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RELATIVITY AND GRAVITATION

por

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Volume I

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INTRODUCTION

In these notes on relativity and gravitation, we will present the theoretical models which have been proposed for explaining the phenomena of gravitation in modern physics. We may as well use the term relativistic theory of gravitation in place of the objective modern used above. The relativistic theoretical models used for describing the gravitational field can be devided into two main parts, the so called flat-space time theories which use as the fundamental concept the overall flat space-time of special relativity, and the so called general theory of relativity which uses as the fundamental concept a four dimensional curved space with a Riemannian structure. In the course of this work we will try to compare the results and concepts of these two different points of view.

It is supposed that the reader is familiar with the special theory of relativity, and with the basic physical arguments which have conducted to this theory, as well as with the allied areas such as the theory of the electromagnetic field. At the same time a knowledge of classical canonical dynamics and quantum mechanics will be necessary. Nevertheless, more involved details which turn out to be necessary, as for instance the theory of canonical dyamics with vanishing Jacobian (sometimes called by Dirac theory) will be discussed. This work will be divided into two volumes, the first will treat with the classical model for gravitation from both points of view stressed before. The second volume will treat with the quantum theory of gravitation as far as we know it.

These notes are intended to be an advanced presentation of gravitation, however, in order to have a more complete treatment, we also included some basic introductory concepts; this is done in the section # 1. In this section the concept of tensor is introduced. It is supposed that such concept is already known with respect to the rotation group in three-dimensional Euclidian space, and also with respect to the four-dimensional Lorentz group. In any case, a further reading supplying all points not covered here is advised [1].

1. INTRODUCTORY GEOMETRICAL CONCEPTS

Similarly to almost all branches of physics, such as the Newtonian mechanics, the special relativity and quantum mechanics, the fundamental concept on which general relativity is based on is the space-time concept (the same obviously holds for all flat space-time models of gravitation).

A space-time point is defined by a given observer as the point where a given event takes place at some instant of time as measured by the observer. The collection of all such points forms the space-time.

The space-time is further characterized by its geometrical properties. This means that the way by which the observer locates a point is not arbitrary, but must be compatible with the overall geometrical structure of the space-time. The fundamental geometrical property of the space-time is its topological structure. The topological properties of a space consist of those properties which are not affected by arbitrary deformations of the space. Any space which by means of arbitrary deformations is made to coincide with another space is topologically equivalent to it.

We will assume that locally the topology of space-time is that of a Euclidian four-plane. Consequently it is possible to map the points of a small but finite region of space-time onto the points of a corresponding region of this Euclidian four-plane, which is done by using quadruplets of real numbers. With this property the space-time becomes a manifold, the points being characterized by four real numbers. It should be noted that this is a local property of the space, it may happen that is impossible to map the whole manifold onto a single Euclidian four-plane. The totality of coordinates corresponding to the above region of space-time is called a coordinate patch.

In this process two details have to be clarified. First, the way by which the above mapping was done it is not unique, we separate each one of all possible mappings by associating a observer to each one of them. Since the point P of the space-time by this process may have several different coordinates, for the several different mappings (observers), we must require that all possible coordinatizations, or all possible observers, should conduct to the same physical result. This is the requirement of coordinate covariance, one of the basic postulates of general relativity.

Second, in assigning coordinates to the points of space-time a difficulty arises if the topology of the manifold on the large is not Euclidian; that means if it is not overall equivalent to a single four-plane. In this case we have to use more than a single coordinate patch for covering the whole manifold. In this process it is not claimed the existence of more than one observer. As example, in spherical coordinates the latitude and the longitude becomes singular at the poles and we need two coordinate patches, one patch corvering the entire surface and another patch used to cover the polar region, the overlapping of these two coordinate patches form a coordinate covering of the whole spherical region.

When we have a coordinate covering onto the manifold (a set of more than one coordinate patch), certain sub-sets of points will be covered by two overlapping coordinate patches. A point of such sub-set will therefore have associated to it two coordinates, x^{μ} and x'^{μ}. When x^{μ} are continuous and differentiable functions of x'^{μ}, and vice-versa, we say that the space-time constitutes a differentiable manifold.

1.1) Space-time Mappings

After having coordinated the space-time manifold, it is possible to consider a mapping of the manifold onto itself. Under this mapping each point of the manifold is associated with some other point of the manifold. In order to preserve the topological properties of the space this mapping has to be one-to-one and continuous.

Presently we treat the mapping from a pure mathematical point of view. Later on, we will use this concept for introducing the notion of geometrical objects, such as tensors (and generalizing the notion of mappings to other spaces than the coordinate space, we will introduce other objects such as the spinors and tetrads). The physical interpretation of a mapping in the space-time manifold is similar to that of a Lorentz transformation in special relativity, however, this is a very peculiar example, in general we may have quite different situations.

Assuming that under the mapping the point x^μ of the manifold goes over x'^μ, also belonging to the manifold, by

$$x'^\mu = x'^\mu(x^\alpha) \quad ,$$

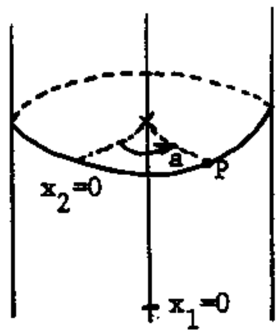
since we have to get a one-to-one transformation which is continuous, this function has to be single-valued and

$$\det\left(\frac{\partial x'}{\partial x}\right) \neq 0$$

If the topology of the space-time is such as to require but a single coordinate patch for its coordinatization, that is, if it has the global topology of a Euclidian four-plane, those are the unique restrictions imposed on the function $x'^\mu = x'^\mu(x^\alpha)$. However, if the topology is such as to require more than one coordinate patch on the space, care must be taken in order to be sure that the mapping function is not mapping a point lying in the overlapping region of two or more patches onto more than one other point of the manifold.

In this case we need more restrictions on the previous mapping function, the form of these extra restrictions will vary in each case depending on the topology of the manifold and the coordinatization employed.

A simple example is given by a two-dimensional manifold with the topology of a cylinder. The simplest coordinatization compatible with this manifold is to use x_1 running along a generator of the cylinder and x_2 running along around the circumference of the cylinder and taking on values in the interval $(0,2\pi)$



Consequently all points lying in the circumferences are characterized by two values of x_2, namely, $x_2 = a$ and $x_2 = a+2\pi$. In order that the allowed mappings be one-to-one, all mapping functions have to be periodic functions of x_2 with period 2π.

If we use a different coordinatization, the conditions on the mapping functions would be different but equivalent to those used above.

1.2) Groups of Mappings

An important property of the set of permissible mappings of the manifold onto itself is that they form a group. This group is defined with respect to the operation of products of mappings. In general this group is not Abelian (commutative). In addition, since the topology of the manifold restricts the allowable mappings, this group will also depend on the topology, and in fact can be used in part to characterize this topology. We will call for short this group by MMG, the manifold mapping group.

Due to the assumed continuity of the mapping functions we can simplify the discussion by considering solely infinitesimal mappings.

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \tag{1-2-1}$$

with the ξ^{μ} small but otherwise arbitrary functions of the x^{μ}. They are called as the descriptors of the mapping. The inverse mapping is, to first order terms,

$$x^{\mu} = x'^{\mu} - \xi^{\mu}(x')$$

Let us now consider the net effect of taking two infinitesimal mappings first in one order and then in the reversed order:

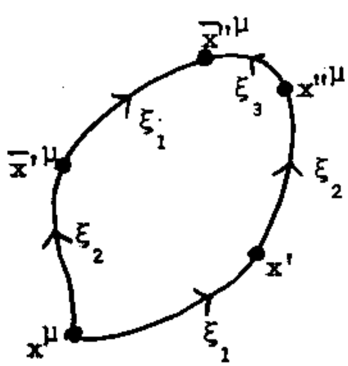
i) The first mapping, with descriptor ξ_1^{μ}, is used to send x^{μ} to x'^{μ} ;

ii) The second mapping, with descriptor ξ_2^{μ}, is used to send x'^{μ} to x''^{μ};

iii) The second mapping is used to transform x^{μ} to $\bar{x}'^{\mu}$;

iv) The first mapping is then used to send $\bar{x}'^{\mu}$ to $\bar{x}''^{\mu}$.

The drawing and the calculations up to second order in the descriptors is



$$x'^{\mu} = x^{\mu} + \xi_1^{\mu}(x)$$

$$x''^{\mu} = x'^{\mu} + \xi_2^{\mu}(x') = x^{\mu} + \xi_1^{\mu}(x) + \xi_2^{\mu}(x) + \xi_1^{\nu}(x)\ \xi_{2,\nu}^{\mu}(x)$$

$$\bar{x}'^{\mu} = x^{\mu} + \xi_2^{\mu}(x)$$

$$\bar{x}''^{\mu} = \bar{x}'^{\mu} + \xi_1^{\mu}(\bar{x}) = x^{\mu} + \xi_2^{\mu}(x) + \xi_1^{\mu}(x) + \xi_2^{\nu}(x)\ \xi_{1,\nu}^{\mu}(x).$$

The difference $\bar{x}''^{\mu} - x''^{\mu}$, is given by

$$\bar{x}''^{\mu} - x''^{\mu} = \xi_2^{\nu}\ \xi_{1,\nu}^{\mu} - \xi_1^{\nu}\ \xi_{2,\nu}^{\mu} = \xi_3^{\mu} \qquad (1\text{-}2\text{-}3)$$

where ξ_3^{μ} is the descriptor of the mapping that transforms x''^{μ} directly over $\bar{x}''^{\mu}$. The mapping represented by the descriptors ξ_3^{μ} is called the commutator of the two mappings with descriptors ξ_1^{μ} and ξ_2^{μ}.

The importance of this commutator lies in the fact that it determines the structure of the group of mappings in the vicinity of the identity element. In particular it serves to check out directly if some given set of mappings form a group, it will be so if the commutator of these mappings is of the same form than each element. Indeed, if some set of mappings form a group, the difference of two mappings is another mapping belonging to the group, as example, consider

$$x'^{\mu} = x^{\mu} + \lambda_1^{\mu}(x)$$

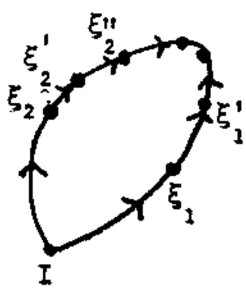
and

$$\bar{x}'^{\mu} = x^{\mu} + \lambda_2^{\mu}(x)$$

for two mappings with descriptors λ_1^{μ} and λ_2^{μ}. The difference is obviously of the same form. Since for the definition of the commutator we take this difference only after computing the products, as done previously, it is necessary that the mappings form a group, since only then is defined the operation of product of mappings, which generates another element of the group.

This property of the commutator will prove to be important when we introduce sub-groups of mappings which form a Lie group. A sub-group of the MMG constitutes a Lie group when the mappings of this sub-group depend on one or more parameters that take on a continuous range of values. Strictly saying this corresponds to Lie groups of first kind, later on we will introduce more general Lie groups which depend on an infinite number of such parameters.

Introducing the abstract manifold where the points are represented by the parameters of the Lie group, which we call the "group space", we see that the mapping associated to a point of this group space which is not close to the origin (the idendity transformation) is independent of the path of integration used to arrive at this point (a path of integration here represents the build up of a finite transformation out of infinitesimal transformations). This property of integrability comes as consequence of the commutator defined before.



Indeed, the difference between two such paths is another element of the group, and therefore is a possible transformation.

An example of Lie group is given by the three-dimensional rotation group, which has infinitesimal mappings of the type

$$x'^r = x^r + \varepsilon^{rs} x^s \qquad r, s = 1, 2, 3 \qquad (1\text{-}2\text{-}4)$$

with $\varepsilon^{rs} = -\varepsilon^{rs}$. This is a three-parameter Lie group, the three parameters are just the three components of the vector around which we carry out the rotation. The commutator of two such mappings has the form

$$\xi_3^r = \varepsilon_3^{rm} x^m = (\varepsilon_1^{rs} \varepsilon_2^{sm} - \varepsilon_2^{rs} \varepsilon_1^{sm}) x^m .$$

The matrix associated to the third mapping is,

$$\varepsilon_3 = \varepsilon_1 \varepsilon_2 - \varepsilon_2 \varepsilon_1 = \left[\varepsilon_1, \varepsilon_2\right]$$

which is antisymmetric, thus proving that the commutator of two rotations is another rotation.

The full linear group in three-space is a nine parameter group with parameters ε^{rs} not subject to any symmetry requirement, the rotation group is a sub-group of it, and as we saw, form a Lie group. The full linear group in three-space my be decomposed into the rotation group, with three parameters, and into a six parameter group

$$x'^r = x^r + \varepsilon^{rs} x^s , \qquad \varepsilon^{rs} = \varepsilon^{sr}$$

which represents a group of deformations in three-space. The commutator of two mappings of this six parameter sub-group of the full linear group, has the form similar to the previous one, now with symmetric 3 × 3 matrices,

$$\varepsilon_3 = \varepsilon_1 \varepsilon_2 - \varepsilon_2 \varepsilon_1 , \ \varepsilon_1^T = \varepsilon_1 , \ \ \varepsilon_2^T = \varepsilon_2 .$$

thus, the matrix associtated to the commutator mapping is antisymmetric, which shows that we get a rotation as result. This shows that the six parameter sub-group is not a Lie group, and cannot, strictly saying, be considered as an individual group either. It may be considered as a part of the full linear group, the commutator of two of these mappings represents a rotation, that is, belongs to the remainder of the linear group. However, the commutator of a deformation and a rotation is a deformation,

$$\varepsilon_3 = \varepsilon_1 \varepsilon_2 - \varepsilon_2 \varepsilon_1$$

where ε_1 is symmetric and ε_2 skew-symmetric. The ε_3 is symmetric.

The table of multiplication for the linear group in three-space is then,

$$R_1 R_2 - R_2 R_1 = R_3$$

$$D_1 D_2 - D_2 D_1 = R$$

$$D_1 R - R D_1 = D_2$$

In four dimensions the full linear group is a sixteen parameter group. The sub-group of this group with ten parameter,

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}{}_{\nu} x^{\nu} + \varepsilon^{\mu}$$

$$\eta_{\mu\rho} \varepsilon^{\rho}{}_{\nu} + \eta_{\nu\rho} \varepsilon^{\rho}{}_{\mu} = 0$$

$$\eta_{\mu\nu} = (1, -1, -1, -1)$$

is the Poincaré, or the inhomogeneous Lorentz group, it forms a ten parameter Lie group. Indeed, the commutator of two elements of the Poincaré group is of the form

$$\bar{x}''^{\mu} - x''^{\mu} = \varepsilon^{\mu}_{(3)\nu} x^{\nu} + \varepsilon^{\mu}_{(3)}$$

with

$$\varepsilon^{\mu}_{(3)\ \nu} = \varepsilon^{\mu}_{(1)\ \lambda} \varepsilon^{\lambda}_{(2)\ \nu} - \varepsilon^{\mu}_{(2)\ \lambda} \varepsilon^{\lambda}_{(1)\ \nu}$$

$$\varepsilon^{\mu}_{(3)} = \varepsilon^{\mu}_{(1)\ \lambda} \varepsilon^{\lambda}_{(2)} - \varepsilon^{\mu}_{(2)\ \lambda} \varepsilon^{\lambda}_{(1)}$$

as it may be easily checked the matrix $\varepsilon_{(3)}$ satisfies

$$\eta_{\alpha\mu} \varepsilon^{\mu}_{(3)\ \nu} + \eta_{\nu\mu} \varepsilon^{\mu}_{(3)\ \alpha} = 0$$

which shows that the Poincaré group is a ten parameter Lie group.

The expression for the $\xi^{\mu}_{(x)}$ given above for the Poincaré group are the cartesian forms for these quantities. They are not the unique form that yield the commutator structure of these groups. Indeed, given a Lie group and one particular form of the $\xi^\mu(x)$ that reproduces the commutator structure of this group (in the previous case, the cartesian form of the ξ^μ), we can construct an infinity of other ξ^μ having this same property. They can be presented in the general form

$$\xi^\mu(x) = \varepsilon^i \; f_i^\mu(x) \;, \; i = 1, \ldots N \tag{1-2-5}$$

where the ε^i are the group parameters. The commutator of two mappings of the form given by (1-2-5) is,

$$\bar{x}''^\mu - x''^\mu = - (\varepsilon_1^i \, \varepsilon_2^j - \varepsilon_2^i \, \varepsilon_1^j) \; f_j^\alpha \, f_{i,\alpha}^\mu = \xi_3^\mu \tag{1-2-6}$$

In order that it be of the form (1-2-5) the functions f_i must satisfy

$$f_j^\alpha \;\; f_{i,\alpha}^\mu \; = c_{ij}^k \;\; f_k^\mu \tag{1-2-7}$$

where the c_{ij}^k are constants independent of the ε^i and of the x^μ. They are called the "structure constants" of the group and serve to characterize the group intrinsically, that is, independently of the particular form taken by the f_i^μ. Consequently, two groups with the same set of structure constants are isomorphic to each other, at least in the vicinity of the identity element.

Substituting (7) into (6) we get,

$$\xi_3^\mu = (\varepsilon_1^i \, \varepsilon_2^j - \varepsilon_2^i \, \varepsilon_1^j) \; c_{ij}^k \; f_k{}^\mu \tag{1-2-8}$$

therefore, the infinitesimal parameters ε_3^i of the commutator are,

$$\varepsilon^k = (\varepsilon_1^i \, \varepsilon_2^j - \varepsilon_2^i \, \varepsilon_1^j) c_{ij}^k \tag{1-2-9}$$

According to the form under which the structure constants have been constructed, they have to satisfy

$$c_{ij}^{k} = - c_{ji}^{k} .$$

They can also be shown to satisfy the Jacobi identity (which is left as an exercise)

$$c_{ij}^{k} \; c_{kl}^{m} + c_{li}^{k} \; c_{kj}^{m} + c_{jl}^{k} \; c_{ki}^{m} = 0 .$$

It is possible reparametrize a given group by introducing a new set of parameters, function of the original set,

$$\varepsilon'^{i} = \alpha_{j}^{i} \varepsilon^{j}$$

where α is a non-singular matrix. One can then express the ξ^{μ} in terms of the new parameters by

$$\xi^{\mu} = \varepsilon'^{i} \, f'^{\mu}_{i}$$

note that this is a change only on the index i, nothing having to do with the index μ.

Then,

$$f'^{\mu}_{i} = \bar{\alpha}_{i}^{j} \; f_{j}^{\mu}$$

where $\bar{\alpha}$ is the inverse of α.

The structure constants change under this reparametrization, a short calculation gives

$$c'^{k}_{ij} = \alpha_{n}^{k} \bar{\alpha}_{i}^{\ell} \bar{\alpha}_{j}^{m} \, c_{\ell m}^{n} \qquad (1\text{-}2\text{-}10)$$

Since a variation in parameters does not change the basic structure of a group, we see that two groups which possess structure constants differing according to (10) are two realizations of a same basic group.

Problem 1.1

Reparametrize the rotation group in three-space by taking, as the new infinitesimal parameters, $\varepsilon^{1} = \varepsilon^{23}$, $\varepsilon^{2} = \varepsilon^{31}$, and $\varepsilon^{3} = \varepsilon^{12}$. Calculate the structure constants for these parameters.

Suppose we have a f_i^μ which satisfies (7) and that we introduce a new set of variables $x'^\mu = x'^\mu$ (x). The descriptors ξ^μ when expressed in function of the new set of variables, take the form

$$\xi'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\alpha} \; \xi^\alpha(x)$$

we may write this as

$$\xi'^\mu(x') = \varepsilon^i \; g_i^\mu(x') \;, \qquad g_i^\mu = x'^\mu_{,\alpha} \; f_i^\alpha$$

We now show that the transformed ξ' have the same commutator structure as the original ξ, only replacing in the original formulas f_i^α by g_i^α. The commutator here takes the form

$$\xi'^\mu_3 = (\varepsilon_1^i \; \varepsilon_2^j \quad \varepsilon_2^i \; \varepsilon_1^j) g^\mu_{i,\nu} \; g_j^\nu$$

Using the above formula for g_i^μ in terms of f_i^μ, we get

$$g^\mu_{i,\nu} \; g_j^\nu = x'^\mu_{,\rho\sigma} f_i^\rho \; f_j^\sigma + x'^\mu_{,\rho} \; f^\rho_{i,\sigma} f_j^\sigma$$

Thus,

$$\xi_3'^\mu = (\varepsilon_1^i \; \varepsilon_2^j \quad \varepsilon_2^i \; \varepsilon_1^j)(x'_{,\rho\sigma} \; f_i^\rho \; f_j^\sigma + x'^\mu_{,\rho} \; f^\rho_{i,\sigma} \; f_j^\sigma)$$

which gives simply,

$$\xi_3'^\mu = (\varepsilon_1^i \; \varepsilon_2^j - \varepsilon_2^i \; \varepsilon_1^j) \; x'^\mu_{,\rho} \; f^\rho_{i,\sigma} \; f_j^\sigma$$

Using (7), we obtain for this relation

$$\xi_3' = (\varepsilon_1^i \; \varepsilon_2^j - \varepsilon_2^i \; \varepsilon_1^j) \; x'^\mu_{,\rho} \; c^k_{ij} \; f_k^\rho$$

From the equation (9), this is equal to

$$\xi_3'^\mu = \varepsilon_3^k \; x'^\mu_{,\rho} \; f_k^\rho = \varepsilon_3^k \; g_k^\mu$$

Which shows that the group structure was preserved by the transformation $x'^\mu =$ $= x'^\mu(x)$.

1.3) Geometrical Objects

Associated to the observables of a given physical system there exists quantities, such as for instance, the force field, the angular momentum or the intrinsic angular momentum. We call those quantities in general as geometrical objects. The term geometrical used here emphasizes the fact that those quantities will be defined according to their transformation law under the action of the MMG.

Allied to the concept of transformation suffered by some object with this structure, there exists the concept of covariance of a given relation involving several quantities of this form. We say that such a relation is covariant if all members of it change in the same way under the MMG. We will turn back to a more clear definition of covariance at the end of this section.

Suppose that a geometrical object is defined at the point x^μ by means of the N functions $y_A(x)$, $A = 1,\ldots N$. The specification of the nature of such object is determined according to the way it transforms under the MMG which transforms the coordinate x^μ into another value x'^μ.

$$y_A^{(x)} \rightarrow y_A'(x') = f_A(y(x),\, x'(x)) \qquad (1\text{-}3\text{-}1)$$

provided that this mapping in the Y-space possess all properties of the mapping into the coordinate space. That is, associated to the product of two elements of the MMG there exists a mapping in the Y-space which is the product of the two mappings generated by each element of the MMG. There exists the identity mapping in the Y-space, and the inverse of each mapping is another mapping of the Y-space onto itself. With these impositions the mapping of the Y-space onto itself is a realization of the MMG.

Associated to the infinitesimal mapping with descriptor $\xi^\mu(x)$ there exists the mapping on the components of y,

$$y'_A(x')=y_A(x) + \delta y_A(x) \tag{1-3-2}$$

$$\delta y_A(x) = \phi_A(y_A(x),\ y_{A,\mu}(x),\ldots,\xi^\alpha(x),\ \ \xi^\alpha_{,\mu}(x),\ldots)$$

With these concepts, we can now define what we meant by covariance of any given relation. Let it be of the general form

$$F_B(y_A^{(1)}(x),\ y_A^{(2)}(x),\ldots) = 0$$

if all members of this relation change in the same way under the mapping induced by any element of the MMG the relation will mantain the same structure in terms of the new variables $y'^{(1)}_A(x')$, $y'^{(2)}_A(x')$,... In this situation we say that the relation is covariant under the mapping induced by the MMG.

1.4) Tensors

There exists a large number of quantities satisfying the previous definitions of what we have called geometrical objects. However, there is a sub-class of such quantities which are specially useful in representing physical observables. This sub-class corresponds to the case where the f_A are linear functions of the y_A

$$y'_A(x') = f_A(y(x),\ x'(x)) = \lambda_A^B(x)\ y_B(x) + \phi_A(x)$$

For those who are already aquainted with differential geometry and with tensor calculus, an example of such objects is given, for instance, by the Christoffel symbols. Even more important are the quantities which possess a homogeneous and linear transformation law, $\phi_A = 0$. In this case the geometrical objects $y_A(x)$ constitutes a representation of the MMG. In this section we will treat with such quantities.

The simplest example of these objects is the scalar, for which,

$$y'(x') = y\ (x) \qquad (1\text{-}4\text{-}1)$$

Next, in order of the crescent number of components, we have the vectors which sub-divide into two types: Contravariant and covariant vectors. A contravariant vector in a n-dimensional manifold is the set of n components $A^{\mu}(x)$ which under any element of the MMG transforms in the same way as the differential of the coordinates.

$$A'^{\mu}(x') = \frac{\partial x'^{\mu}}{x^{\nu}} A^{\nu}(x) \qquad (1\text{-}4\text{-}2)$$

therefore

$$\delta A^{\mu}(x) = \xi^{\mu}_{,\nu}(x)\ A^{\nu}(x) \qquad (1\text{-}4\text{-}3)$$

A covariant vector is one which transforms in the same way as the gradient of a scalar,

$$A'_{\mu}(x') = \frac{\partial x^{\nu}}{\partial x'^{\mu}} A_{\nu}(x) \qquad (1\text{-}4\text{-}4)$$

thus,

$$\delta A_{\mu}(x) = -\ \xi^{\nu}_{,\mu}(x)\ A_{\nu}(x) \qquad (1\text{-}4\text{-}5)$$

Scalars, contravariant and covariant vectors are special cases of a general class of geometrical objects with linear homogeneous transformation laws, called tensors. In a n-dimensional manifold, a tensor of rank r has n^r components $T^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_k}(x)$, p+k=r. For our case we take n=4, representing the dimensions of the space-time manifold. It satisfies the transformation law,

$$T'^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_k}(x') = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_p}}{\partial x^{\alpha_p}}\ \frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_k}}{\partial x'^{\nu_k}}\ T^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_k}(x) \qquad (1\text{-}4\text{-}6)$$

It is possible to perform several operations with tensors, we just list those operations and give some examples. A further reading is advised for those which are not familiar with tensor calculus.

i) Addition of tensors

One can add or subtract tensors of the same type and variance, provided they are all defined at the same point of the manifold. As result another tensor of the same kind is obtained.

ii) Multiplication of tensors

The product of a tensor of rank r by another of rank s generates a tensor with rank r+s.

iii) Contraction

From any tensor of rank r with contra and covariant indices one can form another tensor of rank r-2 by means of the operation,

$$T^{\mu_1 \ldots \mu_p}_{\mu_1 \ldots \nu_k} = B^{\mu_2 \ldots \mu_p}_{\nu_2 \ldots \nu_k} , \quad p + k = r$$

called the contraction of $T^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_k}$. This operation may be repeated whenever we have at our disposal pairs of contra and co-variant indices. A property of tensors which is very important is the symmetry or skew symmetry character of their components. A tensor is symmetric or anti-symmetric depending if the components change, or do not change, under a change of two indices of the same kind. This property is independent of the choice of coordinates. If this happens for any pair of indices we say that the tensor is completely symmetric, or completely anti-symmetric. As example,

$$T_{\mu\nu\alpha} = \phi_\mu \phi_\nu \phi_\alpha .$$

is completely symmetric, and the Levi-Civita permutation symbol is completely anti-symmetric.

Any tensor may be decomposed into a symmetric and a skew-symmetric parts.

$$T_{\mu\nu} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) + \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}) = T_{(\mu\nu)} + T_{|\mu\nu|}$$

This result may be generalized to higher order tensors, for instance

$$T_{(\mu\nu\rho)} = \frac{1}{3!}(T_{\mu\nu\rho} + T_{\rho\mu\nu} + T_{\nu\rho\mu} + T_{\nu\mu\rho} + T_{\mu\rho\nu} + T_{\rho\nu\mu})$$

$$T_{[\mu\nu\rho]} = \frac{1}{3!}(T_{\mu\nu\rho} + T_{\rho\mu\nu} + T_{\nu\rho\mu} - T_{\nu\mu\rho} - T_{\mu\rho\nu} - T_{\rho\nu\mu})$$

Turning back to the case of $T_{\mu\nu}$, it may be verified that under a transformation induced by the MMG, the symmetric part $T_{(\mu\nu)}$ transforms as function of the symmetric part in the new coordinate system,

$$T'_{(\mu\nu)}(x') = f_{\mu\nu}(T_{(\alpha\beta)}(x), \xi_\rho(x))$$

The same occuring for the antisymmetric part,

$$T'_{[\mu\nu]}(x') = \phi_{\mu\nu}(T_{|\mu\nu|}(x), \xi_\rho(x))$$

Whenever a geometrical object can be broken up into parts that transform among themselves, we say that we have a reducible object. If no such decomposition is possible, we have an irreducible object. We see that $T_{(\mu\nu)}$ and $T_{[\mu\nu]}$ are both irreducible objects.

Problem 1.2 – Show that T^μ_ν is reducible and that is irreducible parts consist of its trace T^μ_μ, and a traceless tensor with components $T^\mu_\nu - \frac{1}{4}\delta^\mu_\nu T^\rho_\rho$.

Of some importance for practical calculations are the generalizations of the Kronecker symbol

$$\delta^{\mu\nu}_{\rho\sigma} = \begin{vmatrix} \delta^\mu_\rho & \delta^\nu_\rho \\ \delta^\mu_\sigma & \delta^\nu_\sigma \end{vmatrix} = \delta^\mu_\rho \; \delta^\nu_\sigma - \delta^\mu_\sigma \; \delta^\nu_\rho$$

and

$$\delta^{\mu\nu\lambda}_{\rho\sigma\tau} = \begin{vmatrix} \delta^{\mu}_{\rho} & \delta^{\nu}_{\rho} & \delta^{\lambda}_{\rho} \\ \delta^{\mu}_{\sigma} & \delta^{\nu}_{\sigma} & \delta^{\lambda}_{\sigma} \\ \delta^{\mu}_{\tau} & \delta^{\nu}_{\tau} & \delta^{\lambda}_{\tau} \end{vmatrix}$$

which, for instance, allow us to rewrite the previous relations in compact form,

$$T_{[\mu\nu]} = \frac{1}{2}\,\delta^{\rho\sigma}_{\mu\nu}\;T_{\rho\sigma}$$

$$T_{[\mu\nu\lambda]} = \frac{1}{3!}\,\delta^{\rho\sigma\tau}_{\mu\nu\lambda}\,T_{\rho\sigma\tau}$$

In the remainder of this section we treat the problem of construction of tensors by differentiation. Given for instance the first rank covariant tensor $A_\mu(x)$, we may form the components

$$B_{\mu\nu}(x) = \frac{\partial A_\mu}{\partial x^\nu}$$

a direct calculation shows that these componentes transform as

$$B'_{\mu\nu}\;(x') = \frac{\partial x^\rho}{\partial x'^\mu}\;\frac{\partial x^\sigma}{\partial x'^\nu}\,B_{\rho\sigma}(x) + \frac{\partial^2\, x^\rho}{\partial x'^\mu \partial x'^\nu}\,A_\rho$$

Thus, in order to calculate $B'_{\mu\nu}$ we need to know $B_{\mu\nu}$ and A_μ at each point. Since the $B_{\mu\nu}$ are independent of the A_μ, we need more than just the $B_{\mu\nu}$ and the mapping function $\xi^\mu(x)$ for calculating the $B'_{\mu\nu}$. This means that the $B_{\mu\nu}(x)$ do not form a geometrical object, according to the previous definition (1-3-1). Consequently, the $B_{\mu\nu}(x)$ do not form a tensor. Nevertheless, there are some operations involving derivatives which give as result new tensors, they are:

i) $F_{\mu\nu} = \dfrac{\partial A_\mu}{\partial x^\nu} - \dfrac{\partial A_\nu}{\partial x^\mu}$, which is a skew-symmetric second rank tensor.

ii) $B_{\mu\nu\rho} = A_{\rho\mu,\nu} + A_{\nu\rho,\mu} + A_{\mu\nu,\rho}$ is a completely antisymmetric third rank tensor. In this relation, the $A_{\mu\nu}$ is antisymmetric.

iii) If $A_{\mu\nu\rho}$ is completely antisymmetric, the combination

$$B_{\mu\nu\rho\sigma} = A_{\mu\nu\rho,\sigma} - A_{\sigma\mu\nu,\rho} + A_{\rho\sigma\mu,\nu} - A_{\nu\rho\sigma,\mu} = \delta^{\alpha\beta\lambda\tau}_{\mu\nu\rho\sigma} A_{\alpha\beta\lambda,\tau}$$

forms a fourth rank completely antisymmetric tensor. This last construction is as far as one can go with such process in four dimensions. It must be noted that all such tensors are antisymmetric, no symmetric tensor can be formed out of derivatives of a tensor. The reason for that lies in the fact that for antisymmetric tensors we can eliminate the extra term which appear in the transformation law, for instance when calculating $A_{\mu,\nu} - A_{\nu,\mu}$, we eliminate the term $\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} A_\rho$ in the transformation law, thus yielding a new tensor. Finally, it must be remarked that the present interpretation holds for general non-linear mappings. If the elements of the MMG under consideration belong to a linear sub-group,

$$\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} = 0$$

all operations of derivatives will give as result new tensors. This happens in special relativity where the MMG is a linear group, the Poincaré group.

1.5) Tensor Densities

There are a type of geometrical object which similarly to the tensors also possess a homogeneous, linear transformation law. Strictly saying, we may consider the tensor as a particular object of this nature. They are called as tensor densities, and transform as

$$\mathfrak{T}'^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_k}(x') = \left| \frac{\partial x}{\partial x'} \right|^w \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_p}}{\partial x^{\alpha_p}} \frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_k}}{\partial x'^{\nu_k}} \mathfrak{T}^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_k}(x) \qquad (1\text{-}5\text{-}1)$$

where W is the weight of the density and is a constant. Tensors may be considered as tensor densities of weight zero.

The expression for $\delta\mathfrak{T}^{\mu_1\dots}_{\nu_1\dots}$ has, with respect to that of a tensor with the same variance, an extra term which arises from the determinant standing on the right hand side of (5). One finds,

$$\delta\mathfrak{T}^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_k}(x) = -W\xi^{\alpha}_{,\alpha}\,\mathfrak{T}^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_k} + \xi^{\mu_1}_{,\alpha_1}\,\mathfrak{T}^{\alpha_1\dots\mu_p}_{\nu_1\dots\nu_k} + \dots\xi^{\mu_p}_{,\alpha_p}\,\mathfrak{T}^{\mu_1\dots\alpha_p}_{\nu_1\dots\nu_k} - \xi^{\beta_1}_{,\nu_1}\,\mathfrak{T}^{\mu_1\dots\mu_p}_{\beta_1\dots\nu_k} - \dots$$

$$\dots - \xi^{\beta_k}_{,\nu k}\,\mathfrak{T}^{\mu_1\dots\mu_p}_{\nu_1\dots\beta_k} .$$

All operations performed on tensors may be performed too on tensor densities. We have only to take care with: The multiplication of a tensor density of weight W_1 by another with weight W_2 yields a third tensor density with weight $W_1 + W_2$.

Related to the problem of constructing new densities out of differentiation of a given density, we have:

i) If $\mathfrak{V}^{\mu}$ is a contravariant vector density of weight +1,

$$\mathfrak{V}'^{\mu}(x') = \left|\frac{\partial x}{\partial x'}\right| \frac{\partial x'^{\mu}}{\partial x^{\nu}}\,\mathfrak{V}^{\nu}(x)$$

then its divergence $\mathfrak{V}^{\mu}_{,\mu}$ is a scalar density of weight +1, that is:

$$\frac{\partial \mathfrak{V}'^{\mu}(x')}{\partial x'^{\mu}} = \left|\frac{\partial x}{\partial x'}\right| \frac{\partial \mathfrak{V}^{\mu}(x)}{\partial x^{\mu}}$$

The proof being,

$$\left|\frac{\partial x}{\partial x'}\right| \quad \frac{1}{4!}\,\varepsilon^{\lambda\rho k\delta}\,\varepsilon_{\alpha\beta\gamma\tau}\,\frac{\partial x^{\alpha}}{\partial x'^{\lambda}}\,\frac{\partial x^{\beta}}{\partial x'^{\rho}}\,\frac{\partial x^{\gamma}}{\partial x'^{k}}\,\frac{\partial x^{\tau}}{\partial x'^{\delta}}$$

thus,

$$\frac{\partial}{\partial x'^{\mu}} \left| \frac{\partial x}{\partial x'} \right| \frac{1}{3!} \varepsilon^{\lambda\rho k\delta} \varepsilon_{\alpha\beta\gamma\tau} \frac{\partial^2 x^{\alpha}}{\partial x'^{\lambda}\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x^{\rho}} \frac{\partial x^{\gamma}}{\partial x'^{k}} \frac{\partial x^{\tau}}{\partial x'^{\delta}}$$

Therefore

$$\frac{\partial \mathfrak{U}'^{\mu}}{\partial x'^{\mu}} \frac{1}{3!} \varepsilon^{\lambda\rho k\delta} \varepsilon_{\alpha\beta\gamma\tau} \frac{\partial^2 x^{\alpha}}{\partial x'^{\lambda}\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x^{\rho}} \frac{\partial x^{\gamma}}{\partial x'^{k}} \frac{\partial x^{\tau}}{\partial x'^{\delta}} \frac{\partial x'^{\mu}}{\partial x^{\nu}} \mathfrak{U}^{\nu} +$$

$$\left| \frac{\partial x}{\partial x'} \right| \frac{\partial^2 x'^{\mu}}{\partial x^{\nu}\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \mathfrak{U}^{\nu} + \left| \frac{\partial x}{\partial x'} \right| \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial \mathfrak{U}^{\nu}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial x'^{\mu}}$$

it is not difficult to check that the two first terms on the right hand side vanish, and the third term conducts to the desired result.

$$\frac{\partial \mathfrak{U}'^{\mu}}{\partial x'^{\mu}} = \left| \frac{\partial x}{\partial x'} \right| \frac{\partial \mathfrak{U}^{\mu}}{\partial x^{\mu}}$$

ii) If $\mathfrak{U}^{\mu\nu}$ is an antisymmetric contravariant tensor density of weight +1, then its divergence is a contrvariant tensor density of first rank and weight +1.

$$\mathfrak{U}'^{\mu\nu}(x') = \left| \frac{\partial x}{\partial x'} \right| \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} \mathfrak{U}^{\rho\sigma}(x)$$

$$\frac{\partial \mathfrak{U}'^{\mu\nu}}{\partial x'^{\nu}} = \left| \frac{\partial x}{\partial x'} \right| \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial \mathfrak{U}^{\rho\lambda}}{\partial x^{\lambda}}$$

iii) If $\mathfrak{U}^{\mu\nu\rho}$ is fully antisymmetric and with weight +1, then $\frac{\partial \mathfrak{U}^{\mu\nu\rho}}{\partial x^{\rho}}$ is also antisymmetric with weight +1.

iv) If $\mathfrak{U}^{\mu\nu\rho\sigma}$ is a completely antisymmetric tensor density, then its divergence $\mathfrak{U}^{\mu\nu\rho\sigma}_{,\sigma}$ has the same weight as the original object, both being +1.

It must be noted from the definition of determinant written before that we have,

$$\varepsilon_{\lambda\rho k\delta} \left| \frac{\partial x}{\partial x'} \right| = \varepsilon_{\alpha\beta\gamma\tau} \frac{\partial x^{\alpha}}{\partial x'^{\lambda}} \frac{\partial x^{\beta}}{\partial x'^{\rho}} \frac{\partial x^{\gamma}}{\partial x'^{k}} \frac{\partial x^{\tau}}{\partial x'^{\delta}}$$

which means that the Levi Civita permutation symbol $\varepsilon_{\lambda\rho k\delta}$ is a fully antisymmetric fourth rank tensor density with weight -1. By the same token we may write,

$$\varepsilon^{\lambda\rho k\delta} \left| \frac{\partial x'}{\partial x} \right| = \varepsilon^{\alpha\beta\gamma\tau} \frac{\partial x'^{\lambda}}{\partial x^{\alpha}} \frac{\partial x'^{\rho}}{\partial x^{\beta}} \frac{\partial x'^{k}}{\partial x^{\gamma}} \frac{\partial x'^{\delta}}{\partial x^{\tau}}$$

which shows that $\varepsilon^{\lambda\rho k\tau}$ is of the same nature than the above $\varepsilon_{\lambda\rho k\tau}$ but has weight +1.

As we have seen, these permutation symbols serve to give compact expressions for the determinant of given matrices. If now we write the determinant of density matrices with weight W, and of the form $\mathfrak{U}^{\mu\nu}$, $\mathfrak{U}^{\mu}$ and $\mathfrak{U}_{\mu\nu}$. We get,

$$|\mathfrak{U}^{\mu\nu}| \quad \frac{1}{4!} \varepsilon_{\alpha\beta\gamma\delta}\, \varepsilon_{\mu\nu\rho\sigma}\, \mathfrak{U}^{\alpha\mu}\, \mathfrak{U}^{\beta\nu}\, \mathfrak{U}^{\gamma\rho}\, \mathfrak{U}^{\delta\sigma} \tag{1-5-2}$$

$$|\mathfrak{U}^{\mu}_{\ \nu}| \quad \frac{1}{4!} \varepsilon_{\alpha\beta\gamma\delta}\, \varepsilon^{\mu\nu\rho\sigma}\, \mathfrak{U}^{\alpha}_{\ \mu}\, \mathfrak{U}^{\beta}_{\ \nu}\, \mathfrak{U}^{\gamma}_{\ \rho}\, \mathfrak{U}^{\delta}_{\ \sigma} \tag{1-5-3}$$

$$|\mathfrak{U}_{\mu\nu}| \quad \frac{1}{4!} \varepsilon^{\alpha\beta\gamma\delta}\, \varepsilon^{\mu\nu\rho\sigma}\, \mathfrak{U}_{\alpha\mu}\, \mathfrak{U}_{\beta\nu}\, \mathfrak{U}_{\gamma\rho}\, \mathfrak{U}_{\delta\sigma} \tag{1-5-4}$$

Which shows that $|\mathfrak{U}^{\mu\nu}|$, $|\mathfrak{U}^{\mu}_{\nu}|$ and $|\mathfrak{U}_{\mu\nu}|$ are respectively scalar densities with weight 4W-2, 4W and 4W+2.

If the matrices $\mathfrak{U}_{\mu\nu}$ and $\mathfrak{U}^{\mu\nu}$ are ansitymmetric, we have the further property,

$$\sqrt{|\mathfrak{U}^{\mu\nu}|} = \frac{1}{8}\, \varepsilon_{\mu\nu\rho\sigma}\, \mathfrak{U}^{\mu\nu}\, \mathfrak{U}^{\rho\sigma}$$

$$\sqrt{|\mathfrak{U}_{\mu\nu}|} = \frac{1}{8}\, \varepsilon^{\mu\nu\rho\sigma}\, \mathfrak{U}_{\mu\nu}\, \mathfrak{U}_{\rho\sigma}$$

The $\varepsilon_{\mu\nu\rho\sigma}$ symbols also permit to go from tensors (or tensor densities) like A^{α}, or $A^{\alpha\beta}$ to $\check{A}_{\alpha\beta\gamma}$ and $\check{A}_{\alpha\beta}$ by,

$$\check{A}_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma\delta} A^{\delta} \tag{1-5-5}$$

$$\check{A}_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma\delta} A^{\gamma\delta} \tag{1-5-6}$$

If the weight of A^{α} and $A^{\alpha\beta}$ is respectively W_1 and W_2, the weights of $\check{A}_{\alpha\beta\gamma}$ and $\check{A}_{\alpha\beta}$ will be W_1-1 and W_2-1. Similarly, with $\varepsilon^{\mu\nu\rho\sigma}$ we may form

$$\check{A}^{\alpha\beta\gamma} = \varepsilon^{\alpha\beta\gamma\delta} A_{\delta} \tag{1-5-7}$$

$$\check{A}^{\alpha\beta} = \varepsilon^{\alpha\beta\gamma\delta} A_{\gamma\delta} \tag{1-5-8}$$

If A_{δ} and $A_{\gamma\delta}$ have weights W_1 and W_2, the $\check{A}^{\alpha\beta\gamma}$ and $\check{A}^{\alpha\beta}$ will have weights W_1+1 and W_2+1.

As it is clear, the relations (6) and (8) may be solved for $A_{\gamma\delta}$ and $A^{\gamma\delta}$ only if they are both antisymmetric.

$$A^{\gamma\delta} = \frac{1}{2} \varepsilon^{\gamma\delta\alpha\beta} \check{A}_{\alpha\beta}$$

$$A_{\gamma\delta} = \frac{1}{2} \varepsilon_{\gamma\delta\alpha\beta} \check{A}^{\alpha\beta}$$

The equations (5) and (7) may be solved for the A^{δ} and A_{δ} without any restriction on these components.

$$A^{\delta} = \frac{1}{3!} \check{A}_{\alpha\beta\gamma} \varepsilon^{\alpha\beta\gamma\delta}$$

$$A_{\delta} = \frac{1}{3!} \check{A}^{\alpha\beta\gamma} \varepsilon_{\alpha\beta\gamma\delta}$$

For finishing this section we call attention to the fact that the tensor densities of the type $\check{A}$ introduced by the previous relations, are not the usually called dual tensors to the A. In general manifolds the dual tensors can be defined only after introducing a metric into the space.

1.6) Integrals and Stoke's Theorem in Curved Spaces

In this section we want to construct an operation of integration on the components of geometrical objects, however, in order that such an operation have meaning it has to give as result another geometrical object. We begin with the simplest situation, where we want to sum up the several values that a scalar field takes on different points of the manifold. Let those points be $x_1^\mu, x_2^\mu \ldots x_n^\mu$. Thus, we form the quantity

$$\phi(x_1) + \phi(x_2) + \ldots \phi(x_n)$$

This quantity may be thought as a single number associated to the n points.

Under a mapping we have,

$$\phi'(x_1') + \phi'(x_2') + \ldots \phi'(x_n') = \phi(x_1) + \phi(x_2) + \ldots \phi(x_n)$$

Therefore, this n-point scalar is a geometrical object. However, we cannot generalize this for vectors or higher order tensors, for instance, the sum

$$\phi^\mu(x_1) + \phi^\mu(x_2) + \ldots \phi^\mu(x_n)$$

does not represent a geometrical object, since at each different point a vector transforms differently under the MMG, therefore in this sum we will have n different mapping functions, as result we get a complicated expression which is not a vector, nor any known geometrical object. Thus, we already know that the sum of values that a scalar field takes on several points of the manifold, does serve as the integrand of an integral which we write as

$$\int \phi(x)\, d_4x$$

in order to know if such operation does have meaning we calculate how the element of volume transforms under a mapping. We get,

$$d_4x' = dx_1'\, dx'^2\, dx'^3\, dx'^4 = \frac{\partial x'^1}{\partial x^\mu} \frac{\partial x'^2}{\partial x^\nu} \frac{\partial x'^3}{\partial x^\sigma} \frac{\partial x'^4}{\partial x^\rho} dx^\mu\, dx^\nu\, dx^\sigma\, dx^\rho$$

which may be written as

$$d_4x' = \frac{1}{4!}\,\varepsilon'_{\mu\nu\rho\sigma}\;\; d\tau'^{[\mu\nu\rho\sigma]} \quad ,$$

where

$$d\tau^{[\mu\nu\rho\sigma]} = \delta^{[\mu\nu\rho\sigma]}_{1234}\; dx^1\; dx^2\; dx^3\; dx^4$$

thus, since $\varepsilon_{\mu\nu\rho\sigma}$ is a tensor density with weight (-1) and $d\tau^{[\mu\nu\rho\sigma]}$ is a tensor

$$d_4x' = \left|\frac{\partial x'}{\partial x}\right|\; d_4x$$

which shows that d_4x is not a scalar, but rather a scalar density with weight -1. Consequently in order to have just the sum of scalars, which as we saw, has a meaning, we need to take $\phi(x)$ as a scalar density with weight +1. If we do not do that, we get an operation which does not give a geometrical object as result. In short, the operation

$$I = \int \phi(x)\; d_4x$$

has a meaning whenever $\phi(x)$ is a scalar density of weight +1.

Usually such integral is extended over a finite region R of the manifold. Often is convenient to represent this region by means of some number of parameters with the obvious condition that we take so many parameters as the number of dimensions of the submanifold R. If this number is $M(\leqslant 4)$, any point within R is given by

$$x^\mu = \phi^\mu(\lambda_1 \ldots \lambda_M) \qquad (1\text{-}6\text{-}1)$$

We now form the element of "area"

$$D^{\mu_1 \ldots \mu_M}\, d\lambda_1 \ldots d\lambda_M = d\tau^{\mu_1 \ldots \mu_M}$$

$$D^{\mu_1 \ldots \mu_M} = \delta^{\mu_1 \ldots \mu_M}_{\nu_1 \ldots \nu_M}\;\; \phi^{\nu_1}_{,1} \ldots \phi^{\nu_M}_{,M}\; ;\; \phi^{\nu_k}_{,\ell} = \frac{\partial \phi^{\nu_k}}{\partial \lambda_\ell}\; . \qquad (1\text{-}6\text{-}2)$$

In order to understand this relation, we turn back to the simple case of three-dimensional spaces, and R is here a surface embedded into this space. The surface is parametrized by two parameters u_1 and u_2,

$$x^i = \phi^i(u_1, u_2), \quad i = 1, \ldots 3$$

we write the two vectors associated to infinitesimal increments du_1 and du_2,

$$A^i = \phi^i_{,1} du_1 = \frac{\partial \phi^i}{\partial u_1} du_1$$

$$B^i = \phi^i_{,2}\ du_2$$

These two vectors span a paralelogram on the surface, the area of this paralelogram may be projected on the three coordinate planes, and will be given by the vector product of $\vec{A}$ and $\vec{B}$. For instance, the area projected on the coordinate plane i-j will be

$$d\tau^{ij} \equiv A^i\ B^j - A^j\ B^i = (\phi^i_{,1}\ \phi^j_{,2} - \phi^j_{,1}\ \phi^i_{,2})\ du_1\ du_2$$

which may be written in the form of (2),

$$d\tau^{ij} = \delta^{ij}_{\ell m}\ \phi^{\ell}_{,1}\ \phi^{m}_{,2}\ du_1\ du_2$$

Thus, the equation (2) is just the generalization of the usual area element for a submanifold with M dimensions.

As it may be verified, which is left as an exercise, the element $d\tau^{\mu_1 \ldots \mu_M}$ is invariant under parameter changes,

$$\lambda'_i = f_i(\lambda_k) \ ; \qquad i, k = 1 \ldots M$$

and is a Mth rank covariant tensor under coordinate mappings. Hence, if $f_{\mu_1 \ldots \mu_M}$ is an Mth rank covariant tensor, the quantity $f_{\mu_1 \ldots \mu_M}\ d\tau^{\mu_1 \ldots \mu_M}$ is a scalar under both parameter changes and coordinate mappings. Thus, we may form the integral

$$\mathcal{J} = \int_{\Omega_M} f_{\mu_1 \ldots \mu_M} d\tau^{\mu_1 \ldots \mu_M} \tag{1-6-3}$$

For M = 1, 2, 3 and 4 this is respectively a line, surface, hypersurface and volume integral. These are the four possible forms of integration at four dimensions.

Exercise:

Determine explicitly the element of integration on a hypersurface of the four-dimensional manifold.

We now state the Stokes' theorem in general form, that is for a M dimensional manifold,

$$\int_{\Omega_{M-1}} f_{\mu_1 \cdots \mu_{M-1}} \, d\tau^{\mu_1 \cdots \mu_{M-1}} = \int_{\Omega_M} \frac{\partial f_{\mu_1 \cdots \mu_{M-1}}}{\partial x^{\mu_M}} \, d\tau^{\mu_1 \cdots \mu_M} \tag{1-6-4}$$

The proof will not be given, since it involves only mathematical concepts, the symbol Ω_M indicates a region of R, and Ω_{M-1} is the M-1 dimensional manifold which forms the boundary of Ω_M.

Of particular importance is the case where M=4. In this situation we can rewrite it in a more familiar form by introducing the vector density

$$\mathcal{F}^{\mu} = \frac{1}{3!} \varepsilon^{\nu\mu\rho\sigma} f_{\nu\rho\sigma}$$

which has weight +1, taking as before the $f_{\nu\rho\sigma}$ with weight zero. Similarly, we define

$$dS_{\mu} = \frac{1}{3!} \varepsilon_{\mu\nu\rho\sigma} \, d\tau^{\nu\rho\sigma}$$

and

$$dS = \frac{1}{4!} \varepsilon_{\mu\nu\rho\sigma} \, d\tau^{\mu\nu\rho\sigma}$$

which allow us to write (4) in the form

$$\int_{\Omega_3} \mathcal{F}^{\mu} \, dS_{\mu} = \int_{\Omega_4} \mathcal{F}^{\mu}{}_{,\mu} \, dS \tag{1-6-5}$$

If, further we take the parameters in Ω_4 as the coordinates itself,

$$\frac{\partial \phi^\mu}{\partial \lambda_k} = \delta^\mu_k$$

we obtain, $d\tau^{\mu\nu\rho\sigma} = \delta^{\mu\nu\rho\sigma}_{1234}\, dx^1\, dx^2\, dx^3\, dx^0 = \varepsilon^{\mu\nu\rho\sigma}\, dx^0\, dx^1\, dx^2\, dx^3$, and since $\varepsilon_{\mu\nu\rho\sigma}\, \varepsilon^{\mu\nu\rho\sigma} = 4\,!$

$$dS = dx^0\ dx^1\ dx^2\ dx^3 = d_4x$$

$$dS_\mu = (dx^1\ dx^2\ dx^3,\ dx^0\ dx^2\ dx^3,\ dx^0\ dx^1,\ dx^3,\ dx^0\ dx^1\ dx^2)$$

And we obtain the four-dimensional form of Gauss' theorem.

$$\int_{\Omega_3} \mathcal{J}^\mu\, dS_\mu \quad \int_{\Omega_4} \mathcal{J}^\mu_{,\mu}\, d_4x$$

1.7) <u>Internal Transformations</u>

In addition to the MMG we sometimes have to deal with transformation groups which are not associated to coordinate mappings. This result holds true whenever we work in a manifold with some dimension, say M, that is, in the M-dimensional manifold equipped with a M-dimensional mapping of symmetry, there may exist some number of internal mappings, depending on what geometrical objects there exist within this manifold. If we enlarge the manifold to a dimension M+K, where K is the total number of descriptors of the internal mapping, we may suceed eventually in representing all existing symmetry mappings as M+K-dimensional MMG. We have used the term "eventually" since this is not a general theorem, but in some cases it has be proven to be correct.

An outstanding example of a internal group is given by the gauge group of electrodynamics, in the four-dimensional manifold of special relativity, it still exists into the four-dimensional manifold of general relativity. This group is characterized by a scalar function of the coordinates which is added to the potentials, thus generating another potential,

$$A'_\mu(x) = A_\mu(x) + \Lambda_{,\mu}(x) \tag{1-7-1}$$

Since to this function there are associated an infinite number of parameters, this is an infinite dimensional Lie group.

It is possible to construct a five dimensional manifold with the local topology of a hyperplane, possessing a MMG characterized by five descriptors $\xi^i(x)$ in such way that a gauge transformation is represented by a translation along the fifth axis. This translation depends in magnitude on the point where it started. This theory is known as the Kaluza-Klein formalism, and serves to give an example of what we called attention in the beginning of this section.

From the point of view of the theory of relativity, there exists an important internal group of transformations defined on a two dimensional complex space. The points of this internal space are the columns of complex elements.

$$(\Psi^A(x)) = \begin{pmatrix} \Psi^1(x) \\ \Psi^2(x) \end{pmatrix}$$

The group of transformation is given by a two-by-two matrix with complex elements and determinant equal to one.

$$(M^A{}_B) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \ \alpha\delta - \beta\gamma = 1$$

Under the action of this transformation, points $\Psi(x)$ are mapped on another points $\Psi'(x)$, for any Ψ. This group is of the type SL_2. As it will be shown later on, this group is associated to coordinate transformations in the four-dimensional manifold of special relativity, we may see that indeed it depends on six real parameters, the same number of parameters as for a Lorentz transformation. In general relativity the complex numbers appearing in the transformation a different transformation of Ψ. We will then, frequently, use the term "local Lorentz transformations". Another possible situation appears when the matrix M is both unimodular (det M = 1) and unitary. In this case we have a realization of SU_2. An easy checking shows that in this case we are left with three independent real parameters in M. As it is natural, this group is associated to three-dimensional rotations in the coordinate space, which is "half" of the full Lorentz group. An SU_3 group may also be imagined, as the group of linear unitary unimodular transformations in complex three-dimensional space. These groups are important in the classification schemes of elementary particles.

Turning back to the transformation on Ψ, we have the mapping

$$\Psi'^{A}(x) = M^{A}{}_{B}(x)\ \Psi^{B}(x) \qquad (1\text{-}7\text{-}2)$$

Since the matrix M has unity determinant, we may construct an infinitesimal mapping, by writing

$$M = I + E$$

where E is an infinitesimal matrix with vanishing trace

$$\mathrm{Tr}\ E = 0 \qquad (1\text{-}7\text{-}3)$$

Under this mapping we will have,

$$\bar{\delta}\Psi^{A}(x) \equiv \Psi'^{A}(x) - \Psi^{A}(x) = E^{A}{}_{B}(x)\Psi^{B}(x) \qquad (1\text{-}7\text{-}4)$$

Since the groups we are discussing here are not generated directly by coordinate mappings. For instance the SL_2 is defined on a space which is not a coordinate space, the geometrical objects like $\Psi(x)$ or whatever any other which may appear, are not yet fully determined, we need to specify how they transform under the MMG. For Ψ we usually have,

$$\Psi'(x') = \Psi(x)$$

That is, $\Psi(x)$ is a coordinate scalar. It may happen that we have to deal with objects like $\Psi^{\mu\nu\cdots}(x)$, which are coordinate tensors on the indices $\mu\nu\ldots$, and internal geometrical objects of the type discussed here.

Since we will turn back again to the group SL_2 after the introduction of a metric into the manifold, we finish here this brief introduction.

2. AFFINE GEOMETRY

We have seen that except for some particular combinations, the ordinary derivatives of tensors and tensor densities did not form the components of new geometrical objects. The reason of such difficulty arises in the first instance from the fact that we cannot add or subtract tensors at separated points, and such operation is needed for introducing the derivatives. Since the concept of derivative is closely related to the evolution of the tensor field as it moves on the manifold, this restriction needs to be overcomed. In the following sections we will see how to do that.

We first introduce the concept of parallel displacement of tensors.

2.1) Covariant Differentiation

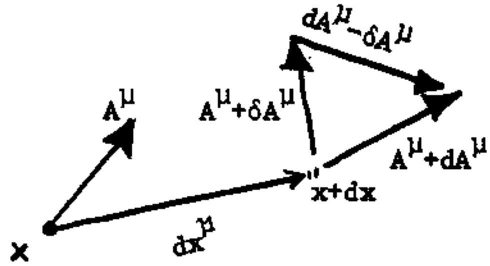
Consider a contravariant vector field $A^\mu(x)$. The components of A^μ at the point $x^\mu + dx^\mu$ are related to the components at the point x^μ by,

$$\begin{aligned} A^\mu(x + dx) &= A^\mu(x) + dA^\mu(x) \\ &= A^\mu(x) + A^\mu{}_{,\nu}(x)\, dx^\nu \end{aligned}$$

The quantities $dA^\mu(x)$ being the difference of two vectors, $A^\mu(x+dx) - A^\mu(x)$, located at different points of the manifold, do not constitute the components of any geometrical object. What is needed for our purposes is another vector at the point

$x^\mu + dx^\mu$, by definition we take this vector to be parallel to $A^\mu(x)$. This vector is denoted by $A^\mu(x) + \delta A^\mu(x)$, as before $\delta A^\mu(x)$ is not a vector, similarly to $dA^\mu(x)$.

However, we shall construct δA^μ in such way that the difference $dA^\mu(x) - \delta A^\mu(x)$ behaves as another vector.



This is possible since it represents the difference of two vectors at the same point,

$$A^\mu(x) + dA^\mu(x) - (A^\mu(x) + \delta A^\mu(x)) = dA^\mu(x) - \delta A^\mu(x)$$

In constructing $\delta A^\mu(x)$ we will require that it vanishes either $A^\mu(x)$ or dx^μ vanishes, so as to conform with the usual notions of parallel vectors in Euclidian geometry. The simplest possibility is when $\delta A^\mu(x)$ is bilinear in $A^\mu(x)$ and dx^μ,

$$\delta A^\mu(x) = - \Gamma^\mu_{\rho\sigma}(x)\, A^\rho(x)\, dx^\sigma \qquad (2\text{-}1\text{-}1)$$

The negative sign is just a matter of convention. The quantities $\Gamma^\mu_{\rho\sigma}(x)$ are the components of a new geometrical object defined on the manifold, called as the affine connection or affinity. It has in all 64 components, which in principle are arbitrary, the unique restriction imposed on them is that $\Gamma^\mu_{\rho\sigma}(x)$ has to transform in such way that the $\delta A^\mu(x)$ of (2-1-1) satisfies the condition that $dA^\mu(x) - \delta A^\mu(x)$ is a vector.

We form the following combination,

$$A^\mu_{;\nu}\, dx^\nu = dA^\mu - \delta A^\mu \qquad (2\text{-}1\text{-}2)$$

Since the right hand side behaves as a vector, and dx^ν is also a vector, we conclude that the quantities $A^\mu_{;\nu}$ form the components of a second rank mixed tensor, which is called the covariant derivatives of the contravariant vector A^μ.

In this way we have solved our initial problem, namely, how to construct new geometrical objects out of derivatives of a tensor field. This was possible by means of two steps:

1) Consider the vector field at the nearby point $x^\mu + dx^\mu$, the $A^\mu(x+dx) = A^\mu(x) + dA^\mu(x)$.

2) Transport the vector $A^\mu(x)$ to the point $x^\mu + dx^\mu$, $A^\mu(x) \to A^\mu(x) + \delta A^\mu(x)$.

3) Calculate the difference of these two vectors, which are located at the same point. This difference is equated to dx^ν times the new derivative $A^\mu{}_{;\nu}(x)$.

Let us now determine how $\Gamma^\mu_{\rho\sigma}(x)$ transforms under the MMG. We have,

$$dA^\mu = \frac{\partial A^\mu}{\partial x^\nu}\, dx^\nu$$

$$= \frac{\partial}{\partial x'^\nu}\left(\frac{\partial x^\mu}{\partial x'^\rho} A'^\rho\right)\frac{\partial x'^\alpha}{\partial x^\nu}\,\frac{\partial x^\nu}{\partial x'^\lambda}\, dx'^\lambda$$

$$= \frac{\partial x^\mu}{\partial x'^\rho} A'^\rho_{,\alpha}\, dx'^\alpha + \frac{\partial^2 x^\mu}{\partial x'^\rho \partial x'^\alpha} A'^\rho\, dx'^\alpha \tag{2-1-3}$$

We also have that

$$\delta A^\mu = -\Gamma^\mu_{\rho\sigma} A^\rho\, dx^\sigma$$

$$= -\Gamma^\mu_{\rho\sigma}\frac{\partial x^\rho}{\partial x'^\alpha} A'^\alpha \frac{\partial x^\sigma}{\partial x'^\lambda}\, dx'^\lambda \tag{2-1-4}$$

Combining the equations (3) and (4) we obtain

$$dA^{\mu} - \delta A^{\mu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} dA'^{\nu} + \left(\frac{\partial^2 \ x^{\mu}}{\partial x'^{\rho} \partial x'^{\alpha}} + \Gamma^{\mu}_{\lambda\sigma} \frac{\partial x^{\lambda}}{\partial x'^{\rho}} \ \frac{\partial x^{\sigma}}{\partial x'^{\alpha}} \right) A'^{\rho} dx'^{\alpha}$$

we can write this last equation as

$$dA^{\mu} - \delta A^{\mu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} \left[dA'^{\nu} + \Gamma'^{\nu}_{\rho\alpha} \ A'^{\rho} \ dx'^{\alpha} \right] \tag{2-1-5}$$

Provided that the transformed $\Gamma'^{\nu}_{\rho\alpha}$ satisfy

$$\frac{\partial x^{\mu}}{\partial x'^{\nu}} \ \Gamma'^{\nu}_{\rho\alpha} = \Gamma^{\mu}_{\lambda\sigma} \ \frac{\partial x^{\lambda}}{\partial x'^{\rho}} \ \frac{\partial x^{\sigma}}{\partial x'^{\alpha}} + \frac{\partial^2 \ x^{\mu}}{\partial x'^{\rho} \partial x'^{\alpha}} , \tag{2-1-6}$$

multiplying both sides of this equation by $\frac{\partial x'^{\beta}}{\partial x^{\mu}}$ we obtain

$$\Gamma'^{\beta}_{\rho\sigma}(x') = \frac{\partial x^{\lambda}}{\partial x'^{\rho}} \ \frac{\partial x^{\sigma}}{\partial x'^{\alpha}} \ \frac{\partial x'^{\beta}}{\partial x^{\mu}} \Gamma^{\mu}_{\lambda\sigma}(x) + \frac{\partial^2 \ x^{\mu}}{\partial x'^{\rho} \partial x'^{\alpha}} \ \frac{\partial x'^{\beta}}{\partial x^{\mu}} \tag{2-1-7}$$

With the choice (6), we may write (5) which in turn shows that the difference $dA^{\mu} - \delta A^{\mu}$ is a vector.

The transformation law for the addinities, the equation (7), is a linear and inhomogeneous law, the transformed affinities depend linearly on the original affinities and there exists a term independent of the original affinity. Thus, the $\Gamma^{\mu}_{\nu\sigma}(x)$ represents a geometrical object, but is not a third rank tensor.

The knowledge of the functions $\Gamma^{\mu}_{\nu\sigma}(x)$ allow us to compare nearby physical events, indeed, if those events are characterized by the geometrical objects $0_1(x)$ and $0_2(y)$, usually tensor fields at the points x and y of a same coordinate system, we in principle know how to transport 0_1 to the point y and there stablish a local comparison between these quantities. The manifolds which possess geometrical objects like the $\Gamma^{\mu}_{\nu\sigma}$, and therefore possess this property are called affine, or equivalently, the geometry of such manifolds is called an affine geometry.

The calculus of affinities is somehow different of the tensor calculus, there are results of the later which do not apply to the calculus of affinities. We quote some of them:

i) The sum of two affinities located at a same point of the manifold, is not a third affinity. Nevertheless, this sum is still a geometrical object.

ii) The sum of an affinity and a tensor with the same index structure, both located at a same point is again an affinity.

iii) The difference of two affinities located at a same point is a tensor with the same index structure. (The inhomogeneous term cancels out).

In order that the properties (i) and (iii) make sound in actual situations, is necessary that we use two, or even more, affinities simultaneously into the manifold. This is mathematically possible [2], and have been used in the literature [3]. One property of importance of the affinities is their symmetry property on the two lower indices, or the lack of such symmetry. We just quote the following results without proof.

i) If $\Gamma^{\mu}_{\rho\sigma}$ is symmetric on ρ, σ the transformed $\Gamma'^{\mu}_{\rho\sigma}$ will be also symmetric in the lower indices.

ii) The antisymmetric part of an affinity is a tensor. Therefore, a general affinity may be written as the sum of the symmetrical part and of a tensor, represented here by its antisymmetric part.

The second result is a direct consequence of the property (iii) written above. A result comes out immediately, in order that the $A^{\mu}_{;\nu}$ of Eq. (2-1-2) be a tensor is necessary that the $\Gamma^{\mu}_{\rho\sigma}$ be not skew symmetric in ρ, σ. Indeed, if

$\Gamma^{\mu}_{\rho\sigma}$ is such that $\Gamma^{\mu}_{\rho\sigma} = -\Gamma^{\mu}_{\sigma\rho}$, the $\delta A^{\mu}(x)$ of (2-1-1) will be a vector, and since $dA^{\mu}(x)$ is not a vector it follows that $A^{\mu}_{;\nu}$ will not be a tensor. Theories where antisymmetric $\Gamma^{\mu}_{\nu\alpha}$ are used together with symmetric $\Gamma^{\mu}_{\rho\sigma}$ do exist, as for instance the unitary theory of Einstein [4]. However, in the theory of general relativity only symmetrical affinities are used.

We have shown that

$$A^{\mu}_{;\nu} = \frac{\partial A^{\mu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\nu\sigma} A^{\sigma} \qquad (2\text{-}1\text{-}8)$$

is a second order mixed tensor formed out of the first partial derivatives of the vector field $A^{\mu}(x)$. * This concept of covariant derivatives of a contravariant vector may be extended to higher order tensors and even for tensor densities. For obtaining this generalization we impose two additional conditions on the operation of covariant differentiation:

i) The covariant derivative of a scalar is identical to the ordinary derivative, $A_{;\mu} = A_{,\mu}$.

ii) The covariant derivative of a product of tensor (or tensor densities) obeys the product rule of ordinary derivatives.

As result of these two conditions we can prove that

$$T^{\mu\ldots}_{\nu\ldots;\rho} = T^{\mu\ldots}_{\nu\ldots,\rho} + \Gamma^{\mu}_{\sigma\rho} T^{\sigma\ldots}_{\nu\ldots} - \Gamma^{\sigma}_{\nu\rho} T^{\mu\ldots}_{\sigma\ldots} + \ldots \qquad (2\text{-}1\text{-}9)$$

Exercises:

1) Prove the equation (2-1-9). First compute $B_{\mu;\nu}$ using $A^{\mu}B_{\mu}$ = scalar, and the

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* Decomposing $\Gamma^{\mu}_{\nu\sigma}$ into symmetrical and antisymmetrical parts, we see that $A^{\mu}_{;\nu}$ decomposes into two tensors, $A^{\mu}_{;\nu} = A^{(1)\mu}_{;\nu} + A^{(2)\mu}_{;\nu}$, where:

$$A^{(1)\mu}_{;\nu} = A^{\mu}_{,\nu} + \Gamma^{\mu}_{(\nu\sigma)} A^{\sigma} \text{ and } A^{(2)\mu}_{;\nu} = \Gamma^{\mu}{}_{[\nu\sigma]} A^{\sigma}$$

items (i) and (ii) of the previous page, along with (2-1-8). After this, form the scalar $T^{\mu\dots}_{\nu\dots} A^{\nu}\dots B_{\mu}\dots$ for arbitrary vector $A^{\mu}\dots B_{\mu}\dots$. Once more use (i), (ii).

2) Show that $T^{\mu\dots}_{\nu\dots;\rho}$ is a tensor of the order indicated by the indices.

3) Show that $\delta^{\mu}_{\nu;\rho} = 0$.

Now, we treat the problem of constructing covariant derivatives of tensor densities. We impose the further condition that the parallel displacement of a scalar density of weight W is, by analogy with (2-1-1), equalt to

$$\delta\ (x) = W\ \Gamma^{\sigma}_{\sigma\rho}\ dx^{\rho}\ . \tag{2-1-10}$$

The weight W appears explicitly as a multiplicative factor because we know that for W = 0, that is for scalars, $\delta\ = 0$ which means that the covariant derivative is equal to the usual partial derivative, $\mathfrak{U}_{;\mu} = \mathfrak{U}_{,\mu}$. Thus, we have similarly as before

$$\begin{aligned}\mathfrak{U}(x)_{;\mu}\ dx^{\mu} &= d\mathfrak{U}(x) - \delta\mathfrak{U}(x)\\ &= \frac{\partial\mathfrak{U}}{\partial x^{\mu}}\, dx^{\mu} - W\,\mathfrak{U}\,\Gamma^{\sigma}_{\sigma\mu}\ dx^{\mu}\end{aligned}$$

which gives for the covariant derivative of the scalar density $\mathfrak{U}$ of weight W,

$$\mathfrak{U}(x)_{;\mu} = \mathfrak{U}_{,\mu}(x) - W\,\mathfrak{U}(x)\ \Gamma^{\sigma}_{\sigma\mu}(x)\ . \tag{2-1-11}$$

As it may be shown, the $\mathfrak{U}_{;\mu}$ of Eq. (2-1-11) is a vector density of weight W. Proceeding now in a way similar to the case of tensors, we determine the covariant derivative of any tensor density as follows: Multiply the tensor density by covariant and contravariant tensors so as to sum on all indices, finally, multiply this by a scalar density of weight $-W$, where W is the weight of the tensor density we started with. The final expression is a scalar and we proceed in the same way as before. As example we calculate $\mathcal{U}_{\mu;\nu}$, with $\mathcal{U}_\mu$ a vector density of weight W. We form the scalar

$$\mathcal{U}_\mu \, A^\mu \mathcal{B}$$

for A^μ a vector and $\mathcal{B}$ a scalar density of weight $-W$. Then

$${}_{\mu;\nu} \, A^\mu \mathcal{B} + \mathcal{U}_\mu \, A^\mu_{;\nu} \mathcal{B} + \mathcal{U}_\mu \, A^\mu \, \mathcal{B}_{;\nu} = \mathcal{U}_{\mu,\nu} \, A^\mu \mathcal{B} + \mathcal{U}_\mu \, A^\mu_{,\nu} \mathcal{B} + \mathcal{U}_\mu \, A^\mu \, \mathcal{B}_{,\nu}$$

Substituting the values of $A^\mu{}_{;\nu}$ and $\mathcal{B}_{;\nu}$, one finds

$$A^\mu \mathcal{B} (\mathcal{U}_{\mu;\nu} + \mathcal{U}_\alpha \, \Gamma^\alpha_{\mu\nu} + \mathcal{U}_\mu \, W \, \Gamma^\rho_{\rho\nu} - \mathcal{U}_{\mu,\nu}) = 0$$

Since this vanishes for arbitrary A^μ and $\mathcal{B}$, we conclude that

$$\mathcal{U}_{\mu;\nu} = \mathcal{U}_{\mu,\nu} - \Gamma^\alpha_{\mu\nu} \, \mathcal{U}_\alpha - W \, \Gamma^\rho_{\rho\nu} \, \mathcal{U}_\mu, \qquad (2\text{-}1\text{-}12)$$

which is the covariant derivative of $\mathcal{U}_\mu$, it may be shown that it forms a second rank tensor density with the same weight W.

Generalizing this method for a tensor density $\mathcal{U}^{\mu\ldots}_{\nu\ldots;\rho}$ of weight W, we find,

$$\mathcal{U}^{\mu}_{\nu\ldots;\rho} = \mathcal{U}^{\mu\ldots}_{\nu\ldots,\rho} + \Gamma^\mu_{\sigma\rho} \, \mathcal{U}^{\sigma\ldots}_{\nu\ldots} - \Gamma^\sigma_{\nu\rho} \, \mathcal{U}^{\mu\ldots}_{\sigma\ldots} + \ldots - W\Gamma^\sigma_{\sigma\rho} \, \mathcal{U}^{\mu\ldots}_{\nu\ldots} \qquad (2\text{-}1\text{-}13)$$

Exercise:

Prove the equation (13) by using the previous method.

In the last section we indicated that some particular operations involving just the ordinary derivatives of the components of tensor densities generate new tensor densities. All those results are equally obtained replacing the ordinary derivatives. That is, in all such cases both derivatives happen to coincide. It may be proven that:

i) For a contravariant vector density of weight $W = +1$, $\mathcal{U}^\mu$

$$\mathcal{U}^{\mu}_{;\mu} = \mathcal{U}^{\mu}_{,\mu}$$

(this holds true only if W = +1, for other values of W this equality is not correct).

ii) For antisymmetric tensor densities with W = +1, $\mathcal{U}^{\mu\nu}$, $\mathcal{U}^{\mu\nu\rho}$ and $\mathcal{U}^{\mu\nu\rho\sigma}$ the divergences $\mathcal{U}^{\mu\nu}_{,\nu}$, $\mathcal{U}^{\mu\nu\alpha}_{,\alpha}$, $\mathcal{U}^{\mu\nu\rho\alpha}_{,\alpha}$ coincide with the covariant divergences. So that they are new tensor densities. This again holds true only for W = +1.

The proof of (i) for instance, follows from (2-1-13) as,

$$\mathcal{U}^{\mu}_{;\mu} = \mathcal{U}^{\mu}_{,\mu} + \Gamma^{\mu}_{\sigma\mu}\ \mathcal{U}^{\sigma} - W\ \Gamma^{\sigma}_{\sigma\mu}\ \mathcal{U}^{\mu}$$

$$\mathcal{U}^{\mu}_{;\mu} = \mathcal{U}^{\mu}_{,\mu}$$

Similarly we may check the item (ii). In the above proof we have used the fact that the affinity $\Gamma^{\mu}_{\sigma\rho}$ is symmetric over the lower indices. The same kind of treatment can be extended to tensors, in the case that ordinary derivatives yields new tensors.

At this point we introduce one of the most important properties of covariant derivatives. Unlike ordinary derivatives, the covariant derivatives are not commutative, that means

$$(\mathcal{A}_{;\mu})_{;\nu} - (\mathcal{A}_{;\nu})_{;\mu} \neq 0$$

for a generic geometrical object, such as a tensor or a tensor density. In the following section we will see that it holds too for "internal geometrical objects" such as tetrads and spinors.

If one calculates this "commutator" for $\mathcal{A} = k^{\mu}$, a given vector field, one finds $(\mathcal{A}_{;\mu\nu} = (\mathcal{A}_{,\mu})_{;\nu})$

$$k^{\rho}_{;\mu\nu} - k^{\rho}_{;\nu\mu} = (\Gamma^{\rho}_{\sigma\mu,\nu} - \Gamma^{\rho}_{\sigma\nu,\mu} - \Gamma^{\rho}_{\lambda\mu}\ \Gamma^{\lambda}_{\sigma\nu}\quad \Gamma^{\rho}_{\lambda\nu}\ \Gamma^{\lambda}_{\sigma\mu})k^{\sigma} - \Gamma^{\lambda}_{[\mu\nu]}\ k^{\rho}_{;\lambda} \quad (2\text{-}1\text{-}14)$$

Even in the case of symmetrical affinities this "commutator" does not vanish, and is equal to

$$k^{\rho}_{;\mu\nu} - k^{\rho}_{;\nu\mu} = R^{\rho}_{\sigma\mu\nu} \; k^{\rho} \; , \qquad (2\text{-}1\text{-}15)$$

with

$$R^{\rho}_{\sigma\mu\nu} = \Gamma^{\rho}_{\sigma\mu,\nu} - \Gamma^{\rho}_{\sigma\nu,\mu} - \Gamma^{\rho}_{\lambda\mu}\,\Gamma^{\lambda}_{\sigma\nu} + \Gamma^{\rho}_{\lambda\nu}\,\Gamma^{\lambda}_{\sigma\mu} \qquad (2\text{-}1\text{-}15)$$

Since $k^{\rho}_{;\mu\nu} - k^{\rho}_{;\nu\mu}$ is a tensor, and k^{ρ} a vector, we conclude at once from the Eq. (2-1-15) that the $R^{\rho}_{\sigma\mu\nu}$ of (2-1-16) forms a fourth rank tensor. This tensor has a dominant part in the geometrical structure of the manifold, and is called the Riemann tensor. Its vanishing, for the case of a symmetric affinity, is the necessary and suficient condition for the covariant derivative to be commutative. We have proved this only for a vector field, however, it may be extended for any commutator of covariant derivatives of any tensor or tensor density, any one of such commutators can be expressed as a linear combination of $R^{\rho}_{\sigma\mu\nu}$ and its contraction $R^{\rho}_{\sigma\rho\nu}$ which appears in the case where we take a tensor density to start with.

Exercises:

1) Compute $\mathfrak{V}^{\rho}_{;\mu\nu} - \mathfrak{V}^{\rho}_{;\nu\mu}$ for a vector density of weight W. Show that this quantity vanishes whenever $R^{\rho}_{\sigma\mu\nu}$ vanishes.

2) Show that $T_{\mu\nu;\rho\sigma} - T_{\mu\nu;\sigma\rho} = -R^{\lambda}_{\mu\rho\sigma}\; T_{\lambda\nu} - R^{\lambda}_{\nu\rho\sigma}\; T_{\mu\lambda}$, for a symmetric affinity.

