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HIGH ENERGY BEHAVIOR OF SCATTERING AMPLITUDES.

by

S. W. McDowell

Notes prepared by:

H. M. Nussenzveig
Prem P. Srivastava
Alberto Vidal

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Av. Wenceslau Braz, 71

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1. INTRODUCTION

The first consideration we have to make concerns the question of what is to be regarded as high energy scattering.

In the theory of potential scattering if a is the range of the potential, our approaches the asymtotic region for energies such that $ka \gg 1$, where k is the wave number of the incident particle. This condition means that very many partial waves, those with angular momentum up to $\ell \simeq ka$, contribute to the scattering amplitude.

In Nuclear Physics, the existence of a large number of resonances characterize reactions in the low (500 - 1000 KeV) and intermediate energies (1 Mev - 20 Mev). These ranges are of course only rough approximations; they depend, funamentally, on the projectile; for instance, neutron induced reactions start at energies much lower than the thresholds for proton or α-particle reactions, because of the Coulomb barier. They also depend on the target nucleus: light nuclei behave quite differently from heavy nuclei. However, generally speaking, one can say that many sharp resonances are found in the lower range and broader resonances in the higher range due to electromagnetic and nuclear excitation, respectively. These resonances reveal the structure of the nuclear levels. In the high energy region (20-100 Mev) the main feature is the appearance of many particle reactions (p, 2n), (α, 2n) and so on. The nucleus behaves like an absorbing medium which can be described by an optical potential. The total cross section approaches the geometrical limit $2\pi R^2$ where the nuclear radius R is the range of the average potential. Reactions above 100 MeV take place essentially through direct collisions with individual nucleons in the nucleus, and practically do not depend on the nuclear structure. They may be described in terms of elementary particle processes. The range of the interaction is of the order of the Compton wave lenght. Thus one might consider the high energy limit for nuclear reactions to be around 100 MeV. This is, roughly, on upper limit for the specific domain of Nuclear Physics.

In Elementary Particle Physics one has an analogous situation. In the low energy region (below $\simeq$ 2 Gev) several resonances of both mesons and baryons have been found. Above this region many meson production processes are dominant, the total cross sections seem to go down smoothly, slowly approaching a constant value and the elastic scattering becomes essentially forward diffraction.

Both in nuclear reactions and in elementary particle processes at high energies, the scattering is dominated by inelastic processes. Hence in addition to the condition $ka \gg 1$ set up for potential scattering, the presence of very many open channels is another characteristic feature of high energy collisions. The energy has to be sufficiently high to give rise to a fair amount of inelastic transitions in states with large total angular momentum J.

We shall take these qualitative criteria as a starting point for a discussion of high energy scattering of elementary particles. According to them the high energy region is expected to be above 20 GeV.

The first attempts to give a theoretical interpretation of high energy collisions of elementary particles were based on the optical model, the nucleon being treated as a "grey or black sphere". Some interesting consequences of this model were derived by Pomerauchuk (1956)[1,2]. He also discussed (1958)[3] some implications of forward scattering dispersion relations on the high energy behaviour of cross sections, in conection with this model.

2. POMERAUCHUK'S MODEL

Pomerauchuk proposed the following picture for high energy scattering.

In the collision of elementary particles with a target nucleon at high energies, very many inelastic channels are open and strong absorption takes place. The nucleon behaves like a dark grey or black body with a size of the order of the pion Compton wave length. The scattering at these energies would then have the following characteristics:

i) The total cross section $\sigma(E)$ approaches a constant limit $\sigma(\infty)$ as the incident energy E increases to infinity.

ii) The total elastic cross section is of the same order of magnitude as the total inelastic cross section. Since there are a great many inelastic channels, one expects on statistical grounds, the elastic cross section to be much larger than the cross section for any particular inelastic channel.

iii) The elastic scattering is predominantly forward diffraction. The reaction being strongly absorptive, the real part of the forward amplitude cannot increase as fast as the imaginary part when $E \longrightarrow \infty$, that is, in this asymptotic limit the scattering amplitude becomes purely imaginary.

Let us introduce kinematical variables in the center of mass system:

S - the square of the total energy in the c.m. system.

k - the momentum of the incoming particles in the c.m. system:

$$k^2 = \frac{1}{4S}\left[S^2 - 2\left(M_2^2 + M_1^2\right)S + \left(M_2^2 - M_1^2\right)^2\right] \qquad (2.1)$$

t - the negative of the square of the momentum transfer. It is related to the scattering angle θ in the c.m. system by:

$$t = -2k^2\,(1 - \cos\theta) + \frac{\left(M_2^2 - M_1^2\right)^2}{S} \qquad (2.2)$$

For equal mass particles $M_1 = M_2 = M$ (2.1) and (2.2) reduce to:

$$k^2 = \frac{1}{4}\,(S - 4M^2) \qquad (2.3)$$

$$t = -2k^2(1 - \cos\theta) \qquad (2.4)$$

The center of mass variables are related to the lab. energy E and momentum $p = \sqrt{E^2 - M_1^2}$ of the incident particle by:

$$S = M_1^2 + M_2^2 + 2 E M_2 \tag{2.5}$$

$$p = \frac{k\sqrt{S}}{M_2} \tag{2.6}$$

Let $f(S, t)$ (or $f(E, \theta)$ be the (relativistic) elastic scattering amplitude. Unitarity relates the forward scattering amplitude to the total cross section by

$$\sigma = \frac{1}{M_2} \frac{4\pi}{p} \operatorname{Im} f(E, 0)$$
$$= \frac{1}{\sqrt{S}} \frac{4\pi}{k} \operatorname{Im} f(S, 0) \tag{2.7}$$

Hence it follows from conditions i) and iii) that $f(S, 0)/S$ (or $f(E, 0)/E$) <u>is bounded</u>.

iv) The model predicts that the width of the elastic forward peak (in the momentum transfer) approaches a constant limit. Indeed let $|t_o(S)|$ be the width of the elastic forward peak. The total elastic cross section will be given by:

$$\sigma_{el.} \simeq \frac{2\pi}{S} \cdot |f(S,0)|^2 \frac{|t_o(S)|}{2\,k^2} \tag{2.8}$$

Since $\sigma_{el.}$ and $f(S, 0)/S$ are bounded it follows that $|t_o(S)| = 0\,(1)$, as $s \to \infty$. It should be noted that the total cross section $\sigma\,(S)$ may have a limit and yet $\sigma_{el.}$ although finite may oscillate; in this case the width of the forward peak would also oscillate. But if $\sigma_{el.} \longrightarrow$ const. then $|t_o(S)| \longrightarrow$ const. as $s \longrightarrow \infty$.

Some interesting consequences can be drawn from these properties. Let us first discuss the consequences of Pomeranchuk's conjecture ii) according to which:

$$\sigma_{el.} \approx \sigma_{in.} \; ; \; \sigma_{el.} \gg \sigma_n \qquad (2.9)$$

where n is a given inelastic channel. If one considers reactions with two outgoing particles only, the elastic scattering amplitude near the forward direction must be much larger than the amplitudes for exchange reactions (helicity exchange, charge exchange, strangeness exchange). Let us take as an example nucleon-antinucleon scattering. We have the following processes in the collisions of antiprotons on a proton target:

$$1)\ \bar{n} + p \longrightarrow \bar{n} + p$$
$$2)\ \bar{p} + p \longrightarrow \bar{p} + p$$
$$3)\ \bar{p} + p \longrightarrow \bar{n} + n$$

There are similar processes on a neutron target which are related to these by "charge symmetry".

For each of these processes there are five helicity amplitudes [4]:

$$\varphi_1 = \left\langle \tfrac{1}{2}, \tfrac{1}{2} \middle| \tfrac{1}{2}, \tfrac{1}{2} \right\rangle - \left\langle \tfrac{1}{2}, \tfrac{1}{2} \middle| -\tfrac{1}{2} - \tfrac{1}{2} \right\rangle$$
$$\varphi_2 = \left\langle \tfrac{1}{2}, \tfrac{1}{2} \middle| \tfrac{1}{2}, \tfrac{1}{2} \right\rangle + \left\langle \tfrac{1}{2}, \tfrac{1}{2} \middle| -\tfrac{1}{2} - \tfrac{1}{2} \right\rangle$$
$$\varphi_3 = \left\langle \tfrac{1}{2}, -\tfrac{1}{2} \middle| \tfrac{1}{2}, -\tfrac{1}{2} \right\rangle - \left\langle \tfrac{1}{2}, -\tfrac{1}{2} \middle| -\tfrac{1}{2}, \tfrac{1}{2} \right\rangle$$

$$\varphi_4 = \left\langle \tfrac{1}{2}, -\tfrac{1}{2} \,\middle|\, \tfrac{1}{2}, -\tfrac{1}{2} \right\rangle + \left\langle \tfrac{1}{2}, -\tfrac{1}{2} \,\middle|\, -\tfrac{1}{2}, \tfrac{1}{2} \right\rangle$$

$$\varphi_5 = 2\left\langle \tfrac{1}{2}, \tfrac{1}{2} \,\middle|\, \tfrac{1}{2}, -\tfrac{1}{2} \right\rangle \qquad (2.10)$$

All other amplitudes corresponding to transitions in different helicity states may be reduced to the above ones by virtue of parity conservation, time-reversal and G-invariance. In particular from G - invariance it follows that there can be no transitions between singlet and triplet spin states. In nucleon-nucleon scattering this selection rule is a consequence of isospin invariance and the Pauli principle. The first amplitude φ_1 in (2.10) corresponds to the transition in the singlet state; the remaining ones are the amplitudes for transitions in triplet states.

