(a) $\Longrightarrow$ (b)

Let $A = \bigcup_{i=1}^{n} B\,\rho_i(x_i)$, where $B\,\rho_i(x_i)$ is the closed ball with center $x_i \in U$ and radius $\rho_i < \tau(x_i)$, $i = 1, 2, \ldots, n$. If $r > 0$ is such that $r < \inf\{\tau(x_i) - \rho_i;\ i = 1, 2, \ldots, n\}$, $\hat{A}_U^\tau + B_r(0) \subset U$ and $d(\hat{A}_U^\tau, \partial U) > 0$. In fact: if $f$ is in $\mathcal{H}_\tau(U)$, it is bounded in $\bar{A} = \bigcup_{i=1}^{n} B_{\rho_i + r}(x_i)$. Hence:

$$\left| \frac{1}{j!} \hat{d}^j f(t) \right| \leq M.r^{-1}$$

for every $t$ in $A$. Here $M = \sup\{|f(x)|;\ \|x-t\| \leq r,\ t\in A\}$. Thus,

$$\left| \frac{1}{j!} \hat{d}^j f(z) \right| \leq M.r^{-1}$$

for every $z$ in $\hat{A}_U^\tau$. Hence the Taylor series of $f$ in $z$ converges uniformly in every closed ball $B_s(z)$, $s < r$. Therefore it defines a function $f_1$ in $\mathcal{H}(\text{int } B_r(z))$ which is equal to $f$ in the connected component of $U \cap \text{int } B_r(z) \neq \phi$ containing $\{z\}$. Since $U$ is a domain of $\tau$-holomorphy, $\text{Int } B_r(z) \subset U$ and $d(z, \partial U) \geq r$ for each $z$ in $\hat{A}_U^\tau$. Thus $d(\hat{A}_U^\tau, \partial U) > 0$. $\hat{A}_U^\tau$ is bounded by Corollary 1 and closed in $U$ by (1). Since $d(\hat{A}_U^\tau, \partial U) > 0$, $\hat{A}_U^\tau$ is closed in $E$.

(b) $\Longrightarrow$ (c)

<u>Lemma 1</u> - Let $X$ be a countable dense set in $U$. Let $(\xi_n)_{n=1}^{\infty}$ be a sequence formed by all elements of $X$ in such a way that each $x$ in $X$ appears an infinite number of times in the sequence. For each $n = 1, 2, \ldots$, set

$$B_n = \bigcup_{j=1}^{n} B_{(1-\frac{1}{j})\,\tau(\xi_j)}(\xi_j)$$

Then $\bigcup_{j=1}^{n} B_j = U$ and every A in $\mathcal{B}_\tau(U)$ is contained in some $B_n$.

Proof - Let $B_\rho(x)$ be such that $\rho < \tau(x)$. Consider $\delta = \tau(x) - \rho > 0$ Given $B_{\delta/4}(x)$, there is $\xi \in X$ in $B_{\delta/4}(x)$ such that $\tau(\xi) > \tau(x) - \frac{\delta}{4}$. Hence there is a subsequence $(\xi_{n_k})_{k=1}^{\infty}$ of $(\xi_n)_{n=1}^{\infty}$ such that $\xi_{n_k} = \xi$, $k = 1, 2, \ldots$, and

$$\tau(\xi_{n_k}) > \tau(x) - \delta/4 = \rho + \delta/2 + \gamma,$$

$\gamma >$ , for all $k = 1, 2, \ldots$ Choose $n_{k_0}$ such that

$$\tau(\xi_{n_{k_0}})\,(1-1/n_{k_0}) > \rho + \delta/2$$

Thus

$$B_\rho(x) \subset B_{\tau(\xi n_{k_0}).(1-1/n_{k_0})}(\xi n_{k_0})$$

and the lemma follows.

Proof of (b) $\Longrightarrow$ (c). Let $(\xi_n)_{n=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ be sequences like those of Lemma 1. By (b) $\hat{B}_{nU}^{\tau}$ is bounded and closed in E. Moreover $d(\hat{B}_{nU}^{\tau}, \partial U) > 0$. For each $\xi_j$ let $B_{\xi_j}$ be the largest open ball with center $\xi_j$ contained in U. Then $B_{\xi_j} \not\subset \hat{B}_{j_U}^{\tau}$ for $j = 1, 2, \ldots$ Let $z_j$ be an element of $B_{\xi_j} - \hat{B}_{j_U}^{\tau}$. Denote by $f_j$ a function in $\mathcal{H}_\tau(U)$ such that $f_j(z_j) = 1$ and $\sup\{|f_j(t)|;\, t \in B_j\} < 1$. Replacing $f_j$ by a power of itself, it is possible to consider $f_j(z_j)$ equal to 1 and $\sup\{|f_j(t)|;\, t \in B_j\} < 1/2^j$. Thus $f_j$ is not the constant function 1 in U. Define $f = \prod_{j=1}^{\infty} (1-f_j)^j$. Since $\sum_{j=0}^{\infty} j/2^j$ converges, it follows that $\sum_{j=1}^{\infty} |(1-f_j)^j - 1|$ converges uniformly on each $B_n$. Hence the infinite product converges uniformly to f on each $B_n$ and f is in $\mathcal{H}_\tau(U)$. On the other hand:

$$\inf_{t \in B_s} \left| \prod_{j=s}^{\infty} (1-f_j(t))^j \right| \geq \prod_{j=s}^{\infty} (1-1/2^j)^j > 0 .$$