2.2) Covariant-Differentiation for Internal Groups

Whenever we have other geometrical objects than tensors into the manifold, we are faced with the problem of calculating new geometrical objects with the derivatives of such quantities. This is possible to be solved by extending the concept of covariant derivatives to those "internal variables". The covariant derivative of any geometrical object $\Psi^A(x)$ is defined as a new geometrical object $\Psi^A_{;\mu}(x)$ which is a covariant first rank tensor under space-time mappings (considering Ψ^A as a scalar, in case contrary, we just aggregate the indices to form a higher order tensor) and transforms like Ψ^A under the internal mappings.

In order to introduce the covariant derivatives, we consider for simplicity a quantity Ψ^A which is a scalar under the space-time mappings, and which transforms under internal mappings as

$$\bar{\delta}\Psi^A(x) = -i\ \varepsilon^r(x)\ L_{r\ B}^{A}\ \ \Psi^B(x)\ , \qquad (2\text{-}2\text{-}1)$$

these later represent an infinitesimal group of transformations, with group parameters ε^r which are arbitrary space-time functions. However, in some particular cases, as for instance in special relativity, they may be constants.

The transformation law for Ψ^A may be extended too for Ψ_A ,that is, for internal spaces we also deal with covariant and contravariant objects. We start with the condition, $\Psi^A\ \Psi_A = u =$ scalar (with respect to internal transformations) and get, on account that $\bar{\delta}u = 0$,

$$\bar{\delta}\Psi^A.\Psi_A + \Psi^A.\ \bar{\delta}\Psi_A = 0\ .$$

Substitution of (1) gives

$$(\bar{\delta}\Psi_A - i\varepsilon^r\ L_{rA}^{B}\ \Psi_B)\ \Psi^A = 0$$

since this vanishes for any Ψ^A, we obtain the counterpart of (1) now written in terms of Ψ_A.

$$\bar{\delta}\Psi_A(x) = i\varepsilon^r(x)\ L_{r\ A}^{B}\ \ \Psi_B(x) \qquad (2\text{-}2\text{-}2)$$

The matrix which conducts to (1) is

$$M^A_{\ B} = \delta^A_B - i\ \varepsilon^r\ L_{r\ B}^{\ A}$$

that is,

$$\Psi'^A(x) = M^A_B(x)\ \Psi^B(x) \qquad (2\text{-}2\text{-}3)$$

We now determine the matrix associated to the transformation of the Ψ_A. For that, again we use the condition that $\Psi^A\ \Psi_A$ is a scalar,

$$\Psi'^A(x)\ \Psi'_A(x) = \Psi^A(x)\ \Psi_A(x)$$

using (3) we write this as

$$(M^A{}_B\ \Psi'_A - \Psi_B)\Psi^B = 0$$

since Ψ^B is arbitrary, we get

$$\Psi'_c = \Psi_B\ M^{-1}{}^B{}_c \qquad (2\text{-}2\text{-}4)$$

Thus, Ψ^A transforms with a matrix M, and Ψ_A with a matrix M^{-1}. For the case of infinitesimal transformation, the matrix M^{-1} has the form

$$M^{-1}{}^A{}_B = \delta^A_B + i\ \varepsilon^r\ L^A_{r\ B}$$

which shows once more that (2) is satisfied.

Since the transformation in internal space is carried out at a fixed point of space-time, we can differentiate all terms of (3),

$$\Psi'^A_{,\mu} = \Psi^A_{,\mu} - i\ \varepsilon^r\ L^A_{r\ B}\ \Psi^B_{,\mu} - i(\varepsilon^r\ L^A_{rB,\mu} + \varepsilon^r_{,\mu} L^A_{r\ B})\Psi^B$$

thus,

$$\bar{\delta}\Psi^A_{,\mu} = -i\varepsilon^r\ L^A_{rB}\ \ \Psi^B_{,\mu} - i(\varepsilon^r\ L^A_{rB})_{,\mu}\Psi^B$$

this means that $\Psi'^A_{,\mu}$ depends both on $\Psi^A_{,\mu}$ and Ψ^A, which shows that the derivative $\Psi_{,\mu}$ do not form a geometrical object like Ψ. Such object will be represented by $\Psi^A_{;\mu}$. For defining $\Psi^A_{;\mu}$, we introduce the "parallel transport" of Ψ^A from the point x^μ to $x^\mu + dx$, which generates on the Ψ^A the variation

$$\delta\Psi^A(x) = -g\ \Gamma^A_\mu(x)_B\ \Psi^B(x)\ dx^\mu \qquad (2\text{-}2\text{-}5)$$

$\Gamma^{A}_{\mu\ B}$ is the affinity for the internal group, and g is some constant which depends on what particular internal structure we deal with. The covariant derivative of Ψ^A is therefore given by

$$\Psi^{A}_{;\mu} dx^{\mu} = d\Psi^{A}(x) - \delta\Psi^{A}(x)$$

$$\Psi^{A}_{;\mu} = \Psi^{A}_{,\mu} + g\ \Gamma^{A}_{\mu\ B}\ \Psi^{B} \tag{2-2-6}$$

For simplifying the formulae we shall use matrix notation, in this notation we indicate Ψ^A by Ψ, and Ψ_A by χ. Thus, the previous formulas now read as

$$\Psi' = M\Psi$$

$$\chi' = \chi M^{-1}$$

$$\delta\Psi = -\ g\ \Gamma_{\mu}\ \Psi\ dx^{\mu}$$

$$\Psi_{;\mu} = \Psi_{,\mu} + \ g\ \Gamma_{\mu}\ \Psi$$

Our basic requirement on $\Psi_{;\mu}$ is that it transforms as Ψ,

$$\Psi'_{;\mu}(x) = M(x)\ \Psi_{;\mu}(x) \tag{2-2-7}$$

under internal mappings. In order that the $\Psi_{;\mu}$ of (2-2-6) satisfy these conditions is necessary that the affinity possess the transformation law

$$\Gamma'_{\mu}(x) = -\frac{1}{g} M_{,\mu}(x)\ M^{-1}(x) + M(x)\ \Gamma_{\mu}(x)\ M^{-1}(x)\ . \tag{2-2-8}$$

For infinitesimal transformations this gives,

$$\Gamma'_{\mu}(x) = \Gamma_{\mu}(x) + \frac{i}{g}\ (\varepsilon^{r}\ L_{r})_{,\mu} + i\ \varepsilon^{r}\left[\Gamma_{\mu},\ L_{r}\right] \tag{2-2-9}$$

where $\left[\Gamma_{\mu},\ L_{r}\right]$ is the matrix representing the commutator of the two matrices Γ_{μ} and L_r.

A particular class of internal affinities is obtained when one takes

$$\Gamma^{A}_{\mu\ B} = i\ L^{A}_{\ r\ B}\ B^{r}_{\mu} \tag{2-2-10}$$

In this case, the commutator standing on the right side of (9) takes the form

$$\left[\Gamma_\mu, L_r\right] = i\, C^m_{sr}\, L_m\, B^s_\mu \tag{2-2-11}$$

we have used the fact that the generators L_r satisfy the group property,

$$\left[L_s, L_r\right] = C^m_{sr}\, L_m$$

For the choice (10) we get for (9)

$$\Gamma'_\mu = i\, L_r\, \{B^r + \frac{i}{g}\,\varepsilon^r_{,\mu} + i\, \varepsilon^m\, C^r_{sm}\, B^s_\mu\} \tag{2-2-12}$$

which on account of (10) may be put as

$$\Gamma'_\mu = i\, L_r\, B'^r_\mu$$

with the transformed B^r equal to

$$B'^r_\mu = B^r_\mu + \frac{i}{g}\,\varepsilon^r_{,\mu} + i\, \varepsilon^m\, C^r_{sm}\, B^s_\mu \tag{2-2-13}$$

An important property now comes out, if we consider another internal space spanned by quantities with indices r,m..., that is, the sapce equipped with geometric objects $U^{rs\ldots}_{mn\ldots}$ transforming under their internal mappings as,

$$\bar{\delta}\, U^r = +\, i\, \varepsilon^s\, C^r_{ms}\, U^m \qquad \text{(contravariant vector)}$$

$$\bar{\delta}\, U_r = -\, i\, \varepsilon^s\, C^m_{rs}\, U_m \qquad \text{(covariant vector)}$$

(the transformation law for $U^{rs\ldots}_{mn\ldots}$ may be obtained by generalizing those two laws as we did for tensors in coordinate space), which means that the generators $\mathcal{L}^m_{sr}$ are here

$$\mathcal{L}^m_{sr} = -\, C^m_{sr}$$

we see that the B^r are a set of four internal vectors of this space (the equation (2-2-13) has the form of (2-2-1) for $\mathcal{L}=-iC$, except for the term $\varepsilon^r_{,\mu}$) with an addition al term $\varepsilon^r_{,\mu}$ in their transformation law.

With respect to the index μ the B^r_μ form a vector under space-time mappings.

Exercises:

1) Prove that the quantities $\mathcal{B}^m_{sr} = - C^m_{sr}$ satisfy the group property

$$\left[\mathcal{L}_s ; \mathcal{L}_r\right]^m_{\ n} \equiv \mathcal{B}^m_{sj} \mathcal{B}^j_{rn} - \mathcal{B}^m_{rj} \mathcal{B}^j_{sn} = C^k_{sr} \mathcal{B}^m_{kn}$$

which was used before for introducing the internal space of the objects $U^{rs\ldots}_{mn\ldots}$. The C^m_{sr} are the structure constants of the group with generators L^A_{SB}. (Hint: use the Jacobi indenty for these structure constants).

2) Prove that B^r_μ is a covariant vector for the index μ. (Hint: consider the expression for $\Psi_{;\mu}$ which is a covariant vector).

The extension of covariant differentiation for objects with both space-time and internal indices is easily obtained, consider for instance the object Ψ^A_μ, we have

$$\Psi^A_{\mu;\nu} = \Psi^A_{\mu,\nu} + g\, \Gamma^A_{\nu\ B}\, \Psi^B_\mu - \Gamma^\rho_{\mu\nu}\, \Psi^A_\rho \tag{2-2-14}$$

For the choice (10) for the affinity, we get

$$\Psi^A_{\mu;\nu} = \Psi^A_{\mu,\nu} + ig\, L^A_{r\ B}\, B^r_\nu\, \Psi^B_\mu - \Gamma^\rho_{\mu\nu}\, \Psi^A_\rho \tag{2-2-15}$$

If the internal group is SU_2, B^r_μ is the Yang-Mills field [5]. If the range of variation of r is restricted to just one parameter $\varepsilon^r = \varepsilon$, then,

$$\overline{\delta}\Psi^A(x) = - i\varepsilon(x)\, L^A_{\ B}\, \Psi^B(x)$$

This group is necessarily an Abelian group, since

$$\left[L^A_{\ B}, L^B_{\ C}\right] = 0$$

that is, all structure constant vanish. Thus, from (13) one gets

$$B'_\mu = B_\mu + \frac{i}{g}\, \varepsilon_{,\mu}$$

which is just the transformation law for the vector potential in electrodynamics under a gauge transformation.

In this case the space of the Ψ^A is a two dimensional internal vector space, the elements of this space are the real and imaginary parts of a complex scalar.

$$\Psi = \Psi^1 + i\,\Psi^2$$

$$\Psi^A = \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix}$$

the group of rotations along an axis perpendicular to this space is an Abelian group. A rotation through an angle $\varepsilon(x)$ may then be written as

$$\begin{pmatrix} \Psi'^1 \\ \Psi'^2 \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & 1 \end{pmatrix} \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix}$$

which is equivalent to multiply the scalar by an arbitrary phase factor,

$$\Psi' = e^{-i\varepsilon}\,\Psi$$

as we know this is just the transformation of the matter field *, which holds independently if Ψ is a scalar or a spinor for instance. In any case Ψ is always a space-time scalar, and this is all that matters.

As in the case of the covariant derivative of space-time objects, the commutator of covariant derivatives of internal objects in general does not vanish. A straightforward calculation gives,

$$\Psi^A_{;\mu\nu} - \Psi_{;\nu\mu} = g\,P^A_{\mu\nu\;B}\,\Psi^B \qquad (2\text{-}2\text{-}16)$$

where

$$P^A_{\mu\nu\;B} = \frac{\partial\,\Gamma^A_{\mu B}}{\partial x^\nu} - \frac{\partial\Gamma^A_{\nu B}}{\partial x^\mu} + g(\Gamma^A_{\nu C}\,\Gamma^C_{\mu\;B} - \Gamma^A_{\mu\;C}\,\Gamma^C_{\nu\;B}) \qquad (2\text{-}2\text{-}17)$$

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* under a gauge transformation of the potentials.

From the form presented by $P^{A}_{\mu\nu\ B}$ we see that it transforms as an antisymmetric second rank tensor under space-time mappings, and under internal mappings we have,

$$\Psi'^{A}_{;\mu\nu} \equiv (\Psi'^{A}_{;\mu})_{;\nu} = M^{A}_{\ B}(\Psi^{B}_{;\mu})_{;\nu}$$

since $(\Psi^{A}_{;\mu})_{;\nu}$ is a covariant derivative on the index ν. Thus,

$$\Psi'^{A}_{;\mu\nu} - \Psi'^{A}_{;\nu\mu} = M^{A}_{\ B}\ (\Psi^{B}_{;\mu\nu} - \Psi^{B}_{;\nu\mu})$$

but the right hand side of (2-2-16) contains Ψ^{B}, which changes as $\Psi'^{B}=M^{B}_{\ C}\ \Psi^{C}$, thus,

$$M^{A}_{\ B}\ (\Psi^{B}_{;\mu\nu} - \Psi^{B}_{;\nu\mu}) = g\ P'^{A}_{\mu\nu B}\ M^{B}_{\ C}\ \Psi^{C}$$

which implies that $P^{A}_{\mu\nu B}$ transforms as a mixed second rank internal tensor,

$$P'^{A}_{\mu\nu\ B}(x) = M^{A}_{\ R}(x) P^{R}_{\mu\nu\ S}(x)\ M^{-1S}_{\ \ B}(x) \qquad (2\text{-}2\text{-}18)$$

Then, under the simultaneous effect of both mappings we have

$$P'^{A}_{\mu\nu}(x')_{\ B} = \frac{\partial x^{\rho}}{\partial x'^{\mu}}\ \frac{\partial x^{\sigma}}{\partial x'^{\nu}}\ M^{A}(x)\ P^{R}_{\rho\sigma}(x)_{S}\ M^{-1S}(x)_{B} \qquad (2\text{-}2\text{-}19)$$

For the choice (2-2-10) of the affinity, we obtain the following particular representation of the $P^{A}_{\mu\nu B}$,

$$P^{A}_{\mu\nu\ B} = i\ L^{A}_{r\ B}\mathcal{F}^{r}_{\mu\nu}$$

$$\mathcal{F}^{r}_{\mu\nu} = \frac{\partial B^{r}_{\mu}}{\partial x^{\nu}} - \frac{\partial B^{r}_{\nu}}{\partial x^{\mu}} + ig\ C^{r}_{ms}\ B^{m}_{\nu}\ B^{s}_{\mu} \qquad (2\text{-}2\text{-}20)$$

in this case the transformation law (2-2-18) takes the form

$$\delta\mathcal{F}^{r}_{\mu\nu} = -\ i\ \epsilon^{s}\mathcal{L}^{r}_{sm}\ \mathcal{F}^{m}_{\mu\nu}\ . \qquad (2\text{-}2\text{-}21)$$

$$\mathcal{L}^{r}_{sm} = -\ C^{r}_{sm}$$

We see that $P^{A}_{\mu\nu\ B}$ is the analogue of the Riemann tensor for the internal space.

Its vanishing is the necessary and sufficient condition for internal covariant differentiation be commutative.

For the case where r = 1, $P^{A}_{\mu\nu\ B}$ reduces to (using (2-2-20)).

$$\mathcal{F}_{\mu\nu} = \frac{\partial B_\mu}{\partial x^\nu} - \frac{\partial B_\nu}{\partial x^\mu}\ ; \quad C^{r}_{ms} = 0\ . \tag{2-2-22}$$

Which is the expression for the electromagnetic field $F_{\mu\nu}$. Thus, the electromagnetic potential is the affinity for the gauge group and the electromagnetic field is the corresponding Riemann tensor.

Finally, we give a general expression for the internal covariant derivative of an object like $\Psi^{AB\ldots}_{CD\ldots}$,

$$\Psi^{AB\ldots}_{CD\ldots;\mu} = \Psi^{AB\ldots}_{CD\ldots,\mu} + g\,\Gamma^{A}_{\mu\ 0}\,\Psi^{0B\ldots}_{CD\ldots} + \ldots - g\,\Gamma^{0}_{\mu\ C}\,\Psi^{AB\ldots}_{0D\ldots} - \ldots \tag{2-2-24}$$

Exercises

1) Using that $\Psi^{A}\,\Psi_{A}$ is an internal scalar, calculate $\Psi_{A;\mu}$ by imposing that internal covariant differentiation shares all properties of the space-time covariant differentiation. Answer: $\Psi_{A;\mu} = \Psi_{A,\mu} - g\Gamma^{R}_{\mu\ A}\,\Psi_{R}$.

2) Prove the equation (24) by using the results of the previous problem and similar results of the conventional tensor calculus.

3) Compute $\Psi_{AB;\mu\nu} - \Psi_{AB;\nu\mu}$. Answer $\Psi_{AB;\mu\nu} - \Psi_{AB;\nu\mu} = -g\,P^{R}_{\mu\nu\ A}\,\Psi_{RB} - g\,P^{R}_{\mu\nu\ B}\,\Psi_{AR}$.

For closing up this section, we give the summary of all important results obtained. These results will be of interest when we come to consider objects like $\Psi^{\mu\ \ldots}_{AB\ldots}$ onto the manifold, and this will be done later on in connection with the spinor formulation of general relativity.

2.1) Summary of the Structure of Internal Objects

(only general formulas are included)

Object	Symbol	Transf. law	Inf. t. law	Covar. Differ.
Contravariant Vector	$\Psi^A(x) = \Psi$:	$\Psi'(x) = M(x)\ \Psi(x)$	$\bar{\delta}\Psi = -i\varepsilon^r L_r \Psi$	$\Psi_{;\mu} = \Psi_{,\mu} + g\Gamma_\mu\ \Psi$:
Covariant Vector	$\Psi_A(x) = \chi$	$\chi'(x) = \chi(x)\bar{M}^1(x)$	$\bar{\delta}\chi = i\ \varepsilon^r \chi\ L_r$	$\chi_{;\mu} = \chi_{,\mu} - g\chi\ \Gamma_\mu$:
Affinity	$\Gamma^A_{\mu\ B} = \Gamma_\mu$	$\Gamma'_\mu = -\frac{1}{g} M_{,\mu}\ M^{-1} + M\ \Gamma_\mu\ M^{-1}$	$\bar{\delta}\Gamma_\mu = \frac{i}{g}\ \varepsilon^r_{,\mu} L_r + i\varepsilon^r \left[\Gamma_\mu, L_r\right]$	-
Analogue of the Riemann Tensor	$P_{\mu\nu} = \Gamma_{\mu,\nu} - \Gamma_{\nu,\mu} + g(\Gamma_\nu \Gamma_\mu - \Gamma_\mu \Gamma_\nu)$	$P'_{\mu\nu}(x) = M(x) P_{\mu\nu}(x) M^{-1}(x)$	$\bar{\delta}P_{\mu\nu} = -i\varepsilon^r \cdot \left[L_r, P_{\mu\nu}\right]$	$P_{\mu\nu;\sigma} = P_{\mu\nu,\sigma} + g\left[\Gamma_\sigma, P_{\mu\nu}\right]$
Mixed multi-order tensors	$\Psi^{AB\ldots}_{CD\ldots}$	Similar to that obtained by product of components $\Psi^A_1 \Psi^B_2 \ldots \chi^1_C\ \chi^2_D \ldots$	Similar to the linearization of the transf. of $\Psi^A_1 \Psi^B_2 \ldots \chi^1_C \chi^2_D \ldots$	Obtained by superposition of derivat. of Ψ and χ.

2.3) Affine Geodesics

Once we have given an affinity over the manifold, and thus an affine geometry is defined on the manifold, we can correspond distant points and introduce curves into the manifold. An important class of curves are the affine geodesics. They are defined as follows: Given a point P lying on the curve, take any vector proportional to the tangent at the curve at P and transport this vector parallel to itself along the curve to another point P' also on the curve. Then, if the paral lel transported vector is proportional to the tangent to the curve at P' for every point P', and every starting point P, the curve is an affine geodesic.

The significance of this may be understood easily if one takes a overall flat manifold, where the affinities $\Gamma^{\mu}_{\nu\sigma}$ vanish at all points, then the parallel transport of a vector does not change their components, and the vector at P is identical to the transported vector at P'; thus, an affine geodesic for flat space is just a straight line joining P to P'. Therefore, one can describe the affine geodesic between two points as the "straightest" line which joins these points.

For determining explicitly the equation for those family of curves let us use a parametric representation for fixing the points on the curve.

$$x^{\mu} = \xi^{\mu}(\tau)$$

where τ is a continuous parameter defined on the curve, which increases monotonically as one proceeds along the curve in a fixed direction. The tangent to the curve at a point with parameter value τ is given by

$$t^{\mu} = \frac{d\xi^{\mu}}{d\tau}$$

since the differentials $d\xi^{\mu}$ transform as a contravariant vector and $d\tau$ is assumed to be a scalar, the tangent is a contravariant vector. If we transport the tangent vector from the point τ to $\tau + d\tau$ we will get a new vector,

$$\bar{t}^{\mu} = t^{\mu} + \delta t^{\mu} = \frac{d\xi^{\mu}}{d\tau} - \Gamma^{\mu}_{\nu\sigma} \frac{d\xi^{\nu}}{d\tau} d\xi^{\sigma}$$

while the tangent at the curve at P' (corresponding to $\tau + d\tau$) has the value

$$t'^{\mu} = \frac{d}{d\tau} \xi^{\mu}(\tau + d\tau) = \frac{d\xi^{\mu}}{d\tau} + \frac{d^2\xi^{\mu}}{d\tau^2} d\tau$$

From the definition of geodesic it follows that these vectors must be proportional to each other,

$$\overline{t}^{\mu} = K(\tau,\ d\tau)t'^{\mu}$$

where the proportionality constant K must be such that $K(\tau,\ 0) = 1$. Hence $K(\tau, d\tau)$ must have the form

$$K(\tau,\ d\tau) = 1 + \alpha(\tau)d\tau$$

we then have

$$\frac{d^2\xi^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\ \frac{d\xi^\rho}{d\tau}\ \frac{d\xi^\sigma}{d\tau} = -\alpha(\tau)\ \frac{d\xi^\mu}{d\tau} \tag{2-3-1}$$

To a large extent the parametrization of the curve is arbitrary. Given one parametrization in terms of τ, we can introduce a new parameter $s(\tau)$ without affecting the geodesic nature of the curve. If we do this, we obtain

$$\frac{d^2\xi^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma}\ \frac{d\xi^\rho}{ds}\ \frac{d\xi^\sigma}{ds} = -\ \frac{\alpha(\tau)s'+s''}{s'^2}\ \frac{d\xi^\mu}{ds} \tag{2-3-2}$$

where s' and s'' are the first and second derivatives of s with respect to τ. We see that is always possible to find a parameter $s(\tau)$ such that the right hand side of (2-3-2) vanishes, that is, a $s(\tau)$ such that

$$\alpha(\tau)s' + s'' = 0\ .$$

Since this equation possess solutions for arbitrary $\alpha(\tau)$. Thus, we may present the equation of a geodesic as

$$\frac{d^2\xi^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma}\frac{d\xi^\rho}{ds}\ \frac{d\xi^\sigma}{ds} = 0 \tag{2-3-3}$$

2.4) Distant parallelism and Affine Flatness. The flat Space-Time Tetrads

The affinities $\Gamma^\mu_{\nu\sigma}(x)$ represent altogether 64 functions of the coordinates, which transform under a mapping according to (2-1-7),

$$\Gamma'^{\mu}_{\nu\sigma}(x') = \frac{\partial x^{\lambda}}{\partial x'^{\nu}} \frac{\partial x^{\beta}}{\partial x'^{\sigma}} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \Gamma^{\alpha}_{\lambda\beta}(x) + \frac{\partial^2 x^{\beta}}{\partial x'^{\nu}\partial x'^{\sigma}} \frac{\partial x'^{\mu}}{\partial x^{\beta}}$$

The $\Gamma^{\mu}_{\nu\sigma}(x)$ and $\Gamma'^{\mu}_{\nu\sigma}(x')$ represent the same affine geometry set down on the manifold in two different ways. We now prove the following result, which may be taken as a theorem: It is always possible to determine a coordinate system, in fact infinitely many, in which the $\Gamma^{\mu}_{\nu\sigma}(x)$ vanish at a given point of the manifold. Indeed, we have

$$\frac{\partial^2 x^{\rho}}{\partial x'^{\nu}\partial x'^{\sigma}} + \Gamma^{\rho}_{\beta\alpha}(x) \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\alpha}}{\partial x'^{\sigma}} = \Gamma'^{\mu}_{\nu\sigma}(x') \frac{\partial x^{\rho}}{\partial x'^{\mu}} \qquad (2\text{-}4\text{-}1)$$

let us take a point P on the manifold, in the coordinate system x it has coordinates x^{μ}_{P}. We consider the coordinate mapping $x^{\mu} = x^{\mu}(x'^{\nu})$ and impose that at P the affinities $\Gamma'^{\mu}_{\nu\sigma}$ vanish

$$\Gamma'^{\mu}_{\nu\sigma}(x'_{P}) = 0 \qquad (2\text{-}4\text{-}2)$$

always there exists a coordinate transformation such that this condition is satisfied. Indeed, the condition (2-4-2) implies from (2-4-1) that at P,

$$\left\{ \frac{\partial^2 x^{\rho}}{\partial x'^{\nu}\partial x'^{\sigma}} + \Gamma^{\rho}_{\beta\alpha}(x) \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\alpha}}{\partial x'^{\sigma}} \right\}_{P} = 0 \qquad (2\text{-}4\text{-}3)$$

a solution of this equation, which at the same time is the coordinate transformation we are looking for, is

$$x^{\alpha}(x') = x^{\alpha}_{P} + x'^{\alpha} - \frac{1}{2} (\Gamma^{\alpha}_{\tau\mu})_{P}\, x'^{\tau} x'^{\mu} , \qquad (2\text{-}4\text{-}4)$$

indeed, from (2-4-4) one gets

$$\frac{\partial x^{\rho}}{\partial x'^{\nu}} = \delta^{\rho}_{\nu} - \frac{1}{2} (\Gamma^{\rho}_{\tau\mu})_P \; (\delta^{\tau}_{\nu} \; x'^{\mu} + \delta^{\mu}_{\nu} \, x'^{\tau})$$

$$\frac{\partial^2 \, x^{\rho}}{\partial x'^{\nu} \partial x'^{\sigma}} = - \; (\Gamma^{\rho}_{\nu\sigma})_P$$

since in the new coordinate system the point P is given by $x'^{\mu} = 0$ (see the equation (2-4-4)), we have

$$\left(\frac{\partial x^{\beta}}{\partial x'^{\nu}}\right)_P = \delta^{\beta}_{\nu} \, , \quad \left(\frac{\partial^2 \, x^{\rho}}{\partial x'^{\nu} \partial x'^{\sigma}}\right)_P = - \; (\Gamma^{\rho}_{\nu\sigma})_P$$

substitution of these two relations into (2-4-3) shows that this equation is satisfied at P.

We remark that there is an extension of this result which shows that exists a coordinate transformation, which is a generalization of our equations (2-4-4), such that in this new representation the affinities vanish along an arbitrarily prescribed geodesic. [6]

There exists a manifold for which the affinities $\Gamma^{\beta}_{\rho\alpha}(x)$ vanish everywhere in a given system of coordinates, this manifold is the flat space-time of special relativity *, and the coordinate system displaying this feature is the cartesian system of coordinates. In any other coordinate system we will get non-vanishing components. These later components may be calculated by the following method, take x' as cartesian coordinates in (2-1-7), so that $\Gamma'^{\beta}_{\rho\alpha}(x') = 0$ which gives for the coordinates x, which may be for instance spherical coordinates,

$$\Gamma^{\beta}_{\rho\alpha}(x) = - \frac{\partial^2 \, x^{\beta}}{\partial x'^{\nu} \partial x'^{\sigma}} \; \frac{\partial x'^{\nu}}{\partial x^{\rho}} \; \frac{\partial x'^{\sigma}}{\partial x^{\alpha}} \qquad (2\text{-}4\text{-}5)$$

* Obviously all present statements hold for n-dimensional spaces, butwe restrict the discussion to four-dimensional spaces.

we may thereby call such type of affinity as the flat space-time affinity. If further we allow just linear mappings onto the manifold, as is the case for special relativity, we get that in cartesian coordinates related one to the other by Lorentz transformations, the affinities $\Gamma^{\beta}_{\rho\alpha}(x)$ vanish for any coordinate system. Since in the next sections we will generalize the special relativistic principle of relativity, we will barely make use of such simple types of manifold, unless for treating the so called flat space-time theories of gravitation, which will be done later on. Presently, in discussing geometrical properties of the manifold, we take this case as a very particular situation with no further comments.

Flat space-time affinities given by (2-4-5) give rise to a flat affine geometry. In what follows we give a method for recognizing this type of geometry, the so called method of distant parallelism of vectors. This method has the further advantage of introducing another important geometrical object into the manifold, the flat space-time tetrad, which in the following section will be generalized to the Rieamnnian space-time tetrad.

The knowledge of the $\Gamma^{\beta}_{\rho\alpha}(x)$ allow us to introduce the notion of local parallelism, used before for introducing the concept of covariant derivatives, which is essentially a local operation, as any other kind of derivative. Thus, an affine geometry supplies us with the notion of local parallelism. However, we cannot in general decide if two vectors on distant separated points are, or are not parallel. To answer this, we would have to parallel-transport one of the vectors along some curve connecting these two points, up to the location of the other vector. In general the transported vector will vary according to the curve upon which it was moved. In other words, the affinity in general is not integrable,

$$\oint_C \delta A^\mu = - \oint_C \Gamma^\mu_{\nu\rho} A^\rho dx^\nu \neq 0$$

If however, the resultant vector is independent of the path of transport, we would have an integrable affinity and we may speak of the notion of distant parallelism.

Suppose that we do have an integrable affinity. Then, given a vector h^μ at a point P, we can construct a unique vector $h^\mu(x)$, that is a unique vector field over the manifold, by transporting h^μ parallel to itself to each point of the manifold. As it is clear, whenever we calculate $h^\mu(x+dx)$ by Taylor's expansion, we obtain another vector $h^\mu(x) + dh^\mu(x)$ which is parallel to $h^\mu(x)$. Then, if we want to impose that the transported vector from the point x to x + dx is parallel to $h^\mu(x)$, we need to impose that $\delta h^\mu = dh^\mu$. But this imposition is equivalent to $h^\mu_{;\nu} = 0$,

$$h^\mu_{;\nu} dx^\nu = dh^\mu - \delta h^\mu = 0$$

or

$$h^\mu_{;\nu} = h^\mu_{,\nu} + \Gamma^\mu_{\rho\nu} h^\rho = 0 \qquad (2\text{-}4\text{-}6)$$

These are the equations for a parallel vector field $h^\mu(x)$. Alternatively, the conditions that these equations possess solutions are the necessary and suficient conditions that the affinity is integrable. Before stablishing those conditions, we generalize the equation (2-4-6) by taking a system of four linearly independent unit vectors $h^\mu_{(\alpha)}(x)$ at each point of the manifold. Any vector B^μ is then written as a combination of the basis vectors $h^\mu_{(\alpha)}$,

$$B^\mu(x) = B^{(\alpha)}(x)\, h^\mu_{(\alpha)}(x) \;, \qquad (2\text{-}4\text{-}7)$$

(the index into round brackets denote the four vectors of the basis, $h_{(1)} \ldots h_{(4)}$). This basis is supposed to be parallel to itself at all points, that is, it satisfies the equation (2-4-6),

$$h^{\mu}_{(\alpha);\nu} = h^{\mu}_{(\alpha),\nu} + \Gamma^{\mu}_{\rho\nu}\, h^{\rho}_{(\alpha)} = 0 \; . \qquad (2\text{-}4\text{-}8)$$

We now study the problem of determination of the necessary and suficient conditions for the existence of solutions of (2-4-8). This equation is of the form

$$\frac{\partial f^{\alpha}}{\partial x^{i}} = - F^{\alpha}_{i}\,(f^{1}\ldots f^{m};\; x^{1}\ldots x^{4}) \; ; \qquad (2\text{-}4\text{-}9)$$

where the F^{α}_{i} are known functions of the f^{α} and x^{i}. For getting the relation (2-4-8) we take

$$f^{\alpha} \rightarrow h^{\mu}_{(\rho)} \; , \qquad m = 1,\ldots 16 \; .$$

we shall assume that the function F_{i} are of class C^{1} over the domain of variation of the f^{α} and the x^{i}. Therefore, the functions f^{α} are of class C^{2} on the four-dimensional manifold, and we have

$$\frac{\partial^{2} f^{\alpha}}{\partial x^{i}\partial x^{j}} = \frac{\partial^{2} f^{\alpha}}{\partial x^{j}\partial x^{i}} \qquad (2\text{-}4\text{-}10)$$

using this result into (2-4-9) one gets

$$\frac{\partial F^{\alpha}_{i}}{\partial x^{j}} + \frac{\partial F^{\alpha}_{i}}{\partial f^{\beta}}\, F^{\beta}_{j} = \frac{\partial F^{\alpha}_{j}}{\partial x^{i}} + \frac{\partial F^{\alpha}_{j}}{\partial f^{\beta}}\, F^{\beta}_{i} \; . \qquad (2\text{-}4\text{-}11)$$

Which form the necessary conditions for integrability of the (2-4-9). If the system of equation (2-4-9) has a solution, then either (2-4-11) are identities in f^{α} and x^{i} or else, there are certain functional relations among the f^{α} and x^{i}. Applying these relations for our particular situation, where

$$f^{\alpha} \rightarrow h^{\mu}_{(\rho)}$$

$$F^{\alpha}_{i} \rightarrow - \Gamma^{\mu}_{\lambda\nu}\, h^{\lambda}_{(\rho)}$$

$$i,j \rightarrow \nu,\gamma$$

we obtain for (2-4-11),

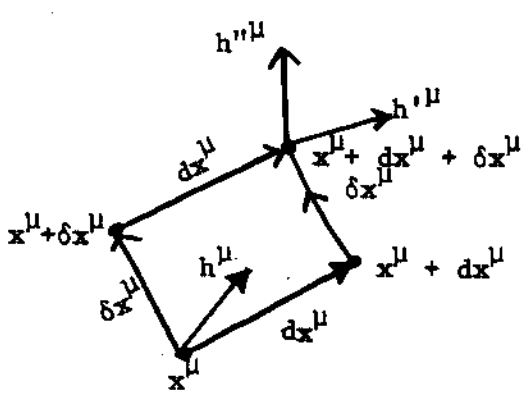
$$(\Gamma^{\rho}_{\sigma\mu,\nu} - \Gamma^{\rho}_{\sigma\nu,\mu} - \Gamma^{\rho}_{\lambda\mu}\,\Gamma^{\lambda}_{\sigma\nu} + \Gamma^{\rho}_{\lambda\nu}\,\Gamma^{\lambda}_{\sigma\mu})\,h^{\sigma}_{(\tau)} \equiv R^{\rho}_{\sigma\mu\nu}\,h^{\sigma}_{(\tau)} = 0\ . \qquad (2\text{-}4\text{-}12)$$

Since the value assigned to $h^{\sigma}_{(\tau)}$ at the starting point P was arbitrary, this condition holds for arbitrary values of $h^{\sigma}_{(\tau)}$, at every point of the manifold. As consequence, we conclude that the necessary condition for the integrability of an affinity is that the associated Riemann tensor vanishes over all points of the manifold.

$$R^{\rho}_{\sigma\mu\nu} = 0 \qquad (2\text{-}4\text{-}13)$$

The vanishing of the Riemann tensor is also a sufficient condition for the integrability of an affinity. To show this we first take a small closed loop. Once having shown that the affinity is integrable along this loop, we can extend this result to a closed finite path connecting P with P', by taking this closed path as formed by two neighbouring curves obtained by adding the necessary number of small closed loops.

Then, we have to prove that the affinity is integrable around a small closed loop when $R^{\rho}_{\sigma\mu\nu} = 0$ on the region contained in this loop. Let us consider the small polygon with vertices at the points x^{μ}, $x^{\mu} + dx^{\mu}$, $x^{\mu} + \delta x^{\mu}$ and $x^{\mu} + dx^{\mu} + \delta x^{\mu}$ as shown in the following picture.



Parallel transport of the basis $h^\mu_{(\alpha)}$ from x^μ to $x^\mu + dx^\mu$ results in the new basis $h^\mu_{(\alpha)} - \Gamma^\mu_{\beta\nu} h^\beta_{(\alpha)} dx^\nu$. In transporting these vectors to the point $x^\mu + dx^\mu + \delta x^\mu$, we must use the value of the affinity at the starting point $x^\mu + dx^\mu$, that is, $\Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\sigma\mu,\nu} dx^\nu$. In this way we construct at $x^\mu + dx^\mu + \delta x^\mu$ the new basis $h'^\mu_{(\alpha)}$ with value

$$h'^\mu_{(\alpha)} = (h^\mu_{(\alpha)} - \Gamma^\mu_{\beta\nu} h^\beta_{(\alpha)} dx^\nu) - (\Gamma^\mu_{\beta\sigma} + \Gamma^\mu_{\beta\sigma,\tau} dx^\tau)(h^\beta_{(\alpha)} - \Gamma^\beta_{\rho\lambda} h^\rho_{(\alpha)} dx^\lambda)\, \delta x^\sigma$$

We could continue to parallel transport this vector back to x^μ via the point $x^\mu_\mu + \delta x^\mu$, and compare its components with the original components at x^μ, the $h^\mu_{(\alpha)}$; however, we can equally test the integrability of the affinity by comparing at the point $x^\mu + dx^\mu + \delta x^\mu$ the two basis $h'^\mu_{(\alpha)}$ and $h''^\mu_{(\alpha)}$ obtained by a similar process but going through the point $x^\mu + \delta x^\mu$,

$$h''^\mu_{(\alpha)} = (h^\mu_{(\alpha)} - \Gamma^\mu_{\beta\nu} h^\beta_{(\alpha)} \delta x^\nu) - (\Gamma^\mu_{\beta\sigma} + \Gamma^\mu_{\beta\sigma\ \tau} \delta x^\tau)(h^\beta_{(\alpha)} - \Gamma^\beta_{\rho\lambda} h^\rho_{(\alpha)} \delta x^\lambda) dx^\sigma$$

(note that we obtain directly $h^{\mu''}_{(\alpha)}$ from $h^{\mu'}_{(\alpha)}$ simply by changing d into δ and viceversa). We get

$$h''^\mu_{(\alpha)} - h'^\mu_{(\alpha)} = R^\mu{}_{\sigma\lambda\tau}\ h^\sigma_{(\alpha)}\, \delta x^\lambda\ dx^\tau = 0\ .$$

If the two basis $h^{\mu'}_{(\alpha)}$ and $h^{\mu''}_{(\alpha)}$ are equal the affinity is integrable. We see that this happens when $R^\mu_{\sigma\lambda\tau} = 0$. Thus, the vanishing of the Riemann tensor is a sufficient condition for the affinity being integrable.

One important point must be understood, the criterion of integrability of an affinity is an invariant property. Namely, such criterion is independent of how one places the affinity on the manifold, if $R^\mu_{\sigma\lambda\tau} = 0$ for some choice of the affinity, it will be zero for any other setting of the affinity. Thus, sets of $\Gamma^\mu_{\beta\sigma}$ which are related by (2-1-7) and which have $R^\mu_{\sigma\lambda\tau} = 0$, are all integrable.

We have said before that one particular set of such integrable $\Gamma^{\mu}_{\beta\sigma}$ were distinguished from the other sets by the fact that they vanish on all points of the manifold, $\Gamma^{\mu}_{\beta\sigma} = 0$, and that the coordinates satisfying this were the cartesian coordinates. We cannot prove this statement in the framework of the affine geometry, only after introducing a metric into the manifold we will be able to prove this, but presently we take this as true and investigate further the structure of the remaining sets of integrable $\Gamma^{\mu}_{\beta\sigma}$. They are given by the equation (2-4-5)". Solving (2-4-8) for the integrable affinity,

$$\Gamma^{\mu}_{\nu\lambda} = - h^{\mu}_{(\alpha),\nu} \bar{h}^{(\alpha)}_{\lambda} \qquad (2\text{-}4\text{-}14)$$

(we have used the fact that $h^{\rho}_{(\alpha)} \bar{h}^{(\alpha)}_{\lambda} = \delta^{\rho}_{\lambda}$, or, $\bar{h}^{(\alpha)}_{\lambda}$ is the inverse matrix to $h^{\lambda}_{(\alpha)}$) *. Comparison of (2-4-5) with (2-4-14) gives

$$h^{\rho}_{(\alpha)} = \frac{\partial x^{\rho}}{\partial x'^{\alpha}} \qquad (2\text{-}4\text{-}15)$$

$$h^{(\alpha)}_{\rho} = \frac{\partial x'^{\alpha}}{\partial x^{\rho}} \qquad (2\text{-}4\text{-}16)$$

where the x' are cartesian coordinates. Indeed, from these later two equations one gets

$$h^{\mu}_{(\alpha),\nu} = \frac{\partial^2 x^{\mu}}{\partial x'^{\alpha} \partial x^{\nu}} = \frac{\partial^2 x^{\mu}}{\partial x'^{\alpha} \partial x'^{\beta}} \frac{\partial x'^{\beta}}{\partial x^{\nu}}$$

so that

$$h^{\mu}_{(\alpha),\nu} \bar{h}^{(\alpha)}_{\lambda} = \frac{\partial^2 x^{\mu}}{\partial x'^{\alpha} \partial x'^{\beta}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} \frac{\partial x'^{\alpha}}{\partial x^{\lambda}}$$

which proves our previous statement. Thus, we conclude that the vanishing of the affinities $\Gamma'^{\mu}_{\nu\lambda}(x')$ which conducted to (2-4-5), implies through (2-4-14) that the

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* Since the $h^{\mu}_{(\alpha)}$ are linearly independent, we have $c^{(\alpha)} h^{\mu}_{(\alpha)} = c^{\mu}$, for $c^{(\alpha)} \neq 0$, and this system of equation has solutions for the $c^{(\alpha)}$ only if $|h^{\mu}_{(\alpha)}| \neq 0$.

$h^{\rho}_{(\alpha)}$ and $\bar{h}^{(\alpha)}_{\rho}$ are given by (2-4-15) and (2-4-16). Equivalently, one can say that the necessary conditions for $\Gamma'^{\mu}_{\nu\lambda} = 0$ is that there exists a set of basis vectors $h^{\rho}_{(\alpha)}$ and $\bar{h}^{(\alpha)}_{\rho}$ such that (2-4-15) and (2-4-16) are satisfied. We prove now that this is also a sufficient condition: The existence of a system of basis vectors satisfying (2-4-15) and (2-4-16) implies that there exists a mapping such that all affinities vanish in the new representation. Obviously we are always referring to integrable sets of affinities. For proving this later statement, we start with (2-4-14). Imposing that the $\Gamma^{\mu}_{\nu\lambda}$ of (2-4-14) is symmetric in the lower indices we get

$$h^{\mu}_{(\alpha),\nu}\,\bar{h}^{(\alpha)}_{\lambda} = h^{\mu}_{(\alpha),\lambda}\,\bar{h}^{(\alpha)}_{\nu} \tag{2-4-17}$$

further, from the fact that $h^{\mu}_{(\alpha)}$ and $\bar{h}^{(\alpha)}_{\mu}$ are reciprocal matrices, we obtain

$$h^{\mu}_{(\alpha)}\,\bar{h}^{(\alpha)}_{\nu,\lambda} = -\,h^{\mu}_{(\alpha),\lambda}\,\bar{h}^{(\alpha)}_{\nu} \tag{2-4-18}$$

adding (2-4-17) and (2-4-18),

$$h^{\mu}_{(\alpha),\nu}\,\bar{h}^{(\alpha)}_{\lambda} + h^{\mu}_{(\alpha)}\,\bar{h}^{(\alpha)}_{\nu,\lambda} = 0 \tag{2-4-19}$$

in the first term of (2-4-19) we use once more a relation like (2-4-18),

$$-\,h^{\mu}_{(\alpha)}\,\bar{h}^{(\alpha)}_{\lambda,\nu} + h^{\mu}_{(\alpha)}\,\bar{h}^{(\alpha)}_{\nu,\lambda} = 0$$

which may be written as

$$h^{\mu}_{(\alpha)}\,(\bar{h}^{(\alpha)}_{\nu,\lambda} - \bar{h}^{(\alpha)}_{\lambda,\nu}) = 0$$

thus

$$\bar{h}^{(\alpha)}_{\nu,\lambda} = \bar{h}^{(\alpha)}_{\lambda,\nu}$$

But this implies that within the same region where the affinity is integrable there exists a scalar $f^{(\alpha)}(x)$ such that

$$\bar{h}^{(\alpha)}_{\nu} = f^{(\alpha)}_{,\nu}$$

Let us use these four scalars $f^{(\alpha)}$ in order to form a mapping

$$x'^{\alpha} = f^{(\alpha)}(x)$$

then

$$\bar{h}^{(\alpha)}_{\nu} = \frac{\partial x'^{\alpha}}{\partial x^{\nu}} = \frac{\partial f^{(\alpha)}}{\partial x^{\nu}}$$

and consequently

$$h^{\nu}_{(\alpha)} = \frac{\partial x^{\nu}}{\partial x'^{\alpha}}$$

which are the similar to the equations (2-4-15) and (2-4-16). For completing the proof, we have to show that the affinity in the new representation vanishes $\Gamma'^{\mu}_{\nu\sigma}(x') = 0$. We have from (2-1-17),

$$\Gamma'^{\mu}_{\nu\sigma}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial x^{\beta}}{\partial x'^{\sigma}} \frac{\partial x'^{\mu}}{\partial x^{\gamma}} \Gamma^{\gamma}_{\alpha\beta}(x) + \frac{\partial^2 x^{\alpha}}{\partial x'^{\nu} \partial x'^{\sigma}} \frac{\partial x'^{\mu}}{\partial x^{\alpha}}$$

substituting the $\frac{\partial x^{\alpha}}{\partial x'^{\nu}}$ and $\frac{\partial x'^{\mu}}{\partial x^{\gamma}}$ by $h^{\alpha}_{(\nu)}$ and $\bar{h}^{(\mu)}_{\gamma}$, and using the expression for $\Gamma^{\gamma}_{\alpha\beta}(x)$ given by (2-4-14), we obtain after some easy steps

$$\Gamma'^{\mu}_{\nu\sigma}(x') = 0$$

This completes the proof. Therefore, we can state: The necessary and suficient condition for $\Gamma^{\mu}_{\nu\sigma} = 0$ is the existence of a parallel system of basis vector over the manifold satisfying (2-4-15) and (2-4-16).

For completeness we have to prove now that indeed the $h^{\mu}_{(\alpha)}$ and $\bar{h}^{(\alpha)}_{\mu}$ of those relations are really a set of four contravariant and four covariant vectors. That is, we have to show that under a mapping $y^{\lambda} = y^{\lambda}(x)$,

$$h'^{\mu}_{(\alpha)}(y) = \frac{\partial y^{\mu}}{\partial x^{\rho}} h^{\rho}_{(\alpha)}(x)$$

(the proof for $\bar{h}^{(\alpha)}_{\mu}$ follows analogously). The proof of (2-4-20) is easily obtained, since

$$h'^{\mu}_{(\alpha)}(y) = \frac{\partial y^{\mu}}{\partial x'^{\alpha}} = \frac{\partial y^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\alpha}} = \frac{\partial y^{\mu}}{\partial x^{\rho}} h^{\rho}_{(\alpha)}(x)$$

The system of parallel basis $h^{\mu}_{(\alpha)}(x)$ and their reciprocal $\bar{h}^{(\alpha)}_{\mu}(x)$ on the manifold, form the so called flat space-time tetrads. They satisfy

$$h^{\rho}_{(\alpha)}(x)\ \bar{h}^{(\alpha)}_{\lambda}(x) = \delta^{\rho}_{\lambda} \ , \tag{2-4-21}$$

$$h^{\rho}_{(\alpha)}(x)\ \bar{h}^{(\beta)}_{\rho}(x) = \delta^{(\beta)}_{(\alpha)} \ . \tag{2-4-22}$$

In the next chapter we will generalize this concept in order to introduce into the curved manifold a set of basis system (which will not form a field of parallel basis anymore). The tetrad field of vectors.

3. THE GEOMETRY ON A METRICAL MANIFOLD

3.1) The Metric Tensor

Following our discussions of the geometrical concepts of space-time we now introduce the idea of distance and angle between vectors, that is, we introduce a metrical geometry into the manifold. According to Riemann we define the distance between the nearby points x and x+dx as

$$ds^2 = g_{\mu\nu}(x)\ dx^{\mu}\ dx^{\nu} \tag{3-1-1}$$

ds is called the line element. A metrical geometry with ds^2 of the form above is called a Riemannian geometry. The norm of a vector is defined by

$$A^2(x) = g_{\mu\nu}(x)\ A^{\mu}(x)\ A^{\nu}(x) \tag{3-1-2}$$

If $A^2 > 0$ (or < 0) for arbitrary nonzero components A^{μ}, the metric $g_{\mu\nu}$ is said to be positive (or negative) definite. Otherwise the metric is said to be indefinite. In the great part of all applications to the relativity theory the metric is indefinite, however, there exist sub-spaces of the four-dimensional manifold which possess definite positive (or definite negative) metrics.

The angle between two vectors A^μ and B^μ is defined by

$$\cos(A,B) = \frac{g_{\mu\nu} \; A^\mu \; B^\nu}{\sqrt{g_{\mu'\nu'} \; A^{\mu'\nu'}} \; \sqrt{g_{\rho\sigma} \; A^\rho \; B^\sigma}} \tag{3-1-3}$$

Care must be taken since for indefinite metrics it may happen that a given vector C^μ has a null norm

$$C^2 = g_{\mu\nu} \; C^\mu \; C^\nu = 0 \tag{3-1-4}$$

vectors satisfying this condition are called null vectors. For them the definition formula (3-1-3) does not apply. Nevertheless, in general we still can define orthogonality even for null vectors and some other vector, by

$$A.B = g_{\mu\nu}(x) \; A^\mu(x) \; B^\nu(x) = 0 \tag{3-1-5}$$

we then said that A^μ and B^μ are perpendicular, in this case A^μ or B^μ may be a null vector, one at a time of course. In a next section we will define more completely the structure of the indefinite metric of general relativity.

The geometrical character of $g_{\mu\nu}$ is fixed by the requirement that ds isa scalar under space-time mappings. This is an extra imposition and is done both due to our previous experience in special relativity and to an heuristic argument of logical simplicity. In this case $g_{\mu\nu}$ is a symmetrical tensor. Given $g_{\mu\nu}$ we can construct several other quantities which will be useful in the subsequent treatment. First, we form the determinant g of $g_{\mu\nu}$,

$$g = |g_{\mu\nu}|$$

with its help we form the inverse matrix to $g_{\mu\nu}$ as

$$g^{\mu\nu} = \frac{1}{g}\frac{\partial g}{\partial g_{\mu\nu}} \qquad (3\text{-}1\text{-}6)$$

since

$$g_{\mu\rho}\, g^{\rho\nu} = \delta^{\nu}_{\mu} \qquad (3\text{-}1\text{-}7)$$

it follows that $g^{\mu\nu}$ is a contravariant symmetric tensor of rank two. The quantity g is a scalar density of weight +2, since

$$g'_{\mu\nu}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\,\frac{\partial x^{\beta}}{\partial x'^{\nu}}\, g_{\alpha\beta}(x)$$

and taking determinants on both sides we obtain

$$g'(x') = \left|\frac{\partial x}{\partial x'}\right|^{2} g(x) .$$

Given $g_{\mu\nu}$ and $g^{\mu\nu}$ we may introduce the operation of raising and lowering indices. For a tensor $T^{\cdots\mu\cdots}_{\cdots\;\cdots}$ we can construct another tensor $T^{\cdots\;\cdots}_{\cdots\mu\cdots}$ according to

$$T^{\cdots\;\cdots}_{\cdots\mu\cdots} = g_{\mu\nu}\, T^{\cdots\nu\cdots}_{\cdots\;\cdots}$$

while for $T^{\cdots\;\cdots}_{\cdots\mu\cdots}$ we can construct $T^{\cdots\mu\cdots}_{\cdots\;\cdots}$ by

$$T^{\cdots\mu\cdots}_{\cdots\;\cdots} = g^{\mu\nu}\, T^{\cdots\;\cdots}_{\cdots\nu\cdots}$$

Similarly we can raise or lower indices on densities. Once we use this operation of raising and lowering indices we have to take care on the correct position of the indices, as example given $T^{\mu\nu}$ which is not symmetrical, the two tensors

$$T_{\rho}{}^{\nu} = g_{\rho\mu}\, T^{\mu\nu}$$

$$T^{\nu}{}_{\rho} = g_{\rho\mu}\, T^{\nu\mu}$$

are not equal.

3.2) Metric Geodesics and Metric Affinity

In the previous section we introduced the concept of geodesics associated to a given affinity as a generalization of one of the properties of the straight line in Euclidian geometry, namely the property that the tangent vector to the straight line forms a field of parallel vectors. Here, we define a metric geodesic also as a generalization of the property of straight lines that they include the shortest distance between two points. Accordingly, we define a metric geodesic between P and P' as the curve joining these points for which the arc element is stationary

$$\delta\, S_{PP'} = \delta \int_P^{P'} ds = \delta \int_P^{P'} \frac{ds}{d\lambda}\, d\lambda = 0$$

for variations which vanish at the boundaries. Here,

$$S_{,\lambda} = \frac{ds}{d\lambda} = \sqrt{g_{\mu\nu}\, \dot{x}^{\mu}\, \dot{x}^{\nu}} \quad , \quad \dot{x}^{\mu} = \frac{dx^{\mu}}{d\lambda}$$

Thus, we have

$$S_{PP'} = \int_P^{P'} S_{,\lambda}\, d\lambda \ , \quad S_{,\lambda} = f_{\lambda}(g_{\mu\nu}(x), \dot{x}^{\mu})$$

hence,

$$\delta S_{PP'} = \int_P^{P'} \left(\frac{\partial f_{\lambda}}{\partial x^{\mu}}\, \delta x^{\mu} + \frac{\partial f_{\lambda}}{\partial \dot{x}^{\mu}}\, \delta \dot{x}^{\mu} \right) d\lambda$$

or,

$$S_{PP'} = \int_{P}^{P'} \left(\frac{\partial f_\lambda}{\partial x^\mu} - \frac{d}{d\lambda} \left(\frac{\partial f_\lambda}{\partial \dot{x}^\mu} \right) \right) \delta x^\mu \, d\lambda + \left[\frac{\partial f_\lambda}{\partial \dot{x}^\mu} \, \delta x^\mu \right]_{P}^{P'}$$

calculating the derivatives, we get

$$\delta S_{PP'} = \int_{P}^{P'} \left\{ \frac{1}{2} \frac{\dot{x}^\beta \dot{x}^\nu}{S_{,\lambda}} \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{d}{d\lambda} \left(\frac{g_{\mu\nu} \dot{x}^\nu}{S_{,\lambda}} \right) \right\} \delta x^\mu \, d\lambda = 0$$

where we used that δx^μ vanish at the boundaries. Since δx^μ is arbitrary inside the region of integration we conclude that

$$\frac{1}{2} \frac{\dot{x}^\beta \dot{x}^\nu}{S_{,\lambda}} \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{d}{d\lambda} \left(\frac{g_{\mu\nu} \dot{x}^\nu}{S_{,\lambda}} \right) = 0 \tag{3-2-1}$$

using the symmetry of $g_{\mu\nu}$ we obtain after some easy steps,

$$- \{\mu\nu,\sigma\} \dot{x}^\mu \dot{x}^\nu - g_{\mu\sigma} \ddot{x}^\mu + \frac{\left(\frac{d^2 s}{d\lambda^2} \right)}{S_{,\lambda}} g_{\mu\sigma} \dot{x}^\mu = 0 \tag{3-2-2}$$

where,

$$\{\mu\nu,\sigma\} = \frac{1}{2} (g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}) \tag{3-2-3}$$

are the so called Christoffel symbols of first kind. The equation (3-2-2) can be greatly simplified by choosing the parameter λ equal to the distance s along the curve. In this case the last term on the right hand side of (3-2-2) is zero, and we get, solving for the $\ddot{x}^\mu$,

$$\frac{d^2 x^\mu}{ds^2} + \left\{ {\mu \atop \rho\sigma} \right\} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} = 0 \tag{3-2-4}$$

where the $\{^{\mu}_{\rho\sigma}\}$ are the Christoffel symbols of second kind, * defined as

$$\{^{\mu}_{\rho\sigma}\} = g^{\mu\nu} \quad \{\rho\sigma,\nu\} = \{^{\mu}_{\sigma\rho}\} \tag{3-2-5}$$

Thus, the equation of the metric geodesic may be put in the simple form (3-2-4). Since s is the arc element along the curve, we must restrict the solutions of (3-2-4) by the supplementary condition

$$g_{\mu\nu} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} = 1 \ . \tag{3-2-6}$$

It can be shown that the equation (3-2-6) is a first integral of (3-2-4), indeed differentiating (3-2-6) one finds,

$$g_{\mu\nu,\sigma} \frac{dx^{\sigma}}{ds} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} + g_{\mu\nu} \left(\frac{d^2x^{\mu}}{ds^2} \frac{dx^{\nu}}{ds} + \frac{dx^{\mu}}{ds} \frac{d^2x^{\nu}}{ds^2} \right) = 0$$

Using (3-2-4) we will get an identity.

For obtaining the previous results we assumed that $ds^2 \neq 0$. But since we deal with indefinite metrics, it may happen that $ds^2 = 0$. As we will see, this case corresponds to the motion of particles with zero rest mass, moving along the light cone. For this situation the previous relations do not apply, and we need to derive other variational principle. This variational principle is obtained directly from Fermat's principle of optics.

Consider the case where g_{oo} is taken negative, as we will see this corresponds to a Riemannian (or pseudo-Riemannian) metric with local signature +2. We have

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* These quantities transform under the MMG as an affinity.

$$ds^2 = g_{ik}\, dx^i\, dx^k + 2g_{oi}\, dx^o\, dx^i + g_{oo}\, dx^o\, dx^o = 0$$

for $g_{oo} < 0$, the positive root of this equation is,

$$dx^o = - g_{oi}\, dx^i/g_{oo} - 1/g_{oo}\, \sqrt{g_{oi}\, g_{oj}\, dx^i\, dx^j - g_{ij}\, dx^i\, dx^j} > 0 \ ,$$

The quantity with the dimension of lenght associated to x^o is

$$du = \sqrt{- g_{oo}}\, dx^o$$

which here is equal to,

$$du = \frac{g_{oi}}{\sqrt{-g_{oo}}}\, dx^i + \sqrt{\gamma_{ij}\, dx^i\, dx^j} \qquad (3\text{-}2\text{-}7)$$

where

$$\gamma_{ij} = g_{ij} - \frac{g_{oi}\, g_{oj}}{g_{oo}} \qquad (3\text{-}2\text{-}8)$$

as it may be easily shown, the three-dimensional tensor γ_{ij} is the reciprocal of the spatial components g^{ij} of the four-dimensional metric. The variational principle for light rays is then,

$$\delta \int_{u_o}^{u_1} \frac{du}{d\lambda}\, d\lambda = 0$$

with

$$u_{,\lambda} = \frac{du}{d\lambda} = \frac{g_{oi}}{\sqrt{-g_{oo}}}\, \dot{x}^i + \sqrt{\gamma_{ij}\, \dot{x}^i\, \dot{x}^j}$$

the Euler-Lagrange equations for this problem are,

$$\frac{\partial u_{,\lambda}}{\partial x^i} - \frac{d}{d\lambda}\, \frac{\partial u_{,\lambda}}{\partial \dot{x}^i} = 0$$

we see that in the case where $g_{oi} = 0$, du has formally the same form as ds, but now written in terms of three-dimensional quantities,

$$du = \sqrt{\gamma_{ij}\ dx^i\ dx^j}, \quad g_{oi} = 0$$

hence, in this case we can write immediately, by analogy with (3-2-4),

$$\frac{d^2x^i}{d\ u^2} + \Delta^i_{k\ell}\frac{dx^k}{du}\ \frac{dx^l}{du} = 0 \qquad (3\text{-}2\text{-}9)$$

and $\Delta^i_{k\ell}$ is the three-dimensional Christoffel symbol build up with γ_{ij} and g^{ij}, it happens that here $\gamma_{ij} = g_{ij}$, and we have

$$\Delta^i_{k\ell} = \frac{1}{2}\ g^{im}\left(\frac{\partial\gamma_{mk}}{\partial x^\ell} + \frac{\partial\gamma_{m\ell}}{\partial x^k} - \frac{\partial\gamma_{k\ell}}{\partial x^m}\right) = \frac{1}{2}\ g^{im}\left(\frac{\partial g_{mk}}{\partial x^\ell} + \frac{\partial g_{m\ell}}{\partial x^k} - \frac{\partial g_{k\ell}}{\partial x^m}\right) \qquad (3\text{-}2\text{-}10)$$

For this situation, we have

$$\left\{{i \atop k\ell}\right\} = \Delta^i_{k\ell}$$

but whenever $g_{oi} \neq 0$, clearly this equallity will not hold. For the general si tuation where g_{oi} do not vanish, we have from the Euler-Lagrange equation,

$$\frac{d}{d\lambda}\left(\frac{g_{oi}}{\sqrt{-g_{oo}}} + \frac{\gamma_{i\ell}\ \dot{x}^\ell}{\sqrt{\gamma_{\ell_j}\ \dot{x}^\ell\ \dot{x}^j}}\right) = \dot{x}^\ell\frac{\partial}{\partial x^i}\left(\frac{g_{o\ell}}{\sqrt{-g_{oo}}}\right) + \frac{\dot{x}^\ell\dot{x}^j\ \dfrac{\partial\gamma_{\ell_j}}{\partial x^i}}{2\sqrt{\gamma_{\ell_j}\ \dot{x}^\ell\ \dot{x}^j}} \qquad (3\text{-}2\text{-}11)$$

The equations (3-2-9) and (3-2-11) are the equations of motion for particles moving along null geodesics, the first equation corresponds to stationary fields. Since $\gamma_{i\ell}$ lowers spatial indices, we have $\dot{x}_i = \gamma_{i\ell}\dot{x}^\ell$. What Eq. (3-2-11) says is that the particle moving on a null geodesic is acted by the force

$$-F_i = \frac{\partial u}{\partial x^i}{}^\lambda$$

and has a momentum equal to

$$p_i = \frac{\dot{x}_i}{\sqrt{\gamma_{ij}\,\dot{x}^i\,\dot{x}^j}} + \frac{g_{oi}}{\sqrt{-g_{oo}}}$$

Note the presence of the momentum $g_{oi}/\sqrt{-g_{oo}}$ which is a contribution coming from the field, similarly to the total momentum $p_i + e/cA_i$ in presence of a magnetic field. Thus, the non-stationary part of the field, the g_{oi} acts as a magnetic field imparting an extra momentum to the particle. Finally, we can take λ equal to the time, in order to be able to speak of momentum and force, as we did. Since we will turn back to these questions later on, we finish here the discussion of null geodesics.

In order that the two geodesic paths introduced in the four-dimensional manifold, the affine geodesic and the metric geodesic coincide, we have to suppose that

$$\Gamma^{\mu}_{\nu\alpha} = \{^{\mu}_{\nu\alpha}\} \cdot \qquad (3\text{-}2\text{-}12)$$

This is an extra imposition since the two structures are unrelated in the sense that they can be introduced independently. For all known physical applications of Riemannian geometry the above assumption is sufficient. With this choice our affinity is symmetric on the lower indices.

With the condition (3-2-12) it follows immediately that

$$g_{\mu\nu;\alpha} = 0 \qquad (3\text{-}2\text{-}13)$$

we may say that with the choice (3-2-12) for the affine connection $\Gamma^{\mu}_{\nu\alpha}$, the $g_{\mu\nu}$ turns out to be a constant under covariant differentiation. It may be verified however that the choice (3.2.12) is not unique, indeed we may take instead of (3-2-12),

$$\Gamma^{\mu}_{\nu\alpha} = \{^{\mu}_{\nu\alpha}\} - \phi_{\alpha}\,\delta^{\mu}_{\nu} - \phi_{\nu}\,\delta^{\mu}_{\alpha} + \phi^{\mu}\,g_{\nu\alpha} \qquad (3\text{-}2\text{-}14)$$

This is correct, since the sum of an affinity with a tensor yields a new affini ty, our $\Gamma^{\mu}_{\nu\alpha}$. The geometry underlined through the condition (3-2-14) is called the Weyl's geometry [7] since it contains besides the $g_{\mu\nu}$ another basic variable, the covector ϕ_{μ}. For $\phi_{\mu} = 0$ we recover the Riemannian structure. The condition (3-2-14) conducts to

$$g_{\mu\nu;\alpha} = g_{\mu\nu,\alpha} - \Gamma^{\beta}_{\mu\alpha} g_{\beta\nu} - \Gamma^{\beta}_{\nu\alpha} g_{\mu\beta} = \phi_{\alpha} g_{\mu\nu} \quad . \tag{3-2-15}$$

According to Weyl's interpretation the vector ϕ_{α} describes the geometrical structure of the electromagnetic potentials. In this way an unitary field theory is constructed for gravitation and electromagnetism by generalizing the Riemannian condition (3-2-12).

Problem: Construct the curvature tensor for Weyl's theory using (3-2-14) and the expression of $R^{\rho}_{\sigma\mu\nu}$ in terms of the affine connections $\Gamma^{\rho}_{\sigma\mu}$,

$$R^{\rho}_{\sigma\mu\nu} = \Gamma^{\rho}_{\sigma\mu\ \nu} - \Gamma^{\rho}_{\sigma\nu\ \mu} - \Gamma^{\rho}_{\lambda\mu} \Gamma^{\lambda}_{\sigma\nu} + \Gamma^{\rho}_{\lambda\nu} \Gamma^{\lambda}_{\sigma\mu} \ .$$

We may still go beyond Weyl's work by writing in place of (3-2-14),

$$\Gamma^{\mu}_{\nu\alpha} = \left\{ {\mu \atop \nu\alpha} \right\} + \text{arbitrary tensor field.} \tag{3-2-16}$$

for a tensor field to be specified later, and we can try to construct this tensor in the more general form as possible. However, such general geometries do not possess an obvious geometrical interpretation as the simple Weyl's geometry has.

3.3) Metric Flatness

The relations (3-2-12), (3-2-14) and (3-2-16) are possible choices for the fourty unknown represented by the affine connections $\Gamma^{\mu}_{\nu\alpha}$, which are given by those formulas as linear functions of the Christoffel symbols of the second type.

We here turn once again to study the properties of a metrical space with the square of the line element given by (3-1-1).

$$ds^2 = g_{\mu\nu}(x)\ dx^\mu\ dx^\nu\ .$$

Consider a point P on the manifold, and let ds^2 be the square of the line element emerging from P. We use two settings on the manifold, the first denoted by coordinates x and the second by x', from the invariance of ds^2,

$$ds^2 = g_{\mu\nu}(x)\ dx^\mu\ dx^\nu = g'_{\mu\nu}\ (x')\ dx'^\mu\ dx'^\nu\ . \qquad (3\text{-}3\text{-}1)$$

Since ds^2 is written for the point P, we should write for completeness

$$ds^2 = g_{\mu\nu}(x_P)\ dx_P^\mu\ dx_P^\nu = g'_{\mu\nu}\ (x'_P)\ dx'^\mu_P dx'^\nu_P$$

now, let us take another point in the vicinity of P, which has coordinates x for the first setting and coordinates x' for the second one. We expand the metric tensor on both settings in power series expansion of the separation,

$$g_{\mu\nu}(x) = g_{\mu\nu}(x_P) + (x^\alpha - x_P^\alpha)(g_{\mu\nu,\alpha})_{x=x_P} \qquad (3\text{-}3\text{-}2)$$

$$g'_{\mu\nu}(x') = g'_{\mu\nu}(x'_P) + (x'^\alpha - x'^\alpha_P)(g'_{\mu\nu,\alpha})_{x'=x'_P} \qquad (3\text{-}3\text{-}3)$$

where we assumed that the square of the separation is of higher order and may be neglected. From the equation (3-2-3) we can find after some easy steps,

$$g_{\mu\nu,\alpha} = g_{\sigma\nu}\ \{^{\sigma}_{\mu\alpha}\} + g_{\sigma\mu}\ \{^{\sigma}_{\nu\alpha}\}$$

using the condition (3-2-12), which equals $\{^{\sigma}_{\mu\nu}\}$ to the $\Gamma^\sigma_{\mu\nu}$, we rewrite the above Taylor's series as

$$g_{\mu\nu}(x) = g_{\mu\nu}(x_P) + (x^\alpha - x_P^\alpha)\ \{g_{\sigma\nu}\ \Gamma^\sigma_{\mu\alpha} + g_{\sigma\mu}\ \Gamma^\sigma_{\nu\alpha}\}_{x=x_P} \qquad (3\text{-}3\text{-}4)$$

$$g'_{\mu\nu}\ (x') = g'_{\mu\nu}(x'_P) + (x'^\alpha - x'^\alpha_P)\{g'_{\sigma\nu}\ \Gamma'^\sigma_{\mu\alpha} + g'_{\sigma\mu}\ \Gamma'_{\nu\alpha}\}_{x'=x'_P} \qquad (3\text{-}3\text{-}5)$$

Now, using the theorem proved on section (2-4), which says that it is always possible to determine a setting such that locally the $\Gamma^{\sigma}_{\mu\alpha}$ vanish. As before, we take this setting as given by the coordinates x', and the point under consideration the point P,

$$\Gamma^{\sigma}_{\mu\alpha}(x'_P) = 0$$

$$\Gamma^{\sigma}_{\mu\alpha}(x_P) = A^{\sigma}_{\mu\alpha}$$

where the $A^{\sigma}_{\mu\alpha}$ are constants. From those relations we get for (3-3-4) and (3-3-5) the values

$$g_{\mu\nu}(x) = g_{\mu\nu}(x_P) + (x^{\alpha} - x^{\alpha}_P)\{g_{\sigma\nu}(x_P)\, A^{\sigma}_{\mu\alpha} + g_{\sigma\mu}(x_P)\, A^{\sigma}_{\nu\alpha}\}\,, \tag{3-3-6}$$

$$g'_{\mu\nu}(x') = g'_{\mu\nu}(x'_P)\,. \tag{3-3-7}$$

Thus, for the coordinate system X', the ten quantities $g'_{\mu\nu}(x')$ have the same value $g'_{\mu\nu}(x'_P)$ for all points x' inside the infinitesimal neighbourhood of P. Therefore, in the setting X' where the $\Gamma'^{\mu}_{\nu\alpha}$ vanish at P, the metric tensor is constant over an infinitesimal volume of the manifold containing the point P. Since we know that a quadratic form with constant coefficients is always reducible to a sum of squares, we have at the vicinity of P, from the Eqs. (3-3-1) and (3-3-7),

$$ds^2_P = a_{\mu\nu}\, dx'^{\mu}\, dx'^{\nu} = \text{algebraic sum of square of } dx'^{\mu}$$

where the $a_{\mu\nu}$ is just $g'_{\mu\nu}(x'_P)$. From here on, for simplicity we drop the lines from the coordinates. The above sum over the square of the dx^{μ} cannot be of the type

$$(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2$$

(nor of the type where only the minus sign appears), because we have seen that the metric has to be indefinite. Thus, we have to get alongside with some number of plus signs, some number of minus signs. The difference between the total number of plus signs and the total number of minus signs (or the reverse, depending on the convention which is used) is called the signature of the metric. Since the above process for obtaining the diagonalization of ds^2, and thus, the signature of $g_{\mu\nu}(x)$, is good for any point P of the manifold, we will obtain the same signature for $g_{\mu\nu}(x)$ at any point on the manifold.

In the next section of this chapter we will give explicitly the two possible signatures one can take for $g_{\mu\nu}(x)$. Once each one of them is chosen, it will remain fixed for the remaining of the treatment.

Thus, for a Riemannian manifold, it is possible to determine a coordinate system such that locally the metric tensor $g_{\mu\nu}(x)$ takes on constant values +1 and -1, with a certain arrangement which defines its signature. That is all one can do, whenever we go off this infinitesimal volume, the $g_{\mu\nu}$ assume arbitrary values.*

We treat now the case for a flat manifold. In this situation we saw that there exists a coordinate system, the cartesian coordinates, where the affinities $\Gamma_{\nu\alpha}^{\ \mu}$ vanish over all space. In any other setting, compatible with the flatness, the $\Gamma^{\mu}_{\nu\alpha}$ take on values in terms of the flat tetrads as

$$\Gamma^{\mu}_{\nu\lambda}(x) = - h^{\ \mu}_{(\alpha),\nu}\ \overline{h}^{(\alpha)}_{\lambda} \tag{3-3-8}$$

with

$$h^{\mu}_{(\alpha)} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}}\ ,\ h^{(\alpha)}_{\mu} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \tag{3-3-9}$$

Since the derivatives of $g_{\mu\nu}$ are related to the $\Gamma^{\mu}_{\nu\sigma}$ through the relation written previously, it is to be expected that $g_{\mu\nu}$ is some function of the flat tetrads

* This can be seen as follows, if we go off the infinitesimal vicinity of P we have to take also the remaining terms in the Taylor's expansion (3-3-3), but the next order term will include the curvature at P which is not zero, and thus, the $g'_{\mu\nu}$ (x') is not constant in the second order approximation.

$h^{\mu}_{(\alpha)}$ and their reciprocal $\bar{h}^{(\alpha)}_{\mu}$. Indeed, if we call by $g_{(\alpha)(\beta)}$ the components of the metric tensor in cartesian coordinates, and by $g_{\alpha\beta}$ the values into the setting where the $\Gamma^{\mu}_{\nu\sigma}$ take on values given by (3-3-8), we get by using the property that the metric is a second rank tensor,

$$g_{\mu\nu}(x) = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g_{(\alpha)(\beta)} , \qquad g^{\mu\nu}(x) = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} g^{(\alpha)(\beta)} \tag{3-3-10}$$

using (3-3-9) we write this as

$$g_{\mu\nu}(x) = \bar{h}^{(\alpha)}_{\mu} \bar{h}^{(\beta)}_{\nu} g_{(\alpha)(\beta)}, \quad g^{\mu\nu}(x) = h^{\mu}_{(\alpha)} h^{\nu}_{(\beta)} g^{(\alpha)(\beta)} \tag{3-3-11}$$

and this is just the relation between the metric $g_{\mu\nu}$ and the flat tetrads $h^{\mu}_{(\alpha)}$ and $\bar{h}^{(\alpha)}_{\mu}$. A metric field satisfying (3-3-11) with the $h^{\mu}_{(\alpha)}$ and $\bar{h}^{(\alpha)}_{\mu}$ given by (3-3-9) is called a flat metric tensor.

We now analyze the structure of the components $g_{(\alpha)(\beta)}$. Once this is known, we are able to get the structure of any other flat metric tensor $g_{\mu\nu}$ by using (3-3-11). As we said before, the $g_{(\alpha)(\beta)}$ refer to the cartesian system of coordinates, where the affinities vanish (and where covariant differentiation is equivalent to the usual partial differentiation). Using once more the theorem (2-4) we get that the derivatives $g_{(\alpha)(\beta),\mu}$ vanish not only at a point, but now at all points of the manifold *, and thus, the $g_{(\alpha)(\beta)}$ are constants. We indicate the constant values assumed by $g_{(\alpha)(\beta)}$ as $\mathring{g}_{\alpha\beta}$,

$$g_{(\alpha)(\beta)} = \mathring{g}_{\alpha\beta}$$

This result is consistent since in the flat manifold the curvature tensor vanishes at all points.

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* Use that $g_{(\alpha)(\beta),\mu} = g_{(\lambda)(\beta)} \Gamma^{(\lambda)}_{(\alpha)(\mu)} + g_{(\lambda)(\alpha)} \Gamma^{(\lambda)}_{(\beta)(\mu)}$, and in cartesian coordinates the Γ vanish at all points, giving as result that the derivatives of $g_{(\alpha)(\beta)}$ are zero for all points.

Problem: Using the formula (3-3-8) for the affinities, compute the components of the curvature tensor for a flat manifold. Answer: they are all null.

As the last comment on this section, we call attention to an important fact: The choice of settings for $\Gamma^{\mu}_{\nu\sigma}$ and $g_{\mu\nu}$ according to (3-3-8) and (3-3-11) is limited by the condition that the curvature tensor vanishes. For instance, a four-dimensional system of spherical coordinates is excluded since it gives a constant curvature for the manifold which contradicts the fact that the curvature tensor vanishes. All coordinate system to be used are those which are topologically flat.

3.4) Tetrads in Curved Spaces and the Limit to Flatness

The equations (3-3-11) can be generalized to a similar looking type of equation, where now the $h^{(\alpha)}_{\mu}$ and $\bar{h}^{\cdot\mu}_{(\alpha)}$ are taken as arbitrary functions.

$$g_{\mu\nu}(x) = h^{(\alpha)}_{\mu}\ h^{(\beta)}_{\alpha\beta}\ g_{\alpha\beta}\ ,\quad g^{\mu\nu}(x) = \bar{h}^{\mu}_{(\alpha)}\ \bar{h}^{\nu}_{(\beta)}\ g^{\alpha\beta} \qquad (3\text{-}4\text{-}1)$$

satisfying the same relationships (2-4-21) and (2-4-22),

$$\bar{h}^{\mu}_{(\alpha)}\ h^{(\beta)}_{\mu} = \delta^{(\beta)}_{(\alpha)}\ , \qquad \bar{h}^{\ \mu}_{(\alpha)}\ h^{(\alpha)}_{\rho} = \delta^{\mu}_{\rho} \qquad (3\text{-}4\text{-}2)$$

The $h^{(\alpha)}_{\mu}$ and $\bar{h}^{\mu}_{(\alpha)}$ have necessarily to satisfy (3-4-2) in order that the two matrices $(g_{\mu\nu})$ and $(g^{\mu\nu})$ of (3-4-1) be reciprocal,

$$g^{\mu\lambda}\ g^{\mu}_{\nu} = \delta^{\mu}_{\nu}$$

A set of quantities $h^{(\alpha)}_{\mu}$ and $\bar{h}^{\mu}_{(\alpha)}$ satisfying such conditions is called by tetrad four-vectors in the Riemannian space. [8] The reason for such term is obtained when we calculate the Riemann curvature tensor with such field of tetrads, and obtain a non-vanishing tensor as result. We will do this in the next section, but presently we can calculate the Christoffel symbols in terms of these tetrad field of vectors,

$$\{^{\alpha}_{\mu\rho}\} = \frac{1}{2}\,\bar{h}^{\alpha}_{(\lambda)}\; S^{(\lambda)}_{\mu\rho} + \frac{1}{2}\,h^{\alpha(\lambda)}\;\bar{h}^{\beta}_{(\lambda)}\left\{\bar{h}_{(\nu)\mu}\,A^{(\nu)}_{\beta\rho} + \bar{h}_{(\nu)\rho}A^{(\nu)}_{\beta\mu}\right\} \qquad (3\text{-}4\text{-}3)$$

where

$$S^{(\lambda)}_{\mu\rho} = \frac{\partial h^{(\lambda)}_{\mu}}{\partial x^{\rho}} + \frac{\partial h^{(\lambda)}_{\mu}}{\partial x^{\mu}}\,, \quad A^{(\lambda)}_{\mu\rho} = \frac{\partial h^{(\lambda)}_{\mu}}{\partial x^{\rho}} - \frac{\partial h^{(\lambda)}_{\rho}}{\partial x^{\mu}}\;. \qquad (3\text{-}4\text{-}4)$$

As we see the equation (3-4-3) goes over the equation (3-3-8) with the identification (3-2-13), in the limit where the $h^{(\alpha)}_{\mu}$ and $\bar{h}^{\mu}_{(\alpha)}$ take the values for a flat manifold, i.e. the values given by (3-3-9). In this case the $A^{(\lambda)}_{\mu\rho}$ vanish. The geometrical objects represented by the $h^{(\alpha)}_{\mu}$ and $h^{\mu}_{(\alpha)}$ (from here on we shall drop out the bar on the $h^{\mu}_{(\alpha)}$) are of two distinct natures. First, they are four-vectors on the index μ, and second, they are quantities which change under a transformation on the index (α), at a fixed point x, as

$$h'^{(\alpha)}_{\mu}(x) = L^{(\alpha)}(x)_{(\beta)}\; h^{(\beta)}_{\mu}(x) \qquad (3\text{-}4\text{-}5)$$

where $L = (L^{(\alpha)}{}_{(\beta)})$ is a matrix satisfying the same pseudo-orthogonallity condition of a Lorentz transformation matrix in special relativity,

$$\overset{T}{L}_{(\beta)}{}^{(\alpha)}\;\overset{\circ}{g}_{\alpha\rho}\;L^{(\rho)}_{(\lambda)} = \overset{\circ}{g}_{\beta\lambda} \qquad (3\text{-}4\text{-}6)$$

The geometrical meaning of (3-4-5) is that of a local rotation of the four-legs at a fixed point x, the point at which is located the origin of the basys system of tetrads. Under a mapping of the MMG we obtain,

$$h'^{(\alpha)}_{\mu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}}\,h^{(\alpha)}_{\rho}(x) \qquad (3\text{-}4\text{-}7)$$

The transformation law (3-4-5) needs further explanation, and we will see that the fact that $L = L(x)$ implies in the impossibility of stablishing a field of parallel basis on the manifold.