Now, according to our discussion, at high energies and near the forward direction the helicity exchange amplitudes must be small as compared with the truly elastic amplitude. This implies:

$$\varphi_1 \simeq \varphi_2 \ ; \ \ \varphi_3 \simeq \varphi_4 \ ; \ \varphi_5 \ll \varphi_{1,3} \qquad (2.11)$$

The expression "near the forward direction" is used here to mean that as the energy is increased, the scattering angle is taken sufficiently small so as to keep the momentum transfer finite. One verifies, however, that in the forward direction ($\theta = 0$) the following identities hold:

$$\Big[\varphi_3 - \varphi_4\Big]_{\theta=0} = 0, \qquad \varphi_5\Big]_{\theta=0} = 0 \qquad (2.12)$$

which simply result from conservation of the component of the total angular momentum in the incident direction. Hence only the relation $\varphi_1 \simeq \varphi_2$ leads to a strong restriction on the high energy behaviour of the covariant amplitudes at fixed momentum transfer. These asymptotic relations imply that at fixed momentum transfer:

$$(\varphi_1/\varphi_2) \longrightarrow 1 \quad ; \; (\varphi_3/\varphi_4) \longrightarrow 1 \; ; \; (\varphi_5/\varphi_{1,3}) \longrightarrow 0. \qquad (2.13)$$

as $E \longrightarrow \infty$. Let us now consider a new set of asymtotic spin states, obtained by rotating the direction of quantization through an angle about an axis normal to the momentum of the particle. If the axis is taken in the plane of scattering we have:

$$\left|\tfrac{1}{2}\right\rangle_\alpha = \cos\tfrac{\alpha}{2}\left|\tfrac{1}{2}\right\rangle + i\sin\tfrac{\alpha}{2}\left|-\tfrac{1}{2}\right\rangle$$

$$\left|-\tfrac{1}{2}\right\rangle_\alpha = i\sin\tfrac{\alpha}{2}\left|\tfrac{1}{2}\right\rangle + \cos\tfrac{\alpha}{2}\left|-\tfrac{1}{2}\right\rangle \qquad (2.14)$$

and

$$\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_\alpha - \left|-\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle_\alpha = \left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle - \left|-\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_\alpha + \left|-\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle_\alpha = \cos\alpha\left(\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + \left|-\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle\right) + i\sin\alpha\left(\left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle + \left|-\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle\right)$$

$$\left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle_\alpha - \left|-\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_\alpha = \left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle - \left|-\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle \qquad (2.15)$$

$$\left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle_\alpha + \left|-\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_\alpha = \cos\alpha\left(\left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle + \left|-\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle\right) + i\sin\alpha\left(\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + \left|-\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle\right)$$

The new amplitudes are then related to the helicity amplitudes by:

$$
\begin{aligned}
\varphi_{1\alpha} &= \varphi_1 \\
\varphi_{2\alpha} &= \cos^2\alpha\ \varphi_2 + \sin^2\alpha\ \varphi_4 \\
\varphi_{3\alpha} &= \varphi_3 \\
\varphi_{4\alpha} &= \cos^2\alpha\ \varphi_4 + \sin^2\alpha\ \varphi_2 \\
\varphi_{5\alpha} &= \varphi_5 + i \sin\alpha \cos\alpha (\varphi_2 - \varphi_4)
\end{aligned}
\tag{2.16}
$$

in an obvious notation. If Pomerauchuk's rule ii) is applied to the new set of amplitudes there results a new asymptotic relation at high energies, namely $\varphi_2 \simeq \varphi_4$. Hence the four amplitudes φ_1, φ_2, φ_3, φ_4 have the same asymptotic behaviour at fixed momentum transfer. The out come of this analysis is expressed in the following theorem:

Theorem 1. "The high energy behaviour of the scattering matrix at fixed momentum transfer is spin independent if spin-flip amplitudes are small as compared with the elastic amplitudes".

Then the total cross sections and differential elastic cross sections near the forward direction are also independent of spin. The implications of the previous result on the high energy behaviour of the covariant amplitudes in nucleon-nucleon and nucleon-antinucleon scattering is discussed in Appendix.

One can carry out a similar analysis to discuss the isotopic-spin dependence of the scattering matrix in the high energy limit, as was originally done by Pomerauchuk. The total isotopic-spin of the nucleon-antinucleon system, for example, is either $I=0$ or $I=1$. Let ϕ^0 and ϕ^1 be the amplitudes for transitions in states with $I = 0$ and $I = 1$ respectively. Then, assuming charge independence, the amplitudes for processes 1), 2) and 3), will be given by:

$$\varphi^1 = \phi^1 \; ; \quad \varphi^2 = \frac{1}{2}(\phi^0 + \phi^1) \quad ; \quad \varphi^3 = \frac{1}{2}(\phi^0 - \phi^1)$$

But process 3) involves charge exchange and according to Pomerauchuk's rule its amplitude must be small as compared with those for the elastic processes 1) and 2). Therefore $\phi^0 \simeq \phi^1$ and $\varphi^1 \simeq \varphi^2$ in the high energy limit, that is, the scattering matrix for nucleon-antinucleon scattering becomes independent of isotopic spin as the energy tends to infinity. In pion-nucleon scattering the same result would obtain. One can in fact establish the following theorem[5,6]:

<u>Theorem 2</u>. "If, at high energies and fixed momentum transfer charge exchange amplitudes are much smaller than elastic amplitudes in the collision of particles of a multiplet I_1 with particles of a multiplet I_2, then the asymptotic behaviour of the scattering matrix in the limit as $E \longrightarrow \infty$ is independent of isotopic spin".

Indeed, according to Pomerauchuk's assumption the transition matrix M in isospin space is diagonal in the representation where I_{1Z} and I_{2Z} are diagonal; it is, therefore, a function of I_{1Z} and I_{2Z} only. However, because of charge independence, it can depend on I_{1Z} and I_{2Z} through the combination $I^2 = (I_{1X} + I_{2X})^2 + (I_{1Y} + I_{2Y})^2 + (I_{1Z} + I_{2Z})^2$ but cannot depend on I_{1Z} and I_{2Z} alone. It follows that M is proportional to the unit matrix, hence independent of the total isotopic spin I^2. A corolary of this theorem is that the total cross sections and the differential elastic cross sections near the forward direction are independent of isotopic spin in the high energy limit. This property has been experimentally verified in nucleon nucleon scattering at an incident momentum $\gtrsim 3$ Gev/c. It should be remarked that Pomerauchuk's original formulation of his rule on the asymptotic behaviour of scattering amplitudes does not refer to the dependence on the spin state. His statements for total cross sections refer to spin averages over the initial states. They apply to the scattering of impolarized beams. It is, however, apparent from the above discussion that the spin independence of the scattering matrix is obtained on the same grounds as the isotopic spin independence. One should then reformulate Pomerauchuk's rule so as to exhibit in a transparent way the common source of these results.

Let us call "one-particle state" any discrete eigenstate of the operator P^2, where P is the total momentum four-vector operator. Evidently the corresponding eigenvalue is the square of the mass of the particle. Let us then assume that:

"All physically measurable asymptotic states are direct products of one-particle states".

We shall illustrate these concepts with an example: The striking difference between a proton and a neutron is a familiar and all too common experience. It is fairly easy to prepare a beam of protons with given momentum; one can quite easily accelerate, stop, or detect them. What is involved in all these experiments is the long range electromagnetic interaction. Neutrons have no charge, hence no Coulomb interaction and the processes used to produce or detect neutrons are of a rather different kind. It is true, that one can produce a beam which contains both neutrons and protons of given momentum; however any experiment set up to identify a particle would allways give, as a result, either a proton or a neutron but never a mixed particle. This means that proton and neutron are truly different particles. However, one knows that in spite of these differences there are also striking similarities between proton and neutron. At short distances, when strong interactions overcome electromagnetic interactions, they have essentially the same properties. The mass difference is very small and most certainly, also of electromagnetic origin. One believes that all the differences between proton and neutron are indeed of electromagnetic nature, but for very small weak interaction effects. If one imagines electromagnetic and weak interactions switched off in the physical world, it would turn out that proton and neutron would essentially be two different (isotopic) states of the same particle, the nucleon. Possible states of a nucleon would then be a proton, a neutron or any superposition of these two states. In other words, one could arbitrarily choose the axis of quantization for the isospin of the one nucleon physical states. The superposition principle applies to states of the same particle but it has no physical meaning to superpose states of different particles. That is the reason why, in the physical world where electromagnetic interactions do exist, one cannot find a mixed nucleon.

Let us now consider states of two particles. An asymptotic two particle state may be constructed by giving the states of both particles. But an arbitrary superposition of states thus constructed might not be a physically possible state. Indeed, one requires that the state of the projectile and that of the target be uncorrelated, since asymptotically the particles are so far apart that no interaction is possible between them so as to produce a correlation. This precludes such combinations as $\frac{1}{\sqrt{3}}(|N\bar{N}\rangle + |P\bar{P}\rangle + |\Lambda\bar{\Lambda}\rangle)$. Let us now return to the original discussion. A particle is labeled by a set of indices which specify the properties of its states under certain groups of transformations: spin for the Lorentz group, isospin for rotations in a three dimensional charge space (when electromagnetic and weak interactions are neglected), baryon number and hypercharge for two kinds of gauge transformations, parity for space inversion, G-parity for charge reflection. The set of possible states of a particle form a multiplet whithin which, each state is labeled by the quantum numbers α_1; of any complete set of commuting variables (components of spin, isospin etc.), associated with these transformations. Let $|\alpha_1\ p_1\rangle$ be the state vector of a particle with quantum numbers α_1 and momentum p_1, and consider the scattering, in the center of mass system, of two particles in the initial state $|\alpha_1 p_1, \alpha_2 p_2\rangle$. A transition into the final state $|\alpha_1' p_1', \alpha_2' p_2'$ is truly elastic if

$$|\alpha_1' p_1', \alpha_2' p_2'\rangle = R(\vec{p}, \vec{p}')|\alpha_1 p_1, \alpha_2 p_2\rangle \tag{2.17}$$

where $R(\vec{p}, \vec{p}')$ is the rotation matrix associated with the transformation $\vec{p} \longrightarrow \vec{p}'$. We shall now state the following rule which is a generalization of Pomerauchuk's conjecture ii):

Generalized Pomerauchuk's Conjecture: "In the high energy limit and for fixed momentum transfer, any amplitude:

$$\langle \beta_1' p_1', \beta_2' p_2'|m|\alpha_1 p_1, \alpha_2 p_2\rangle$$

is (vanishingly) small as compared with the truly elastic amplitude:

$$\langle \alpha_1' p_1', \alpha_2' p_2'|m|\alpha_1 p_1, \alpha_2 p_2\rangle$$

if $|\beta_1' p_1', \beta_2' p_2'\rangle$ is orthogonal to $|\alpha_1' p_1', \alpha_2' p_2'\rangle$".