Therefore, if $V = B_\rho(x)$ with $\rho<\tau(x)$, there is s such that V is contained in $B_s$ and

$$\inf \left\{ \left| \prod_{j=s}^{\infty} (1 - f_j(x))^j \right| ; x \in V \right\} > 0 .$$

Thus f is not identically zero since U is connected and $\mathcal{H}_\tau(U)$ is an integrity domain. It is true that $\hat{d}^k f(z_j) = 0$ if $k < j$. Since each $\xi \in X$ appears an infinite number of times in $(\xi_n)_{n=1}^{\infty}$, the following fect follows: if $\xi \in X$ and N is a positive integer, there are points in $B_\xi$, of the form $z_j$, where all the differentials of f of order m less than N vanish. Thus f can not be extended to a holomorphic function in a neighbourhood of $B_\xi$. In fact! if this were true, f would have all of its differentials at some point of the boundary $\partial B_\xi$ of $B_\xi$ equal to zero. Hence f would be 0 in $B_\xi$ and, consequently f would be identically zero in U. Suppose that there are open connected subsets $U_1$ and $U_2$ of E such that $U_1 \not\subset U$, $U_1 \cap U \supset U_2 \neq \phi$ and exists $f_1$ in $\mathcal{H}(U_1)$ such that f is equal to $f_1$ in $U_2$. Call $\tilde{U}_2$ the connected component of $U_1 \cap U$ containing $U_2$. Let $\xi$ be a point of $\partial U \cap U_1$ and of the closure of $\tilde{U}_2$. If $r > 0$ is such that $B_r(\xi)$ is a subset of $U_1$, choose $\xi_o$ in $X \cap B_{r/2}(\xi)$. The radius $\rho$ of $B_{\xi_o}$ is such that $\rho < d(\xi, \xi_o)$. Hence $B_{\xi_o} \subset B_r(\xi)$ and $\overline{B_{\xi_o}} \subset B_r(\xi) \subset U_1$, which is a contradiction to the fact that f cannot be extended to a holomorphic function in a neighbourhood of $B_{\xi_o}$. Thus f is a function as in (c)

Q.E.D.

Remark - A similar result was proved by Dineen in ref. 12 for the case in which $\mathcal{H}_b(U)$ replaces $\mathcal{H}_\tau(U)$.

# CHAPTER V

# A MALGRANGE - GUPTA THEOREM

## 1. Nuclear Entire Functions of Bounded Type

Throughout this chapter $E, E_1, E_2, \ldots, E_n$ are locally convex spaces whose strong duals are Fréchet spaces.

In $\mathcal{L}(E_1, E_2, \ldots, E_n)$ we consider the topology of the uniform convergence over bounded subsets of $E_1 \times E_2 \times \ldots \times E_n$. Let $\mathbb{L}(E_1, \ldots, E_n)$ be the completion of this space. $\mathcal{L}_f(E_1, \ldots, E_n) = \mathbb{L}_f(E_1, \ldots, E_n)$ is the vector subspace of $\mathbb{L}(E_1, \ldots, E_n)$ generated by all n-linear mappings $\phi_1 \times \ldots \times \phi_n$, where $\phi_i \in E_i'$, $i = 1, 2, \ldots, n$ and $\phi_1 \times \ldots \times \phi_n(x_1, \ldots, x_n) = \phi_1(x_1) \ldots \phi_n(x_n)$ for all $(x_1, \ldots, x_n)$ in $E_1 \times \ldots \times E_n$. The mapping $\alpha_n$: $(\phi_1, \ldots, \phi_n) \in E_1' \times \ldots \times E_n' \longrightarrow \phi_1 \times \ldots \times \phi_n \in \mathbb{L}(E_1, \ldots, E_n)$ is n-linear and continuous when it is given to $E_1' \times \ldots \times E_n'$ the product topology of the strong topologies $\beta(E_i', E_i)$ in $E_i'$, $i = 1, 2, \ldots, n$. There is a continuous linear mapping $\chi_n$ from the projective tensor product $E_1' \otimes_\pi E_2' \otimes_\pi \ldots \otimes_\pi E_n'$ of the spaces $E_i'$, $i = 1, 2, \ldots n$ into $\mathbb{L}(E_1, \ldots, E_n)$. $\chi_n$ is $1 - 1$ and its image is $\mathbb{L}_f(E_1, \ldots, E_n)$. $\chi_n$ may be extended to a continuous linear mapping from the completion $E_1' \hat{\otimes}_\pi E_2' \hat{\otimes}_\pi \ldots \hat{\otimes}_\pi E_n'$ of $E_1' \otimes_\pi E_2' \otimes_\pi \ldots \otimes_\pi E_n'$ into $\mathbb{L}(E_1, \ldots, E_n)$. This extension is also denoted by $\chi_n$. It is not known if $\chi_n$ is 1-1. Here this is supposed to be true and, at the end of this chapter, some remarks are made in order to deal with the other case.