Since in (3-4-5) the index μ is fixed, that is, the transformation (3-4-5) applies equally to all components of the four vectors $h^{(1)} \dots h^{(4)}$, we may write this relation without the index μ.

$$L'^{(\alpha)}(x) = L^{(\alpha)}(x)_{(\beta)}\ h^{(\beta)}(x) \qquad (3\text{-}4\text{-}8)$$

where by $h^{(\alpha)}$ we mean the α-th vector of the basis. What (3-4-8) says is that the vectors $h^{(1)} \dots h^{(4)}$ change as a linear combination of themselves. Obviously, this implies that at the fixed point x we are getting another basis $h'^{(1)} \dots h'^{(4)}$ which is rotated respect to the original basis by some angle. Since L depends on x, we get at each different point x a different choice for the orientation of the new field of basis vectors $h'^{(\alpha)}$.

Now, comes the fundamental point, we impose that all relevant relations are covariant with respect to this choice of orientation (or are invariant, as it happens frequently), in making this statement we are puting the transformations (3-4-5) in the same footing of the transformations (3-4-7), thus obtaining a complete symmetry on all existing groups. We left for a future discussion the meaning of such amplification of the invariance group of general relativity, so as to include the (3-4-5) as a part. The reason for this is simply because we have not as yet introduced the basic postulates of general relativity. With the imposition of symmetry under the choice of any one of the several basys systems $h^{(\alpha)}$ and $h'^{(\alpha)}$ related through (3-4-5), we are immediately conducted to the result that any choice of local orientation for the basis is equally good, and thus, we lost completely the concept of a parallel field of four-legs onto the manifold, as we had for a flat manifold. As result, we have to expect that the Riemann tensor does not vanish over the manifold, and this is indeed the situation.

The inverse components $h^{\mu}_{(\alpha)}$ transform under the internal mapping as,

$$h'^{\mu}_{(\alpha)}(x) = \bar{L}^{1(\beta)}(x)_{(\alpha)}\ h^{\mu}_{(\beta)}(x) \tag{3-4-9}$$

so that the contracted product, the scalar product in the internal space, is invariant.

$$h^{(\alpha)}_{\mu}\ h_{(\alpha)\nu} = \text{internal invariants.}$$

Similarly, any scalar product in coordinate space, such as $h^{\mu}_{(\alpha)}\ h_{\mu(\beta)}$ is also invariant.

The expressions for the covariant derivatives of $h^{(\alpha)}_{\mu}$ and $h^{\mu}_{(\alpha)}$ are obtained taking into account the geometrical properties of those objects as

$$h^{(\alpha)}_{\mu;\nu} = h^{(\alpha)}_{\mu,\nu} - \Gamma^{\lambda}_{\mu\nu} h^{(\alpha)}_{\lambda} + \Lambda^{(\alpha)}{}_{\nu(\beta)}\ h^{(\beta)}_{\mu} \tag{3-4-10}$$

$$h^{\mu}_{(\alpha);\nu} = h^{\mu}_{(\alpha),\nu} + \Gamma^{\mu}_{\nu\lambda}\ h^{\lambda}_{(\alpha)} - \Lambda^{(\beta)}{}_{\nu(\alpha)}\ h^{\mu}_{(\beta)} \tag{3-4-11}$$

Where the existence of the internal affinity $\Lambda^{(\alpha)}{}_{\mu}(\beta)$ is due to the fact that the derivatives of $h^{(\alpha)}_{\mu}{}_{(\alpha)}$ (or $h^{\mu}_{(\alpha)}$) do not transform as $h^{(\alpha)}_{\mu}$ (or $h^{\mu}_{(\alpha)}$) since $L = L(x)$, only the combination $h^{(\alpha)}_{\nu,\mu} + \Lambda^{(\alpha)}{}_{\mu(\beta)}\ h^{(\beta)}_{\nu}$ does have such property.

Now, we know that the Riemannian structure was characterized by the condition $g_{\mu\nu;\rho} = 0$. As it can be proved from (3-4-1), the unique possibility conducting to this is

$$h^{\mu}_{(\alpha);\nu} = 0 \tag{3-4-12}$$

$$\mathring{g}_{\mu\nu;\rho} = 0\ . \tag{3-4-13}$$

Where $\mathring{g}_{\mu\nu;\rho}$ is given by

$$\mathring{g}_{\mu\nu;\rho} = -\Lambda^{(\lambda)}_{\rho(\mu)}\ \mathring{g}_{\lambda\nu} - \Lambda^{(\lambda)}_{\rho(\nu)}\ \mathring{g}_{\mu\lambda}$$

Using (3-4-11) and (3-4-12) we obtain $\Lambda^{(\alpha)}_{\mu}{}_{(\beta)}$ as function of $\Gamma^{\mu}_{\nu\sigma}$,

$$\Lambda_{\mu}^{(\alpha)}{}_{(\beta)} = h^{\lambda}_{(\beta),\mu} \; h^{(\alpha)}_{\lambda} + \Gamma^{\lambda}_{\mu\rho} \; h^{\rho}_{(\beta)} h^{(\alpha)}_{\lambda} \qquad (3\text{-}4\text{-}14)$$

As it is easily seen, the two equations (3-4-10) and $h^{(\alpha)}_{\mu;\nu} = 0$ also conduct to the equation (3-4-14). Now, from (3-4-13) we obtain a further restriction on the components of $\Lambda^{(\alpha)}{}_{(\beta)}$,

$$\Lambda_{\rho(\nu)(\mu)} + \Lambda_{\rho(\mu)(\nu)} = 0 \qquad (3\text{-}4\text{-}15)$$

Which means that the matrix $\Lambda_{\rho} = (\Lambda_{\rho(\nu)(\mu)})$ is skew symmetric. A direct calcula tion using the formula (3-4-14) shows that Λ_{ρ} do not automatically satisfy this condition. Thus, we have to antissymetrize the internal affinity.

$$\Lambda_{\rho(\nu)(\mu)} \rightarrow \frac{1}{2} (\Lambda_{\rho(\nu)(\mu)} - \Lambda_{\rho(\mu)(\nu)})$$

We can obtain further information regarding the equations (3-4-11) and (3-4-12); indeed, from these relations one gets

$$h^{\mu}_{(\alpha),\nu} = - \Gamma^{\mu}_{\nu\lambda} h^{\lambda}_{(\alpha)} + \Lambda^{(\beta)}{}_{\nu\;(\alpha)} h^{\mu}_{(\beta)} \qquad (3\text{-}4\text{-}16)$$

the left hand side of this equation is a gradient, so that we identically get, $h^{\mu}_{(\alpha),\nu\tau} = h^{\mu}_{(\alpha),\tau\nu}$, and as consequence the curl of the right hand side vanishes

$$(-\Gamma^{\mu}_{\nu\lambda} h^{\lambda}_{(\alpha)} + \Lambda^{(\beta)}{}_{\nu(\alpha)} h^{\mu}_{(\beta)})_{,\tau} - (-\Gamma^{\mu}_{\tau\lambda} h^{\lambda}_{(\alpha)} + \Lambda^{(\beta)}{}_{\tau(\alpha)} h^{\mu}_{(\beta)})_{,\nu} = 0 \qquad (3\text{-}4\text{-}17)$$

a straightforward calculation from (3-4-17) leads to,

$$R^{\mu}_{\lambda\tau\nu} h^{\lambda}_{(\alpha)} + S^{(\beta)}_{\nu\tau\;(\alpha)} h^{\mu}_{(\beta)} = 0 \qquad (3\text{-}4\text{-}18)$$

where $R^{\mu}_{\lambda\tau\nu}$ is the Riemann curvature tensor, and $S^{(\beta)}{}_{\nu\tau(\alpha)}$ is

$$S^{(\beta)}{}_{\nu\tau(\alpha)} = \Lambda^{(\beta)}{}_{\nu(\alpha),\tau} - \Lambda^{(\beta)}{}_{\tau(\alpha),\nu} + \Lambda^{(\rho)}{}_{\nu\;(\alpha)} \Lambda^{(\beta)}{}_{\tau(\rho)} - \Lambda^{(\rho)}{}_{\tau(\alpha)} \Lambda^{(\beta)}{}_{\nu\;(\rho)} \qquad (3\text{-}4\text{-}19)$$

But this quantity is just the internal curvature tensor obtained from the equations

$$V^{(\alpha)}_{;\mu\nu} - V^{(\alpha)}_{;\nu\mu} = S^{(\alpha)}_{\mu\nu(\beta)} V^{(\beta)}$$

The equation (3-4-19) may be written in matrix notation as

$$S_{\nu\tau} = \Lambda_{\nu,\tau} - \Lambda_{\tau,\nu} + \Lambda_\tau \Lambda_\nu - \Lambda_\nu \Lambda_\tau \ . \qquad (3\text{-}4\text{-}20)$$

Therefore, the two curvatures are related to each other according to (3-4-18), solving these equations for the $R^{\mu}_{\lambda\tau\nu}$ we get

$$R_{\lambda\tau\nu}^{\ \ \ \ \mu} = h^{\mu}_{(\beta)} h^{(\alpha)}_{\lambda} \ S^{(\beta)}_{\tau\nu(\alpha)} \ , \quad S^{(\beta)}_{\tau\nu(\alpha)} = R^{\mu}_{\lambda\tau\nu} h^{\lambda}_{(\alpha)} h^{(\beta)}_{\mu} \ . \qquad (3\text{-}4\text{-}21)$$

We will turn back to these relations when treating the relationships of this method with the formalism in the complex two-dimensional symplectic space S_2 studied before.

The constant values assumed by the $\overset{\circ}{g}_{\mu\nu}$ will be now interpreted, in consistent way, by turning back to the imposition (3-4-6) on the transformation matrix L. What this relation means, is that the components $\overset{\circ}{g}_{\mu\nu}$ given by

$$\overset{\circ}{g}_{\mu\nu} = h^{\lambda}_{(\mu)} h^{\sigma}_{(\nu)} g_{\lambda\sigma}$$

are unaffected whenever we carry out a local rotation of the "legs". But this is just what happens with the metric tensor in special relativity under the Lorentz group. Thus, we can identify the $\overset{\circ}{g}_{\mu\nu}$ with the special relativistic components of the metric, and the group of local rotations of the legs as a local Lorentz transformation. With this identification we are just saying that in the neighbourhood of any point x of the curved manifold, there exists a tangent hyperplane defined by the set of four pseudo-Euclidian vectors $h^{\mu} = (h^1, \ldots h^4)$ with components $h^1 = (h^1{}_{(1)}, \ldots h^1{}_{(4)})$ etc, which span this local hyperplane. The covariant or contravariant components of those vectors are related through the pseudo-Euclidian metric $\overset{\circ}{g}$,

$$h^{\mu}_{(\alpha)} = \overset{\circ}{g}_{\alpha\beta} h^{(\beta)\mu} \ , \quad h^{(\beta)}_{\mu} = \overset{\circ}{g}^{\beta\alpha} h_{(\alpha)\mu}$$

Thus, in a certain sense, a tetrad formalism is a two-metric formalism, since we simultaneously use the two tensors $g_{\mu\nu}$ and $\overset{\circ}{g}_{\mu\nu}$. The signature of $g_{\mu\nu}$ is, in this method, given by the signature of $\overset{\circ}{g}_{\mu\nu}$, and thus may be +2 or -2 according to any one of the choices (3,1) or(1,3), where the number at the left on the parenthesis indicates the total number of positive components, and the other number the total number of negative components. Through the relation (3-4-14) we have expressed the $\Lambda^{(\alpha)}{}_{\mu(\beta)}$ as function of the $\Gamma^{\mu}_{\nu\sigma}$. Since we already know the $\Gamma^{\mu}_{\nu\sigma}$ in terms of the tetrad, according to (3-4-3), we can express the internal affinity in terms of their own variables, the tetrad vectors. A direct calcula tion gives

$$\Lambda^{(\alpha)}{}_{\mu(\beta)} = h^{(\alpha)}{}_{\rho}\, h^{\rho}_{(\beta),\mu} + \frac{1}{2} h^{\rho}_{(\beta)}\, h^{(\alpha)}{}_{\mu,\rho} + \frac{1}{2} g_{\mu\rho}(h^{(\alpha)\gamma}\, h^{\rho}_{(\beta),\gamma} - - h^{(\alpha)}{}_{,\gamma} h^{\gamma}_{(\beta)}) + \frac{1}{2} h^{(\alpha)\rho}\, h^{\lambda}_{(\beta)}\, g_{\rho\lambda,\mu} - \frac{1}{2} h^{(\alpha)\rho}\, h_{(\beta)\mu,\rho} \,. \tag{3-4-22}$$

Using (3-4-19) we may write the internal curvature entirely in function of the tetrad. We will not give here the details of such calculation which is rather lenghty. We just note that a similar type of calculation may be carried out from the second of the relations (3-4-21) by expressing $R^{\mu}{}_{\nu\rho\sigma}$ in terms of tetrad. This is possible since $R^{\mu}{}_{\nu\rho\sigma}$ is given in terms of the $\Gamma^{\mu}_{\nu\sigma}$ and these later are functions of the tetrad according to (3-4-3).

The flat space-time limit of the tetrad theory is obtained for $h^{\mu}_{(\nu)} = \delta^{\mu}_{\nu}$, since then the $g_{\mu\nu}$ goes over its flat space-time components $\overset{\circ}{g}_{\mu\nu}$.

3.5) The Riemann Tensor in Terms of the Metric

Turning back to the Riemann curvature tensor introduced at the chapter on affine geometry according to the relation

$$R^{\rho}_{\sigma\mu\nu} = \Gamma_{\sigma\mu,\nu} - \Gamma_{\sigma\nu,\mu} - \Gamma^{\rho}_{\lambda\mu}\Gamma^{\lambda}_{\sigma\nu} + \Gamma^{\rho}_{\lambda\nu}\Gamma^{\lambda}_{\sigma\mu}$$

we presently study the symmetries and further properties of this tensor.

Obviously this tensor is skew-symmetric and satisfies the following set of conditions

$$R^{\rho}{}_{\sigma\mu\nu} = - R^{\rho}_{\sigma\nu\mu} \tag{3-5-1}$$

$$R^{\rho}_{\sigma\mu\nu} + R^{\rho}_{\nu\sigma\mu} + R^{\rho}_{\mu\nu\sigma} = 0 \tag{3-5-2}$$

As consequence of these two conditions the $R^{\rho}_{\mu\nu\sigma}$ has eighty independent components. These conditions belong to the affine curvature tensor, that is they are consequence of the affine equation written at the beginning of this section. Now, we use the choice (3-2-12) for the affinity $\Gamma^{\mu}_{\nu\sigma}$, in doing so we are specializing the choice of the $R^{\rho}_{\sigma\mu\nu}$ for the metrical Riemannian geometry. In this case the $R^{\rho}_{\sigma\mu\nu}$ take over the form

$$R^{\rho}_{\sigma\mu\nu} = \{^{\rho}_{\sigma\mu}\}_{,\nu} - \{^{\rho}_{\sigma\nu}\}_{,\mu} - \{^{\rho}_{\lambda\mu}\}\{^{\lambda}_{\sigma\nu}\} + \{^{\rho}_{\lambda\nu}\}\{^{\lambda}_{\sigma\mu}\} \tag{3-5-3}$$

which defines the metrical curvature in the Riemannian geometry. As we said before, a different choice might be done for the $\Gamma^{\mu}_{\nu\sigma}$ which conducts to a different form of the $R^{\rho}_{\sigma\mu\nu}$, as for instance in Weyl's theory.

A rather characteristic difference arises from the $R^{\rho}_{\sigma\mu\nu}$ of (3-5-3) and the $R^{\rho}_{\sigma\mu\nu}$ of the affine geometry, namely, the metrical curvature may be written with all indices down according to

$$R_{\sigma\rho\mu\nu} = g_{\rho\lambda} R^{\lambda}{}_{\sigma\mu\nu} = \frac{1}{2} (g_{\rho\mu,\sigma\nu} + g_{\sigma\nu,\rho\mu} - g_{\rho\nu,\sigma\mu} - g_{\sigma\mu,\rho\nu}) +$$
$$+ g^{\alpha\beta}(\{\rho\mu,\alpha\}\{\sigma\nu\ ,\beta\} - \{\rho\nu\ ,\alpha\}\{\sigma\mu,\ \beta\}) \tag{3-5-4}$$

a property which is not shared by the affine curvature tensor since in the affine geometry the operation of lowering and raising of indices is not defined.

The metrical curvature tensor has besides the symmetries of the affine curvature tensor new symmetry properties. For finding out these extra symmetries a trick is of real value: We make a mapping such that locally the $\{\rho\mu,\alpha\}$ vanish, and then study at this point the symmetries of the non-vanishing part of $R_{\rho\sigma\mu\nu}$. Since we know that a symmetry property is an absolute property of the geometrical object, the symmetry which is found in this way will hold in general, that is for any other system of coordinates.

Thus, for the coordinate system which has vanishing components for the $\{\rho\mu,\alpha\}$ at some point with coordinates x^{α}, we have

$$R_{\rho\sigma\mu\nu}(x^{\alpha}) = \frac{1}{2}\,(g_{\rho\mu,\sigma\nu} + g_{\sigma\nu,\rho\mu} - g_{\rho\nu,\sigma\mu} - g_{\sigma\mu,\rho\nu})_{x=x^{\alpha}} \qquad (3\text{-}5\text{-}5)$$

at the point x^{α}. One may use the term "the principal part of the curvature" for the expression (3-5-5), since this is the part which does not vanish at the point where the Christoffel symbols vanish. From (3-5-5) we easily obtain the further symmetries.

$$R_{\rho\sigma\mu\nu} = - R_{\sigma\rho\mu\nu} \qquad (3\text{-}5\text{-}6)$$

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \qquad (3\text{-}5\text{-}7)$$

As result of these additional symmetries the number of independent components of the curvature tensor $R_{\mu\nu\rho\sigma}$ is reduced to twenty.

In addition, $R_{\mu\nu\rho\sigma}$ also satisfy differential identities,

$$R_{\mu\nu\rho\sigma;\lambda} + R_{\mu\nu\lambda\rho;\sigma} + R_{\mu\nu\sigma\lambda;\rho} \equiv 0 \qquad (3\text{-}5\text{-}8)$$

which are called the differential Bianchi identities. Again one can prove (3-5-8) by the same method used for obtaining (4-5-6) and (3-5-7). Since all identities outlined previously are automatically satisfied by any curvature tensor, they do not mean any extra imposition on the metric.

Of special interest are the new geometrical objects one can form with the curvature tensor. Obviously we have affine objects and metrical objects, the later being of more practical interest. The first important metrical object one can construct with $R_{\mu\nu\rho\sigma}$ is the metric Ricci tensor.

$$R_{\nu\sigma} = g^{\mu\rho} R_{\mu\nu\rho\sigma} = \{^{\rho}_{\nu\rho}\}_{,\sigma} - \{^{\rho}_{\nu\sigma}\}_{,\rho} - \{^{\beta}_{\nu\sigma}\}\{^{\rho}_{\beta\rho}\} + \{^{\rho}_{\beta\sigma}\}\{^{\beta}_{\nu\rho}\} \qquad (3\text{-}5\text{-}9)$$

which is a symmetric second rank tensor. This is the unique independent form by which one can contract the metrical curvature tensor. A further contraction yields the scalar curvature R,

$$R = g^{\nu\sigma} R_{\nu\sigma} = g^{\nu\sigma}\{^{\rho}_{\nu\rho}\}_{,\sigma} \quad \{^{\rho}_{\nu\sigma}\}_{,\rho}\, g^{\nu\sigma} - g^{\nu\sigma}\{^{\beta}_{\nu\sigma}\}\{^{\rho}_{\beta\rho}\} + g^{\nu\sigma}\{^{\rho}_{\beta\sigma}\}\{^{\beta}_{\nu\rho}\} \qquad (3\text{-}5\text{-}10)$$

This scalar has the very important property of being, aside from a trivial constant factor, the only scalar that depends on the $g_{\mu\nu}$ and their first and second derivatives and is linear in the second derivatives.

At this point it is interesting to ask if there are other possibilities of constructing scalars out of the components of the curvature. The answer is positi ve, we can form new scalars besides the R. In what will follow we shall turn back to this possibility.

Besides $R_{\mu\nu}$ there exists too the symmetric second rank tensor $Rg_{\mu\nu}$ which depends only up to second order derivatives of $g_{\alpha\beta}$ and is linear in these quantities. These are the only second rank tensors sharing such property. A linear combination of these two tensors of great importance is the so called Einstein's tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \qquad (3\text{-}5\text{-}11)$$

Which thereby is linear in the second order derivatives of $g_{\alpha\beta}$. This tensor satisfies the contracted Bianchi identities

$$G_{\nu;\lambda}^{\lambda} = (g^{\lambda\mu} G_{\mu\nu})_{;\lambda} = 0 \qquad (3\text{-}5\text{-}12)$$

as can be proven directly from (3-5-8).

3.6) The Weyl Tensor and the Algebraic Classifications

An important tensor may be formed up with the components of the Riemann tensor as

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{1}{2} g_{\rho\mu} R_{\nu\sigma} + \frac{1}{2} g_{\rho\nu} R_{\mu\sigma} + \frac{1}{2} g_{\sigma\mu} R_{\nu\rho} - \\ - \frac{1}{2} g_{\sigma\nu} R_{\mu\rho} - \frac{1}{6} g_{\rho\nu} g_{\mu\sigma} R + \frac{1}{6} g_{\rho\mu} g_{\nu\sigma} R \qquad (3\text{-}6\text{-}1)$$

and is called as the Weyl conformal tensor. It satisfies all symmetries of $R_{\rho\sigma\mu\nu}$, and satisfies also the further property,

$$C_{\sigma\nu} \equiv g^{\rho\mu} C_{\rho\sigma\mu\nu} \equiv 0 \qquad (3\text{-}6\text{-}2)$$

which means that the Ricci tensor associated to the Weyl tensor vanishes. Associated to the Weyl tensor there exists some useful properties which we outline now. For spaces where the Ricci tensor vanishes, $R_{\mu\nu} = 0$, the Weyl tensor is equal to the remaining non null components of the Riemann tensor,

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu}$$

In the above circumstance $C_{\rho\sigma\mu\nu}$ does not vanish, but it may exist the case where $C_{\rho\sigma\mu\nu}$ vanishes, in this situation the space is conformally flat. The proof being as follows, first one can prove that the $C_{\rho\sigma\mu\nu}$ of (3-6-1) may be written in a form exactly equal to the Riemann tensor but replacing $g_{\mu\nu}$ by $\tilde{g}_{\mu\nu} = \frac{g_{\mu\nu}}{g^{\frac{1}{4}}}$,

$$C^{\rho}_{\sigma\mu\nu} = \widetilde{\{^{\rho}_{\sigma\mu}\}}_{,\nu} - \widetilde{\{^{\rho}_{\sigma\nu}\}}_{,\mu} - \widetilde{\{^{\rho}_{\lambda\mu}\}}\widetilde{\{^{\lambda}_{\sigma\nu}\}} + \widetilde{\{^{\rho}_{\lambda\nu}\}}\widetilde{\{^{\lambda}_{\sigma\mu}\}} \tag{3-6-3}$$

$$\widetilde{\{^{\rho}_{\sigma\mu}\}} = \frac{1}{2}\,\tilde{g}^{\rho\alpha}\,(\tilde{g}_{\alpha\sigma,\mu} + \tilde{g}_{\alpha\mu,\sigma} - \tilde{g}_{\mu\sigma,\alpha}) \tag{3-6-4}$$

$$\tilde{g}^{\mu\nu} = g^{\frac{1}{4}}\,g^{\mu\nu} \tag{3-6-5}$$

(the proof of (3-6-3) will not be given since it is rather lenghty). Now, we know from the previous chapter that the vanishing of the Riemann tensor over all the space implies that there exists a mapping such that $g_{\mu\nu}$ takes on the Galilean values $\mathring{g}_{\mu\nu}$. In our case what vanishes is the $C^{\rho}_{\sigma\mu\nu}$ of the equation (3-6-3), and thus the $\tilde{g}_{\mu\nu}$ take the values $\mathring{g}_{\mu\nu}$ in some coordinate system,

$$\tilde{g}_{\mu\nu} = \mathring{g}_{\mu\nu}$$

over all points of the space. But, then, the $g_{\mu\nu}$ take on values

$$g_{\mu\nu}(x) = g^{1/4}(x)\,\mathring{g}_{\mu\nu} = \phi(x)\,\mathring{g}_{\mu\nu}$$

on all points, and this is the mathematical condition for the space being conformally flat. This is the reason for calling $C_{\rho\sigma\mu\nu}$ as the conformal curvature tensor. In summary, for $R^{\rho}_{\sigma\mu\nu} = 0$, on all points, the space is flat; and for $C^{\rho}_{\sigma\mu\nu} = 0$, also on all points, the space is conformally flat.

Before going on with the study of the Weyl tensor, is interesting to turn back to the initial equation (3-6-1). This relation may be clarified by means of an equivalent decomposition of the several components of the Riemann tensor [9]

$$R_{\rho\sigma\mu\nu} = E_{\rho\sigma\mu\nu} + C_{\rho\sigma\mu\nu} + \frac{R}{12}\,g_{\rho\sigma\mu\nu} \tag{3-6-6}$$

$$E_{\rho\sigma\mu\nu} = \frac{1}{2}\,(g_{\rho\sigma\lambda\nu}\,S^{\lambda}_{\mu} - g_{\rho\sigma\lambda\mu}\,S^{\lambda}_{\nu}) \tag{3-6-7}$$

$$S^{\lambda}_{\mu} = R^{\lambda}_{\mu} - \frac{R}{4}\,\delta^{\lambda}_{\mu} \tag{3-6-8}$$

$$g_{\rho\sigma\lambda\nu} = g_{\rho\lambda}\,g_{\sigma\nu} - g_{\rho\nu}\,g_{\sigma\lambda} \tag{3-6-9}$$

The S^{λ}_{μ} is the reduced Ricci tensor, that is $S^{\lambda}_{\lambda} = 0$, the $E_{\rho\sigma\mu\nu}$ is algebraically equivalent to the reduced Ricci tensor, and thus has in all nine independent components. The quantities $g_{\rho\sigma\lambda\nu}$ possess just one independent component, g_{1203} for instance. Thus, the $C_{\rho\sigma\mu\nu}$ has in all ten independent components, a direct comparison of (3-6-6) with (3-6-1) shows that the $C_{\rho\sigma\mu\nu}$ of (3-6-6) is just the Weyl tensor. Then, the Eq. (3-6-6) consistently decomposes the 20 components of $R_{\rho\sigma\mu\nu}$ into three geometrical objects, the S^{λ}_{μ}, $C_{\rho\sigma\mu\nu}$ and R , with respectively nine, ten and one components. We see that for $R_{\mu\nu} = 0$, both S^{λ}_{μ} and $E_{\rho\sigma\mu\nu}$ vanish and $R_{\mu\nu\rho\sigma}$ is equal to $C_{\mu\nu\rho\sigma}$.

From the Formula (3-6-3) we can obtain a further property of invariance of the Weyl tensor. Indeed, if we consider the conformal transformation on the metric, [10]

$$g'_{\mu\nu} = e^{2\,u(x)}\, g_{\mu\nu}$$

under which the $\tilde{g}_{\mu\nu}$ is invariant,

$$\tilde{g}'_{\mu\nu} = \tilde{g}_{\mu\nu}$$

we obtain directly as consequence of this invariance in $\tilde{g}_{\mu\nu}$ that the Weyl tensor is also invariant under the above conformal mapping, $C'^{\rho}{}_{\sigma\mu\nu} = C^{\rho}{}_{\sigma\mu\nu}$. Thus, all possible Riemannian manifolds obtained for all possible values of the function u(x) possess the same conformal curvature $C^{\rho}{}_{\sigma\mu\nu}$. We may interpret this invariance property as a gauge like type of invariance of the geometry of the manifold.

We now analyze the structure of the several components of the conformal tensor. For doing this we use the symmetry properties of those components,

$$C_{\rho\sigma\mu\nu} = C_{\mu\nu\rho\sigma}$$

$$C_{\rho\sigma\mu\nu} = -C_{\sigma\rho\mu\nu} = -C_{\rho\sigma\nu\mu}$$

Which allow us to interpret the $C_{\rho\sigma\mu\nu}$ as a real symmetric 6 × 6 matrix,

$$C_{\rho\sigma\mu\nu} = C_{ab} = C_{ba}$$

where the indices a, b range from 1 to 6. From (3-6-2) we have,

$$C^{\mu\sigma}{}_{\mu\nu} = 0 \qquad (3\text{-}6\text{-}10)$$

A real 6 × 6 matrix has in all 36 components, however as consequence of the symmetries of $C_{\rho\sigma\mu\nu}$ only ten components are different of zero.

It is interesting to draw an analogy with the electromagnetic tensor $F_{\mu\nu}$ which may be identified with a real six-vector N_a,

$$F_{\mu\nu} = N_a$$

Presently, our $C_{\rho\sigma\mu\nu}$ is equivalent to the point of view of its symmetries to a direct product of two tensors like $F_{\mu\nu}$, that is, $C_{\rho\sigma\mu\nu}$ has the same symmetries of the quantity $F_{\rho\sigma}\, F_{\mu\nu}$, and thus it corresponds to $N_a N_b$ which is a particular type of a symmetric 6 × 6 matrix. This analogy is only formal but it will be useful in what will follow.

We can consider the following eigenvalue equation,

$$C_{ab}\, V^b = \lambda\, V_a \qquad (3\text{-}6\text{-}11)$$

which holds for the vector V of the real vector space with six dimensions. This six-dimensional vector space has a metric g_{ab} given by (3-6-9),

$$g_{ab} = g_{\mu\nu\rho\sigma} = g_{\mu\rho}\ g_{\nu\sigma} - g_{\mu\sigma}\ g_{\nu\rho} \qquad (3\text{-}6\text{-}12)$$

so that

$$V_a = g_{ab}\ V^b$$

Thus, we can write, for the condition that (3-6-11) has solutions different from the trivial solution,

$$\left| C_{ab} - \lambda g_{ab} \right| = 0 \qquad (3\text{-}6\text{-}13)$$

which is a sixth-degree polinomial equation in the eigenvalue λ. According to the several solutions of this equation it is possible to stablish a classification of the number of distinct eigenvalues and eigenvectors of C_{ab}. This work is credited to Petrov [11] and is known in the literature as the Petrov classes. We list bellow the results obtained

Petrov type	L_I	D(degenerate)	II	N(null)	III	Conformally flat
Distinct eigenvectors	3	3	2	2	1	zero
Eigenvalues	distinct	two equal	distinct	equal (both null)	null	zero

There exists still another possibility which is associated to manifolds where there are no topological symmetries, we may call this case as the class completely asymmetric. In this case there are four independent eigenvalues for C_{ab}. This is the largest possible number of eigenvalues. As example, we give the following cases which will be treated in more detail later on: a manifold with the spatial topology of a sphere (spherically symmetric) belongs to the Petrov class D. The Petrov class N is associated to a manifold with the local topology of the light-cone, that is, the radiation field of gravitation. As we know from field theory (see for instance Landau and Lifschitz's book), the eigenvalues of $C_{\rho\sigma\mu\nu}$ are the invariants one can form with the components of this tensor. According to the Petrov classification we see that it is possible according to the topology of the space, to construct from one such invariant up to four invariants at each space-time point. There exists too, the well known case where all such invariants vanish at each point, this corresponds to a space where exists just gravitational radiation, with similarity with the electromagnetic field of waves where the two invariants of the field are both null. Thus, we have arrived at the result previously referred of constructing other invariants with $R_{\rho\sigma\mu\nu}$ besides the scalar curvature of the space. It turns out that such invariants are all necessarily quadratic and cubic in the components of $C_{\rho\sigma\mu\nu}$. In the case where the space is asymmetric at the neighbourhood of a given point, there exists there four of such invariants. They can be presented in the compact notation,

$$A^1 = Tr(CgCg) = C_{ab}\, g^{bc}\, C_{cd}\, g^{da}$$

$$A^2 = Tr(CgC\varepsilon)$$

$$A^3 = Tr(CgCgCg)$$

$$A^4 = Tr(CgCgC\varepsilon)$$

The A^α are functions of the coordinate x^μ for the point under consideration, and all matrices standing above are six by six matrices, ε_{ab} is the Levi-Civita symbol in the six-dimensional notation. These four invariants will be of special importance in our future discussion of the role of the initial value problem for gravitation and the associated problem of obtaining geometrical objects with a well prescribed behaviour with respect to the initial Cauchy data.

Obviously, besides the above A^α one can construct other invariant such as for instance $A^\alpha R$, where R is the scalar curvature. In general, one can form A^α f(R) with f(R) a polinomial in R. All such invariants are functionally dependent on the basic set A^α and the R. Note that for the case where $R_{\mu\nu}$ vanish, and so R vanishes too, the A^α do not vanish in general, since for $R_{\mu\nu}$ and R both null the $C_{\rho\sigma\mu\nu}$ it is not zero. Therefore, in general we are to expect that the A^α are more fundamental than the other invariants for the description of the system. Of course, such interpretation holds good if the A^α do exist, that is, if the space is asymmetric at the vicinity of the region under study. This same interpretation holds for the Weyl tensor, which is clearly more fundamental than the Riemann tensor for describing the system, in the sense that their components are always different of zero even when the Ricci tensor and the scalar curvature go to zero.

3.7) The Internal SL_2 Group in the Metrical Geometry

In this section we will relate the method of tetrad calculus with the variables belonging to the complex two-dimensional internal vector space. In this space, which we call S_2, it is defined a skew symmetric bilinear non degenerate inner product. That is, given two vectors of S_2 (two-component spinors), say u and v, we have

$$u.v = -v.u$$

The realization of this operation is obtained by introducing into S_2 a skew symmetric matrix ε_{AB} with components

$$\varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

such that

$$u.v = \varepsilon_{AB}\, u^A\, v^B \qquad (3\text{-}7\text{-}1)$$

The matrix ε_{AB} corresponds in S_2 to the symmetric matrix $g_{\mu\nu}$ in the coordinate space. Thus, we can introduce operations of lowering and raising of indices of geometrical objects in S_2. With this end we define a matrix ε^{AB} such that $\varepsilon_{AB}\, \varepsilon^{BC} = -\, \delta^C_A$,

$$\varepsilon^{AB} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \qquad (3\text{-}7\text{-}2)$$

Then, given $u = (u^A)$ we form the variable u_A as

$$u_A = u^B\, \varepsilon_{BA} = -\, \varepsilon_{AB}\, u^B$$

and given $v = (v_A)$, we form the variable v^A by

$$v^A = \varepsilon^{AB}\, v_B = -\, \varepsilon^{BA}\, v_B$$

which is characteristic of the antisymmetry of the metric. In S_2 we define besides of the vectors u with covariant or contravariant components, the vectors u*, where the star means complex conjugation. These later possess too covariant as well as contravariant components. We indicate these components by $u_{\dot{A}}$ and $u^{\dot{A}}$. Under the group SL_2 the u^A, v_A, $w^{\dot{A}}$ and $y_{\dot{A}}$ vary as

$$u'^A = M^A{}_B\, u^B$$
$$v'_A = v_B\, M^{-1\,B}{}_A$$
$$w'^{\dot{A}} = M^{\dot{A}}{}_{\dot{B}}\, w^{\dot{B}}$$
$$y'_{\dot{A}} = y_{\dot{B}}\, M^{-1\,\dot{B}}{}_{\dot{A}}$$

A Hermitian matrix in S_2 is represented by $T_{A\dot{B}}$ satisfying

$$T_{A\dot{B}} = T_{\dot{B}A}$$

A very important set of four Hermitian matrices is given by the three Pauli matrices together with the two-by-two identity matrix.

$$\mathring{\sigma}_1{}^{A\dot{B}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathring{\sigma}_2{}^{A\dot{B}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathring{\sigma}_3{}^{A\dot{B}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathring{\sigma}_0{}^{A\dot{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

They are of special interest since they relate each real fourvector V^μ to a Hermitian matrix in S_2, given by a linear combination of the above matrices.

$$V^{A\dot{B}} = V^\mu\, \mathring{\sigma}_\mu^{A\dot{B}} \tag{3-7-3}$$

By the other hand, the three Pauli matrices satisfy the anticommutation rules,

$$\mathring{\sigma}_{i\;A}{}^{\dot{B}}\, \mathring{\sigma}_j{}^{A\dot{R}} + \mathring{\sigma}_{j\;A}{}^{\dot{B}}\, \mathring{\sigma}_i{}^{A\dot{R}} = 2\delta_{ji}\, \varepsilon^{\dot{B}\dot{R}} \tag{3-7-4}$$

the addition of the fourth matrix $\mathring{\sigma}_o$ allow us to generalize (3-7-4) to

$$\mathring{\sigma}_{\mu\;A}{}^{\dot{B}}\, \sigma_\nu^{A\dot{R}} + \mathring{\sigma}_{\nu\;A}{}^{\dot{B}}\, \mathring{\sigma}_\mu{}^{A\dot{R}} = 2\, \mathring{g}_{\mu\nu}\, \varepsilon^{\dot{B}\dot{R}} \tag{3-7-5}$$

where $\mathring{g}_{\mu\nu}$ has signature +2.

From the usual operations of lowering and raising of indices we have,

$$\overset{\circ}{\sigma}{}^{\nu}_{A\dot{B}} = \overset{\circ}{g}{}^{\nu\lambda} \; \overset{\circ}{\sigma}{}^{R\dot{S}}_{\lambda} \; \varepsilon_{\dot{S}\dot{B}} \; \varepsilon_{RA} \tag{3-7-6}$$

$$\overset{\circ}{\sigma}_{\mu}{}^{A\dot{B}} \; \overset{\circ}{\sigma}{}^{\nu}_{A\dot{B}} = 2\,\delta^{\nu}{}_{\mu} \tag{3-7-7}$$

this later equation comes from (3-7-5). As result, we may invert (3-7-3),

$$V^{\mu} = \frac{1}{2} \; \overset{\circ}{\sigma}{}^{\mu}_{A\dot{B}} \; V^{A\dot{B}} \tag{3-7-8}$$

From (3-7-8) we may study the structure of the fourvector V^{μ}. For doing this, we compute the norm of V^{μ},

$$V^{\mu} \; V_{\mu} = -\frac{1}{2}\,\mathrm{Tr}(V\varepsilon \; \overset{*}{V\varepsilon}), \quad V = (V_{A\dot{B}}) \tag{3-7-9}$$

We see that for the present choice of signature the fourvector V^{μ} is time-like if $V^{A\dot{B}}$ is a Hermitian negative definite matrix. V^{μ} is space-like if $V^{A\dot{B}}$ is a Hermitian positive definite matrix; and is a null vector if $V^{A\dot{B}}$ is a Hermitian singular matrix. This discussion holds only for real fourvectors. Indeed, for complex fourvectors, we see from (3-7-3) that $V^{A\dot{B}}$ is not a Hermitian matrix.

The equation (3-7-3) or (3-7-8) may be generalized for tensors of arbitrary rank as,

$$T^{A\dot{B}\;R\dot{S}\;\ldots} = \overset{\circ}{\sigma}{}^{A\dot{B}}_{\mu} \; \overset{\circ}{\sigma}{}^{R\dot{S}}_{\nu} \; \ldots \; T^{\mu\nu\ldots} \tag{3-7-10}$$

$$T^{\mu\nu\ldots} = \left(\frac{1}{2}\right)^{n} \overset{\circ}{\sigma}{}^{\mu}_{A\dot{B}} \; \overset{\circ}{\sigma}{}^{\nu}_{R\dot{S}} \; \ldots \; T^{A\dot{B}\;R\dot{S}\;\ldots} \tag{3-7-11}$$

where n is the rank of the tensor $T^{\mu\nu\ldots}$. Thus, the $\sigma_{\mu}{}^{A\dot{B}}$ are similar to projection operators which transform each tensor index into a pair of spinor indices, one undotted, the other dotted.

We may apply, for instance the above equations to the electromagnetic four-potentials A_{μ}, as well as to the skew symmetric field strenghts $F_{\mu\nu}$. In this case we obtain,

$$A_{\mu} = \frac{1}{2} \; \overset{\circ}{\sigma}{}^{A\dot{B}}_{\mu} \; A_{A\dot{B}} \tag{3-7-12}$$

$$F_{\mu\nu} = \frac{1}{4} \overset{\circ}{\sigma}_{\mu A\dot{B}} \overset{\circ}{\sigma}_{\nu R\dot{S}} F^{A\dot{B}\ R\dot{S}} \tag{3-7-13}$$

Since $F_{\mu\nu}$ is skew symmetric, we have the further condition,

$$F^{A\dot{B}\ R\dot{S}} = - F^{R\dot{S}\ A\dot{B}} \tag{3-7-14}$$

It is an easy matter to verify that a $F^{A\dot{B}R\dot{S}}$ satisfying the constraints (3-7-14) is of the form,

$$F^{A\dot{B}\ R\dot{S}} = \frac{1}{2} \left[\phi^{(AR)} \epsilon^{\dot{B}\dot{S}} + \phi^{(\dot{S}\dot{R})} \epsilon^{AR}\right] \tag{3-7-15}$$

That means, is given entirely in terms of a symmetric second rank spinor $\phi^{(AB)}$. Such object has three complex independent components, or equivalently six real independent components, the same total number of independent components present in $F_{\mu\nu}$.

Problem: Write up the Maxwell field equations in spinor notation.

So far we have treated the situation where the metric tensor assumes its canonical value over all points on the four-space. Thus, all results so far derived belong to the formalism of special relativity. However, it is possible to generalize the four constant Hermitian matrices $\overset{\circ}{\sigma}_\mu$ to a set of four new Hermitian matrices $\sigma_\mu(x)$ such that the rule (3-7-5) gets generalized to

$$\sigma_{\mu\ A}{}^{\dot{B}} \sigma_\nu{}^{A\dot{R}} + \sigma_{\nu A}{}^{\dot{B}} \sigma_\mu{}^{A\dot{R}} = 2\ g_{\mu\nu}(x)\ \epsilon^{\dot{B}\dot{R}} \tag{3-7-16}$$

with a metric $g_{\mu\nu}$, an arbitrary function of the coordinates. Locally these $g_{\mu\nu}$ go over the $\overset{\circ}{g}_{\mu\nu}$. All previous formulas still hold, writing σ_μ in place of the $\overset{\circ}{\sigma}_\mu$.

Since the four constant matrices $\overset{\circ}{\sigma}_\mu$ form a basis for the vector space of all two-by-two matrices, we may write for any such matrix, say N,

$$N = a^\mu \overset{\circ}{\sigma}_\mu \tag{3-7-17}$$

a particular case is obtained when we take the two-by-two matrices σ_μ and write for each one of them a relation like (3-7-17).

$$\sigma_\mu(x) = h_\mu^{(\alpha)}(x)\dot{\sigma}_\alpha \tag{3-7-18}$$

Since the σ_μ are Hermitian the quantities $h_\mu^{(\alpha)}$ are real functions. Note that for arbitrary matrices, such as the above N, these coefficients are not necessarily real.

Now, it is simple to verify that the $h_\mu^{(\alpha)}$ introduced in (3-7-18) are just the tetrad vectors studied previously. Indeed, from (3-7-18) and (3-7-16) we obtain.

$$h_\mu^{(\alpha)} \quad h_\nu^{(\beta)} \overset{o}{g}_{\alpha\beta} = g_{\mu\nu}$$

As it obvious, it is important to look for the spinor representation of the Riemann tensor and of the Weyl tensor. First of all, from (3-7-11) one may ask what should be the spinor representation for the metric $g_{\mu\nu}$. A direct calculation gives,

$$g_{\mu\nu} = \frac{1}{4}\, \sigma_{\mu A\dot{B}}\ \sigma_{\nu R\dot{S}}\ g^{A\dot{B}\ R\dot{S}}$$

with (as it follows from (3-7-16)),

$$g^{A\dot{B}\ R\dot{S}} = \varepsilon^{AR}\ \varepsilon^{\dot{R}\dot{S}}$$

similarly,

$$g_{A\dot{B}\ R\dot{S}} = \varepsilon_{AR}\ \varepsilon_{\dot{B}\dot{S}}$$

so that such spinor representation for $g_{\mu\nu}$ does not tell us any new result.

For $R_{\mu\nu\rho\lambda}$ we have the formula,

$$R_{\mu\nu\rho\lambda} = \left(\frac{1}{2}\right)^4 \sigma_\mu^{A\dot{B}}\ \sigma_\nu^{R\dot{S}}\ \sigma_\rho^{C\dot{M}}\ \sigma_\lambda^{N\dot{K}}\ R_{A\dot{B}\ R\dot{S}\ C\dot{M}\ N\dot{K}} \tag{3-7-19}$$

Due to the symmetries of $R_{\mu\nu\rho\lambda}$ we have the restrictions

$$R_{A\dot{B}\,R\dot{S}\,C\dot{M}\,N\dot{K}} = -R_{R\dot{S}\,A\dot{B}\,C\dot{M}\,N\dot{K}} \tag{3-7-20}$$

$$R_{A\dot{B}\,R\dot{S}\,C\dot{M}\,N\dot{K}} = -R_{A\dot{B}\,R\dot{S}\,N\dot{K}\,C\dot{M}} \tag{3-7-21}$$

$$R_{A\dot{B}\,R\dot{S}\,C\dot{M}\,N\dot{K}} = R_{C\dot{M}\,N\dot{K}\,A\dot{B}\,R\dot{S}} \tag{3-7-22}$$

Which allow us to write similarly to (3-7-15),

$$R_{A\dot{B}\,R\dot{S}\,C\dot{M}\,N\dot{K}} = \frac{1}{2}\Big[\chi_{ARCN}\,\varepsilon_{\dot{B}\dot{S}}\,\varepsilon_{\dot{M}\dot{K}} + \phi_{AR\dot{M}\dot{K}}\,\varepsilon_{CN}\,\varepsilon_{\dot{B}\dot{S}} + \phi_{\dot{B}\dot{S}CN}\,\varepsilon_{AR}\,\varepsilon_{\dot{M}\dot{K}} + \chi_{\dot{B}\dot{S}\dot{M}\dot{K}}\,\varepsilon_{AR}\,\varepsilon_{CN}\Big] \tag{3-7-23}$$

where χ_{ARCN} and $\phi_{AR\dot{M}\dot{K}}$ satisfy the conditions

$$\chi_{ARCN} = \chi_{RACN} = \chi_{ARNC} = \chi_{CNAR} \tag{3-7-24}$$

$$\phi_{AR\dot{M}\dot{K}} = \phi_{RA\dot{M}\dot{K}} = \phi_{AR\dot{K}\dot{M}} = \phi_{\dot{M}\dot{K}AR} \tag{3-7-25}$$

These relations have been first derived by Witten [12], subsequently a spinor formalism for general relativity was constructed by Penrose [13]. Thus, the Riemann tensor is represented by two types of spinors, the χ_{ABCD} and the $\phi_{AB\dot{C}\dot{D}}$. The Ricci tensor is represented by

$$R_{B\dot{F}\,D\dot{H}} = \sigma^{\mu}_{B\dot{F}}\,\sigma^{\nu}_{D\dot{H}}\,R_{\mu\nu} \tag{3-7-26}$$

where

$$R_{B\dot{F}\,D\dot{H}} = \frac{1}{2}\,\varepsilon^{AC}\varepsilon^{\dot{E}\dot{G}}\,R_{A\dot{E}\,B\dot{F}\,C\dot{G}\,D\dot{H}}$$

the scalar curvature is given by

$$R = \frac{1}{2}\,\varepsilon^{BD}\,\varepsilon^{\dot{F}\dot{H}}\,R_{B\dot{F}\,D\dot{H}} \tag{3-7-27}$$

and the Einstein tensor by

$$G_{A\dot{C}\,B\dot{D}} = -\frac{1}{2}\Big[R\,\varepsilon_{AB}\,\varepsilon_{\dot{C}\dot{D}} + \phi_{AB\dot{C}\dot{D}}\Big] \tag{3-7-28}$$

Thus, we see that whenever $R_{\mu\nu} = 0$, which implies that R and $G_{\mu\nu}$ are both equal to zero, we have

$$\phi_{AB\dot{C}\dot{D}} = 0 \tag{3-7-29}$$

Therefore, we conclude that the Weyl tensor is given entirely in terms of the spinor χ_{ABCD}. A further analisys shows that χ_{ABCD} is entirely symmetric when R vanishes. The final conclusion is that the spinor representation of the Weyl tensor $C_{\mu\nu\rho\sigma}$ is obtained in terms of a fully symmetric fourth rank spinor. This is deeply connected with the spin 2 of the gravitation.

3.8) <u>The Relations Between Internal and the Space-Time Curvatures</u>

So far we have obtained the components of the Riemann and Weyl tensors and we also have obtained the components of the internal curvature $P^{A}_{\mu\nu\ B}$. According to the method outlined in the preceding section we know how to write the curvatures $R_{\mu\nu\rho\sigma}$ and $C_{\mu\nu\rho\sigma}$ in the spinor representation. Actually, the $P^{A}_{\mu\nu R}$ and $R_{\mu\nu\rho\sigma}$ may be related by simple formulas. To get started we introduce the covariant derivatives of the Hermitian σ_μ matrices

$$\sigma_{\mu;\nu} = \sigma_{\mu\ \nu} - \{^{\lambda}_{\mu\nu}\}\ \sigma_\lambda + \Gamma_\nu\ \sigma_\mu + \sigma_\mu\ \Gamma^{\dagger}_{\nu}, \quad \sigma_\mu = (\sigma^{A\dot{B}}_{\mu}) \tag{3-8-1}$$

Since the covariant derivatives of $g_{\mu\nu}$ vanish in Riemannian geometry, it is natural to require that $\sigma_{\mu;\nu} = 0$. From (3-7-18) this implies that $h^{(\alpha)}_{\mu;\nu} = 0$ and $\overset{\circ}{\sigma}_{\alpha;\nu} = 0$. This later is similar in structure to our previous conditions $\overset{\circ}{g}_{\alpha\beta;\nu} = 0$. The significance of this condition is also similar to that of the tensor case. We thus get,

$$\sigma_{\mu,\nu} - \{^{\lambda}_{\mu\nu}\}\ \sigma_\lambda + \Gamma_\nu\ \sigma_\mu + \sigma_\mu\ \Gamma^{\dagger}_{\nu} = 0\ . \tag{3-8-2}$$

This equation can be solved for the internal affinity Γ_ν in terms of the Christoffel symbol [14]. Furthermore, we may write (3-8-2) in the form

$$\sigma_{\mu,\nu} = \{^{\lambda}_{\mu\nu}\}\ \sigma_\lambda - \Gamma_\nu\ \sigma_\mu - \sigma_\mu\ \Gamma^{\dagger}_{\nu}$$

Since the left hand side is a gradient of σ_μ, we have $\sigma_{\mu,\nu\alpha} = \sigma_{\mu,\alpha\nu}$. This condition implies that the curl of the right hand side vanishes,

$$\partial_\alpha\left[\{^{\lambda}_{\mu\nu}\}\ \sigma_\lambda + \Gamma_\nu\sigma_\mu + \sigma_\mu\ \Gamma^{\dagger}_\nu\right] = \partial_\nu\left[\{^{\lambda}_{\mu\alpha}\}\ \sigma_\lambda + \Gamma_\alpha\ \sigma_\mu + \sigma_\mu\ \Gamma^{\dagger}_\alpha\right]$$

working out explicitly this condition, we obtain,

$$P_{\mu\nu\ A}{}^{B} = \frac{1}{4}\ \sigma^{\lambda}_{A\dot{R}}\ \sigma^{P\dot{R}B}\ R_{\mu\nu\lambda\rho} \tag{3-8-3}$$

Which is the relationship between the curvature $P_{\mu\nu}$ and $R_{\mu\nu\lambda\rho}$. [15] Besides this type of relationship, we may also get formulas connecting the $P_{\mu\nu}$ with the Penrose curvatures χ_{ABCD} and $\phi_{AB\dot{C}\dot{D}}$. [16] They are,

$$P_{\mu\nu\ C}{}^{V} = \frac{1}{16}\ \varepsilon^{VD}\left\{\sigma_\mu{}^{A}{}_{\dot{F}}\ \sigma_\nu{}^{B\dot{F}}\ \chi_{ABCD} + \sigma^{\dot{E}}_{\mu\ B}\ \sigma^{B\dot{F}}_\nu\ \phi_{CD\dot{E}\dot{F}}\right\} \tag{3-8-4}$$

The inverse relationships may be also written.

FOUNDATIONS OF GENERAL RELATIVITY

4.1) Introduction

In the year of 1915 Einstein proposed a modification of the space-time description of physical systems in special relativity. In this description, as in the older Newtonian description, the geometry of space-time was fixed once for all, that means, it was not affected by the presence, or absence, of other physical systems existing in space-time. In other words, the geometry of space-time in special relativity appeared as an absolute element in all theories where the covariant group was the MMG. The modification proposed by Einstein consisted in removing the geometry of space-time from the realm of the absolute, and placing it as a dynamical element of the theory. The theory which was formed alongside with such proposal is today known as the general theory of relativity.

The geometry of space-time in the general theory was assumed by Einstein to be characterized by a Riemannian metric. Being a dynamical element, it was to be determined by a set of dynamical laws, similarly for instance to the way that the electromagnetic field is determined by the Maxwell equations.

Due to the dyamical nature of the metric, Einstein was able to propose a second fundamental property. He assumed that in the general theory the gravitational field was not to be described by a separate new geometrical object but was to be described by the metric tensor.

The reasons which conducted Einstein to a search of more general world pictures than that afforded by special relativity was related to certain aspects of these theories which did not satisfy him. In particular, he was disturbed by the inability of either Newtonian mechanics or special relativity to explain the universal constancy of the ratio of inertial to gravitational mass of material bodies. In addition he objected to the existence of certain types of absolute motions, namely, accelerated motions, in these two theories. Following some arguments put forth by Mach, he felt that all motion should be relative, not just uniform motions as is the case in special relativity.

In the search that ultimately conducted to the general theory, Einstein was guided by three general principles. The first was Mach's principle, the second was the principle of equivalence, and the third was the principle of general invariance. In what follows we examine each of these principles and show how they led Einstein to the general theory. We then introduce the field equations proposed by him for gravitation and discuss their general properties.

Before doing this, we turn back again to the discussion of absolute and relative motions. In order to clarify the previous conclusions we show how is possible to analize the several types of motion from the point of view of special relativity and of the general theory. The discussion which follows will further clarify the significance of the geodesics of the Riemannian manifold.

First one starts from the special theory. In this case it is possible to obtain relative uniform motions as consequence of the principle of relativity. It might be thought that such restriction should enter in contradiction with the geometrical fact that the four-velocity is a vector. Actually this is not the case, what happens is that the four-velocity is a very particular example of a four-vector, it satisfies simultaneously both requirements. We show this explicitly: consider a particle moving uniformly along the direction X of some inertial frame. We may consider the particle itself as another inertial frame, and the Lorentz transformation which connects both systems. Call by K the reference system at rest and K' the frame travelling along with the particle. For K the particle moves along the direction X with a four-velocity with components

$$u^{\nu} = \frac{dx^{\nu}}{ds} = \left(\frac{dx}{dx} , 0, 0, \frac{dx^{o}}{dx} \right)$$

where

$$\frac{dx}{ds} = \frac{v}{C\sqrt{1-\frac{v^2}{c^2}}} , \frac{dx^{o}}{ds} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

We prove now that for K' the particle is at rest, and that circumstance do not contradicts the property that the four-velocity is a vector (which being null in one frame should vanish in all other frames). Consider the Lorentz transformation which leds K to K',

$$u'^{\nu} = L^{\nu}{}_{\lambda} u^{\lambda}$$

for the spatial components we get

$$u'^i = L^i{}_j\, u^j + L^i{}_o\, u^o \tag{4-1-1}$$

From the equations of the Lorentz transformation we obtain

$$L^1{}_1 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad L^1{}_2 = 0, \quad L^1{}_3 = 0, \quad L^1{}_0 = \frac{-\frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Since all motion occurs along the common direction X, we may write just (4-1-1) for $i = 1$,

$$u'^1 = L^1{}_1\, u^1 + L^1{}_o\, u^o$$

using the values for $u^1 = \frac{dx}{ds}$, $u^o = \frac{dx^o}{ds}$, $L^1{}_1$ and $L^1{}_o$ written before, we see that u'^1 vanishes, proving that indeed the particle is at rest in K'. Note that this does not contradicts the fact that u^ν is a four-vector and its vanishing in one frame should imply in its overall vanishing. What vanishes in K' is not a consequence of $u^\nu = 0$, but instead all right hand side is what vanishes,

$$L^\nu{}_\lambda\, u^\lambda = 0 \ .$$

Thus, we see that the four-velocity is a particular type of vector, one which allows a direct construction of the matrix elements $L^\nu{}_\lambda$ in terms of it.

Now let's go to the four-acceleration ω^ν, defined by,

$$\omega^\nu = \frac{du^\nu}{ds} = \frac{d^2x^\nu}{ds^2}$$

As it is clear, for ω^ν we do not have any relation which allow us to put the ω^ν equal to zero in K', and this follows simply from the fact that the matrix elements $L^\nu{}_\lambda$ depend just on the velocities, not on the accelerations. Thus, the particle which is accelerated for K will be accelerated for K' as well, with value

$$\omega'^{\nu} = L^{\nu}{}_{\lambda}\, \omega^{\lambda}$$

This means that accelerated motions are absolute motions. As consequence, the equation of motion of a charged particle has a physical meaning for all inertial observers,

$$\frac{d^2x^{\nu}}{ds^2} = \frac{e}{m} F^{\nu}{}_{\lambda} u^{\lambda} \qquad (4\text{-}1\text{-}2)$$

(the right hand side is the Lorentz force). We will turn back to this equation in what follows.

Now we analize the motion from the point of view of the general theory, that is, for a Riemannian manifold where the motion is represented by its geodesics. The case of motion along straight lines, realized with constant four-velocity $u^{\nu} = \frac{dx^{\nu}}{ds}$, is in the general theory without a physical significance since this later theory deals with a force field, the field of gravitation, nevertheless we may consider such type of free motion in certain systems of coordinates, such as the geodesic system of coordinates, where $\Gamma^{\mu}_{\nu\sigma} = 0$. In these coordinates the motion proceeds as if the particle was free,

$$\frac{d^2x^{\nu}}{ds^2} = 0$$

The particular choice of such coordinates does not mean at all that the description so far obtained is a priviledged one. This is due to the fact that the description in geodesics coordinates possess the same physical significance as in any other coordinates. The physically prevailing components of the curvature tensor do not vanish in geodesic coordinates. However, from the pure mathematical point of view these coordinates give us a constant value for the four-velocity along the geodesic of the particle. With this constant four-velocity we may imagine to do what we did before in special relativity, that is, to go over the rest frame of the particle. Since the choice of geodesics coordinates is just a mathematical trick, we know that the space has an intrinsic curvature, so that we may consider at each point of the four-space the pseudo-Euclidian tangent four-space. On this tangent space we may consider a Lorentz transformation which leds u^{ν} to the value zero on that point,

$$u'^{\nu}(P) = L^{\nu}_{\lambda}(P)\, u^{\lambda}(P) = 0$$

For each different point along the geodesic we have to consider a new such rest frame. That is all one can do now. Thus, we arrive too in a principle of relati ve uniform motions.

What about accelerated motions ? For considering these motions, which now have a prevailing physical meaning, we start with the equation of geodesics.

$$u^{\nu}\, u^{\alpha}_{;\nu} = 0 \tag{4-1-3}$$

This equation tells us that the motion possess a covariant significance, that is, the equation of motion is covariant under the MMG of the curved space. But this equation splits into two terms, the first being the four-acceleration $\omega^{\nu} = \frac{d^2x^{\nu}}{ds^2}$ and the second the term $\Gamma^{\nu}_{\alpha\lambda}\, u^{\alpha}\, u^{\lambda}$. Thus, as a trivial analysis shows, none of the two above terms are separately covariant. This conclusion implies just in the type of analysis done before. Consider the w^{ν} in separate, since this is not a fourvector we may state a law of relative accelerated motions directly. Indeed, under a coordinate transformation the w^{ν} vary as

$$w' = \frac{\partial x'^{\nu\nu}}{\partial x^{\alpha}}\, w^{\alpha} + \frac{\partial^2\, x'^{\nu}}{\partial x^{\alpha}\partial x^{\beta}}\, u^{\alpha}\, u^{\beta} \tag{4-1-4}$$

And we may consider the coordinate transformation such that w^{α} vanishes, in this case w'^{ν} have the values

$$w'^{\nu} = \frac{\partial^2 x'^{\nu}}{\partial x^{\alpha}\partial x^{\beta}}\, u^{\alpha}\, u^{\beta} \tag{4-1-5}$$

Therefore, all accelerated motions are now relative motions. It is important to keep in mind the two different arguments leading to relative motions. The first refers to relative uniform motions and comes from the suitable choice of coordinates such that the u^ν in the new coordinates are constants,

$$u'^\nu = \frac{\partial x'^\nu}{\partial x^\alpha} u^\alpha = \text{constant} \qquad (4\text{-}1\text{-}6)$$

following with a local Lorentz transformation we arrive at a local rest frame. But the u^ν are four-vectors, so that what is constant, or zero, is the full right hand side of (4-1-6), not the u^α alone. The second type of relative motions refers to the accelerated motions, here the w^ν are not the components of a four-vector so that the argument which leds to the choice of a rest frame is entirely similar to that which gave us the local vanishing of the affinities $\Gamma^\mu_{\nu\alpha}$.

Hence, the u^ν and the w^ν are geometrical objects of distinct nature, and thus need different type of analysis.

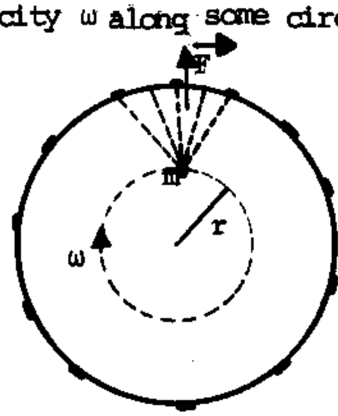
The fact that the w^ν are not four-vectors in the general theory implies that the Lorentz equation (4-1-2) is wrong and needs to be corrected. This is done by introducing the extra term $\Gamma^\nu_{\alpha\lambda}\, u^\alpha\, u^\lambda$, which transforms this equation to the equation of a geodesic

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\lambda}\, u^\alpha\, u^\lambda = \frac{e}{m} F^\mu{}_\lambda\, u^\lambda \, .$$

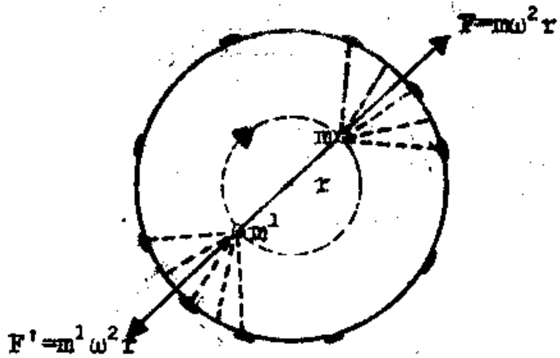
Only this full equation possess a covariant character under the MMG of the curved space.

4.2) Mach's principle

According to Ernst Mach [17] all inertial effects are due to mutual interaction of matter. He says in his book: When a body rotates relatively to the fixed stars, centrifugal forces arise. Such forces appear due to the interaction of the body with the rest of the universe. As example, if we take two bodies connected by a rope and let then rotate through their common center, a tension arises in the rope and according to him this is due to the interaction of the rotating masses with all other bodies in the universe. Another example is the following: consider the fixed stars as an uniform mass distribution on the surface of a sphere, and take another mass inside of this sphere rotating with constant angular velocity ω along some circular path.



It is seen that at all points of the orbit the mass m is closer to some part of the mass distribution than to masses on the opposite side. Thus, an effective force $m\omega^2 r$ arises on the particle, accelerating it towards the nearby masses on the celestial sphere, the centrifugal force as interpreted by Mach. For r going to zero all effects tend to cancel since then the overall distance is the same for all points on the celestial sphere, and the force tends to zero. If we take two masses m and m' connected by a rope and let then rotate inside this celestial sphere we will observe a tension on the rope, such tension arises due to the interaction of m and m' with the nearby masses on the celestial sphere.



Thus, Mach proposes that all inertial effects were due to an action-at-distance interaction with other material bodies of the universe. He says in his book that no relative motion exists, only motion with respect to other bodies do exist in nature. What Mach intends to say for relative motion is just that: any motion respect to another body. He was contrary to the existence of motions respect to the fixed absolute space of the Newtonian theory.

This type of interpretation impressed very much Einstein and indeed it represented the initial motivation for his search towards the general theory. Recently Dicke [18] has advanced the hypothesis that Mach's principle was connected to the existence of a relativistic scalar field of atractive forces into the universe. As we will see, the principle of Mach in spite of its heuristic appeal is not verified as a consequence of the field equations of general relativity. Many persons believe that this principle is just a philosophical point of view, other believe that it is indeed a possible interpretation explaining the role of inertial forces. In this connection, both theoretical and experimental work has been done based on possible detections of this principle. Of such works we have to give separate reference of the works of Dicke [19], Wheeler [20] and Gursey [21] in the theoretical domain, and those of Cocconi and Salpeter [22], more recently treated by Hughes [23].

First we treat slightly the work of Dicke. A more complete treatment of this work will be given when discussing advanced research topics later on.

According to Dicke the Mach program should be realized through the existence of an atractive long range scalar interaction acting on matter, this later possessing a similar to a "mesonic charge" depending in general of the particular body on consideration. The action function for the whole system is then of the form

$$S = \int d_4x(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}-\alpha^2\phi^2)-\sum_i m_i\int_{\sigma_1}^{\sigma_2}\sqrt{\dot{Z}_i^2}\ d\lambda_i -\sum_i g_i \int_{\sigma_1}^{\sigma_2}\int\phi(x)\delta_4(x-z_i)\sqrt{\dot{Z}_i^2}\ d_{\ i}\ d_4\ x_i \quad (4\text{-}2\text{-}1)$$

where the second and third integrals are carried over the line of universe of the bodies. The field equation and the equations of motions are,

$$\frac{\delta S}{\delta\phi} = (\Box + \alpha^2)\ \phi(x) + \rho(x) = 0 \qquad (4\text{-}2\text{-}2)$$

$$\frac{\delta S}{\delta Z_{i\mu}} = \frac{d}{d\tau_i}\ \left\{(m_i + g_i\ \phi(z_i))\dot{Z}_{i\mu}\right\} - g_i\ \frac{\partial\phi}{\partial Z_\mu^i} = 0 \qquad (4\text{-}2\text{-}3)$$

with

$$\rho(x) = \sum_i\ g_i\int_{\sigma_1}^{\sigma_2}\delta_4(x-z_i)\ \sqrt{\dot{Z}_i^2}\ \ d\lambda_i$$

A first implication coming from this formulation is that the effective mass of all bodies possessing the "mesonic charge" become a function of the scalar field, and thus become dependent on the location along the line of universe of these particles. This result, which is given by the equation

$$m_i^{effect} = m_i + g_i\ \phi(Z_i^\nu)$$

may also be interpreted as aggregating to the inertial mass m_i a new mass term $g_i\phi(z_i^\nu)$ due to the presence of the field ϕ, this new mass is generated by the interaction of m_i with all other bodies in the universe. This extra term explains for the effects of inertia on the body, this explanation is in full agreement with Mach's hypothesis. The total force acting on the body is then

$$F_{i\mu}^{\ast} = g_i \frac{\partial\phi}{\partial Z^i_\nu} - \dot{z}_i^\mu \frac{d}{d\tau_i} (g_i \phi) = g_i \frac{\partial\phi}{\partial Z^i_\nu} (\delta_{\nu\mu} - \dot{z}_{i\mu} \dot{z}_{i\nu})$$

which depends on the velocities along the orbit of the body, in agreement with the observed fact that all inertial forces depend on the body velocities. For simplifying the calculations we may assume that the field ϕ has its origin in the distant matter distributed homogeneously and isotropically into the universe, similarly to our previous celestial sphere, in this case all relevant force terms are due to the nearby masses in the universe, similarly to our previous situation.

The experimental work on the possible verifications of Mach's principle are mainly due to Cocconi, Salpeter and Hughes. The idea of Cocconi and Salpeter is: If inertial effects are due to the distribution of matter in the universe, we should observe small anisotropy of the inertial effects on the Earth surface since the solar system is embeded in our galaxy in such way that the distribution of masses respect to it is not isotropic.

Cocconi and Salpeter assumed that the contribution Δm to the inertial mass of a body, due to a mass ΔM located at a distance r away from the body is proportional to $\frac{\Delta M}{r^\nu}$. Then, the total contribution to the inertial mass of the body due to all other masses in the universe, that is going outside of our galaxy, is

$$m_o = K \int_0^R 4\pi\, r^2 \frac{\rho}{r^\nu}\, dr = K \frac{4\pi\rho\, R^{3-\nu}}{3 - \nu}$$

where R is the radius of the universe, ρ the average density of matter, assumed distributed homogeneously and isotropically, and K a proportionality constant. The range of values for the exponent ν can be restricted by the considerations: If $\nu < 0$ the more distant bodies should give larger contribution to m_o which is unreasonable. For $\nu > 1$, the nearby bodies should contribute strongly to m_o, for example the sun would dominate in determining the effective mass $m + m_o$, which is contrary on the observed facts of planetary motions. Thus, they take $0 < \nu < 1$. Since they look for an effect due to the anisotropy in mass distribution around the planetary system, they consider the increment Δm to m due to the mass M_o in the galaxy, assumed concentrated at a distance R_o from m,

$$\Delta m = K \frac{M_o}{R_o^\nu}$$

the ratio of this contribution to m_o would give an effect of mass anisotropy, it gives,

$$\frac{\Delta m}{m_o} = \frac{M_o\ (3-\nu)}{R_o^\nu\, 4\pi\rho\ \ R^{3-\nu}}$$

Using the presently accepted values,

$$M_o = 3 \times 10^{44}\ \text{g}$$

$$R_o = 2.5 \times 10^{22}\ \text{cm}$$

$$R = 3 \times 10^{27}\ \text{cm}$$

$$\rho = 10^{-29}\ \text{g/cm}^3$$

(for comparison we recall that:

Approximation mass of the sun = 1.9×10^{33} g ,

Mean distance Earth to Sun = 1.5×10^{13} cm ,

Mean density of Earth = 5.5 g/cm^3

which give an idea of the order of magnitude of the above numbers).

One finds for $\nu = 0$,

$$\frac{\Delta m}{m_o} = 3 \times 10^{-10}$$

$$\frac{\Delta m}{m_o} = 2 \times 10^{-5}$$

Hughes has tested the Cocconi-Salpeter hypothesis by a series of standard nuclear magnetic - resonance experiments, using a Li^7 nucleus in the ground state. The nuclear spin in the ground state is 3/2, so that in a magnetic field the Li^7 nucleus will have four energy levels. They should be equally spaced if there were no mass anisotropy, and a single nuclear resonance line should be observed. If there is a mass anisotropy the spacing will no longer be uniform and one should observe a triplet nuclear resonance line if the structure is resolved, or a single broadned line if the strucutre is not resolved. Hughes observed the Li^7 resonance over a 12-hr. period. A single line of width 1.2 cps was observed, which changed by less than 0.2 cps over this period. Using a simple shell model for this nucleus Hughes calculated the value for $\frac{\Delta m}{m_o}$ to the nuclear mass of Li^7. He obtained as a limit $\frac{\Delta m}{m_o} < 10^{-22}$. This rules out any effect of mass anisotropy according to the Cocconi-Salpeter values.

However, Dicke has argued that Mach's principle should apply not only to inertial mass but to fields as well, so that the propagation of photons would be influenced by the distribution of matter in the universe. Likewise, nuclear forces should also possess an anisotropy,it may happen that including such factor of anisotropy in the model for the nuclear interaction one can get a value for $\frac{\Delta m}{m_o}$ compatible with the Cocconi-Salpeter values. The interpretation of Hughes'experiment, at least as it bears on Mach's principle must therefore be considered as an open question.

Now, we comment how Mach's principle leds Einstein towards its theory. Einstein remarks [24]: "The theory of relativity makes it appear probable that Mach was on the right road in his thoughts that inertia depends upon a mutual action of matter". He was able to go beyond Mach's proposition by saying that the effect of mutual interaction of matter was translated in terms of the geometry of the space containing those masses. Thus follows that at most the geometry of space must be influenced by the distribution of masses, and must be regarded as a dynamical element of the theory and not as an absolute element. Thus we may say that the crucial step of removing the geometry of space from the realm of the absolute to the realm of a dynamical element was essentially due to the influence exerced on him by Mach's thoughts.

After he developed his general theory, Einstein came to realize that it did not satisfy Mach's principle in the strict sense.

Wheeler has suggested that Mach's principle provides boundary conditions for the general theory but does not bear directly on the field equations of the theory. By the other hand, Dicke thoughts differently, he argues that possibly Mach's principle is contained into another type of field, the scalar long range interactions. All such suggestions are actually open questions.

4.3) The Equivalence Principle

Mach's principle suggested to Einstein that the geometry of space-time should be a dynamical element in all physical theories. He follows with a second principle which he calls as the principle of equivalence. This principle is the outgrowth of an experimental fact known since the time of Galileo which says that the ratio of inertial to gravitational mass is a universal constant. Let us call such constant by Q, Galileo noted that in the gravitational field all bodies fall with the same acceleration, which gives rise to this universal constancy. Before going on with Einstein's point of view we comment on the implications of this observed fact in physics, prior to the proposal of the general theory.

In Newtonian mechanics and in all gravitational theories in special relativity such constancy of Q have to be taken as an additional hypothesis rather than being a consequence of the theory. Since the time of Galileo the constancy of Q has been tested with ever-increasing accuracy. In his Principia, Newton described a pendulum experiment that fixed the constancy of Q to one part in a thousand; later, Bessel [25] also using a pendulum experiment improved the accuracy to one part in 60.000. The next significant improvement in accuracy was done by Eötvös [26] and subsequently repeated by Eötvös, Pekar and Fekete [27]. They stablished the constancy of Q to one part in 10^8. Later Southerns [28] working with radioative uranium oxide, fixed Q to one part in 2×10^{15}. While not so accurate as Eötvös measurements, Southerns' experiment was sufficient to stablish the constancy of Q for the mass equivalent of the binding energy of nuclei. More recently Dicke [29] working with aluminium and gold was able to extend the accuracy of the Eötvös result to one part in 10^{11}. On the basis of the available data, Wapstra and Nijgh [30] were able to argue for the constancy of Q for the various particles which constitute the matter. Thus, the Q ratio for a proton plus electron is shown to be equal to Q for a neutron to about one part in 10^7. Thus, we may suppose that Q is a universal constant for matter of all kind.

This universal constancy is a characteristic property of the gravitational field, since for instance not all charged bodies move equally into an electric field.

If Q has the same value for all bodies, they should all behave in the same manner in an arbitrary, but given, gravitational field. It may be shown that the equation of motion for those bodies has the form

$$\frac{d^2 z^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dz^\rho}{d\lambda} \frac{dz^\sigma}{d\lambda} = 0 \qquad (4\text{-}3\text{-}1)$$

for a suitable choice of the parameter λ.

Indeed, we can construct a theory for gravitational fields in special relativity, by representing tentatively the field by a symmetric Lorentz tensor $\psi_{\mu\nu}$ and by a Lorentz scalar ϕ. If we require that the resulting particle equations of motion contain no higher than quadratic terms in the particle velocities, the total action for the system is of the form

$$S = \int_R \frac{1}{2} \left\{ \overset{\circ}{g}{}^{\rho\sigma}\, \psi_{\mu\nu,\rho}\, \psi^{\mu\nu}_{\ ,\sigma} + \overset{\circ}{g}{}^{\rho\sigma}\, \phi_{,\sigma}\, \phi_{,\rho} \right\} d^4x - \sum_i m_{oi} \int_R \int_{\sigma_1}^{\sigma_2} \sqrt{g_{\mu\nu}(x)\, \dot{z}_i^\mu \dot{z}_i^\nu}\, \delta_4(x - z_i)\, d\lambda_i \; d^4x$$

$$g_{\mu\nu}(x) \equiv g_{\mu\nu} + \alpha g_{\mu\nu}\, \phi + \beta\, \psi_{\mu\nu}$$

From which follows the equations of motion

$$\ddot{z}_i^\mu + \{^\mu_{\rho\sigma}\} \dot{z}_i^\rho \dot{z}_i^\sigma = 0$$

for path parameters fixed by the condition

$$g_{\mu\nu}(\dot{z}_i) \dot{z}_i^\mu \dot{z}_i^\nu = 1$$

The $\{^\mu_{\rho\sigma}\}$ is exactly the Christoffel symbol build up from $g_{\mu\nu}(z_i)$. The equations for $\psi_{\mu\nu}$ and ϕ being

$$\Box\psi_{\mu\nu} = -\frac{\beta}{2} \sum_i m_{oi} \int_{\sigma_1}^{\sigma_2} \dot{z}_{i\mu} \dot{z}_{i\nu} \delta^4(x-z_i) \, d\lambda_i$$

$$\Box\phi = -\frac{\alpha}{2} \sum_i m_{oi} \int_{\sigma_1}^{\sigma_2} g_{\mu\nu} \dot{z}_i^\mu \dot{z}_i^\nu \delta^4(x-z_i) \, d\lambda_i$$

Thus, the form of the equation of motion (4-3-1) is not a prescribed property of a geometrical theory, but rather appears naturally in any relativistic formulation.

We conclude from this analysis that the motion of all types of particles into the field will serve once for all to fix the metric affinity $\{^\mu_{\rho\sigma}\}$ for the field; this holding if we neglect the reaction of the test particles onto the field, since such reaction clearly depends on the particular test-particle one chooses.

We now enunciate the Principle of Equivalence of Einstein, one of the basic postulates of the general theory of relativity:

"On the extent that we can neglect the reaction on the field by the test particle moving in it, measurements made on any physical test system will give the same affinity in the space-time region under consideration."

What this principle means is the following: Take any type of particle and let it move into the field. If the interaction of the particle with the field is neglected, which may be obtained for small masses of the test particle, we determinate the affinity $\Gamma^{\mu}_{\rho\sigma}$ from (4-3-1). This affinity will serve for the subsequent motion of any other test particle, in particular it will serve for the motion of low energy light rays into the field. Thus, this principle gives us a well prescribed form for fixing once for all the underline field which will act on all kind of matter moving in it.