Now it follows from this assumption that:

<u>Theorem 3</u>. "In the high energy limit and for finite momentum transfer the scattering matrix is proportional to the unit matrix in the space of the variables which label the states".

Indeed the scattering matrix is diagonal, whatever the choice of a set of basic orthonormal states in the space of these variables. Therefore it is proportional to the unit matrix in this space.

Theorems 1 and 2 are then immediate consequences of Theorem 3.

For instance, the scattering matrix cannot depend on the total isotopic spin I^2, because $\vec{I} = \vec{I}_1 + \vec{I}_2$ and the scattering matrix, according to the theorem, does not depend on the components of $\vec{I}_1$ and $\vec{I}_2$.

One can speculate on whether particle and anti-particle should always be regarded as different states of the came object. Actually they have the same mass and there exists a unitary transformation $\mathscr{C}$(charge conjugation) which transforms one into the other. It is therefore plausible to interpret them as states. If so, a further result will obtain:

Theorem 4. "In the high energy limit the scattering amplitudes near the forward direction, are invariant under the substitution of one particle by its antiparticle. The total cross sections for particle and antiparticle collisions on the same target will have the same limit".

Let us make, as a final remark, an important qualification. In the framework of the optical model, the results thus for obtained should be valid for all finite values of the momentum transfer. However, in a more realistic approach to the problem some inferences of the optical model should perhaps be disregarded. The theory has to be confronted with experimental evidence, which is the ultimate test to sanction its validity. At the moment, there is quite definite indication of departures from some qualitative predictions of the model. One such evidence is the shrinking of the diffraction peak in proton-proton scattering, which by now seems to be quite well established[7,8]. If this is indeed true, one has to reject prediction IV) of the model and the first part of ii) since the total elastic cross section would then be small as compared with the total inelastic cross section. However, the second part of ii) may still hold true, since the number of inelastic channels increases fast with the energy (at least linearly), whereas the total elastic cross section decreases slowly, perhaps like $1/\ln S$. The conditions on the scattering amplitudes would then be valid only for momentum transfers below a certain value depending on the process considered, and only for this range of values of the momentum transfer, would the theorems be valid. The same applies to what was said about differential elastic cross sections. On the other hand, since the total cross sections are related by unitarity to the imaginary part of the forward elastic amplitudes, the statements concerning them remain valid.

An essentially different approach to high-energy scattering is based upon the possible existence of a higher symmetry in strong interactions. If the symmetry is broken in a convenient way, as for instance by mass splitting terms, then it would only manifest itself at sufficiently high energies, when the effects of mass differences become negligible. Under these circumstances, the isotopic spin independence of the scattering matrix at high energies can be derived from the assumption that all elementary particles belong to irreducible representations of a semi-simple compact Lie group. This group would be at least of rank two in order to include the comuting observables T_Z and S among its generators. Rotations in isospin space would form a subgroup of this larger group. The proof of isotopic spin independence at high energies is based on the fact that the scattering matrix will depend only on the "Casimir operators" of the group. A Casimir operator is an operator which comutes with all the generators of the group. For example I^2 is a Casimir operator for the isotopic spin subgroup but not for the whole group. Hence the theorem follows. The interest of this approach stems from the fact that, if such an underlying symmetry actually exists, one would expect its consequences to show up in a range of energies lower than that required for statistical considerations, as invoked in Pomerauchuk's model, to apply. If so, the scattering matrix would become isotopic-spin-independent at energies much lower than those for which spin independence (and particle-antiparticle invariance) would also hold. The experimental verification of this prediction would be good evidence for the existence of a higher symmetry in strong interactions.

3. POMERAUCHUK'S MODEL AND DISPERSION RELATIONS

We turn now to an investigation of items i) and iii) of Pomerauchuk's model, in connection with forward scattering dispersion relations[3].

Let f(E, 0) be the forward scattering amplitude for elastic collision of a particle of momentum $p = \sqrt{E^2 - m^2}$ with a target at rest, D(E) and A(E) the dispersive and absorptive parts of f(E,0):

$$f(E,\ 0) = D(E) + iA(E) \tag{3.1}$$

In the physical region A(E) is related to the total cross section by:

$$A(E) = \text{Im}\ f(E,0) = \frac{p}{4\pi}\ \sigma(E) \tag{3.2}$$

According to i) and iii) we have the following asymptotic conditions as $E \longrightarrow \infty$:

$$\text{i)}\quad A(E)/E \longrightarrow \text{const.} = \frac{\sigma(\infty)}{4\pi}$$

$$\text{ii)}\quad D(E)/A(E) \longrightarrow 0$$

When this information is introduced in the dispersion relations it turns out that the total cross sections for scattering of a particle and its antiparticle by a given target approach the same limit.

To fix ideas let us consider the scattering of protons and antiprotons by protons and let $f_+(E)$ and $f_-(E)$ be, respectively, the elastic forward amplitudes for the processes:

$$\text{I.}\quad p + p \longrightarrow p + p$$
$$\text{II.}\quad p + \bar{p} \longrightarrow p + \bar{p}$$

We shall take, for simplicity, protons with positive helicity and antiprotons with negative helicity. In this case there will be no contribution from the one pion exchange term. Under the assumption that $f_{\pm}(E)/E$ is bounded at large E, one can write dispersion relations for $D_{\pm}(E)$ with two subtractions, which we make at $E = 0$. We have then:

$$D_{\pm}(E) = D(0) \pm ED'(0) + \frac{E^2}{\pi}\int_{-M+4\mu^2/2M}^{M} \frac{dE'}{E'^2}\frac{A_-(E')}{E'\pm E} + \frac{E^2}{4\pi^2}\int_M^\infty \frac{p'dE'}{E'^2}\left[\frac{\sigma_+(E')}{E'\pm E} + \frac{\sigma_-(E')}{E'\pm E}\right] \tag{3.3}$$

These integrals are principal values.

The first integral extends down to an "unphysical region" corresponding to annihilation channels of the antiproton-proton pair, below the physical threshold. We take the even and odd combinations $G_{\pm}(E) = \frac{1}{2}\,(D_+(E) \pm D_-(E))$:

$$G_+(E) = D(0) + \frac{E^2}{\pi}\int_{-M+4\mu^2/2M}^{M} \frac{dE}{E'}\frac{A_-(E\ )}{E'^2 - E^2} + \frac{E^2}{4\pi^2}\int_M^\infty \frac{p'dE'}{E'}\left[\sigma_+(E') + \sigma_-(E')\right]\left(\frac{1}{E'^2 - E^2}\right) \tag{3.4}$$

$$G_-(E) = E\left\{D'(0) - \frac{E^2}{\pi}\int_{-M+4\mu^2/2M}^{M} \frac{dE'}{E'}\frac{A_-(E')}{E'^2 - E^2} + \right.$$

$$\left. + \frac{E^2}{4\pi^2}\int_M^\infty \frac{p'dE'}{E'^2}\left[\sigma_+(E') - \sigma_-(E')\right]\frac{1}{E'^2 - E^2}\right\} \tag{3.5}$$

The first result that energies, when the asymptotic conditions implied by Pomerauchuk's model are put in these relations, may be expressed in the following way:

<u>Theorem 1</u>. If the total cross sections $\sigma_{\pm}(E)$ have a limit $\sigma_{\pm}(\infty)$ and $G_-(E)/E$ is bounded, then they have the same limit $\sigma_+(\infty) = \sigma_-(\infty)$".

Indeed the first two terms of (3.5), inside the brakets, approach a constant as $E \longrightarrow \infty$. The integration in the last term may be split into two parts, from M to E_o and from E_o to ∞, where E_o is so chosen that, for $E > E_o$, $|\sigma_+(E) - \sigma_-(E) - \Delta| < \epsilon$ where $\Delta = \sigma_+(\infty) - \sigma_-(\infty)$. The first part again contributes a constant in the limit $E \to \infty$; in the second part one replaces $\frac{p'}{E'}(\sigma_+ - \sigma_-)$ by its limit Δ, whereupon this term behaves asymptotically like $\frac{\Delta}{4\pi^2} \ln E$. Since $G_-(E)/E$ is bounded it follows that one must have $\Delta = 0$. A certain amount of care is required in dealing with a principal value integral specially if one recalls that the point at infinity is an accumulation of branch points of $f_{\pm}(E)$. However, the proof we have outlined above can be made rigorous in the following way.