The following result is true. (See ref. 13 for a proof).

Proposition 1.1 - If $F_1$ and $F_2$ are Fréchet spaces, every element u of $F_1 \hat{\otimes}_\pi F_2$ is the sum of an absolutely summable series

$$u = \sum_{n=0}^{\infty} \lambda_n x_n \otimes y_n$$

where $(\lambda_n)_{n=0}^{\infty}$ is a sequence of complex numbers such that $\sum_{n=0}^{\infty} |\lambda_n|$ is smaller than 1 and $(x_n)_{n=0}^{\infty}$ (resp. $(y_n)_{n=0}^{\infty}$) is a sequence in $F_1$ (resp. $F_2$) converging to zero. Moreover, the topology in $F_1 \hat{\otimes}_\pi F_2$ is given by the seminorms:

$$\|u\|_{p,q} = \inf\left\{ \sum_{n=0}^{\infty} p(x_n)\, q(y_n);\ u = \sum_{n=0}^{\infty} x_n \otimes y_n,\ x_n \in F_1,\ y_n \in F_2 \right\}$$

where p and q are continuous seminorms in $F_1$ and $F_2$ respectively.

Define in $\chi_n(E_1' \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n')$ the locally convex topology generated by the seminorms:

$$\|A\|_{N,p_1,\dots,p_n} = \inf\left\{ \sum_{j=0}^{\infty} p_1(\phi_{1j})\dots p_n(\phi_{nj});\ A = \sum_{j=0}^{\infty} \phi_{1j} \times \dots \times \phi_{nj} \right\}$$

for each $A \in \chi_n(E_1' \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n')$, where $p_i$ is a continuous seminorm in $E_i'$, $i = 1, 2, \dots, n$. This subspace of $\mathbb{L}(E_1,\dots, E_n)$ with the topology just defined is called the space of the n-linear nuclear mappings from $E_1 \times \dots \times E_n$ into $\mathbb{C}$ and it is denoted by $\mathbb{L}_N(E_1,\dots, E_n)$. If all the spaces $E_i$ are equal to E the notation is $\mathbb{L}_N(^nE)$. Let $\mathbb{P}(^nE)$ be the set of all mappings $P_A: x \in E \longrightarrow P_A(x) = A(x,\dots,x) \in \mathbb{C}$ for A in $\mathbb{L}(^nE)$. $\mathbb{P}_N(^nE)$ and $\mathbb{P}_f(^nE)$ are defined by the same process with $\mathbb{L}(^nE)$ replaced by $\mathbb{L}_N(^nE)$ and $\mathbb{L}_f(^nE)$ respectively. Consider in $\mathbb{P}_f(^nE)$ the locally convex topology generated by the seminorms:

$$\|P\|_{N,q} = \inf\left\{ \sum_{j=1}^{m} q(\phi_j)^n;\quad P = \sum_{j=1}^{m} \phi_j \times \dots \times \phi_j = \sum_{j=1}^{m} \phi_j^m \right\}$$

for all $P \in \mathbb{P}_f({}^nE)$, where q is a continuous seminorm in E'.

$\mathbb{L}_{fs}({}^nE)$ denotes the subspace of $\mathbb{L}_f({}^nE)$ formed by the symmetric n-linear mappings. Similar definitions are given for $\mathbb{L}_s({}^nE)$ and $\mathbb{L}_{Ns}({}^nE)$. If $P \in \mathbb{P}_f({}^nE)$ is defined by A in $\mathbb{L}_{fs}({}^nE)$, it is denoted by $P_A$. Same notation is used for P in $\mathbb{P}({}^nE)$ and $\mathbb{P}_N({}^nE)$.

Proposition 1.2 - If A is in $\mathbb{L}_{fs}({}^nE)$, then

$$\|A\|_{N,q,q,\ldots,q} = \|A\|_{N,q} \leq \|P_A\|_{N,q} \leq n^n(n!)^{-1}\|A\|_{N,q}$$

for each continuous seminorm q in E'.

Remark - In this chapter, a result left without a proof has formally the same demonstration as the corresponding one in ref.14.

Since it is known that $\mathbb{L}_{fs}({}^nE)$ is dense in $\mathbb{L}_{Ns}({}^nE)$ and that Proposition 1.2 is true, it is possible to define the seminorms $\|.\|_{N,q}$ in $\mathbb{P}_N({}^nE)$ by: if $P_A$ is in $\mathbb{L}_N({}^nE)$ and $(A_j)_{j=0}^{\infty}$ is a sequence in $\mathbb{L}_{fs}({}^nE)$ converging to A in $\mathbb{L}_N({}^nE)$, $\|P_A\|_{N,q}$ is the limit of the sequence $(\|P_{A_j}\|_{N,q})_{j=0}^{\infty}$, for each continuous seminorm q in E'. It is not difficult to see that $\mathbb{P}_N({}^nE)$ with the locally convex topology generated by the seminorms $\|.\|_{N,q}$, as q varies in the collection of all continuous seminorms in E', is a Fréchet space and

$$\|A\|_{N,q} \leq \|P_A\|_{N,q} \leq n^n(n!)^{-1}\|A\|_{N,q}$$

for each A in $\mathbb{L}_N({}^nE)$ and all seminorms $\|.\|_{N,q}$.