We note that this principle is compatible with a special relativistic formulation of the gravitational field theory, as follows from (4-3-1) which is also an equation of the special relativistic theory of gravitation. Thus, in spite of not being a consequence of this theory, the equality of $m_{in.}$ to $m_{gr.}$ is coherent with the results of the special relativistic theories.

It should be noted however, that a special relativistic equation bearing the Lorentz invariance, and having $m_{in.}$ different of $m_{gr.}$ can be written,

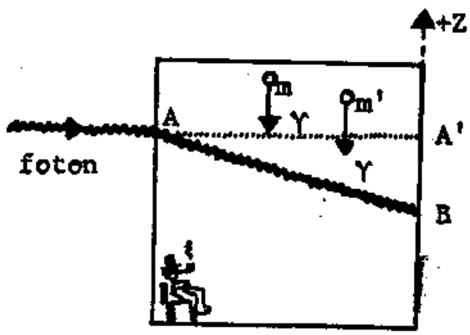
$$m_{in.} \frac{d^2 z^{\mu}}{d\lambda^2} + m_{gr.} \left\{ {\mu \atop \rho\sigma} \right\} \frac{dz^{\rho}}{d\lambda} \frac{dz^{\sigma}}{d\lambda} = 0 \qquad (4\text{-}3\text{-}2)$$

since both $\frac{d^2 z}{d\lambda^2}$ and $\left\{ {\mu \atop \rho\sigma} \right\} \frac{dz^{\rho}}{d\lambda} \frac{dz^{\sigma}}{d\lambda}$ are Lorentz vectors. Thus, the special relativistic equations of motion in a field of gravitation still allow for a possible violation of the Principle of Equivalence.

A rather different situation will appear in the general theory of relativity. If one imposes general covariance of (4-3-1) necessarily $m_{in.}$ has to be set equal to $m_{gr.}$ in the (4-3-2) since none $\left\{ {\mu \atop \rho\sigma} \right\} \frac{dz^{\rho}}{d\lambda} \frac{dz^{\sigma}}{d\lambda}$ or $\frac{d^2 z^{\mu}}{d\lambda^2}$ are vectors in the Riemannian space, and in order to maintain the general covariance of this equation the extra non-tensor terms coming from $\frac{d^2 z^{\mu}}{d\lambda^2}$ have to be elimminated by the similar non-tensor terms coming from $\left\{ {\mu \atop \rho\sigma} \right\} \frac{dz^{\rho}}{d\lambda} \frac{dz^{\sigma}}{d\lambda}$, and this is possible only if both terms have the same coefficient, thus implying in $m_{in.} = m_{gr.}$

For the time being we will not enter in further comments on this fact since this will be treated in the realm of the Postulate of General Covariance.

For introducing his Principle of Equivalence Einstein gave as example the nowaday famous falling elevator idealized experiment. He imagined an elevator that was freely falling in a gravitational field, that means that was falling with the acceleration imposed by this field. To an observer inside this elevator, a material body that was in also in freely falling would appear to be in a state of uniform motion. Likewise, in an elevator accelerated in the direction +Z, in gravity free region, a material body appears to be accelerated towards the bottom of the elevator with this same acceleration. This is exactly the same what occurs as if the body were falling in an uniform gravitational field. Also, light rays would appear to travel along curved trajectories in this accelerated elevator. In the drawing which follows we give the informations which an observer at rest with respect to the accelerated elevator should observe



The first part of this observation, namely, that all bodies fall with an uniform acceleration is exactly the same as if the observations made inside an uniform accelerated region of space, were made in a fixed region where an uniform gravitational field exists. On the basis of this idealized experiment Einstein concluded two things: First, all motions are relative, second, on the basis of purely local observations, to distinguish between uniform motion of a body in a gravitational field and uniform accelerated motion in a gravity free region was impossible. This was possible to be imagined due to the fact that $m_{in} = m_g$. Going beyond this, we immediately have that from this total local equivalence it follows that light rays should bend in presence of gravitational fields. This was later detected for light rays travelling near the surface of the sun.

4.4) The Principle of General Invariance

We come now to the third principle which leds Einstein to the theory of general relativity. This is the Principle of general invariance, or as usually is referred in the literature, the Principle of general covariance. There is still a good deal of confusion concerning just what content Einstein implied by this principle, due in part to his own way of writing. He says that the laws of physics are independent of any particular reference system we choose to write explicitly those laws.

This requirement is equivalent to demanding that the MMG is a covariance group of all physical theories. The MMG here means the manifold mapping group for curved spaces.

Before discussing the consequences of accepting the principle of general invariance as a fundamental principle of nature let us discuss what we should accept in the first place. Let's first of all enumerate some arguments which favour this principle:

1) What little experimental evidence we have, tends to support it. Indeed the observed universal constancy of the ratio of m_{in} to m_g is a direct consequence of this principle, as remarked in the previous section.

2) The principle of general covariance leds us to a new physics. While it is possible to describe all known gravitational phenomena within the framework of special relativity, one learns very little more by doing so. Most of the new effects predicted by this theory are too much small to be experimentally detected by present-day-techniques. On the other hand, the principle of general invariance force on us theories which predict essentially new types of phenomena having no counterpart in the flat, linear world of special relativity. This is a positive point for the general theory.

3) The principle of invariance rules out the possibility of existence of absolute objects in the space-time description of the universe. In a world where exist absolute objects, parts of this world, the absolute objects, influence all other remainder elements of the space without being influenced in turn. Such description therefore lacks reciprocity. It seems reasonable, however, and in order to keep alongside with experience that there is a operative law in nature relating the action and reaction of the several elements of the universe. While the equivalence principle suggests that the space-time affinity should be a dynamical object, it does not rule out completely the possibility of the existence of a flat affinity. But it says nothing regarding the existence of the Newtonian planes of absolute simultaneity. The principle of general invariance on the other hand does eliminate all such objects from the space-time description of the universe. Thus the principle of equivalence can be considered to be a consequence of the principle of general invariance. In fact, the principle of general invariance is a stronger statement than all previous principles, and contains, or implies, in the principle of equivalence and in Mach's principle. If one accepts the principle of general invariance, one can dispense both the principle of equivalence and the principle of Mach as foundations of the general

theory of relativity. Adopting this type of view, one may say that the verified constant ratio of m_{in} to m_g is an experimental verification of the principle of general invariance.

By the other hand, one can accept the principle of equivalence as the basic postulate of the gravitational phenomena, a theory based on this postulate may be twofold, it may be a theory of gravitation in flat space, where it happens that $m_{in} = m_g$, that is, the postulate of equivalence is just an observed fact, without any decisive implication on the foundations of the theory. Or it may be a theory where we adopt the point of view that the equivalence principle has to bear a deeper implication on the foundations of the theory. Such theory has necessarily to consider the similarity between the four-acceleration and the four-force as a fundamental fact.

Since the four-acceleration transforms as

$$w'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\rho}} w^{\rho}(x) + \frac{\partial^2 x'^{\mu}}{\partial x^{\lambda}\partial x^{\rho}} u^{\lambda}(x)\ u^{\rho}(x) \qquad (4\text{-}4\text{-}1)$$

under a coordinate transformation allowing for non-linear terms. Thus, if the equation of motion is covariant, the four-force $\Gamma^{\mu}_{\nu\sigma} u^{\nu} u^{\sigma}$ has to transform as,

$$(\Gamma^{\mu}_{\nu\sigma}u^{\nu}\ u^{\sigma})'(x') = \frac{\partial x'^{\mu}}{\partial x^{\rho}} (\Gamma^{\rho}_{\alpha\beta}u^{\alpha}\ u^{\beta})(x) = \frac{\partial^2 x'^{\mu}}{\partial x^{\lambda}\partial x^{\rho}} u^{\lambda}(x)\ u^{\rho}(x)\ . \qquad (4\text{-}4\text{-}2)$$

In the flat space-time formulation the quadratic term in the law of transformation vanishes.

$$\frac{\partial^2 x'^{\mu}}{\partial x^{\lambda}\partial x^{\rho}} = 0$$

But for a more general group of transformations this will not happen, and the identity between (4-4-1) and (4-4-2) gives an information on how the $\Gamma^{\mu}_{\nu\sigma}$ transform. Since the u^{ν} are four-vectors, from (4-4-2) we obtain that the $\Gamma^{\mu}_{\nu\sigma}$ transform as an affinity. Then, we rearrive at the geometrical formulation of Einstein. For arriving to this result one used as a given fact that ds is a scalar. We know that indeed ds^2 is such an object, but ds itself has to be taken as an invariant.

Following with our finality, we try to set forth all possible generally covariant field theories. For doing this, we require that a single irreducible field should be used. The simplest possible object is a scalar field $\phi(x)$. Since the metric of the underlying space is a dynamical object of the theory, we arrive at the result that a field equation of the form

$$\phi^{;\mu}_{;\mu} = j \ , \qquad (4\text{-}4\text{-}2)$$

where j is the source functions, for instance the trace of the stress tensor of matter and energy in space

$$j = g^{\mu\nu} T_{\mu\nu},$$

is not possible. Indeed, this equation is generally covariant, but it contains besides of the scalar $\phi(x)$ also the metric $g_{\mu\nu}$ which is to be determined by the field equations. Thus, we get two negative results:

1) We have one field equation for determining altogether eleven unknown, the ϕ and the $g_{\mu\nu}$.

2) The field is described by two irreducible geometrical objects ϕ and $g_{\mu\nu}$ contrarily to our previous remark.

The next geometrical object is a vector field f_{μ}. This case has against it several negative results. First of all it will give a field which may be both attractive, as well as repulsive. This is contrary to the observed fact that only attractive gravitational forces are known. Besides this, with f_μ we can only form generally covariant equations like

$$f_{\mu;\nu}^{\ ;\nu} = j_\mu \qquad (4\text{-}4\text{-}3)$$

$$f_{\mu\nu}^{\ ;\nu} = \lambda_\mu \qquad (4\text{-}4\text{-}4)$$

where $f_{\mu\nu}$ is the field intensity tensor

$$f_{\mu\nu} = f_{\mu,\nu} - f_{\nu,\mu}$$

Both such equations contain besides f_μ, also $g_{\mu\nu}$ and present the same negative result pointed out for ϕ.

However, it is possible to set up field equations for an antisymmetric second rank tensor $F_{\mu\nu}$, or for a tensor density $\mathfrak{F}_{\mu\nu}^{w=1}$

$$\partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} = j_{\sigma\mu\nu}, \qquad (4\text{-}4\text{-}5)$$

$$\mathfrak{F}^{\mu\nu}_{,\nu} = \sqrt{\det \mathfrak{F}^{\mu\nu}}\, j^\mu, \qquad (4\text{-}4\text{-}6)$$

which are independent of the metric $g_{\mu\nu}$. This avoids the problem of reducibility of the variables which appear in the field equations. But we still have problems, since $F_{\mu\nu}$, or $\mathfrak{F}^{\mu\nu}$, possess six independent components. From (4-4-5) or (4-4-6), we have only four field equations for fixing the dynamical behaviour of these six functions. We should think in introducing again equations like

$$F_{\mu\nu}^{\ ;\nu} = 0 \qquad (4\text{-}4\text{-}7)$$

for $F_{\mu\nu}$. This nevertheless introduces $g_{\mu\nu}$ into the field equations, and we are left once again with sixteen field variables and just eight field equations, the (4-4-5) and (4-4-7). For $\mathfrak{F}^{\mu\nu}$ we have no possibility of obtaining two new relationships independent of the metric. We might take

$$\det \mathfrak{F}^{\mu\nu} = 0 \qquad (4\text{-}4\text{-}8)$$

But this is just one condition (which implies in taking the source term in the (4-4-6) as zero), and is not enough. Consequently, a theory using antisymmetric second rank tensors does not fixes all field variables, we are always left with some number of arbitrary components.

The next simple formulation is obtained for symmetric second rank tensors. This formulation is the most natural one, since the metric itself, which is a dynamical element, is such type of geometrical object. As a natural imposition we take the metric $g_{\mu\nu}$ itself as the field variables. This is not essential from the mathematical point of view, we may use any other symmetric second rank tensor such that $g_{\mu\nu}$ is a function of this tensor. * However, the choice of $g_{\mu\nu}$ directly as the field variables is the simplest possible choice. For this case, we have three possible generally covariant second-order differential equations. For a better presentation, we give a number to each one of those formalisms.

1) We require that

$$R_{\mu\nu\rho\sigma} = 0 \qquad (4-4-9)$$

where $R_{\mu\nu\rho\sigma}$ is the metrical Riemann tensor. This equation in spite of being generally covariant, introduces a direct difficulty, the affinity which corresponds to (4-4-9), is a flat affinity $\Gamma^{(o)\mu}_{\nu\sigma}$ (the symbol (o) denotes the flat nature of this affinity). Thus, there exists a mapping such that $g_{\mu\nu}$ takes

* Of these theories, where $g_{\mu\nu} = F_{\mu\nu}(s_{\alpha\beta})$, for $s_{\alpha\beta} = s_{\beta\alpha}$ the field variable, we have to exclude the cases where $g_{\mu\nu} = \phi_{;\mu\nu}$, and $g_{\mu\nu} = \phi_{\mu;\nu} + \phi_{\nu;\mu}$. Both of these cases give symmetric tensors, but their expression contains a relation among $g_{\mu\nu}$, $\frac{\partial g_{\mu\nu}}{\partial x^\lambda}$ and the derivatives of ϕ(or ϕ_μ) and the ϕ_μ itself. These relations cannot be solved for $g_{\mu\nu}$ directly, and have to be taken as a set of first order differential equations for $g_{\mu\nu}$ in terms of $s_{\alpha\beta}$ known from the field equations.

the Galilean values $\overset{\circ}{g}_{\mu\nu}$ over all space time. We arrive in this way to the construction of an absolute object into the manifold, the $g_{\mu\nu}$. This is contrary to the principle of general invariance. Thus, we have to avoid this type of equation.

2) We require that the Weyl tensor constructed with $g_{\mu\nu}$ vanishes.

$$C^{\mu}_{\nu\rho\sigma} = 0 \qquad (4\text{-}4\text{-}10)$$

In this case there exists a mapping such that $g^{*}_{\mu\nu} = (-g)^{-1/4} g_{\mu\nu}$ takes on Galilean values over all space-time,

$$g^{*}_{\mu\nu} = \overset{\circ}{g}_{\mu\nu}$$

Therefore $g^{*}_{\mu\nu}$ is an absolute object. This implies that $g_{\mu\nu}$ itself is given as a dynamical object, entirely in function of just a scalar density of weight + 1/2, the quantity $(-g)^{1/4}$.

$$g_{\mu\nu} = (-g)^{1/4}\, \overset{\circ}{g}_{\mu\nu} = \phi(x)\, \overset{\circ}{g}_{\mu\nu}$$

This is still a possible theory to be considered. Nordström, in 1912 formulated such type of theory, what Nordström tried to set up was a scalar theory of gravitation, as a natural generalization of Newton's theory. As we said, a scalar theory of gravitation is related to the equation (4-4-10), but for Nordström this was not considered since he did not claim for general covariance, which was not motivated in his time. Later on in 1914 Einstein and Fokker in one of his first attempts to set forth a relativistic theory of gravitation, have generalized Nordström's theory, so as to turn it into a general covariant formulation. In the Einstein-Fokker theory, the scalar density $g = \det g_{\mu\nu}$ was to be determined by the equation

$$R = k\, g_{\mu\nu}\, T^{\mu\nu}$$

where $T_{\mu\nu}$ is the stress-energy tensor of matter or of any other field present. Comparing this equation with (4-4-2), we see that the D'Alembertian of the scalar is here replaced by the scalar curvature R formed with $g_{\mu\nu}$. However, since for several facts this theory was wrong *, it was substituted two years later by the correct formulation, today called the general theory of relativity.

3) The only other possible, general covariant equation, which is of the second differential order, and is written for the metric tensor $g_{\mu\nu}$ is of the form,

$$R_{\mu\nu} + \alpha R g_{\mu\nu} + \Lambda g_{\mu\nu} = K T_{\mu\nu} \quad (4\text{-}4\text{-}11)$$

where $R_{\mu\nu}$ and R are respectively the Ricci tensor and the scalar curvature calculated with $g_{\mu\nu}$. The α, Λ and K are constants which may be taken initially with arbitrary values. If we require that (4-4-11) follow from a variational principle, then $\alpha = - 1/2$. These equations with $\alpha = - 1/2$ were the equations proposed by Einstein in 1916. As we have remarked before, from the strict point of view of the principle of invariance, it is not necessary that $g_{\mu\nu}$ solution of (4-4-11) be the metric tensor, nor $R_{\mu\nu}$ the Ricci tensor. Only it is enough that $R_{\mu\nu}$ have the form similar to that of a Ricci tensor, this is all that is necessary for ascertaining that this equation is a general covariant second order equation, involving a symmetric tensor $g_{\mu\nu}$. It is important to note that in principle one can form a continuous infinity of generally covariant equations for a symmetric tensor $g_{\mu\nu}$. Only the extra imposition of obtaining a second order differential equation, which is an additional requirement over and above the principle of invariance, uniquely fixed the equation in the form of (4-4-11).

It is interesting to draw an analogy with the case of electrodynamics. In

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* It predicts the wrong sign for the advance of the planetary perihelia, it does not predict a bending for light rays in a gravitational field.

this situation the principle of invariance is replaced by the principle of Poincaré invariance plus gauge invariance. Only on the basis of these two invariance principles, one can form a continuous infinity of field equations, by using as Lagrangian density any function of the two Maxwell scalars

$$u_1 = F^{\mu\nu} F_{\mu\nu}$$

$$u_2 = \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

that is,

$$\mathcal{L} = F(u_1, u_2)$$

The field equations are the infinite set of equations

$$\frac{\delta\mathcal{L}}{\delta A_\mu} = \partial_\alpha \frac{\partial\mathcal{L}}{\partial A_{\mu,\alpha}} = \frac{\partial^2 F}{\partial u_1^2} \frac{\partial u_1}{\partial x^\alpha} \frac{\partial u_1}{\partial A_{\mu,\alpha}} + \frac{\partial^2 F}{\partial u_1\, \partial u_2} \frac{\partial u_2}{\partial x^\alpha} \frac{\partial u_1}{\partial A_{\mu,\alpha}} + \frac{\partial F}{\partial u_1} \frac{\partial}{\partial x^\alpha} \frac{\partial u_1}{\partial A_{\mu,\alpha}}$$

$$+ \frac{\partial^2 F}{\partial u_2\, \partial u_1} \frac{\partial u_1}{\partial x^\alpha} \frac{\partial u_2}{\partial A_{\mu,\alpha}} + \frac{\partial^2 F}{\partial u_2^2} \frac{\partial u_2}{\partial x^\alpha} \frac{\partial u_2}{\partial A_{\mu,\alpha}} + \frac{\partial F}{\partial u_2} \frac{\partial}{\partial x^\alpha} \frac{\partial u_2}{\partial A_{\mu,\alpha}} = 0$$

where

$$\frac{\partial}{\partial x^\alpha} \frac{\partial u_1}{\partial A_{\mu,\alpha}} = F_{\mu\lambda}{}^{,\lambda}$$

$$\frac{\partial}{\partial x^\alpha} \frac{\partial u_2}{\partial A_{\mu,\alpha}} = \varepsilon^{\mu\nu\rho\sigma} F_{\nu\rho,\sigma} \equiv 0$$

Here the situation is a bit different since all above equations are of the second differential order. The extra requirement which will reduce the infinity of all possible equations, is the requirement of linearity, which reduces the field equations to the unique form

$$\frac{\delta u_1}{\delta A_\mu} \quad \frac{\partial}{\partial x^\alpha} \quad \frac{\partial u_1}{\partial A_{\mu,\alpha}} = 0$$

This analogy serves to point out that we always have to use some other type of extra requirements besides the invariance principle. This extra imposition may be different for each type of theory. If we require further that linearity also holds for the gravitational field equations, which sums up two extra requirements besides the general covariance, we get the result that no field equation do exists, which is altogether of second order and linear, being generally covariant. This means that the requirement of linearity is not compatible with the general covariance for second order differential equations. The only possible linear second order differential equations, are those covariant with respect to the Poincaré plus spin-two gauge transformations of the form

$$\phi'_{\mu\nu}(x) = \phi_{\mu\nu}(x) + \Lambda_{\mu,\nu}(x) + \Lambda_{\nu,\mu}(x)$$

and we get again the so called flat space-time formulations of gravitation.

We now comment on a point very important. Let us suppose that we face the following problem: To construct a theory which is generally covariant involving some tensor field. From what we have seen, the only meaningful of such theories are those involving a symmetric tensor field of second rank.

Regardless of what field it describes, if we impose that the field equations, are of the second differential order and follow from a variational principle, we get a unique possible choice. Later on, by means of consequences of this field equation, which are compared with observable phenomena, we learn that this theory describes the gravitational field. Comparing with a similar situation respect to the restricted invariance of the Poincaré group, we see that the imposition of Poincaré invariance does not pick up any particular type of field variable, we have several possible theories, scalar, vector, tensor and so on. This means that the Poincaré invariance does not force us to introduce any tensor field, which a posteriori will describe the gravitational field, as a necessary element of the theory. If we do so, we still have a lot of arbitrariness in the field equations. We have to impose besides the correct differential order, that the equations are linear (similarly to the case seen before for the electrodynamics) and the positive definiteness of the energy density of the field. Only then, the field equations will be unique.

A comparison of this situation with those presented in the generally invariant formulation, shows us how the principle of general invariance is formally stronger than the restricted principle of relativity.

5. THE THEORY OF GENERAL RELATIVITY

5.1) The Einstein Field Equations

For obtaining his equation for the metric field $g_{\mu\nu}$ Einstein was guided by the Principle of invariance together with the extra requirement that in an apropriate limit, his equations should coincide with the Newtonian gravitational theory. If we look upon this later theory as a field theory, then the gravitational potential $\phi(x)$ satisfies the Poisson equation.

$$\nabla^2\phi(x) = 4\pi G\rho(x) \qquad (5\text{-}1\text{-}1)$$

a solution of this equation is,

$$\phi(x) = -\sum_i G \frac{g_i}{S_{xi}} \qquad (5\text{-}1\text{-}2)$$

where s_{xi} is the distance between the ith body and the field point. g_i is the gravitational charge of the ith particle of an N-body system of gravitating particles. In this case

$$\rho(x) = \sum_i \rho_i(x) = \sum_i g_i \; \delta(\vec{x}-\vec{x}_i) \qquad (5\text{-}1\text{-}3)$$

Was just the requirement that in an apropriate limit the generally covariant equations should coincide with (5-1-1), which indicates that the field equations for $g_{\mu\nu}$ should be of the second differential order. Besides this, the equations should contain in the right hand side the stress-energy tensor $T_{\mu\nu}$ linearly, in order to insure a correct limit to the Newtonian gravitational charge density ρ.

Einstein takes the equations in the form (4-5-11) with $\alpha = -1/2$. Then, since

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

satisfies

$$g^{\nu\lambda} G_{\mu\nu;\lambda} = 0$$

we get, as consequence of the field equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = k\, T_{\mu\nu} \qquad (5\text{-}1\text{-}4)$$

that $T_{\mu\nu}$ satisfies the covariant continuity equation,

$$g^{\nu\lambda} T_{\mu\nu;\lambda} = 0 \qquad (5\text{-}1\text{-}5)$$

where one used that $g_{\mu\nu}$ also satisfies $g_{\mu\nu;\lambda} = 0$, in order to take into account the term $\Lambda g_{\mu\nu}$. In what will follow we shall consider $\Lambda = 0$. This was also initially done by Einstein. The term $\Lambda g_{\mu\nu}$, called the cosmological term will be considered later.

In the same year that Einstein proposed this field equation, [31] Hilbert [32] derived a similar equation from a variational principle. The comparison of both approaches served to show up that really the form of a generally covariant equation of the second differential order involving a symmetric second rank tensor was indeed unique, on the basis of the two extra impositions.

1) It follows from a variational principle.

2) It has to be of the second differential order.

We give now the derivation of (5-1-4) from the variational principle. [33] First we consider the equations for $g_{\mu\nu}$ in the absence of matter and other fields. The two invariants (in strict sense they are not scalars, both rather scalar densities with weight $W = +1$)

$$q_1 = \sqrt{-g}\ R$$

$$q_2 = \sqrt{-g}$$

are of the correct differential order for generating second order differential equations, upon variation in $g_{\mu\nu}$ (for q_1 we need a little more of care, since it involves besides the $g_{\mu\nu}$ and the $g_{\mu\nu,\lambda}$, also the $g_{\mu\nu,\lambda\sigma}$. We might think that due to this the field equations $\frac{\delta q_1}{\delta g_{\mu\nu}} = 0$ should include third order derivatives of $g_{\mu\nu}$. But this will not happen because all terms in q_1 which depend on $g_{\mu\nu,\lambda\sigma}$ form a divergence, and thus do not give contribution to the variational problem). Thus, we may put the most general Action function as

$$S_G = -\frac{1}{2K}\int \sqrt{-g}\ (R-2\Lambda)\ d^4x \qquad (5\text{-}1\text{-}6)$$

where K and Λ are constants to be determined. R is the scalar curvature

$$R = g^{\mu\nu}\left[\Gamma^{\rho}_{\mu\rho,\nu} - \Gamma^{\rho}_{\mu\nu,\rho} + \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu} - \Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma}\right] \qquad (5\text{-}1\text{-}7)$$

For deriving the Euler-Lagrange equations for (5-1-6), it is customary to take the affinity $\Gamma^{\rho}_{\mu\nu}$ identical to the Christoffel symbols.

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2}\, g^{\mu\lambda} \left(\frac{\partial g_{\nu\lambda}}{\partial x^{\sigma}} + \frac{\partial g_{\nu\lambda}}{\partial x^{\nu}} - \frac{\partial g_{\nu\sigma}}{\partial x^{\lambda}} \right)$$

and hence as known functions of the $g_{\nu\lambda}$. However, one can also treat both the $g_{\nu\lambda}$ and the $\Gamma^{\mu}_{\nu\lambda}$ as the independents variables in the variational principle, and thus obtaining equations that determine both objects. Such procedure is known as the Palatini variational technique. This procedure for electrodynamics implies in using both the potentials A_{μ} and the fields $F_{\mu\nu}$ as variants in the Action principle. Presently, we use Palatini's method.

We have,

$$\delta\sqrt{-g} = \frac{\partial\sqrt{-g}}{\partial g_{\mu\nu}}\, \delta g_{\mu\nu} = \frac{1}{2}\sqrt{-g}\; \frac{1}{g}\; \frac{\partial g}{\partial g_{\mu\nu}}\, \delta g_{\mu\nu} = \frac{1}{2}\sqrt{-g}\; g^{\mu\nu}\; \delta g_{\mu\nu} \qquad (5\text{-}1\text{-}8)$$

$$\delta g^{\mu\nu} = -g^{\mu\sigma}\, g^{\nu\lambda}\; \delta g_{\sigma\lambda} \qquad (5\text{-}1\text{-}9)$$

and thus, a variation in the integrand of S_G gives,

$$\sqrt{-g}\;\; \delta R + R\, \delta\sqrt{-g}\; - 2\,\Lambda\, \delta\sqrt{-g}$$

since $R = g^{\mu\nu}\; R_{\mu\nu}$, we get

$$\delta R = g^{\mu\nu}\, \delta R_{\mu\nu} + \delta g^{\mu\nu}\cdot R_{\mu\nu} = g^{\mu\nu}\, \delta R_{\mu\nu} - R^{\mu\nu}\, \delta g_{\mu\nu}$$

substitution of this into the above equation gives

$$\sqrt{-g}\;\; \delta R + R\, \delta\sqrt{-g} - 2\Lambda\, \delta\sqrt{-g} = \sqrt{-g}\, (-R^{\mu\nu}\, \delta g_{\mu\nu} + g^{\mu\nu}\, \delta R_{\mu\nu} +$$

$$+ \frac{1}{2}\, g^{\mu\nu}\, \delta g_{\mu\nu}\, .R) - \Lambda\, \sqrt{-g}\, g^{\mu\nu}\, \delta g_{\mu\nu}.$$

Thus,

$$\delta S_G = \frac{1}{2K} \int \sqrt{-g} \left[(R^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu}\, R\; + \Lambda\, g^{\mu\nu})\, \delta g_{\mu\nu} - g^{\mu\nu}\, \delta R_{\mu\nu} \right] d^4x \qquad (5\text{-}1\text{-}11)$$

From (5-1-7) we see that $R_{\mu\nu}$ involves only the affinities $\Gamma^{\mu}_{\nu\lambda}$. If we take first an arbitrary variation in $g_{\mu\nu}$ that vanish on the boundary of the region of integration, for fixed $\Gamma^{\mu}_{\nu\lambda}$, we obtain the equations.

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} = 0 \tag{5-1-12}$$

which are part of the complete field equations (5-1-4). The remainder of these equations are fixed by considering variations in $\Gamma^{\mu}_{\nu\lambda}$ vanishing at the boundaries of integration, for fixed $g_{\mu\nu}$. We get from the last term in the integrand of (5-1-11).

$$\sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu} = \{\sqrt{-g}(g^{\mu\rho}\, \delta\Gamma^{\nu}_{\mu\nu} - g^{\mu\nu}\, \delta\Gamma^{\rho}_{\mu\nu})\}_{,\rho} + \{(\sqrt{-g}\, g^{\mu\nu})_{;\rho} - (\sqrt{-g}\, g^{\mu\sigma})_{;\sigma}\, \delta^{\nu}_{\rho}\}\, \delta\Gamma^{\rho}_{\mu\nu} \tag{5-1-13}$$

The first term in (5-1-13) is a divergence, and does not contribute to δS_G, since the $\delta\Gamma^{\mu}_{\nu\lambda}$ vanish on the surface which bounds the region of integration in S_G. We are left with

$$\delta S_G = \frac{-1}{2K} \int d_4 x\, \{(\sqrt{-g}\, g^{\mu\nu})_{;\rho} - (\sqrt{-g}\, g^{\mu\sigma})_{;\sigma}\, \delta^{\nu}_{\rho}\}\, \delta\Gamma^{\rho}_{\mu\nu} = 0$$

from the Euler-Lagrange equation for the variation in the affinities, we obtain,

$$(\sqrt{-g}\, g^{\mu\nu})_{;\rho} - (\sqrt{-g}\, g^{\mu\sigma})_{;\sigma}\, \delta^{\nu}_{\rho} = 0$$

But these equations in turn imply that $g^{\mu\nu}_{;\rho} = 0$, and hence $\Gamma^{\mu}_{\rho\nu} = \{^{\mu}_{\rho\nu}\}$. With this identification, the equations (5-1-12) form the left hand side of the field equations (5-1-4).

Let us consider now the interaction of matter or fields, or both, with the gravitational field. This amounts to adding to S_G an Action integral S_M describing such interaction. The total Action functional is then

$$S = S_G + S_M$$

Since in the Palatini variational method, both $g_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$ are variants, if we want to keep the simple form $\Gamma^{\alpha}_{\mu\nu} = \{^{\alpha}_{\mu\nu}\}$ unchanged, we have to restrict the form of S_M so as that it does not depend upon the $\Gamma^{\alpha}_{\mu\nu}$. This is the same as requiring the principle of minimal coupling: all interactions depend just on the potentials of the gravitational fields, the $g_{\mu\nu}$, not on the $\Gamma^{\alpha}_{\mu\nu}$. Since then S_M will depend just on the $g_{\mu\nu}$, besides its own variables, we get by variating the $g_{\mu\nu}$ in S_M,

$$\delta S_M = \frac{1}{2}\int \sqrt{-g}\; T^{\mu\nu}\; \delta g_{\mu\nu}\; d^4x \tag{5-1-14}$$

where the contravariant second rank tensor $T^{\mu\nu}$ is defined by this relation. This may be put simbolically as,

$$\frac{\delta S_M}{\delta g_{\mu\nu}^{(x)}} = T^{\mu\nu}(x) \tag{5-1-15}$$

Thus, the field equation for the gravitational field, obtained by variating the total Action integral $S_G + S_M$, will have the form (5-1-4). As an application of (5-1-14), we give the following examples.

1) Determine the expression for $T^{\mu\nu}$ for the electromagnetic field, where

$$S_M = -\frac{1}{8\pi}\int d^4x\, \sqrt{-g}\, g^{\mu\rho}\, g^{\nu\sigma}\, \frac{1}{2} F_{\mu\nu}\, F_{\rho\sigma} \tag{5-1-16}$$

Effecting the variation on the $g_{\mu\nu}$ we obtain

$$\delta S_M = -\frac{1}{8\pi}\int d^4x\, F_{\mu\nu\rho\sigma}\, \{g^{\mu\rho}\, g^{\nu\sigma}\, \delta\sqrt{-g} + \sqrt{-g}\, g^{\nu\sigma}\, \delta g^{\mu\rho} + \sqrt{-g}\, g^{\mu\rho}\, \delta g^{\nu\sigma}\}$$

where $F_{\mu\nu\rho\sigma}$ is a short for $\frac{1}{2} F_{\mu\nu} F_{\rho\sigma}$. Using (5-1-8) and (5-1-9) we get

$$\delta S_M = -\frac{1}{8\pi}\int d^4x\, \sqrt{-g}\, \{F_{\mu\nu\rho\sigma}\, \frac{1}{2}\, g^{\mu\rho}\, g^{\nu\sigma}\, g^{\lambda\beta} - F_{\mu\nu\rho\sigma}\, g^{\mu\lambda} g^{\rho\beta} g^{\nu\sigma} - F_{\mu\nu\rho\sigma}\, g^{\mu\rho} g^{\nu\lambda} g^{\sigma\beta}\}\delta g_{\lambda\beta}$$

which gives

$$T^{\lambda\beta} = \frac{1}{4\pi}\, (F^{\lambda\sigma}\, F^{\beta}{}_{\sigma} - \frac{1}{4} F_{\rho\sigma}\, F^{\rho\sigma}\, g^{\lambda\beta}) \tag{5-1-17}$$

which is the Maxwell stress-energy tensor [34], satisfying the properties,

$$T^{\lambda\beta} = T^{\beta\lambda}$$

$$g_{\lambda\beta}\, T^{\lambda\beta} = 0$$

2) Derive the expression for $T^{\mu\nu}$ for the scalar field, with

$$S_M = \frac{1}{2}\int d^4x \sqrt{-g}\,(g^{\mu\nu}\,\phi_{,\mu}\,\phi_{,\nu} - m^2\phi^2) \tag{5-1-18}$$

we have,

$$\delta S_M = \frac{1}{2}\int d^4x \left[\phi_{,\mu}\,\phi_{,\nu}(\delta\sqrt{-g}.g^{\mu\nu} + \sqrt{-g}\,\delta g^{\mu\nu}) - m^2\phi^2\,\delta\sqrt{-g}\right]$$

which gives,

$$\delta S_M = \frac{1}{2}\int d^4x \sqrt{-g}(\frac{1}{2}\,g^{\mu\nu}\,g^{\alpha\beta}\,\phi_{,\mu}\,\phi_{,\nu} - g^{\mu\alpha}\,g^{\nu\beta}\,\phi_{,\mu}\,\phi_{,\nu} - \frac{m^2}{2}\,\phi^2\,g^{\alpha\beta})\,\delta g_{\alpha\beta}$$

The expression for $T^{\alpha\beta}$ is then,

$$T^{\alpha\beta} = (g^{\mu\alpha}\,g^{\nu\beta} - \frac{1}{2}\,g^{\mu\nu}\,g^{\alpha\beta})\,\phi_{,\mu}\,\phi_{,\nu} + \frac{m^2}{2}\,\phi^2\,g^{\alpha\beta} \tag{5-1-19}$$

3) Derive the $T^{\alpha\beta}$ for a system of neutral particles with rest mass m_o, with

$$S_M = -\,m_o\int\sqrt{-g}\,d^4x\,\sum_i\int d\,\lambda_i\,\delta^4(x-z_i)\,\sqrt{g_{\mu\nu}(x)\,\dot{z}_i^\mu\,\dot{z}_i^\nu} \tag{5-1-20}$$

doing the variation in $g_{\mu\nu}{}^{(x)}$ we get

$$\delta S_M = -\,m_o\int d^4x\,\sum_i\int d\lambda_i\,\delta^4(x-z_i)\left\{\sqrt{-g}\,\frac{\dot{z}_i^\alpha\,\dot{z}_i^\beta\,\delta g_{\alpha\beta}}{2\sqrt{g_{\mu\nu}\dot{z}_i^\mu\,\dot{z}_i^\nu}} + \sqrt{g_{\mu\nu}\dot{z}_i^\mu\,\dot{z}_i^\nu}\,\delta\sqrt{-g}\right\}$$

thus

$$\delta S_M = -\,\frac{m_o}{2}\int d^4x\,\sqrt{-g}\left\{\sum_i\int d\lambda_i\,\delta^4(x-z_i)\,\frac{\dot{z}_i^\alpha\,\dot{z}_i^\beta}{\sqrt{g_{\mu\nu}\dot{z}_i^\mu\,\dot{z}_i^\nu}} + \right.$$

$$\left. + \sum_i\int d\lambda_i\,\delta^4(x-z_i)\,\sqrt{g_{\mu\nu}\,\dot{z}_i^\mu\,\dot{z}_i^\nu}\,g^{\alpha\beta}\right\}\delta\,g_{\alpha\beta}$$

and the $T^{\alpha\beta}$ has the value

$$T^{\alpha\beta}(x) = m_o\,\sum_i\int d\lambda_i\,\delta^4(x-z_i)\left\{\frac{\dot{z}_i^\alpha\,\dot{z}_i^\beta}{\sqrt{g_{\mu\nu}(x)\,\dot{z}_i^\mu\,\dot{z}_i^\nu}} + \sqrt{g(x)\,\dot{z}_i^\mu\,\dot{z}_i^\nu}\,g^{\alpha\beta}(x)\right\}. \tag{5-1-21}$$

5.2) Stationary and Static Gravitational Fields

Before introducing this class of kinematically possible gravitational fields, we will need a further concept, called by symmetries associated to the behaviour of the metric tensor $g_{\mu\nu}$. Under an infinitesimal mapping of the MMG, the metric $g_{\mu\nu}$ varies as

$$\bar{\delta} g_{\mu\nu} = - g_{\mu\rho}\, \xi^{\rho}_{,\nu} - g_{\rho\nu}\, \xi^{\rho}_{,\mu} - g_{\mu\nu,\rho}\, \xi^{\rho} \qquad (5\text{-}2\text{-}1)$$

where

$$\bar{\delta} g_{\mu\nu} = g'_{\mu\nu}(x) - g_{\mu\nu}(x)$$

if it happens that the transformation with descriptors $\xi^{\rho}(x)$ is such that this variation vanishes.

$$\bar{\delta} g_{\mu\nu}(x) = 0$$

we say that this ξ^{ρ} is a Killing vector field. By the same argument, we call the equation

$$g_{\mu\rho}\, \xi^{\rho}_{,\nu} + g_{\rho\nu}\, \xi^{\rho}_{,\mu} + g_{\mu\nu,\rho}\, \xi^{\rho} = 0 \qquad (5\text{-}2\text{-}2)$$

a Killing equation. The existence of Killing vectors on the space-time serves to characterize the symmetries which these spaces possess. These symmetries are translated in the form assumed by the metric of the space, which satisfy (5-2-2). As example, consider the flat space-time of special relativity. Then, $g_{\mu\nu} = \eta_{\mu\nu}$ at all points, in cartesian coordinates. In this case, we get simply

$$\eta_{\mu\rho}\, \xi^{\rho}_{,\nu} + \eta_{\rho\nu}\, \xi^{\rho}_{,\mu} = 0$$

the most general solution of this equation is of the form

$$\xi^{\mu} = \varepsilon^{\mu} + \varepsilon^{\mu}{}_{\nu}\, x^{\nu}$$

where ε^{μ} and $\varepsilon_{\mu\nu} = \eta_{\mu\rho}\, \varepsilon^{\rho}{}_{\nu} = -\varepsilon_{\nu\mu}$ are ten infinitesimal parameters. The four Killing vectors, or better saying, the four components of the Killing vector in this case, are associated to the descriptors of the Poincaré group.

After some calculations, we may put (5-2-1) in the form

$$\bar{\delta} g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} \quad (5\text{-}2\text{-}3)$$

where

$$\xi_\mu = g_{\mu\lambda}\, \xi^\lambda$$

We prove now the following property: A gravitational field is called stationary if it possess an everywhere timelike Killing vector. The proof is as follows, let τ_μ be such vector, we then have from (5-2-3),

$$\tau_{\nu;\mu} + \tau_{\mu;\nu} = 0 \quad (5\text{-}2\text{-}4)$$

writing explicitly this equation one finds

$$\tau_{\mu,\nu} + \tau_{\nu,\mu} - 2\, \Gamma^\lambda_{\mu\nu}\, \tau_\lambda = 0 \quad (5\text{-}2\text{-}5)$$

Consider now a transformation of the reference system, such that in the new frame the components of τ^μ, which satisfy in all points the property

$$\tau^2(x) = g_{\mu\nu}(x)\; \tau^\mu(x)\tau^\nu(x) > 0 \quad (5\text{-}2\text{-}6)$$

have the canonical value

$$\tau^\mu = \delta^\mu_o \quad (5\text{-}2\text{-}7)$$

What such transformation do exists at each point of the manifold is seen from its equations

$$\begin{cases} \dfrac{\partial x^o}{\partial x'^o}\; \tau'^o + \dfrac{\partial x^o}{\partial x'^i}\, \tau'^i = \tau^o = 1 \\[2ex] \dfrac{\partial x^i}{\partial x^o}\, \tau'^o \quad \dfrac{\partial x^i}{\partial x'^j}\; \tau'^j = \tau^i = 0 \end{cases}$$

they always possess a solution. Now, for the reference system where (5-2-7) holds, we have

$$\tau_o = g_{o\lambda}\; \tau^\lambda = g_{oo},\; \tau_i = g_{i\lambda}\; \tau^\lambda = g_{io} \quad (5\text{-}2\text{-}8)$$

or, $\tau_\mu = g_{\mu o}$. For these values of τ_μ the Killing equation (5-2-5) takes the form

$$
\begin{cases}
\dfrac{\partial g_{oo}}{\partial x^o} - \dfrac{1}{2}\Gamma_{o,oo} = 0 \\[2ex]
\dfrac{\partial g_{oo}}{\partial x^i} + \dfrac{\partial g_{io}}{\partial x^o} - \Gamma_{o,oi} = 0 \\[2ex]
\dfrac{\partial g_{io}}{\partial x^j} + \dfrac{\partial g_{jo}}{\partial x^i} - \Gamma_{o,ij} = 0
\end{cases}
$$

which gives

$$\frac{\partial g_{\mu\nu}}{\partial x^o} = 0 \tag{5-2-9}$$

Besides this, from (5-2-7) and (5-2-8) one gets,

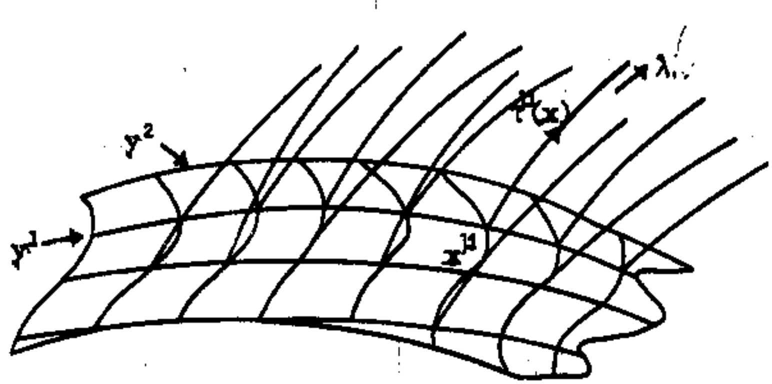
$$\tau^2 = \tau_\mu \tau^\mu = g_{oo} > 0 \tag{5-2-10}$$

The two relations (5-2-9) and (5-2-10) define a stationary gravitational field in general relativity.

Given a timelike vector $\tau^\mu(x)$, not necessarily a Killing field, defined on the space-time manifold, we may introduce a three-parameter family of curves whose tangent are equal to the $\tau^\mu(x)$. The points of such curves are given by

$$x^\mu = \xi^\mu(\lambda, y^i) \tag{5-2-11}$$

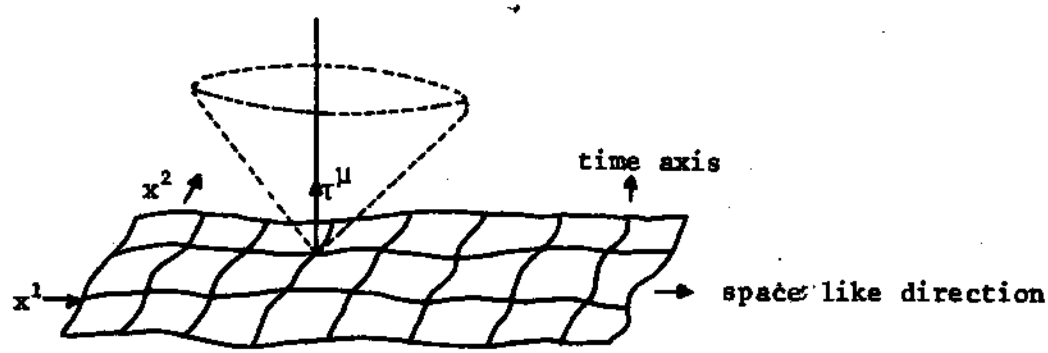
where λ is a parameter along the curves and the y^i, $i = 1, 2, 3$ are the parameters characterizing a given curve of the family.



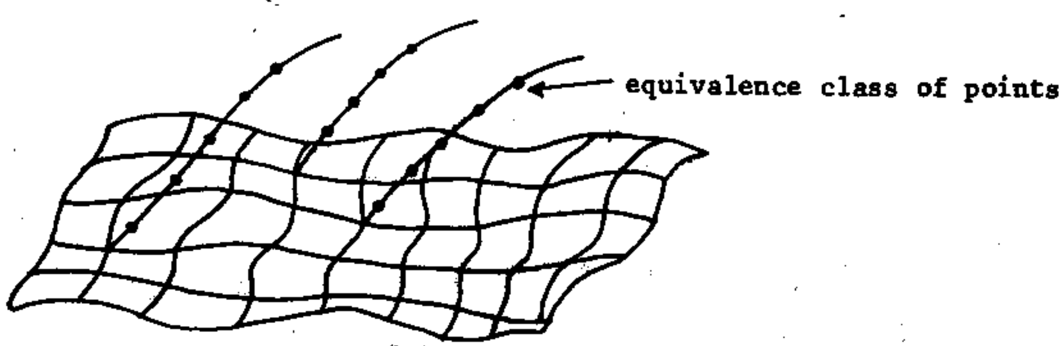
Then,

$$\frac{\partial \xi^\mu}{\partial \lambda} = \tau^\mu(\xi(\lambda, y^i)) \qquad (5\text{-}2\text{-}12)$$

are the defining equations of these curves. As we have seen, any timelike vector may be put in the form $\tau^\mu = \{1,\vec{0}\}$ by adequate choice of coordinates. In this coordinate system, the curves are given by x^r = const. and $\lambda = x^0$. In this case the y^i may be identified to the spatial coordinates x^i,



Thus, a timelike vector field furnishes us with a method of defining a space S_3, similar to the absolute space of Newtonian mechanics. A point of this space consist of an equivalence class of points of S_4, two points being considered as equivalent for the purpose of constructing this space S_3 if they lie on the same curve (5-2-11).



The parameters y^i can then be taken as the coordinates of a point in S_3. A trajectory in S_4 gets projected onto a curve in S_3. This curve will be refered to as the orbit associated with the trajectory.

We may decompose an arbitrary displacement dx^μ at a point of S_4 into a component parallel to τ^μ and a component normal to τ^μ, thus belonging to S_3, according to

$$dx^\mu = \tau^\mu \, d\alpha + d\beta^\mu \tag{5-2-13}$$

where $\tau_\mu \, d\beta^\mu = 0$. Then,

$$d\alpha = \frac{\tau_\mu \, dx^\mu}{\tau^2} \tag{5-2-14}$$

and

$$d\beta^\mu = dx^\mu - \frac{\tau_\beta \, \tau^\mu \, dx^\beta}{\tau^2} = \mathcal{B}^\mu{}_\nu \, dx^\nu \tag{5-2-15}$$

where

$$\mathcal{B}^\mu{}_\nu = \delta^\mu{}_\nu - \frac{\tau^\mu \, \tau_\nu}{\tau^2} \tag{5-2-16}$$

The tensor $\mathcal{B}^\mu{}_\nu$ is a projection operator for the hyperplane normal to τ^μ. It satisfies,

$$\mathcal{B}^\mu{}_\nu \, \mathcal{B}^\nu{}_\lambda = \mathcal{B}^\mu{}_\lambda$$

$$\mathcal{B}^\mu{}_\nu \, \tau^\nu = 0$$

The length dl, which is the projection onto the place normal to τ^μ of a displacement dx^μ, is

$$d\ell^2 \equiv d\beta_\mu \, d\beta^\mu = g_{\mu\nu} \, d\beta^\mu \, d\beta^\nu$$

from (5-2-15),

$$d\ell^2 = g_{\mu\nu} \, \mathcal{B}^\mu{}_\lambda \, \mathcal{B}^\nu{}_\sigma \, dx^\lambda \, dx^\rho = e_{\lambda\sigma} \, dx^\lambda \, dx^\rho \tag{5-2-17}$$

with

$$e_{\lambda\rho} = \mathcal{B}^\mu{}_\lambda \, \mathcal{B}^\nu{}_\rho \, g_{\mu\nu} = g_{\lambda\rho} - \frac{\tau_\lambda \, \tau_\rho}{\tau^2} \tag{5-2-18}$$

The tensor $e_{\mu\nu}$ is thus the projection of $g_{\mu\nu}$ onto the plane normal to τ^μ and plays the role of a metric in this plane, as is shown through the relation

(5-2-17). When $\tau^{\mu} = \{1,\vec{0}\}$, we get for this tensor the canonical values

$$e_{\lambda\rho} = g_{\lambda\rho} - \frac{g_{\lambda o}\, g_{\rho o}}{g_{oo}}\,; \quad \begin{cases} e_{oo} = e_{or} = 0 & \text{(5-2-19-1)} \\ e_{rs} = g_{rs} - \dfrac{g_{os}\, g_{or}}{g_{oo}} & \text{(5-2-19-2)} \end{cases}$$

In a similar way we may define the differential of a displacement along the direction of τ^{μ}. Its square being

$$du^2 \equiv (d\alpha\ \tau_{\mu})(d\alpha\ \tau^{\mu}) = g_{\mu\nu}\ (d\alpha\ \tau^{\nu})(d\alpha\ \tau^{\mu})$$
$$= p_{\mu\nu}\ dx^{\mu}\ dx^{\nu} \tag{5-2-20}$$

where

$$p_{\mu\nu} = \frac{\tau_{\mu}\ \tau_{\nu}}{\tau^2} \tag{5-2-21}$$

In the canonical representation, where $\tau_{\mu} = g_{\mu o}$ we obtain

$$p_{\mu\nu} = \frac{g_{\mu o}\ g_{\nu o}}{g_{oo}} \tag{5-2-22}$$

which means that all their components are different from zero. It should be observed that both $e_{\mu\nu}$ and $p_{\mu\nu}$ are tensors at four-dimensions, as can be seen from (5-2-18) and (5-2-21). They satisfy the orthogonallity condition,

$$e_{\mu\nu}\ p^{\nu}{}_{\lambda} = 0$$

In the "canonical frame" we get simply

$$d\ell^2 = e_{rs}\ dx^r\ dx^s = \left(g_{rs} - \frac{g_{ro}\ g_{so}}{g_{oo}}\right)\ dx^r\ dx^s \tag{5-2-23}$$

$$du^2 = \frac{g_{\mu o}\ g_{\nu o}}{g_{oo}}\ dx^{\mu}\ dx^{\nu} \tag{5-2-24}$$

From these relations we see that the $g_{\mu o}$ are variables which are associated to displacements normal to the hyperplane S_3 which has an arc element given by (5-2-23).

All those results hold good for any arbitrarily chosen timelike vector τ^μ. At this point it is important to draw attention to the conventions used. According to (5-2-6) we are defining a timelike vector in the case that the signature of $g_{\mu\nu}$ is -2. If we chose the opposite signature, that is +2, we have to define a timelike vector by

$$\tau^2(x) = g_{\mu\nu}(x)\ \tau^\mu(x)\ \tau^\nu(x) < 0$$

instead of by (5-2-6). In this case g_{oo} is negative, and in the "canonical frame" we get

$$\tau^2(x) = g_{oo}(x) < 0 \tag{5-2-25}$$

instead of (5-2-10). In any case, we may introduce a unit vector along the direction of τ^μ by,

$$\ell^\mu = \frac{\tau^\mu}{\sqrt{\tau^2}}\ ,\ \ell^\mu\ \ell_\mu = 1 \tag{5-2-26}$$

For signature -2, and in the "canonical frame" we have

$$\ell^\mu = \frac{\delta^\mu_o}{\sqrt{g_{oo}}}\ ,\ \ell_\mu = \frac{g_{\mu o}}{\sqrt{g_{oo}}} \tag{5-2-27}$$

For the signature +2, and in the "canonical frame" we will get

$$\ell^\mu = \frac{\delta^\mu_o}{\sqrt{-g_{oo}}}\ ,\quad \ell_\mu = \frac{g_{\mu o}}{\sqrt{-g_{oo}}} \tag{5-2-28}$$

Using ℓ^μ instead of τ^μ we may rewrite $\mathcal{B}^\mu{}_\nu$ as

$$\mathcal{B}^\mu{}_\nu = \delta^\mu{}_\nu - \ell^\mu\ \ell_\nu \tag{5-2-29}$$

and for $e_{\mu\nu}$ and $p_{\mu\nu}$ we have

$$e_{\mu\nu} = g_{\mu\nu} - \ell_\mu \ell_\nu \equiv \mathcal{B}_{\mu\nu} \tag{5-2-30}$$

$$p_{\mu\nu} = \ell_\mu \ell_\nu \tag{5-2-31}$$

So that du^2 and $d\ell^2$ take on the form

$$d\ell^2 = (g_{\mu\nu} - \ell_\mu \ell_\nu)\, dx^\mu\, dx^\nu \tag{5-2-32}$$

$$du^2 = (\ell_\mu\, dx^\mu)^2 \tag{5-2-33}$$

From (5-2-33) we clearly see that du is the measure of an interval outside of the hyperplane S_3. All these relations will be of importance later on when we will study the Dirac's formulation of the gravitational field in terms of a Hamiltonian.

Now, we turn attention to another detail. We have fixed the choice of a "canonical frame" from the conditions (5-2-7) on the τ^μ. However, we may chose to fix such coordinates by the four conditions

$$\tau_\mu = \delta^o_\mu \tag{5-2-34}$$

on the τ_μ, instead of the (5-2-7). Both types of settings are equally correct. However, in Dirac's theory we chose (5-2-34) instead of (5-2-7), so that it is important to rewrite all important relations in this setting. Since the method of obtention of these quantities is exactly the same, we just write the results.

$$\tau^\mu = g^{\mu\nu}\, \tau_\nu = g^{\mu o}$$

$$\tau^2 = g^{oo}$$

$$\ell^\mu = \frac{g^{\mu o}}{\sqrt{g^{oo}}}\,,\quad \ell_\mu = \frac{\delta^o_\mu}{\sqrt{g^{oo}}}$$

$$e_{rs} = g_{rs}\,,\quad e^{rs} = g^{rs} - \frac{g^{os}\, g^{or}}{g^{oo}}$$

$$e^{rs}\, g_{sk} = \delta^r_k$$

These relations referring to the signature -2. For opposite signature we change signs in g^{oo}.

In the case that the τ^{μ} is a Killing field,

$$\tau_{\mu;\nu} + \tau_{\nu;\mu} = 0$$

the $e_{\mu\nu}$ and $p_{\mu\nu}$ are independent of x^{o}, corresponding to stationary fields.

While we can define a hyperplane normal to τ^{μ} at each point of the manifold where the τ^{μ} is defined, we cannot in general form a family of hypersurfaces with the τ^{μ} as normals. What this is so, we may see from the following facts: Let these hypersurfaces be given by the relation

$$\phi(x) = \phi_{o}$$

If τ^{μ} is orthogonal to this family,

$$\tau_{\mu}(x) = \xi(x)\ \phi_{,\mu}(x) \qquad (5\text{-}2\text{-}35)$$

that is, τ_{μ} is proportional to the gradient of s scalar. Up to this point everything is correct, but if we map to the canonical frame where $\tau_{\mu} = g_{\mu o}$ we obtain

$$g_{\mu o}(x) = \xi(x)\ \phi_{,\mu}(x) \qquad (5\text{-}2\text{-}36)$$

which in general is not satisfied for all metrics $g_{\mu\nu}$. It is important to fix attention in one detail: The equation (5-2-36) is not covariant as it stands, however this does not mean that it is wrong. We obtained this equation from the covariant relation (5-2-35) by picking up a particular system of coordinates. By the same token, when we put $\tau_{\mu} = g_{\mu o}$, we also get a non-covariant relation, since in general $g_{\mu o}$ is not a vector. This does not mean that such relation is wrong, it just means that τ_{μ} takes this particular form in this coordinate system, and since we always know how to pass to a general system of coordinates, and thus obtaining covariant equations, we are free to work in these particular coordinates.

A special class of stationary gravitational fields are those for which the associated timelike Killing field is also hypersurface orthogonal, that is, there exists a family of hypersurfaces possessing these vectors as normals. Thus, (5-2-35) is verified and one gets

$$\left(\frac{\tau_\mu}{\xi}\right)_{,\nu} - \left(\frac{\tau_\nu}{\xi}\right)_{,\mu} = 0 \qquad (5\text{-}2\text{-}37)$$

From this equation one gets

$$\tau_{\mu,\nu} - \tau_{\nu,\mu} = (\xi_{,\nu}\,\tau_\mu - \xi_{,\mu}\,\tau_\nu)\xi^{-1} \qquad (5\text{-}2\text{-}38)$$

We now prove that from (5-2-38) it follows that $\tau_{\mu,\nu} - \tau_{\nu,\mu}$ has a null projection onto the plane normal to τ^μ.

$$\mathcal{B}^\mu{}_\rho\,\mathcal{B}^\nu{}_\sigma(\tau_{\mu,\nu} - \tau_{\nu,\mu}) = 0 \qquad (5\text{-}2\text{-}39)$$

Indeed, a straightforward calculation using (5-2-16) and (5-2-38) shows that (5-2-39) is verified.

Thus, we can mathematically translate the fact that τ^μ is a timelike hypersurface orthogonal Killing field by means of the set of equations,

$$\tau^2 = \tau_\mu\,\tau^\mu > 0$$

$$\tau_{\mu;\nu} + \tau_{\nu;\mu} = 0$$

$$\mathcal{B}^\mu{}_\rho\,\mathcal{B}^\nu{}_\sigma\,(\tau_{\mu,\nu} - \tau_{\nu,\mu}) = 0$$

A gravitational field satisfying all such impositions is called a static gravitational field in general relativity. They are particular cases of stationary gravitational fields.

In the "canonical frame", where

$$\mathcal{B}^\mu{}_\rho \equiv \delta^\mu{}_\sigma - \ell^\mu\,\ell_\sigma = \delta^\mu{}_\sigma - \frac{\delta^\mu_o\,g_{\rho o}}{g_{oo}}$$

and $\tau_\mu = g_{\mu o}$, the (5-2-39) takes the form

$$\mathcal{B}^\mu{}_\rho \mathcal{B}^\nu{}_\lambda(\tau_{\mu,\nu}-\tau_{\nu,\mu}) = g_{\rho\sigma,\lambda}-g_{\lambda\sigma,\rho}+g_{oo,\rho}\frac{g_{\lambda o}}{g_{oo}} - g_{oo,\lambda}\frac{g_{\rho o}}{g_{oo}} = 0 \qquad (5\text{-}2\text{-}40)$$

where we have also taken under consideration that τ^μ is a Killing field, and thus satisfies $g_{\mu\nu,o} = 0$ in this coordinate system.

For $\lambda = 0$, and $\rho = 0$, the (5-2-40) is identically satisfied. For $\lambda = i$, we get

$$g_{\rho o,i} - g_{io,\rho} + g_{oo,\rho}\frac{g_{io}}{g_{oo}} - g_{oo,i}\frac{g_{\rho o}}{g_{oo}} = 0$$

again for $\rho = 0$ this is an identity, but for $\rho = k$, we obtain

$$g_{ko}(x) = 0 \qquad (5\text{-}2\text{-}41)$$

at all points of the manifold. Thus, the condition fixing the static field in the canonical frame is the equation (5-2-41).

As we will see when studying the solutions of the field equations, the knowledge of the existence of Killing fields onto the manifold will serve in an invariant fashion for characterizing the properties of symmetry of the manifold, and thus relating all found solutions, and eventually proving that several of such solutions are merely the restatement of a same solution in simply another form, that is in a different coordinatization. As an example of this, we have seen that a stationary field is characterized by the equations

$$\tau^2 = \tau_\mu \tau^\mu > 0$$

$$\tau_{\mu;\nu} + \tau_{\nu;\mu} = 0$$

in the canonical coordinate system they assume the form

$$g_{oo} > 0$$

$$g_{\mu\nu,o} = 0$$

But in other system of coordinates they can have a different form. If we were unaware of the fact that both metrics describe the same type of fields we might consider these two different forms as two different types of solutions of the field equations. Since these are non-linear differential equations, it is hard to see if this is true or not. Only the knowledge of the fact that both are consequence of the Killing equation shows us that indeed they are two different forms of the same geometry.

We finish this section with a table of the results obtained.

Type of field	Geometrical character.	Mathematical Condit.	Canonical Frame (signature -2)
Stationary	There exists a timelike Killing vector field.	$\tau^2 > 0$ $\tau_{(\mu;\nu)} = 0$	$g_{oo} > 0$ $g_{\mu\nu,o} = 0$
Static	There exists a hypersurface orthogonal timelike Killing vector field.	$\tau^2 > 0, \quad \tau_{(\mu;\nu)} = 0$ $\mathscr{B}^{\mu}{}_{\lambda}\, \mathscr{B}^{\nu}{}_{\sigma}\, \tau_{[\mu,\nu]} = 0$	$g_{oo} > 0$ $g_{\mu\nu,o} = 0$ $g_{ro} = 0$
General	There exists a timelike vector field.	$\tau^2 > 0$	$e_{rs} = g_{rs} - \dfrac{g_{os}\, g_{or}}{g_{oo}}$ $p_{\mu\nu} = \dfrac{g_{\mu o}\, g_{\nu o}}{g_{oo}}$

5.3) Interaction Between Gravitation and Other Physical System

According to our previous presentation, we have introduced in the Action principle a term giving account of the interactions of external systems with the gravitational field. They were obtained from their counterparts in special relativity by replacing everywhere the Minkowskian matrix $\overset{o}{g}_{\mu\nu}$ by the Riemannian metric tensor. The justification for doing so rests on the principle of minimal coupling.

By this principle the dynamical laws should reduce at each point to their special relativistic form by means of s suitable space-time mapping. If we did not invoke this principle but only the principle of general invariance, then there would be no reason for ruling out terms proportional to $R_{\mu\nu\rho\sigma}$, such as for instance $F^{\mu\nu} F^{\rho\sigma} R_{\mu\nu\rho\sigma}$ for the interaction with a electromagnetic field. In this case we should add to S_M a term

$$\int \sqrt{-g}\, d^4x\, F^{\mu\nu} F^{\rho\sigma} R_{\mu\nu\rho\sigma}$$

However, such types of interactions may generate higher order derivatives of $g_{\mu\nu}$ in the field equations for the gravitational field, this will turn these equations in a complicated form. All contributions to the study of interactions with the gravitational field in general relativity accept as a initial imposition that the principle of minimal coupling is verified, and thus, they start by postulating that all interactions are obtained by the above replacement. In what follows we give several examples of interactions.

5-3-1) Motion of Particles in a Gravitational Field

We start with the simplest type of interaction, that exists between a massive particle and a gravitational field, which initially is taken as an external field, that is, not generated by the massive body under study. Since however, the massive body is also a source of a gravitational field, and it acts on the sources of the original field, we get that the body will moves into the total field represented by the original field modified by the changes in its sources due to the presence of the body plus the own gravitational field generated by the body itself. The motion satisfies the equation of a geodesic of the total field.

$$\ddot{x}^{\mu} + \{^{\mu}_{\rho\sigma}\} \dot{x}^{\rho} \dot{x}^{\sigma} = 0 \qquad (5\text{-}3\text{-}1.1)$$

with the sypplementary condition

$$g_{\mu\nu}(x) \dot{x}^{\mu} \dot{x}^{\nu} = 1 \qquad (5\text{-}3\text{-}1.2)$$

Only in the limit where we can neglect the influence of the particle on the sources of the initial gravitational field can we subtract the effects of self-interaction, and thus treat the motion of the particle in a given field of gravitation. For instance, the motion of a small body in the gravitating field of a massive star.

Of special interest are the cases where the gravitational field, that is, the total field or the initial field depending on the structure of the approximation taken, possesses some symmetries, that is, there exists some Killing fields onto the manifold.

$$k_{\mu;\nu} + k_{\nu;\mu} = 0 \qquad (5\text{-}3\text{-}1.3)$$

where $k_{\mu} = g_{\mu\nu}k^{\nu}$. Then it is possible to show that $\dot{x}^{\mu} k_{\mu}$ is a constant of the motion along the trajectory of the particle.

$$\frac{d}{ds} (\dot{x}^{\mu} k_{\mu}) = \ddot{x}^{\mu} k_{\mu} + \dot{x}^{\mu} k_{\mu,\nu} \dot{x}^{\nu} = 0 \qquad (5\text{-}3\text{-}1.4)$$

as consequence of (5-3-1.1) and (5-3-1.3). If the Killing vector field is time-like, that is, if the field $g_{\mu\nu}$ is stationary, then the quantity $\mathfrak{E} = m \tau_{\mu} \dot{x}^{\mu}$ where τ_{μ} is the time-like Killing vector associated to the field, can be taken as the energy of the particle in analogy with the definition of this quantity in special relativity. The energy of the particle is then a constant of the motion.

We may re-express $\mathfrak{E} = m \tau_{\mu} \dot{x}^{\mu}$ in such way that brings out its similarity with the expression giving the energy of a charged particle in a constant electro magnetic field.

$$\mathfrak{E}' = \frac{m_o}{\sqrt{1 - \frac{\vec{x}^2}{c^2}}} + e\phi \qquad (5\text{-}3\text{-}1.5)$$

To this end we introduce the "normal" velocity of the particle as the change in the displacement normal to τ^μ relative to its displacement parallel to τ^μ.

$$u^\mu = \frac{\mathcal{B}^\mu{}_\nu \, dx^\nu}{\ell_\lambda \quad dx^\lambda}$$

However, before going on with this proof, let us turn back in order to show that (5-3-1.5) is constant of the motion for a charged particle in a constant electro magnetic field in special relativity. This will be done as an exercise.

Problem: Show that (5-3-1.5) is conserved for constant electromagnetic fields in special relativity.

Solution: Starting from the variational principle for a charged particle in an electromagnetic field in special relativity.

$$S = \int_{\lambda_1}^{\lambda_2} \left\{ -m_o \sqrt{\dot{z}_\mu \, \dot{z}^\mu} - e \, A_\mu(z) \, \dot{z}^\mu \right\} d\lambda$$

where $\dot{z}^\mu = \frac{dz^\mu}{d\lambda}$. We have

$$\sqrt{\dot{z}_\mu \, \dot{z}^\mu} = \sqrt{\overset{o}{g}_{\mu\nu} \, \dot{z}^\mu \, \dot{z}^\nu} = \frac{ds}{d\lambda}$$

Taking a variation in the coordinates z^ν of the particle, such that δz^ν vanish on the boundaries λ_1 and λ_2.

$$\delta S = \int_{\lambda_1}^{\lambda_2} \left\{ \frac{-m_o \, \dot{z}_\mu \, \delta \dot{z}^\mu}{\sqrt{\dot{z}_\nu \, \dot{z}^\nu}} - e \, A_\mu(z) \, \delta \dot{z}^\mu - e \, \dot{z}^\mu \, A_{\mu,\nu}(z) \, \delta z^\nu \right\} d\lambda$$

Integrating by parts the first two terms standing on the right hand side,

$$\delta S = \int_{\lambda_1}^{\lambda_2} \frac{d}{d\lambda} \left\{ \frac{m_o \, \dot{z}_\mu}{\sqrt{\dot{z}_\nu \dot{z}^\nu}} + e \, A_\mu(z) \right\} \delta z^\mu \, d\lambda - \int_{\lambda_1}^{\lambda_2} e \, \dot{z}^\mu \, A_{\mu,\nu}(z) \, \delta z^\nu \, d\lambda$$

This is a general relation, we now look for some variation under which the field $A_\mu(z)$ is symmetric. An important type of variation is a translation of the coordinates z^μ in the direction of a time-like unit vector n^μ,

$$\overset{o}{g}_{\mu\nu}\, n^\mu\, n^\nu = 1$$

$$\delta z^\mu = \varepsilon\, n^\mu$$

f we impose that the field $A_\mu(z)$ has the symmetry property

$$A_{\mu,\nu}(z)\; \delta z^\nu = \varepsilon\, A_{\mu,\nu} n^\nu = 0$$

Then

$$\delta S = \int_{\lambda_1}^{\lambda_2} \frac{d}{d\lambda} \left\{ \frac{m_o\, \dot{z}_\mu}{\sqrt{\dot{z}_\nu \dot{z}^\nu}} + e\, A_\mu(z) \right\} \varepsilon\;\; n^\mu\, d\lambda$$

and invariance under the translation along the time-like direction n^μ requires $\delta S = 0$

$$\int_{\lambda_1}^{\lambda_2} \frac{d}{d\lambda} \left\{ \frac{m_o\, \dot{z}_\mu}{\sqrt{\dot{z}_\nu\, \dot{z}^\nu}} + e\, A_\mu(z) \right\}\; n^\mu\, d\lambda = 0$$

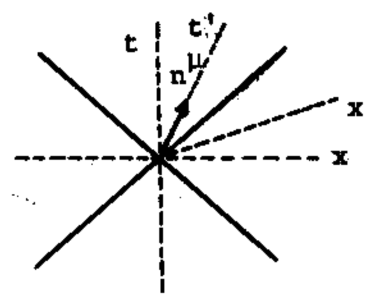
or

$$\left\{ \frac{m_o\, \dot{z}_\mu}{\sqrt{\dot{z}_\nu \dot{z}^\nu}} + e\, A_\mu(z) \right\}_{z_1(\lambda_1)} n_1^\mu = \left\{ \frac{m_o\, \dot{z}_\mu}{\sqrt{\dot{z}_\nu \dot{z}^\nu}} + e\, A_\mu(z) \right\}_{z_2(\lambda_2)} n_2^\mu$$

which means that this quantity is a constant of the motion along the trajectory of the particle. Since we can always take a reference system such that

$$n^\mu = \{1, 0, 0, 0\}\;,\;\; \delta z^\mu = \{\varepsilon, 0, 0, 0\}$$

which may be seen from the graph



we get that $\left\{\frac{m_o}{\sqrt{1-\dot{z}^2/c^2}} + e\phi\right\}$ is a constant along the motion of the particle. In this case the symmetry property of the field is simply the statement that the field is constant,

$$A_{\mu,\nu}\, n^\nu = A_{\mu,o} = 0$$

the conserved quantity $\left\{\frac{m_o}{\sqrt{1-\dot{z}^2/c^2}} + e\phi\right\}$ is the total energy of the particle in this field. Note that the quantity

$$\left\{\frac{m_o\ \dot{z}_\mu}{\sqrt{\dot{z}_\nu\ \dot{z}^\nu}} + e\ A_\mu(z)\right\} n^\mu$$

may be written as

$$\{ m\ \dot{z}_\mu + e\ A_\mu(z)\}\ n^\mu$$

which is similar to our previous formula $\mathcal{E} = m\dot{x}^\mu \tau_\mu$ which we called by energy. As we saw, this quantity is constant along the trajectory of the particle, for stationary fields. For electromagnetic fields it reduces to the total energy of the particle in the "rest frame" of the time-like vector n^μ.

Turning back to (5-3-1.6), the dx^μ means the displacement along the path of the particle. For $\tau^\mu = \{1,\vec{0}\}$ the u^μ take the value

$$u^r = \frac{dx^r}{\sqrt{h}\,(dx^o + g_s\ dx^s)}, \qquad u^o = \frac{-g_s\quad dx^s}{\sqrt{h}\,(dx^o + g_r\ dx^r)} \tag{5-3-1.7}$$

with

$$h = g_{oo}$$

$$g_r = \frac{g_{ro}}{g_{oo}}$$

Thus, we may write

$$u^\mu = \{u^o,\ \vec{u}\ \} = \{-\vec{g}.\vec{u},\ \vec{u}\ \} \tag{5-3-1.8}$$

The actual velocity of the particle is $\dot{x}^\mu$, which from (5-3-1.6) has the value

$$\dot{x}^\mu = \frac{1}{\sqrt{1-u^2}} (u^\mu + \ell^\mu) \qquad (5\text{-}3\text{-}1.9)$$

where

$$u^2 = e_{\mu\nu}\, u^\mu\, u^\nu = g_{\mu\nu}\, u^\mu\, u^\nu, \quad \ell_\mu\, u^\mu = 0$$

In the reference system where $\tau^\mu = \{1,\vec{0}\}$ we get

$$\dot{x}^\mu = \frac{1}{\sqrt{1-u^2}} \left\{ -\vec{g}.\vec{u} + \frac{1}{\sqrt{h}}, \vec{u} \right\} \qquad (5\text{-}3\text{-}1.10)$$

By comparison of these components with the special relativistic components for the four-velocity,

$$\frac{dx^\mu}{ds} = \left\{ \frac{\vec{u}}{\sqrt{1-\vec{u}^2}}, \frac{1}{\sqrt{1-\vec{u}^2}} \right\}$$

where here $\vec{u}$ denotes the three-dimensional velocity, $\vec{u} = \frac{d\vec{x}}{dt}$, and we put c=1. We therefore see that the $\vec{u}$ of (5-3-1.7),

$$\vec{u} = \frac{\left(\frac{d\vec{x}}{dt}\right)}{\sqrt{h}\left(1 + \vec{g} \cdot \frac{d\vec{x}}{dt}\right)}$$

is now the analogue of the particle three-velocity. For $\mathcal{E}$ we get, using the relation (5-3-1.9)., and $\tau_\mu\, u^\mu = 0$

$$\mathcal{E} = \frac{m}{\sqrt{1-u^2}} \sqrt{g_{\mu\nu}\, \tau^\mu\, \tau^\nu} \qquad (5\text{-}3\text{-}1.11)$$

For $\tau^\mu = \{1,\vec{0}\}$ we have

$$\mathcal{E} = \frac{m\sqrt{h}}{\sqrt{1-u^2}} \qquad (5\text{-}3\text{-}1.12)$$

In the next section we will see that in the limit of a weak gravitational field

$$g_{oo} \approx 1 + 2\,\phi$$

where ϕ is the Newtonian potential of this field. Hence, in this limit,

$$\mathcal{E} \approx \frac{m}{\sqrt{1-u^2}} + \frac{m\phi}{\sqrt{1-u^2}} \qquad (5\text{-}3\text{-}1.13)$$

Which is the gravitational analogue of the relation

$$\mathcal{E} = \frac{m}{\sqrt{1-u^2}} + e\,\phi$$

holding for a charged particle in an electromagnetic field in special relativity. Finally, replacing m, u^2 and ϕ by mc^2, u^2/c^2 and ϕ/c^2 and letting $c \to \infty$ we obtain

$$\mathcal{E} \approx mc^2 + \frac{1}{2}\,mu^2 + m\phi$$

which asides from the rest-energy term mc^2 is the usual Newtonian expression for the energy of a particle in a gravitational field.

5-3-2) The Interaction of Electromagnetic and Gravitational Fields

The dynamical laws for an electromagnetic field in interaction with a gravitational field can be obtained from (5-1-16) by variation on the A_μ. The Action integral (5-1-16) may be re-expressed so as to allow for a variational principle of Palatini similar to that used for the gravitational field previously. We put

$$S = -\frac{1}{8\pi}\int d^4x\,\sqrt{-g}\,g^{\mu\rho}\,g^{\nu\sigma}\left\{\frac{1}{2}\,F_{\mu\nu}\,F_{\rho\sigma} - F_{\mu\nu}(A_{\sigma,\rho} - A_{\rho,\sigma})\right\} \qquad (5\text{-}3\text{-}2.1)$$

and take independent variations on the $F_{\mu\nu}$ and the A_μ. The resulting field equations being

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \qquad (5\text{-}3\text{-}2.2)$$

$$(\sqrt{-g}\,F^{\mu\nu})_{,\nu} = 0 \quad . \qquad (5\text{-}3\text{-}2.3)$$

The former equations being just the definition of $F_{\mu\nu}$ in terms of A_μ. The second set of equations are the Maxwell equations in presence of gravitational fields in general relativity.

If in addition there is a source term

$$-\int A_\mu(x)\; j^\mu(x)\; d^4x$$

in the Action functional, ~~the equations (5-3-2.3) become~~

$$(\sqrt{-g}\; F^{\mu\nu})_{,\nu} = -\,4\pi j^\mu \qquad (5\text{-}3\text{-}2.4)$$

and the (5-3-2.2) keep unchanged. From (5-3-2.4) we see that $(\sqrt{-g}\; F^{\mu\nu})$ is a skew-symmetric second rank tensor density with weight W = +1, this implies that j^μ is a vector density with the same weight. Thus, the divergence of j^μ is a scalar density of weight +1. This may also be seen from the integral $\int d_4x\, A^\mu\, j_\mu$ which being a scalar implies that $A^\mu\, j_\mu\, d_4x$ is a scalar and thus j^μ is a vector density with weight +1. Computing the divergence of j^μ from (5-3-2.4) we get

$$j^\mu_{\;,\mu} = 0 \qquad (5\text{-}3\text{-}2.5)$$

Which is the law of conservation of charges and currents for the system. This equation is covariant since the divergence of a vector density of weight +1 coincides with its covariant divergence.

The j^μ has the form

$$j^\mu(x) = \sum_i\; e_i \int d\lambda_i\;\; \delta_4(x-z_i)\,\dot{z}_i^\mu \qquad (5\text{-}3\text{-}2.6)$$

that is, we have a system of point charges moving with four-velocity $\dot{z}_i^\mu$ into the region where there exists the gravitational and electromagnetic fields. These particles in turn are sources for the gravitational field, with an integral

$$\mathcal{J}_1 = -\sum_i m_i \int d_4x \int d\lambda_i\;\; \delta_4(x-z_i)\;\sqrt{g_{\mu\nu}(x)\,\dot{z}_i^\mu\; \dot{z}_i^\nu} \qquad (5\text{-}3\text{-}2.7)$$

Therefore, the full Action integral involving the particle variables has the form

$$\mathcal{J} = - \sum_i m_i \int d_4x \int d\lambda_i \ \delta_4(x-z_i) \sqrt{g_{\mu\nu}(x) \dot{z}_i^{\mu} \dot{z}_i^{\nu}} + \sum_i e_i \int d_4x \int d\lambda_i \ \delta_4(x-z_i) A_{\mu}(x) \dot{z}_i^{\mu} \qquad (5\text{-}3\text{-}2.8)$$

Variating the coordinate of the i-th particle, the z_i, we get the equations of motion

$$\ddot{z}_i^{\mu} + \{^{\mu}_{\rho\sigma}\} \dot{z}_i^{\rho} \ \dot{z}_i^{\sigma} = \frac{e_i}{m_i} F_{\nu}{}^{\mu} \ \dot{z}_i^{\nu} \qquad (5\text{-}3\text{-}2.9)$$

where at the right hand side we have the Lorentz four-force. This equation is a generalization of the usual geodesic equation, which holds for pure gravitational fields. Note that the electromagnetic field is treated separately of the gravitational field in the sense that the later is the geometry of the space, as is seen from the first integral in (5-3-2.8), whereas the former is introduced a posteriori by postulating the principle of minimal coupling with the system of particles, which gives rise to the second integral in (5-3-2.8). In a complete unitary field theory for gravitation and electromagnetism both fields should appear as the geometry of the space, as for instance in Kaluza-Klein's theory [35] where (5-3-2.9) is obtained from a geodesic at a five-dimensional manifold, the geometry of this manifold is determined from both fields.