Consider the last integral inside brakets in (3.5)

$$g(E) = \frac{E}{\pi} \int_M^\infty dE' \left(\frac{1}{E'-E} - \frac{1}{E'+E} \right) \frac{h(E')}{E'} \tag{3.6}$$

where:

$$h(E) = \frac{p}{2E} (\sigma_+(E) - \sigma_-(E))/4\pi \tag{3.7}$$

Since $h(E)$ is bounded and continuous, $\frac{h(E)}{E}$ and $\frac{g(E)}{E}$ are Hilbert transforms of class $L^2(-\infty, \infty)$ so that one can take the inverse relation[9]:

$$\frac{h(E)}{E} = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dE'}{E'-E} \frac{g(E')}{E'} \tag{3.8}$$

Now let us assume that for $E > M$, $g(E)$ satisfies a Lipschitz condition * :

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* **Actually it would be sufficient to require a Lipschitz condition for $E > E_o > M$ and with $|h| < h_o$.**

$$|g(E+h)-g(E)| \leqslant K\,|h|^{\alpha} \qquad (\alpha > 0) \tag{3.9}$$

Then, in

$$\int_M^E \frac{h(E')}{E'}\,dE' = -\frac{1}{\pi}\int_M^E dE'' \int_{-\infty}^{+\infty} \frac{dE'}{E'-E''}\,\frac{g(E')}{E'} \tag{3.10}$$

one can interchange the order of integrations, so that:

$$\int_M^E \frac{h(E')}{E'}\,dE' = \frac{1}{\pi}\int_{-\infty}^{+\infty} g(E')\,\frac{dE'}{E'}\,\ln\left|\frac{E'-E}{E'-M}\right| \tag{3.11}$$

Since g(E) is an even function of E, (3.11) may be cast into the form:

$$\int_M^E \frac{h(E')}{E'}\,dE' = \frac{1}{\pi}\int_0^{\infty} g(E')\,\frac{dE'}{E'}\,\ln\left|\frac{E'+E}{E'-E}\;\frac{E'+M}{E'-M}\right|$$

$$= \frac{1}{\pi}\int_0^{\infty} g(E')\,\frac{dE'}{E'}\,\ln\frac{E'-E}{E'+E} + \text{const.} \tag{3.12}$$

But assuming that $G_{-}(E)/E$ is bounded, then g(E) must be bounded and since $\frac{1}{E}\ln\left|\frac{E-E}{E+E}\right|$, for positive E and E^1, is always negative, it follows that

$$\left|\int_0^{\infty} g(E)\,\frac{dE}{E}\,\ln\left|\frac{E'-E}{E'+E}\right|\right| \leqslant |g|_{max.}\int_0^{\infty}\frac{dE'}{E'}\,\ln\left|\frac{E'+E}{E'-E}\right| =$$

$$= |g|_{max}\int_0^{\infty}\frac{dx}{x}\,\ln\left|\frac{x+1}{x-1}\right| = \text{const.} \tag{3.13}$$

Hence $\int_M^E \frac{h(E')}{E'}\,dE'$ is bounded. We can express this result in the following way:

<u>Theorem Ia</u>. "If the dispersive amplitude $G_-(E)/E$ is bounded and satisfies a Lipschitz condition (3.9) for $E \geqslant M$, then the integral

$$\int_M^E \frac{p'}{E'} \left(\sigma_+(E') - \sigma_-(E')\right) \frac{dE'}{E'} \qquad (3.14)$$

is bounded and the integrand satisfies a Lipschitz condition with the same α".

The last statement follows from a theorem on Hilbert transforms, satisfying a Lipschitz condition (3.9).[10] Condition (3.14) was obtained by Amati, Fierz and Glaser[11]. It is a <u>necessary</u> but, in general, not a sufficient condition for the boundedness of $G_-(E)/E$. As a consequence of this theorem we have the following:

Corolary: "If, as in Pomerauchuk's model, the cross sections have a limit when $E \longrightarrow \infty$, then the boundedness of (3.14) requires that $\sigma_+(E)$ and $\sigma_-(E)$ have the same limit."

This is precially the statement of theorem 1.

In order to establish sufficient conditions for the boundedness of $G_-(E)/E$, let us write:

$$g(E) = \frac{E}{\pi} \int_M^\infty dE' \left(\frac{1}{E - E} + \frac{1}{E + E} \right) \frac{h(E')}{E'} - \frac{2E}{\pi} \int_M^\infty \frac{dE'}{E' + E} \frac{h(E')}{E'}$$

$$= \frac{E}{\pi} \int_M^\infty \frac{dE'^2}{E'^2 - E^2} \frac{h(E')}{E'} - \frac{2E}{\pi} \int_M^\infty \frac{dE'}{E' + E} \frac{h(E')}{E'} = g_1(E) - g_2(E) \qquad (3.15)$$

Let us assume that the integral:

$$\int_M^{\infty} \left(\frac{h(E)}{E}\right)^2 dE^2 = 2 \int_M^{\infty} \frac{h(E)^2}{E} dE \qquad (3.16)$$

exists. Hence the function $\frac{h(E)}{E}$, where h(E) is defined by (3.7) for $E^2 \geqslant E_o^2$ and $h(E) = 0$ for $E^2 < E_o^2$, belongs to L^2 $(-\infty, \infty)$ in the variable E^2. Then according to a fundamental theorem on Hilbert transforms [9], $g_1(E)/E$ also belongs to $L^2(-\infty, \infty)$, that is:

$$\int_{-\infty}^{+\infty} \left(\frac{g_1(E)}{E}\right)^2 dE^2 = 4 \int_0^{+\infty} g_1(E)^2 \frac{dE}{E} \qquad (3.17)$$

exists, which implies that $g_1(E)^2 \longrightarrow 0$ "in mean", for $E \longrightarrow \infty$; but, it does not necessarily follow that $g_1(E)$ is bounded*. How

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* Since h(E) is a continuous function of E, $g_1(E)$ is continuous almost everywhere and the points for which $|g_1(E)| > \delta$, for $E > E_o$, form a set of intervals. The length of any such an interval tends to zero when $E_o \to \infty$. As an example consider the function:

$$h(E) = 0, \ (E<M); \ h(E) = \left(\frac{M}{E}\right)^p \left[\left(\frac{E}{M} - n\right)\left(n + 1 - \frac{E}{M}\right)\right]^{1/n(n+1)}, \ \left(n < \frac{E}{M} < n+1\right)$$

which is continuous and positive in the interval (M, ∞) and tends to zero when $E \longrightarrow \infty$. Now:

$$g(E) = \frac{1}{\pi} \int_M^{\infty} dE' \frac{h(E)}{E'-E}$$

behaves, for large E, like $\sim\left(\frac{M}{E}\right)^p$ almost everywhere, but for $E = n M$, $g(E) \sim n^{1-p}$. For $\frac{1}{2} < p < 1$, $g(E)$ is not bounded but since $\int_M^{\infty} h(E')^2 dE'$ exists, then $\int_{-\infty}^{+\infty} g(E')^2 dE'$ also exists, that is $g(E) \longrightarrow 0$ "in mean".

ever if one assumes, as before, that $h(E)$ satisfies a Lipschitz condition, then $g_1(E)$ also satisfies a Lipschitz condition [10] and in this case the convergence of the integral (3.17) plainly implies that $g_1(E) \rightarrow 0$.

On the other hand, for $g_2(E)$ we write:

$$g_2(E) - \frac{2}{\pi}\int_M^E dE' \frac{h(E')}{E'} = \frac{2}{\pi}\int_E^\infty dE' \, h(E') \left(\frac{1}{E'} - \frac{1}{E'+E}\right) - \frac{2}{\pi}\int_M^E dE' \frac{h(E')}{E'+E} \tag{3.18}$$

Since $h(E)$ is bounded then:

$$g_2(E) - \frac{2}{\pi}\int_M^E dE' \frac{h(E')}{E'} = O(1) \tag{3.19}$$

When $\lim_{\epsilon \to \infty} (\sigma_+(E) - \sigma_-(E))$ exists, one can actually show, without difficulty that (3.19) is $O(1)$. Then the boundedness of (3.14) is a necessary and sufficient condition for $g_2(E)$ to be bounded.

We have thus proved the following theorem:

Theorem 2. "If the integral (3.14) is bounded and (3.16) exists, $h(E)$ being given by (3.7) and satisfies a Lipschitz condition of the form (3.9) then $G_-(E)/E$ is bounded:"

We remark that a sufficient condition for the existence of (3.16) is that $|h(E) \ln E|$ be bounded, that is:

$$(\sigma_+(E) - \sigma_-(E)) = 0\left[(\ln E)^{-1}\right].$$

So far we have dealt with the odd combination of amplitudes $G_-(E)$.

Let us turn to the even combination $G_+(E)$. An inspection of (3.4) shows that the asymptotic behaviour of $G_+(E)/E$ is determined by the behaviour of the last integral. Let us assume that $(\sigma_+(E) + \sigma_-(E))$ approach a limit 2σ. One can write:

$$G_+(E)/E = \frac{\text{[illegible]}}{4\pi^2} \int_M^\infty \frac{p'}{E'} \frac{dE'}{E'^2 - E^2} (\sigma_+(E') + \sigma_-(E') - 2\sigma) + 0(E^{-1}) \quad (3.20)$$

where the integral is of the same form as in $g_1(E)$. Therefore by the same arguments as used before [9,10], if $(\sigma_+(E) + \sigma_-(E))$ approaches the limit 2σ in such a way that:

$$\int_M^\infty \frac{p'}{E'} (\sigma_+(E') + \sigma_-(E') - 2\sigma)^2 \frac{dE'}{E'} \quad (3.21)$$

exists, and the integrand satisfies a Lipschitz condition then $G_+(E)/E \to 0$ as $E \to \infty$. Hence:

<u>Theorem 3</u>. "If $(\sigma_+(E) + \sigma_-(E))$ satisfies a Lipschitz condition of the form (3.9), and approaches a limit 2σ so that the integral (3.21) exists, then $G_+(E)/E \to 0$ when $E \to \infty$."