Proposition 1.3 - If A is in $\mathbb{L}_N({}^nE)$, $x_i \in E_i$, $i = 1, 2, \ldots k$, $1 \leq k \leq n$, then $A.x_1\ldots x_k$ defined by

$$A.x_1 \ldots x_k\ (x_{k+1},\ldots,x_n) = A(x_1, x_2,\ldots, x_n)$$

for each $(x_{k+1},\ldots, x_n)$ in $E^{n-k}$, is an element of $\mathbb{L}_N({}^{n-k}E)$ and

$$\|A.x_1 \ldots x_k\|_{N,q} \leqslant \|A\|_{N,q_{x_1,\ldots x_k}}$$

where $q_{x_1},\ldots, x_k$ is the continuous seminorm on $E'$ given by

$$q_{x_1,\ldots, x_k}(\phi) = \sup\,\{q(\phi),\ |\phi(x_1)|,\ldots,\ |\phi(x_k)|\}$$

for each $\phi$ in $E'$. Moreover:

$$\|P_{A.x_1,\ldots,x_k}\|_{N,q} \leqslant \|P_A\|_{N,q_{x_1,\ldots,\ x_k}}$$

Proof - If A is in $\mathbb{L}_f({}^nE)$, so that $A = \sum_{j=1}^{m} \phi_{1j} \times \ldots \times \phi_{nj}$, it follows that

$$\|A.x_1 \ldots x_k\|_{N,q} \leqslant \sum_{j=1}^{m} |\phi_{1j}(x_1)\ldots\ \phi_{kj}(x_k)|\,q(\phi_{k+1j})\ldots q(\phi_{nj}) \leqslant$$

$$\leqslant \sum_{j=1}^{m} q_{x_1,\ldots,x_k}(\phi_{1j})\ldots\ q_{x_1,\ldots,\ x_k}(\phi_{nj})$$

Hence

$$\|A.x_1 \ldots x_k\|_{N,q} \leqslant \|A\|_{N,q_{x_1,\ldots,\ x_k}} \qquad (1)$$

The density of $\mathbb{L}_f({}^nE)$ in $\mathbb{L}_N({}^nE)$ and (1) imply the first part of the proposition. The second part follows analogously.

A mapping f from E into $\mathbb{C}$ is <u>entire of bounded nuclear type</u> in E if:

(1) $f \in \mathcal{H}_b(E)$

(2) $\hat{d}^n f(0) \in \mathbb{P}_N({}^nE)$, $n = 0, 1, 2, \ldots$

(3) $\lim_{n\to\infty} \left[(n!)^{-1}\|\ \hat{d}^n f(0)\|_{N,q}\right]^{1/n} = 0$, for each continuous seminorm q in $E'$.

In the space $\mathcal{H}_{Nb}(E)$ of all entire mappings of bounded nuclear type in E it is considered the locally convex topology $\tau_N$ generated by the seminorms

$$\|f\|_{N,q,\rho} = \sum_{n=0}^{\infty} \rho^n (n!)^{-1} \|\hat{d}^n f(0)\|_{N,q}$$

as q varies in the collection of all continuous seminorms in E' and ρ varies in the set of all positive real numbers.

Proposition 1.4 - The completion $\mathbb{H}_{Nb}(E)$ of $(\mathcal{H}_{Nb}(E), \tau_N)$ is the set of all mappings f from E into $\mathbb{C}$ such that there are $P_n \in \mathbb{P}_N({}^nE)$, n = 0, 1, ... satisfying the following conditions:

(1) $f(x) = \sum_{n=0}^{\infty} (n!)^{-1} P_n(x)$, for each x in E;

(2) $\sum_{n=0}^{\infty} \rho^n (n!)^{-1} \|P_n\|_{N,q} < +\infty$

$\Updownarrow$

(3) $\lim_{n\to\infty} \left[ (n!)^{-1} \|P_n\|_{N,q} \right]^{1/n} = 0$

for each $\rho > 0$ and each continuous seminorm q on E'. $P_n$ is denoted by $\hat{d}^n f(0)$ and $\|f\|_{N,q,\rho}$ is defined by the same series as before. $(\mathbb{H}_{Nb}(E), \tau_N)$ is a Fréchet space.