Summarizing, the full Action principle for gravitational plus electromagnetic fields and a system of point charges, including all interactions has the form

$$I = \frac{-1}{2K} \int \sqrt{-g} \ d_4x (R-2\Lambda) - \frac{1}{8\pi} \int \sqrt{-g} \ d_4x \ g^{\mu\rho} \ g^{\nu\sigma} \left\{ \frac{1}{2} F_{\mu\nu} F_{\rho\sigma} - \right.$$

$$\left. - F_{\mu\nu}(A_{\sigma,\rho} - A_{\rho,\sigma}) \right\} - \sum_i m_i \int d_4x \int d\lambda_i \ \delta_4(x-z_i) \sqrt{g_{\mu\nu}(x) \ \dot{z}_i^{\mu} \ \dot{z}_i^{\nu}} +$$

$$+ \sum e_i \int d_4x \int d\lambda_i \ \delta_4(x-z_i) \ A_{\mu}(x) \ \dot{z}_i^{\mu} .$$

5.3.3) The Dirac Equation in a Gravitational Field

From the special relativistic formalism we know that the Dirac wave function transforms as a spinor representation of the Poincaré group. However, there is no similar result for the general transformations of the Riemannian geometry of general relativity. It is possible to write a general relativistic wave equation which in many aspects resembles the Dirac equation, but which nevertheless differs fundamentally from the Dirac equation in special relativity. For understanding the nature of this difference it will be instructive to discuss the invariance of the Dirac equation in special relativity.

The Dirac equation

$$\gamma^\mu \frac{\partial\Psi}{\partial x^\mu} - m\Psi = 0 , \qquad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\, g^{\mu\nu} \tag{5-3-3.1}$$

will be form invariant under a Lorentz transformation $x'^\mu = L^\mu_{\ \nu}\, x^\nu$, if in the new inertial frame we get

$$\gamma'^\mu \frac{\partial\Psi'}{\partial x'^\mu} - m\Psi' = 0$$

In order to determine the transformation law for Ψ and γ^μ which gives rise to this form invariance, we write explicitly also the spinor indices. The equation (5-3-3.1) takes the form

$$\gamma^\mu_{ij} \frac{\partial\Psi_j}{\partial x^\mu} - m\, \Psi_i = 0$$

We now show that the most general transformation for Ψ_i is of the form

$$\Psi'_i(x) = S_{ij}\, \Psi_j(x) \tag{5-3-3.2}$$

with a constant unitary four-by-four matrix S. The unitarity of S is imposed for conserving the expression for the density of probability $\Psi^*_i\, \Psi_i$ under the above transformation. We then have for (5-3-3.2) in the new inertial coordinate system

$$\gamma'^{\mu}_{ij} \frac{\partial \Psi'_j}{\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} - m \Psi'_i = 0$$

or

$$\gamma'^{\mu}_{ij} L^{-1\,\sigma}{}_{\mu} S_{jk} \frac{\partial \Psi_k}{\partial x^{\sigma}} - m S_{ik} \Psi_k = 0$$

this equation will be form invariant if

$$\gamma'^{\mu}_{ij} L^{-1\,\sigma}{}_{\mu} = S_{im} \gamma^{\sigma}_{m\ell} S^{+}_{\ell j} \qquad (5\text{-}3\text{-}3.3)$$

Thus, the relativistic invariance of the Dirac equation under the Lorentz group is achieved when Ψ and γ^{μ} transform according to (5-3-3.2) and (5-3-3.3). It is simple to verify that the anticommutation relation for the γ^{μ} is also form invariant when γ^{μ} transform according to (5-3-3.3).

From (5-3-3.3) we obtain

$$\gamma'^{\mu}_{ij} = S_{ik} \gamma^{\sigma}_{k\ell} S^{+}_{\ell j} L^{\mu}{}_{\sigma} \qquad (5\text{-}3\text{-}3.4)$$

Taking an infinitesimal transformation $x'^{\mu} = x^{\mu} + \varepsilon^{\mu}{}_{\nu} x^{\nu}$ and $S = e^{i\varepsilon_A \Lambda_A} =$ $= 1 + i\,\varepsilon_A \Lambda_A$ we get for these relations.

$$\delta\Psi_i(x) \equiv \Psi'_i(x) - \Psi_i(x) = i\,\varepsilon^{A} \Lambda_{A,ij} \Psi_j(x) \qquad (5\text{-}3\text{-}3.5)$$

$$\delta\gamma^{\mu}_{ij} \equiv \gamma'^{\mu}_{ij} - \gamma^{\mu}_{ij} = \varepsilon^{\mu}{}_{\sigma} \gamma^{\sigma}_{ij} + i\varepsilon^{A}(\Lambda_{A,ik} \gamma^{\mu}_{kj} - \Lambda_{A,kj} \gamma^{\mu}_{ik}) \qquad (5\text{-}3\text{-}3.6)$$

Now, the transformation law (5-3-3.2) for $\Psi_i(x)$ is such that S is a function of the Lorentz matrix $L^{\mu}{}_{\nu}$, which amounts to say that Ψ is a spinor representation of the Lorentz group. For infinitesimal transformations, it follows that $\varepsilon^{A} \Lambda_{A}$ is a function of the $\varepsilon_{\mu\nu}$. Since both ε_A and $\varepsilon_{\mu\nu}$ are of the first order, the only possible way for satisfying this is to put $\varepsilon^{A} = \varepsilon^{\mu\nu}$. Then

$$S = 1 + i\,\varepsilon^{\mu\nu} \Lambda_{\mu\nu}\,, \quad \Lambda^{\dagger}_{\mu\nu} = \Lambda_{\mu\nu}$$

Therefore, the operator $\Lambda_{\mu\nu}$ which acts on the Ψ is not a function of the Lorentz parameters $\varepsilon_{\alpha\beta}$, and the only way for geting an explicit form for the $\Lambda_{\mu\nu}$ is to take them as dependent on the γ_{α}. The explicit form for the $\Lambda_{\mu\nu}$ in terms of the γ_α is obtained by requiring that the γ^μ have the same form in all Lorentz frames, $\delta\gamma^\mu = 0$. Imposing this condition into (5-3-3.6) we get

$$\varepsilon^\mu{}_\sigma \gamma^\sigma_{ij} + i\,\varepsilon^{\beta\nu} (\Lambda_{\beta\nu,ik}\,\gamma^\mu_{kj} - \gamma^\mu_{ik}\,\Lambda_{\beta\nu,kj}) = 0 \qquad (5\text{-}3\text{-}3.7)$$

The vector space of all four-by-four matrices is spanned by the set of sixteeen basis matrices 1, γ^μ, $\gamma^5\gamma^\mu$, $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$ and $\frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Thus

$$\Lambda_{\beta\nu,ij} = a_{\beta\nu}\,\delta_{ij} + b_{\mu,\beta\nu}\,\gamma^\mu_{ij} + e_{\mu,\beta\nu}(\gamma^5\gamma^\mu)_{ij} + e_{\beta\nu}\,\gamma^5{}_{ij} + f_{\lambda\sigma,\beta\nu}\,\frac{i}{2}\left[\gamma^\lambda,\gamma^\sigma\right]_{ij}$$

Substituting this linear combination into (5-3-3.7) we obtain

$$(\varepsilon^\mu{}_\lambda - 2\,\varepsilon_{\beta\nu} f^{\beta\nu\varrho\mu\sigma}_{\lambda\sigma} g + 2\,\varepsilon_{\beta\nu} f^{\beta\nu}_{\sigma\lambda}\, g^{\varrho\mu\sigma})\gamma^\lambda + 2\,\varepsilon_{\beta\nu} b^{\beta\nu}_\lambda\,\frac{i}{2}\left[\gamma^\lambda, \gamma^\mu\right] +$$

$$+ 2\,i\,\varepsilon_{\beta\nu}\;c^{\beta\nu,\mu}\,\gamma^5 + 2\,i\,\varepsilon_{\beta\nu}\,e^{\beta\nu}\,\gamma^5\gamma^\mu = \theta \qquad (5\text{-}3\text{-}3.8)$$

where use have been made of the commutation relation

$$\gamma^\lambda\,\sigma^{\rho\alpha} - \sigma^{\rho\alpha}\,\gamma^\lambda = 2i(g^{\varrho\lambda\rho}\,\gamma^\alpha - g^{\varrho\lambda\alpha}\,\gamma^\rho)\,, \sigma^{\rho\alpha} = \frac{i}{2}\left[\gamma^\rho, \gamma^\alpha\right]$$

The equation (5-3-3.8) involves only linearly independent matrices, therefore each one of the several coefficients must be put equal to zero,

$$\varepsilon_{\beta\nu}\, b^{\beta\nu}_\lambda = 0$$

$$\varepsilon_{\beta\nu}\, c^{\beta\nu,\mu} = 0$$

$$\varepsilon_{\beta\nu}\, e^{\beta\nu} = 0$$

$$\varepsilon^\mu{}_\lambda - 2\,\varepsilon_{\beta\nu}\, f^{\beta\nu}_{\lambda\sigma}\, g^{\varrho\mu\sigma} + 2\,\varepsilon_{\beta\nu}\, f^{\beta\nu}_{\sigma\lambda}\, g^{\varrho\mu\sigma} = 0$$

the last of those relations is now used for obtaining the value of $f^{\beta\nu}_{\lambda\sigma}$, we find

$$f^{\beta\nu}_{[\sigma\lambda]} = \frac{1}{4}\,(\delta^\beta_\sigma\,\delta^\nu_\lambda - \delta^\beta_\lambda\,\delta^\nu_\sigma)$$

Hence,

$$\Lambda_{\beta\nu} = a_{\beta\nu}.1 + f_{\beta\nu}^{[\sigma\lambda]}\,\sigma_{\sigma\lambda} = \Lambda_{\beta\nu}^{\dagger}$$

With this $\Lambda_{\beta\nu}$ we calculate the matrix S by using that $S = 1 + \varepsilon^{\mu\nu}\,\Lambda_{\mu\nu}$. For the term involving the four-by-four identity matrix we will get a term $\varepsilon^{\mu\nu}\,a_{\mu\nu}$, so as the $a_{\mu\nu}$ has to be antisymmetric. By the other hand, the $a_{\mu\nu}$ cannot depend on the $\varepsilon_{\mu\nu}$ in this first order approximation, it cannot depend too on the γ_μ matrices, in other terms, it has to be a number. Since no such number is available, we have to put $a_{\mu\nu} = 0$, thus getting

$$\Lambda_{\mu\nu} = f_{\mu\nu}^{[\sigma\alpha]}\sigma_{\sigma\alpha} = \frac{i}{4}\left[\gamma_\mu,\ \gamma_\nu\right] \qquad (5\text{-}3\text{-}3.9)$$

which completes our proof, the Dirac spinor wave function transforms as

$$\delta\Psi(x) = i\ \varepsilon^{\mu\nu}\Lambda_{\mu\nu}\ \Psi = -\ \frac{1}{4}\ \varepsilon^{\mu\nu}\left[\gamma_\mu,\gamma_\nu\right]\Psi(x) \qquad (5\text{-}3\text{-}3.10)$$

under a Lorentz transformation.

For writing a Dirac type of equation in general relativity we generalize the anticommutation relation for the γ_μ matrices as was done before for the Pauli matrices.

$$\gamma_\mu\ \gamma_\nu + \gamma_\nu\ \gamma_\mu = 2\ g_{\mu\nu}(x) \qquad (5\text{-}3\text{-}3.10)$$

where now the γ_μ are dependent on the coordinates. The Ψ still transforms by (5-3-3.9) but with point dependent matrices. In the tetrad method the $\gamma_\mu(x)$ is given in function of the γ_μ of Dirac by

$$\gamma_\mu(x) = h_\mu{}^{\alpha}(x)\ \gamma_{(\alpha)}$$

Therefore, $\Psi_{,\mu}$ will not transform as Ψ under an internal mapping with a transformation matrix $S = S(x)$. Using the tetrad formalism we write the transformation of Ψ as

$$\delta\Psi(x) = \frac{1}{2}\,\varepsilon^{(\alpha)(\beta)}(x)\sigma_{(\alpha)(\beta)}\Psi(x) \qquad (5\text{-}3\text{-}3.11)$$

$$\sigma_{(\alpha)(\beta)} = \frac{i}{2}\left[\gamma_{(\alpha)}, \gamma_{(\beta)}\right]$$

where

$$\varepsilon^{(\alpha)(\beta)}(x) = \varepsilon^{\mu\nu}\, h_{\mu}^{(\alpha)} \quad h_{\nu}^{(\beta)}$$

and the $\sigma_{(\alpha)(\beta)}$ are usual matrices of special relativity, and thus are independent of the coordinates. A quantity transforming as Ψ under an internal mapping is the covariant derivative of Ψ,

$$\Psi_{;\mu} = \Psi_{,\mu} + \Gamma_{\mu}\,\Psi$$

that is,

$$\delta\Psi_{;\mu}(x) = \frac{1}{2}\,\varepsilon^{(\alpha)(\beta)}(x)\,\sigma_{(\alpha)(\beta)} \quad \Psi_{;\mu}(x)$$

This implies, similarly to what we saw before that Γ_{μ}, the internal affinity, transforms under internal mappings as

$$\delta\Gamma_{\mu} = \frac{1}{2}\,\varepsilon^{(\alpha)(\beta)}(x)\left[\sigma_{(\alpha)(\beta)}\Gamma_{\mu} - \Gamma_{\mu}\,\sigma_{(\alpha)(\beta)}\right] - \frac{1}{2}\,\varepsilon^{(\alpha)(\beta)}_{\;\;,\mu}\;\sigma_{(\alpha)(\beta)} \qquad (5\text{-}3\text{-}3.12)$$

Now, it is simple to verify that the γ_{μ} matrices may be written in terms of the σ_{μ} matrices of the preceding sections by

$$\gamma_{\mu} = \begin{pmatrix} 0 & \sigma_{\mu} \\ \sigma_{\mu} & 0 \end{pmatrix}$$

since

$$\gamma_{\mu}\,\gamma_{\nu} + \gamma_{\nu}\,\gamma_{\mu} = \begin{pmatrix} \sigma_{\nu}\,\sigma_{\mu} + \sigma_{\mu}\,\sigma_{\nu} & 0 \\ 0 & \sigma_{\nu}\,\sigma_{\mu} + \sigma_{\mu}\,\sigma_{\nu} \end{pmatrix} = 2\,g_{\mu\nu}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Further, the Dirac equation (5-3-3.1) may be decomposed into two equations for the spinors ϕ_A and χ^A by means of

$$\psi = \begin{pmatrix} \phi_A \\ \chi^{\dot{A}} \end{pmatrix} \quad , \quad \gamma_{\mu} = \begin{pmatrix} 0 & \sigma^{\mu}{}_{\dot{A}B} \\ \sigma^{\mu A\dot{B}} & 0 \end{pmatrix}$$

we get

$$\sigma^{\mu}_{A\dot{B}}\ \chi^{\dot{A}}_{\ ,\mu} = m\ \phi_B \qquad (5\text{-}3\text{-}3.13)$$

$$\sigma^{\mu A\dot{B}}\ \phi_{A,\mu} = m\ \chi^{\dot{B}} \qquad (5\text{-}3\text{-}3.14)$$

Replacing the derivatives by covariant derivatives with affinity $\Delta^{A}_{\mu\ B}$ (a different symbol is used for the case of two-spinors, we will get a relation between the Δ_μ and the previous Γ_μ), we obtain

$$\sigma^{\mu}_{A\dot{B}}(x)\,(\chi^{\dot{A}}_{\ ,\mu} + \Delta^{\dot{A}}_{\mu\ \dot{B}}\ \chi^{\dot{B}}) = m\ \phi_B \qquad (5\text{-}3\text{-}3.15)$$

$$\sigma^{\mu A\dot{B}}(x)\,(\phi_{A,\mu} - \Delta^{B}_{\mu\ A}\ \phi_B) = m\ \chi^{\dot{B}} \qquad (5\text{-}3\text{-}3.16)$$

As we know, the value for the Δ_μ is

$$\Delta^{A}_{\mu\ B} = -\frac{1}{4}\,\sigma_{\beta B\dot{R}}\left\{\sigma^{\beta\dot{R}A}_{\ \ ,\mu} + \Gamma^{\beta}_{\alpha\mu}\ \sigma^{\alpha\dot{R}A}\right\} \qquad (5\text{-}3\text{-}3.17)$$

If we write the equations (5-3-3.15) and (5-3-3.16) in the notation of four-by-four matrices as

$$\gamma^{\mu}(\Psi_{,\mu} + \Gamma_\mu\ \ \Psi) = m\Psi \qquad (5\text{-}3\text{-}3.18)$$

we will find

$$\Gamma_\mu = \begin{pmatrix} -\Delta_\mu & 0 \\ 0 & \Delta_\mu \end{pmatrix}, \qquad = (\Delta^{A}_{\mu\ B})$$

which is the relation between the affinities on both notations. A direct calculation permits to rewrite this equation also as

$$\Gamma_\mu = -\frac{1}{4}\ \gamma_\alpha(\gamma^{\alpha}_{\ ,\mu} + \Gamma^{\alpha}_{\beta\mu}\ \gamma^{\beta}) \qquad (5\text{-}3\text{-}3.19)$$

where we used the formula (5-3-3.17) along with the formula for the γ_μ in terms of the σ_μ. This later formula may also be presented in terms of the tetrad as

$$\Gamma_\mu = -\frac{1}{4}\,\gamma^{(\lambda)}\ \gamma^{(\rho)}\ h_{\alpha(\lambda)}\ (h^{\alpha}_{(\rho),\mu} + \Gamma^{\alpha}_{\beta\mu}\ h^{\beta}_{(\rho)})$$

Thus, in spite of being possible to write a Dirac type of equation, covariant under the full MMG plus internal group, this equation differs from the Dirac equation in special relativity since there is anymore no relation between the internal mappings with the MMG as was the case in special relativity. The most we have now is a local relationship between these groups.

5.4) Non-Relativistic Limit of the Einstein Theory

The basic structure of the metric $g_{\mu\nu}$ as describing the gravitational field in general relativity is determined by imposing that the field equations of this theory as well as the equation of motion for all the bodies in this field have the correct non-relativistic limit, in the same sense that for instance quantum mechanics degenerates in the classical mechanics in the adequate limit. The limit for transition to the non-relativistic region holds good for some types of gravitational fields, the weak fields, or even for strong fields as far as we move far away from its sources. Doing such type of approximation of the Einstein's equations, we will obtain just the linearization of these equations. As it turns out clear, if we do just this approximation we are not obtaining a non-relativistic limit since we obtain a linear field equation similar in structure to all other theories of gravitation in special relativity. The correct non-relativistic limit is obtained when we further impose that the velocity of all bodies are much smaller than the speed of light. We recall that in the theory of general relativity all bodies are source of fields one acting on all other through non-linear interactions. If we impose slow speeds on these bodies we are imposing on the generated field a limitation, this limitation is indeed just the imposition that the static part of the field will be the largest one. All other parts due to the relative motion of the bodies will be much smaller than the static field.

Having doing this, we interpret the resulting static potential as the Newtonian potential.

The linear approximation for weak gravitational fields is obtained by writing

$$g_{\mu\nu}(x) = \overset{0}{g}_{\mu\nu} + \phi_{\mu\nu}(x) \ , \qquad \det\ (\phi_{\mu\nu}) << 1$$

consider the motion of a small mass (in order to maintain the weak field approximation) in this field. Then, we can neglect all effects of this mass on the sources of the weak field. This allows us to take the geodesic equation as the dynamical law governing the motion of this mass in the given weak gravitational field. This equation takes the form

$$\frac{d^2 z^i}{ds^2} + \frac{1}{2} \overset{0}{g}{}^{\mu\lambda} \left(\phi_{\lambda\nu,\alpha} + \phi_{\lambda\alpha,\nu} - \phi_{\nu\alpha,\lambda}\right) \frac{dz^\nu}{ds} \frac{dz^\alpha}{ds} = 0 \qquad (5\text{-}4\text{-}1)$$

Taking the approximation of a static gravitational field, which for our present limit of weak fields take the form

$$\phi_{\mu\nu,o} = 0 \ , \qquad \phi_{oi} = 0$$

we will get for (5-4-1) for $\mu = i$,

$$\frac{d^2 z^i}{dt^2} - \frac{1}{2} \left(\phi_{ik,\ell} + \phi_{i\ell,k} - \phi_{k\ell,i}\right) \frac{dz^k}{dt} \frac{dz^\ell}{dt} + \phi_{oo,i} \frac{c}{2} = 0$$

imposing that $\left| \frac{dz^i}{dt} \right| << c$, we can neglect the medium term, and obtain

$$\frac{d^2 z^i}{dt^2} = - \frac{c^2}{2} \frac{\partial}{\partial x^i} \phi_{oo}$$

Which is the Newtonian equation of motion for the potential

$$\phi = \frac{c^2}{2} \phi_{oo}$$

in the above derivation we have taken the metric $\overset{0}{g}_{\mu\nu}$ with signature -2, so that $\overset{0}{g}_{ij} = - \delta_{ij}$, $\overset{0}{g}_{oo} = 1$

$$g_{oo} = 1 + \phi_{oo} = 1 + \frac{2\phi}{c^2} \tag{5-4-2}$$

In the static approximation $g_{oi} = 0$, and we still have g_{ij} which is smaller in magnitude than g_{oo}. The largest part of a weak static gravitational field is g_{oo}.

The equation obtained by putting $\mu = 0$ in (5-4-1) is the law of the conservation of total energy of the particle in this Newtonian field,

$$E = 1/2\ m\ (dz^i/dt)^2 + m\phi = \text{constant}$$

For completing the discussion of the non-relativistic limit, we have to prove that the Einstein equations degenerate in the Poisson equation for the potential $\frac{c^2}{2}\ \phi_{oo} = \phi$. Since the Newtonian theory assumes that only massive bodies are sources of the gravitational field, we have to use in the right hand side of Einstein's equations the stress energy tensor

$$T^{\mu\nu}(x) = c \sum_i m_i \int \delta^4(x - z_i(\lambda_i))\ \frac{\dot{z}_i^{\mu}\ \dot{z}_i^{\nu}}{\sqrt{\dot{z}_i^2}}\ d\lambda_i$$

This expression is the Minkowskian stress energy tensor. It comes similarly to (5-1-21) but supressing the term $\sqrt{-g}$ in S_m. For our purposes it is sufficient to take this tensor instead of (5-1-21). In the limit of slow velocities the preponderant part is T^{oo}, with value

$$T^{oo}(x) = c \sum_i m_i \int \delta^4(x - z_i)\ \frac{c^2}{c\sqrt{1 - \frac{\dot{z}_i^2}{c^2}}}\ d\lambda_i \tag{5-4-3}$$

or

$$T^{oo}(x) \approx c^2 \sum_i m_i \int \delta^4(x - z_i)\ d\lambda_i = c \sum_i m_i\ \delta^3(x - z_i) = c^2\ \rho(\vec{x}) \tag{5-4-4}$$

Multiplying the Einstein tensor $G_{\mu\nu}$ by $g^{\mu\nu}$ we get $-R$, therefore the field equations may be written also as

$$R = -k\ g_{\mu\nu}\ T^{\mu\nu} \approx -k\ T^{oo} \tag{5-4-5}$$

But since $R = g_{\mu\nu} R^{\mu\nu}$, we get in this approximation

$$R \approx R^{oo} \approx - k\ T^{oo}\ , \qquad R^{oo} = \frac{1}{2} k\ c^2 \rho(\vec{x}) \tag{5-4-6}$$

all other components of the Ricci tensor, R^{ij} and R^{oi}, will depend on the velocities of the source particles and thus are of small magnitude and are neglected.

Computing R^{oo} to first order terms in the deviation $\phi_{\mu\nu}(\vec{x})$, and imposing that the field is static, $\phi_{\mu\nu,o} = 0$ and $\phi_{oi} = 0$, we obtain after some calculations

$$R^{oo} \cong \{ {r \atop oo} \}_{,r} = - \frac{1}{c^2} \nabla^2 \phi \tag{5-4-7}$$

Exercise: Prove the equation (5-4-7).

So that Eq. (5-4-6) becomes

$$\nabla^2 \phi = - \frac{1}{2} k\ c^4\ \rho(\vec{x}) \tag{5-4-8}$$

Therefore the constant k takes the value, in terms of the gravitational constant G,

$$k = - \frac{8\pi G}{c^4} \tag{5-4-9}$$

and thus, for a point mass $\rho(\vec{x}) = m\ \delta^3(\vec{x})$

$$\phi = - \frac{Gm}{r} = \frac{kc^4}{8\pi} \frac{m}{r}$$

and g_{oo} is

$$g_{oo} = 1 + \frac{kc^2}{4\pi} \frac{m}{r} = 1 - \frac{2Gm}{c^2 r} \tag{5-4-10}$$

all other g's being of second order in this approximation.

5.5) Structure of the Einstein Equations

Presently we begin to discuss the necessary topics for application in the initial value problem, the Cauchy-Kowalewski problem, to be treated in the following section. The matter then covered will be of importance in our future presentation of the Hamiltonian formulation of general relativity. Thus, the importance of the present discussion may be outlined through two different points. First, it has importance from the point of view of a quantization problem which starts from the classical Hamiltonian formulation, as is the case for all spin integers fields. Second, the present discussion will also bear importance in the discussion of conservation laws for the field, since it treats directly with the symmetry properties of the system which leds to the statement of a correct problem of initial data for the field. In rather general terms, what we will cover presently is a general problem of structure of the Einstein tensor, which comes from the application of the second Noether theorem [36] to this theory.

Consider a system described by the functions $y_A(x)$, $A = 1\ldots N$, by means of the Lagrangian density

$$L(x;y_A(x),\ y_{A,\rho}(x)) = L(x,y(x))$$

$$x = (x\ \ldots\ x^n), \qquad \rho = 1\ldots\ n$$

$$W = \int_\Omega L\ dx\ .$$

Consider the transformations

$$y'_A(x') = f_A(x;y)$$
$$x'^\rho = f^\rho(x)\ .$$

which are assumed to be continuous with the identity transformation. These transformations defined on the Y and X spaces may be, or may be not, correlated one to the other.

The infinitesimal form of these transformations will be written as

$$y'_A(x') = y_A(x) + \delta y_A(x)\ ,$$
$$x'^\rho = x^\rho + \delta x^\rho(x)\ .$$

Such transformation groups may be divided into two classes, the first depending on a finite number of parameters: $\{G_p\}$ (for p parameters). The other depending on a set of given functions $\{G_{\infty q}\}$, this notation was used since a set of functions, in the case q functions, depend on a infinite number of parameters.

What the Noether's theorems say is: The invariance of W under G_p gives the conservation of a set of p functionals, $\{\mathcal{S}_p\}$, of $y \in Y$. In every case this set may be identified to some physical variable of the system. Second, the invariance of W under $\{G_{\infty q}\}$ amounts to a set $\{I_q\}$ of identities which involve the field functions $y \in Y$ on each fixed point $x \in X$. This later situation is what is called the second Noether theorem. We will restrict to this later case.

The $\{G_{\infty q}\}$ may be more properly described by some set of q-functions $\varepsilon^i(x)$, $i = 1 \ldots q$, by means of

$$\delta x^{\beta} = \varepsilon^i(x)\, \xi_i^{\beta}(x)$$

$$\delta y_A(x) = \varepsilon^i(x)\, \eta_{Ai}(x) + \varepsilon^i_{,\rho}(x)\, \gamma^{\rho}_{Ai}(x) \qquad (5\text{-}5\text{-}1)$$

The fact that we have restricted the transformation law for $\delta y_A(x)$ in such form that only the first derivatives of $\varepsilon^i(x)$ appear, is characteristic of the tensor law of transformation. This case is sufficient for all known applications of this theorem.

Invariance of the field equations, the Euler-Lagrange equations for W, implies that at most we can sum a surface integral to W in the transformed frame.

$$\int_{\Omega} L(x;y(x))\, dx \equiv \int_{\Omega} L(x';y'(x'))\, dx' + \oint Q^{\rho}\, d\Sigma_{\rho}$$

The Q^{ρ} being in general functions of x and y(x), they are of the same order than δx and δy. Presently we take $Q^{\rho} = 0$ since our discussion will independ on the value assumed by Q^{ρ}. To first order we get

$$(\partial^A L)\, \bar{\delta} y_A + (\partial^{A\rho} L)\, \bar{\delta} y_{A,\rho} + (L\delta\, x^{\rho})_{,\rho} \equiv 0$$

where

$$\partial^A L = \frac{\partial L}{\partial y_A}\ , \quad \partial^{A\rho} L = \frac{\partial L}{\partial y_{A,\rho}}\ , \quad \bar{\delta} = \delta - \delta x^\alpha \frac{\partial}{\partial x^\alpha}$$

We may write this relation as

$$L^A\, \bar{\delta} y_A + \Gamma^\rho{}_{,\rho} \equiv 0 \qquad (5\text{-}5\text{-}2)$$

where L^A is the variational derivative of L with respect to y_A,

$$L^A = \partial^A L - (\partial^{A\rho} L)_{,\rho}$$

and Γ^ρ is a short for

$$\Gamma^\rho = L\ \delta x^\rho + (\partial^{A\rho}\ L)\ \bar{\delta} y_A \qquad (5\text{-}5\text{-}3)$$

Substitution of (5-5-1) into the (5-5-3) gives

$$\varepsilon^i \Big[L^A (\eta_{Ai} - y_{A,\mu} \xi^\mu_i) - (L^A\ \gamma^\rho_{Ai})_{,\rho} \Big] + \Big[(\partial^{A\rho}\ L) \bar{\delta} y_A + L^A\ \varepsilon^i\ \gamma^\rho_{Ai} + L\ \delta x^\rho \Big]_{,\rho} \equiv 0$$

Since $\varepsilon^i(x)$ are arbitrary functions, the identity sign holds only if each one of the coefficients of ε^i, $\varepsilon^i_{,\mu}$ and $\varepsilon^i_{,\mu\sigma}$ vanish separately.

$$(L^A\ \gamma^\rho_{Ai} + \eta_{Ai}\ \partial^{A\rho}\ L - T^\rho{}_\beta\ \xi^\beta_i)_{,\rho} \equiv 0 \qquad (5\text{-}5\text{-}4)$$

$$L^A\ \gamma^\rho_{Ai} + \eta_{Ai}\ \partial^{A\rho}\ L - T^\rho{}_\beta\ \xi^\beta_i + (\gamma^\rho_{Ai}\ \partial^{A\beta}\ L)_{,\beta} \equiv 0 \qquad (5\text{-}5\text{-}5)$$

$$\gamma^\rho_{Ai}\ \partial^{A\beta}\ L + \gamma^\beta_{Ai}\ \partial^{A\rho}\ L \equiv 0 \qquad (5\text{-}5\text{-}6)$$

We now specialize these general results to the case where the field variables y_A are the components of the metric $g_{\mu\nu}$, and the transformation group in the X-space is the group of general coordinate transformations.

$$\delta x^\alpha\ (x) = \varepsilon^\alpha (x)\ ,$$

that is, the MMG for general relativity. Then q = 4, and $\xi^\beta_\nu = \delta^\beta_\nu$. We have in place of δ_{y_A} the $\delta g_{\mu\nu}$ with values

$$\bar{\delta} g_{\mu\nu}(x) = (g_{\lambda\mu}\ \delta^\alpha_\nu + g_{\lambda\nu}\ \delta^\alpha_\mu) \varepsilon^\lambda_{,\alpha} - \varepsilon^\alpha\ g_{\mu\nu,\alpha}$$

Hence

$$\eta_{Ai} = \eta_{\rho\sigma\lambda} = 0$$
$$\gamma^{\rho}_{Ai} = \gamma^{\rho}_{\mu\nu\lambda} = -\,(g_{\mu\lambda}\,\delta^{\rho}_{\nu} + g_{\lambda\nu}\,\delta^{\rho}_{\mu})$$

The Lagrangian density for gravitation, $\sqrt{-g}\,R$, contains up to second order derivatives of $g_{\mu\nu}$ but is linear in these derivatives. This allows us to separate the second order derivatives as a divergence in the Lagrangian density.

$$L = \sqrt{-g}\;R = L' + K^{\sigma}_{,\sigma}$$

with

$$L' = \sqrt{-g}\,g^{\mu\nu}(\{{\alpha \atop \mu\lambda}\}\{{\lambda \atop \nu\alpha}\} - \{{\lambda \atop \mu\nu}\}\{{\alpha \atop \lambda\alpha}\}) \qquad (5\text{-}5\text{-}7)$$

$$K^{\sigma} = \sqrt{-g}\,(g^{\lambda\alpha}\{{\sigma \atop \lambda\alpha}\} - g^{\lambda\sigma}\{{\alpha \atop \lambda\alpha}\}) \qquad (5\text{-}5\text{-}8)$$

L' depends just on $g_{\mu\nu}$ and the first partial derivatives of $g_{\mu\nu}$. What really plays the role of a Lagrangian density in the variational principle is L', not L. Thus, we have to rewrite our formulas (5-5-4) through (5-5-6) for L'. However,

$$W' = \int_{R_4} L'\,d_4x$$

is not invariant under $G_{\infty 4}$ since depends only on $g_{\mu\nu}$ and $\{{\lambda \atop \mu\nu}\}$, and no invariant with such dependence does exist. But since

$$\delta W = \delta W' + \oint_{\Sigma} d\sigma_{\mu}\;\delta K^{\mu}$$

where Σ is the boundary of integration limiting R_4. Then, the surface integral gives

$$\oint_{\Sigma} d\sigma_{\mu}\;\delta K^{\mu} = \oint_{\Sigma}\left(\frac{\partial K^{\mu}}{\partial g_{\alpha\beta}}\,\delta g_{\alpha\beta} + \frac{\partial K^{\mu}}{\partial\{{\lambda \atop \nu\alpha}\}}\,\delta\{{\lambda \atop \nu\alpha}\}\right) d\sigma_{\mu}$$

If we restrict to groups which are the identity transformation onto the boundary Σ, that is,

$$\{\bar{G}_{\infty 4}\}: \begin{cases}\varepsilon_{\alpha}\;(x) \to 0 \\ \varepsilon_{\alpha\,\mu}\;(x) \to 0\end{cases};\quad |x| \to \Sigma$$

we will get $\delta W = \delta W'$; so that in spite of W' being not an invariant, the variation $\delta W'$ is an invariant for this type of groups.

Restricting to such groups, we try to see what happens to the three identities (5-5-4) through (5-5-6). Such identities came from the expression

$$\varepsilon^i\left[L^A(\eta_{Ai} - y_{A,\mu}\xi^\mu_i) - (L^A\,\gamma^\rho_{Ai})_{,\rho}\right] + \left[(\partial^{A\rho}\,L)\bar{\delta}y_A + L^A\,\varepsilon^i\,\gamma^\rho_{Ai} + L\,\delta x^\rho\right]_{,\rho} \equiv 0$$

which contains $\varepsilon_i(x)$ and $\varepsilon_{i,\mu}(x)$ inside a divergence. Thus, they will not contribute to our case, since $\varepsilon_\alpha(x)$ and $\varepsilon_{\alpha,\mu}(x)$ vanish at the boundaries. We are left just with the term containing ε^i undifferentiated,

$$\varepsilon^i\left[L^A(\eta_{Ai} - y_{A,\mu}\,\xi^\mu_i) - (L^A\,\gamma^\rho_{Ai})_{,\rho}\right] \equiv 0$$

this simplified identity, in our case is

$$-\,L'^{\mu\nu}\,g_{\mu\nu,\rho} + \left[L'^{\mu\nu}(g_{\mu\rho}\,\delta^\sigma_\nu + g_{\nu\rho}\,\delta^\sigma_\mu)\right]_{,\sigma} \equiv 0 \qquad (5\text{-}5\text{-}9)$$

where

$$L'^{\mu\nu} = g^{\mu\lambda}\,g^{\nu\alpha}\,L'_{\lambda\alpha}$$

and $L'_{\lambda\alpha}$ is just the Einstein tensor $\sqrt{-g}\,G_{\lambda\alpha}$. The identity (5-5-9) may be brought to a familiar form, if we write the explicit value for $L'_{\mu\nu}$,

$$G^\mu_{\nu;\mu} \equiv (R^\mu{}_\nu - \tfrac{1}{2}\,\delta^\mu_\nu\,R)_{;\mu} \equiv 0 \qquad (5\text{-}5\text{-}10)$$

Therefore, as effect of the invariance of the Action principle under the general group of coordinate transformations the field equations satisfy, or strictly saying, the left hand side of the field equations (The Einstein's tensor $G_{\mu\nu}$) satisfy four identities. These later may be looked as if they were four conditions on the possible solutions of those equations.

Further, if we write the variational equation giving the left hand side of the field equations, for empty spaces,

$$\sqrt{-g}\; G^{\alpha\beta} \equiv \partial_\mu \left(\frac{\partial L'}{\partial g_{\alpha\beta,\mu}} \right) - \frac{\partial L'}{\partial g_{\alpha\beta}}$$

we will have

$$\frac{\partial^2 L'}{\partial g_{\alpha\beta,\mu}\, \partial g_{\lambda\sigma}} g_{\lambda\sigma,\mu} + \frac{\partial^2 L'}{\partial g_{\alpha\beta,\mu}\, \partial g_{\lambda\sigma,\nu}} g_{\lambda\sigma,\nu\mu} - \frac{\partial L'}{\partial g_{\alpha\beta}} \equiv G^{\alpha\beta} \sqrt{-g}$$

Expanding the sums over the $g_{\lambda\sigma,\mu}$ and the $g_{\lambda\sigma,\nu\mu}$ we obtain

$$\sqrt{-g}\; G^{\alpha\beta} \equiv \frac{\partial^2 L'}{\partial g_{\alpha\beta,o}\, \partial g_{\lambda\sigma,o}} g_{\lambda\sigma,oo} + \frac{\partial^2 L'}{\partial g_{\alpha\beta,o}\, \partial g_{\lambda\sigma}} g_{\lambda\sigma,o} + \left(\frac{\partial^2 L'}{\partial g_{\alpha\beta,o}\, \partial g_{\lambda\sigma,i}} + \right.$$

$$\left. + \frac{\partial^2 L'}{\partial g_{\alpha\beta,i}\, \partial g_{\lambda\sigma,o}} \right) g_{\lambda\sigma,oi} + \frac{\partial^2 L'}{\partial g_{\alpha\beta,i}\, \partial g_{\lambda\sigma,j}} g_{\lambda\sigma,ij} + \frac{\partial^2 L'}{\partial g_{\alpha\beta,i}\, \partial g_{\lambda\sigma}} g_{\lambda\sigma,i} - \frac{\partial L'}{\partial g_{\alpha\beta}}$$

(5-5-11)

so that the highest order time-derivative occuring in $G^{\alpha\beta}$ is $g_{\lambda\sigma,oo}$

Since

$$G^{\alpha\beta}_{\ ;\beta} = G^{\alpha\beta}_{\ ,\beta} + \left\{ {\alpha \atop \beta\lambda} \right\} G^{\lambda\beta} + \left\{ {\beta \atop \beta\lambda} \right\} G^{\alpha\lambda} \equiv 0$$

we see that indeed the left hand side of the field equations satisfy four conditions, the explicit value of these conditions is obtained by replacing (5-5-11) in this later equation. Later on we shall turn back to these conditions.

Before closing this section, we shall prove in a clear and direct fashion the fact that the contracted Bianchi identity follow from the general invariance of the Action principle. Consider the variation of W,

$$\bar{\delta} W = \int_{R_4} \sqrt{-g}\; G^{\mu\nu}\, \bar{\delta} g_{\mu\nu}\, d_4 x$$

This is a general formula from the variational calculus. We specialize it for variations induced by mappings of the MMG. In this case, we have from (5-2-3)

$$\bar{\delta}W = \int_{R_4} \sqrt{-g}\, G^{\mu\nu} (\xi_{\mu;\nu} + \xi_{\nu;\mu})\, d_4x$$

using the distributive property of covariant derivatives, we have

$$\bar{\delta}W = -\int_{R_4} d_4x\, \{(\sqrt{-g}\, G^{\mu\nu})_{;\nu}\, \xi_\mu + (\sqrt{-g}\, G^{\mu\nu})_{;\mu}\, \xi_\nu\} + \int_{R_4} d_4x \{(\sqrt{-g}\, G^{\mu\nu} \xi_\mu)_{;\nu} + (\sqrt{-g}\, G^{\mu\nu} \xi_\nu)_{;\mu}\}$$

Now, we use the fact that $\sqrt{-g}$ is a scalar density with weight +1, so that from (2-1-11) we have

$$(\sqrt{-g})_{;\mu} = (\sqrt{-g})_{,\mu} - \sqrt{-g}\, \{^{\lambda}_{\lambda\mu}\} = 0$$

thus

$$\bar{\delta}W = -\int_{R_4} \sqrt{-g}\, d_4x \{G^{\mu\nu}_{;\nu}\, \xi_\mu + G^{\mu\nu}_{;\mu}\, \xi_\nu\} + \int_{R_4} d_4x \{(\sqrt{-g}\, G^{\mu\nu}\, \xi_\mu)_{;\nu} + (\sqrt{-g}\, G^{\mu\nu}\, \xi_\nu)_{;\mu}\}$$

Taking the symmetry group as $\mathcal{G}_{\infty 4}$, so that ξ_μ and $\xi_{\mu,\nu}$ vanish on the boundary of integration, we will get (note that $\sqrt{-g}\, G^{\mu\nu}\, \xi_\nu$ is a vector density with weight +1)

$$\int_{R_4} d_4x (\sqrt{-g}\, G^{\mu\nu} \xi_\mu)_{;\nu} = \oint_{\Sigma} \sqrt{-g}\, G^{\mu\nu}\, \xi_\mu d\sigma_\nu + \int_{R_4} \sqrt{-g}\, G^{\lambda\alpha}\, \xi_\alpha \{^{\nu}_{\lambda\nu}\}\, d_4x -$$

$$- \int_{R_4} \sqrt{-g}\, \{^{\lambda}_{\lambda\nu}\}\, G^{\mu\nu}\, \xi_\mu\, d_4x = \oint_{\Sigma} \sqrt{-g}\, G^{\mu\nu}\, \xi_\mu\, d\sigma_\nu + 0$$

Then

$$\bar{\delta}W = -\int_{R_4} \sqrt{-g}\, \{G^{\mu\nu}_{;\nu}\, \xi_\mu + G^{\mu\nu}_{;\mu}\, \xi_\nu\}\, d_4x$$

and $\bar{\delta}W = 0$ implies in the contracted Bianchi identity $G^{\mu\nu}_{;\nu} = 0$. What we have proved is nothing but the Noether theorem, we just used a more direct proof.

5.6) The Initial Value Problem for the Einstein Equations

In this section we shall study the problem of initial conditions for the Einstein differential field equations. This problem is of importance for the determination of solutions of this equation in the following sense: suppose one gives a set of components of $g_{\mu\nu}$ which at a given value of the x^{μ} represent a solution of these equations, one then argues if these $g_{\mu\nu}$ at a point in the future with respect to x^{o} are still a solution of the field equations. Therefore, the knowledge of the correct statement of this problem permits us with a prescribed way for propagating given solutions in time. Besides this, the initial value problem will be important in the laydown of a canonical formulation for the field, the initial point towards the field quantization. Usually in physics, one gives the discussion of the Cauchy-Kowalewski problem in the framework of the canonical formalism. However, from the mathematical point of view, this problem may be presented simply as the study of the initial conditions for a given partial differential equation. We shall do in this later method.

Prior to the statement of this problem, is the determination of the region upon which we rest the initial values for the system. In special relativity this represents no problem since space-time is absolute. In general relativity we shall run in principle in difficulties. We need to specify a space-like hypersurface upon which the Cauchy data is given, but the specification of such hypersurface is conditioned to the knowledge of the metric on the four-space, that is, we apparently have first to solve the equations and only afterwards to obtain the initial data implying in such solution. This should be a rather disappointing feature if it indeed occured. However, it may be shown that the specification of a space-like hypersurface may be done entirely in terms of general arguments on the behaviour of $g_{\mu\nu}$ independently of the fact that $g_{\mu\nu}$ is a solution of the field equations.

In particular one may choose such hypersurface as given by the set of coordinates satisfying $x^o = c$, with time-like normal $l_\mu = \delta^o_\mu$. For the signature -2,

$$g^{\mu\nu}\,\ell_\mu\,\ell_\nu > 0$$

so that $g^{oo} > 0$. Also $g < 0$. Thus, we choose the hypersurface S in such form that in our coordinate system it has $x^o = c$ at all points. We may further take $c = 0$. On S we specify $g_{\mu\nu} = g_{\mu\nu}(x^r, 0)$ and $\dot{g}_{\mu\nu} = \left(\frac{\partial g_{\mu\nu}(x^r,x^o)}{\partial x^o}\right)_o$. Thus, on S we know $g_{\mu\nu,i}$ and $\dot{g}_{\mu\nu,i}$.

Higher order spatial derivatives may be obtained similarly on S. The initial data $g_{\mu\nu}$ (S) and $\dot{g}_{\mu\nu}$(S) together with the differential equation $R_{\mu\nu} = 0$ form our Cauchy-Kowalewski initial value problem. Suppose we expand $g_{\mu\nu}(x^r, x^o)$ around $x^o = 0$,

$$g_{\mu\nu}(x^r, x^o) = g_{\mu\nu}(x^r, 0) + x^o\left(\frac{\partial g_{\mu\nu}(x^r, x^o)}{\partial x^o}\right)_o + \frac{(x^o)^2}{2}\left(\frac{\partial^2 g_{\mu\nu}(x^r,x^o)}{\partial x^o\ \partial x^o}\right)_o + \ldots$$

if we can get the ten quantities $g_{\mu\nu,oo}(x^r,x^o)$ from the differential equations $R_{\mu\nu} = 0$, in terms of $g_{\mu\nu}(x^r,x^o)$ and $g_{\mu\nu,o}(x^r,x^o)$ plus their spatial derivatives.

$$\frac{\partial^2 g_{\mu\nu}(x^r,x^o)}{\partial x^o\quad \partial x^o} = \phi_{\mu\nu}\left(g_{\alpha\beta}(x^r,x^o),\quad \frac{\partial g_{\alpha\beta}(x^r,x^o)}{\partial x^o},\ + \text{space derivat.}\right) \tag{5-6-1}$$

then by further differentiations in time we may get all the coefficients of the previous series expansion. This will gives $g_{\mu\nu}(x^r,x^o)$ uniquely in terms of the initial data on S, and we have solved our problem. Then, in order to solve the problem we have to determine the form of the relations (5-6-1).

A direct calculation gives

$$R_{\mu\nu} = \frac{1}{2}\,g^{\rho\sigma}(g_{\mu\nu,\rho\sigma} + g_{\rho\sigma,\mu\nu} - g_{\mu\sigma,\nu\rho} - g_{\nu\rho,\mu\sigma}) + K_{\mu\nu} \tag{5-6-2}$$

$$K_{\mu\nu} = f_{\mu\nu}(g_{\alpha\beta},\ g_{\alpha\beta,\rho}) \qquad (5\text{-}6\text{-}3)$$

Thus, on S the $K_{\mu\nu}$ are known since they contain just first order derivatives of $g_{\alpha\beta}$.

An easy calculation gives

$$R_{ij}(x^r,x^o) = \frac{1}{2}\, g^{oo}\, g_{ij,oo} + M_{ij} \qquad (5\text{-}6\text{-}4.1)$$

$$R_{oi}(x^r,x^o) = -\,\frac{1}{2}\, g^{oj}\, g_{ij,oo} + M_{io} \qquad (5\text{-}6\text{-}4.2)$$

$$R_{oo}(x^r,x^o) = \frac{1}{2}\, g^{ij}\, g_{ij,oo} + M_{oo} \qquad (5\text{-}6\text{-}4.3)$$

where

$$M_{\mu\nu} = \Psi_{\mu\nu}(g_{\alpha\beta}(x^r,x^o),\ g_{\alpha\beta,\rho}(x^r,x^o),\ g_{\alpha\beta,ij}(x^r,x^o)) \quad ,$$

The $M_{\mu\nu}$ are given on S from the initial data. Therefore we conclude that our differential equations, the (5-6-4), are ten equations in just six unknown, the $g_{ij,oo}$. The $g_{o\mu,oo}$ are not present in the equations, and thus cannot be determined from the initial data. The results obtained are:

1) $g_{o\mu,oo}$ is left undetermined by the field equations, and consequently $g_{o\mu}(x^r,x^o)$ is undetermined as function of x^o.

2) The $g_{ij,oo}$ are determined by the field equations in terms of the remaining variables, compatible with the initial data. But the six $g_{ij,oo}$ are in lower number than the total number of field equations, as consequence we have four relations among the $g_{\alpha\beta}$, $g_{\alpha\beta,\rho}$ and $g_{\alpha\beta,ij}$. For instance, if we use (5-6-4.1) for obtaining $g_{ij,oo}$,

$$g_{ij,oo} = -\,M_{ij}(2/g^{oo})$$

we still have four relations, the (5-6-4.2) and (5-6-4.3) which represent four relations among the $g_{\alpha\beta}$, $g_{\alpha\beta,\rho}$ and $g_{\alpha\beta,ij}$. They are

$$-\frac{1}{2} g^{oj} g_{ij,oo} + M_{io} = 0 , \qquad \frac{1}{2} g^{ij} g_{ij,oo} + M_{oo} = 0$$

In particular, on S we also have four relations among the twenty elements forming up the Cauchy data. This means that this data cannot be chosen arbitrarily but must be consistent with these four conditions. These consistency conditions are not the result of the property that the field equations satisfy four identities (the differential Bianchi identities), but is related to this fact. Later on, when we study the Hamiltonian formulation we will see that the four Bianchi identities give rise to the so called primary constraints on the canonical set of variables, whereas these four relations presently referred give rise to the secondary constraints. The presence of sources for the field will not modify the content of the differential Bianchi identities, since $T^{\mu\nu}_{\ ;\nu} = 0$, thus the primary constraints are not modified by the presence of sources. The field equations are modified by the presence of sources, therefore the secondary constraints are also modified by the presence of sources. This applies too for the statement of the Cauchy problem, presently we consider spaces free of sources, the presence of these extra variables will just modify the right-hand side of the field equations, case where we get

$$g_{ij,oo} = (T_{ij} - M_{ij}) \frac{2}{g^{oo}} - \frac{1}{g^{oo}} g_{ij} T$$

all the remaining being unchanged in the mathematical formulation of the initial value problem.

The geometrical interpretation of the fact that the Einstein's equations fail in giving a dynamical solution for the components $g_{o\mu}$ may be seen from the following argument: If we consider a coordinate transformation which on S is the identity transformation, for instance an infinitesimal mapping,

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x^{r}, x^{o})$$

with

$$\xi^{o}(x^{r},x^{o}) = x^{o} f(x^{r})$$

whereas for all points outside S the mapping is arbitrary. The field equations being invariant under such mapping cannot contain any variable which changes under the transformation with no compensation coming from other terms. It happens that $g_{o\mu}$ is exactly such type of variable since it measures an interval with a component normal to S, the dx^{o},

$$ds = (g_{o\mu}\, dx^{o}\, dx^{\mu})^{1/2}$$

and thus will change under the mapping. In other words, by taking such types of mappings we may give any arbitrary value to the $g_{o\mu}$, thus showing that no prescribed value may be set for these variables. The only possibility for fixing the $g_{o\mu}$ is to choose a coordinate condition once for all. For instance, we may choose the coordinate systems such that for them $g_{o\mu} = \delta^{o}_{\mu}$. However, we do not intend to do this now, since this gives up the principle of invariance, we rather take the $g_{o\mu}$ as undetermined variables in the Cauchy problem.

Lichnerowski has given a presentation of the Cauchy problem which shows more clearly the form assumed by the four relations on the Cauchy data [37]. He calls this method by the Normal Form of the Field Equations. He uses the Einstein tensor instead of the Ricci tensor, case where we obtain

$$G^{o}_{i} \equiv g^{oj} R_{ij} + g^{oo} R_{io} \qquad (5\text{-}6\text{-}5)$$

$$G^{o}_{o} \equiv \frac{1}{2} (g^{oo} R_{oo} - g^{ij} R_{ij}) \qquad (5\text{-}6\text{-}6)$$

Using the previous expressions for the several components of the Ricci tensor we write this as

$$G^{o}_{i} \equiv g^{oj} M_{ij} + g^{oo} M_{oo} \qquad (5\text{-}6\text{-}7)$$

$$G^{o}_{o} \equiv \frac{1}{2} (g^{oo} M_{oo} - g^{ij} M_{ij}) \qquad (5\text{-}6\text{-}8)$$

Then, $G_{o\mu}$ is a function only of the Cauchy data and of their spatial derivatives. The normal form of the field equations is to take for the first six equations R_{ij} = 0, and for the remaining four equations $G_{o\mu} = 0$. This shows that the six equations $R_{ij} = 0$ (or $R_{ij} = T_{ij} = \frac{1}{2} g_{ij} g^{kl} T_{kl}$) determine the $g_{ij,oo}$ by

$$g_{ij,oo} = - \frac{2 M_{ij}}{g^{oo}}$$

or by

$$g_{ij,oo} = (-M_{ij} + T_{ij} - \frac{1}{2} g_{ij} T) \frac{2}{g^{oo}}$$

and the remaining equations are consistency conditions (on S) upon the Cauchy data.

$$G^{o}_{\mu} = f_{\mu}(g_{\alpha\beta}, g_{\alpha\beta,\rho}, g_{\alpha\beta,ij}) = 0$$

In the case of presence of external sources,

$$G^{o}_{\mu} - T^{o}_{\mu} = 0$$

these will be the secondary constraints present in the Hamiltonian formulation for gravitation.

In order to clarify further the content of this section, we will give the corresponding problem for electrodynamics, and will stablish the similarities between these two situations. In electrodynamics, which is a linear vector field theory with null rest mass, the field variables are the potentials

$$A_i = A_i(x^r, x^o)$$

$$A_o = \phi(x^r, x^o)$$

with field equations

$$\Box A_{\mu} = 0 \qquad (5\text{-}6\text{-}9)$$

$$A^{\mu}_{,\mu} = 0 \qquad (5\text{-}6\text{-}10)$$

The field equations (5-6-9) and (5-6-10) are a step beyond the gravitational field equations due to the fact that the later equation, namely the (5-6-10) is equivalent to a "coordinate condition" in the sense we used for the gravitational problem. This equation is possible to be set since the A_μ are arbitrary up to a gauge transformation, so that we may impose on them this condition. Similarly we might impose on $g_{\mu\nu}$ four conditions, the coordinate conditions. Let us study the Cauchy problem with the Lorentz condition, and determine its analogies with the gravitational problem even if there we did not use any coordinate condition.

The Cauchy data is given by the set of eight quantities on a given space-like hypersurface, taken as $x^o = 0$,

$$A_i(x^r, 0) = f_i(x^r)$$

$$A_{i,o}(x^r, 0) = g_i(x^r)$$

$$\phi(x^r, 0) = h(x^r)$$

$$\phi_{,o}(x^r, 0) = p(x^r)$$

Expanding the potentials in power series of x^o

$$A_\mu(x^r, x^o) = A_\mu(x^r, 0) + x^o(A_{\mu,o}(x^r, x^o))_{x^o=0} + \frac{(x^o)^2}{2}(A_{\mu,oo}(x^r, x^o))_{x^o=0} + \ldots$$

The solution for this problem is obtained when we get the four quantities $A_{i,oo}(x^r, x^o)$ and $\phi_{,oo}(x^r, x^o)$ from the field equations. Since now we have five field equations it follows that the Cauchy data is subjected to one condition, which is just the (5-6-10)

$$A_{i,}{}^{i} + \phi^{,o} = A_{i,i} - \phi_{,o} = 0 \qquad (5\text{-}6\text{-}11)$$

or, on the hypersurface $x^o = 0$,

$$p(x^r) = f_{i,i}(x^r) \qquad (5\text{-}6\text{-}12)$$

So that the imposition of the Lorentz condition as a gauge condition has the effect of dropping out an element of the Cauchy data, the $p(x^r)$, which is given as function of $f_i(x^r)$. A similar situation for general relativity should eliminate, for instance, the $g_{o\mu}$ by setting them equal to δ^o_μ by means of four coordinate conditions. The same applies for the checking of the number of equations, we have ten equations $G_{\mu\nu} = 0$, plus four coordinate conditions which gives fourteen relations, and we have just ten unknown, the $g_{\mu\nu,oo}$, thus we will get four conditions of the type (5-6-11). They are used for elimminating the non-physical variables $g_{o\mu}$.

Thus, after dropping the $p(x^r)$ from the Cauchy data, we are left with $f_i(x^r)$, $g_i(x^r)$ and $h(x^r)$. A similar reduced set for general relativity might be $g_{ij}(x^r)$, $g_{ij,o}(x^r)$ and $g_{\mu o,o}(x^r)$. Note that $g_{\mu o,o}$ which depends arbitrarily on the choice of coordinates outside of S is still present. Similarly in electrodynamics we eliminate $p = \phi_{,o}$ but still have $h = \phi$ which is mathematically similar. This means that our solution for the Cauchy problem will involve gauge invariant as well as gauge variant quantities.

The four equations (5-6-9) give $A_{i,oo}$ and $\phi_{,oo}$ in terms of the elements of the Cauchy data,

$$A_{i,oo}(x^r, x^o) = A_{i,kk}(x^r, x^o) \qquad (5\text{-}6\text{-}13)$$

$$\phi_{,oo}(x^r, x^o) = \phi_{,kk}(x^r, x^o) \qquad (5\text{-}6\text{-}14)$$

(we are using a metric $\overset{o}{g}_{\mu\nu}$ with signature +2) By sucessive time differentiation on (5-6-11) and using (5-6-13), we get at $x^o = 0$,

$$\phi_{,o} = A_{i,i} = f_{i,i}(x^r)$$

$$\phi_{,oo} = A_{i,io} = g_{i,i}(x^r)$$

$$\phi_{,ooo} = A_{i,ioo} = A_{i,ikk} = \nabla^2 f_{i,i}(x^r)$$

and so on. Therefore, the solution for ϕ according to Cauchy's problem is

$$\phi(x^r,x^o) = h(x^r) + x^o\, f_{i,i}(x^r) + \frac{(x^o)^2}{2}\, g_{i,i}(x^r) + \frac{(x^o)^3}{6}\,\nabla^2\, f_{i,i}(x^r)+\ldots \tag{5-6-15}$$

For the vector potential we find similarly

$$A_i(x^r,x^o) = f_i(x^r) + x^o\, g_i(x^r) + \frac{(x^o)^2}{2}\,\nabla^2\, f_i(x^r) + \frac{(x^o)^3}{6}\,\nabla^2\, g_i(x^r)+\ldots \tag{5-6-16}$$

Under a gauge transformation

$$A_i'(x^r,x^o) = A_i(x^r,x^o) + \Lambda_{,i}(x^r,x^o) \tag{5-6-17}$$

$$\phi'(x^r,x^o) = \phi(x^r,x^o) + \Lambda_{,o}(x^r,x^o) \tag{5-6-18}$$

where $\Lambda(x^\mu)$ is a solution of the scalar wave equation $\Box\Lambda = 0$. We take the set of gauge transformations vanishing onto the hyperplane $x^o = 0$,

$$\Lambda(x^r,\ x^o = 0) = 0, \quad \text{for all } x^r \in S$$

but otherwise arbitrary. Expanding this function in power series of x^o,

$$\begin{aligned}\Lambda(x^r,x^o) &= x^o\,\Lambda_{,o}(x^r,0) + \frac{(x^o)^2}{2}\,\Lambda_{,oo}(x^r,0) + \ldots\\ &= x^o\, a(x^r) + \frac{(x^o)^2}{2}\, b(x^r) + \ldots\end{aligned} \tag{5-6-19}$$

the functions $a(x^r)$, $b(x^r)$... being restricted by the scalar wave equation. Then, on S

$$A_i'(x^r,\ 0) = A_i(x^r,\ 0) \tag{5-6-20}$$

$$\phi'(x^r,0) = \phi(x^r,\ 0) + a(x^r) \tag{5-6-21}$$

as consequence, the vector potential $A_i(x^r,x^o)$ is gauge invariant on the hyperplane $x^o = 0$. In other terms, A_i is a function only of variables associated to the hyperplane, and doesn't depend on the choice of gauge outside S, and is invariant. The scalar potential is not gauge invariant on the hyperplane $x^o = 0$, and pairs with

$g_{o\mu}$, whereas A_i corresponds to g_{ij}. We have from (5-6-20) and (5-6-21)

$$f_i'(x^r) = f_i(x^r) \tag{5-6-22}$$

$$h'(x^r) = h(x^r) + a(x^r) \tag{5-6-23}$$

Taking the time derivative of equations (5-6-17) and (5-6-18) and setting $x^o = 0$, we find

$$g_i'(x^r) = g_i(x^r) + a_{,i}(x^r) \tag{5-6-24}$$

$$p'(x^r) = p(x^r) + b(x^r) \tag{5-6-25}$$

the equations (5-6-22) through (5-6-25) give the variance of the Cauchy data on S under the gauge transformations considered. However, the (5-6-25) as it stands is not yet the complete story. Indeed, from (5-6-12) the $p(x^r)$ being the divergence of $f_i(x^r)$ has to be invariant on account of (5-6-22). Indeed, we have not used, as yet, the fact that $\Box\Lambda = 0$. From this equation we obtain limitations on the functions $a(x^r)$, $b(x^r)$..., we have

$$\Box\Lambda = (\partial^2_{kk} - \partial^2_o)\,\Lambda = x^o\, a_{,kk} + \frac{(x^o)^2}{2}\, b_{,kk} - b(x^r) + x^o\, c(x^r) + \dots = 0$$

since this equation holds at all points (x^r, x^o), taking $x^o = 0$ we get, at S

$$b(x^r) = 0$$

so that from (5-6-25) we obtain

$$p'(x^r) = p(x^r) \tag{5-6-26}$$

Our conclusion is as follows: The Cauchy series (5-6-15) and (5-6-16) giving the fields on all space-time points from the initial conditions on S, represented by the functions $f_i(x^r)$, $g_i(x^r)$ and $h(x^r)$, on a manifold where all gauge transformation are solutions of the scalar wave equation, and vanish on the hyperplane S, will contain gauge invariant as well gauge dependent functions. The gauge invariant parts are those containing the $f_i(x^r)$, the gauge dependent parts will be those depending on $g_i(x^r)$ and on $h(x^r)$. It will be important for subsequent discussions, when we will study the Hamiltonian formulation of Dirac, to separate these two different terms. We indicate by $A_i^{(I)}$ and by $A_i^{(II)}$ the gauge invariant and gauge variant parts of A_i. The same for the scalar potential ϕ.

$$A_i = A_i^{(I)} + A_i^{(II)}$$

$$\phi = \phi^{(I)} + \phi^{(II)}$$

$$A_i^{(I)}(x^r,x^o) = f_i(x^r) + \frac{(x^o)^2}{2} \nabla^2 f_i(x^r) + \ldots \qquad (5\text{-}6\text{-}27)$$

$$A_i^{(II)}(x^r,x^o) = x^o g_i(x^r) + \frac{(x^o)^3}{6} \nabla^2 g_i(x^r)+\ldots \qquad (5\text{-}6\text{-}28)$$

$$\phi^{(I)}(x^r,x^o) = x^o f_{i,i}(x^r) + \frac{(x_o)^3}{6} \nabla^2 f_{i,i}(x^r) + \ldots \qquad (5\text{-}6\text{-}29)$$

$$\phi^{(II)}(x^r,x^o) = h(x^r) + \frac{(x^o)^2}{2} g_{i,i}(x^r) + \ldots \qquad (5\text{-}6\text{-}30)$$

We shall use here a definition which later on will be of importance. We call any quantity which is invariant under the function group of the theory which leaves unchanged the hypersurface S, but which changes arbitrarily the region outside S, by D-invariant. Examples of these quantities for electrodynamics, where the function group is the gauge group, are the $A_i^{(I)}(x^r, x^o)$, * another such quantity might

* We might aggregate the fact that the whole $A_i(x^r,x^o)$ is D-invariant, since $A_i = A_i^{(I)} + A_i^{(II)}$, and on S the $A_i^{(II)}$ vanish. This shows that indeed A_i pairs with g_{ij}.

be the $\phi^{(I)}(x^r,x^o)$, but this quantity is zero on the hypersurface S, as is seen from (5-6-29). In general relativity example of D-invariants are the $g_{ij}(x^r,x^o)$ and its reciprocal $e^{ij}(x^r,x^o)$.

Still another form for characterizing the field variables is frequently used, specially in connection with the exhaustive work by Arnowitt Deser and Misner on the dynamical content of general relativity. This notation is an extension of the usual vector notation which decomposes a vector into longitudinal and transversal components,

$$\vec{V} = \vec{V}^{\,T} + \vec{V}^{\,L}$$

$$V^T_{i,i}(x^r,\ x^o) = 0$$

$$\varepsilon_{ijk}\ V^L_{j,k}(x^r,\ x^o) = 0$$

Since a gauge transformation on A_i just adds a gradient,

$$A_i' = A_i + \nabla_i\ \Lambda$$

we may put

$$\vec{A} = \vec{A}^{\,T} + \vec{A}^{\,L}$$

which transform

$$\vec{A}'^{T}(x^r,x^o) = \vec{A}^{\,T}(x^r,x^o)$$

$$\vec{A}'^{L}(x^r,x^o) = \vec{A}^{\,L}(x^r,x^o) + \vec{\nabla}\Lambda(x^r,x^o)$$

so that A_i^T is gauge invariant and A_i^L is gauge dependent. This identifies the transversal components with our $A_i^{(I)}$ and A_i^L with the $A_i^{(II)}$. By extension we might call $\phi^{(I)}$ by ϕ^T and the $\phi^{(II)}$ by ϕ^L.

As it follows from (5-6-27), (5-6-30), on S the A_i is of the type "T",

$$A_i(x^r,\ x^o = 0) = A_i^{(I)}(x^r,0) = A_i^{\,T}$$

and that on S the scalar potential is of the type "L", that is gauge variant.

$$\phi(x^r,\ 0) = \phi^{(II)}(x^r,\ 0) = \phi^L$$

This shows that on S only the vector potential is a physical quantity, the scalar potential is non-physical since depends on the choice of gauge. Similarly, in general relativity, on S only $g_{ij} = g_{ij}(x^r, x^o = 0)$ is a physical variable, the $g_{o\mu}(x^r,0)$ will depend on the choice of the coordinates in the neighbourhood of the normal to S at the point $x^r \epsilon S$. In the Hamiltonian formulation this will pose difficulties, and we are guided mostly by the results which are obtained for the simple case of electrodyamics. In this case, what we do is to eliminate the scalar potential by choosing an appropriate gauge, $\phi = 0$. We still have the $\vec{A}^L$ into the description of the system. We will not enter into further discussions of this subject since we will turn back again to these questions when treating the Hamiltonian formulation. We deserve to this point a more detailed discussion of these topics.

5.7) The Linearized Einstein Field Equations

In this section we will study the weak gravitational fields, that is, the fields which are characterized by a samll, first order, deviation from the Galilean flat metric tensor $\overset{o}{g}_{\mu\nu}$. Such fields were first considered by Einstein himself in 1916 [38]. Because of the smallness of the gravitational coupling constant the linearized form of the field equations resulting for these fields is applicable to a large class of gravitational phenomena. As long as we accept the full non linear theory as correct, to consider the linear field theory corresponds to take a very strong approximation. Indeed, we know that there exist solutions of the non-linear equations which cannot be approximate by series of perturbations starting with the linear equations. This means that all solutions presently taken of the linear equations, may bear little or no relation to solutions of the full theory. However, there exists an interest towards the linear approximation as long as we do not know in detail the behaviour of the solutions of the full non-linear equations. Still exists the hope of determine the necessary conditions in order to approximate correctly a solution of the non-linear equations by a series of solutions of the method of perturbations starting with the linear equations as the first order approximation.

Thus, taking in mind that this is just an approximation, which must be carefully taken into serious account with respect to the classifications of the solutions for the field equations, we may without further difficulties to undertaken the linearization of the Einstein equations. The theory which emerges in this approximation is a spin 2 theory for a field which does not interact with himself, similarly to the spin 1 fields of Maxwell or Procca. This theory is indeed similar to an usual theory for gravitation in special relativity, a result to be expected since the non-linearity of the field equations came essentially from the imposition of general invariance of these equations, or equivalently, of the Lagrangian density. The simplest possible form for the Lagrangian satisfying this invariance was found to be non-linear in the field variables and their derivatives.

We start by saying that for weak gravitational fields there always exist a coordinate system where the metric tensor $g_{\mu\nu}$ is the sum of $\overset{o}{g}_{\mu\nu}$ and a first order deviation $\varepsilon\ \phi_{\mu\nu}$. Let us call this coordinate system by W.

$$g_{\mu\nu}(x) = \overset{o}{g}_{\mu\nu} + \varepsilon\ \phi_{\mu\nu}(x) \tag{5-7-1}$$

With this $g_{\mu\nu}$ we start the necessary calculations for introducing the Einstein tensor and then obtaining the field equations to first order in ε. First of all we need the contravariant $g^{\mu\nu}$ to first order in ε. Writing,

$$g^{\mu\nu}(x) = \overset{o}{g}{}^{\mu\nu} + \varepsilon\ \psi^{\mu\nu}(x)$$

and imposing that this is the inverse matrix to the $g_{\mu\nu}$ of (5-7-1), we find

$$g^{\mu\nu}(x) = \overset{o}{g}{}^{\mu\nu} - \varepsilon\, \overset{o}{g}{}^{\mu\rho}\, \overset{o}{g}{}^{\nu\sigma}\, \phi_{\rho\sigma}(x) \qquad (5\text{-}7\text{-}2)$$

Since the Christoffel symbols are of the first order in ε,

$$\left\{{\mu \atop \rho\sigma}\right\} \cong \frac{\varepsilon}{2}\, \overset{o}{g}{}^{\mu\nu}(\phi_{\rho\nu,\sigma} + \phi_{\sigma\nu,\rho} - \phi_{\rho\sigma,\nu})$$

Therefore the Ricci tensor is

$$R_{\mu\nu} \cong \frac{\varepsilon}{2}\{\phi_{,\mu\nu} + \overset{o}{g}{}^{\rho\sigma}(\phi_{\mu\nu,\rho\sigma} - \phi_{\mu\rho\ \nu\sigma} - \phi_{\nu\rho\ \mu\sigma})\}$$

with

$$\phi = \overset{o}{g}{}^{\mu\nu}\, \phi_{\mu\nu}$$

Calculating the scalar curvature R from this equation, we can form the components of the Einstein tensor, it is found the value

$$G_{\mu\nu} \cong \frac{\varepsilon}{2}\ (\phi_{,\mu\nu} + \overset{o}{g}{}^{\rho\sigma}\ (\phi_{\mu\nu,\rho\sigma} - \phi_{\mu\rho,\nu\sigma} - \phi_{\nu\rho,\mu\sigma})) - \frac{\varepsilon}{2}\, \overset{o}{g}_{\mu\nu}\, \overset{o}{g}{}^{\rho\sigma}(\phi_{,\rho\sigma} - \overset{o}{g}{}^{\alpha\beta}\, \phi_{\rho\sigma,\alpha\beta}) \qquad (5\text{-}7\text{-}3)$$

By convenience, we introduce the new variables

$$\gamma_{\mu\nu} \equiv \phi_{\mu\nu} - \frac{1}{2}\, \overset{o}{g}_{\mu\nu}\ \phi \qquad (5\text{-}7\text{-}4)$$

This equation may be solved for the $\phi_{\mu\nu}$ in terms of $\gamma_{\mu\nu}$ as

$$\phi_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\, \overset{o}{g}_{\mu\nu}\ \gamma$$

In terms of the $\gamma_{\mu\nu}$ the Einstein tensor takes the form

$$G_{\mu\nu} \cong \frac{\varepsilon}{2}\ (\overset{o}{g}{}^{\rho\sigma}\ \gamma_{\mu\nu,\rho\sigma} - (\overset{o}{g}{}^{\rho\sigma}\ \gamma_{\mu\rho,\sigma})_{,\nu} - (\overset{o}{g}{}^{\rho\sigma}\ \gamma_{\nu\rho,\sigma})_{,\mu} + \overset{o}{g}_{\mu\nu}\ \overset{o}{g}{}^{\rho\sigma}(\overset{o}{g}{}^{\lambda\beta}\ \gamma_{\rho\lambda,\beta})_{,\sigma} \qquad (5\text{-}7\text{-}5)$$

So that the field equations are

$$\frac{\varepsilon}{2}\left[\overset{o}{g}{}^{\rho\sigma}\ \gamma_{\mu\nu,\rho\sigma} - (\overset{o}{g}{}^{\rho\sigma}\ \gamma_{\mu\rho,\sigma})_{,\nu} - (\overset{o}{g}{}^{\rho\sigma}\ \gamma_{\nu\rho,\sigma})_{,\mu} + \overset{o}{g}_{\mu\nu}\overset{o}{g}{}^{\rho\sigma}(\overset{o}{g}{}^{\lambda\beta}\gamma_{\rho\lambda,\beta})_{,\sigma}\right] = k\ T_{\mu\nu} \qquad (5\text{-}7\text{-}6)$$

These equations may be simplified by a convenient interpretation of the symmetries presented by the metric $g_{\mu\nu}$ of the weak field approximation. Indeed, from (5-7-1) we verify that $g_{\mu\nu}$ may be subjected to arbitrary Poincaré mappings

$$x'^{\mu} = x^{\mu} + \ell^{\mu}{}_{\nu}\, x^{\nu} + \ell^{\mu} , \quad \ell_{\mu\nu} = - \ell_{\nu\mu}$$

since the new $g_{\mu\nu}$ still has the form of a weak gravitational field.

$$g'_{\mu\nu}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}(x) = (\delta^{\alpha}_{\mu} - \ell^{\alpha}_{\mu})(\delta^{\beta}_{\nu} - \ell^{\beta}_{\nu})\ g_{\alpha\beta}$$

to first order terms this takes the form

$$g'_{\mu\nu}\ (x) = \overset{o}{g}_{\mu\nu} + \varepsilon\ \phi_{\mu\nu} - \ell_{\mu\nu} - \ell_{\nu\mu} + 0(\ell.\varepsilon)$$

but the infinitesimal Lorentzian matrix $l_{\mu\nu}$ is antisymmetric. so that

$$g'_{\mu\nu}(x) \cong \overset{o}{g}_{\mu\nu} + \varepsilon\ \phi_{\mu\nu} = g_{\mu\nu}(x)$$

which proves our statement. Besides these transformations, we may perform gauge transformations (these transformations will be seen to be just the gauge transformations for spin 2 fields),

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

where the $\xi^{\mu}(x)$ are first order infinitesimals. In this case

$$g'_{\mu\nu}\ (x') = (\delta^{\alpha}_{\mu} - \xi^{\alpha}{}_{,\mu})(\delta^{\beta}_{\nu} - \xi^{\beta}{}_{,\nu})(\overset{o}{g}_{\alpha\beta} + \varepsilon\ \phi_{\alpha\beta})$$
$$= \overset{o}{g}_{\mu\nu} + \varepsilon\ \phi_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu\ \mu} + 0(\xi^{2}, \xi.\varepsilon)$$

expanding the left hand side in Taylor's series around x,

$$g'_{\mu\nu}\ (x) \cong \overset{o}{g}_{\mu\nu} + \varepsilon\ \phi_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \qquad (5\text{-}7\text{-}7)$$

which has the form of a weak gravitational field (according to (5-7-1)) for

$$\varepsilon\ \phi'_{\mu\nu}(x) = \varepsilon\ \phi_{\mu\nu}(x) - \xi_{\mu,\nu}(x) - \xi_{\nu,\mu}(x) \qquad (5\text{-}7\text{-}8)$$

This last equation says that the point dependent part of the weak field is determin ed up to a term $\xi_{\mu,\nu} + \xi_{\nu,\mu}$ where the ξ_{μ} (x) are arbitrary infinitesimal functions.

The Riemann tensor $R_{\mu\nu\rho\sigma}$ is invariant under the gauge transformations. Indeed, the components of $R_{\mu\nu\rho\sigma}$ up to first order terms are

$$R_{\mu\nu\rho\sigma} \cong \frac{1}{2}(g_{\mu\sigma,\nu\rho} + g_{\nu\rho,\mu\sigma} - g_{\mu\rho,\nu\sigma} - g_{\nu\sigma,\mu\rho}) \quad (5\text{-}7\text{-}9)$$

a direct inspection shows that under $x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$, we get

$$R'_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}$$

Thus, the Ricci tensor and the scalar curvature will be too invariants in the first order approximation,

$$R_{\mu\nu} = g^{\lambda\beta} R_{\mu\lambda\nu\beta} \simeq \overset{o}{g}{}^{\lambda\beta} R_{\mu\lambda\nu\beta}$$

since $R_{\mu\lambda\nu\beta}$ is already of the first order. Then

$$R'_{\mu\nu} = \overset{o}{g}{}'^{\lambda\beta} R'_{\mu\lambda\nu\beta} = \overset{o}{g}{}^{\lambda\beta} R_{\mu\lambda\nu\beta} = R_{\mu\nu}$$

similar proof holds for the scalar curvature R. Therefore, the Einstein tensor is gauge invariant, $G_{\mu\nu}' = G_{\mu\nu}$, and the field equations will be gauge invariant as long as the source tensor $T_{\mu\nu}$ is also gauge invariant.

Besides this, the field equations obviously display covariance with respect to the Poincaré mappings. The property that the field equations for the ten unknown $\phi_{\mu\nu}$ are gauge invariant shows that the field equations (5-7-6) may be further simplified by imposing a gauge condition. In order to show that the present situation is entirely similar to what happens in electromagnetism, and in the spin 2 flat formulation, we give the correspondent treatment for those cases as an example.

For electromagnetism, we have in place of $R_{\mu\nu\rho\sigma}$ the gauge invariant field strength $F_{\mu\nu}$ and the field equations are

$$F^{\mu\nu}_{\ \ ,\nu} = j^{\mu}$$

the potentials A_μ are not gauge invariant, similarly to ours $\phi_{\mu\nu}$ but $F_{\mu\nu}$ may be thought as if it were an operator acting on A_μ and giving as result a gauge invariant quantity.