One should emphazise the fact the one cannot prove a similar result for the difference of the dispersive amplitudes. Actually it would be in no way inconsistent with dispersion relations to assume that $G_-(E)/E$ remained finite. On the other hand if one believes that at high energies the scattering amplitude becomes purely absorptive and since $(\sigma_+(E) - \sigma_-(E) \to 0$, then one can write down the dispersion relation for $G_-(E)/E$, without subtractions. This unsubtracted form seems to fit well with experimental results. One can understand why it is the sum of the dispersive amplitudes which can be shown to vanish at infinite energy, whereas it is the difference of the absorptive parts which display this behaviour. The reason stems from the fact that dispersion relations are satisfied by the causal rather than by the Feynman amplitudes. Now, for the causal amplitudes crossing symmetry gives:

$$f_-^c(E) = f_+^c(-E)^* \tag{3.22}$$

whereas for the Feynman amplitudes one has:

$$f_-(E) = f_+(-E). \tag{3.23}$$

Recalling that in the physical region for a given process the causal and Feynman amplitudes coincide one has:

$$D_+^c(-E) \pm D_-^c(-E) = \pm (D_+(E) \pm D_-(E))$$

$$A_+^c(-E) \pm A_-^c(-E) = \mp (A_+(E) \pm A_-(E)). \tag{3.24}$$

The results so far obtained for forward amplitudes may be immediately generalized to the case of fixed momentum transfer. Dispersion relations for fixed, limited values of the momentum transfer ($|t| < t_1$) have been rigorously proved for certain processes, as for instance, pion-nucleon scattering. The assumption of Mandelstam's representation for a given elementary process, enables one to derive dispersion relations for fixed values of the momentum transfer on the whole complex plane, with branch cuts on the real axis. Introducing the variable:

$$\xi = E + \frac{t}{4M} \qquad (3.25)$$

one can write down dispersion relations for processes I and II with fixed, real momentum transfer, in the form:

$$D_{\pm}(\xi) = D(0) \pm \xi D'(0) + \frac{\xi^2}{\pi} \int_{-M+4\mu^2/2M+t/4M}^{M+t/4M} \frac{d\xi'}{\xi'^2} \frac{A_-(\xi',t)}{\xi' \pm \xi} +$$

$$+ \frac{\xi^2}{\pi} \int_{M+t/4M}^{\infty} \frac{d\xi'}{\xi'^2} \left(\frac{A_+(\xi',t)}{\xi' \mp \xi} + \frac{A_-(\xi',t)}{\xi' \pm \xi} \right) \qquad (3.26)$$

where we have made two subtractions (at $\xi = 0$) and the integrals are again principal values.* Contrariwise to the case of forward amplitudes, the last integral here, has also an unplysical region for values of ξ' below the physical threshold $\xi_0 = M - \frac{t}{4M}$. Moreover, if $|t|$ is sufficiently large the dispersive and absorptive amplitudes are no longer real for all values of ξ. But, for $\xi \geq \xi_0$ (or $\xi' \geq \xi_0$ under the integral), they are, of course, real and physical. Therefore, in the high

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* A dispersion relation of this form holds, for instance, for a pair of amplitudes $f_+(s, u, t)$ and $f_-(u, s, t)$, which, in terms of the covariant amplitudes defined in (A.4), are given by:

$$f_+ = \frac{1}{\pi} \left[M^2 (F_1 + F_2 + F_3 + F_4) - S\, F_3 \right]$$

$$f_- = \frac{1}{\pi} \left[M^2 (\bar{F}_1 + 2\bar{F}_2 - 2\bar{F}_3 - \bar{F}_5) + \frac{u}{4} (-\bar{F}_1 + 2\bar{F}_3 + 2\bar{F}_4 + \bar{F}_5) \right]$$

For forward scattering they coincide with $(\varphi_2 - \varphi_1)$ and $(\bar{\varphi}_4 - \bar{\varphi}_3)$.

energy region, once the condition $\xi \gg \xi_0$ is satisfied, one can apply the same arguments used for forward scattering, to show that:

Theorem 4. "If, for fixed t, the physical amplitudes are bounded and the absorptive amplitudes tend to a limit when $E \to \infty$, then this limit is the same for the scattering, on a given target, of particle and anti-particle, with the same momentum transfer (and spins reversed)". All these results were already obtained at the end of last section (Sec. 2, Th. 4), from the consideration of the smallness of inelastic channels as compared with the truly elastic ones. It is remarkable that one can also derive them from quite a different standpoint, based on the independent assumption of boundedness of the scattering amplitude and the validity of dispersion relations. It should be pointed out, that the assumptions involved here, from the physical point of view weaker than those required in the former treatment.

Let us, finally, take into account the isotopic spin invariance of the scattering matrix. It will be convenient to reformulate the preceding results in terms of cross sections and amplitudes for transitions in states of given total isotopic spin. Let us consider the processes:

$$\text{Ia)}\quad A_1 + A_2 \longrightarrow A_3 + A_4$$

$$\text{IIa)}\quad A_1 + \bar{A}_4 \longrightarrow A_3 + \bar{A}_2$$

where A_1, A_2, A_3, A_4 belong to different multiplets with isotopic spins I_1, I_2, I_3, I_4, respectively. Let f^I and $\bar{f}^I$ denote the amplitudes for transition Ia) and IIa) in states of total isotopic spin I. The crossing relations for these amplitudes, with fixed momentum transfer between the pair (A_1, A_3) are derived in Appendix B. The result is: *

$$f^I_{\mu_1\mu_2\mu_3\mu_4}(S,\mu,t) = \sum_{I'} O_{II'}\, \bar{f}_{\mu_1-\mu_4\mu_3-\mu_2}(\mu,S,t) \tag{3.28}$$

where:

$$O_{II'} = (2I'+1)(-1)^{2(I_1+I_2)} \begin{Bmatrix} I_1 & I_2 & I \\ I_3 & I_4 & I' \end{Bmatrix} \tag{3.29}$$

and $\begin{Bmatrix} I_1 & I_2 & I \\ I_3 & I_4 & I' \end{Bmatrix}$ is the six-j symbol[12]. If $I_2 = I_4$ as in the case of elastic scattering, $O_{II'}$ satisfies the orthogonality condition:

$$\sum_I O_{I''I}\, O_{II'} = \delta_{I''I'} \tag{3.30}$$

Using the unitarity relation between the total cross section and the imaginary part of the forward elastic amplitude, equation (3.28) enables us to formulate theorem 1 in the following way[5]:

<u>Theorem 5</u>. "The total cross sections for reactions (A_1, A_2) and $(A_1, \bar{A}_2)$, where A_1 and A_2 belong, respectively, to multiplets with isotopic spin I_1 and I_2, are, in the asymptotic limit $E \longrightarrow \infty$, related by:

$$\sigma^I_{A_1A_2}(\lambda_1,\lambda_2) = \sum_{I'} (2I'+1)(-1)^{2(I_1+I_2)} \begin{Bmatrix} I_1 & I_2 & I \\ I_1 & I_2 & I' \end{Bmatrix} \sigma^{I'}_{A_1\bar{A}_2}(\lambda_1,-\lambda_2) \tag{3.31}$$

where the helicities refer to the center of mass system."

* The interchange $S \rightleftarrows u$, leads to $\xi \longrightarrow -\xi$.

An analogous formulation may be given for theorem 4.

4. GENERAL ASYMPTOTIC PROPERTIES OF THE SCATTERING AMPLITUDES

The method of dispersion relations has proved very useful and extensively used, for the analysis of elementary processes, giving definite information on the nature and strength of strong interactions. Its subsequent and fundamental development into the theory of the Mandelstam representation opened the possibility of a dynamical description of the scattering in terms of the analytical properties of the scattering amplitudes and a few coupling parameters. However, it was soon realized, from the begining, that a major difficulty which always arise in connection with the application of this method is the question of subtractions required for convergence of the dispersion integrals. This question is directly related to the asymptotic behaviour of the amplitudes. The problem was first tackled in relativistic field theory by Froissart [13]. Assuming Mandelstam's representation, he derived bounds imposed by unitarity on the asymptotic behaviour of forward and non forward amplitudes. Later the problem was more generally discussed by Martin [14] who considered the limitations imposed on the scattering amplitudes by the requirement of unitarity alone.

In this section we shall essentially follow Martin's algebraic approach in the investigation of some restrictions imposed by unitarity on the elastic scattering amplitude and its asymptotic behaviour.

For simplicity we take the scattering of two scalar particles. Let $|\alpha_i\rangle$ and $|\alpha_f\rangle$ be the initial and final state of an elastic collision with total energy $\sqrt{S}$ and momentum transfer $-t$. The elastic scattering amplitude is:

$$f(s,t) = \frac{(2p_{10}2p_{20}2p_{10}'2p_{20}')^{\frac{1}{2}}}{8\pi} \langle\alpha_f|T|\alpha_i\rangle \qquad (4.1)$$

where p_1, p_2 are the initial momenta and p_1', p_2' the final momenta of particles 1 and 2 respectively.