Proof - It is clear that each f satisfying (1) and (2) is the limit of a sequence of elements of $\overline{\mathcal{H}_{Nb}(E)}$. If $(f_k)_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{H}_{Nb}(E)$, $((n!)^{-1} \hat{d}^n f_k(0))_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{P}_N({}^nE)$ and converges to $(n!)^{-1} P_n \in \mathbb{P}_N({}^nE)$, for n = 0, 1, 2, ... For each continuous seminorm q on E' and each $\rho > 0$, there is $M_{q,\rho} > 0$ such that $\|f_k\|_{N,q,\rho} \leq M_{q,\rho}$ for k = 1,2,... Hence, for all k and all n, $(n!)^{-1} \|\hat{d}^n f_k(0)\|_{N,q} \leq \rho^{-n} M_{q,\rho}$. Now

$$(n!)^{-1} \|P_n\|_{N,q} \leq (n!)^{-1} \|P_n - \hat{d}^n f_k(0)\|_{N,q} + (n!)^{-1} \|\hat{d}^n f_k(0)\|_{N,q} \leq$$

$$\leq (n!)^{-1} \|P_n - \hat{d}^n f_k(0)\|_{N,q} + \rho^{-n} M_{q,\rho} .$$

Passing to the limit as k goes to $\infty$:

$$(n!)^{-1} \|P_n\|_{N,q} \leq \rho^{-n} M_{q;\rho} \qquad \text{for each } n$$

This implies that

$$\lim_{n\to\infty} \sup \left[(n!)^{-1}\|P_n\|_{N,q}\right]^{1/n} \leq \rho^{-1} \quad \text{for each } \rho.$$

As $\rho$ goes to $\infty$ condition (3) is obtained. Define

$$f(x) = \sum_{n=0}^{\infty} (n!)^{-1} P_n(x)$$

for each x in E. Such f is in $\mathcal{H}_{Nb}(E)$ and it is the limit of the sequence $(f_k)_{k=1}^{\infty}$.

Q.E.D.

## 2. Convolution Operators and Borel Transforms in $\mathcal{H}_{Nb}(E)$.

Proposition 2.1 - Let $a \in E$ and $f \in \mathcal{H}_{Nb}(E)$. Then

(1) $\hat{d}^n f(.)a \in \mathcal{H}_{Nb}(E)$ and

$$\hat{d}^n f(x)a = \sum_{i=0}^{\infty} (i!)^{-1} \widehat{d^{i+n} f(0)x^i}\, a$$

in the sense of $\mathcal{H}_{Nb}(E)$.

(2)

$$(\tau_{-a}f)(x) = f(x+a) = \sum_{n=0}^{\infty} (n!)^{-1} \hat{d}^n f(x)a$$

in the sense of $\mathcal{H}_{Nb}(E)$.

Proof - Observe that

$$\hat{d}^{\,i}f(x) = \sum_{n=i}^{\infty} [(n-i)!]^{-1}\, \widehat{d^n f(0)x^{n-i}} = \sum_{n=0}^{\infty} (n!)^{-1}\, \widehat{d^{i+n} f(0)x^n}$$

for each x in E. Now the proof is formally equal to the proof of Lemma 8 in ref. 14 with Lemma 6 replaced by Proposition 3.1.

Remark - If f ins an element of $\mathbb{H}_{Nb}(E) - \mathscr{H}_{Nb}(E)$ it is the limit of a sequence $(f_k)_{k=1}^{\infty}$ of elements of $\mathscr{H}_{Nb}(E)$. Thus, for each x in E, $\hat{d}^{\,n}f_k(x)$ exists for all k and

$$\|\hat{d}^{\,n}f_k(x) - \hat{d}^{\,n}f_m(x)\|_{N,q} = \sum_{i=0}^{\infty} (i!)^{-1} \|\widehat{d^{i+n} f_k(0)\, x^i} - \widehat{d^{i+n} f_m(0)x^i}\|_{N,q}$$

$$\leq \sum_{i=0}^{\infty} (i!)^{-1} \|\hat{d}^{\,i+n} f_k - \hat{d}^{\,i+n} f_m\|_{N,q_{x,\dots,x}} \leq$$

$$\leq n! \sum_{i=0}^{\infty} [(i+n)!]^{-1}\, 2^{i+n} \|\hat{d}^{\,i+n} f_k - \hat{d}^{\,i+n} f_m\|_{N,q_x} \longrightarrow 0 \quad \text{as } k, m \longrightarrow \infty$$

(here $q_x$ denotes $q_{x,\dots,x}$). Hence the sequence $(\hat{d}^{\,n}f_k(x))_{k=1}^{\infty}$ converges to a point $P_n$ of $\mathbb{P}_N(^nE)$ and $P_n$ is denoted by $\hat{d}^{\,n}f(x)$.

Proposition 2.2 - Proposition 2.1 is true when $\mathscr{H}_{Nb}(E)$ is replaced everywhere by $\mathbb{H}_{Nb}(E)$.

Proof - (1) Let $(f_k)_{k=1}^{\infty}$ be a sequence in $\mathscr{H}_{Nb}(E)$ converging to f in $\mathbb{H}_{Nb}(E)$. Then, for each a in E,

$$\|\hat{d}^{\,n}f_k(.)a - \hat{d}^{\,n}f_m(.)a\|_{N,q,\bar{\rho}} =$$

$$= \sum_{i=0}^{\infty} (i!)^{-1} \rho^i \|\widehat{d^{i+n}f_k(0)a^n} - \widehat{d^{i+n}f_m(0)a^n}\|_{N,q} \leq$$

$$\rho^{-n}\, n! \sum_{i=0}^{\infty} [(i+n)!]^{-1} (2.\rho)^{i+n} \|\hat{d}^{\,i+n} f_k(0) - \hat{d}^{\,i+n} f_m(0)\|_{N,q_a} =$$

$$\rho^{-n}.\, n!\, \|f_k - f_m\|_{N,q_a,\, 2\rho} \longrightarrow 0 \text{ as } k, m \longrightarrow \infty$$