Indeed, we may put

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} = (\delta^\alpha_\mu \partial_\nu - \delta^\alpha_\nu \partial_\mu)A_\alpha = 0^\alpha_{\mu\nu} A_\alpha$$

similarly we may present $R_{\mu\nu\rho\sigma}$ of (5-7-9) as an operator equation on the gauge variant $g_{\mu\nu}$,

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\{\delta^\lambda_\mu \delta^\tau_\sigma \partial^2_{\nu\rho} + \delta^\lambda_\nu \delta^\tau_\rho \partial^2_{\mu\sigma} - \delta^\lambda_\mu \delta^\tau_\rho \partial^2_{\nu\sigma} - \delta^\lambda_\nu \delta^\tau_\sigma \partial^2_{\mu\rho}\} g_{\lambda\tau} = 0^{\lambda\tau}_{\mu\nu\rho\sigma} g_{\lambda\tau}$$

From the above expression for $F_{\mu\nu}$ we have for the Maxwell equations

$$(\delta^\mu_\alpha \partial^\nu \partial_\nu - \delta^\alpha_\nu \partial^\nu \partial_\mu)A_\alpha = j_\mu$$

or

$$(\delta^\mu_\alpha \Box - \partial_\mu \partial^\alpha)A_\alpha = j_\mu$$

The A_μ are determined up to the gradient of an arbitrary function, so that we may impose on them one condition. The field equations being entirely given in terms of the gauge invariant $F_{\mu\nu}$ are unaware of this arbitrarity in the A_μ. We thus impose the subsidiary condition of Lorentz.

$$\partial^\alpha A_\alpha = 0$$

case where the above field equation goes to the D'Alembertian equation with a source function j_μ.

$$\Box A_\mu = j_\mu$$

The gauge functions $\xi(x)$ which are allowed are now those which satisfy the scalar wave equation.

$$\Box \xi = 0$$

For spin 2 and null rest mass, similarly to photon equation seen, we have a gauge variant potential $A_{\mu\nu} = A_{\nu\mu}$, and introduce a differential operator which acting on

$A_{\mu\nu}$ yields a gauge invariant quantity. This operator may be

$$0^{\lambda}_{\rho\sigma} = \delta^{\lambda}_{\rho}\, \partial_{\sigma} - \delta^{\lambda}_{\sigma}\, \partial_{\rho}$$

since

$$\mathcal{R}_{\mu[\rho\sigma]} = (\delta^{\lambda}_{\rho}\, \partial_{\sigma} - \delta^{\lambda}_{\sigma}\, \partial_{\rho}) A_{\mu\lambda} = 0^{\lambda}_{\rho\sigma}\, A_{\mu\lambda}$$

is gauge invariant. However, we still can introduce the operator

$$0^{\lambda\tau}_{\mu\nu\rho\sigma} = \delta^{\lambda}_{\mu}\, \delta^{\tau}_{\sigma}\, \partial^2_{\nu\rho} - \delta^{\lambda}_{\mu}\, \delta^{\tau}_{\rho}\, \partial^2_{\nu\sigma} + \delta^{\lambda}_{\nu}\, \delta^{\tau}_{\rho}\, \partial^2_{\mu\sigma} - \delta^{\lambda}_{\nu}\, \delta^{\tau}_{\sigma}\, \partial^2_{\mu\rho}$$

which gives too a gauge invariant quantity.

$$\mathcal{R}_{[\mu\nu][\rho\sigma]} = 0^{\lambda\tau}_{\mu\nu\rho\sigma}\; A_{\lambda\tau}$$

This later object, the $\mathcal{R}_{[\mu\nu][\rho\sigma]}$ has all the symmetries of the Riemann tensor $R_{\mu\nu\rho\sigma}$. Really, the $\mathcal{R}_{[\mu\nu][\rho\sigma]}$ gets identical to $R_{\mu\nu\rho\sigma}$ if we make $A_{\mu\nu}$ identical to the metric $g_{\mu\nu}$. The field equations are the Fierz-Pauli [38] equations for spin 2 massless fields.

$$\mathcal{G}_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2}\, A_{\mu\nu}\, \mathcal{R} = T_{\mu\nu}$$

$$\mathcal{R}_{\mu\nu} = \overset{o}{g}{}^{\lambda\beta}\, \mathcal{R}_{[\mu\lambda][\nu\beta]}\, , \quad \mathcal{R} = \overset{o}{g}{}^{\lambda\beta}\, \mathcal{R}_{\lambda\beta}$$

since the $A_{\mu\nu}$ are determined up to the knowledge of four arbitrary functions $\xi_{\mu}(x)$, we may impose on them four conditions. These conditions are just a generalization of the Lorentz condition for spin 1,

$$\partial^{\alpha}\, A_{\mu\alpha} = 0 \tag{5-7-10}$$

so that the field equations simplify to

$$\Box(A_{\mu\nu} - \frac{1}{2}\, \overset{o}{g}_{\mu\nu}\, A) = T_{\mu\nu} \tag{5-7-11}$$

$$A = \overset{o}{g}{}^{\alpha\beta}\, A_{\alpha\beta}$$

As we note, the Fierz-Pauli theory for a symmetric second rank flat tensor is indeed very similar to the linearized Einstein equations, we may therefore impose too in the case of general relativity, in the weak field approximation, conditions similar to the (5-7-10). In this case they are called as the Harmonic coordinate conditions, suggested first by de Donder and by Fock. They are written for the $\gamma_{\mu\nu}(x)$,

$$\overset{o}{g}^{\rho\sigma}\, \gamma_{\mu\rho,\sigma}(x) = 0 \qquad (5\text{-}7\text{-}12)$$

The field equations (5-7-6) then simplify to the D'Alembertian type of equation with a source function.

$$\frac{\varepsilon}{2}\, \overset{o}{g}^{\rho\sigma}\, \gamma_{\mu\nu,\rho\sigma}(x) = k\, T_{\mu\nu}(x) \qquad (5\text{-}7\text{-}13)$$

These will be the equations for the gravitational field in the linear approximation. The Harmonic condition is then interpreted as a gauge condition, a fact largely used in all the applications of those equations. All indices are lowered and raised by $\overset{o}{g}$. Let us now see how the $\gamma_{\mu\nu}(x)$ transform under a gauge transformation, from (5-7-4) and (5-7-8) one gets

$$\gamma'_{\mu\nu}(x) \simeq \gamma_{\mu\nu}(x) - \overset{o}{g}_{\mu\lambda}\, \xi^{\lambda}_{\ ,\nu}(x) - \overset{o}{g}_{\lambda\nu}\, \xi^{\lambda}_{\ ,\mu}(x) + \overset{o}{g}_{\mu\nu}\, \xi^{\lambda}_{\ ,\lambda}(x)$$

multiplying this by the operator $\overset{o}{g}^{\nu\alpha}\, \partial_{\alpha}$ we obtain

$$\overset{o}{g}^{\nu\alpha}\, \gamma'_{\mu\nu,\alpha}(x) = \overset{o}{g}^{\nu\alpha}\, \gamma_{\mu\nu,\alpha}(x) - \overset{o}{g}_{\mu\lambda}\, \overset{o}{g}^{\nu\alpha}\, \xi^{\lambda}_{\ ,\nu\alpha}$$

in order that the transformed $\gamma'_{\mu\nu}$ still satisfy the Harmonic condition it is necessary that the gauge transformations be such that

$$\overset{o}{g}^{\nu\alpha}\, \xi^{\lambda}_{\ ,\nu\alpha}(x) \equiv \Box \xi^{\lambda}(x) = 0 \qquad (5\text{-}7\text{-}14)$$

Which is entirely similar to the preserving of the Lorentz condition in Electrodynamics, case where the gauge transformations are subjected to satisfy the wave equation. Thus, we use only gauge transformations which preserve the Harmonic coordinate conditions. They are the transformations satisfying (5-7-14) and form a sub-group of all possible gauge transformations.

6. SOLUTIONS OF THE EINSTEIN EQUATIONS

6.1) Solutions of the Linearized Einstein Field Equations

We begin this section by discussing the solutions of the linearized field equations of Einstein. The reason for that is twofold, first we begin by the simplest possible situation, which for didactic reasons is quite logical. Second, the knowledge of these simple solutions will serve as guide marks towards the understanding of the content of the full solutions. Besides the obvious argument that due to the smallness of the gravitational coupling constant, these solutions serve for characterizing a large class of gravitational fields of interest.

As we have seen, the linearized Einstein equations may be put in the form

$$\frac{\varepsilon}{2} \overset{o}{g}{}^{\rho\sigma} \gamma_{\mu\nu,\rho\sigma} = k\ T_{\mu\nu} \tag{6-1-1}$$

plus the supplementary condition

$$\overset{o}{g}{}^{\rho\sigma} \gamma_{\mu\rho,\sigma} = 0 \tag{6-1-2}$$

The general solution of (6-1-1) is

$$\varepsilon\ \gamma_{\mu\nu}(x) = \varepsilon \overset{(1)}{\gamma}_{\mu\nu}(x) + \varepsilon \overset{(2)}{\gamma}_{\mu\nu}(x)$$

where $\overset{(1)}{\gamma}_{\mu\nu}$ is a solution of the equation with the source term, a particular solution,

$$\varepsilon \overset{(1)}{\gamma}_{\mu\nu}(x) = \frac{k}{2\pi} \int d^4x'\ \delta((x-x')^2)\ T_{\mu\nu}(x') \tag{6-1-3}$$

and $\overset{(2)}{\gamma}_{\mu\nu}$ is a general solution of the equation without the right hand side, the homogeneous equations. In discussing these solutions we shall consider several special situations.

(a) Stationary Mass Distributions

Using the general formula $T_{\mu\nu} = \rho\ u_\mu\ u_\nu$ where u^μ is the four-velocity of the source particle, we get for a particle at rest at the origin of the spatial coordinates

$$T_{oo}(x) = m\ \delta_3(x)$$

all other components being null. Then

$$\varepsilon \overset{(1)}{\gamma}_{oo}(x) = \frac{km}{2\pi}\int dx'_o \int d_3x'\ \delta((x-x')^2)\ \delta_3(x')$$

$$= \frac{km}{2\pi}\int dx'_o \int d_3x'\ \delta((x'_o-x_o)^2 - (\vec{x}'-\vec{x})^2)\ \delta_3(\vec{x}')$$

recalling that in the linear approximation the role of the metric is taken over by the flat $\overset{o}{g}_{\mu\nu}$. Integration over $\vec{x}'$ gives

$$\varepsilon \overset{(1)}{\gamma}_{oo}(x) = \frac{km}{2\pi}\int dx'_o\ \delta((x'_o-x_o)^2 - \vec{x}^2)$$

Using the formula

$$\delta(x^2-a^2) = \frac{1}{2|a|}\{\delta(x+a) + \delta(x-a)\}$$

which in our case is

$$\delta((x'_o-x_o)^2 - \vec{x}^2) = \frac{1}{2|x|}\{\delta(x'_o-x_o + |\vec{x}|) + \delta(x'_o-x_o-|\vec{x}|)\}$$

we get

$$\varepsilon \overset{(1)}{\gamma}_{oo}(x) = \frac{km}{4\pi}\ \frac{1}{|\vec{x}|}\left\{\int dx'_o\ \delta(x'_o-x_o+|\vec{x}|) + \int dx'_o\ \delta(x'_o-x_o-|\vec{x}|)\right\}$$

each one of the above integrals give one, so that their sum is equal to 2, therefore

$$\varepsilon \overset{(1)}{\gamma}_{oo}(x) = \frac{km}{2\pi|\vec{x}|} \tag{6-1-4}$$

Rewriting this in terms of the $\phi_{\mu\nu}$ by using the formula

$$\phi_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\overset{o}{g}_{\mu\nu}\ \gamma$$

and recalling that the only non-vanishing $\overset{(1)}{\gamma}_{\mu\nu}$ is the $\overset{(1)}{\gamma}_{oo}$, we get

$$\phi_{oi} = 0$$

$$\phi_{oo} = \frac{1}{2}\,\gamma_{oo}$$

$$\phi_{rs} = \frac{1}{2}\,\delta_{rs}\,\gamma_{oo}$$

Thus,

$$g_{oo} = 1 + \frac{\varepsilon}{2}\,\gamma_{oo} = 1 + \frac{km}{4\pi|\vec{x}|}$$

$$g_{oi} = \varepsilon\ \phi_{oi} = 0$$

$$g_{rs} = \delta_{rs}(-1 + \frac{\varepsilon}{2}\,\gamma_{oo}) = \delta_{rs}\left(-1 + \frac{km}{4\pi|\vec{x}|}\right)$$

substituting k by its explicit value $-\frac{8\pi G}{c^4}$, and using dimensions where the velocity of light is c, one finds

$$g_{oo} = 1 - \frac{2\,G\,m}{c^2|\vec{x}|}$$

$$g_{oi} = 0 \qquad (6\text{-}1\text{-}5)$$

$$g_{rs} = \delta_{rs}\left(-1 - \frac{2\,G\,m}{c^2|\vec{x}|}\right)$$

We see that the linear approximation holds good only if $2GM/rc^2 \ll 1$, where $r = |\vec{x}|$. This represents a limitation on the possible values for the mass M source of the field (in this example the field is static). At the surface of the earth this holds good since there

$$2GM/Rc^2 \approx 10^{-9}$$

Even for the static gravitational field of the Sun, case where $M \approx 1.9\times10^{33}$ g, $R \approx 864\ 000$ miles, we obtain

$$2GM/Rc^2 \approx 10^{-6}$$

What happens is that the Sun being in gaseous state, its mass is distributed in a large region, thus R is large as compared with the mass of its several parts. This tends to make the above ratio smaller than it should be if for instance the Sun were not into gaseous state. For the stars of the type called as "white dwarfs" which are small in size yet are very massive bodies (for instance they are about 150.000 to 800.000 times the mass of the Earth) with all the mass concentrated in a region of the size of the Earth or even smaller, the above ratio may go up to the value 10^{-3}. For the Neutronic stars this is even lower than this value, eventually reaching the upper limit of one. This gives an idea of the wide range of application of the linear approximation.

For general stationary mass distributions characterized by $T_{oo}(x)$ with all other components of $T_{\mu\nu}$ zero, we have

$$\varepsilon \overset{(1)}{\gamma}_{oo}(x) = \frac{k}{2\pi}\int d_3x' \frac{T_{oo}(x')}{|\vec{x}-\vec{x}'|} \tag{6-1-6}$$

A multipole expansion is obtained by expanding the denominator into power series of x'^i. $(r = |\vec{x}|)$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r} - x'^r\left(\frac{1}{r}\right)_{,r} + \frac{1}{2}x'^r x'^s\left(\frac{1}{r}\right)_{,rs} + \ldots$$

Then

$$\varepsilon \overset{(1)}{\gamma}_{oo}(x) = \frac{k}{2\pi}\left\{\frac{M}{r} - D_s\left(\frac{1}{r}\right)_{,s} + \frac{1}{6}Q_{rs}\left(\frac{1}{r}\right)_{,rs} - \ldots\right\} \tag{6-1-7}$$

Taking into account that $T_{oo}(x)$ is just equal to the mass density $\rho(x)$, the first term is the field of a total mass M as if it were at the origin (gravitational monopole field). The second term represents the dipole field and the third the quadrupole field, and so on,

$$M = \int T_{oo}(x)\ d_3x$$

$$D_s = \int x_s\ T_{oo}(x)\ d_3x$$

$$Q_{rs} = \int (3\ x_r x_s - r^2\ \delta_{rs})\ T_{oo}(x)\ d_3x$$

By a suitable translation of the axes D_s may be set equal to zero since $T_{oo}(x)$ is positive (no negative mass), as result no dipole field of gravitational origin does exist.

In regard to how a linear field solution may approximate a full solution, we may say here the following: In the previous expansion only the first term, describing the monopole field M/r, exist as a first term of an expansion of a full solution, the Schwarzschild solution. At present is not known if the quadrupole and higher moment terms correspond to expansions of exact solutions. Such terms correspond to fields produced by nonspherically symmetric mass distributions, for instance an oblate sun or earth. Their existence in general relativity is thus not fully determined.

(b) <u>More General Solutions</u>

We may allow g_{or} to be nonzero by introducing a source $T_{or}(x)$, that is, a stationary source with a more general structure than the previous one. We then have for the extra component $\varepsilon \overset{(1)}{\gamma}_{os}(\vec{x})$,

$$\varepsilon \overset{(1)}{\gamma}_{os}(\vec{x}) = \frac{k}{2\pi} \int \frac{T_{os}(\vec{x}')\ d_3x'}{|\vec{x} - \vec{x}'|} \qquad (6\text{-}1\text{-}8)$$

Expanding in Taylor's series of $\vec{x}'$

$$\varepsilon \overset{(1)}{\gamma}_{os}(\vec{x}) = \frac{k}{2\pi} \left\{ \frac{P_s}{|\vec{x}|} + S_{sj} \left(\frac{1}{|\vec{x}|} \right)_{,j} + \ldots \right\} \qquad (6\text{-}1\text{-}9)$$

where

$$P_s = \int T_{os}(\vec{x})\ d_3x$$

$$S_{sj} = \int x_s\ T_{oj}(\vec{x})\ d_3x$$

....

This solution must satisfy the gauge condition (6-1-2). Since presently the field is time independent, this condition reads as

$$\overset{(1)}{\gamma}_{\mu i}{}^{,i} = -\gamma_{\mu i,i} = 0$$

applied to the $\overset{(1)}{\gamma}_{oi}$ this gives $\overset{(1)}{\gamma}_{os,s} = 0$. From (6-1-9) we get

$$P_s = 0$$

$$S_{sj} = \frac{1}{2}\alpha_{sj} + \beta\ \delta_{sj}$$

where $\alpha_{rs} = -\alpha_{sr}$ and β are four arbitrary constants. Note that the β is associated to the symmetric part of S_{rs}, whereas the α_{rs} correspond to the skew symmetric part of S_{rs}.

The symmetric part of S_{rs} may be set equal to zero by suitable choice of gauge, indeed, by taking the transformation

$$x'^{\nu} = x^{\nu} + \xi^{\nu}(x)$$

with gauge function

$$\xi^{o}(x) = \frac{k}{2\pi}\ \frac{\beta}{|\vec{x}|}$$

$$\xi^{r}(x) = 0$$

we have (from here on we shall supress the ε denoting infinitesimal part since this is clearly understood)

$$\gamma'_{\mu\nu} = \phi'_{\mu\nu} - \frac{1}{2}\overset{o}{g}_{\mu\nu}\ \phi'$$

in the new frame, where the $\phi'_{\mu\nu}$ are

$$\phi'_{\mu\nu} = \phi_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$$

we have,

$$\phi'_{os} = \phi_{os} - \xi_{o,s} - \xi_{s,o} = \phi_{os} - \xi_{o,s}$$

or

$$\gamma'_{os} = \phi'_{os} - \frac{1}{2}\overset{o}{g}_{os}\ \phi' = \phi'_{os}$$

which gives

$$\gamma'_{os} = \phi'_{os} - \xi_{o,s} = \gamma_{os} - \xi_{o,s} = \gamma_{os} - \frac{k}{2\pi}\beta\left(\frac{1}{|\vec{x}|}\right)_{,s}$$

and according to the expression for $\overset{(1)}{\gamma}_{os}$, we get

$$\overset{(1)}{\gamma'}_{os} = \frac{k}{4\pi}\alpha_{sr}\left(\frac{1}{|\vec{x}|}\right)_{,r} + \dots \qquad (6\text{-}1\text{-}10)$$

But $T^{\mu\nu}x^{\sigma} - T^{\mu\sigma}x^{\nu}$ is just the four-dimensional angular momentum tensor for the linearized gravitational field. Its spatial part is the total angular momentum of the sources.

$$M^{rs} = \int (T^{or}\, x^{s} - T^{os}\, x^{r})\, d_3x = \alpha^{rs} \qquad (6\text{-}1\text{-}11)$$

thus, the field $\overset{(1)}{\gamma}_{os}$ describes the behaviour of an uniform rotating mass distribution with angular momentum M^{rs}. Such type of contribution to the relativistic gravitational field was considered in relation to the effect of rotation of the sun over its planetary orbits. [39]

(c) Gravitational Waves in the Linear Approximation

For empty space-time regions, $T_{\mu\nu} = 0$, the field equations for $\gamma_{\mu\nu}$, or for the $\phi_{\mu\nu}$, reduce to the D'Alembertian equations.

$$\Box\gamma_{\mu\nu} = 0 \qquad (6\text{-}1\text{-}12)$$

Let us consider the case where $\gamma_{\mu\nu}$ depends only on x and $x^{o} = ct$, that is a plane wave. The wave equations possess solutions given by arbitrary functions of $x \pm x^{o}$.

Taking a solution representing a wave propagation along the positive direction x, we have

$$\gamma_{\mu\nu} = \gamma_{\mu\nu}(x - x^o)$$

These solutions are subjected to the Harmonic gauge conditions $\gamma_{\mu\nu}{}^{,\nu} = 0$, which in the present case read as

$$\gamma_{\mu o,o} - \gamma_{\mu 1,1} = 0$$

(our signature is taken as -2). Then, indicating differentiation with respect to x-x^o by a dot,

$$- \dot{\gamma}_{\mu o} - \dot{\gamma}_{\mu 1} = 0. \qquad (6\text{-}1\text{-}13)$$

This equation is integrated, and we get

$$\gamma_{\mu o} + \gamma_{\mu 1} = 0$$

we took the constant of integration as zero since we are interested just in the varying parts of the field. We thus have

$$\begin{aligned} \gamma_{oo} &= - \gamma_{o1} \\ \gamma_{1o} &= \gamma_{o1} = - \gamma_{11} \\ \gamma_{2o} &= \gamma_{o2} = - \gamma_{21} \\ \gamma_{3o} &= \gamma_{o3} = - \gamma_{31} \end{aligned} \qquad (6\text{-}1\text{-}14)$$

We may subject all variables to any gauge transformation which satisfy the wave equation, that is any $\xi^{\mu}(x^{\alpha})$ such that $\Box\xi^{\mu} = 0$. Taking $\xi^{\mu} = \xi^{\mu}(x\text{-}x^o)$ we have this condition satisfied. The $\gamma_{\mu\nu}$ transform as

$$\gamma'_{\mu\nu} = \gamma_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \overset{o}{g}_{\mu\nu} \xi^{\lambda}{}_{,\lambda}$$

which presently take the form

$$\gamma'_{\mu\nu} = \gamma_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \overset{o}{g}_{\mu\nu}(-\dot{\xi}_o - \dot{\xi}_1)$$

The $\xi^{\mu}(x-x^{o})$ are four quantities to be chosen as we want to. We pick them out such that the four equations hold

$$\begin{aligned} \gamma'_{o1} &= 0 \\ \gamma'_{o2} &= 0 \\ \gamma'_{o3} &= 0 \\ \gamma'_{22} + \gamma'_{33} &= 0 \end{aligned} \tag{6-1-15}$$

Since

$$\gamma'_{o1} = \gamma_{o1} - \dot{\xi}_{o} + \dot{\xi}_{1}$$

$$\gamma'_{o2} = \gamma_{o2} + \dot{\xi}_{2}$$

$$\gamma'_{o3} = \gamma_{o3} + \dot{\xi}_{3}$$

$$\gamma'_{22} + \gamma'_{33} = \gamma_{22} + \gamma_{33} - 2\dot{\xi}_{o} - 2\dot{\xi}_{1}$$

a direct integration gives for the $\xi^{\mu}(x-x^{o})$ the values

$$\xi_{1}(x-x^{o}) = \frac{1}{4}\int(\gamma_{22} + \gamma_{33})\, d(x-x^{o}) - \frac{1}{2}\int \gamma_{o1}(x-x^{o})\, d(x-x^{o})$$

$$\xi_{2}(x-x^{o}) = -\int \gamma_{22}(x-x^{o})\, d(x-x^{o})$$

$$\xi_{3}(x-x^{o}) = -\int \gamma_{o2}(x-x^{o}]\, d(x-x^{o})$$

$$\xi_{o}(x-x^{o}) = \frac{1}{4}\int(\gamma_{22} + \gamma_{33})\, d(x-x^{o}) + \frac{1}{2}\int \gamma_{o1}(x-x^{o})\, d(x-x^{o})$$

In this new gauge frame (we drop the lines for brevity) we get, from (6-1-14) and (6-1-15), *

* The eqs. (6-1-14) are satisfied in this new frame since the ξ^{μ} satisfy the wave equation $\Box\xi^{\mu} = 0$, which as we saw is the necessary condition for preservation of the Harmonic gauge relation $\gamma_{\mu\nu}{}^{,\nu} = 0$.

$$(\gamma_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \gamma_{22} & \gamma_{23} & 0 \\ 0 & \gamma_{23} & -\gamma_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

That means, the $\gamma_{\mu\nu}$ are represented by a symmetric second rank tensor with null trace in the plane YZ. This plane is orthogonal to the direction of propagation of the wave, and since this tensor has just two independent components, γ_{22} and γ_{23}, we conclude that the plane gravitational wave is a transverse wave with polarization given by two functions $\gamma_{22}(x-x^o)$ and $\gamma_{23}(x-x^o)$. That is, two different states of polarization may occur.

(d) Time Varying Sources

For the situation where the source term $T_{\mu\nu}$ is present and depends on x^o, we may take directly as solution of the field equation the integral (6-1-3). For sources which are bounded into the three-space this solution satisfies the gauge condition (6-1-2) provided $T_{\mu\nu}$ satisfies

$$T^{\mu\nu}_{\ ,\nu} = 0 \qquad (6\text{-}1\text{-}16)$$

where

$$T^{\mu\nu} = \overset{o}{g}{}^{\mu\lambda} \ \overset{o}{g}{}^{\nu\alpha} \ T_{\lambda\alpha}$$

We thereby obtain

$$\frac{\partial}{\partial x^o} \int T^{\mu o} \ x^s \ d_3x = - \int T^{\mu r}_{\ ,r} \ x^s \ d_3x$$

$$= - \int (T^{\mu r} \ x^s)_{,r} \ d_3x + \int T^{\mu s} \ d_3x$$

since $T^{\mu\nu}$ is bounded on the three-space, the first integral standing on the right hand side vanishes as result of the Gauss theorem. Thus

$$\frac{\partial}{\partial x^o}\int T^{\mu o}\, x^s\, d_3x = \int T^{\mu s}\, d_3x \qquad (6\text{-}1\text{-}17)$$

Similarly we prove that

$$\frac{\partial}{\partial x^o}\int T^{oo}\, x^r\, x^s\, d_3x = \int (T^{ro}\, x^s + T^{so}\, x^r)\, d_3x \qquad (6\text{-}1\text{-}18)$$

Taking $\mu = r$ in (6-1-17) and nothing that T^{rs} is symmetrical we obtain

$$\int T^{rs}\, d_3x = \frac{1}{2}\, \frac{\partial}{\partial x^o}\left(\int (T^{ro}\, x^s + T^{so}\, x^r)\, d_3x\right)$$

From (6-1-18) this reads as

$$\int T^{rs}\, d_3x = \frac{1}{2}\, \frac{\partial^2}{\partial x^{o2}}\int T^{oo}\, x^r\, x^s\, d_3x \qquad (6\text{-}1\text{-}19)$$

If we now use the relation

$$\delta\,(x^2) = \delta(x_o^2 - \vec{x}^2) = \frac{1}{2|\vec{x}|}\, \{\delta(x_o + |\vec{x}|) + \delta(x_o - |\vec{x}|)\}$$

which for our case should be written for $\delta((x-x')^2) = \delta((x_o-x_o')^2 - (\vec{x}-\vec{x}')^2)$ as

$$\delta((x-x')^2) = \frac{1}{2|\vec{x}-\vec{x}'|}\, \{\delta(x_o - x_o' + |\vec{x}-\vec{x}'|) + \delta(x_o - x_o' - |\vec{x}-\vec{x}'|)\}$$

expanding $|\vec{x}-\vec{x}'|^{-1}$ in power series of $\vec{x}'$,

$$\begin{aligned}\delta(x-x')^2) = {} & \frac{1}{2|\vec{x}|}\left[\delta(x_o - x_o' + |\vec{x}-\vec{x}'|) + \delta(x_o - x_o' - |\vec{x}-\vec{x}'|)\right] - \\ & - \frac{1}{2}\left(\frac{1}{|\vec{x}|}\right)_{,u} x'^u \left[\delta(x_o - x_o' + |\vec{x}-\vec{x}'|) + \delta(x_o - x_o' - |\vec{x}-\vec{x}'|)\right] + \\ & + \frac{1}{4}\left(\frac{1}{|\vec{x}|}\right)_{,uv} x'^u\, x'^v \left[\delta(x_o - x_o' + |\vec{x}-\vec{x}'|) + \delta(x_o - x_o' - |\vec{x}-\vec{x}'|)\right] + \dots\end{aligned}$$

Thus, the solution for the field equation will be

$$\varepsilon\ \gamma_{\mu\nu}(x) = \frac{k}{2\pi}\int d_4x'\ \delta((x-x')^2)\ T_{\mu\nu}(x'_o, \vec{x}')$$

Let us take $\mu = r$, $\nu = s$. We get

$$\varepsilon\ \gamma_{rs}(x) = \frac{k}{2\pi}\left\{ \frac{1}{2|\vec{x}|}\int dx'_o \int d_3x' \left[\delta(x_o-x'_o+|\vec{x}-\vec{x}'|) + \delta(x_o-x'_o-|\vec{x}-\vec{x}'|)\right] T_{rs}(x'_o,\vec{x}') - \right.$$

$$- \frac{1}{2}\left(\frac{1}{|\vec{x}|}\right)_{,u} \int dx'_o \int d_3x'\ x'^u \left[\delta(x_o-x'_o+|\vec{x}-\vec{x}'|) + \delta(x_o-x'_o-|\vec{x}-\vec{x}'|)\right]\ T_{rs}(x'_o, \vec{x}') +$$

$$\left. + \frac{1}{4}\left(\frac{1}{|\vec{x}|}\right)_{,uv} \int dx'_o \int d_3x'\ x'^u\ x'^v \left[\delta(x_o-x'_o+|\vec{x}-\vec{x}'|)+\delta(x_o-x'_o-|\vec{x}-\vec{x}'|)\right] T_{rs}(x'_o,\vec{x}') \ldots \right\}$$

Integration on x'^o yields,

$$\varepsilon\ \gamma_{rs}(x) = \frac{k}{2\pi}\left\{ \frac{1}{2|\vec{x}|}\int d_3x' \left[T_{rs}(x_o+|\vec{x}-\vec{x}'|,\ x') + T_{rs}(x_o-|\vec{x}-\vec{x}'|,\ \vec{x}')\right] - \right.$$

$$- \frac{1}{2}\left(\frac{1}{|\vec{x}|}\right)_{,u} \int d_3x' \left[x'^u\ T_{rs}(x_o+|\vec{x}-\vec{x}'|,\ \vec{x}') + x'^u\ T_{rs}(x_o-|\vec{x}-\vec{x}'|,\ \vec{x}')\right] +$$

$$\left. + \frac{1}{4}\left(\frac{1}{|\vec{x}|}\right)_{,uv} \int d_3x' \left[x'^u\ x'^v\ T_{rs}(x_o+|\vec{x}-\vec{x}'|,\vec{x}') + x'^u\ x'^v\ T_{rs}(x_o-|\vec{x}-\vec{x}'|,\vec{x}')\right] + \ldots \right\}$$

Using (6-1-19) this takes the form

$$\varepsilon\ \gamma_{rs}(x) = \frac{k}{4\pi}\left\{ \frac{I''_{rs}}{|\vec{x}|} + \frac{n^k\ I'''_{rsk}}{|\vec{x}|^2} + \ldots . \right\} \qquad (6\text{-}1\text{-}20)$$

with

$$n^k = \frac{x^k}{|\vec{x}|}$$

Where I are the moments of the matter density,

$$I^{\pm}_{rs} = \int \rho(x_o \pm |\vec{x}-\vec{x}'|,\ \vec{x}')\ x'_r\ x'_s\ d_3x'$$

$$I^{\pm}_{rsk} = \int \rho(x_o \pm |\vec{x}-\vec{x}'|,\ \vec{x}')\ x'_r\ x'_s\ x'_k\ d_3x'$$

the primes indicate differentiation with respect to x^o. In explicit notation, the (6-1-20) is

$$\varepsilon\,\gamma_{rs}(x) = \frac{k}{4}\left\{\frac{1}{|\vec{x}|}\left(\overset{''}{I}{}^{+}_{rs}(x_o,\vec{x}) + \overset{''}{I}{}^{-}_{rs}(x_o,\vec{x})\right) + \right.$$

$$\left. + \frac{n^k}{|\vec{x}|^2}\left(\overset{'''}{I}{}^{+}_{rsk}(x_o,\vec{x}) + \overset{'''}{I}{}^{-}_{rsk}(x_o,\vec{x})\right) + \ldots\right\} \qquad (6\text{-}1\text{-}21)$$

The integral with the + indicates the contribution of the advanced value of the matter density at the time x^o, and with sign - means the contribution of the retarded part of the matter density at x^o. A similar calculation for the remaining components of the $\varepsilon\,\gamma_{\mu\nu}$ gives.

$$\varepsilon\,\gamma_{ro}(x) = \frac{k}{2\pi}\left\{-\frac{I'^{+}_{r}}{|\vec{x}|} - \frac{I'^{-}_{r}}{|\vec{x}|} - \frac{1}{2}\,\frac{n^s}{|\vec{x}|^2}\left(I''^{+}_{rs} + I''^{-}_{rs}\right) + \ldots\right\} \qquad (6\text{-}1\text{-}22)$$

$$\varepsilon\,\gamma_{oo}(x) = \frac{k}{2}\left\{\frac{I^{+} + I^{-}}{|\vec{x}|} + \frac{n^r}{|\vec{x}|}\left(I'^{+}_{r} + I'^{-}_{r}\right) + \frac{1}{2}\,\frac{n^r\,n^s}{|\vec{x}|^3}\left(I''^{+}_{rs} + I''^{-}_{rs}\right) + \ldots\right\}$$

$$(6\text{-}1\text{-}23)$$

where according to the same notation

$$I^{\pm}(x_o,\vec{x}) = \int \rho(x_o \pm |\vec{x}-\vec{x}'|,\, \vec{x}')\, d_3x'$$

$$I^{\pm}_{r}(x_o,\vec{x}) = \int \rho(x^o \pm |\vec{x}-\vec{x}'|,\, \vec{x}')\, x'_r\, d_3x' .$$

These solutions will be of importance later on when discussing the problem of interaction of the radiation with the sources.

6.2) Solutions with Spherical Symmetry - The Schwarzschild Field

One of the first, and perhaps still the most important, exact solution of the Einstein equations was obtained by Schwarzschild [40] which imposed the condition of spherical symmetry on the $g_{\mu\nu}$ requiring at the same time that it be static.

Initially we will work out the most general form for a symmetric second rank tensor displaying symmetry with respect to the three-dimensional rotation group. If this happens, we have from the Killing equation

$$\bar{\delta}\, g_{\mu\nu}(x) \equiv - g_{\mu\rho}\, \xi^{\rho}_{\ ,\nu} - g_{\rho\nu}\, \xi^{\rho}_{\ ,\mu} - g_{\mu\nu,\rho}\, \xi^{\rho} = 0$$

for the Euclidian Killing vector *

$$\xi^{\rho} = (\xi^{o}, \xi^{r}) = (0, \varepsilon^{rk}\, x^{k})$$

$$\varepsilon^{rk} = - \varepsilon^{kr}$$

for $\mu = 0$, $\nu = 0$ we obtain

$$- g_{oo,i}\, \varepsilon^{ik}\, x^{k} = 0$$

as it may be easily checked this equation becomes identically satisfied if g_{oo} is a function of the distance r and of x^{o}. Taking $\mu = 0$, $\nu = r$ we get

$$- g_{oi}\, \varepsilon^{ir} - g_{or,i}\, \varepsilon^{ik}\, x^{k} = 0$$

again, this equation becomes identically satisfied for g_{or} of the form

$$g_{or} = \beta(r, x^{o})\, \frac{x^{r}}{r}$$

Of course, what we obtained from the previous relations is that g_{or} must be a function of r and x^{o} times the x^{s}. By convenience we wrote this function in the form above. For $\mu = r$, $\nu = r$, we get

$$- g_{ru}\, \varepsilon^{ur} - g_{us}\, \varepsilon^{ur} - g_{rs,u}\, \varepsilon^{uv}\, x^{v} = 0$$

which similarly as before implies in

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* That is, $\xi_{r,s} + \xi_{s,r} = 0$.

$$g_{ik}(x) = \gamma(r, x^o)\delta_{ik} + \lambda(r, x^o) \frac{x^i x^k}{r^2}$$

In summary, the imposition of spherical symmetry on the $g_{\mu\nu}$ implies that the components of this tensor are given in terms of four arbitrary function of r and x^o as

$$\begin{aligned} g_{oo}(x) &= \alpha(r, x^o) \\ g_{oi}(x) &= \beta(r, x^o)\, x^i/r \\ g_{ik}(x) &= \gamma(r, x^o)\, \delta_{ik} + \lambda(r, x^o)\, x^i x^k/r^2 \end{aligned} \tag{6-2-1}$$

We now look for more general mappings, consistent with the curvilinear structure of the manifold, which leave the (6-2-1) form invariant. It may be proved that under the mapping

$$\begin{aligned} x^s &= f_1(r', x'_o)\, x'^s \\ x^o &= f_2(r', x'_o) \end{aligned}$$

the relations (6-2-1) keep the same form in the new coordinate system. We prove this as an exercise.

Exercise: Prove that (6-2-1) are form invariant under the above mapping.

Solution: Take for instance the g_{ik}, we have

$$g'_{ik}(x') = \frac{\partial x^\ell}{\partial x'^i} \frac{\partial x^m}{\partial x'^k} g_{\ell m}(x) + \left(\frac{\partial x^\ell}{\partial x'^i} \frac{\partial x^o}{\partial x'^k} + \frac{\partial x^\ell}{\partial x'^k} \frac{\partial x^o}{\partial x'^i} \right) g_{\ell o}(x) + \frac{\partial x^o}{\partial x'^i} \frac{\partial x^o}{\partial x'^k} g_{oo}(x)$$

by noting that up to now both the x^s and the x'^s are cartesian coordinates for the three-space.

$$\frac{\partial x^\ell}{\partial x'^i} = f_1\, \delta^\ell_i + \frac{\partial f_1}{\partial r'} \frac{x'^i x'^\ell}{r'}$$

$$\frac{\partial x^o}{\partial x'^i} = \frac{\partial f_2}{\partial r'} \frac{x'^i}{r'}$$

so that

$$g'_{ik}(x') = f_1^2\, g_{ik} + f_1 \frac{\partial f_1}{\partial r'} \frac{x'^m x'^k}{r'} g_{im} + f_1 \frac{\partial f_1}{\partial r'} \frac{x'^i x'^\ell}{r'} g_{\ell k} +$$

$$+ \left(\frac{\partial f_1}{\partial r'}\right)^2 \frac{1}{r'^2}\, x'^i x'^\ell x'^m x'^k\, g_{\ell m} + f_1 \frac{\partial f_2}{\partial r'} \frac{x'^k}{r'} g_{io} +$$

$$+ f_1 \frac{\partial f_2}{\partial r'} \frac{x'^i}{r'} g_{ko} + 2 \frac{\partial f_1}{\partial r'} \frac{\partial f_2}{\partial r'} \frac{x'^i x'^\ell x'^k}{r'^2} g_{\ell o} \left(\frac{\partial f_2}{\partial r'}\right)^2 \frac{x'^i x'^k}{r'^2} g_{oo}$$

An inspection on the structure of the equations of the mapping shows that

$$r = \sqrt{x^s x^s} = f_1\, r' \tag{6-2-2}$$

thus

$$\frac{x^r}{r} = \frac{f_1\, x'^r}{f_1\, r'} = \frac{x'^r}{r'} \tag{6-2-3}$$

From (6-2-2.3) and (6-2-1) we get

$$g_{oo} = \alpha(r, x^o) = \alpha(f_1\, r', f_2(r', x'_o)) = \phi(r', x'_o)$$

$$g_{oi} = \beta(r, x^o) \frac{x^i}{r} = \beta(f_1 r', f_2(r', x'_o)) \frac{x'^i}{r'} = \Psi(r', x'^o) \frac{x'^i}{r'}$$

$$g_{ik} = \gamma(r, x^o)\delta_{ik} + \lambda(r, x^o) \frac{x^i x^k}{r^2} = \gamma(f_1\, r', f_2(r', x'_o))\delta_{ik} +$$

$$+ \lambda(f_1 r', f_2(r', x'_o)) \frac{x'^i x'^k}{r'^2} = \chi(r', x'_o)\, \delta_{ik} + \tau(r', x'_o) \frac{x'^i x'^k}{r'^2}$$

Substituting the later of these relations into the previous form for $g'_{ik}(x')$ we get

$$g'_{ik}(x') = \chi\, f_1^2\, \delta_{ik} + \{\tau\, f_1^2 + f_{1,r'}^2\, \chi\, r'^2 + f_{1,r'}^2\, \tau\, r'^2 + 2\, f_1\, f_{1,r'}\, r'(\chi+\tau)$$

$$+ 2\, f_1\, f_{2,r'}\, \Psi + 2\, f_{1,r'}\, f_{2,r'}\, r'^3\, \Psi + f_{2,r'}^2\, \phi\,\} \frac{x'^i x'^k}{r'^2} =$$

$$= F(r_1', x'^o)\ \delta_{ik} + M(r', x'^o)\ \frac{x'^i\ x'^k}{r'^2}$$

which proves our initial proposition. The same proof may be done for g'_{oi} and g'_{oo}.

We can use the previous transformations which depend on two arbitrary functions, the f_1 and f_2 in order to eliminate two of the four arbitrary functions which appear in the definition (6-2-1) for $g_{\mu\nu}$. This is obtained by adequate choice of the f_1 and f_2. First we rewrite the transformation law for g_{rs} by using directly the expression given by (6-2-1),

$$g'_{ik}(x') = \left(f_1\ \delta_i^\ell + \frac{\partial f_1}{\partial r'}\ \frac{x'^i}{r'}\ x'^\ell\right)\left(f_1\ \delta_k^m + \frac{\partial f_1}{\partial r'}\ \frac{x'^m}{r'}\ x'^k\right)\left(\gamma\ \delta_{\ell m} + \lambda\ \frac{x^\ell\ x^m}{r^2}\right) + \left(\frac{\partial x^\ell}{\partial x'^i}\frac{\partial x^o}{\partial x'^k} + \frac{\partial x^o}{\partial x'^i}\frac{\partial x^\ell}{\partial x'^k}\right)g_{o\ell} + \frac{\partial x^o}{\partial x'^i}\frac{\partial x^o}{\partial x'^k}\ g_{oo}$$

This gives, after some easy steps

$$g'_{ik}(x') = \gamma\ f_1^2\ \delta_{ik} + \left[\left(f_1 + r'\ \frac{\partial}{\partial r'}\ f_1\right)^2\lambda - r'\ \frac{\partial}{\partial r'}\left(2f_1 + r'\ \frac{\partial}{\partial r'}\ f_1\right)\gamma\right]\frac{x'^i\ x'^k}{r'^2} + \left(\frac{\partial x^\ell}{\partial x'^i}\frac{\partial x^o}{\partial x'^k} + \frac{\partial x^o}{\partial x'^i}\frac{\partial x^\ell}{\partial x'^k}\right)g_{o\ell} + \frac{\partial x^o}{\partial x'^i}\frac{\partial x^o}{\partial x'^k}\ g_{oo}$$

Choosing $f_1 = -\ \gamma^{-1/2}$ and $f_2 = f_2(x'^o) = x'^o$, we get $\frac{\partial x^o}{\partial x'^i} = 0$, and thus

$$g'_{ik}(x') = \delta_{ik} + \mathcal{M}(r', x'^o \equiv x^o)\ \frac{x'^i\ x'^k}{r'^2}$$

However, to be correct with our signature −2 would require that a minus sign be present in the first term of the right side of this relation. But since the choice for $\gamma(r, x^o)$ in (6-2-1) is arbitrary, one may choose $-\gamma(r, x^o)$ there, which implies in writing the above equation with the minus sign in the first term. Dropping the lines, which are now unecessary we can ascertain that $g_{rs}(x)$ may be written as

$$g_{rs}(x) = -\delta_{rs}+\lambda(r,\ x^{o})\ \frac{x^{r}\ x^{s}}{r^{2}} \qquad (6\text{-}2\text{-}4)$$

thus, we already eliminate the $\gamma(r,\ x^{o})$. We still have the freedom to make mappings which do not change (6-2-4) but which change the g_{or}, or the g_{oo}. They are of the form * (this is a second possible choice for f_1 and f_2)

$$x^{s} = x'^{s} \ ,$$

$$x^{o} = F_2(r',\ x'^{o}) \ ,$$

under this mapping g_{ok} change as

$$g'_{ok}(x') = \frac{\partial x^{o}}{\partial x'^{o}}\left(g_{ok} + \frac{\partial x^{o}}{\partial x'^{k}}\, g_{oo}\right) = \frac{\partial f_2}{\partial x'^{o}}\left(\frac{\beta x'^{k}}{r'} + \frac{\partial f_2}{\partial x'^{k}}\,\alpha\right) ,$$

taking

$$\frac{\partial f_2}{\partial x'^{k}} = -\frac{\beta}{\alpha}\,\frac{x'^{k}}{r'}$$

We will obtain $g'_{ok} = 0$, and therefore we eliminate $\beta'(r',\ x'^{o})$, in the new frame. We now have to show that this new mapping will not change the results of the first mapping, that is, will not change the value -1 for $\gamma(r,\ x^{o})$. We have,

$$g'_{ik}(x') = g_{ik}(x) + \frac{\partial f_2}{\partial x'^{k}}\, g_{io} + \frac{\partial f_2}{\partial x'^{i}}\, g_{ko} + \frac{\partial f_2}{\partial x'^{i}}\,\frac{\partial f_2}{\partial x'^{k}}\, g_{oo}$$

$$= -\delta_{ik} - \frac{\beta^{2}}{\alpha}\ \frac{x'^{k}\ x'^{i}}{r'^{2}} + \mathcal{M}(r',\ x'^{o})\ \frac{x'^{i}\ x'^{k}}{r'^{2}}$$

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* This mapping is of the previous general form for $f_1 = 1$, and is permissible as was shown. The change in g_{ok} under this mapping is compatible with (6-2-1) since it gives $g'_{ok} = F_{1k}(r',\ x'^{o}) + F_2(r',\ x'^{o})\ \frac{x'^{k}}{r'}$.

Therefore, the g_{ik} in the new frame have the same general form of (6-2-4), the only change is in the arbitrary function multiplying $x^r x^s/r^2$, but this is not relevant. Similarly, we can show too that g_{oo} is not relevantly affected by any one of the two previous mappings, that means, under any one of these transformations we will get

$$g'_{oo}(x') = \Psi(r', x'^o)$$

Thus, after all possible recalibrations, we arrive at the result that it is always possible to reduce the most general spherically symmetric second rank tensor to a form depending on two arbitrary functions of the distance r and of the time coordinate x^o.

$$g_{oo}(x) = \alpha(r, x^o)$$

$$g_{os}(x) = 0 \qquad (6\text{-}2\text{-}5)$$

$$g_{rs}(x) = -\delta_{rs} + \lambda(r, x^o)\,\frac{x^r x^s}{r^2}$$

It is of obvious interest to use spherical coordinates for the three-space (the metric in (6-2-5) is given in cartesian coordinates x^s)

$$x^1 = r \sin\phi \sin\theta$$

$$x^2 = r \cos\phi \sin\theta$$

$$x^3 = r \cos\theta$$

Besides this we also make the substitutions

$$\alpha = e^{\nu}, \quad \lambda = 1 - e^{\Lambda}$$

Then,

$$g_{oo} = e^{\nu(r, x^o)}$$

$$g_{os} = 0$$

$$g_{11} = -e^{\Lambda(r, x^o)}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta \qquad (6\text{-}2\text{-}6)$$

The symbols of Christoffel for this metric are

$$\left\{\begin{matrix}0\\00\end{matrix}\right\} = \frac{\dot\nu}{2} \qquad \left\{\begin{matrix}0\\10\end{matrix}\right\} = \frac{\nu'}{2} \qquad \left\{\begin{matrix}0\\11\end{matrix}\right\} = \frac{\dot\Lambda}{2}\, e^{\Lambda-\nu} \qquad \left\{\begin{matrix}1\\22\end{matrix}\right\} = - re^{-\Lambda}$$

$$\left\{\begin{matrix}1\\00\end{matrix}\right\} = \frac{\nu'}{2}\, e^{\nu-\Lambda} \qquad \left\{\begin{matrix}1\\10\end{matrix}\right\} = \frac{\dot\Lambda}{2} \qquad \left\{\begin{matrix}1\\11\end{matrix}\right\} = \frac{\Lambda'}{2} \qquad \left\{\begin{matrix}1\\33\end{matrix}\right\} = - r \sin^2\theta\, e^{-\Lambda}$$

$$\left\{\begin{matrix}2\\12\end{matrix}\right\} = \frac{1}{r} \qquad \left\{\begin{matrix}2\\33\end{matrix}\right\} = - \sin\theta\cos\theta \qquad \left\{\begin{matrix}3\\23\end{matrix}\right\} = \operatorname{ctn}\theta$$

where the dot means differentiation with respect to x^0, and the prime denotes the same with respect to r. The nonvanishing components of the Einstein's tensor G^{μ}_{ν} are

$$G^1_1 = e^{-\Lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \tag{6-2-7}$$

$$G^2_2 = G^3_3 = \frac{1}{2}\, e^{-\Lambda}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu'-\Lambda'}{r} - \frac{\nu'\Lambda'}{2}\right) - \frac{1}{2}\, e^{-\nu}\left(\ddot\Lambda + \frac{\dot\Lambda^2}{2} - \frac{\dot\Lambda\dot\nu}{2}\right) \tag{6-2-8}$$

$$G^0_0 = e^{-\Lambda}\left(\frac{1}{r^2} - \frac{\Lambda'}{r}\right) - \frac{1}{r^2} \tag{6-2-9}$$

$$G^1_0 = e^{-\Lambda}\,\frac{\dot\Lambda}{r} \tag{6-2-10}$$

Since we consider the field equations for empty regions, the two functions $\nu(r, x^0)$ and $\Lambda(r, x^0)$ will be determined by the equations $G^{\mu}_{\nu} = 0$. As result of the Bianchi identities, $G^{\mu}_{\nu;\mu} = 0$, it may be shown that G^2_2 is a linear combination of the remaining components of G^{μ}_{ν}, therefore the field equation $G^2_2 = 0$ is a consequence of the other field equations.

Exercise: Prove this property.

Thus, we are left with the equations

$$e^{-\Lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 0 \qquad (6\text{-}2\text{-}11)$$

$$e^{-\Lambda}\left(\frac{\Lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0 \qquad (6\text{-}2\text{-}12)$$

$$\dot{\Lambda} = 0 \qquad (6\text{-}2\text{-}13)$$

For determining ν and Λ. Summing up (6-2-11) and (6-2-12) we get

$$\nu' + \Lambda' = 0$$

then

$$\nu + \Lambda = \phi(x^o) \qquad (6\text{-}2\text{-}14)$$

As we have seen, we have the freedom to use any one of the above mappings which preserve the spherical symmetry of $g_{\mu\nu}$. Of these mappings we have selected two types which have served as simplifying frames for the specification of the $g_{\mu\nu}$. It is still possible a further simplification by choosing a third mapping which is a simplified transformation as the type used for making the g_{or} vanish,

$$x^r = x'^r$$

$$x^o = f(x'^o)$$

under this transformation we get,

$$g'_{rs} = g_{rs}$$

$$g'_{or} = \frac{\partial x^o}{\partial x'^o}\, g_{or} = 0$$

$$g'_{oo} = \left(\frac{\partial x^o}{\partial x'^o}\right)^2 g_{oo} = \dot{f}^2 g_{oo}$$

where we used that g_{or} is zero in the initial frame. Consequently, only the g_{oo} changes, but this change is not inconsistent with our previous requirements. Indeed, this change is just a multiplicative factor $\dot{f}^2$ in front of $g_{oo} = \alpha$, transforming this function to $\dot{f}^2(x^o)\ \alpha(r, x^o)$, or

$$g'_{oo} = \alpha'(r, x^o) = \dot{f}^2(x^o)\ \alpha(r, x^o)$$

But since $\alpha = e^{\nu}$, this is equivalent to add to ν an arbitrary function of x^o.

$$g'_{oo} = e^{\nu'} = \dot{f}^2(x^o)\ \alpha(r, x^o) = \dot{f}^2(x^o)\ e^{\nu} = e^{\nu + \phi(x^o)}$$

We use this arbitrarity in the specification of ν, which says that both ν and $\nu + \phi(x^o)$ are equally good, for writing (6-2-14) as

$$\nu + \Lambda = 0$$

Then, as consequence of (6-2-13) we get $\dot{\nu} = 0$, and thus both ν and Λ are independent of x^o. What we have obtained is the statement that all spherically symmetric solutions of the field equations of general relativity have necessarily to be static: $g_{or} = 0$ and g_{rs}, g_{oo} are independent of the time coordinate x^o. In other words, the fact that the field is time independent is a direct consequence of its spherical symmetry, and is not a further imposition on the system. The solution which we are discussing refer to regions where $T_{\mu\nu} = 0$, for instance it means the field of a static mass distribution with finite size, with center at the origin of the spherical coordinates. The conclusion, that if $T_{\mu\nu}$ is spherically symmetric and is zero for r greater than some value a, then $g_{\mu\nu}$ is the Schwarzschild field for $r > a$, is known as the Birkhoff's theorem [41].

Now it remains just to determine Λ as function of r. Using (6-2-12), which possess as solution

$$e^{-\Lambda} = e^{\nu} = 1 - \frac{2c}{r}$$

where c is a constant of integration, from (6-2-6) we get

$$g_{oo}(r) = 1 - \frac{2c}{r} \, ,$$

$$g_{os} = 0 \, ,$$

$$g_{11} = - \frac{1}{1 - \frac{2c}{r}} \, , \; g_{22} = - r^2, \quad g_{33} = - r^2 \; \text{sen}^2 \theta$$

Comparison of this solution with the solution for the linearized equations of a static point mass located at $r = 0$, the equation (6-1-5), we see that both have to coincide at large values of r, since asymptotically the weak field approximation applies, then

$$g \quad = g_{rr} \approx - 1 - \frac{2c}{r}$$

this relation is equal to the g_{11} of (6-1-5) if we set

c = Gm/square of the velocity of light

where m is the mass of the point source. To close this section we give the value for the Schwarzschild field in the two types of coordinates used. In spherical coordinates plus the time x^o,

$$g_{\mu\nu} = \begin{pmatrix} 1 - \frac{2Gm}{c^2 r} & & & 0 \\ & \frac{-1}{1 - \frac{2Gm}{c^2 r}} & & \\ & & - r^2 & \\ 0 & & & - r^2 \; \text{sen}^2 \theta \end{pmatrix}$$

for a point source at $r = 0$, and in general

$$g_{\mu\nu} = \begin{pmatrix} 1 - \frac{2c}{r} & & & 0 \\ & \frac{-1}{1 - \frac{2c}{r}} & & \\ & & -r^2 & \\ 0 & & & -r^2 \sin^2\theta \end{pmatrix}$$

For cartesian coordinates, we get from (6-2-5)

$$g_{oo} = 1 - \frac{2c}{r}$$

$$g_{os} = 0$$

$$g_{rs} = -\delta_{rs} - \left(\frac{\frac{2c}{r}}{1 - \frac{2c}{r}} \right) \frac{x^r x^s}{r^2}$$

where c may be given by the previous relation. It is also useful for some purposes to use the so-called isotropic form of $g_{\mu\nu}$, obtained by the mapping

$$x^r = \left(1 + \frac{c}{2r'} \right) x'^r , \qquad x^o = x'^o$$

and given by

$$g_{oo} = \left(\frac{1 - \frac{c}{2r}}{1 + \frac{c}{2r}} \right)^2$$

$$g_{os} = 0 \tag{6-2-15}$$

$$g_{rs} = - \left(1 + \frac{c}{2r} \right)^4 \delta_{rs}$$

Finally we note that for $r \to \infty$ the field tends to the Minkowskian metric $\overset{0}{g}_{\mu\nu}$ in any one of the above coordinate systems. This comes out naturally, that is, we did not impose this asymptotic behaviour for obtaining our solution.

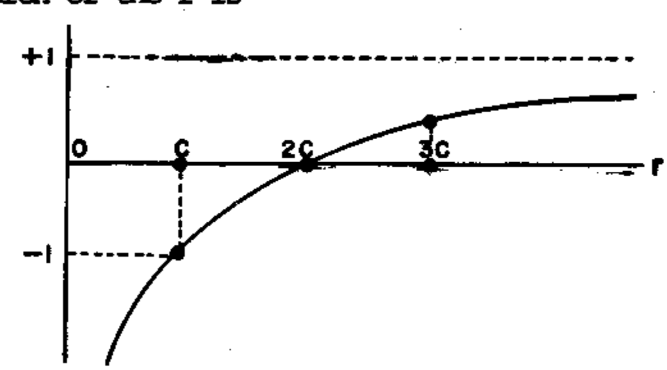
It should to be noted how the symmetry of the field was a decisive property, indeed, we have solved the non-linear field equations without any approximation, and such integration involved no problem; this was possible due to the several simplifications coming from the symmetry as well as from the use of the adequate coordinate system.

6.3) The Schwarzschild Singularity; The Topology of the Schwarzschild Field

Let us study the behaviour of the Schwarzschild solution in spherical coordinates. Later we will use other types of coordinates. First we pay attention to $g_{00}(r) = 1 - 2c/r$. Due to the fact that in particular the constant c for a pointlike distribution takes on positive values $c = Gm/c^2$, we shall take it as positive for physically possible distributions. Then, it comes that $g_{00}(r)$ is smaller or equal to one,

$$g_{00}(r) \leqslant 1$$

the value 1, the upper bound, being reached asymptotically for $r \to \infty$. The graph for g_{00} as function of the r is



As one sees, the value of c is an important one for stablishing the several points of interest for g_{oo}. At $r = 2c$ the g_{oo} vanishes.

As we know from special relativity, the proper time of an event is given by setting the dx^r equal to zero in ds^2. Thus

$$c^2 \begin{pmatrix} \text{differential of} \\ \text{proper time} \end{pmatrix}^2 = g_{oo}(r) \; (dx^o)^2$$

which gives,

$$\begin{pmatrix} \text{differential of} \\ \text{proper time} \end{pmatrix} = \sqrt{g_{oo}(r)} \;\; dt$$

where dt is the differential of time for the external observer. Therefore, from $g_{oo}(r) \leqslant 1$, we get that $d\tau \leqslant dt$, where $d\tau$ is the differential of proper time. This means that at finite distances from the gravitating masses there is a "slowing down" of time as measured by the observer moving in this region (the proper observer) in comparison with the time of the external observer, which for instance may be placed at $r = 0$, the origin of the field. The $d\tau$ is equal to dt at $r = \infty$, that is, asymptotically where there is no field, the "slowing down" in time disappears. It tends to get smaller when the proper observer moves away from the sources of the field. This gives as result the observed effect of deviation towards smaller frequencies of the light observed, or emitted, by the observer inside the field region as compared with the same phenomena in a region free of fields. Since we will not turn back to this type of effect during these lectures, we give now the explanation of this effect. Consider the following idealized experiment: We have two observers, one is free of the presence of fields, the other is the proper observer for the Schwarzschild field, and moves in a finite distance from the gravitating sources. This later observer receives a light ray coming, according to him with a frequency ν_p (p stands for proper), measured as a certain number of vibrations per second of its local time (proper time). The same light ray as observed by the first observer, which we may pictorially call as the "external observer", will have a frequency given as a certain number of vibrations per second of its time. Now, since one second for the "external observer" is equivalent to several seconds of proper time, we have by putting 1 sec for the external observer as N sec of proper time,

$$\nu = \frac{\text{K vibrations}}{\text{sec}}$$

$$\nu_p = \frac{\text{K vibrations}}{\text{N sec of properties}} = \frac{1}{N}\nu$$

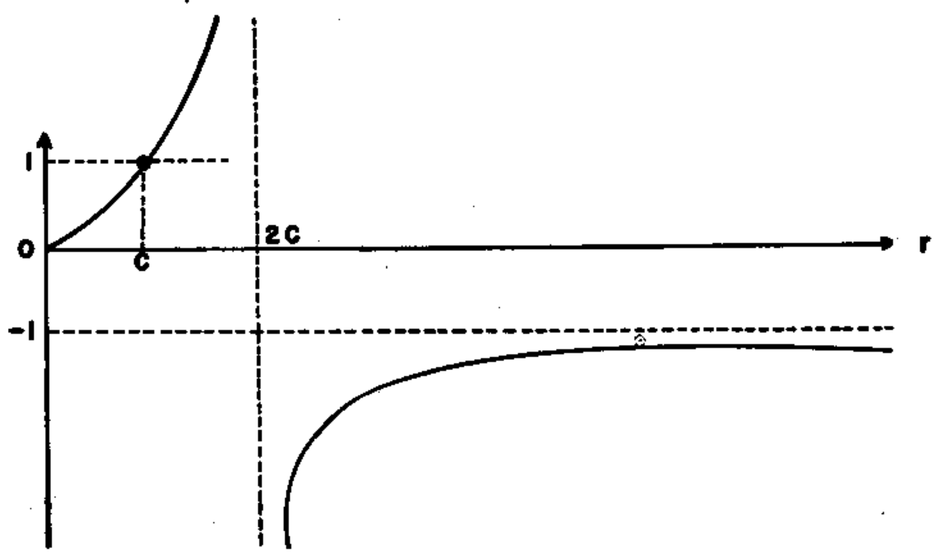
that means, the light observed by the proper observer as compared with the same spectrum for the free observer, appears to be shifted towards the red. The same argument holds good for emitted rays. All light emitted by the observer inside the field is deviated towards the red as compared with the same light emitted by the free observer. In particular for emitting some light ray with frequency ν, observed at spatial infinity, we need initially a photon with a higher frequency, part of its energy being lost for crossing the gravitational barrier. Nevertheless, this is an idealized experiment. What really happens is just a similar situation, what we have idealized is an experiment where there exists a variation in the gravitational potentials given by: $g_{00}(r)$ for the Schwarzschild field minus zero, for the external observer. We can consider now a difference $g_{00}(r)$ for a strong field minus $g_{00}(r)$ for a weak field, for instance the fields of the sun and the earth. Our arguments still hold, since the difference in potentials is not essential, and thus, light emitted at the surface of the sun and observed at the surface of earth appears shifted towards the red as compared with the same light emitted at the surface of the earth. This is the so-called gravitational red shift of light. This effect is a direct consequence of the slowing down of time in presence of gravitating bodies.

Turning back to the discussion of the form of $g_{00}(r)$, we see that the distance $r = 2c$ is not possible for physical observers since it implies that $g_{00} = 0$, and this in turn gives a null proper time at this point, which is unphysical. Also all region for $r < 2c$ is not possible since there g_{00} is negative, and again we get in trouble for defining $d\tau$ in this region. Thus, only the region $r > 2c$ has a good behaviour, in this region g_{00} is bounded as

$$0 < g_{00}(r) \leqslant 1, \quad \text{for } r > 2c \; .$$

The 2c is called as the radius of Schwarzschild. For the sun it has the value of 1.47 km, for the earth is 4.9 mm.

Let us now consider the $g_{11}(r)$, the graph of this component, which is the radial component of $g_{\mu\nu}$, is



An interesting fact arises, for the region $r < 2c$ the signs of g_{00} and g_{11} are opposite to the correspondent signs for $r > 2c$. That is, it looks how if the signature changes sign when we proceed towards the Schwarzschild radius. Nevertheless, this argument is not correct since we cannot cross this radius with a physical reference system, since at $r = 2c$ the g_{11} is divergent and the g_{00} vanishes. Both results being unphysical. Again the region $r > 2c$ presents a correct structure for g_{11},

$$g_{11} \leqslant -1, \quad r > 2c$$

the value -1 being reached asymptotically for $r \to \infty$. The distance from the center to any point in space is $\int_0^r \sqrt{-g_{11}}\, dr$, and since $-g_{11} \geqslant 1$,

$$\int_0^r \sqrt{-g_{11}}\, dr \geqslant r$$

(the equality sign holds for points at infinity). Consequently in the presence of fields the ratio of the circumference of the circle drawn through the origin to the radius is less than 2π.

The value $r = 2c$ is called the Schwarzschild singularity. We will see that this singularity is not an intrinsic property of the Schwarzschild solution but rather a consequence of the coordinates used to obtain this solution. One direct indication is given by the value of the determinant g which in spherical coordinates is regular at $r = 2c$, $g = -r^4 \sin^2\theta$. Furthermore, the scalar quadratic in the curvature, $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$, is also regular there. Since the structure of the gravitational field is described more properly by $R_{\mu\nu\rho\sigma}$ than by $g_{\mu\nu}$ itself, it follows that the above singularity cannot be of physical significance.

Exercise: Compute $R_{\mu\nu\rho\sigma}$ for the Schwarzschild field in spherical coordinates. Calculate $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$, and get its value at $r = 2c$.

Eddington in 1924 [42] showed that the Schwarzschild singularity may be removed by the mapping

$$x'^{r} = x^{r} , \quad x^{0} = x'^{0} \pm 2c \log (r'/2c-1) . \tag{6-3-1}$$

The metric tensor in the new coordinates being (these coordinates are x'^{0}, r, θ, ϕ)

$$g'_{\mu\nu} = \begin{pmatrix} 1 - \frac{2c}{r} & \pm \frac{2c}{r} & 0 & 0 \\ \pm \frac{2c}{r} & -\left(1+ \frac{2c}{r}\right) & 0 & 0 \\ 0 & 0 & - r^{2} & 0 \\ 0 & 0 & 0 & - r^{2} \sin^{2} \theta \end{pmatrix} \tag{6-3-2}$$

This $g'_{\mu\nu}$ is no longer singular at $r = 2c$. To avoid this singularity we have taken a point dependent translation along the timelike axis,

$$x'^{0} = x^{0} \pm f(r) , \qquad f(r) = 2c \log (r/2c-1)$$

Thus, the singularity was transported to the mapping function, since $f(r)$ is singular at $r = 2c$. Therefore, x'^{0} is singular at the Schwarzschild surface. From the previous formulae we see that two singularities are present for $g_{\mu\nu}$, one at $r = 0$, the other for $r = 2c$. The first being really a singularity in the metric, a fact to be expected, the second being just a singularity due to the choice of coordinates, such as spherical or cartesian. Several authors have studied the behaviour of these singular points [43], the results obtained may be summarized as follows,

r	invariant representation of curvature	singularity in metric ?	singularity in coordinate system ?
0	infinite as c/r^3	yes	yes
2c	finite	no	yes

Kruskal [44] has shown that the removal of the Schwarzschild singularity may be achieved by a choice of coordinates which is simpler than that done by Eddington. At the same time the use of these coordinates allow a maximal singularity-free extension of Schwarzschild's field.

In order to clarify some technical terms used, we give here some definitions: A manifold is said maximal if either every geodesic emanating from a given point has an infinite length in both directions or this geodesic ends on a physical singularity of the geometry (a singularity which cannot be mapped away). If all geodesics from a given point have infinite length in both directions, the manifold is maximal and complete. Clearly, a manifold which is maximal but not complete possess singularities which cannot be mapped away. Thus the study of these topological properties of a manifold are important since in last instancy they serve for characterizing the existence, or absence, of physical singularities. This as compared with our previous way for knowing if a given singularity was physical, or not, is a greater technical improvement of the theory.

Let's turn back again to Schwarzschild's line element in spherical coordinates.

$$ds^2 = +\left(1 - \frac{2c}{r}\right) dx_o^2 - \frac{1}{1 - \frac{2c}{r}} dr^2 - r^2 \, d\Omega^2 \tag{6-3-3}$$

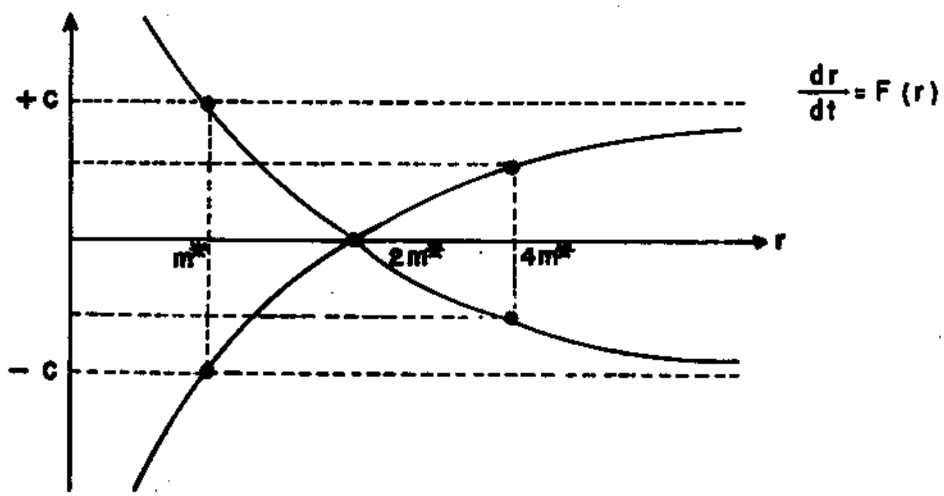
The radial light rays moving on the direction $\phi = \phi_o$ on the plane $\Theta = \Theta_o$ are given by taking $ds^2 = 0$ and $d\Omega^2 = 0$.

$$-\frac{1}{1 - \frac{2c}{r}} dr^2 + c^2_{light} \left(1 - \frac{2c}{r}\right) dt^2 = 0$$

therefore, the slope dr/dt for them is $\begin{pmatrix} c \to m^* \\ c_{light} \to c \end{pmatrix}$

$$\frac{dr}{dt} = \pm \; c \left(1 - \frac{2m^*}{r}\right)$$

since this is the effective velocity of the light rays in the field. If we draw the graphic of dr/dt as function of r we may see the actual distribution of velocities along the radial separation to the center of the field in this coordinate system.



the above velocity is a three-velocity and does not have a covariant significance. Only the four-velocity defined by $u^\mu = dx^\mu/ds$ has such significance. This means that the above distribution may appear different for another coordinate system. However, it is interesting to note that a photon in this coordinate system cannot reach the Schwarzschild singularity $r = 2m^*$ (we indicate our previous c by m^* for avoiding misunderstanding with the letter c which indicates the velocity of light in flat spaces) proceeding from the region $r < 2m^*$, since there it gets an infinite inertia. By the same argument the photon cannot start from the Schwarzschild singularity $r = 2m^*$ and travel towards the region $r > 2m^*$ since at $r = 2m^*$ it has a null velocity, since the field is attractive it cannot go away starting at rest in $r = 2m^*$. Thus, two regions are possible for the photons, but the surface which connects these two regions cannot be reached by them. The region $r < 2m^*$ acts as if it were a bounding region for the radiation field of photons, and all photons in the region $r > 2m^*$ tends to come towards the region $r < 2m^*$ in the direction of decreasing r.

We have seen that the most general spherically symmetric line element has the form.

$$ds^2 = f(r,t)dx_0^2 - h(r,t)dr^2 - r^2\, d\Omega^2$$

Kruskal proposes to take $f = F^2(r)$, and $h = f$, case where we get (he uses signature opposite to ours)

$$ds^2 = F^2(r)(du^2 - dv^2) - r^2\, d\Omega^2 \tag{6-3-4}$$

where u, v, θ and ϕ are the new coordinates, where the ds has this form. Identifying (6-3-3) with (6-3-4) we may derive the transformation equations connecting the spherical coordinates plus time with the u, v, Θ and ϕ. One finds,

$$u = \left(\frac{r}{2m^*} - 1\right)^{1/2} e^{\frac{r}{4m^*}} \cosh\left(\frac{x^o}{4m^*}\right),$$

$$v = \left(\frac{r}{2m^*} - 1\right)^{1/2} e^{\frac{r}{4m^*}} \operatorname{senh}\left(\frac{x^o}{4m^*}\right),$$

with inverse

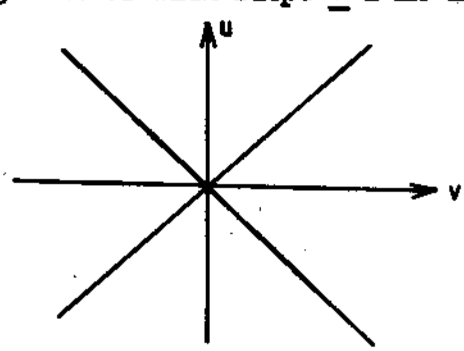
$$\left(\frac{r}{2m^*} - 1\right) e^{\frac{r}{2m^*}} = u^2 - v^2$$

$$\frac{x^o}{4m^*} = \operatorname{arctg} h\left(\frac{v}{u}\right) = \frac{1}{2} \operatorname{arctg} h\left(\frac{2\,uv}{u^2 + v^2}\right)$$

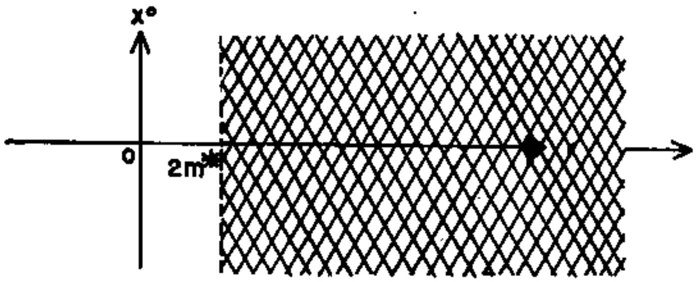
and

$$F^2(r) = \frac{32\ m^{*3}}{r}\ e^{-\frac{r}{2m^*}}.$$

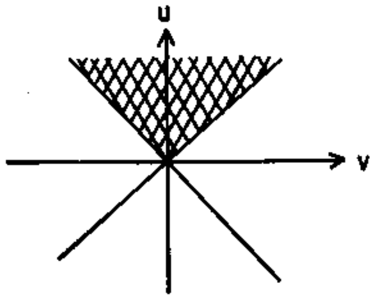
These transformation equations hold good for the singularity-free region $r > 2m^*$. At $r = 2m^*$ both u and v are equal, and for $r < 2m^*$ the u and v are imaginary. For $r > 2m^*$ the equations written above imply that $u > |v|$. Thus, the singularity-free region $r > 2m^*$ corresponds to the quadrant $u > |v|$ in the u-v plane. Light rays moving radially in the r-x^o plane will correspond to straight lines with slope $\pm$ 1 in the u-v plane.



The singularity-free region $r > 2m^*$ is represented as



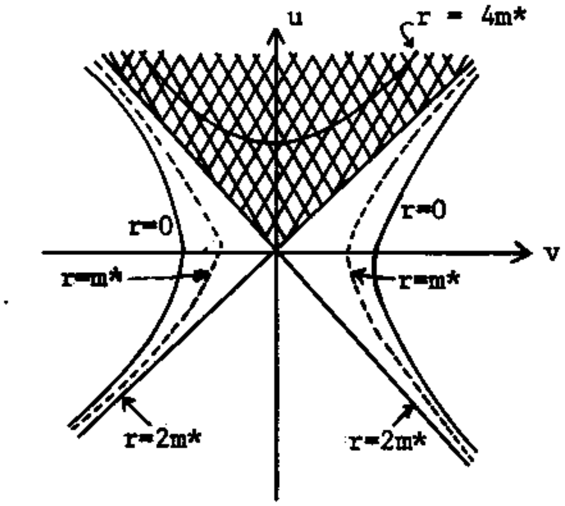
this entire region is mapped into the quadrant $u > |v|$,

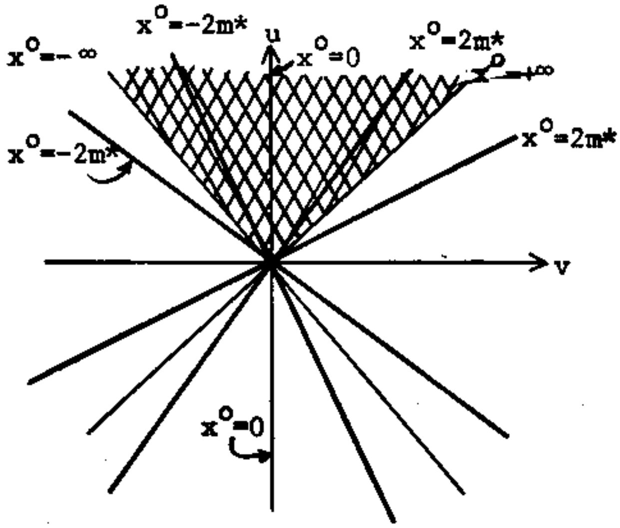


The curves r = const. are mapped into the hyperbolae $u^2 - v^2$ = const., and the curves x^0 = const. are mapped into the curves

$$\operatorname{arctg\,h}\left(\frac{v}{u}\right) = \frac{1}{2}\log\frac{u+v}{u-v} = \frac{\pm|c|}{4m^*}$$

where $\pm|c|$ indicates the constant value for x^0. Several curves are indicated in the next page.



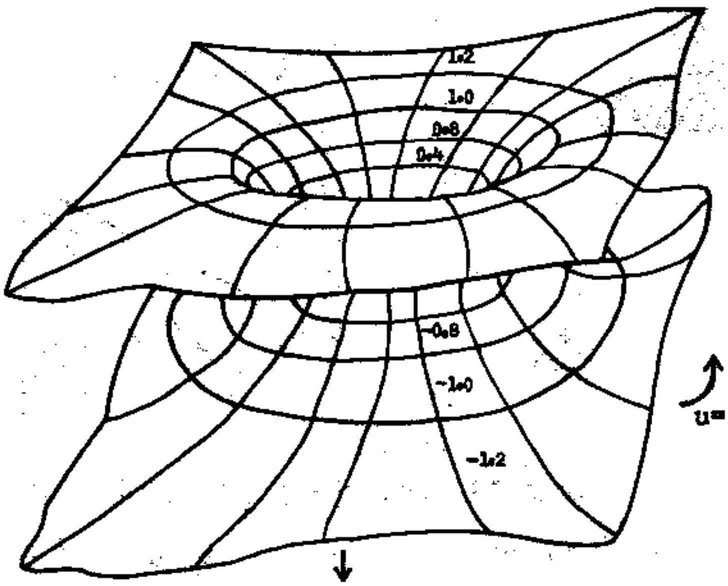


The region of the u, v manifold where the $g_{\mu\nu}$ of (6-3-4) is regular is the part bounded by the two hyperbolae $r = 0$; that is, $u^2 - v^2 = -1$. The curvature invariants (in particular the scalar curvature) becomes infinite along these bounding curves, which represent a physical singularity of the field.

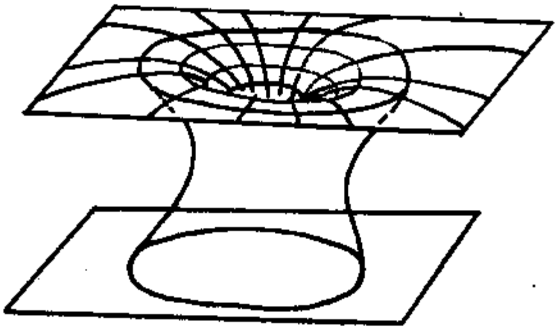
The complete manifold is represented by the coordinates u, v, Θ and ϕ, so as we have represented above only a part of this manifold. To get an idea of the structure of the full manifold on which Kruskal's metric is regular, let us consider the submanifold $v = 0$, that is the axis u, as we move along this axis from $+\infty$ down to the origin, r decreases to a minimum value 2m* at $u = 0$. As we proceed crossing the origin towards $u = -\infty$, r increases going asymptotically to $+\infty$. We can draw a picture of a cross section of this manifold corresponding to some constant value for Θ, say $\Theta = \frac{\pi}{2}$, by constructing a two-dimensional surface embedded in a flat three-dimensional space in such way that the metric on this surface is

$$ds^2 = g_{uu}\, du^2 + g_{\phi\phi}\, d\phi^2 = f^2\, du^2 + r^2 \operatorname{sen}^2 \frac{\pi}{2}\, d\phi^2$$

(the u here corresponds to the role of the v respect to the choice of signature). This surface thus corresponds to $v = 0$, $\Theta = \frac{\pi}{2}$. We obtain

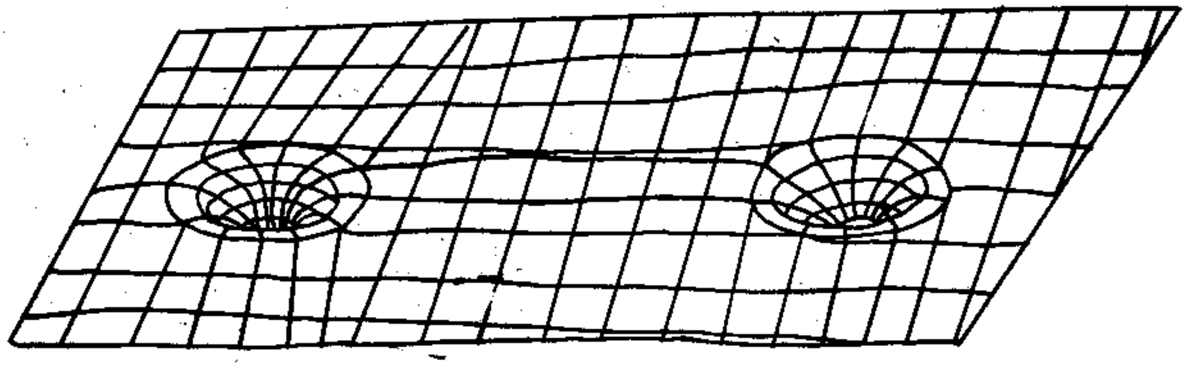


On this surface various u = const. curves are traced out as ϕ runs from 0 to 2π. The ϕ = const. curves run from the edges of the upper surface through the throat, and out toward the edge of the bottom surface. The same kind of picture holds for all Schwarzschild sub-manifolds for x^0 = const., these sub -manifolds are parts of the shaded area represented before. Thus, taking into consideration the drawings of the page 228 and 229, we see that for instance $x^0 = 0$ corresponds to the axis u, that is, is represented by the full set of values for u. This means that we may interpret a drawing similar to the above as a connection or bridge in the sense of Einstein and Rosen[43] between two otherwise Euclidian spaces. Wheeler and Fuller [46] have proposed a different interpretation for this maximally extended singularity-free manifold. Retaining just one Euclidian space, they stablish a connection be-



Bridge between two Euclidian spaces. It represents the maximally extended Schwarzschild metric at $x^0 = 0$.

tween two particles by specifying these particles by the throat of the hole corresponding to each one of them. The connection is then stablished with all geodesics running without crossing any singularity.