The unitarity conditions gives:

$$\mathrm{Im}\, f(s,t) = \frac{1}{16\pi} (2p_{10}^2\ 2p_{20}^2\ 2p_{10}'^2\ 2p_{20}'^2)^{\frac{1}{2}}$$

$$\sum_n \langle\alpha_f|T^+|n\rangle \langle n|T|\alpha_i\rangle\ (2\pi)^4\ \delta(p_n - p_i) \qquad (4.2)$$

where the sum is extended to all open channels in the intermediate states. Taking absolute value one has:

$$|\mathrm{Im}\, f(s,t)| \leqslant \frac{1}{16\pi} (2p_{10}^2\ 2p_{20}^2\ 2p_{10}'^2\ 2p_{20}'^2)^{\frac{1}{2}}$$

$$\sum_n | \langle\alpha_f|T^+|n\rangle \langle n|T|\alpha_i\rangle |(2\pi)^4\ \delta(p_n - p_i) \qquad (4.3)$$

and applying Schwartz inequality to the terms in the sum one obtains:

$$|\mathrm{Im}\, f(s,t)| \leqslant \frac{1}{16\pi} (2p_{10}^2\ 2p_{20}^2\ 2p_{10}'^2\ 2p_{20}'^2)^{\frac{1}{2}} \sum_n \frac{1}{2} (|\langle\alpha_f|T^+|n\rangle|^2 +$$

$$+ |\langle n|T|\alpha_i\rangle|^2) \times (2\pi)^4\ \delta(p_n - p_i)$$

$$= \frac{1}{16\pi} (2p_{10}^2\ 2p_{20}^2\ 2p_{10}'^2\ 2p_{20}'^2)^{\frac{1}{2}} \sum_n |\langle n|T|\alpha_i\rangle|^2 (2\pi)^4 \delta(p_n - p_i) = \text{Im } f(s,0) \tag{4.4}$$

Therefore our first result is:

Theorem 1. - For all physical values of s and t one has:

$$|\text{Im } f(s,t) \leqslant \text{Im } f(s,0) \tag{4.5}$$

Next, in the forward direction $t = 0$, in the center of mass system, (4.2) gives:

$$|\text{Im } f(s,0)| = \frac{1}{16\pi}(2p_{10}2p_{20})\sum|\langle\alpha_n|T|\alpha_i\rangle|^2(2\pi)^4\delta(p_n-p_i) + \text{inelastic}$$

$$\geqslant \frac{1}{4\pi}\frac{k}{\sqrt{S}}\int|\langle\alpha_n|T|\alpha_i\rangle|^2\left(\frac{p_{10}p_{20}}{2\pi}\right)^2 d\Omega = \frac{1}{4\pi}\frac{k}{\sqrt{S}}\int|f(s,t)|^2 d\Omega$$

$$\geqslant \frac{1}{4\pi}\frac{k}{\sqrt{S}}\int(\text{Im } f(s,t))^2 d\Omega = \frac{k}{2\sqrt{S}}\int_{-4k^2}^{0}(\text{Im } f(s,t))^2\frac{dt}{2k^2} \tag{4.6}$$

In the last step we have used the relation (2.4) between the scattering angle and the momentum transfer.

Let us now write:

$$|\text{Im } f(s,t)| = a^2 k\sqrt{S}\left(\frac{s}{s_o}\right)^{\alpha(s,t)} \tag{4.7}$$

where a is the s-wave scattering lenght and $s_o = (m_1 + m_2)^2$. From theorem 4.1 it follows that, for all physical values of s and t:

$$\alpha(s,t) \leqslant \alpha(s,0) \tag{4.8}$$

Taking (4.7) into (4.6) one obtains:

$$\left(\frac{s}{s_o}\right)^{\alpha(s,0)} \geqslant \frac{a^2k^2}{2}\int_{-4k^2}^{0}\left(\frac{s}{s_o}\right)^{2\alpha(s,t)}\frac{dt}{2k^2}$$

or

$$1 \geqslant \frac{a^2}{4}\left(\frac{s}{s_o}\right)^{\alpha(s,0)} \int_{-4k^2}^{0} \left(\frac{s}{s_o}\right)^{2[\alpha(s,t)-\alpha(s,0)]} dt \qquad (4.9)$$

Let us assume that for sufficiently large s, say $s > s'$, $\alpha(st)$ converges uniformly, with respect to s, to $\alpha(s,0)$, as $t \to 0^-$. Then given an ϵ one can find a positive to, <u>independent of s</u>, such that for $-4\,to < t < 0$, one has:

$$\alpha(s,0) - \alpha(s,t) < \frac{\epsilon}{4} \qquad (4.10)$$

Then (4.9) gives

$$1 \geqslant \frac{a^2}{4}\left(\frac{s}{s_o}\right)^{\alpha(s,0)} \int_{-4t_o}^{0} \left(\frac{s}{s_o}\right)^{2[\alpha(s,t)-\alpha(s,0)]} dt$$

$$> \frac{a^2}{4}\left(\frac{s}{s_o}\right)^{\alpha(s,0)} \int_{-4t_o}^{0} \left(\frac{s}{s_o}\right)^{-\frac{\epsilon}{2}} dt = a^2\, t_o \left(\frac{s}{s_o}\right)^{\alpha(s,0)-\frac{\epsilon}{2}} \qquad (4.11)$$

Hence

$$\alpha(s,0) - \frac{\epsilon}{2} < -\ln\,(a^2\, t_o)/\ln\,(s/s_o) \qquad (4.12)$$

Now one can find an $s'' > s'$ such that for $s > s''$ the right hand side of (4.12) becomes less than $\frac{\epsilon}{2}$. Then for $s > s''$:

$$\alpha(s,0) < \epsilon \qquad (4.13)$$

This result may be expressed in the following way:

Theorem 4.2.- "If for sufficiently large s,

$$\ln|\operatorname{Im} f(s,t)|/\ln(s/s_o)$$

converges uniformly to $\ln \operatorname{Im} f(s,0)/\ln\,(s/s_o)$ as $t \to 0^-$, then

$\frac{\text{Im } f(s,t)}{s}$ is bounded by any positive power of s, however small."
As an immediate consequence of this theorem we have:

Corolary. "If Im f(s,t) has for large s, Regge's asymptotic behaviour $\beta(t)\left(\frac{s}{s_o}\right)^{\alpha(t)}$ where $\beta(t)$ and $\alpha(t)$ are continuous functions of t, then $\alpha(0) \leqslant 1$."

This important result follows from unitarity alone without any assumption about the analytic properties of Im f(s,t). It was obtained by Martin, using the partial wave espansion of the scattering amplitude. It has also been derived by Froissart, but starting from the Mandelstam representation, therefore, under the restrictive assumption of analicity.

Let us now consider the partial wave expansion:

$$f(s,t) = \frac{\sqrt{s}}{k}\sum_{\ell}(2\ell+1)\, f_{\ell}(s)\, P_{\ell}(\cos\theta) \qquad (4.14)$$

and
$$\text{Im } f(s,t) = \frac{\sqrt{s}}{k}\sum_{\ell}(2\ell+1)\, a_{\ell}(s)\, P_{\ell}(\cos\theta) \qquad (4.15)$$

where $a_{\ell}(s) = \text{Im } f_{\ell}(s)$. Unitarity for partial waves implies that:

$$|f_{\ell}|^2 \leqslant \text{Im } f_{\ell} \leqslant 1 \qquad (4.16)$$

Since $|P_{\ell}(\cos\theta)| \leqslant 1$, theorem 1 can readily be obtained from (4.15). Let us investigate the behaviour of Im f(s,t) near t=0. Taking logarithmic derivative of (4.15) one obtain:

$$\frac{d}{dt}\ln \text{Im } f(s,t)\Big|_{t=0} = \frac{1}{\text{Im } f(s,0)}\, \frac{\sqrt{s}}{k}\sum_{\ell}(2\ell+1)\, a_{\ell}(s)\, \frac{\ell(\ell+1)}{2}\, \frac{1}{2k^2} \qquad (4.17)$$

where:

$$\text{Im } f(s,0) = \frac{\sqrt{s}}{k} \sum (2\ell+1)\, a_\ell(s) \tag{4.18}$$

The expression (4.17) is positive definite as expected from (4.15). Since the factor $\frac{\ell(\ell+1)}{2}$ is an increasing function of ℓ, it is clear that, for a given s, a lower bound of (4.17), with a subject to the condition (4.16), is obtained when:

$$\begin{aligned} a_\ell &= 1 & \ell &< L \\ a_L &< 1 & \ell &= L \\ a_L &= 0 & \ell &> L \end{aligned} \tag{4.19}$$

where L and a_L are determined by (4.18), which gives:

$$\text{Im } f(s,0) = \frac{\sqrt{s}}{k}\left(\sum_{\ell=0}^{L-1} (2\ell+1) + (2L+1)\, a_L = \frac{\sqrt{s}}{k}\,(L^2 + (2L+1)a_L) = \frac{\sqrt{s}}{k}\,\bar{L}^2\right. \tag{4.20}$$

where:

$$(2L+1)\, a_L = \bar{L}^2 - L^2 = (\bar{L} - L)(\bar{L} + L). \tag{4.21}$$

The sum in (4.17) gives:

$$\begin{aligned} \sum_\ell (2\ell+1)a_\ell \frac{\ell(\ell+1)}{2} &= \sum_{\ell=0}^{L-1} \left[\left(\ell + \frac{1}{2}\right)^3 - \frac{1}{4}\left(\ell + \frac{1}{2}\right)\right] + L\left(L + \frac{1}{2}\right)(L+1)a_L = \\ &= \frac{1}{4} L^2(L^2 - 1) + L\left(L + \frac{1}{2}\right)(L+1)a_L \\ &\leqslant \frac{1}{4}\bar{L}^2(\bar{L}^2 - 1) \end{aligned} \tag{4.22}$$

The last step can easily be proved with the help of (4.21). Taking (4.20) and (4.22) into (4.17) and using the optical relation:

$$\frac{k}{\sqrt{s}}\, \text{Im } f(s,0) = k^2 \frac{\sigma(s)}{4\pi} \tag{4.23}$$

where $\sigma(s)$ is the total cross section, one obtains:

$$\frac{d}{dt} \ln \operatorname{Im} f(s,t)\Big|_{t=0} \geq \frac{1}{8}\left(\frac{\sigma}{4\pi} - \frac{1}{k^2}\right) \qquad (4.24)$$

which gives a lower bound of (see (4.7)):

$$\frac{d}{dt} \ln \operatorname{Im} f(s,t)\Big|_{t=0} = \ln\left(\frac{s}{s_o}\right)\alpha'(s,0) \qquad (4.25)$$

provided that $\sigma > \frac{4\pi}{k^2}$. At high energies this expression is the inverse of the width of the diffraction peak. Therefore (4.23) gives a lower bound for the inverse of the width of the diffraction peak at a given energy in terms of the total cross section.