Thus $(\hat{d}^n f_k(.)a)_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{H}_{Nb}(E)$ and converges to a $g \in \mathbb{H}_{Nb}(E)$. Since $(\hat{d}^n f_k(x))_{k=1}^{\infty}$ converges to $\hat{d}^n f(x)$ for each x in E, it is easy to see that $g = \hat{d}^n f(.)a$ and

$$\hat{d}^n f(x)a = \sum_{i=0}^{\infty} (i!)^{-1} \widehat{d^{i+n} f(0) x^i}.a$$

for each $a \in E$ and x in E. Use the same argument of part (1) of the preceding proposition to complete the proof.

(2) Let $(f_k)_{k=1}^{\infty}$ be a sequence of elements of $\mathcal{H}_{Nb}(E)$ converging to f in $\mathbb{H}_{Nb}(E)$. Then

$$\|\tau_{-a} f_k - \tau_{-a} f_m\|_{N,q,\rho} =$$

$$\left\| \sum_{n=0}^{\infty} (n!)^{-1} \left[\hat{d}^n f_k(.)a - \hat{d}^n f_m(.)a\right] \right\|_{N,q,\rho} \leqslant$$

$$\sum_{n=0}^{\infty} \left\| \sum_{i=0}^{\infty} (i!)^{-1} \left[\widehat{d^{i+n} f_k(0) a^n} - \widehat{d^{i+n} f_m(0) a^n}\right] \right\|_{N,q,\rho} \leqslant$$

$$\sum_{n=0}^{\infty} \rho^{-n} \sum_{i=0}^{\infty} \left[(i+n)!\right]^{-1} (2\rho)^{i+n} \|\hat{d}^{i+n} f_k(0) - \hat{d}^{i+n} f_m(0)\|_{N,q_a} =$$

$$\|f_k - f_m\|_{N,q_a,2\rho} \left[\sum_{n=0}^{\infty} \rho^{-n}\right] \longrightarrow 0, \text{ as } k,m \longrightarrow \infty, \text{ if } \rho > 1.$$

Since $\lim_{k\to\infty} \tau_{-a} f_k(x) = \tau_{-a} f(x)$ for each x in E, $g = \tau_{-a} f \in \mathbb{H}_{Nb}(E)$. On the other hand (1) of Proposition 2.1 holds for each $f_k$. Moreover $\hat{d}^i(\tau_{-a} f_k)(0) = \hat{d}^i f_k(a)$ and, passing to the limit, $\hat{d}^i(\tau_{-a} f)(0)$ - is equal to $\hat{d}^i f(a)$. Now, the rest of the proof follows as in Proposition 2.1.

A mapping $\mathcal{O}$ from $\mathbb{H}_{Nb}(E)$ into itself is called a <u>convolution operator</u> on $\mathbb{H}_{Nb}(E)$ if (i) $\mathcal{O}$ is linear ; (ii) $\mathcal{O}$ is continuous; (iii) $\mathcal{O}$ is translation invariant, that is: $\mathcal{O}\tau_{-a} f = \tau_{-a} \mathcal{O} f$ for each a in E and each f in $\mathbb{H}_{Nb}(E)$. $\mathcal{A}$ denotes the collection of all convolution operators on $\mathbb{H}_{Nb}(E)$. It is an algebra with unity under composition of mappings as multiplication and the usual vector space operations. Consider the mapping $\gamma: \mathcal{O} \in \mathcal{Q} \longrightarrow \gamma\mathcal{O} = T \quad \mathbb{H}'_{Nb}(E)$, where $T(f) = \mathcal{O}f(0)$ for each f in $\mathbb{H}_{Nb}(E)$.

Proposition 2.3 - $\gamma$ is a linear 1-1 mapping from $\mathcal{Q}$ onto $\mathbb{H}'_{Nb}(E)$.

The following two lemmas are used in the proof of Proposition 2.3 and they are interesting in themselves.

Lemma 2.1 - Let $T \in \mathbb{H}'_{Nb}(E)$ so that there are $\rho > 0$, $c > 0$ and a continuous semi-norm q on E' such that $|T(f)| \leqslant c\|f\|_{N,q,\rho}$ for each f in $\mathbb{H}_{Nb}(E)$. Then, for every $P \in \mathbb{P}_N(^nE)$, with A in $\mathbb{L}_{Ns}(^nE)$ such that $P_A = P$, the polynomial $y \in E \longrightarrow T_x(Ax^k y^{n-k}) \in \mathbb{C}$, denoted by $T_x(\widehat{Ax^k})$, is in $\mathbb{P}_N(^{n-k}E)$ for each $k \leqslant n$. Moreover

$$\|T_x(\widehat{Ax^k})\|_{N,q} \leqslant c\,\rho^k\,\|P_A\|_{N,q}$$

Lemma 2.2 - If $T \in \mathbb{H}'_{Nb}(E)$ is as in Lemma 2.1, then $(T * f)(x) = T(\tau_{-x} f)$, for each x in E, defines a function $T * f \in \mathbb{H}_{Nb}(E)$ for each f in $\mathbb{H}_{Nb}(E)$. For every $\rho_1 > 0$ there is $c_1 > 0$ such that

$$\|T * f\|_{N,q,\rho_1} \leqslant c_1\,\|f\|_{N,q,2(\rho+\rho_1)}$$

for each f in $\mathbb{H}_{Nb}(E)$.