Since the Kruskal's manifold contains singularities at $r = 0$, the intrinsic physical singularity of the point-like solution of the field equations, the maximal extension is singularity free in the region $r \geq 2m^*$ but is not complete since we cannot include the region $r < 2m^*$. If we include this region, no singularity will appear at $r = 2m^*$, since there the metric coefficient

$F^2(r)$ of (6-3-4) is finite, but $F^2(r)$ goes to infinity at $r = 0$ as $1/r$. Therefore the Kruskal manifold is not complete in the sense we defined before.

If we compute the geodesics for the Kruskal manifold, for instance by considering the familiar r, x^0 plane and using the metric as given by (6-3-4), we will find that every geodesic followed in any direction, either runs into the barrier of intrinsic singularities at $r = 0$ $(v^2 - u^2 = 1)$, or is continuable infinitely with respect to its "natural length". This shows according to our previous definitions that the Kruskal manifold is not complete.

There are several interesting features of Kruskal's geometry which will not be discussed here. Further information may be found on a work of Wheeler in Geometrodynamics and the Issue of the Final State, in Relativity Groups and Topology (Gordon and Breach, N. Y. 1964).

6.4) Gravitational Fields With Cylindrical Symmetry

Similarly to the work done in the section (6.2), we begin by looking to the most general form taken by a second rank symmetric tensor invariant under rotations about a fixed axis, that is, that has cylindrical symmetry. We take this axis as the Z-axis. An infinitesimal rotation by an angle Θ about the Z-axis is given by

$$x' = x - \Theta y$$

$$y' = y + \Theta x$$

$$z' = z$$

$$x'^0 = x^0$$

This transformation will generate a symmetry Killing field ξ^ρ if

$$\bar{\delta} g_{\mu\nu}(x) = - g_{\mu\rho}\,\xi^\rho_{\ ,\nu} - g_{\rho\nu}\,\xi^\rho_{\ ,\mu} - g_{\mu\nu,\rho}\,\xi^\rho = 0 \qquad (6\text{-}4\text{-}1)$$

In the present case the Killing field will be

$$\xi^{\rho} = (-\Theta y,\ \Theta x,\ 0,\ 0)\ .$$

Taking $\mu = 0$, $\nu = 0$ in (6-4-5), we find, by noting that ξ^{o} is null,

$$g_{oo,i}\,\xi^{i} = 0$$

or,

$$\frac{\partial g_{oo}}{\partial x} y - \frac{\partial g_{oo}}{\partial y} x = 0$$

this relation implies that x and y are contained within g_{oo} only through the combination $\rho = \sqrt{x^2 + y^2}$. Thus,

$$g_{oo} = f(\rho,\ z,\ x^{o}) \tag{6-4-2}$$

Taking now $\mu = 0$, $\nu = i$ we get

$$-\ g_{ok}\ \xi^{k}_{,i} - g_{oi,k}\,\xi^{k} = 0$$

for $i = 1$, this takes the form

$$-\ g_{02} + \frac{\partial g_{01}}{\partial x} y - \frac{\partial g_{01}}{\partial y} x = 0$$

writing

$$\begin{cases} g_{01} = \phi(\rho,\ z,\ x^{o})\ \dfrac{x}{\rho} & (6\text{-}4\text{-}3) \\ g_{02} = \phi(\rho,\ z,\ x^{o})\ \dfrac{y}{\rho} & (6\text{-}4\text{-}4) \end{cases}$$

in analogy with the case of spherical symmetry, we can easily prove that this relation is satisfied. The relation for $i = 2$,

$$g_{01} + \frac{g_{02}}{\partial x} y - \frac{g_{02}}{\partial y} x = 0$$

will be satisfied too. The equation for $i = 3$, simplifies to $g_{03,k}\,\xi^{k} = 0$, or

$$\frac{\partial g_{03}}{\partial x} y - \frac{\partial g_{03}}{\partial y} x = 0$$

which similarly to the case for g_{00} is satisfied for

$$g_{03} = F(\rho, z, x^0) \quad (6\text{-}4\text{-}5)$$

Now, take $\mu = 3$, $\nu = 1$ and 2,

$$- g_{32} + \frac{\partial g_{31}}{\partial x} y - \frac{\partial g_{31}}{\partial y} x = 0$$

$$g_{31} + \frac{\partial g_{32}}{\partial x} y - \frac{\partial g_{32}}{\partial y} x = 0$$

these relations are satisfied for g_{31} and g_{32} of the form

$$g_{31} = \phi(\rho, z, x^0)\, zx/\rho^2 \quad (6\text{-}4\text{-}6)$$

$$g_{32} = \phi(\rho, z, x^0) zy/\rho^2 \quad (6\text{-}4\text{-}7)$$

It remains out of the diagonal only the element g_{12} to be calculated. For this element we have by taking $\mu = 1$, $\nu = 2$.

$$g_{11} - g_{22} + \frac{\partial g_{12}}{\partial x} y - \frac{\partial g_{12}}{\partial y} x = 0 \quad (6\text{-}4\text{-}8)$$

and for the three diagonal metric components we get

$$- 2\, g_{12} + \frac{\partial g_{11}}{\partial x} y - \frac{\partial g_{11}}{\partial y} x = 0 \quad (6\text{-}4\text{-}9)$$

$$2\, g_{12} + \frac{\partial g_{22}}{\partial x} y - \frac{\partial g_{22}}{\partial y} x = 0 \quad (6\text{-}4\text{-}10)$$

$$\frac{\partial g_{33}}{\partial x} y - \frac{\partial g_{33}}{\partial y} x = 0 \quad (6\text{-}4\text{-}11)$$

these last relationships split into two groups, the first involving the g_{12}, g_{11} and g_{22} formed by (6-4-8) through (6-4-10), and the other involving only g_{33} as given by (6-4-11). The solutions being

$$g_{nm} = -\delta_{nm}\Psi(\rho, z, x^o) + \chi(\rho, z, x^o)x_n x_m/\rho^2$$

for n, m = 1, 2. And

$$g_{33} = -\Lambda(\rho, z, x^o) .$$

In summary, from what was seen, we have six groups which divide the several components of $g_{\mu\nu}$ as

1) Involves g_{oo}, as given by the relation written previously with solution

$$g_{oo} = f(\rho, z, x^o)$$

2) formed with g_{01} and g_{02} with solutions

$$g_{01} = \phi(\rho, z, x^o)x/\rho$$

$$g_{02} = \phi(\rho, z, x^o)y/\rho$$

3) formed with g_{03} as

$$g_{03} = F(\rho, z, x^o)$$

4) formed with g_{32} and g_{31} as

$$g_{32} = \phi(\rho, z, x^o)zy/\rho^2$$

$$g_{31} = \phi(\rho, z, x^o)zx/\rho^2$$

5) involving the g_{12}, g_{11} and g_{22} as

$$g_{nm} = -\delta_{nm}\Psi(\rho, z, x^o) + \chi(\rho, z, x^o)x_n x_m/\rho^2$$

6) involving the g_{33} as

$$g_{33} = -\Lambda(\rho, z, x^o)$$

As consequence we have in all seven arbitrary functions of $\rho = \sqrt{x^2+y^2}$, z and x^o. Similarly to the case of spherical symmetry seen before, we may introduce the full group of mappings which maintain this form for $g_{\mu\nu}$ invariant. After this we select appropriate elements of this group in order to simplify the form of $g_{\mu\nu}$. Instead of going through all details we just quote some possible transformations. First, on the coordinate plane x-y we get a similar situation of spherical symmetry, so that there it holds the previous discussion. Taking a transformation which is the identity transformation on this plane but corresponds to a new x^o axis as

$$x^o = f(\rho, x'^o)$$

we may similarly to before set the g_{01} and g_{02} as zero. By taking a transformation which is a translation along the symmetry direction

$$x = y = 0$$

given by

$$z = z' + f(\rho, z', x^o)$$

all other coordinates fixed, we can set the g_{03} equal to zero, by selecting the function f as

$$f = - z' + \phi(\rho, x^o) \ .$$

By considering transformations on the symmetry plane x-y we may still set equal to zero the g_{32} and g_{31}. As result, we are left with just five metric elements. It may be shown that these remaining components may be mapped to the simpler form involving only two arbitrary functions (for static fields).

$$g_{00} = e^{\mu}, \quad g_{om} = 0, \quad g_{03} \quad = 0, \ m, \ n = 1, \ 2$$

$$g_{33} = - e^{\nu-\mu}, \ g_{3m} = 0, \quad \mu = \mu(\rho,z), \ \nu = \nu(\rho,z)$$

$$g_{nm} = e^{-\mu}\left\{-\delta_{nm} + (1-e^{-\nu}) \ x^m \ x^n\right\}$$

Weyl [45] has investigated the solution for the field equations for $g_{\mu\nu}$ of this form. In the exterior region (where $T_{\mu\nu}$ is null), the field equations are

$$K_1 \equiv \frac{\partial^2\mu}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\mu}{\partial\rho} + \frac{\partial^2\mu}{\partial z^2} = 0 \tag{6-4-13}$$

$$K_2 \equiv \frac{\partial\nu}{\partial\rho} - \frac{\rho}{2}\left[\left(\frac{\partial\mu}{\partial\rho}\right)^2 - \left(\frac{\partial\mu}{\partial z}\right)^2\right] = 0 \tag{6-4-14}$$

$$K_3 \equiv \frac{\partial\nu}{\partial z} - \rho\frac{\partial\mu}{\partial\rho}\frac{\partial\mu}{\partial z} = 0 \tag{6-4-15}$$

$$K_4 \equiv \frac{\partial^2\nu}{\partial\rho^2} + \frac{\partial^2\nu}{\partial z^2} + \frac{1}{2}\left[\left(\frac{\partial\mu}{\partial\rho}\right)^2 + \left(\frac{\partial\mu}{\partial z}\right)^2\right] = 0 \tag{6-4-16}$$

Due to the Bianchi identity the last equation is a consequence of the first three:

$$K_4 \equiv \frac{\partial K_2}{\partial\rho} + \frac{\partial K_3}{\partial z} + \rho\frac{\partial\mu}{\partial\rho}K_1 \tag{6-4-17}$$

in addition it also implies in

$$\frac{\partial K_2}{\partial z} - \frac{\partial K_3}{\partial\rho} \equiv \rho\, K_1\frac{\partial\mu}{\partial z} \tag{6-4-18}$$

The equation (6-4-13) is just Laplace's equation in cylindrical coordinates for a function with spherical symmetry. In order to get the correct asymptotic bevaviour at spatial infinity, the solutions of this equation must have the property that $\mu(\rho, z) \to 0$ for $|\vec{r}| \to \infty$. In addition it must be finite and well behaved outside some finite spatial region where $T_{\mu\nu} \neq 0$, (i.e., in the exterior region). Such type of solutions may be found. From (6-4-14) and (6-4-15) one finds by a further differentiation

$$\frac{\partial}{\partial z}\left(\frac{\partial \nu}{\partial \rho}\right) - \frac{\partial}{\partial \rho}\left(\frac{\partial \nu}{\partial z}\right) = \frac{\partial}{\partial z}\left[-\frac{\rho}{2}\left\{\left(\frac{\partial \mu}{\partial \rho}\right)^2 - \left(\frac{\partial \mu}{\partial z}\right)^2\right\}\right] - \frac{\partial}{\partial \rho}\left[-\rho\,\frac{\partial \mu}{\partial \rho}\,\frac{\partial \mu}{\partial z}\right] = 0\,, \tag{6-4-19}$$

in the exterior region. Thus, the solutions of Laplace's equation are subjected to satisfy too the (6-4-19). A solution of Laplace's equation satisfying all this is the well known potential-like solution

$$\mu = (\rho^2 + z^2)^{-1/2} = \frac{1}{r} \tag{6-4-20}$$

Representing a point singularity at the origin $\rho = 0$, $z = 0$. Recall that $\nabla^2 \frac{1}{r} = -4\pi\delta(r)$ and for the exterior region $r \neq 0$, this gives $\delta(r) = 0$. This solution satisfies (6-4-19) at all points except at the origin.

Replacing back the value (6-4-20) for μ into (6-4-14) and (6-4-15), one finds by integration

$$\nu(\rho, z) = -\frac{\rho^2}{4(\rho^2 + z^2)^2} \tag{6-4-21}$$

Thereby we have gotten a cylindrically symmetric solution of Einstein's equations in the exterior region corresponding to a point singularity at $\rho = 0$, $z = 0$. Its explicit form being obtained by replacing (6-4-20) and (6-4-21) into (6-4-12).

$$g_{00} = e^{(\rho^2+z^2)^{-1/2}} = e^{1/r}, \quad g_{01} = g_{02} = g_{03} = 0$$

$$g_{33} = -e^{-\frac{\rho^2}{4r^4} - \frac{1}{r}}, \quad g_{31} = g_{32} = 0 \tag{6-4-22}$$

$$g_{nm} = e^{-1/r}\left\{-\delta_{nm} + \left(1 - e^{\frac{\rho^2}{4r^4}}\right) x^m x^n\right\}; \; m, n = 1, 2$$

In spite of representing a point singularity, this solution does not correspond to the Schwarzschild field. Indeed, a glance on the form of $\nu(\rho, z)$ shows that this function is not spherically symmetric since it contains only ρ^2 in the numerator, and not $r^2 = \rho^2 + z^2$ as it should be for having spherical symmetry. Therefore, the $g_{\mu\nu}$ of (6-4-22) is not spherically symmetric. This means that this solution corresponds to a particle with multipole structure. The case where two singularities exist on the z axis, that is, for

$$\mu = \frac{a_1}{r} + \frac{a_2}{r-b}$$

which for $\rho = 0$ is singular at $z = 0$ and at $z = b$, was studied [47]. In this case we get the result that if $\nu(\rho,z)$ is an exact differential, that means, if (6-4-19) holds, then this equation will not, in general, be satisfied on the line joining these two singularities. Thus, not all possible regular solutions of Laplace's equation (going to zero in spatial infinity and finite and continuous on the exterior region) are allowed. If we intend to consider these solutions we have to exclude the line joining the two singularities from the exterior region.

6.5) Null Gravitational Fields-Exact Solutions Representing Fields of Radiation

A large class of exact solutions of the empty-space Einstein's equations have been found for which the corresponding Riemann tensor is everywhere of the type N in the Petrov classification scheme. The interest of these solutions lies in the formal analogies they have with the plane-wave solutions of the empty-space Maxwell equations. Similarly to the electromagnetic plane-wave solutions, these so called null gravitational fields must be considered as an idealization since both type of solutions are unrelated to any source structure, as they should be for real wave solutions. The first type N field was discovered by Brinckmann[48] in 1925, but he did not associate his solution with the radiation field. Later, Rosen [49] rediscovered a special N field, but rejected its interpretation by means of arguments which are now considered incorrect. In 1956 Robinson [50] rediscovered, independently, the Brinckmann solutions and attributed to them the actually accepted interpretation of "plane fronted gravitational waves".

The guiding ideas in most discussions of wave-like solutions of the Einstein field equations is that the Riemann tensor $R_{\mu\nu\rho\sigma}$ plays the role of a field strength in the gravitational theory analogous to that played by the $F_{\mu\nu}$ in electromagnetism (note that as we remarked before, in the weak field approximation $R_{\mu\nu\rho\sigma}$ possess a gauge invariant interpretation similarly to $F_{\mu\nu}$ in electromagnetism), and that $g_{\mu\nu}$ is the potential of this type of field similarly to the A_μ in electromagnetism. The analogy is not complete since $R_{\mu\nu\rho\sigma}$ depends on the second derivatives of $g_{\mu\nu}$ while $F_{\mu\nu}$ depends only on the first derivatives of A_μ. We must recall that besides the possible formal analogies between these two types of fields, such as the transport of energy and momentum, which exist for electromagnetic waves and has to be shown for gravitational waves to exist too, gravitational waves are essentially distinct from electromagnetic waves in relation to the behaviour of the generating sources (however, for plane waves this may be put in a secondary place since we are in the radiation zone), namely, the electromagnetic waves are originated in the charges but they do not transport charge away from the source, whereas gravitational waves are originated in the masses and transport mass away from the emitter, since they transport energy out of the source.

In order to stablish the analogy with the plane electromagnetic wave it will be necessary to recall the properties of these fields, we do this now. Starting from the empty space Maxwell equations for $F_{\mu\nu}$.

$$g^{\mu\alpha}\, \partial_\alpha\, F_{\mu\nu} = 0 \tag{6-5-1}$$

$$\varepsilon^{\mu\nu\rho\sigma}\, F_{\nu\rho,\sigma} = 0 \tag{6-5-2}$$

The plane-wave solution for these equations is

$$F_{\mu\nu}(x) = \overset{o}{F}_{\mu\nu}\, e^{ik_\lambda x^\lambda}, \tag{6-5-3}$$

as long as the four-vector k_λ satisfies the conditions

$$k^\mu\, \overset{o}{F}_{\mu\nu} = 0 \tag{6-5-4}$$

$$\varepsilon^{\mu\nu\rho\sigma}\, k_\sigma\, \overset{o}{F}_{\nu\rho} = 0 \tag{6-5-5}$$

Expressed in terms of the vectors $\vec{E}_o$ and $\vec{H}_o$ these conditions are

$$\vec{E}_o.\vec{k} = 0, \quad \vec{H}_o.\vec{k} = 0 \tag{6-5-6}$$

$$\vec{H}_o \times \vec{k} - \vec{E}_o\, \omega = 0 \tag{6-5-7}$$

$$\vec{E}_o \times \vec{k} - \vec{H}_o\, \omega = 0 \tag{6-5-8}$$

where

$$k^\mu = (\omega, \vec{k})$$

ω is the frequency of the wave, and $\vec{k}$ is its wave-number vector. The equations (6-5-4) and (6-5-5) for $\overset{o}{F}_{\mu\nu}$, are linear and homogeneous in both $\overset{o}{F}_{\mu\nu}$ and k_μ, we will show that they possess non-trivial solutions for $\overset{o}{F}_{\mu\nu}$ only if both $\overset{o}{F}_{\mu\nu}$ and k_μ satisfy the following conditions,

$$\overset{o}{F}_{\mu\nu}\, \overset{o}{F}^{\mu\nu} = \vec{E}^2_o - \vec{H}^2_o = 0 \tag{6-5-9}$$

$$\frac{1}{6}\, \varepsilon^{\mu\nu\rho\sigma} \overset{o}{F}_{\mu\nu}\, \overset{o}{F}_{\rho\sigma} = \vec{E}_o.\vec{H}_o = 0 \tag{6-5-10}$$

$$k^\mu\, k_\mu = \omega^2 - \vec{k}^2 = 0 \tag{6-5-11}$$

For proving this we start from the fact that $\overset{o}{F}_{\mu\nu}$ is a second rank tensor and k_μ a covariant vector with respect to Poincaré mappings. Thereby the k_μ can be characterized invariantly according to its geometrical behaviour as a time-like, space-like or null vector. Each case has to be separately studied. We start supposing that k_μ is time-like. In this case there always exists a mapping such that in the new frame the k_μ takes on value $k'_\mu = (k'_o, \vec{0})$. Direct substitution of these values into (6-5-4) and (6-5-5) gives $\overset{o}{F}{}'_{\mu\nu} = 0$. But since $\overset{o}{F}_{\mu\nu}$ is a tensor, it will vanish for any other Lorentz frame, thus showing that the (6-5-4) and (6-5-5) have only trivial solutions if k_μ is time-like. For k_μ space-like we can mapp to a frame where $k'_\mu = (0, k'_1, k'_2, k'_2)$, we can choose the orientation of the spatial axis such that $k'_2 = k'_3 = 0$, or $k' = (0, k'_1, 0, 0)$. Replacing this in (6-5-4) and (6-5-5) we again obtain $\overset{o}{F}{}'_{\mu\nu} = 0$. Once more, due to the fact that $\overset{o}{F}_{\mu\nu}$ is a tensor, it will vanish for any space-like k_μ. If k_μ is a null vector, $k^2 = 0$, by a Lorentz mapping it may be put as $k'_\mu = (k_o, k_1, 0, 0)$, with $k_1 = \pm k_o$. In this case the (6-5-4) and (6-5-5) will have non trivial solutions. Such solutions satisfy

$$E_{01} = H_{01} = 0 \,, \qquad E_{02} = \mp H_{03} \,, \qquad E_{03} = \pm H_{02}$$

where $\vec{E}_o \equiv \overset{o}{\vec{E}}$, the same for $\vec{H}$. As it may be seen these non-trivial solutions satisfy the conditions (6-5-9), (6-5-10) and (6-5-11) as we wanted to prove. By an arbitrary Lorentz transformation which transforms the particular null vector $k'_\mu = (k_o, \pm k_o, 0, 0)$ into an arbitrary null vector the previous solution will transform into an arbitrary solution satisfying the consistency conditions (6-5-9) and (6-5-10), since these are invariant equations. It is also instructive to note that the plane wave is characterized by two independent components of $\vec{E}_o$ and $\vec{H}_o$, in our previous frame they were $\overset{o}{E}_2$ and $\overset{o}{E}_3$ (or $\overset{o}{H}_2$ and $\overset{o}{H}_3$), showing that the wave has two states of circular polarization.

Therefore, the plane electromagnetic waves are the fields satisfying the two conditions (6-5-9) and (6-5-10). Since these two quantities are the field invariants we can characterize an electromagnetic plane wave by the property that the two field invariants are zero. By this reason, we call this field as a null field. Null fields have associated to them a family of null or light-like vectors k_μ satisfying (6-5-4) and (6-5-5). For our purposes it is better to rewrite (6-5-5) as

$$F_{\mu[\nu} k_{\rho]} = 0 \tag{6-5-12}$$

In general relativity null gravitational fields are those belonging to the Petrov class N, that is, with all curvature invariants equal to zero, and with a family of light-like directions k_μ satisfying *

$$R^{\mu}{}_{\nu\rho\sigma} k_\mu = 0 \tag{6-5-13}$$

$$R_{\mu\nu[\rho\sigma} k_{\lambda]} = 0 \tag{6-5-14}$$

In addition [51] the Riemann tensor of such field, in empty space, can suffer discontinuities only across a null surface. The physical meaning of this fact lies in the property that the plane fronted wave represents the propagation of a field singularity along light-like directions.

The solutions found by Brinckmann and rediscovered by Robinson can be represented by a metric $g_{\mu\nu}$ of the form

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2H & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6-5-15}$$

* in analogy with the equations (6-5-4) and (6-5-12).

with H a function of x^1, x^2, x^3. The empty space Einstein equations reduce in this case to a single equation

$$\frac{\partial^2 H}{\partial x_2^2} + \frac{\partial^2 H}{\partial x_3^2} = 0 \qquad (6\text{-}5\text{-}16)$$

while the $R_{\mu\nu\rho\sigma}$ is given by

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\,\delta^{oa}_{\mu\nu}\,\delta^{ob}_{\rho\sigma}\,\frac{\partial^2 H}{\partial x^a\,\partial x^b}, \quad (a,\ b = 2,\ 3) \qquad (6\text{-}5\text{-}17)$$

which may be seen to satisfy (6-5-13) and (6-5-14). Thus, $g_{\mu\nu}$ is flat only if H is linear or independent of x^2 and x^3.

In order to shed more light on the way that a metric of the type of (6-5-15) is introduced, we consider now the problem of specifying the geometry of the space by means of null hypersurfaces. In all applications of the formalism of relativity the space-like hypersurfaces appear to have a dominant role, is on these submanifolds that we specify the Cauchy problem, the first step towards the canonical formalism and subsequent quantization. They are as result intimately connected to causality. The fundamental reason for such apparent dominant role comes from the fact that a space-like hypersurface is the relativistic term for the Newtonian "instant of time". However, recently,it emerged the fact that the null hypersurfaces are also very important in explaining the physics of the gravitational field. Indeed, if causality is associated to an ordering in time of the phenomena, the agent transporting the informations, the light ray, propagates along null hypersurfaces which will cross two space-like hypersurfaces in a certain order of crescent time, so giving the causal effect. Several efforts carried out in these last years have proven that the behaviour of the system may be explained by means of null hypersurfaces in a form which is not more complicated than it were in the conventional treatment. In the coming lines we give a short introduction on this [52].

Let us consider the hypersurface $\sigma(x^{\alpha})$ = constant, in the four-dimensional hyperbolic space of general relativity. Then,

$$d\sigma = \sigma_{,\mu}\, dx^{\mu} = 0$$

introducing an affine parameter ρ we rewrite this as

$$d\sigma = \sigma_{,\mu} \frac{dx^{\mu}}{d\rho}\, d\rho = \sigma_{,\mu}\, U^{\mu}\, d\rho = 0$$

the four-vector $U^{\mu} = dx^{\mu}/d\rho$ is tangent to some curve lying on the hypersurface and $\sigma_{,\mu}$ is normal to the hypersurface. Let us consider that this hypersurface is null, that means, the interval between two arbitrarily given events on it is light-like,

$$ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = g_{\mu\nu}\, U^{\mu}\, U^{\nu}\, d\rho^2 = 0$$

which is equivalent to take the line joining x^{μ} to $x^{\mu} + dx^{\mu}$ as a null curve. As result, U^{μ} is a null four-vector, $U_{\mu}\, U^{\mu} = 0$, but this vector is also orthogonal to $\sigma_{,\mu}$; $\sigma_{,\mu}\, U^{\mu} = 0$. This implies that $\sigma_{,\mu}$ is also a null four-vector,

$$U^{\mu} = \frac{dx^{\mu}}{d\rho} = g^{\mu\nu}\, \sigma_{,\nu} \qquad (6\text{-}5\text{-}18)$$

$$g^{\mu\nu}\, \sigma_{,\mu}\, \sigma_{,\nu} = 0 \qquad (6\text{-}5\text{-}19)$$

This result is characteristic of null hypersurfaces. Next we prove that the U^{μ} form a family of null geodesics on the hypersurface, that is, a family of geodesics on the null hypersurface. With this end we calculate the covariant derivative of U^{μ}. From (6-5-18),

$$U^{\mu}_{;\nu} = (g^{\mu\alpha}\, \sigma_{,\alpha})_{;\nu} = g^{\mu\alpha} \left\{ \sigma_{,\alpha\nu} - \left\{ {\lambda \atop \alpha\nu} \right\} \sigma_{,\lambda} \right\}$$

and form the combination $U^{\mu}_{;\nu}\, U^{\nu}$. A direct computation then shows that

$$U^{\mu}_{;\nu}\, U^{\nu} = g^{\mu\alpha} \left\{ \sigma_{,\alpha\nu} - \left\{ {\lambda \atop \alpha\nu} \right\} \sigma_{,\lambda} \right\} g^{\nu\rho}\, \sigma_{,\rho} = 0 \qquad (6\text{-}5\text{-}20)$$

which proves that the curves $x^{\mu} = x^{\mu}(\rho)$ are a family of geodesics on the null hypersurface. We now take new coordinates as follows: for x^1 we take σ, for x^2 we take ρ. The two remaining coordinates x^3, x^4 will label the geodesics on each hypersurface σ = constant. That is, the sub-space $\sigma = x^1$ = constant is the locus of points x^2, x^3, x^4 and on this locus the relations $x^3 = f(x^2)$ and $x^4 = h(x^2)$ define a set of curves on this sub-manifold with coordinates x^4, x^2, x^3. These are the geodesics on the null hypersurface. Which this is the case it is easily seen from (6-5-20),

$$U^{\mu}_{;\nu}\, U^{\nu} = 0$$

here $U^{\mu} = \frac{dx^{\mu}}{d\rho} = \delta^{\mu}_{2} = g^{\mu\nu}\, \sigma_{,\nu} = g^{\mu\nu}\, \delta^{1}_{\nu} = g^{\mu 1}$. Also $U_{\mu} = \sigma_{,\mu} = \delta^{1}_{\mu}$. Then,

$$U^{\mu}_{;\nu}\, U^{\nu} = U^{\mu}_{;\nu}\, \delta^{\nu}_{2} = U^{\mu}_{;2} = \delta^{\mu}_{2;2} = g^{\mu 1}_{;2} = 0$$

therefore the $U^{\mu} = \delta^{\mu}_{2} = g^{\mu 1}$ is tangent to the geodesic and the $U_{\mu} = \delta^{1}_{\mu}$ is normal to it. These choice of coordinates generate a metric of the form (6-5-15),

$$g^{\mu 1} = \delta^{\mu}_{2}$$

or

$$g^{\mu\nu} = \left(\begin{array}{c|c|cc} 0 & 1 & 0 & 0 \\ \hline 1 & g^{22} & g^{2m} & \\ \hline 0 & g^{2m} & g^{mn} & \\ 0 & & & \end{array} \right) \quad ;\ m,\ n = 3,\ 4$$

and is interesting to note that it starts with $g^{11} = 0$, quite differently of the usual Minkowskian $g^{\mu\nu}$ which locally reduces to 1 or -1 along the diagonal.

Clearly this happens in the original coordinates x^1, x^2, x^3, x^4.

Let us investigate now how these metric components can be build up from the concept of tetrad. The first tetrad vector is chosen as the normal to the hyper surface.

$$\ell_\mu = \sigma_{,\mu} \quad . \tag{6-5-21}$$

Since it is a general notation to denote the tetrad by letters 1_μ, $\bar{m}_\mu$, $\bar{m}_\mu$, n_μ we used a different letter to denote $\sigma_{,\mu}$ which was called before by U_μ. Next we introduce another null vector n^μ normalized by $\ell_\mu n^\mu = 1$, and two unit space-like vectors ξ^μ and η^μ orthogonal to ℓ^μ, n^μ and orthogonal to each other.

$$\xi^\mu \ell_\mu = \xi^\mu n_\mu = 0 , \qquad \eta^\mu \ell_\mu = \eta^\mu n_\mu = 0$$
$$\xi^\mu \xi_\mu = -1 , \quad \xi^\mu \eta_\mu = 0 , \quad \eta^\mu \eta_\mu = -1$$

Instead of the real space-like vectors ξ^μ and η^μ, it is convenient to use the following complex vectors

$$m^\mu = \frac{1}{\sqrt{2}} (\xi^\mu - i\eta^\mu) \tag{6-5-22}$$

$$\bar{m}^\mu = \frac{1}{\sqrt{2}} (\xi^\mu + i\,\eta^\mu) \tag{6-5-23}$$

The four tetrad vectors ℓ_μ, n_μ, m_μ and $\bar{m}_\mu$ are null, and satisfy the ortho-normality relations

$$\ell_\mu n^\mu = - m_\mu \bar{m}^\mu = 1$$

$$\ell_\mu \ell^\mu = n_\mu n^\mu = m_\mu m^\mu = \bar{m}_\mu \bar{m}^\mu = \tag{6-5-24}$$

$$= \ell_\mu m^\mu = \ell_\mu \bar{m}^\mu = n_\mu m^\mu = n_\mu \bar{m}^\mu = 0$$

Since we constructed these tetrad vectors for building up with them the $g_{\mu\nu}$ of the form written before, we are in the coordinate sys em used before, or $x^1 = \sigma$,

$x^2 = \rho$, x^3, x^4 labelling the null geodesics. In this frame we can satisfy the conditions (6-5-24) by putting

$$m^\mu = \omega\, \delta_2^\mu + \beta^k\, \delta_k^\mu \qquad (6\text{-}5\text{-}25)$$

$$n^\mu = \delta_1^\mu + \alpha\, \delta_2^\mu + X^k\, \delta_k^\mu\,, \quad k = 3,\, 4 \qquad (6\text{-}5\text{-}26)$$

with ω, β^k, α and $X^k = (X^3, X^4)$ arbitrary functions of the coordinates σ, ρ, x^3, x^4.

In these coordinates we also have

$$\ell_\mu = \sigma_{,\mu} = \delta_\mu^1, \; \ell^\mu = \delta_2^\mu \qquad (6\text{-}5\text{-}27)$$

as consequence of (6-5-24) one gets for the metric in terms of the complete set of vectors of the tetrad.

$$g^{\mu\nu} = \ell^\mu\, n^\nu + \ell^\nu\, n^\mu - m^\mu\, \bar{m}^\nu - \bar{m}^\mu\, m^\nu \qquad (6\text{-}5\text{-}28)$$

since

$$\ell = (\ell^\mu) = (0,1,0,0);\; n = (n^\mu) = (1,\, \alpha,\, X^3,\, X^4)$$

$$m = (m^\mu) = (0,\, \omega,\, \beta^3, \beta^4);\; \bar{m} = (\bar{m}^\mu) = (0,\, \bar{\omega},\, \bar{\beta}^3,\, \bar{\beta}^4)$$

we have

$$g^{\mu\nu} = \left(\begin{array}{c|c|cc} 0 & 1 & 0 & 0 \\ \hline 1 & g^{22} & \multicolumn{2}{c}{g^{2m}} \\ \hline 0 & g^{2m} & \multicolumn{2}{c}{g^{mn}} \\ 0 & & & \end{array}\right)$$

where

$$g^{22} = 2(\alpha - |\omega|^2);\quad g^{2m} = X^m - \omega\bar{\beta}^m - \bar{\omega}\beta^m;\; g^{mn} = -\,(\beta^n\, \bar{\beta}^m + \bar{\beta}^n\, \beta^m) \qquad (6\text{-}5\text{-}29)$$

The $g^{\mu\nu}$ of (6-5-28) is form invariant under the transformations

$$\ell'^{\mu} = \ell^{\mu},\ n'^{\mu} = n^{\mu}\ ,\quad m'^{\mu} = m^{\mu}\ e^{iC} \tag{6-5-30}$$

for real C. We also have invariance under

$$\ell'^{\mu} = \ell^{\mu},\ n'^{\mu} = n^{\mu} + \bar{B}m^{\mu} + B\bar{m}^{\mu} + B\bar{B}\ell^{\mu},\ m'^{\mu} = m^{\mu} + B\ell^{\mu} \tag{6-5-31}$$

The transformation (6-5-30), in the tetrad, corresponds to a rotation of the spatial axes as fixed by the two linearly independent space-like unit vectors ξ^{μ} and η^{μ}. Indeed, from (6-5-22) and (6-5-30) one gets

$$\xi'^{\mu} = \xi^{\mu} \cos C + \eta^{\mu} \text{ sen } C$$

$$\eta'^{\mu} = \eta^{\mu} \cos C - \xi^{\mu} \text{ sen } C$$

but always we can mapp so as $\xi^{\mu} = (0, 1, 0, 0)$ and $\eta^{\mu} = (0, 0, 1, 0)$, for example; and thus obtain that this transformation is a rotation in the coordinate plane X-Y. The transformation (6-5-31) represents a rotation around the direction of the tetrad vector ℓ^{μ}, therefore we call it by a null rotation. It depends on two real parameters since B is complex. Thus, we have a three parameter group of symmetries for the metric (6-5-28).

It is of interest to get a more concise notation, with this finallity the following notation is introduced

$$z_{(m)\mu} = (\ell_{\mu}, n_{\mu}, m_{\mu}, \bar{m}_{\mu})\ ,\qquad m = 1, 2, 3, 4\ .$$

for the null tetrad. Then, the relation (6-5-28) can be put in the form familiar to the calculus of tetrads

$$g_{\mu\nu} = z_{(m)\mu}\, z_{(n)\nu}\, \eta^{(m)(n)}$$

and

$$\eta_{(m)(n)} = z_{(m)\mu}\, z_{(n)\nu}\, g^{\mu\nu}$$

where $\eta_{(n)(m)}$ is the flat-space metric in these coordinates,

$$\eta_{(n)(m)} = \eta^{(n)(m)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

The results of this section are of importance in the exhaustive discussion of the radiation field and of the asymptotic conditions on the curvature tensor, or as it turns better, on the tetrad components of the curvature. We will not go into further details, but refer the reader to the literature [53].

6.6) Solutions with Sources

So far we have considered solutions of the empty-space Einstein's equations. There exists several solutions for fields in presence of sources. Of this type is the Schwarzschild field in the region exterior to the sources, so that we have to consider two regions for complete specification of the solutions. These are usually named "exterior solution" and "interior solution". In the section (6.2) we have treated the "exterior Schwarzschild solution". In comparison to this exact solution there exists the approximate solutions describing the "interior Schwarzschild field". Another important exact solution is that describing the "exterior field" for a charged massive particle. In this case the field may be approximated to that of a pointlike charge at rest, and is thereby a generalization of the "exterior Schwarzschild field". In the literature this is called the Reissner-Nordstrom field [54]. Also of some interest is the gravitational field of a spherically symmetric distribution of incompressible matter. In this case which presents a maximum of symmetries it is possible to obtain the "interior" and "exterior" fields without recuring to approximation methods.

6.6.1) The Reissner-Nordstrom Exterior Solution

For $T_{\mu\nu}$ we use the Maxwell stress energy tensor

$$T_{\mu\nu} = \frac{1}{4\pi}\left\{\frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} - F_{\mu\rho} F_{\nu}^{\ \rho}\right\}$$

since this field also presents spherical symmetry the metric tensor is described by two functions of the radial distance r as was seen in the section (6.2). A method of integration similar to that done for the exterior Scharzschild field, will give the solution

$$g_{oo} = 1 - \frac{2Gm}{c^2 r} + \frac{Ge^2}{c^4 r^2} ,$$

$$g_{rs} = - \delta_{rs} + \left(1 - \frac{1}{g_{oo}}\right) \frac{x^r x^s}{r^2}$$

$$g_{os} = 0 ,$$

in cartesian coordinates. The g_{oo} written in terms of the Schwarzschild radius, here denoted by r_o, is

$$g_{oo} = 1-r_o/r + e^2 r_o/2mc^2 r^2$$

therefore the electrostatic energy which gave rise to the last term in g_{oo} has the effect of removing out the singularity in g_{rs}, at $r = r_o$, since there the g_{oo} is different from zero

$$g_{oo}(r_o) = e^2/2mc^2 (1/r_o)$$

the electrostatic term $e^2 r_o/2mc^2 r^2$, is of the order of the Newtonian term r_o/r for distances equal to

$$r = 1/2 e^2/mc^2 ,$$

for the electron this is of the order of the classical radius $e^2/mc^2 = 10^{-13}$ cm. If one tries to find out the zeros for g_{oo}, that is, the values for r from

$$1 - r_o/r + e^2\, r_o/2mc^2\, r^2 = 0$$

or

$$4\, r/r_o = 1 \pm (1 - 8e^2/mr_o\, c^2)^{1/2}$$

we see that no real solutions occur for $1-8e^2/mr_o c^2 < 0$. Thus, no real zeros for g_{oo} exist if $e^2/Gm^2 > \frac{1}{8}$. Since for an electron this ratio is of the order of 10^{40}, $e^2/Gm^2 \approx 10^{40}$, we see that in this case no real zero will occur for g_{oo}. As consequence the Reissner-Nordstrom solution is regular all the way down to the intrinsic singularity at $r = 0$.

6.6.2) Gravitational Field of an Incompressible Ball of Fluid

The first solution also considering a finite source distribution is found on the Schwarzschild's paper. We have treated before the exterior solution. In this section we treat also the interior solution. Let us take a sphere of incompressible fluid. The stress-energy density $T_{\mu\nu}$ is in this situation given by the special relativistic expression replacing $\overset{o}{g}_{\mu\nu}$ by $g_{\mu\nu}$.

$$T^{\mu\nu} = (c^2\rho + p)\, u^\mu u^\nu - p\, g^{\mu\nu}$$

$$= (\varepsilon + p)\, u^\mu u^\nu - p\, g^{\mu\nu}$$

We look for a solution of the Einstein's equations which is both static and spherically symmetric. Therefore, the $g_{\mu\nu}$ may be mapped to a form depending on just two functions of r.

$$g_{\mu\nu} = \begin{pmatrix} e^{\nu(r)} & & & 0 \\ & -e^{+\lambda(r)} & & \\ & & -r^2 & \\ 0 & & & -r^2\, \mathrm{sen}^2\Theta \end{pmatrix}$$

Furthermore, the field considered is that generated in the rest frame of the observer, that means, for the observer moving together with the mass center of the distribution. For it $u^i = 0$. We might have considered the case where $u^i =$ $= c^i$, where the c^i are constants, since this still generates static solutions. However we will take these constants as zero by passing to the rest frame. For the observer at rest in a certain point we have

$$ds^2 = g_{oo}\ dx_o^2$$

but

$$u^o = dx^o/ds = 1/(g_{oo})^{1/2}$$

and since $u_\mu\ u^\mu = 1$, $u^o = u_o^{-1}$. The field equations have the form of the equations for the Schwarzschild field, with the source term, and simplified by the condition $\dot{\nu} = 0$, $\dot{\lambda} = 0$. This will cancel out one of the equations, the $G_0^1 =$ $= e^{-\lambda}\ \frac{\dot{\lambda}}{r}$.

$$G_0^0 = e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) - \frac{1}{r^2} = k\ \rho\ c^2 \tag{6-6-2.1}$$

$$G_1^1 = e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) - \frac{1}{r^2} = -\ kp \tag{6-6-2.2}$$

$$G_2^2 = G_3^3 = \frac{1}{2}\ e^{-\lambda}\left[\nu'' + \frac{1}{2}\nu'^2 + \frac{\nu' - \lambda'}{r} - \frac{1}{2}\nu'\ \lambda'\right] = -\ kp \tag{6-6-2.3}$$

where as before a prime denotes differentiation with respect to r. Due to the Bianchi identities for $G^\mu{}_\nu$, it follows that $T^\mu{}_{\nu;\mu} = 0$. This represents a condition on $\rho(r)$ and $p(r)$. We have,

$$T^\mu{}_{\nu;\mu} = T^\mu{}_{\nu,\mu} + \{^{\ \mu}_{\mu\lambda}\}\ T^\lambda{}_\nu - \{^{\ \lambda}_{\mu\nu}\}\ T^\mu{}_\lambda =$$

$$= \left[(c^2\rho + p)\ u^\mu\ u_\nu\right]_{,\mu} + p_{,\nu} + \{^{\ \mu}_{\mu\lambda}\}\ (c^2\rho + p)u^\lambda\ u_\nu - \{^{\ \lambda}_{\mu\nu}\}(c^2\rho + p)u_\lambda\ u^\mu$$

which gives after some calculations

$$p' = -\frac{1}{2}\nu'(c^2\rho + p) \qquad (6\text{-}6\text{-}2.4)$$

we can use this equation in place of any of the above field equations, we will do this later. However, we cannot use all above four conditions simultaneously since this is redundant (for obtaining the (6-6-2.1) through (6-6-2.3) we used the Bianchi identities). Thus, we have three field equations for determining four unknowns, the ν, λ, ρ and p. Therefore, it is necessary to know the equation of state which relates ρ and p for the fluid. Our procedure however, will be the most general one, namely, we will assume that $\rho = \rho(r)$ and then compute ν and λ as function of ρ, and finally compute p(r). Clearly this will be possible essentially due to the simple form of the above differential equations. Indeed, multiplying (6-6-2.1) by r^2 we have,

$$kc^2\rho r^2 = (e^{-\lambda} r)' - 1$$

which can be integrated to

$$e^{-\lambda} = 1 + \frac{k}{4\pi r}\varepsilon(r) + \frac{C}{r} \qquad (6\text{-}6\text{-}2.5)$$

$$\varepsilon(r) = 4\pi c^2\int_0^r \rho(r')\, r'^2\, dr'$$

where C is a constant of integration. It is natural to require that $\rho(r) = 0$ for $r > r_o$. In this situation if we compute $e^{-\lambda(r)}$ for $r > r_o$ we get

$$e^{-\lambda(r)} = 1 + \frac{k\ \varepsilon(r_o)}{4\pi r} + \frac{C}{r}, \qquad r > r_o$$

putting

$$m^* = -\frac{k\ \varepsilon(r_o)}{8\pi} \qquad (6\text{-}6\text{-}2.6)$$

we obtain $e^{-\lambda}$ in a form identical to that of the external Schwarzschild field plus the term C/r. In regard to dimensions, we note that k has the dimension of T^2/ML, $\varepsilon(r)$ has dimension of energy, ML^2/T^2, so that $k\varepsilon(r)$ has dimension of length which shows that the dimensions in (6-6-2.5) are correct. The constant C has dimension of length. What (6-6-2.6) says is that the length $k\,\varepsilon(r_o)$ as seen for the observer outside of the distribution is just the Schwarzschild radius of the distribution. We may interpret $\varepsilon(r)$ as the total internal energy of the fluid contained within a sphere of radius r. Thus, $\varepsilon(r_o)$ is the total energy of the fluid (supposing our previous condition of boundness on $\rho(r)$). The quantity $\frac{\varepsilon(r_o)}{c^2}$ is then the total mass of the fluid, similarly to the mass for the exterior Schwarzschild field.

For the determination of $\nu(r)$ we make the Schwarzschild's assumption that $\rho(r)$ is constant for $r < r_o$. Of course this is not be expected to hold true for a star, however with this simplifying assumption we will be able to get the general features of a more realistic solution, even if we lost some other details. For a discussion of the integration of the equations (6-6-2.1) and (6-6-2.2) for various equations of state that might exist in the interior of a star the reader may refer to I. Iben, in the Astro. J., 138, 1090 (1963). Detailed discussion of this would conduct outside of the scope of our lectures.

The suposition that ρ is constant allow us to integrate directly the equation (6-6-2.4).

$$c^2\rho + p(r) = \text{const. } e^{-\frac{\nu(r)}{2}} \qquad (6\text{-}6\text{-}2.7)$$

Subtracting (6-6-2.1) from (6-6-2.2),

$$\frac{e^{-\lambda}}{r}(\lambda' + \nu') = -\,k(c^2\rho + p)$$

from (6-6-2.7) we obtain

$$e^{\frac{1}{2}\nu - \lambda}\left(\frac{\lambda'}{r} + \frac{\nu'}{r}\right) = \text{const.} \tag{6-6-2.8}$$

For ρ constant the equation (6-6-2.5) becomes

$$e^{-\lambda} = 1 + kc^2 \rho \frac{r^2}{3} + \frac{C}{r}$$

writing

$$\rho c^2 = -\frac{3}{k R^2}$$

we have

$$e^{-\lambda} = 1 - \frac{r^2}{R^2} + \frac{C}{r}$$

This expression for $e^{-\lambda}$ is substituted into (6-6-2.8),

$$e^{\frac{1}{2}\nu}\left\{\nu'\left(\frac{1}{r} - \frac{r}{R^2} + \frac{C}{r^2}\right) + \frac{2}{R^2} + \frac{C}{r^3}\right\} = \text{const.}$$

Integrating one finds

$$e^{\frac{1}{2}\nu} = A - B\sqrt{1 - \frac{r^2}{R^2}} \tag{6-6-2.9}$$

The term containing the constant C is divergent at the origin, so that we have taken $C = 0$. From (6-6-2.9) we calculate $\nu(r)$, we find

$$\frac{1}{2}\nu = \log\left(A - B\sqrt{1 - \frac{r^2}{R^2}}\right) \tag{6-6-2.10}$$

so that

$$\nu' = \frac{2 B r}{R^2\left(A\sqrt{1 - \frac{r^2}{R^2}} - B\left(1 - \frac{r^2}{R^2}\right)\right)}$$

Replacing this value for ν' together with the expression for $e^{-\lambda}$ from (6-6-2.5) (where ρ is taken constant and $C = 0$, the result being $e^{-\lambda} = 1 - r^2/R^2$) into (6-6-2.2), we can calculate the pressure p inside the fluid. After some computations one gets the value

$$- kp = \frac{1}{R^2}\left\{\frac{3\,B\sqrt{1-\frac{r^2}{R^2}} - A}{A - B\sqrt{1-\frac{r^2}{R^2}}}\right\} \qquad (6\text{-}6\text{-}2.11)$$

The two constants A and B are fixed by imposing that p = 0 on the surface of the sphere (all particles forming the body do not move normally to the surface but only tangentially to it, since in the opposite case the surface would not form a boundary for the components of the body), and that e^{ν} joins on smoothly to the Schwarzschild field on the surface *. One finds,

$$A = \frac{3}{2}\sqrt{1-\frac{r_0^2}{R^2}}$$

$$B = \frac{1}{2}$$

Replacing this into (6-6-2.9), we get for the field functions inside of the sphere of fluid

$$e^{\frac{\nu}{2}} = \frac{3}{2}\sqrt{1-\frac{r_0^2}{R^2}} - \frac{1}{2}\sqrt{1-\frac{r^2}{R^2}} \qquad (6\text{-}6\text{-}2.12)$$

$$e^{-\lambda} = 1 - \frac{r^2}{R^2} \qquad (6\text{-}6\text{-}2.13)$$

and for the pressure inside the body we find,

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* According to our conditions on the source distribution $\rho(r) = 0$, $p(r) = 0$ for $r > r_o$, so that $T_{\mu\nu} = 0$ outside of the body. Since $T_{\mu\nu}$ is spherically symmetric this implies by Birkhoff's theorem that $g_{\mu\nu}$ is the Schwarzschild field for $r > r_o$.

$$p = \rho \left\{ \frac{\sqrt{1 - \frac{r^2}{R^2}} - \sqrt{1 - \frac{r_0^2}{R^2}}}{3\sqrt{1 - \frac{r_0^2}{R^2}} - \sqrt{1 - \frac{r^2}{R^2}}} \right\} \qquad (6\text{-}6\text{-}2.14)$$

Since r varies from 0 to r_o, in order that this solution be real is necessary that

$$r_0^2 < R^2 = -3/k\,\rho c^2 \,. \qquad (6\text{-}6\text{-}2.15)$$

If we require that the pressure on the fluid is everywhere finite, we obtain from (6-6-2.14) the condition

$$r_0^2 < \frac{8}{9} R^2 \,. \qquad (6\text{-}6\text{-}2.16)$$

Indeed, the singular point for p(r) is r such that the denominator of (6-6-2.14) vanishes,

$$3\sqrt{1 - \frac{r_0^2}{R^2}} - \sqrt{1 - \frac{r^2}{R^2}} = 0$$

which gives

$$r^2 = -8R^2 + 9r_0^2 \qquad (6\text{-}6\text{-}2.17)$$

if it happens that $9r_0^2 - 8R^2 < 0$, no such r does exist and thus p(r) is finite for all values of r. The condition following from this inequallity is just the (6-6-2.16). It is also very easy to see that $p > 0$ on account that $r < r_o$. This just says that the above p(r) is indeed physically reasonable in spite of being obtained without a deeper analysis on the structure of the body. The equation (6-6-2.17) gives the several values for r on which the pressure may become infinite. In particular if $r_0^2 = 8/9\,R^2$, the pressure becomes infinite at $r = 0$. For any distance d away from the center, p will be infinity if $-8R^2 + 9r_0^2 - d^2 = 0$. Thereby, the validity of (6-6-2.16) is a condition of stability for the distribution. This condition being an upper bound for the radius of the sphere; the r_o, represents an upper bound for the possible amount of matter which can be packed for generating a static and stable solution of the Einstein's equations. This maximum amount of matter will be

$$m_{crit} = 4/3\pi\rho\ (r_o^3)_{crit} ,$$

where,

$$(r_o)_{crit} = (8/9\ R^2)^{1/2} .$$

that is, for stability $m < m_{crit}$.

Using (6-6-2.16) together with the explicit value for k we find for m_{crit},

$$m_{crit} = \left(\frac{4}{3}\pi\right)\rho\left(\frac{8}{9}\right)^{3/4}\left(\frac{3\ c^4}{8\pi\rho G}\right)^{3/2}$$

this value for the maximum amount of matter will depend on the value for the density of matter ρ. For masses greater than m_{crit} the fluid begin to collapse as result of the unsupported gravitational attraction of its various parts. Once begun, such a contraction would continue untill all fluid became concentrated on a point.

While the assumption of incompressibility, as was used here, is unrealistic from the intuitive point of view, the use of more realistic equations of state does not modify the existence of unstable collapsing states. Oppenheimer and Volkov [56] have considered the case of a cold neutron gas. Matter in this state might be imagined to exist in a large star after all thermonuclear burning had taken place and gravitational forces had overhelmed the pressure of the electron gas formed by beta-decay of the neutron gas. If the gravitational forces are sufficiently strong, inverse beta-decay would take place and eventually all electrons would be combined with protons to form neutrons [57]. Oppenheimer and

Volkoff integrated the Einstein equations when such matter acts as source of the gravitational field and showed that no stable solution exists for a total mass exceeding 0.7 solar masses, the critical mass for this case. For larger masses no static solutions of these equations exist and the star would undergo gravitational collapse. What happens to matter as it is compressed into an ever-decreasing volume is an open question. Even the assumption of a hard nuclear core will not inhibit the collapse. Indeed, at best it would imply in a state of incompressibility and even in this case an infinite pressure will became an infinite source of energy for the gravitational field, which in turn will produce an infinite gravitational field. Some suggestions have been put forward, as for instance Wheeler's proposal that for exceedingly high densities, mass is completely converted into radiation which is strong enough for beating the gravitational attraction and move away. Alternatively, one may imagine that beyond a certain small region it is no longer permissible to treat the gravitational field classically, but it have to be treated as a quantized system. In any event, essentially new and at present unknown physical laws must come into play in this region.

7. CONSERVATION LAWS IN GENERAL RELATIVITY

7.1) Introductory Concepts

The concepts of energy, momentum and angular momentum have a fundamental importance in both classical and quantum physics. In the Newtonian mechanics we need to know the force field acting on the system in order to determine its state of motion. Nevertheless, frequently, we really do not know in all details the structure of these forces, or it may happen that the individual description of the system in terms of its components turns out too much involved. In such cases the knowledge of general conservation laws such as conservation of charge, energy or angular momentum serve to characterize the system as a whole.

From the point of view of its historical appearance, the first of such quantity was the kinetic energy introduced by Leibnitz under the name of "vis viva". Conservation laws for the kinetic energy and for the linear momentum allow us to solve problems of collisions between particles, even if we do not know the real mechanism of the forces for colliding particles. In general the total kinetic energy of the particles forming up the system is not conserved. In the case where the forces have origin in a potential, we can generalize the concept of conservation for the total energy represented by the sum of the total kinetic energy with the total potential energy. As example of this process is the motion of a projectile neglecting the air resistance. This process of generalizing the conservation laws by introducing new quantities so as the total sum is conserved is characteristic of the development in the knowledge of the dynamics of the system. With the end of retaining conservation laws in presence of electromagnetic radiation we have to recognize that the electromagnetic field transports energy and momentum.

In special relativity energy and momentum form together a sole physical quantity, the fourvector of momentum, so that it has to be conserved as a whole. This result is consequence of the homogeinity of the four-dimensional Minkowski space. This same conclusion may be stated in terms of the fact that the density of energy and of momentum, together with the stress components form a second rank tensor $T_{\alpha\beta}$ with respect to the group of Poincaré. This tensor is divergence-free.

$$T^{\beta}_{\alpha,\beta} = 0$$

If all relevant forms of energy have been taken into consideration, total energy and momentum are obtained from this tensor by suitable integrations over the three-dimensional space. Similarly a third rank tensor $M_{\alpha[\beta\lambda]}$ skew-symmetric on a pair of indices, represents the angular momentum in special relativity.

Difficulties will appear in connection to the notion of energy in the general theory of relativity. The space-time in this theory is not a flat symmetric space as was in special relativity, in which energy and momentum were associated to the homogeneity of all four directions. The finallity of this chapter is to present the relationships between conservation theorems and invariance properties, with special emphasys on the definition of energy in general relativity.

For illustrating the connection between conservation theorems and properties of invariance associated to physical observables of the system, let us consider the following example: the motion of a particle in a static spherically symmetric field of forces. There exists four quantities which are constant of the motion, the three components of the angular momentum of the particle plus the energy of the particle (in the non-relativistic mechanics).

$$d/dt\ \vec{M} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F}$$

which is zero since $\vec{F} = -\operatorname{grad} V(r) = -dV/dr.\vec{r}/r$ is parallel to $\vec{r}$. For the energy,

$$dE/dt = d/dt(\vec{p}^2/2m + V) = \frac{\vec{p}.\dot{\vec{p}}}{m} + \frac{dV}{d\vec{r}} . \dot{\vec{r}} = 0 .$$

The last conclusion was the result that V depends only on r, but not on time.

The linear momentum of the particle, $\vec{p} = m\dot{\vec{r}}$, is not conserved since there is a force field acting, having the origin of coordinates as the center of the field. If we make a translation of the coordinate system the field equations will not be form invariant. We can translate the existence of these four conserved quantities by saying that in a static spherically symmetric field of forces the conservation laws are consequence of invariance of the Lagrangian under a group of continuous transformations with four parameters, formed by the rotations around the center of the field and the transformation $t' = t+a$. These later form the theorems of conservation associated to the four invariants $\vec{M}$ and E.

The connection between invariance properties and theorems of conservation are contained in two general theorems of Emmi Noether. Since in the chapter five we have already proved those theorems presently we will use them directly.

We frequently shall use the concept of a weak conservation law, such a law is one which is satisfied only on the path followed by the system. In other words, given the Lagrangian (or the Lagrangian density) of the system, if the variation on it,generated by some symmetry group,has the form of a divergence along the path for the system,

$$\delta\mathcal{L} = \varepsilon^{\mu_1\ldots\mu_p} \frac{\partial}{\partial x^\nu} \mathcal{J}^\nu{}_{\mu_1\ldots\mu_p}$$

We obtain p weak conservation laws. This is just the statement of the first Noether's theorem. As example of this let us consider the Lagrangian

$$\mathcal{L} = \mathcal{L}\,(q^i(x),\ q^i_{,j}(x))$$

for some physical system. Taking a translation of the coordinates,

$$x'^j = x^j + a^j\ ,$$

we have the variation in $\mathcal{L}$.

$$\mathcal{L}' = \mathcal{L}(q^i(x+a),\ q^i_{,j}(x+a)) = \mathcal{L}(q^i(x) + a^j q^i_{,j}(x), q^i_{,j}(x),\ q^i_{,j}(x) + a^\ell q^i_{,j\ell}(x))$$

or,

$$\mathcal{L}' = \mathcal{L}(q^i(x),\ q^i_{,j}(x)) + a^j\ q^i_{,j} \frac{\partial\mathcal{L}}{\partial q^i} + a^\ell\ q^i_{,j\ell}\ \frac{\partial\mathcal{L}}{\partial q^i_{,j}}$$

but

$$\delta\mathcal{L} \equiv \mathcal{L}'-\mathcal{L}=\mathcal{L}(q^i(x+a),\ q^i_{,j}(x+a)) - \mathcal{L}(q^i(x),\ q^i_j(x)) = a^\ell\ \frac{\partial\mathcal{L}}{\partial x^\ell}$$

thus, we get

$$\delta\mathcal{L} \equiv a^\ell\ \frac{\partial\mathcal{L}}{\partial x^\ell} = a^j\ q^i_{,j} \frac{\partial\mathcal{L}}{\partial q^i} + a^\ell\ q^i_{,j\ell} \frac{\partial\mathcal{L}}{\partial q^i_{,j}}$$

Using the Euler-Lagrange equations of motion (this is just the point where the weak character of the conservation law appears), we write this as

$$\delta\mathcal{L} \equiv a^\ell \frac{\partial\mathcal{L}}{\partial x^\ell} = a^j\ q^i_{,j}\ \frac{\partial}{\partial x^\ell}\ \frac{\partial\mathcal{L}}{\partial q^i_{,\ell}} + a^\ell\ q^i_{,j\ell} \frac{\partial\mathcal{L}}{\partial q^i_{,j}}$$

which may be put as

$$\delta\mathcal{L} \equiv a^\ell \frac{\partial\mathcal{L}}{\partial x^\ell} = a^\ell\ \frac{\partial}{\partial x^j} \left[q^i_{,\ell} \frac{\partial\mathcal{L}}{\partial q^i_{,j}}\right].$$

Therefore, we have the weak conservation law for the stress-energy tensor of the system

$$a^\ell\ \frac{\partial}{\partial x^j}\ T^j_\ell = 0$$

$$T^j_\ell = q^i_{,\ell}\ \frac{\partial\mathcal{L}}{\partial q^i_{,j}} - \delta^j_\ell\, \mathcal{L}$$

This is the relativistic conservation law of energy and momentum if we have four coordinates x^i, as is the case in relativity.

The second Noether's theorem states that invariance of the Lagrangian under a group involving q arbitrary functions implies in the existence of q identities among the left-hand side of the Euler-Lagrange equations.

First of all we need to clarify further the difference between weak conservation laws and identities. This will be done by means of the following example. Consider the two Action integrals

$$I_1 = \int a_{ik}(x) \frac{dx^i}{d\lambda} \frac{dx^k}{d\lambda} d\lambda$$

$$I_2 = \int \sqrt{a_{ik}(x) \frac{dx^i}{d\lambda} \frac{dx^k}{d\lambda}} d\lambda$$

where $a_{ik}(x)$ is a symmetrical matrix independent of the parameter λ. We have, indicating derivative with respect to λ by a prime,

$$\frac{1}{2} \delta I_1 = \int \left(\frac{1}{2} x'^i x'^k \frac{\partial a_{ik}}{\partial x^m} \delta x^m + a_{ik} x'^i \frac{d}{d\lambda} \delta x^k \right) d\lambda$$

considering variations vanishing on the boundaries,

$$\frac{1}{2} \delta I_1 = \int \left(\frac{1}{2} x'^i x'^k \frac{\partial a_{ik}}{\partial x^m} \delta x^m - \frac{d}{d\lambda} (a_{ik} x'^i) \delta x^k \right) d\lambda$$

which gives.

$$\frac{1}{2} I_1 = \int \left(\frac{1}{2} x'^i x'^k \frac{\partial a_{ik}}{\partial x^m} - \frac{\partial a_{im}}{\partial x^\ell} x'^\ell x'^i - a_{im} x''^i \right) \delta x^m d\lambda$$

thus,

$$\frac{1}{2} x'^i x'^k \frac{\partial a_{ik}}{\partial x^m} - \frac{\partial a_{im}}{\partial x^\ell} x'^\ell x'^i - a_{im} x''^i = 0$$

since the second term is symmetric on (1,i) we get

$$\frac{1}{2}\left(\frac{\partial a_{ik}}{\partial x^m} - \frac{\partial a_{im}}{\partial x^k} - \frac{\partial a_{km}}{\partial x^i}\right) x'^k\, x'^i - a_{im}\, x''^i = 0$$

multiplying by the inverse of a_{im} we obtain

$$x''^i + \{^{\,i}_{k\ell}\}\, x'^k\, x'^\ell = 0 \tag{7-1-1}$$

where $\{^{\,i}_{k\ell}\}$ indicates the combination defining the Christoffel symbol if a_{ij} is the metric tensor g_{ij}. However this is not claimed.

$$\{^{\,i}_{k\ell}\} = \frac{1}{2}\, a^{ir}\left(\frac{\partial a_{rk}}{\partial x^\ell} + \frac{\partial a_{r\ell}}{\partial x^k} - \frac{\partial a_{k\ell}}{\partial x^r}\right) \tag{7-1-2}$$

These are the Euler-Lagrange equations for the Action integral I_1. For the other integral we have.

$$\delta\sqrt{a_{ik}\, x'^i\, x'^k} = \frac{1}{2}\, \frac{\delta(a_{ik}\, x'^i\, x'^k)}{\sqrt{a_{rs}\, x'^r\, x'^s}}$$

which gives

$$\delta\sqrt{a_{ik}\, x'^i\, x'^k} = \frac{1}{2}\, \frac{1}{\sqrt{a_{rs}\, x'^r\, x'^s}}\left\{\frac{\partial a_{ik}}{\partial x^\ell}\, x'^i\, x'^k\, \delta x^\ell + 2\, a_{ik}\, x'^i\, \frac{d}{d\lambda}\, \delta x^k\right\}$$

The Euler-Lagrange equations then take the form.

$$\frac{1}{\sqrt{a_{rs}\, x'^r\, x'^s}}\left\{\frac{1}{2}\, x'^i\, x'^k\, \frac{\partial a_{ik}}{\partial x^m} - \frac{\partial a_{im}}{\partial x^\ell}\, x'^\ell\, x'^i - a_{im}\, x''^i\right\} = 0 \tag{7-1-3}$$

We have,

$$x'^k = \frac{dx^k}{d\lambda} = \frac{dx^k}{ds}\, \frac{ds}{d\lambda}$$

where

$$ds = \sqrt{a_{mn}\, x'^m\, x'^n}$$

then,

$$\frac{dx^k}{ds} = x'^k \frac{d\lambda}{ds} .$$

Multiplying (7-1-3) by $\frac{d\lambda^2}{ds}$, and noting that a factor 1/ds is already present in this equation as a multiplicative common factor, we get

$$\frac{d^2x^i}{ds^2} + \{^{\ i}_{k\ell}\} \frac{dx^k}{ds} \frac{dx^\ell}{ds} = 0 . \qquad (7\text{-}1\text{-}4)$$

This is the Euler-Lagrange equation for I_2. Is this last integral which gives the Action integral for the motion of a test particle in a given gravitational field when $a_{mn} = g_{mn}$. The two Euler-Lagrange equations in spite of looking similar have certain fundamental differences. First let us consider I_1. This integral is invariant under the particular parameter change

$$\lambda' = \lambda + \varepsilon$$

with constant ε. Indeed, under this change

$$\tilde{x}'^i = \frac{dx^i}{d\lambda'} = x'^i \frac{d\lambda}{d\lambda'} = x'^i .$$

Considering variations of I_1 which do not necessarily vanish on the boundaries

$$\delta I_1 = \int \mathscr{L}_m\, \delta x^m\, d\lambda + \int \frac{d}{d\lambda} (a_{ik}\, x'^i\, \delta x^k)\, d\lambda$$

where $\mathscr{L}_m$ are the left hand side of the Euler-Lagrange equations. On the path followed by the system $\mathscr{L}_m = 0$, so that

$$\delta I_1 = \int \frac{d}{d\lambda} (a_{ik}\, x'^i\, \delta x^k) d\lambda \; .$$

Considering that δx^k is induced by a parameter change of the type considered before.

$$\delta x^k = \frac{dx^k}{d\lambda} \delta\lambda = x'^k\, \varepsilon \; .$$

Since ε is constant we get,

$$\delta I_1 = \varepsilon \int \frac{d}{d\lambda} (a_{ik}\, x'^i\, x'^k)\; d\lambda \; .$$

Introducing a Lagrangian L_1

$$\delta L_1 = \varepsilon \; \frac{d}{d\lambda} \; (a_{ik}\, x'^i\, x'^k)$$

Invariance of I_1 under the parameter change then implies in

$$\varepsilon \; \frac{d}{d\lambda} (a_{ik}\, x'^i\, x'^k) = 0$$

but since ε is necessarily different from zero,

$$\frac{d}{d\lambda} (a_{ik}\, x'^i\, x'^k) = 0$$

This is an example of a weak conservation law (holding along the path for the system) associated to invariance under the parameter change in I_1. In our initial definition of I_1 we have not given any explicit value for a_{mn}, or for λ. If one takes λ equal to time, the parameter change will be a translation of the time origin, and invariance of I_1 under this transformation will represent the law of conservation of energy. By the other hand, I_2 is invariant under the general parameter change

$$\lambda' = \phi(\lambda)$$

under which the x'^k vary as

$$\tilde{x}'^k = x'^k \frac{d\lambda}{d\lambda'}$$

no reference whatever is done with respect to the form of the function $\phi(\lambda)$. Again we consider variation in I_2, since now δx^k will be a function of λ,

$$\delta x^k = x'^k \, \delta\lambda$$

no conservation law will be obtained by considering a variation along the motion for the system as was done for I_1. Due to this, we instead take variations vanishing on the limits of integration.

$$\delta I_2 = \int \mathcal{L}_m \, \delta x^m \, d\lambda$$

and suppose that they are generated by δx^k of the form given above. In this case the variation δx^k is tangent to the curve followed by the system, and thus, according to the calculus of variations we have that no variation at all is gotten, since we do not have two paths to compare. This in turn implies that δI_2 is the difference between the I_2 with itself, and therefore vanishes identically.

$$\delta I_2 \equiv 0$$

which implies in

$$x'^m \mathcal{L}_m \equiv 0 \,, \qquad (7\text{-}1\text{-}5)$$

or,

$$a_{km} \, x'^m \left\{ \frac{d^2 x^k}{ds^2} + \left\{ {k \atop ij} \right\} \frac{dx^i}{ds} \frac{dx^j}{ds} \right\} \equiv 0 \qquad (7\text{-}1\text{-}6)$$

Recalling that δI_2 is a variation along the same path, it does not imply that $\mathcal{L}_m$ is necessarily zero. Thus, the equation (7-1-5) holds identically, even if $\mathcal{L}_m$ is not null. This example shows how identities relating the left hand side of the

Euler-Lagrange equations can be obtained. They will appear whenever the Action integral is invariant under a function group. In this example I_2 is invariant under the function group with function $\phi(\lambda)$. One gets so many identities as available functions exist, in this example we got just one identity. By the other hand I_1 does not present a functional invariance, but just a finite type of invariance under the one-parameter group $\lambda' = \lambda + \varepsilon$. As consequence one weak conservation law is obtained.

7.2) Weak Conservation laws and Identities for a General Classical Field Theory.

In this section we shall apply directly the two Noether theorems to the problem of stating conservation laws for a given field theory in the classical stage. Let the field functions be the continuous differentiable functions $y_A(x)$, $A = 1 \ldots N$ and $x = (x^1 \ldots x^n)$, possessing the Action integral

$$W = \int_\Omega L(x; y_A(x), y_{A,i}(x), y_{A,ij}(x))\, dx$$

L being the Lagrangian density. The field equations are

$$L^A(y(x),x) = \frac{\delta W}{\delta y_A(x)} = 0 .$$

Consider general gauge transformations,

$$y'_A(x) = Y_A(x;y) , \qquad (7\text{-}2\text{-}1)$$

$$x' = X(x) , \qquad (7\text{-}2\text{-}2)$$

which leave the field equations form invariant. These transformations are called isometries. We consider isometries which are continuous with the identity transformation.

$$y'_A(x') = y_A(x) + \delta y_A(x)$$

$$x' = x + \delta x(x)$$

The Noether theorems then imply that W is invariant if

$$L^A \bar{\delta} y_A + \bar{\delta} t^i_{,i} \equiv 0 \tag{7-2-3}$$

$$\bar{\delta} t^i \equiv L\, \delta x^i + \left(\frac{\partial L}{\partial y_{A,i}} - \partial_k \left(\frac{\partial L}{\partial y_{A,ik}} \right) \right) \bar{\delta} y_A + \frac{\partial L}{\partial y_{A,ik}} \bar{\delta} y_{A,k} \tag{7-2-4}$$

Two situations are of interest. The first is obtained when the isometries form a group G_p, depending on p parameters.

$$\delta x^i(x) = \varepsilon^\mu \xi^i_\mu(x)$$

$$\bar{\delta} y_A(x) = \varepsilon^\mu \eta_{A\mu}(x)$$

$$\bar{\delta} t_i = \varepsilon^\mu t^i_\mu$$

$$\mu = 1 \ldots p, \quad i = 1 \ldots n$$

In this case the (7-2-3) and (7-2-4) are simply

$$L^A \varepsilon^\mu \eta_{A\mu} + \varepsilon^\mu t^i_{\mu,i} \equiv 0$$

Which represent p weak conservation laws, since for $L^A = 0$ we get $t^i_{\mu,i} = 0$.

The second possibility is for a group $G_{\infty q}$, depending on q arbitrary functions $\varepsilon^\nu(x)$, $\nu = 1 \ldots q$. For our purposes it will be sufficient to write

$$\delta x^i(x) = \varepsilon^\nu(x)\, \xi^i_\nu(x) \tag{7-2-5}$$

$$\bar{\delta} y_A(x) = \varepsilon^\nu(x)\, \eta_{A\nu}(x) - \varepsilon^\nu_{,i}\, \gamma^i_{A\nu} \tag{7-2-6}$$

Substituting these relations into (7-2-3) we get an identity involving the ε^ν,

$\varepsilon^{\nu}_{,i}$ and the $\varepsilon^{\nu}_{,ij}$ linearly.

$$F_{\nu}\, \varepsilon^{\nu}(x) + F^{i}_{\nu}\, \varepsilon^{\nu}_{,i}(x) + F^{ij}_{\nu}\, \varepsilon^{\nu}_{,ij}(x) \equiv 0 .$$

Due to the fact that the $\varepsilon^{\nu}(x)$ are arbitrary, the coefficients of ε^{ν}, $\varepsilon^{\nu}_{,i}$ and $\varepsilon^{\nu}_{,ij}$ have to be set equal to zero separately.

$$F_{\nu} \equiv 0 ,$$

$$F^{i}_{\nu} \equiv 0 ,$$

$$F^{ij}_{\nu} \equiv 0 .$$

Nevertheless such relations are still too much general for being useful. We look for identities which are homogeneous and linear in L^{A}. The (7-2-3) are not homogeneous due to the term $\overline{\delta} t^{i}_{,i}$ which depends on the L^{A}. Such type of identities are obtained when $\varepsilon^{\nu}(x)$ is such that it vanishes asymptotically. In other terms, we shall restrict to the sub-group $\tilde{G}_{\infty q}$ of $G_{\infty q}$ formed by the functions $\varepsilon^{\nu}(x)$ in (7-2-5), (7-2-6), satisfying

$$\varepsilon^{\nu}(x) \to 0 , \qquad |x| \to \infty$$

$$\varepsilon^{\nu}_{,i}(x) \to 0$$

$$\cdots\cdots\cdots\cdots$$

For this sub-group we have for (7-2-3),

$$\int_{\Omega} L^{A}\, \overline{\delta} y_{A}\, d\Omega + \int_{\Omega} \overline{\delta} t^{i}_{,i}\, d\Omega \equiv 0$$

But

$$\int_{\Omega} \overline{\delta} t^{i}_{,i}\, d\Omega = \oint_{\Omega-1} \overline{\delta} t^{i}\, n_{i}\, d\sigma$$

which vanishes. Thus,

$$\int_{\Omega} L^{A}\, \overline{\delta} y_{A}\, d\Omega \equiv 0$$

Since the region of integration is arbitrary,

$$L^A \; \bar{\delta} y_A \equiv 0$$

substitution of (7-2-6) gives

$$L^A(\varepsilon^\nu \eta_{A\nu} - \varepsilon^\nu_{,i} \gamma^i_{A\nu}) \equiv 0$$

Partial integration dropping the surface term gives

$$\varepsilon^\nu(x)(L^A \eta_{A\nu} + \partial_i(L^A \gamma^i_{A\nu})) \equiv 0$$

which implies, on account of the arbitrariness in the $\varepsilon^\nu(x)$,

$$L^A \eta_{A\nu} + (L^A \gamma^i_A)_{,i} \equiv 0 \tag{7-2-7}$$

These are the general Bianchi identities. For the case where the $y_A(x)$ are the metric components $g_{\mu\nu}(x)$, and $i = 1...4$, with $q = 4$ they will be just the contracted Bianchi identities of general relativity.

The existence of these identities will imply in ambiguity in the solution of the initial value problem for the $y_A(\vec{x}, x^o)$. This problem was already treated previously and its limitations were interpreted as a prescription for separating physical variables from the other variables. Here, for completeness we include the proof that (7-2-7) will limit the number of possible initial Cauchy data. Let us denote the highest time derivative of y_A inside L^A by $y_A^{(n)}$. Then, if they appear linearly in L^A, which is the case of all known applications, viz for electrodynamics or general relativity,

$$L^A = \alpha^{AB} y_B^{(n)} + \Lambda^A(y, y^{(m)}), \quad m < n \tag{7-2-8}$$

replacing this into (7-2-7) and collecting the term with the highest order time derivative.

$$\alpha^{AB} \gamma^o_{A\nu} \; y_B^{(n+1)} + ... \equiv 0 \tag{7-2-9}$$

The exact form of the terms represented by the dots is not important, all that matters is that they depend on derivatives of y_A with respect to x^o of the order n, or lower. Since all derivatives of y_A at a point are arbitrary, it follows that

$$\alpha^{AB}\ \gamma^{o}_{A\nu} = 0 \qquad (7\text{-}2\text{-}10)$$

Therefore the matrix α^{AB} possess so many null eigenvectors as is the range of variation for the ν. We have called one of the several coordinates x^i by "time", however, from the mathematical point of view this is just one of the n possible values for x^i. In relativity indeed x^o is a coordinate-time. In this case we get four null eigenvectors for α^{AB}, and this matrix is of the form $\alpha^{\mu\nu,\rho\sigma}$. In summary, the matrix α^{AB} is singular, with n linearly dependent null eigenvectors. As consequence we cannot solve (7-2-8) for all $y_A^{(n)}$ in terms of the lower order derivatives. Part of them will not admit such solutions. In general relativity they are the four $g^{(2)}_{o\mu}$. The remaining variables, the $g^{(2)}_{ij}$ admit solutions. The Cauchy problem can be formulated completely only for these later variables. This fact was interpreted before.

7.3) Continuity Equations in General Relativity

Following with our treatment let us consider again Noether's identity (7-2-3). We take here the situation where the mapping functions $\varepsilon^\nu(x)$ are zero on the boundaries of the four-space (the present treatment is specific for general relativity, so as n = 4, ν = 4 and the y_A is just the $g_{\mu\nu}$, but we continue to use the y-notation). The $\varepsilon^\nu(x)$ arise from the symmetry mappings of the theory,

$$x'^\nu = x^\nu + \varepsilon^\nu(x)$$

We will write the identity (7-2-3) by dropping the $\overline{\delta}$ in the divergence term involving the t^i. This has nothing of profound and is just a matter of convenience in the notation . Alongside with (7-2-3) we use too the local identity (7-2-7), the Bianchi identities. For $\tilde{G}_{\infty 4}$ they are consistent with the identities (7-2-3).

The equation (7-2-3) for the choice (7-2-6) may be put in the form

$$t^{\mu}{}_{,\mu} + L^{A}\,\varepsilon^{\nu}(x)\,\eta_{A\nu}(x) - L^{A}\,\varepsilon^{\nu}{}_{,\alpha}\,\gamma^{\alpha}{}_{A\nu} \equiv 0 \;. \qquad (7\text{-}3\text{-}1)$$

Using the identity (7-2-7) we write this as

$$Ⓗ^{\mu}{}_{,\mu} \equiv 0 \qquad (7\text{-}3\text{-}2)$$

where

$$Ⓗ^{\mu} = t^{\mu} - L^{A}\,\varepsilon^{\nu}\,\gamma^{\mu}_{A\nu} \;. \qquad (7\text{-}3\text{-}3)$$

From (7-3-2) we may infer that $Ⓗ^{\mu}$ may be written as the curl of a skew-symmetric third rank tensor, a "superpotential".

$$Ⓗ^{\mu} \equiv U^{[\mu\sigma]}{}_{,\sigma} \;. \qquad (7\text{-}3\text{-}4)$$

Historically the first superpotential was introduced by Freud [58] in conection with the Einstein's pseudo-tensor. From (7-3-3) we obtain, solving for t^{μ},

$$t^{\mu} = L^{A}\,\varepsilon^{\nu}(x)\,\gamma^{\mu}_{A\nu}(x) + U^{[\mu\sigma]}{}_{,\sigma} \;. \qquad (7\text{-}3\text{-}5)$$

However, we have also the explicit expression for t^{μ} which is given from Noether's theorem by (7-2-4). For Einstein's Lagrangian density Komar [59] has shown that it reduces to

$$t^{\mu} = \left\{ \frac{\sqrt{-g}}{k}\,(g^{\sigma\nu}\,\varepsilon^{\mu}_{;\nu} - g^{\mu\nu}\,\varepsilon^{\sigma}_{;\nu}) \right\}_{,\sigma} \qquad (7\text{-}3\text{-}6)$$

where we recall that here $\bar{\delta} y_A = \bar{\delta} g_{\mu\nu} = \varepsilon_{\mu;\nu} + \varepsilon_{\nu;\mu}$. In the proof of this relation we have taken that $g_{\mu\nu}$ is solution of the field equations, so that $L^A = 0$. Then $t^\mu = U^{[\mu\sigma]}_{,\sigma}$ and from (7-3-6) one gets

$$U^{[\mu\sigma]} = \frac{\sqrt{-g}}{k} (g^{\sigma\nu} \varepsilon^\mu_{;\nu} - g^{\mu\nu} \varepsilon^\sigma_{;\nu}) \qquad (7\text{-}3\text{-}7)$$

The nonuniqueness in $t^\mu = U^{[\mu\sigma]}_{,\sigma}$ is obvious since the $\varepsilon^\mu(x)$ are arbitrary functions. This arbitrarity is similarly verified for the superpotential $U^{[\mu\sigma]}$. Besides this we can also sum to t^μ an arbitrary curl, $V^{[\mu\sigma]}_{,\sigma}$, thus obtaining a new t'^μ,

$$t'^\mu = t^\mu + V^{[\mu\sigma]}_{,\sigma}$$

and corresponding to this a new superpotential $U'^{[\mu\sigma]}$

$$U'^{[\mu\sigma]} = U^{[\mu\sigma]} + V^{[\mu\sigma]}$$

By convenient choices of $\varepsilon^\nu(x)$ and $V^{[\mu\sigma]}(x)$ one can obtain all the various stress-energy pseudo-tensors and corresponding superpotentials *. Due to this we cannot really ascribe to any one of them a definite meaning, as the real stress-energy tensor for the gravitational field. They all share the same ground in the treatment.