Let us now derive an upper bound for

$$R(s,t) = \frac{|\operatorname{Im} f(s,t)|}{\operatorname{Im} f(s,0)}$$

From (4.15) one obtains:

$$|\operatorname{Im} f(s,t)| \leq \frac{\sqrt{s}}{k} \sum_{\ell} (2\ell+1)\, a_{\ell}(s) |P_{\ell}(\cos\theta)| \qquad (4.26)$$

But the Legendre polynomials are bounded by:

$$|P_{\ell}(\cos\theta)| \leq \sqrt{\frac{2}{\pi\left(\ell+\frac{1}{2}\right)\operatorname{sen}\theta}} \qquad (4.27)$$

Since this bound is a decreasing function of ℓ, then, by an argument enterily similar to that used before, an upper bound of R(s,t) is obtained when one replaces (4.27) into (4.25) and takes the a_ℓ's given by (4.19). Transforming the sum into an integral one obtains:

$$|\text{Im } f(s,t)| \leqslant \frac{\sqrt{s}}{k}\sqrt{\frac{2}{\pi L \text{ sen } \theta}}\left\{\frac{4}{3}\left(L-\frac{1}{2}\right)^{3/2}+\left(L-\frac{1}{2}\right)^{\frac{1}{2}}+\frac{1}{3}+ + 2\left(L+\frac{1}{2}\right)^{\frac{1}{2}} a_\ell\right\} \quad (4.28)$$

At high energies L is large and (4.28) gives:

$$R(s,t) \leqslant \frac{4}{3}\sqrt{\frac{2}{\pi L \text{ sen } \theta}} \quad (4.29)$$

Using (4.20) and (4.23) one obtains:

$$R(s,t) \leqslant \frac{4}{3}\sqrt{\frac{2}{\pi}}\left(\frac{\sigma}{4\pi} k \text{ sen}\theta\right)^{-\frac{1}{4}} \quad (4.30)$$

$$\simeq \frac{4}{3}\sqrt{\frac{2}{\pi}}\left|\frac{\sigma t}{4\pi}\right|^{-1/4},$$

Since at high energies and fixed momentum transfer θ is small and $|t| \simeq k^2 \text{ sen}^2\theta$. Taking $|\text{Im } f(s,t)|$ as given by (4.7) one has:

$$\left(\frac{s}{s_o}\right)^{\alpha(s,t)-\alpha(s,0)} \leqslant \frac{4}{3}\sqrt{\frac{2}{\pi}}\left|\frac{\sigma t}{4\pi}\right|^{-1/4} \quad (4.31)$$

and if the conditions of theorem 2 are fulfilled then:

$$\sigma \leqslant \frac{4\pi}{|t|}\left(\frac{4}{3}\sqrt{\frac{2}{\pi}}\right)^4\left(\frac{s}{s_o}\right)^{2c} \quad (4.32)$$

that is, the total cross section cannot increase as fast as any positive power of s, which is again the result expressed in theorem 2. If the total cross section σ (s) remain finite as the energy increases, (4.30) will not be a useful bound for small values of t. One has, in this case, to take for the Legendre polynomials, at small angles, a better found than (4.27). Martin has discovered that the function:

$$B_\ell(\cos\theta) = \left[1 + \ell(\ell+1)\sin\theta\right]^{-1/4} \qquad (4.33)$$

is an upper bound for the Legendre polynomial $P_\ell(\cos\theta)$, which, like (4.27), is also a decreasing function of ℓ. It has the following properties:

i) $B_\ell(\cos\theta) \geqslant |P_\ell(\cos\theta)|$

ii) $B_\ell(1) = P_\ell(1) = 1$

iii) $B_\ell'(1) = P_\ell'(1) = \frac{1}{2}\,\ell(\ell+1)$

iv) $B_{\ell_1}(\cos\theta) \geqslant B_{\ell_2}(\cos\theta)$ for $\ell_2 > \ell_1$

v) $B_\ell(\cos\theta) \sim \frac{1}{\sqrt{\ell\sin\theta}}$ for $\ell\sin\theta \to \infty$.

Hence $B_\ell(\cos\theta)$ is a tight bound of $P_\ell(\cos\theta)$ for small θ and, for large ℓ, has, apart from a constant factor, the same asymptotic behaviour as (4.27).

Since $B_\ell(\cos\theta)$ is a decreasing function of ℓ, then upon substitution of $|P_\ell|$ by B_ℓ in (4.26). $|\mathrm{Im}\, f(s,t)|$ would be maximum if the a_ℓ's are given by (4.19). Replacing then the sum by an integral one obtains:

$$|\mathrm{Im}\, f(s,t)| \leqslant \frac{\sqrt{s}}{k}\left\{ \frac{4}{3}\left(\left[1 + L(L-1)\sin^2\theta\right]^{3/4} - 1\right)/\sin^2\theta + \right.$$

$$\left. + \frac{1}{2}\left[(2L-1)B_{L-1} + 1\right] + (2L+1)\, a_L\, B_L \right\} \qquad (4.34)$$

wherefrom one can show, after some algebraic manipulation, that:

$$R(s,t) \leqslant \frac{4}{3}\left(\left[1+\bar{L}(\bar{L}-1)\sin^2\theta\right]^{3/4}-1\right)/\bar{L}(\bar{L}-1)\sin^2\theta \qquad (4.35)$$

which is Martin's result. For high energies and fixed momentum transfer (θ small) one has from (4.20) and (4.23):

$$\bar{L}(\bar{L}-1)\,\sin^2\theta \simeq \frac{\sigma(s)}{4\pi}\,k^2\,\sin^2\theta \quad \frac{\sigma(s)}{4\pi}\,|t|$$

Therefore (4.35) becomes:

$$R(s,t) \leqslant \frac{4}{3}\left(\left[1+\frac{\sigma(s)|t|}{4\pi}\right]^{3/4}-1\right)\Big/\frac{\sigma(s)|t|}{4\pi} \qquad (4.36)$$

In contrast with (4.30), the bounds (4.35) or (4.36) approach one, when θ or t tends to zero.

Thus far we have made no use of the analytical properties of the scattering amplitude. Let us proceed further by investigating the limitations on the high energy behaviour of the scattering amplitude which result from analiticity in momentum transfer. We shall assume that f(s,t) is analytic inside an ellipse in the $\cos\theta$ - plane, with focci at $\cos\theta = \pm 1$ and semi-major axis $a = 1 + c/2k^{2(n+1)}$. The Legendre polynomial expansion is convergent for $z = \cos\theta$ inside the ellipse and may be used to define the function f(s,t) in this domain. Then for t real and in the interval $0 < t < t_o = c/k^{2n}$ one can write:

$$\mathrm{Im}\, f(s,t) = \frac{\sqrt{s}}{k}\sum\,(2\ell+1)\ \mathrm{Im}\, f_\ell(s)\ P_\ell\left(1+\frac{t}{2k^2}\right) \qquad (4.37)$$

Let us assume, as usual, that Im f(s,t) is bounded by a polynomial N(s). Since $P_\ell(z)$ is positive definite for z real and larger than one, it immediatly follows from (4.37) that:

$$\text{Im } f_\ell < \frac{1}{2} \frac{N(s)}{(2\ell+1)\, P_\ell(1+t_o/2k^2)} \qquad (4.38)$$

But for $x > 1$ we have [16]

$$P_\ell(x) > \sqrt{\frac{2}{\pi(2\ell+1)}}\,(x+\sqrt{x^2-1})^\ell$$

Hence $\text{Im } f_\ell$ is bounded by

$$\text{Im } f_\ell < \frac{N(s)}{4}\sqrt{\frac{\pi}{\ell+1/2}}\, U^{-\ell} \qquad (4.39)$$

where

$$U = x_o + \sqrt{x_o^2 - 1} \qquad (4.40)$$

and

$$x_o = 1 + t_o/2k^2 = 1 + c/2k^{2(n+1)} \qquad (4.41)$$

From (4.16) one obtains:

$$|f_\ell| < \frac{\sqrt{N(s)}}{2}\left(\frac{\pi}{\ell+1/2}\right)^{1/4} U^{-\ell/2} \qquad (4.42)$$

Relations (4.39) and (4.41) exhibit the exponential decrease of the partial wave amplitudes. They are effective for $\ell \gg L$ where L is determined by the condition:

$$\frac{N(s)}{4}\sqrt{\frac{\pi}{L+\frac{1}{2}}}\, U^{-L} = 1 \qquad (4.43)$$

For simplicity we shall take for L:

$$N(s)\, U^{-L} = 1 \qquad (4.44)$$

or

$$L = \ln N(s)/\ln U \qquad (4.45)$$

One can use these results to deduce bounds for the amplitude in the physical region. We have:

$$|f(s,t)| \leqslant \frac{\sqrt{s}}{k} \sum (2\ell+1)\, f_\ell(s)|\,|P_\ell(\cos\theta)|$$

$$\leqslant \frac{\sqrt{s}}{k}\sqrt{\frac{2}{\pi \sin\theta}} \left\{ \sum_{\ell=0}^{L-1} 2\left(\ell+\frac{1}{2}\right)^{\frac{1}{2}} + \sum_{\ell=L}^{\infty} \left(\ell+\frac{1}{2}\right)^{\frac{1}{2}} N^{\frac{1}{2}} \left(\frac{\pi}{\ell+\frac{1}{2}}\right)^{1/4} U^{-\ell/2} \right\} \tag{4.46}$$

where we have used the bound (4.27) for $P_\ell(\cos\theta)$. For large s, $u \simeq 1 + \sqrt{t_o}/k$, and (4.45) gives:

$$L \simeq \ln N / \ln\,(1 + \sqrt{t_o}/k) \simeq \frac{k \ln N}{\sqrt{t_o}}$$

Hence the two sums in (4.46) will asymptotically give:

$$\sum_{\ell=0}^{L-1} 2\left(\ell+\frac{1}{2}\right)^{\frac{1}{2}} \simeq \frac{4}{3} L^{3/2} \simeq \frac{4}{3} \frac{k^{3/2} \ln^{3/2} N}{t_o^{3/4}}$$

and

$$\sum_{\ell=L}^{\infty} \left(\ell+\frac{1}{2}\right)^{\frac{1}{2}} N^{\frac{1}{2}} \left(\frac{\pi}{\ell+\frac{1}{2}}\right)^{1/4} U^{-\ell/2} < \frac{\pi^{1/4} N^{1/2}}{\left(L+\frac{1}{2}\right)^{3/4}} \sum_{\ell=L}^{\infty} \left(\ell+\frac{1}{2}\right) U^{-\ell/2} =$$

$$= \pi^{1/4} \left(L+\frac{1}{2}\right)^{-3/4} \left(\frac{L-\frac{1}{2}}{\sqrt{U}-1} + \frac{\sqrt{U}}{(\sqrt{U}-1)^2} + \frac{1}{2} \right) \simeq 2\pi^{1/4} \left(\frac{k^2}{t_o}\right)^{s/8} \ln^{1/4} N\left(1 + O(\ln^{-1} N)\right)$$