The proofs of the last three results are formally equal to the proofs of Lemma 9, Lemma 10 and Proposition 10 in ref. 14.

Notation - The inverse mapping $\gamma'$ of $\gamma$ is given by $\gamma'T(f) = T{*}f$ for each T in $\mathbb{H}'_{Nb}(E)$ and each f in $\mathbb{H}_{Nb}(E)$. Now $\gamma'T$ is denoted by $T^*$, for each T in $\mathbb{H}'_{Nb}(E)$.

If $T_1$ and $T_2$ are in $\mathbb{H}'_{Nb}(E)$ and $\mathcal{O}_1 = T_1 * \in \mathcal{A}, \mathcal{O}_2 = T_2 * \in \mathcal{A}$, $\gamma(\mathcal{O}_1 \circ \mathcal{O}_2) \in \mathbb{H}'_{Nb}(E)$ is denoted by $T_1 * T_2$. Also, for every f in $\mathbb{H}_{Nb}(E)$, $(T_1 * T_2) * f = (\mathcal{O}_1 \circ \mathcal{O}_2)(f) = \mathcal{O}_1(\mathcal{O}_2 f) = T_1 * (T_2 * f)$. $T_1 * T_2$ is called the convolution of $T_1$ and $T_2$.

Remark - $\mathbb{H}'_{Nb}(E)$ is an algebra under the usual vector space operations and with convolution as multiplication. This algebra has a unity $\delta$, defined by $\delta(f) = f(0)$ for each f in $\mathbb{H}_{Nb}(E)$. The mapping $\gamma$ is an algebra isomorphism.

The mapping $\phi \in E' \longmapsto \widehat{T}(\phi) = T(\exp \phi) \in \mathbb{C}$ is called the Borel transform $\widehat{T}$ of T in $\mathbb{H}'_{Nb}(E)$.

Lemma 2.3 - The vector subspace of $\mathbb{H}_{Nb}(E)$ generated by the set $\{\exp \phi;\ \phi \in E'\}$ is dense in $\mathbb{H}_{Nb}(E)$.

Lemma 2.4 - The mapping $\beta$: $T \in \mathbb{P}_N(^nE)' \longmapsto \beta T \in \mathbb{P}(^nE')$ defined by $\beta T(\phi) = T(\phi^n)$ for each $\phi$ in E' establishes an isomorphism between the two spaces. Moreover:

$$\|T\|_{N,q} \overset{\text{def}}{=} \sup_{\|P\|_{N,q} \neq 0} \frac{|T(P)|}{\|P\|_{N,q}} = \sup_{q(\phi)\neq 0} \frac{|\beta T(\phi)|}{q(\phi)^n} \overset{\text{def}}{=} \|\beta T\|_q$$

for each continuous seminorm q on E' such that

$$|T(P)| \leq c(q)\ \|P\|_{N,q}$$

for every P in $\mathbb{P}_N(^nE)$.

A function f of $\mathcal{H}(E')$ is of exponential type if there are $c > 0$ and a continuous seminorm q on E' such that

$$|f(\phi)| \leq c.\exp\ [q(\phi)]$$

for each $\phi$ in E'. Denote by Exp E' the set of all functions of exponential type on E'. It is an algebra under pointwise multiplication and the usual vector space operations.

<u>Proposition 2.4</u> - The mapping $\alpha: T \in H'_{Nb}(E) \longrightarrow \alpha T = \hat{T} \in \text{Exp } E'$ is an algebra isomorphism.

<u>Proposition 2.5</u> - Let $f_1, f_2, f_3 \in \mathcal{H}(E')$ be such that $f_1 = f_2 f_3$ with $f_1$ and $f_2$ of exponential type and $f_2$ not identically zero. Then $f_3$ is of exponential type.

<u>Proof</u> - If $f_1$ is identically zero, then $f_3$ is identically zero and $f_3$ is of exponential type. If $f_1$ is not identically zero, it is possible to suppose $f_1(0) = f_2(0) = 1$ and the existence of positive constants $C_1$, $C_2$ and of a continuous seminorm q on E' such that (1) $|f_i(x)| \leq C_i.\exp[q(x)]$, $i = 1, 2$, for each x in E'; (2) $|f_3(x)| \leq M$ if $q(x) < \delta$, for some $\delta > 0$. If $x \in E'$ and $q(x) \neq 0$, set $x_1 = x/q(x)$, $g_i(z) = f_i(zx_1)$ for $i = 1, 2, 3$ and z in $\mathbb{C}$. Then $g_i \in \mathcal{H}(\mathbb{C})$, $i = 1, 2, 3$, $g_1(0) = g_2(0) = 1$, $|g_i(z)| \leq C_i.\exp\left[q(x_1)|z|\right]$ $i = 1, 2$ and $g_1 = g_2 g_3$. By a Malgrange theorem (see Ref. 15) there are $c_3 > 0$, $C_3 > 0$ such that $|g_3(z)| \leq C_3.\exp\left[c_3|z|\right]$ for all z in $\mathbb{C}$. If $z = q(x)$, then it follows that:

$$|f_3(x)| = |f_3(q(x).x_1)| \leq C_3.\exp\left[c_3 q(x)\right]$$

for all x in E' such that $q(x) \neq 0$. If $q(x) = 0$ it follows that $|f_3(x)| < M$. Hence, if $d = \max\{M, C_3\}$,

$$|f_3(x)| \leq d.\exp\left[c_3 q(x)\right]$$

for each x in E'.

Proposition 2.6 - Let U be a non-empty open connected subset of E'. Let f and g be elements of $\mathcal{H}(E')$, g not identically zero, such that for every affine subspace S, of E', of dimension 1 and for any connected component S' of $S \cap U$, in which g is not identically zero, the restriction $f|S'$ is divisible by $g|S'$ with the quotient as a holomorphic mapping in S'. Then f is divisible by g and the quotient is holomorphic in U.

Proof - It is enough to proof the proposition locally. Let $\xi \in U$. There is $\Theta \in E'$ such that $g(\xi+\Theta) \neq 0$ and $\xi + t\,\Theta \in U$ for $|t| \leq 1$. There is $0 < r < 1$ such that $|g(\xi+t\Theta)| > 0$ for $|t| = r$, because the zeros of a holomorphic function of one complex variable are isolated. Since $\{t \in \mathbb{C};\ |t| = r\}$ is compact, there is a neighborhood V of $\xi$, $V \subset U$ such that $|g(x+t\Theta)| > \delta > 0$ for each x in V and each $|t| = r$. Define

$$h(x) = (2\pi i)^{-1}\int_{|t|=r} t^{-1}\, f(x+t\Theta).\left[g(x+t\Theta)\right]^{-1} dt$$

for each x in V. It is easy to see that h is locally bounded in V. By the hypothesis of this proposition, there is a mapping $\phi_x$ from $\{t \in \mathbb{C};\ |t| < 1\}$ into $\mathbb{C}$, holomorphic on its domain of definition, such that $f(x+t\Theta) = g(x+t\Theta).\phi_x(t)$ for all $|t| < 1$. On the other hand

$$(2\pi i)^{-1}\int_{|t|=r} \phi_x(t).t^{-1}\, dt = (2\pi i)^{-1}\int_{|t|=r} t^{-1}\, f(x+t\Theta).\left[g(x+t\Theta)\right]^{-1} dt$$

for each x in V. Hence $\phi_x(0) = h(x)$ for each x in V and $f(x) = g(x).h(x)$ for each x in V. Thus h is G-holomorphic in V. Since h is also locally bounded in V, it follows that h is holomorphic in V.

Proposition 2.7 - Let $T_1$ and $T_2$ be elements of $\mathbb{H}'_{Nb}(E)$, $T_2$ not identically zero, such that if $P \in \mathbb{P}_N(^nE)$, $\phi \in E'$, $T_2 * P \exp \phi = 0$ then $T_1(P \exp \phi) = 0$. Then $\hat{T}_1$

is divisible by $\hat{T}_2$ with the quotient as an entire function of exponential type on E'.

Theorem 2.1 - Let $\mathcal{O}$ be a convolution operator on $\mathbb{H}_{Nb}(E)$. The vector subspace of $\mathbb{H}_{Nb}(E)$ generated by $\{P \exp \phi;\ \phi \in E',\ P \in \mathbb{P}_N(^nE),\ n = 0, 1, \ldots\ \mathcal{O}(P \exp \phi) = 0\}$ is dense for the topology $\tau_N$ in the closed subspace $\mathcal{K} = \{f \in \mathbb{H}_{Nb}(E);\ \mathcal{O}f = 0\}$ of $\mathbb{H}_{Nb}(E)$.

Theorem 2.2 - Let $\mathcal{O}$ be a non-zero convolution operator on $\mathbb{H}_{Nb}(E)$ and let ${}^t\mathcal{O}$ be its transpose. Then ${}^t\mathcal{O}\mathbb{H}'_{Nb}(E)$ is equal to the orthogonal of $\{f \in \mathbb{H}_{Nb}(E);\ \mathcal{O}f = 0\}$ in $\mathbb{H}'_{Nb}(E)$ and is closed for the weak topology of $\mathbb{H}'_{Nb}(E)$ defined by $\mathbb{H}_{Nb}(E)$.

Theorem 2.3 - The image of a non-zero convolution operator in $\mathbb{H}_{Nb}(E)$ is equal to $\mathbb{H}_{Nb}(E)$.

Remark - If the mappings $\chi_n$ considered in the first section of this chapter are not 1-1, some modifications have to be done in the preceding material. The reading of the Appendix in Ref. 14 will show clearly what are the modifications that must be done.

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