For completing this section we write down the various pseudo-tensors which have been proposed in the literature. The first was the Einstein's pseudo-tensor defined as [60]

$$\sqrt{-g}\,{}_E t_\nu^{\ \mu} \equiv -\delta^\mu_\nu L' + g_{\rho\sigma,\nu} \frac{\partial L'}{\partial g_{\rho\sigma,\mu}} \qquad (7\text{-}3\text{-}8)$$

where L' is the reduced gravitational Lagrangian density (a function only of $g_{\mu\nu}$ and $g_{\mu\nu,\alpha}$).

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* which have been proposed in the literature.

$$-2k\, L' = \sqrt{-g}\ R - \left(\sqrt{-g}\ g_{\mu\nu,\rho} \frac{\partial \mathfrak{L}}{\partial g_{\mu\nu,\rho\sigma}} \right)_{,\sigma}$$

The conservation law for the system represented by the sources of the field and the field itself being

$$_E\mathfrak{J}^{\mu}_{\nu,\mu} = 0 \qquad (7\text{-}3\text{-}9)$$

$$_E\,{}_{\nu}\mathfrak{J}^{\mu} = \sqrt{-g}\ (T_{\nu}^{\ \mu} + {}_E t_{\nu}^{\ \mu}) \qquad (7\text{-}3\text{-}10)$$

von Freud showed that the superpotential for it was

$$_E\mathfrak{J}_{\nu}^{\ \mu} = {}_F U_{\nu}^{[\mu\sigma]}{}_{,\sigma} \qquad (7\text{-}3\text{-}11)$$

$$2k\ \sqrt{-g}\ {}_F U_{\nu}^{[\mu\rho]} = g_{\nu\sigma}\ \{g(g^{\mu\sigma}\ g^{\rho\lambda} - g^{\mu\lambda}\ g^{\rho\sigma})\}_{,\lambda}$$

For obtaining this expression, for ${}_F U_{\nu}^{[\mu\rho]}$ from (7-3-10), we used that $T^{\mu}_{\nu} = \frac{1}{k} G^{\mu}_{\nu}$ and for ${}_E t_{\nu}^{\ \mu}$ used the (7-3-8). The Einstein pseudo-tensor suffers from a very serious drawback, being a function only of $g_{\mu\nu}$ and $g_{\mu\nu,\alpha}$ it may be set zero locally by a choice of coordinates. Therefore it cannot mean the stress-energy tensor for the field. For solving this difficulty Møller has introduced his stress-energy pseudo-tensor as [61]

$$_M\mathfrak{J}_{\nu}^{\ \mu} \equiv {}_M U_{\nu}^{[\mu\sigma]}{}_{,\sigma} = \sqrt{-g}\ (T_{\nu}^{\ \mu} + {}_M t_{\nu}^{\ \mu})$$

with

$$_M U_{\mu}^{[\nu\sigma]} = 2\ {}_F U_{\mu}^{[\nu\sigma]} - \delta^{\nu}_{\mu}\ {}_F U_{\rho}^{[\rho\sigma]} + \delta^{\sigma}_{\mu}\ {}_F U_{\rho}^{[\rho\nu]}$$

$$= \frac{\sqrt{-g}}{k}\ g^{\nu\alpha}\ g^{\sigma\tau}\ (g_{\mu\alpha,\tau} - g_{\mu\tau,\alpha}) \qquad (7\text{-}3\text{-}12)$$

the Møller's pseudo-tensor ${}_{M}t_{\nu}{}^{\mu}$ being dependent also on the second derivatives of $g_{\mu\nu}$ is free from the previous difficulty.

Another proposed stress-energy pseudo-tensor is due to Landau and Lifschitz [62]

$$
{}_{L}\mathcal{T}^{\mu\nu} \equiv {}_{L}U^{\mu[\nu\sigma]}{}_{,\sigma} = (-g)(T^{\mu\nu} + {}_{L}t^{\mu\nu})
$$

$$
{}_{L}U^{\mu[\nu\sigma]}{}_{,\sigma} = \sqrt{-g}\, g^{\mu\rho}\, {}_{F}U_{\rho}{}^{[\nu\sigma]}{}_{,\sigma}
$$

Besides those pseudo-tensors there exist various other candidates, a general reference to this may be found on papers by Goldberg and by Bergmann [63].

Let us now prove how to obtain the Møller's superpotential from the general superpotential given by the equation (7-3-7). For this is sufficient to put $\varepsilon^{\nu}(x)$ equal to constants, $\varepsilon^{\nu} = c^{\nu}$. Then, we have in (7-3-7),

$$
U^{[\mu\sigma]} = \frac{\sqrt{-g}}{k}\left(g^{\sigma\nu} \{{}^{\mu}_{\nu\alpha}\} - g^{\mu\nu} \{{}^{\sigma}_{\nu\alpha}\} \right) c^{\alpha}
$$

a direct and simple calculation shows that this equation is just

$$
U^{[\mu\sigma]} = \frac{\sqrt{-g}}{k}\; g^{\sigma\nu}\, g^{\mu\lambda}\; (g_{\lambda\alpha,\nu} - g_{\nu\alpha,\lambda})\, c^{\alpha}
$$

Using (7-3-12) we write this as

$$
U^{[\mu\sigma]}(\varepsilon^{\nu} = c^{\nu}) = {}_{M}U_{\alpha}{}^{[\mu\sigma]}\; c^{\alpha}
$$

which shows how to recover the Møller's superpotential by choosing the $\varepsilon^{\nu}(x)$ as four constants. Similarly all other superpotentials may be reconstructed out of the general superpotential of (7-3-7) by other choices for $\varepsilon^{\nu}(x)$. This also serves to show that really we have an infinity of possible superpotentials, and associate pseudo-tensors $t_{\nu}{}^{\mu}$ depending on the value taken for the arbitrary functions $\varepsilon^{\nu}(x)$ standing in (7-3-7).

Since the interpretation of this seems at the moment rather questionable, and it is an open subject of research we finish here this section.

8. THEORY OF GRAVITATIONAL RADIATION

8.1) Globally Conserved Quantities for Manifolds Possessing Killing Fields

In the case where the manifold possess certain number of Killing fields it is possible to set up integral conservation laws and interpret them as physical quantities associated to the gravitational field. These integrals are important in the discussion of gravitational radiation. We start from the general super-potential $U^{[\mu\sigma]}$ of Eq. (7-3-7), as it is seen from this equation $U^{[\mu\sigma]}$ is a skew-symmetric second rank tensor for the arbitrary vector $\varepsilon^{\nu}(x)$. Take ε^{ν} to be a Killing vector of the geometry, $\varepsilon^{\nu} = \tau^{\nu}$ (we take τ^{ν} as a time-like Killing vector defining a stationary gravitational field). Let σ be a space-like hypersurface and S its boundary. Form the integral

$$P = \oint_S {}_{\tau}U^{[\mu\sigma]} \, d\Sigma_{\mu\sigma} \tag{8-1-1}$$

we indicate ${}_{\tau}U^{[\mu\sigma]}$ as the superpotential taken for $\varepsilon^{\nu} = \tau^{\nu}$. This integral can be taken as the total energy contained within the volume bounded by S. Indeed, mapping so as $\tau^{\mu} = (1,\vec{0})$ and taking σ as the hypersurface x^{o} = const., we obtain

$$P = \oint_S {}_{\tau}U^{[or]} \, d\Sigma_r \, ; \qquad d\Sigma_r = \varepsilon_{rim} \, dx^i \, dx^m$$

$$= \frac{1}{k} \oint_S \sqrt{-g} \left(g^{r\nu} \{^{o}_{\nu o}\} - g^{o\nu} \{^{r}_{\nu o}\} \right) d\Sigma_r$$

since for a constant τ^{ν} we get

$$_{\tau}U^{[\mu\sigma]} = \frac{\sqrt{-g}}{k}\left(g^{\sigma\nu}\{^{\mu}_{\nu\alpha}\} - g^{\mu\nu}\{^{\sigma}_{\nu\alpha}\}\right)\tau^{\alpha} = \frac{\sqrt{-g}}{k}\left(g^{\sigma\nu}\{^{\mu}_{\nu o}\} - g^{\mu\nu}\{^{\sigma}_{\nu o}\}\right)$$

For giving a definite example of this method we further simplify by considering a static gravitational field, $g_{or} = 0$. Then

$$P = \frac{1}{k}\oint_S \frac{\sqrt{-g}}{g_{oo}}\, g^{rs}\, g_{oo,s}\, d\Sigma_r$$

Taking into account that in this case

$$\sqrt{-g} = \sqrt{g_{oo}}\,\sqrt{-\,^3g}$$

we have

$$P = \frac{2}{k}\oint_S \sqrt{-\,^3g}\; g^{rs}\,\frac{\partial}{\partial x^s}\sqrt{g_{oo}}\, d\Sigma_r$$

Applying this for a Schwarzschild field generated by a bounded source distribution, and taking S as a sphere with radius "a" greater than the radius of the distribution. Assuming that the total mass of this later is m, we have.

$$g_{oo} = 1/g^{oo} = 1 - 2c/r$$

$$g_{11} = 1/g^{11} = -\left(1 - \frac{2c}{r}\right)^{-1}$$

$$g_{22} = 1/g^{22} = -\,r^2$$

$$g_{33} = 1/g^{33} = -\,r^2\,\mathrm{sen}^2\Theta$$

in spherical coordinates. Substituting these values into the above integral, we find P = m. This shows that the appelation of energy for the integral (8-1-1) is in certain sense correct. This can also be proven directly from (8-1-1).

Indeed, writing

$$ {}_{\tau}U^{[\mu\sigma]} = \mathcal{U}^{[\mu\sigma]}_{\alpha}\, \tau^{\alpha} $$

where τ^{α} is the time-like Killing field, and applying Gauss' theorem we get

$$ P = \oint_S {}_{\tau}U^{[\mu\sigma]}\, d\Sigma_{\mu\sigma} = \int_{\sigma} {}_{\tau}U^{[\mu\sigma]}_{\ ,\sigma}\, d\sigma_{\mu} $$

thus, for the reference system where the τ^{α} assume constant values, or even if one retains the possibility of τ^{α} varying with the coordinates,

$$ P = \int_{\sigma} \mathcal{U}^{[\mu\sigma]}_{\alpha,\sigma}\, \tau^{\alpha}\, d\sigma_{\mu} + \int_{\sigma} \tau^{\alpha}_{,\sigma}\, \mathcal{U}^{[\mu\sigma]}_{\alpha}\, d\sigma_{\mu} $$

introducing the pseudo tensor t^{μ}_{α} of energy-momentum

$$ t^{\mu}_{\alpha} = \mathcal{U}^{[\mu\sigma]}_{\alpha,\sigma} $$

we get

$$ P = \int_{\sigma} \tau^{\alpha}\, t^{\mu}_{\alpha}\, d\sigma_{\mu} + \int_{\sigma} \tau^{\alpha}_{,\sigma}\, \mathcal{U}^{[\mu\sigma]}_{\alpha}\, d\sigma_{\mu} $$

mapping to the frame where τ^{α} assume the canonical values written before,

$$ P = \int_{\sigma} t^{\mu}_{o}\, d\sigma_{\mu} = \int_{V} t_{o}^{\ o}\, d_3 x $$

where we took σ as the hypersurface x^{o} = const. This shows in general that P behaves as the energy inside the volume V. From the calculations done before we have for the superpotential associated to the Killing field,

$$ \mathcal{U}^{[\mu\sigma]}_{\alpha} = \frac{\sqrt{-g}}{k} \left(g^{\sigma\nu} \left\{ {\mu \atop \nu\alpha} \right\} - g^{\mu\nu} \left\{ {\sigma \atop \nu\alpha} \right\} \right) $$

a short calculation will give

$$\mathcal{U}^{[\mu\sigma]}_{\alpha} = \frac{\sqrt{-g}}{k}\; g^{\sigma\nu}\; g^{\mu\lambda}\; (g_{\lambda\alpha,\nu} - g_{\nu\alpha,\lambda}) \tag{8-1-2}$$

and this, as was to be expected, is the Møller's superpotential. Therefore, this type of superpotential is associated to the energy inside V. The case which we just finished to discuss presents an exact Killing symmetry field. Sometimes it is meaningful to construct an integral like (8-1-1) for situations where we can at most introduce asymptotical Killing fields. Such fields will exist if one can find a mapping such that asymptotically .

$$g_{\mu\nu} = \overset{o}{g}_{\mu\nu} + 0\left(\frac{1}{r}\right)$$

where r is the distance from some point on a space-like surface that is asymptotically parallel to an x^o = const. surface. In this case the Killing's equation is,

$$\overset{o}{g}_{\mu\rho}\, \xi^{\rho}_{\ ,\nu} + \overset{o}{g}_{\rho\nu}\, \xi^{\rho}_{\ ,\mu} + 0\left(\frac{1}{r}\right) = 0 \ .$$

Consequently the Killing field is given by the ten parameters of the Poincaré group, $\xi^{\rho} = \varepsilon^{\rho}_{\ \lambda}\, x^{\lambda} + \varepsilon^{\rho}$, corresponding to this asymptotically flat manifold.

In this case we get for (7-3-7), by putting $\xi^{\rho} = \varepsilon^{\rho}_{\ \lambda}\, x^{\lambda} + \varepsilon^{\rho}$,

$$_{\xi}U^{[\mu\sigma]} = \varepsilon^{\alpha}_{\ \lambda}\, \mathcal{U}^{[\mu\sigma]\lambda}_{\alpha} + \varepsilon^{\alpha}\, \mathcal{U}^{[\mu\sigma]}_{\alpha} \tag{8-1-3}$$

where

$$\mathcal{U}^{[\mu\sigma]\lambda}_{\alpha} = \frac{\sqrt{-g}}{k}\left(g^{\sigma\nu}\, A^{\mu\lambda}_{\alpha\nu} - g^{\mu\nu}\, A^{\sigma\lambda}_{\alpha\nu}\right) \tag{8-1-4}$$

$$\mathcal{U}^{[\mu\sigma]}_{\alpha} = \frac{\sqrt{-g}}{k}\left(g^{\sigma\nu}\, B^{\mu}_{\nu\alpha} - g^{\mu\nu}\, B^{\sigma}_{\nu\alpha}\right) \tag{8-1-5}$$

and $A^{\mu\lambda}_{\alpha\nu}$ and $B^{\mu}_{\nu\alpha}$ are a short for

$$A^{\mu\lambda}_{\alpha\nu} = \delta^{\mu}_{\alpha}\, \delta^{\lambda}_{\nu} + \left\{{\mu \atop \nu\alpha}\right\} x^{\lambda}$$

$$B^{\mu}_{\nu\alpha} = \{^{\mu}_{\nu\alpha}\} \quad .$$

We see that in this case we have ten objects $\mathcal{U}^{[\mu\sigma]}$. The integral (8-1-1) will take the form, where S is the surface on the spatial infinity.

$$P = \int_{\sigma} \xi \, U^{[\mu\sigma]}_{\ ,\sigma} \ d\sigma_{\mu} = \varepsilon^{\alpha}_{\ \lambda} \, I^{\lambda}_{\ \alpha} + \varepsilon^{\alpha} \, J_{\alpha} \tag{8-1-6}$$

with

$$I^{\lambda}_{\alpha} = \int_{\sigma} \mathcal{U}^{[\mu\sigma]\lambda}_{\alpha,\sigma} \ d\sigma_{\mu} \, , \qquad J_{\alpha} = \int_{\sigma} \mathcal{U}^{[\mu\sigma]}_{\alpha,\sigma} \ d\sigma_{\mu}$$

These integrals are independent of the parameters $\varepsilon^{\alpha}_{\ \lambda}$, ε^{α} inside the hypersurface σ, and are independent of the hypersurface of integration since both integrands have null divergence. Taking σ as the surface x^{o} = const. we get the result that the ten integrals

$$I^{\lambda}_{\ \alpha} = \int_{V} \mathcal{U}^{[o\sigma]\lambda}_{\alpha,\sigma} \ d_3 x \tag{8-1-7}$$

$$J_{\alpha} = \int_{V} \mathcal{U}^{[o\sigma]}_{\alpha,\sigma} \ d_3 x = \int_{V} t^{o}_{\alpha} \ d_3 x \tag{8-1-8}$$

are independent of x^{o}. The second integral is just the total momentum for the gravitational field inside the volume V. The first integral represents the conservation law of the total angular momentum pseudo-tensor of the gravitational field inside V. The explicit expressions for the correspondent densities being

$$\mathcal{U}^{[\mu\sigma]}_{\alpha,\sigma} = \left\{ \frac{\sqrt{-g}}{k} \left(g^{\sigma\nu} \{^{\mu}_{\nu\alpha}\} - g^{\mu\nu} \{^{\sigma}_{\nu\alpha}\} \right) \right\}_{,\sigma} = {}_{M}t^{\mu}_{\alpha} \tag{8-1-9}$$

$$\mathcal{U}^{[\mu\sigma]\lambda}_{\alpha,\sigma} = \left\{ \frac{\sqrt{-g}}{k} \left(g^{\sigma\lambda} \, \delta^{\mu}_{\alpha} - g^{\mu\lambda} \, \delta^{\sigma}_{\alpha} + g^{\sigma\nu} \{^{\mu}_{\nu\alpha}\} \, x^{\lambda} - g^{\mu\nu} \{^{\sigma}_{\nu\alpha}\} \, x^{\lambda} \right) \right\}_{,\sigma} \tag{8-1-10}$$

That is, $\mathcal{U}^{[\mu\sigma]}_{\alpha}$ is the Moller's superpotential and thus $t_{\alpha}^{\ \mu}$ is the Moller's pseudo-tensor of momentum-energy. $\mathcal{U}^{[\mu\sigma]\lambda}_{\alpha}$ will be the superpotential for the density of the angular momentum pseudo-tensor, note that we have two terms in this tensor, an intrinsic angular momentum density (the spin density for the field) and the orbital part. These results which hold in the asymptotic region are entirely similar to what happens for any field in special relativity, that means in flat space-times

8.2) Gravitational Radiation

In a previous section we have already discussed some topics related to this problem. Presently we give a somewhat more detailed exposition of the general situation for this subject. In discussing the problem of gravitational radiation we are guided essentially by the knowledge we got in studying the electromagnetic radiation. Unfortunately, the gravitational theory differs in several respects from electrodynamics and such differences reduce significantly the value of this analogy. By the other hand there is no experimental data to guide one in defining the concept of gravitational radiation, as consequence we have to use this type of analogy.

We review briefly the concepts which lead to the existence of electromagnetic radiation. First of all there exists source-free plane-wave solutions characterized by the conditions that

$$|\vec{E}| = |\vec{H}|, \ \vec{E}.\vec{H} = 0$$

or equivalently, by the condition that there exists a null vector k^μ such that

$$F_{\mu\nu} k^\nu = 0, \ \varepsilon^{\alpha\mu\nu\rho} F_{\mu\nu} k_\rho = 0$$

We have seen that similar plane-wave solutions exist for the Einstein equations. However, the most general solution of Maxwell's equations in the presence of sources may be put as the sum of the general solution of the homogeneous equation, the plane waves or superposition of them, and a particular solution associated to the sources. Thus we may write,

$$A_\mu(x) = \overset{o}{A}_\mu \, e^{ik.x} + \overset{+}{\underset{-}{A}}_\mu(x)$$

$$\overset{o}{A}_\mu k^\mu = 0 \;, \quad \overset{+}{\underset{-}{A}}_\mu(x) = \int \delta^{\pm}((x-x')^2) j_\mu(x') \, d_4x'$$

$$\delta^{\pm}(x^2) = \frac{1}{|\vec{x}|}\left\{\delta(|\vec{x}| \mp x^o)\right\} \;, \Box\delta^{\pm}(x^2) = 4\pi \, \delta_4(x)$$

and for the fields,

$$F_{\mu\nu}(x) = \overset{o}{F}_{\mu\nu} \, e^{ik.x} + F^{\pm}_{\mu\nu}(x)$$

$$\overset{o}{F}_{\mu\nu} = k_{[\mu} \overset{o}{A}_{\nu]} ; \; F^{\pm}_{\mu\nu}(x) = A^{\pm}_{[\mu,\nu]}(x)$$

This complete solution clearly possess more physical interest than the pure homo geneous solution. Indeed, these solutions lead to Poynting vectors whose integral over a closed surface surrounding the sources gives the energy flux out of the sources, and this may be directly confronted with the experimental measurements.

We can put the expression for $A^{\pm}_\mu(x)$ in the familiar form by integration over x'^o,

$$A^{\pm}_\mu(x) = \int \frac{j_\mu(x^o \pm |\vec{x}-\vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|} \, d_3x'$$

The structure of this potential for large values of $r = |\vec{x}|$ will be

$$A^{\pm}_\mu(x) \cong \frac{1}{r}\int j_\mu\left(x^o \pm r \mp \frac{\vec{x}.\vec{x}'}{r}, \vec{x}'\right) d_3x'$$

for the retarded solution only the signs written on the bottom should be consider ed *. At large distances from the source, the field over a not too large region of space may be considered as a plane wave. In this case the fields $\vec{E}$ and $\vec{H}$ are related to each other by $\vec{E} = \vec{H} \times \vec{n}$, where $\vec{n} = \vec{k}/k$. From the integral written above $\vec{A}_{\pm} = \frac{1}{r}\vec{f}(x^o_{\pm})$, where $\vec{f}(x^o_{\pm})$ is the function

$$\vec{f}(x^o_{\pm}) = \int \vec{j}\left(x^o \pm r \mp \frac{\vec{x}.\vec{x}'}{r}, \vec{x}'\right) d_3x' = \vec{f}(x^o \pm r)$$

Thus, $\vec{H} = \text{curl}\,\frac{1}{r}\vec{f}(x^o_{\pm})$. In differentiating this expression we may take r as constant. Indeed, differentiation on r would generate a term 1/r which is small as compared with the original term. However r will contribute to the derivatives as factor inside $f(x^o_{\pm})$. So

$$\text{curl}\,\frac{1}{r}\vec{f}(x^o_{\pm}) = \frac{1}{r}\,\text{curl}\,\vec{f}(x^o_{\pm}) .$$

which gives, on account of, $\text{curl}\,\vec{f}(x^o_{\pm}) = \nabla x^o_{\pm} \times \frac{d\vec{f}}{dx^o_{\pm}}$, and by taking the retarded solution,

$$\nabla x^o_- = - \nabla r = - \vec{n}$$

$$\vec{H} = \frac{-1}{r}\vec{n} \times \frac{d\vec{f}}{dx^o_-} = \frac{1}{r}\frac{d\vec{f}}{dx^o_-} \times \vec{n}$$

which may be put as

$$\vec{H} = \dot{\vec{A}} \times \vec{n} \tag{8-2-1}$$

therefore,

$$\vec{E} = (\dot{\vec{A}} \times \vec{n}) \times \vec{n} \tag{8-2-2}$$

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* At large distances from the source $|\vec{x}| >> |\vec{x}'|$. Therefore we may expand $f(\vec{x}') = |\vec{x}-\vec{x}'|$ for fixed $\vec{x}$ in power series of $\vec{x}'$ and stop with the first term. Thus $f(\vec{x}') = f(0) + \vec{x}'.\nabla f(0)$, and an easy calculation gives

$$f(\vec{x}') = r - \frac{\vec{x}.\vec{x}'}{r}, \quad r = |\vec{x}| .$$

We note that the fields at large distances are proportional to the first power of the inverse of the distance to the radiating source. Therefore the energy flux as given by the Poynting vector will be proportional to the square of 1/r.

We have said before that this class of solutions are approximately plane waves over small regions of the space. The conditions fixing this behaviour are those written at the beginning of this section. Presently they will hold up to terms of the order of 1/r ,

$$F_{\mu\nu}k^{\nu} + 0\left(\frac{1}{r^2}\right) = 0$$

$$\varepsilon^{\alpha\mu\nu\rho} F_{\mu\nu}k_{\rho} + 0\left(\frac{1}{r^2}\right) = 0$$

Indeed, applying Maxwell equations in the asymptotic region where there is no sources, for the complete solution presently considered, we obtain

$$F_{\mu\nu}{}^{,\nu} = \overset{o}{F}_{\mu\nu}\, k^{\nu} + F^{-}_{\mu\nu}{}^{,\nu} = 0$$

$$F^{-}_{\mu\nu} = A^{-}_{\mu,\nu} - A^{-}_{\nu,\mu} = (\vec{E}_{-}, \vec{H}_{-})$$

Since the fields go with 1/r their derivatives will go with $1/r^2$, thus

$$\overset{o}{F}_{\mu\nu}k^{\nu} + 0_{\mu}\left(\frac{1}{r^2}\right) = 0$$

$$\varepsilon^{\alpha\mu\nu\rho}\, \overset{o}{F}_{\mu\nu}k_{\rho} + 0^{\alpha}\left(\frac{1}{r^2}\right) = 0$$

These are the concepts which one tries to apply directly to the gravitational radiation field. In doing this one faces immediately with the crucial problem that no exact solution for time-varying sources is known. So that one is immediately forced to use approximate procedures, such as weak field solutions or multipole expansion methods. In doing that, one is usually forced to make use of coordinate conditions (which in the weak field approximation are just gauge conditions for the radiation field). Up to what extension the use of such coordinate conditions will influence all relevant physical conclusions is not known, therefore we do not know how to separate in a clear fashion the results which are independent of these coordinate conditions from those which depend on them.

In addition to this difficulty, one is also faced with the problem that a definite, geometrical formulation of conservation laws in general relativity is possible only when there exists one or more Killing vectors, at least in the asymptotic region. However, conservation laws are necessary if we are to be able to calculate the amount of energy radiated by a source. As we said before, the absence of exact time-vaying source solutions forces us to use approximation methods. In what follows we give a short summary of such methods.

8.2.1) The Approximation of Weak Gravitational Radiating Field

Einstein was the first person which discussed the possibility of gravitational radiation. He used the weak field approximation for doing this [64]. Suppose we have a continuity equation of the type $t^{\mu}{}_{,\mu} = 0$, by integration over a volume V contained inside a surface S which independs on x^o, we get

$$\frac{d}{dx^o} \int_V t^o \, dV + \int_V t^r{}_{,r} \, dV = 0$$

then,

$$\frac{d}{dx^o} \int_V t^o \, dV + \oint_S t^r \, n_r \, dS = 0 \tag{8-2-1.1}$$

Then, the rate of change of $\int_V t^o \, dV$ in V is equal to the flux of this quantity across S. For ascertainting that this is correct, we have to make sure that S is really not varying with time. In general, for a Riemannian manifold this is only possible when there exists time-like Killing fields for the geometry. For arbitrary gravitational fields in V it turns out questionable the existence of such vectors. Therefore, for applying this simple procedure we have to consider simple models for the gravitational field. One of the simplest models is the weak (or quasi-weak) gravitational field. In this case there exists the Killing vectors of the flat-space metric (or for quasi-weak fields will exist the asymptotical Killing vectors of the flat-space metric) and a x^o = const. surface can be constructed. The second step is the fixation of the value for t^μ.

Einstein used his expression ${}_E t_\nu{}^\mu$ for this. This is reasonable to be done for the linearized approximation since then ${}_E t_\nu{}^\mu$ is a mixed Poincaré second rank tensor. Using in the expression for ${}_E t_\nu{}^\mu$ the fields as given by time-varying source solution of the linearized field equations (the Harmonic gauge condition is used) seen before,

$$g_{\mu\nu} = \overset{o}{g}_{\mu\nu} + + \varepsilon h_{\mu\nu}$$

$$h_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\overset{o}{g}_{\mu\nu}\,\gamma,\ \overset{o\lambda\nu}{g}\,\gamma_{\mu\lambda,\nu} = 0$$

$$\varepsilon\gamma_{rs} = \frac{k}{4\pi}\,\frac{I''_{rs}}{r} + \dots$$

$$\varepsilon\gamma_{ro} = \frac{k}{4\pi}\left\{-\frac{I'_r}{r} - \frac{1}{2}n^s\,\frac{I''_{rs}}{r^2} + \dots\right\},\ n^r = \frac{x^r}{r}$$

$$\varepsilon\gamma_{oo} = \frac{k}{4\pi}\left\{\frac{I}{r} + \frac{n^r\,I'_r}{r^2} + \frac{1}{2}\,\frac{n^r\,n^s\,I''_{rs}}{r^3} + \dots\right\}$$

where the primes denote differentiation with respect to time, and the several I's represent the moments of the matter density calculated at their retarded value.

$$I = \int \rho(\vec{x},\ x^o - \frac{r}{c})\ dV$$

$$I_r = \int \rho(\vec{x},\ x^o - \frac{r}{c})\ x^r \quad dV$$

$$I_{rs} = \int \rho(\vec{x},\ x^o - \frac{r}{c})\ x^r\ x^s\ dV$$

substituting these values into ${}_Et_\nu{}^\mu$ and computing $E = \int_V {}_Et_o{}^o\ dV$, we will get for (8-2-1.1)

$$\frac{dE}{dt} = - \frac{G}{45c^5} \left(\frac{d^3 Q_{rs}}{dt^3}\right)\left(\frac{d^3\ Q_{rs}}{dt^3}\right) \qquad (8\text{-}2\text{-}1.2)$$

where,

$$Q_{rs} = \int_V \rho\left(\vec{x},\ t - \frac{|\vec{x}|}{c}\right) \left(3x^r\ x^s - r^2\ \delta^{rs}\right) \quad dV$$

is the quadrupole moment of the source. Taking the system earth-sun as generating this quadrupole moment, it may be shown that it yields a rate of energy loss of about 200 watts. Einstein, and later Eddington, calculated the energy radiated by a spinning rod, and found it to be

$$P = \frac{32}{5}\ \frac{G\ I^2\ \omega^6}{c^5}$$

where I is the moment of inertia of the rod. A rod 1 meter long, spinning as fast as it can without beaking apart due to internal stresses, will radiate about 10^{-30} ergs per second. These considerations make it appear unlikely that gravitational radiation will play any role in energy transfer in physical processes except under the most sensitive extreme conditions.

8.2.2) Asymptotical Conditions for the Radiation Field

Even if the field is not treated as weak, one might hope to use asymptotical methods for discussing the gravitational radiation. Here the problem is the formulation of boundary conditions on the field, so as the integral of (8-2-1.1) be defined, and simultaneously not excluding the possibility of energy radiation. The first important contribution in this direction was due to Trautman [65]. He supposes that the gravitational field in question defines a scalar field u(x) on the manifold whose gradient, $k_\mu = u_{,\mu}$, is a null vector, $k^2 = 0$. This vector field is used for constructing a congruence of rays on the manifold by requiring that the tangent to the rays passing through a given point is equal to k^μ at that point. To give an idea of how this may be imagined, let us take a flat space-time and form the functions

$$u(x) = f(x^0 \pm |\vec{x}|) = f(\eta_\pm)$$

then, the contravariant vector k^μ will be, for signature -2,

$$k^\mu = \frac{df}{d\eta_\pm}\left(1, \mp \frac{\vec{x}}{|\vec{x}|}\right) = \overset{o}{g}{}^{\mu\nu} u_{,\nu}$$

and the covariant k_μ will be

$$k_\mu = \frac{df}{d\eta_\pm}\left(1, \pm \frac{\vec{x}}{|\vec{x}|}\right)$$

therefore

$$k^2 = 0$$

the radial distance from the origin, $r = |\vec{x}|$, satisfies, by taking $f(\eta_\pm) = x^0 - |\vec{x}|$

$$\left(\frac{k^\mu}{r^2}\right)_{,\mu} = 0 \, , \quad k^\mu = \left(1, \frac{\vec{x}}{|\vec{x}|}\right)$$

for curved manifolds this later condition gets generalized to

$$\left(\frac{k^\mu}{r^2}\right)_{;\mu} = 0 \, , \quad k^\mu = g^{\mu\nu} k_\nu = g^{\mu\nu} u_{,\nu}$$

In this case we call r as the "luminosity distance". Let us assume that asymptotically there exists a mapping such that

$$g_{\mu\nu} = \overset{o}{g}_{\mu\nu} + \lambda_{\mu\nu}\left(\frac{1}{r}\right) \qquad (8\text{-}2\text{-}2.1)$$

In this case the asymptotical form for the field equations imply that $\overset{*}{\lambda}_{\mu\nu} = \lambda_{\mu\nu} - \frac{1}{2}\overset{o}{g}_{\mu\nu}\lambda$ satisfies the wave equation plus the de Donder condition.

$$\Box \lambda^*_{\mu\nu} = 0\left(\frac{1}{r^2}\right) \qquad (8\text{-}2\text{-}2.2)$$

$$\lambda^*_{\mu\nu}{}^{,\nu} = 0\left(\frac{1}{r^2}\right) \qquad (8\text{-}2\text{-}2.3)$$

The (8-2-2.2) implies that a derivative is obtained to the order 1/r by multiplication with k^μ.

$$g_{\mu\nu,\rho} = \lambda_{\mu\nu,\rho} + 0\left(\frac{1}{r^2}\right) = i_{\mu\nu}\, k_\rho + 0\left(\frac{1}{r^2}\right) \qquad (8\text{-}2\text{-}2.4)$$

similarly *

$$\overset{*}{\lambda}_{\mu\nu,\rho} = i^*_{\mu\nu}\; k_\rho + 0\left(\frac{1}{r^2}\right)$$

Indeed, a further differentiation gives $\overset{*}{\lambda}_{\mu\nu,\rho\sigma} = \overset{*}{i}_{\mu\nu}\, k_\rho\, k_\sigma + 0\left(\frac{1}{r^2}\right)$ and since k is null, this satisfies the wave equation (8-2-2.2). The condition (8-2-2.3) takes the form

$$\overset{*}{i}_{\mu\nu}\, k^\nu \equiv \left(i_{\mu\nu} - \frac{1}{2}\overset{o}{g}_{\mu\nu}\,\overset{o}{g}{}^{\lambda\sigma}\, i_{\lambda\sigma}\right) k^\nu = 0\left(\frac{1}{r^2}\right) . \qquad (8\text{-}2\text{-}2.5)$$

With the values (8-2-2.1) and (8-2-2.4) we compute the Einstein's pseudo tensor in the asymptotic region. The result is

$${}_E t^\mu{}_\nu = \not{t}\, k^\mu\, k_\nu + 0\left(\frac{1}{r^3}\right) \qquad (8\text{-}2\text{-}2.6)$$

where

$$\not{t} = \frac{1}{32\pi G}\; i^{\mu\nu}\left(i_{\mu\nu} - \frac{1}{2}\overset{o}{g}_{\mu\nu}\,\overset{o}{g}{}^{\rho\sigma}\, i_{\rho\sigma}\right) \qquad (8\text{-}2\text{-}2.7)$$

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* Both $i_{\mu\nu}$ and $i^*_{\mu\nu}$ are of the order 1/r.

the quantity $\not{k}$ is of the order of $1/r^2$. A surface integral of ${}_E t_o^r$ of the type (8-2-1.1) will be definite and could represent the total energy radiated by the sources of the gravitational field. Cornish [66] has shown that the value of this integral is insensitive to which of the various stress-energy pseudo-tensors are used in its calculation, provided that the boundary conditions (8-2-2.1), (8-2-2.4) and (8-2-2.5) are satisfied.

For further justification of these boundary conditions as being appropriate to fields representing gravitational radiation, let us compute the asymptotic form of the curvature tensor of this field, since

$$g_{\mu\nu,\rho\sigma} = j_{\mu\nu}\, k_\rho\, k_\sigma + 0\left(\frac{1}{r^2}\right) \qquad (8\text{-}2\text{-}2.8)$$

with $j_{\mu\nu}$ of the order $1/r$. The asymptotical de Donder condition will be

$$\left(j_{\mu\nu} - \frac{1}{2}\, \overset{o}{g}_{\mu\nu}\, \overset{o}{g}{}^{\rho\sigma}\, j_{\rho\sigma}\right)k^\nu = 0\left(\frac{1}{r^2}\right) \qquad (8\text{-}2\text{-}2.9)$$

a straightforward computation using this condition will give as result

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\, k_{[\mu}\, j_{\nu][\rho}\, k_{\sigma]} + 0\left(\frac{1}{r^2}\right) \qquad (8\text{-}2\text{-}2.10)$$

the explicit form for this is

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(k_\mu\, j_{\nu\rho}\, k_\sigma - k_\mu\, j_{\nu\sigma}\, k_\rho - k_\nu\, j_{\mu\rho}\, k_\sigma + k_\nu\, j_{\mu\sigma}\, k_\rho\right) + 0\left(\frac{1}{r^2}\right)$$

Let us compute from this formula the quantity $R_{\mu\nu\rho\sigma}\, k^\mu$. We get, using that $k^2 = 0$,

$$R_{\mu\nu\rho\sigma}\, k^\mu = \frac{1}{2}\left(k^\mu\, j_{\mu\sigma}\, k_\nu\, k_\rho - k^\mu\, j_{\mu\rho}\, k_\nu\, k_\sigma\right) + 0\left(\frac{1}{r^2}\right)$$

But from the de Donder condition (8-2-2.9), we have

$$k^{\mu}\, \dot{j}_{\mu\rho} = \frac{1}{2}\, \overset{o}{g}_{\rho\mu}\, \overset{o}{g}^{\lambda\tau}\, \dot{j}_{\lambda\tau}\, k^{\mu} + 0\left(\frac{1}{r^2}\right)$$

therefore, up to the order 1/r we have

$$R_{\mu\nu\rho\sigma}\, k^{\mu} = \frac{1}{4}\, \overset{o}{g}^{\lambda\tau}\, \dot{j}_{\lambda\tau}\, (k_{\nu}k_{\rho}k_{\sigma} - k_{\nu}k_{\sigma}k_{\rho}) + 0\left(\frac{1}{r^2}\right) = 0\left(\frac{1}{r^2}\right) \quad . \qquad (8\text{-}2\text{-}2.11)$$

Similarly we can show that

$$R_{\mu\nu[\rho\sigma}\, k_{\lambda]} = 0\left(\frac{1}{r^2}\right) \quad . \qquad (8\text{-}2\text{-}2.12)$$

According to (8-2-2.11) and (8-2-2.12), in the asymptotic region $R_{\mu\nu\rho\sigma}$ is type N in the Petrov classification. We have seen before that plane-fronted waves were all of type N, so that asymptotical gravitational fields satisfying the Trautman's boundary conditions have properties characteristic of plane fronted gravitational waves.

8.3) Characteristic Initial Value Problem and the Radiation Field

Bondi and his group [67] were the first to look for solutions of the Einstein's field equations that satisfy the Trautman boundary conditions and which describe radiation from a bounded source. To solve these equations they give the initial data on a null hypersurface instead of giving this data, as is usually done, on a space-like hypersurface. These null datum is then expected to characterize the radiative field. The possibility of giving data on characteristic hypersurfaces arises in connection with the simplest (one-dimensional) hyperbolic differential equation

$$\frac{\partial^2 \phi}{c^2 \partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0$$

Usually, for solving this equation one gives ϕ and $\frac{\partial \phi}{\partial t}$, at $t = t_o$. The general solution,

$$\phi(x,t) = f(x + ct) + g(x-ct)$$

is then characterized by the Cauchy data, taking $t_o = 0$,

$$A(x) = f(x) + g(x)$$

$$B(x) = c\left(\frac{df}{d(x+ct)}\right)_{t=0} - c\left(\frac{dg}{d(x-ct)}\right)_{t=0}$$

And this solution may be represented by the Cauchy series in powers of ct,

$$(x,t) = A(x) + t\, B(x) + \frac{c^2t^2}{2}\frac{d^2A}{dx^2} + \frac{c^3t^3}{3!}\frac{d^2B}{dx^2} + \ldots.$$

However, this general solution may also be characterized by giving the values of f(x+ct) and g(x-ct) on the two characteristic lines $x + ct = 0$ and $x-ct = 0$. Indeed, the differential equation may be put in the form

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) = 0$$

calling $x + ct$ by ξ and $x - ct$ by η, we have

$$\frac{\partial^2 \phi}{\partial\xi\partial\eta} = 0$$

But from this equation one gets immediately that

$$\frac{\partial\phi}{\partial\xi} = F(\xi)$$

$$\frac{\partial\phi}{\partial\eta} = G(\eta)$$

which are first order differential equations with solution

$$\phi = f(\xi) + g(\eta)$$

The complete system can be characterized by the two datum f(0) and g(0), since we have just two first order differential equations in the variables ξ and η.

These two datum are just another form of giving the two necessary initial conditions for solving the hyperbolic second order wave equation.

For the case of general relativity we begin by constructing a family of characteristic hypersurfaces u = const. and an associated ray congruence with tangent vectors $k_\mu = u_{,\mu}$ such that $k^\mu_{;\mu} \neq 0$. In addition to the null vector k^μ one constructs another null vector n^μ normalized by $n.\ k=1$. Both are real null vectors. To complete the null tetrad one forms the combinations

$$m_\mu = \frac{1}{\sqrt{2}} (q_\mu - ir_\mu)$$

$$\overline{m}_\mu = \frac{1}{\sqrt{2}} (q_\mu + ir_\mu)$$

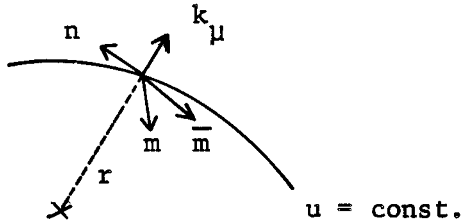
with two real unit space-like vectors q_μ and r_μ which are orthogonal. The m_μ and $\overline{m}_\mu$ are two complex null vectors. The four null vectors k_μ, n_μ, m_μ, $\overline{m}_\mu$ form a tetrad of null vectors. We have

$$n.k = 1, \quad m.\overline{m} = 1 , \qquad k.k = n.n = m.m = \overline{m}.\overline{m} = 0 .$$

We also require that q_μ and r_μ be orthogonal to k^μ and n^μ, which gives

$$k.m = k.\overline{m} = n.m = n.\overline{m} = 0$$

One then defines a luminosity distance r along each ray, and mapps to coordinates $x^\mu = (u,r,\Theta,\phi)$. The tetrad corresponding to these assignements of coordinates is just the set of four null vectors of the type considered.



They can be written in tetrad rotation as $z^{\mu}_{(\alpha)} = (k^{\mu}, n^{\mu}, m^{\mu}, \bar{m}^{\mu})$ with reciprocal $z^{(\alpha)}_{\mu} = (n_{\mu}, k_{\mu}, \bar{m}_{\mu}, m_{\mu})$, satisfying the orthogonality condition

$$z^{(\alpha)}_{\mu} \; z^{\mu}_{(\lambda)} = \delta^{(\alpha)}_{(\lambda)}$$

plus the completicity condition

$$z^{(\alpha)}_{\mu} \; z^{\lambda}_{(\alpha)} = \delta^{\lambda}_{\mu} \; .$$

Bondi then shows that the field equations may be divided into three sets, similarly to the decomposition in $G^{o}_{\mu} = 0$ and $G_{r}^{s} = 0$ correspondent to the initial value problem on space-like hypersurfaces.

(1) $k^{\mu} \; G_{\mu\nu} = 0, \quad m^{\mu} \; m^{\nu} \; G_{\mu\nu} = 0$ (main equations)

(2) $m^{\mu} \; \bar{m}^{\nu} \; G_{\mu\nu} = 0$ (trivial equation)

(3) $m^{\mu} \; n^{\nu} \; G_{\mu\nu} = 0 \; , \quad n^{\mu} \; n^{\nu} \; G_{\mu\nu} = 0$ (supplementary conditions)

In all we have six main equations (similarly to the six $G_{rs} = 0$), one trivial equation and three supplementary conditions (all these later four correspond to the four $G_{\mu}^{o} = 0$). One can then show if the main equations are satisfied everywhere in a space-time region, the trivial equation is also satisfied everywhere and the supplementary conditions are satisfied everywhere if they are satisfied at one point on each ray.

In his search for a solution of the field equations that permitted the construction of this structure, Bondi restricted himself to axially symmetric fields,

$$ds^2 = C \; du^2 + 2D \; du \; dr - r^2 \left[e^{\alpha}(d\Theta - A \; du)^2 + e^{-\alpha}(\mathrm{sen}^2\Theta \; d\phi^2) \right] \qquad (8\text{-}3.1)$$

with A, C, D and α independent of ϕ. The restriction to an axially symmetric field is not essential to the method but it simplifies the calculations. If we treat such a field as the metric, the area of a wave front $u = \text{const}$, $r = \text{const}$ is equal to $4\pi \; r^2$. Furthermore, such a field is the natural generalization of the form of a flat geometry obtained by mapping to coordinates ($u = x^o - r, r, \Theta, \phi$). Note that here one takes $c = 1$, and the retarded time is put as $x^o - r$, so as to conform to the signature -2. Indeed, a flat geometry written in spherical coordinates plus a time $x^o = t$, is given by the line element.

$$ds^2 = dt^2 - (dr^2 + r^2\, d\Theta^2 + r^2\, \mathrm{sen}^2\Theta\, d\phi^2)$$

mapping to the retarded time u,

$$u = x^o - r = t - r$$

one gets for ds ,

$$ds^2 = du^2 + 2\, du\, dr - (r^2\, d\Theta^2 + r^2\, \mathrm{sen}^2\Theta\, d\phi^2) \qquad (8\text{-}3.2)$$

By comparing the line elements given by (8-3.1) and (8-3.2) we easily see that we must set up the following perturbative series expansion in the parameter r.

$$\begin{aligned} \alpha &= \frac{n}{r} + 0\left(\frac{1}{r^2}\right) \\ A &= \frac{a}{r} + 0\left(\frac{1}{r^2}\right) \\ C &= 1 - \frac{2m}{r} + 0\left(\frac{1}{r^2}\right) \\ D &= 1 + \frac{d}{r} + 0\left(\frac{1}{r^2}\right) \end{aligned} \qquad (8\text{-}3.3)$$

(note that the term $du d\Theta$ in (8-3.1) has no counterpart in the flat expression (8-3.2) so that A begins with a term 1/r). The quantities n, a, m and d are functions of u and Θ alone, and describe the presence of the field of gravitation. The metric field of (8-3.1) and (8-3.2) satisfies the asymptotical Trautman conditions.

With these metric components we set the field equations in the form given before. They represent differential equations in A, C, D and α. Bondi then shows that given on a u = const hypersurface, the remaining quantities appearing in (8-3.1) are determined by the main equations up to three functions of integrations, that means three functions of u and Θ. The choice for these functions is then restricted by using the supplementary conditions. As result one finds a = d = 0 and m is determined from the knowledge of n, that is from the knowledge of α. This later quantity is an arbitrary function of u and Θ, and thereby n is an arbitrary function of the same coordinates. Bondi calls $\frac{\partial n}{\partial u}$ as the News Function. To understand the reason of this appelation let us assume that up to some u = const hypersurface a field is independent of u. Then the only way in which this field can vary beyond this hypersurface is if the change in n is taken to be non zero beyond it. The news function can therefore be interpreted as describing the radiation due to an initially static source.* On the other hand the function m is closely related to the total energy of the system. In the static case $\frac{\partial m}{\partial u} = 0$ (since u is a time) and also from the field equations it follows that $\frac{\partial m}{\partial \Theta} = 0$. One can, in fact, show that in static case m is just equal to the total mass of the system as calculated using one of the superpotentials. For the non-static case Bondi defined the mass of the system as the average of m on all angles, $M(u) = \frac{1}{2}\int_0^\pi m(u,\Theta)\ \text{sen}\Theta\ d\Theta$. With the help of the supplementary conditions we get

$$\frac{dM}{du} = -\frac{1}{2}\int^{\pi} \left(\frac{\partial n}{\partial u}\right)^2 \text{sen}\Theta\ d\Theta \ . \qquad (8\text{-}3.4)$$

Hence M decreases when there is news. This decrease in M can be interpreted as a loss in the total energy of the system due to radiation (what this is a

* $\left(\frac{\partial n}{\partial u}\right)_\theta = \left(\frac{\partial n}{\partial u}\right)_{u=u_o,\theta} + (u-u_o)\left(\frac{\partial^2 n}{\partial u^2}\right)_{u=u_o,\theta}$, taking $\left(\frac{\partial n}{\partial u}\right)_{u=u_o,\theta} = 0$, we get $\left(\frac{\partial n}{\partial u}\right)_\theta \neq 0$ for all $u \neq u_o$. Besides this, the source which is static at $u = u_o$, does not generate at this retarded time a flat geometry since $n(u,\theta) = n(u_o,\theta) + (u-u_o)\left(\frac{\partial n}{\partial u}\right)_{u=u_o,\theta} = n(u_o,\theta) \neq 0$.

radiative field is seen from the fact that it satisfies the Trautman radiation conditions). Sachs [68] has shown that the Riemann tensor associated to the Bondi geometry has the general form

$$R = \frac{N}{r} + \frac{III}{r} + \frac{D}{r} + \dots \qquad (8\text{-}3.5)$$

where the indices were supressed for the sake of simplicity. The letters N, III and D denote, respectively, tensors of Petrov type null, III and degenerate. These tensors are covariantly constant along a ray and all have k_μ as their single eigenvector. We see from (8-3.5) that to the order 1/r the Bondi-type field leads to a Riemann tensor of type N, as was the case if it satisfies the asymptotical Trautman conditions. This again indicates the presence of gravitational radiation. This radiation is then directly related to the existence of a news function, simbolically we may write $N = \frac{\partial^2 n}{\partial u^2}$.

The expansion (8-3.5) is a particular example of a general theorem proven by Newman and Penrose [69], and usually referred to in the literature as the peeling theorem. They have shown that given the tetrad of null vectors k, n, m and $\overline{m}$ satisfying the properties seen before, the field $g_{\mu\nu}$ can be put as

$$g_{\mu\nu} = 2\, k_{(\mu}\, n_{\nu)} + 2\, m_{(\mu}\, \overline{m}_{\nu)}$$

Using this relation (which is just the basic equation of the tetrad calculus) they compute, for this general null tetrad, the Riemann tensor and get (again supressing indices)

$$R = N(k) + III(k) + D(k,m) + III(m) + N(m)$$

between parenthesis we have placed the eigenvectors associated to the several types of curvatures. In empty spaces they show that $N(m) = 0(r^{-5})$ implies in

$III(k) = 0(r^{-2})$, $III(m) = 0(r^{-4})$, $D(k,m) = 0(r^{-3})$ and $N(k) = 0(r^{-1})$ in agreement with Sachs' result given by (8-3.5).

To finish this section we discuss the role of existence of gravitational radiation. When discussing this problem from the point of view of general relativity we have just treated the field asymptotically, that is in the wave zone characterized by adequate radiation conditions. In this region, as was seen, mathematical solutions can be constructed with the correct behaviour of fields of radiation. Besides this, we can also construct plane wave solutions of the field equations. However, this is not the entire solution of the problem. To make sure that such type of radiation does exist we have to study its interactions with the sources. Indeed, if we would be able to correlate secular changes in the motion of the sources with the gravitational radiation produced in the wave zone, we could have the best procedure for the experimental detection of this radiation. Unfortunately we do not know an exact solution of the field equations in presence of sources, and as we said before we have to use perturbation expansions. These are of two types, the slow approximation in powers of 1/c or the fast approximation in powers of the gravitational constant G. Both types of solutions, along with the chosen coordinate condition do not always lead to the same physical results (the variation in mass of the radiating sources). On top of this, we might think in using a Minkowskian theory for gravitation, and there do these same perturbations. The final results will in general depend on the theory chosen as well as on the type of perturbative series used. However, if we avoid all this treatment by using the equation (8-3.4) we have a well prescribed formula for calculating the loss in energy of the source. Nevertheless, we do not have any experimental support to enforce this.

APPENDIX

EXPERIMENTAL TESTS OF THE GENERAL RELATIVITY THEORY

A - I) Motion of Test Bodies in a Schwarzschild Field

A test body is by definition a body that, when present in the field of another one, does not disturb the configuration of the original field. The motion of test bodies in a gravitational field according to general relativity is governed by the equations of the metric geodesics determined by the metric $g_{\mu\nu}$, obeying the Einstein field equations.

The motion of the planets around the Sun is well described as geodesics of the Schwarzschild field. We have already seen that the metric can, in this case, be written as

$$g_{\mu\nu} = \begin{pmatrix} 1 - \dfrac{2GM/c^2}{r} & & & 0 \\ & \dfrac{-1}{1 - \dfrac{2GM/c^2}{r}} & & \\ & & -r^2 & \\ 0 & & & -r^2 \sin^2\Theta \end{pmatrix} \quad (1)$$

The equations of the geodesics are obtained by the variational principle $\delta S = 0$, as we know:

$$\delta S_{PP'} = \delta \int_P^{P'} \left[g_{\mu\nu} \dot{\xi}^\mu \dot{\xi}^\nu \right]^{1/2} d\lambda = \delta \int_P^{P'} \frac{ds}{d\lambda} d\lambda = \int_P^{P'} \frac{1}{2} \frac{d\lambda}{ds} \delta (g_{\mu\nu} \dot{\xi}^\mu \dot{\xi}^\nu) \, d\lambda$$

$$= \int_P^{P'} \frac{1}{2} \frac{d\lambda}{ds} \left[g_{\mu\nu,\alpha}\, \delta\xi^\alpha\, \dot{\xi}^\mu \dot{\xi}^\nu + 2 g_{\mu\nu}\, \dot{\xi}^\mu\, \delta\dot{\xi}^\nu \right] d\lambda = \int_P^{P'} \left[\frac{d\lambda}{ds} \cdot \frac{1}{2} g_{\mu\nu,\alpha}\, \dot{\xi}^\mu \dot{\xi}^\nu \delta\xi^\alpha d\lambda + \right.$$

$$\left. + \frac{d\lambda}{ds} g_{\mu\nu}\, \dot{\xi}^\mu\, d(\delta\xi^\nu) \right] = \int_P^{P'} \left[\frac{d\lambda}{ds} g_{\mu\nu,\alpha}\, \dot{\xi}^\mu \dot{\xi}^\nu \delta\xi^\alpha d\lambda - \frac{d}{d\lambda} \left(\frac{1}{s'(\lambda)} g_{\mu\nu}\, \dot{\xi}^\mu \right) \delta\xi^\nu\, d\lambda \right] +$$

$$+ \frac{1}{s'(\lambda)} g_{\mu\nu}\, \dot{\xi}^\mu \delta\xi^\nu \Big|_P^{P'}$$

Then,

$$\int_P^{P'} \left[\frac{1}{2s'(\lambda)} g_{\rho\sigma,\nu}\, \dot{\xi}^\rho \dot{\xi}^\sigma - \frac{d}{d\lambda} \left(\frac{1}{s'(\lambda)} g_{\rho\nu}\, \dot{\xi}^\rho \right) \right] \delta\xi^\nu\, d\lambda = 0$$

$$\frac{1}{2s'(\lambda)} g_{\rho\sigma,\nu}\, \dot{\xi}^\rho \dot{\xi}^\sigma - \frac{d}{d\lambda} \left(\frac{1}{s'(\lambda)} g_{\rho\nu}\, \dot{\xi}^\rho \right) = 0 \tag{2}$$

Now, if we use s as the parameter, we set

$$g_{\mu\nu}\, \dot{\xi}^\mu\, \dot{\xi}^\nu = 1 \tag{3}$$

and

$$\frac{d}{ds} \left[g_{\rho\nu}\, \dot{\xi}^\rho \right] - \frac{1}{2} g_{\rho\sigma,\nu}\, \dot{\xi}^\rho\, \dot{\xi}^\sigma = 0 \tag{4}$$

Substituting (1) into (4), and remembering that $(\xi^\rho) = (\xi^0, r, \Theta, \phi)$,

$$\frac{d}{ds} \left[g_{oo}\, \dot{\xi}^o \right] = 0 \tag{5-a}$$

$$\frac{d}{ds} \left[g_{11}\, \dot{r} \right] - \frac{1}{2} g_{oo,1}\, (\dot{\xi}^o)^2 - \frac{1}{2} g_{11,1}\, \dot{r}^2 + r\dot{\Theta}^2 + r \sin^2\Theta\, \dot{\phi}^2 = 0 \tag{5-b}$$

$$\frac{d}{ds} \left[r^2\, \dot{\Theta} \right] + r^2 \sin\Theta \cos\Theta\, \dot{\phi}^2 = 0 \tag{5-c}$$

(5-d) $$\frac{d}{ds}(r^2 \sin^2\Theta\ \dot{\phi}) = 0$$

If we orient our frame of reference in such way that $\Theta = \frac{\pi}{2}$ and $\dot{\Theta} = 0$ initially, (5-c) gives us $\frac{d}{ds}\left[r^2\ \dot{\Theta}\right] = 0$,

$$r^2\dot{\Theta} = \text{const.} \tag{6}$$

Then, the motion is confined to the plane $\Theta = \frac{\pi}{2}$.

(5-d) $\Longrightarrow$ $$r^2\dot{\phi} = h = \text{const.} \tag{7}$$

(5-a) $\Longrightarrow$ $$g_{oo}\ \dot{\xi}^o = \alpha \tag{8}$$

(5-b) is too complicated. Let us use the first integral (3).

$$g_{oo}\left(\frac{d\dot{\xi}^o}{d\phi}\right)^2 + g_{11}\left(\frac{dr}{d\phi}\right)^2 - r^2 = \frac{1}{\dot{\phi}^2}.$$

Now we can write, using (5-a) and (5-d),

$$\alpha^2\ r^4/h^2\ g_{oo} + g_{11}\left(\frac{dr}{d\phi}\right)^2 - r^2 = r^4/h^2$$

$$\frac{h^2}{r^4}\, g_{oo}\, g_{11}\left(\frac{dr}{d\phi}\right)^2 - \frac{h^2}{r^2}\ g_{oo} + \alpha^2 - g_{oo} = 0$$

$$-\left(\frac{h}{r^2}\ \frac{dr}{d\phi}\right)^2 + \alpha^2 - \left(1 + \frac{h^2}{r^2}\right)\left(1 - \frac{2GM/c^2}{r}\right) = 0$$

Now let's use $\frac{h}{r(\phi)} = u(\phi)$; $-\frac{hr'(\phi)}{r^2} = u'(\phi)$, which gives

$$-\left(\frac{du}{d\phi}\right)^2 + \alpha^2 - (1+u^2)(1-2\lambda u) = 0\,, \qquad \text{where} \quad \lambda = \frac{GM}{h\ c^2}$$

Then, we have the equation

$$(u')^2 = (\alpha^2-1) + 2\,\lambda u - u^2 + 2\lambda u^3$$

Now let us review the same kind of problem in classical mechanics. Using the fact that the motion proceeds in a plane $\left(\Theta = \frac{\pi}{2}\right)$, the kinetic energy will be

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) \tag{12}$$

where the dot means now differention with respect to time. Considering a central potential of the form

$$V = - \frac{A}{r} - \frac{B}{r^3} \tag{13}$$

We get for the Lagrangian,

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{A}{r} + \frac{B}{r^3}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \; ; \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \tag{14}$$

$$r^2 \dot{\phi} = \ell \tag{15}$$

We now use the energy integral:

$$\frac{2E}{m} = \dot{r}^2 + \frac{\ell^2}{r^2} - \frac{2A}{mr} - \frac{2B}{mr^3} \tag{16}$$

But

$$\frac{d}{dt} = \dot{\phi} \frac{d}{d\phi} = \frac{\ell}{r^2} \frac{dr}{d\phi} \quad ;$$

$$\frac{2E}{m} = \left(\frac{\ell}{r^2} \frac{dr}{d\phi}\right)^2 + \frac{\ell^2}{r^2} - \frac{2A}{mr} - \frac{2B}{mr^3}$$

With $\frac{h}{r(\phi)} = u(\phi)$, we get

$$\frac{2E}{m} = \frac{\ell^2}{h^2} (u')^2 + \frac{\ell^2}{h^2} u^2 - \frac{2A}{mh} u - \frac{2B}{mh^3} u^3$$

$$(u')^2 = \frac{2Eh^2}{m \ell^2} + \left(\frac{2 \; Ah}{m\ell^2}\right) u - u^2 + \frac{2B}{m \ell^2 h} u^3 \tag{17}$$

Comparing (11) with (17), we see that we can make the following identifications:

$$\frac{2Eh^2}{m\,\ell^2} = \alpha^2 - 1 \; ; \qquad \frac{Ah}{m\,\ell^2} = \lambda = \frac{B}{m\,\ell^2\,h}$$

with $A = GmM$, and $\lambda = \frac{GM}{h\,c^2}$ we have $h = \frac{\ell}{c}$,

and then

$$\alpha^2 - 1 = \frac{2E}{mc^2}$$

$$B = \frac{GmM\;\ell^2}{c^2}$$

This means that the general relativistic effect may be interpreted as being equivalent to a perturbation potential of the form $-\frac{B}{r^3}$, over the Newtonian potential, with B given above.

To solve the problem, the term u^3 will be treated as a perturbation because of its smallness.

In first order, we have already solved the problem.

The apsidal distances may be obtained from $u'(\phi) = 0$. Then, in first order,

$$(u')^2 = -(1-\alpha^2) + 2\lambda u - u^2 = 0$$

$$u = \lambda \mp \left[\lambda^2 - (1-\alpha^2)\right]^{1/2} \tag{18}$$

or

$$\begin{cases} u_P = \lambda + \left[\lambda^2 - (1-\alpha^2)\right]^{1/2} \\ u_A = \lambda - \left[\lambda^2 - (1-\alpha^2)\right]^{1/2} \end{cases}$$

(We know that $E < 0$)

The difference in ϕ during an increase from u_A to u_P can be easily obtained: *

$$(u')^2 = (u_P - u)(u-u_A); \qquad \frac{du}{d\phi} = \mp \left[(u_P-u)(u-u_A)\right]^{1/2}$$

$$\phi_P - \phi_A = \int_{u_A}^{u_P} \frac{du}{\left[(u-u_A)(u_P-u)\right]^{1/2}} = \text{arc sin} \left. \frac{u - \frac{1}{2}(u_A+u_P)}{\frac{1}{2}(u_P - u_A)} \right|_{u_A}^{u_P} = \pi \qquad (19)$$

Returning to the exact Equation (11), and setting $u'(\phi) = 0$, we have three roots u_1, u_2 and u_3. The equation is

$$u^3 - \frac{1}{2\lambda} u^2 + u + (\alpha^2 - 1) = 0 \qquad (20)$$

A property of this equation is that

$$u_1 + u_2 + u_3 = \frac{1}{2\lambda} \qquad (21)$$

For small values of λ, we can assume that $u_1 \simeq u_A$ and $u_2 \simeq u_P$. Then, (21) shows us that u_3 is very large, and

$$2\lambda u_3 = 1 - 2\lambda(u_A + u_P) .$$

Then,

$$u'(\phi) = \pm \left[2\lambda(u-u_A)(u-u_P)(u-u_3)\right]^{1/2}$$

$$u'(\phi) = \pm \left[-(u-u_A)(u-u_P) \{1 - 2\lambda(u_A+u_P) - 2\lambda u\}\right]^{1/2}$$

$$u'(\phi) = \pm \left[(u-u_A)(u_P-u) \{1 - 2\lambda(u_A + u_P)\} \left\{ 1 - \frac{2\lambda u}{1-2\lambda(u_A+u_P)} \right\}\right]^{1/2}$$

* The following analysis is found in Møller's book - Theory of Space-Time and Gravitation.

We can now express the equation as

$$d\phi = \pm \frac{du}{\left[(u-u_A)(u_P-u)\right]^{1/2}} \frac{1}{\left[1-2\lambda(u_A+u_P)\right]^{1/2}\left[1-\frac{2\lambda u}{1-2\lambda(u_A+u_P)}\right]^{1/2}}$$

which can be expanded in powers of λ. In keeping just the zero and the first order term, we obtain

$$d\phi = \frac{du}{\left[(u-u_A)(u_P-u)\right]^{1/2}} \left[1 + \lambda(u_A + u_P)\right]\left[1 + \lambda u\right]$$

$$\phi_P - \phi_A = \left[1 + \lambda(u_A + u_P)\right] \int_{u_A}^{u_P} \frac{1 + \lambda u}{\left[(u-u_A)(u_P-u)\right]^{1/2}} du \qquad (22)$$

This integral gives us $\phi_P - \phi_A = \left[1 + \lambda(u_A + u_P)\right]\left[1 + \frac{\lambda}{2}(u_A + u_P)\right]\pi$, which, after linearization, gives for $\Delta\phi = 2(\phi_P - \phi_A) - 2\pi$ the formula

$$\Delta\phi = 3\pi\lambda(u_A + u_P)$$

As we know, u_A and u_P here are only slightly different from the corresponding roots from (18). Then, up to first order we can write $u_A + u_P = 2\lambda$ by (18). Then finally we get for the perihelion advance $\Delta\phi$, per revolution

$$\Delta\phi = 6\pi\lambda^2$$

$$\Delta\phi = 6\pi\left(\frac{GM}{\ell c}\right)^2$$

where ℓ is the angular momentum.

The following table gives some values for $\Delta\phi$ per century: The experimental values are obtained as differences between observed values and those calculated from classical celestial mechanics taking into account the perturbations of other planets.

Planet	Exp. value	Gen. Relativity
Mercury	43".11 ± 0".45	43".03
Venus	8".4 ± 4".8	8".6
Earth	5".0 ± 1".2	3".8

The large error in the observed value for Venus is due to the small eccentricity of its orbit, and the fact that $e\Delta\phi$ is the observed quantity. (e = eccentricity).

It is important to remark here that this is the only classical test where we get an effect of second order in G.

A-II) THE RED SHIFT OF SPECTRAL LINES

A - II.1) The Necessity of a Frequency Shift in Gravitational Theories

If we admit as a postulate of all gravitational theories that all fields interact with the gravitational field, and if we assume that electromagnetic radiation in particular couples with the gravitational field of a particle of mass M by a coulomb type potential, the most obvious guess is a potential energy for a photon of the form

$$V = - \alpha h\nu(r) \frac{GM}{c^2 r}$$

where α is some constant and $h\nu(r)$ is the energy of the photon at a distance r from the mass M.

Then, if a photon of energy $h\nu(r)$ is emitted at a distance r from M, it would arrive at infinity with energy $h\nu_\infty$ given by the energy conservation equation

$$h\nu_\infty = h\nu(r) - \alpha h\nu(r)\frac{GM}{c^2 r}$$

and then

$$\nu(r) - \nu_\infty = \alpha\nu(r)\frac{GM}{c^2 r}$$

The constant α in this empirical law should be determined by experiment, or the entire law should be a consequence of a general theory if it gives account of the experimental results. We will see below that general relativity confirms the above formula with $\alpha = 1$ and $\frac{GM}{c^2 r} << 1$.

Then, we see that a photon emitted in the Sun with a frequency ν will arrive at the Earth with the redshifted frequency ν_E given by

$$\frac{\nu - \nu_E}{\nu} = \frac{\Delta\nu}{\nu} = \frac{GM}{c^2 R_\odot} \quad . \tag{4-a}$$

The redshift is usually given in terms of wavelength difference:

$$z = \frac{\lambda_E - \lambda}{\lambda} = \frac{1}{1 - \frac{\Delta\nu}{\nu}} - 1 \approx \frac{GM}{c^2\ R_\odot} \tag{4-b}$$

if $\frac{GM}{c^2\ R_\odot} <<1$.

Also if we admit the principle of equivalence we can infer qualitatively the necessity of the redshift. Let us analyse the situation described in fig. 1. Clock A sends a signal to clock B, that lasts $t = \frac{\ell}{c}$ to arrive at B. The clock B has a velocity $v = gt = \frac{g\ell}{c}$ at the

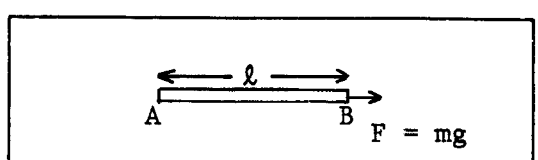


Fig. 1 - Two identical clocks on a rigid rod one at A the other at B, the rod being accelerated by force F.

arrival time. Comparing the rates of the clocks as seen from B,

$$\tau_A = \tau_B\left(1 + \frac{v}{c}\right)$$

$$\tau_A \simeq \tau_B\left(1 + \frac{g\ell}{c^2}\right) \tag{5}$$

By the principle of equivalence this same situation happens when two clocks at rest distant ℓ from each other are stationary in a gravitational field $g = -\nabla\phi$, and by (5) we get

$$\tau_A \simeq \tau_B\left(1 + \frac{\Delta\phi}{c^2}\right) \tag{6}$$

Formula (6) shows that the clock at the lower potential will have a longer observed period. If we consider the case for atomic clocks with a period for one of the spectral lines, we get from (6)

$$\frac{\lambda_A - \lambda_B}{\lambda_s} = z = \frac{\Delta\phi}{c^2} \tag{7}$$

which is equivalent to the formulas (4).

A-II.2) <u>The General Relativistic Redshift Between Stationary Bodies</u>

We consider a set of null geodesics representing the history of the wave crests of an electromagnetic wave transmitted between stationary positions corresponding to different potentials.

If there are n such crests and ds and ds_E are the intervals measured in each position between the first and the last one, we have *

$$n = \nu ds = \nu_E\, ds_E \tag{8}$$

* For more details, see Synge, J. L. - Relativity: The General Theory, North Holland Publishing Co. (1960), p. 122.

where ν is the frequency of emission and ν_E is the frequency of reception.

$$\frac{\nu_E}{\nu} = \frac{ds}{ds_E} \tag{9}$$

Then,

$$\frac{\Delta\nu}{\nu} = \frac{\nu - \nu_E}{\nu} = 1 - \frac{ds}{ds_E}$$

In the case that source and observer are at rest in a stationary universe, their world lines are t-lines. Now we face the central problem when we speak of identical clocks. Experiment tells us, within the possibilities of present day equipment, that if we pick up a standard clock and sinchronize it with another one located at a neighbouring point (the neighbourhood should be so small that the space-time may be regarded as locally flat there and the clocks should be at the same state relative to the gravitational field), this sinchronization is independent of the location of the neighbour in the space-time. Although this assumption seems logical, we must take care when we want to apply these concepts to a region where we are not certain if any "device" we can use as a clock could survive when it passes into that region. For example, what can happen to a "nuclear clock" when it enters the world of a collapsed body ?

Returning to the case of source and receiver of a light signal, if they are at rest in our stationary universe, we have

$$dt = dt_E \tag{11}$$

But $ds = \sqrt{g_{oo}}\ dt$ and $ds_E = \sqrt{g^E_{oo}}\ dt_E$, which together with (11) and (10) gives

$$\frac{\Delta\nu}{\nu} = 1 - \left[\frac{g_{oo}}{g^E_{oo}}\right]^{1/2} \tag{12}$$

When we apply this to the Sun-Earth system,

$$\frac{\Delta\nu}{\nu} = 1 - \left[1 - \frac{2GM}{R_{\odot}cc^2}\right]^{1/2} \simeq \frac{GM}{R_{\odot}\ c^2} \qquad (13)$$

gives the redshift in terms of the frequencies.

In Astrophysics it is common to use the redshift formula in terms of wavelength difference:

$$z = \frac{\Delta\lambda}{\lambda} = \left(1 - \frac{2GM}{R_{\odot}\ c^2}\right)^{-1/2} - 1 \simeq \frac{GM}{R_{\odot}\ c^2} \qquad (14)$$

This formula has been tested in several ways, indicating evidence for the gravitational redshift. Among the most interesting and conclusive experiments we may list the following:

1. Redshift of the Fraunhoffer lines of the Sun: *

   The problems faced when one uses a spectral line of the Sun to test the above formula are:

   a) One must study the spectrum coming from the limb of the Sun, because in central positions there appear spurious Doppler shifts due to granulation (convective currents, mostly radial).

   b) One must choose symmetrical lines, because the effect will be estimated by a measurement of the distortion of the line-shape. The asymmetry in the line shape caused by the relativity effect should also be singled out from the various broadening mechanisms that may also distort the line shape.

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* For more details, see Bertotti, Brill and Krotkov in Gravitation: An Introduction to Current Research - Louis Witten (Ed.), Wiley (N.Y.) (1962) and for details of spectroscopy, see Griem - Plasma Spectroscopy, Mc Graw Hill, N.Y. (1964).

c) Better estimation could be obtained with a better quantitative understanding of the solar atmospheric plasma dynamics.

As an example of a typical line, we take the following:

λ = 5890 Å (Sodium); $\Delta\lambda_R$ = 13 mÅ (relativity); natural width $\simeq 10^{-1}$ mÅ

Stark width; $W_s = 1.2 \times 10^{-2}$ mÅ and $\Delta\lambda_s = 2 \times 10^{-2}$ mÅ

Van der Waals: $W_w = 1.2 \times 10^{-1}$ mÅ and $\Delta\lambda_w = 0.8 \times 10^{-1}$ mÅ

Doppler (Thermal) width; W_D = 36 mÅ

Then we see that the only problem that can hide the relativity effect is the existence of macroscopic streaming giving rise to a Doppler width of the order of 36 mÅ.

This type of experiments do not contradict general relativity, but also do not consist in an experimental confirmation of the formula (14).

2. The study of the spectrum of white dwarf stars (of the same nature of the above study, but with $\frac{M}{R}$ much greater).

3. With the aid of the Mössbauer effect, which uses very precise measurements of gamma ray lines emitted and received at different levels in the gravitational field of the Earth. This is presently the most important method of testing the relativistic formula, and the reader should consult the paper of Pound and Rebka, Phys. Rev. Letters 4, 337 (1960).

A-III) Bending of Light in a Schwarzschild Field

Although we can infer the bending effect to be possible by some empirical arguments, we will not discuss those here. We proceed with the same method of section A-1, with the single modification on the first integral (3) of the geodesic equations, giving account of the massless nature of the photons. We point out here that we may neglect the photon spin because it is always parallel or antiparallel with the direction of motion, and for this reason it does not affect the geodesic equation. *

$$g_{oo}(\dot{\xi}^o)^2 + g_{11}\,\dot{r}^2 - r^2\,\dot{\phi}^2 = 0 \tag{1}$$

$$\dot{\xi}^o = \alpha/g_{oo} \tag{2}$$

$$r^2\,\dot{\phi} = h/r^2 \tag{3}$$

Then, using the same notation as in A-1, we get the analog of equation (A-1-11) for the photon orbit:

$$(u')^2 = \alpha^2 - u^2 + 2\lambda u^3 \tag{4}$$

For analysing the solution of this equation, we can follow an usual perturbation method.

Taking the derivative of (4) with respect to ϕ, we get

$$u'' + u = 3\,\lambda\,u^2 \tag{5}$$

$$u = u^{(o)} + u^{(1)}$$

The zero order equation gives $u^{(o)} = u_o^{(o)}\cos\phi = \frac{h}{R}\cos\phi$ and then we get in first order the equation

$$(u^{(1)})'' + u^{(1)} = \frac{3\lambda h^2}{R^2}\cos^2\phi \tag{6}$$

We look for a first integral of (6), and it can be shown that using the initial conditions we get

$$u^{(1)} = \frac{\lambda h^2}{R^2}(\cos^2\phi + 2\sin^2\phi) \tag{7}$$

and then, with $u = \frac{h}{r}$ and $x = r\cos\phi$, $y = r\sin\phi$, we get

$$X = R - \frac{\lambda h}{R}\,\frac{x^2 + 2y^2}{\sqrt{x^2 + y^2}} \tag{8}$$

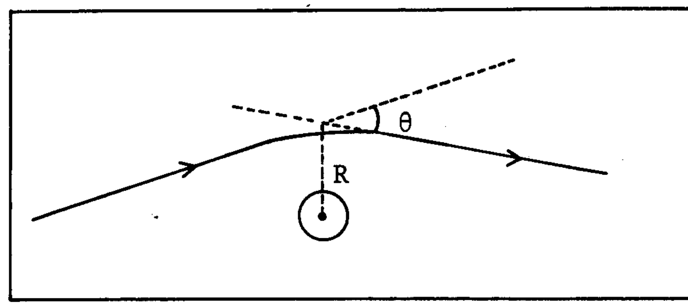
* Corinaldesi, E. and Papapetrou, A., Proc. Roy. Soc., A209, 259 (1951).

For very large y, (8) becomes

$$X \simeq R - \frac{2\lambda h}{R} |y| \qquad (9)$$

Using $\lambda = \frac{GM}{hc^2}$, we get for the deviation angle Θ

$$\Theta = \frac{4\,GM}{R\,c^2} \qquad (10)$$



This formula first derived by Einstein gives the relativistic effect of the bending of a light ray in a central gravitational field.

The experimental verification of the above formula is a very difficult task. Until 1969 the only way to do this was by means of the Solar total eclipse method (see Bertotti et al., loc. cit.). For rays grazing the limb the deflection angle has the value of 1".75 for the Sun.

But with the development of the long base-line interferometry technique for use in Astrophysics *, giving very high angular resolution of radio sources in the centimeter region of the spectrum, it is expected that a more meaningful test will be made. In fact the Sun passes close to two of the most intense quasars, 3C273 and 3C279, and their separation is 4.5 deg., so the measurements of the deflections may be made differentially. This type of experiment was recently carried on by the MIT-Lincoln Lab Group but the results have not yet been analysed.

A-IV) The Problem of Existence of Gravitational Radiation

This very important test will not be discussed here. The search for gravitational radiation is associated with the name of Joseph Weber, who claims

* Burke, B. F. - Long Base-Line Interferometry, Phys. Today (July 1969).

that finally gravitational radiation was detected by his long base-line system of aluminium cylinder antennas. * Since the gravitational waves alter the whole geometry of the space surrounding the experimental apparatus, the only way we can detect it is by the measurement of the Riemann curvature tensor that appears in the equation of the geodesic deviation

$$\frac{\delta^2 \xi^\mu}{\delta \tau^2} + R^\mu_{\nu\rho\sigma} u^\mu \xi^\rho u^\sigma = 0$$

where u^μ is the tangent unit vector along the geodesic and ξ^μ is the displacement driven by the time-varying Riemann tensor. This displacement should be felt by a macroscopic body as a stress, and so the antenna should convert gravitational radiation energy in stress wave energy in the elastic body. To detect the very small strains caused by so small stresses, piezoelectric quartz strain gauges are used in Weber's equipment. Using aluminium cylinders as antennas, detection is only efficient in the lower longitudinal (acoustic) mode of 1657 cps. But of course many other geometries could be used in the laboratory, although all of them have small absorption cross-sections. The seismological study of the Moon seems to give some hope in using it as a antenna. The first studies have shown that it behaves very well as an elastic body, and when its normal modes of vibration are well studied it will perhaps constitute a very good device for measuring Riemann tensor caused by gravitational waves.

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* Weber, J., Phys. Rev. Letters 22, 1320 (1969). The theory and description of the experimental method is listed in the bibliography of this important paper.

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