Therefore $|f(s,t)|$ is asymptotically bounded by:

$$|f(s,t)| < \frac{1}{\sqrt{\pi \sin\theta}} \frac{4}{3} \left(\frac{s}{t_o}\right)^{3/4} \ln^{3/2} N(s) =$$

$$= \frac{C_1}{\sqrt{\sin\theta}} \left(\frac{s}{t_o}\right)^{3/4} \left(\ln^{3/2} s + O(\ln^{1/2} s) \right) \tag{4.48}$$

where c_1 is a constant. One can obtain a similar bound for $|f(s,0)|$, from:

$$|f(s,0)| \leqslant \frac{\sqrt{s}}{k} \sum (2l+1) | f_l(s)|$$

The result is:

$$|f(s,0)| < \frac{s}{2t_o} \ln^2 N = C_2 \frac{s}{t_o} \ln^2 s \qquad (4.49)$$

The bound (4.48) applies to the amplitude for scattering at angles $\theta \neq 0$ or π. One can rewrite (4.48) in terms of fix momentum transfer t:

$$|f(s,t)| < C_1' \frac{s}{|t|^{1/4} \; t_o^{3/4}} \ln^{3/2} s \qquad (4.50)$$

Evidently (4.49) is also a bound for $|f(s,t)|$.

In (4.48, 49, 50), we have explicitly exhibit the dependence on to, which is inversely proportional to k^{2n}. In field theory analiticity in the $\cos\theta$ - plane has only been proved inside the Lheman ellipses [17] whose semi-major axis behave, for large s, like $1 + C/2k^2 s$. This gives $t_o = c/s$. In this case the forward amplitude could increase as fast as $\sim s^2 \ln^2 s$ and the total cross section could behave like $\sim s \ln^2 s$. On the other hand the assumptions of analyticity in momentum transfer and the boundedness of Im f(s,t) are both contained in the Mandelstam representation, in which $t_o = 4\mu^2$ is the square of the total mass of the least massive state in the crossed channel. Thus Mandelstam's representation [18] implies the limitations (4.48-50) with $t_o = 4\mu^2$ for the asymptotic behaviour of the scattering amplitude, and from unitarity and (4.49) the total cross section will be bounded by:

$$\sigma(s) < \frac{4\pi}{t_0} \ln^2 N = \frac{C_3}{t_0} \ln^2 s \qquad (4.51)$$

that is, it cannot increase faster than $\ln^2 s$.

These results were first obtained by Froissart, assuming Mandelstam's representation. Actually, as observed by Greenberg and Low [19] and is apparent from the above deduction, one does not need analyticity in the whole t cut-plane, but only inside an ellipse which collapses to the real axis $(-\infty, 0)$ by terms of order $1/k^2$.

5. MANDELSTAM'S REPRESENTATION AND ASYMPTOTIC BEHAVIOUR

In this section we shall be concerned with the restrictions upon the spectral function in Mandelstam's representation which obtain from the consideration of asymptotic properties and unitarity.

In the first place we consider the following problem: As one writes down Mandelstam's representation for the scattering amplitude a certain number of subtractions has to be made in order to ensure the convergence of the double integrals. In so doing a number of single spectral functions are introduced in the representation. Now one might ask whether these single spectral functions could be chosen arbitrarily or to what extent are they determined by the double spectral functions. Let us write, following Froissart [13], the general form of the Mandelstam representation for the amplitude f(s,t,u), with an arbitrary number of subtractions sufficient for convergence of all the integrals:

$$f(s,t,u) = \frac{1}{\pi^2} \iint \frac{S^N \, t^N}{S'^N t'^N} \, \frac{P_{13}(S',t')}{(S'-S)(t'-t} \, ds' \, dt' + P_{s\,t\,u}$$

$$+ \sum_{p=0}^{M} \frac{t^p}{\pi} \int \frac{S^M}{S'^M} \, \frac{\sigma_{1p}(s')}{s'-s} \, ds' + P_{s\,tu} + \sum_{p_1 q=0}^{L} C_{pq} \, t^p \, s^q \qquad (5.1)$$

where N, M, L are arbitrary, sufficiently large integers, C_{pq} are constant coefficients and $P_{s\,t\,u}$ means a cyclic permutation of (s, t, u)*. Let us consider two possible scattering amplitudes $f_1(s,t,u)$ and $f_2(s,t,u)$ both satisfying Mandelstam's representation with the same double spectral functions P_{ij}. Their difference $\Delta f = f_2 - f_1$ will have the following representation:

$$f(s,t,u) = \sum_{p=0}^{M} \frac{t^p}{\pi} \int \frac{S^M}{S'^M} \, \frac{\Delta\sigma_{1p}(s')}{S'-S} \, ds' + P_{s\,t\,u} +$$

$$+ \sum_{p_1 q=0}^{1} \Delta C_{pq} \, t^p S^q \qquad (5.2)$$

where $\Delta\sigma = \sigma_2 - \sigma_1$ and $\Delta C = C_2 - C_1$.

In the physical region of the s-energy channel one has:

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* In order to comply with the boundedness conditions implied by unitarity these must be cancelations among the various terms in this expression in the high energy limit of each channel. Thus one expects to exhist relationships connecting the coefficient of the polynomials and the single spectral functions to the double spectral functions.

$$\text{Im}\,\Delta f(s,t,u) = \sum_p t^p \Delta\sigma_{1p}(s) = \sum (2k^2)^p \Delta\sigma_{1p}(s)(1-\cos\theta)^p \qquad (5.3)$$

Since $0 \leqslant \text{Im}\, f_{1,2\ell}(s) \leqslant 1$ it follows that $|\text{Im}\,\Delta f_\ell| \leqslant 1$. Hence $(2k^2)^p \Delta\sigma_{1p}(s)$ must be bounded, that is, for $s \to \infty$:

$$\Delta\sigma_{1p}(s) = 0(s - p). \qquad (5.4)$$

Therefore one can undo the subtractions by dropping the factors $\frac{S^M}{S'^M}$ and changing accordingly the coefficients of the polynomials, with the exception of the term with $p = 0$ which still requires one subtraction. Actually one can go further by making the expansion:

$$\int \frac{\Delta\sigma_{1p}(s')}{s'-s}\, ds' \equiv - \sum_{q=0}^{p-2} \int \frac{S'^q}{S^{q+1}} \Delta\sigma_{1p}(s')ds' + \int \frac{S'^{p-1}}{S^{p-1}} \frac{\Delta\sigma_{1p}(s')}{s'-s} ds' \qquad (5.5)$$

when the convergence of the integrals on the right is ensured by (5.4). Therefore one can write:

$$\Delta f(s,t,u) = \sum_{p=1}^{M} \left\{ \frac{t^p}{\pi} \int \frac{S'^{p-1}}{S^{p-1}} \frac{\Delta\sigma_{1p}(s')}{s'-s} ds' - \sum_{q=0}^{p-2} \frac{t^p}{S^{q+1}} I_{pq} \right\} +$$

$$+ \frac{1}{\pi} \int \frac{S}{S'} \frac{\Delta\sigma_{10}(s')}{(s'-s)} ds' + P_{s\,t\,u} + \sum_{p_1 q=0}^{L} \Delta C_{pq}\, t^p S^q . \qquad (5.6)$$

where

$$I_{pq} = \frac{1}{\pi} \int S'^q \, \Delta\sigma_{1p}(s')ds' .$$

Now the amplitudes $f_1(s,t,u)$ and $f_2(s, t, u)$ are asymptotically bounded by (see (4.48), (4.50)):

$$|f(s, t)| < C_1\, S(\ln S)^{3/2} \qquad (t \neq 0) \qquad (5.7)$$

$$|f(s,\cos\theta)| < C_2\, S^{3/4} (\ell n\, S)^{3/2} \qquad (\theta \neq 0,\pi) \qquad (5.8)$$

Then $\Delta f(s, t, u)$ must also satisfy such inequalities. For fixed values of $\theta \neq 0,\pi$, the dispersion integrals in (5.6) ou $O(s)$ for large S. Then one can easily be convinced that in order to have $\Delta f(s, \cos\theta)$ bounded by (5.8) it is necessary that:

$$I_{pq} = 0 \quad \text{and} \quad \Delta C'_{pq} = 0 \qquad (5.9)$$

with the exception of $\Delta C'_{oo}$.

The representation (5.6) of $f'(s,t,u)$ is thus reduced to:

$$\Delta f(s,t,u) = \sum_{p=1}^{M} \frac{t^p}{\pi} \int \frac{s'^{p-1}}{s^{p-1}} \frac{\Delta\sigma_{1p}(s')}{s' - s}\, ds' + \frac{1}{\pi}\int \frac{s}{s'} \frac{\Delta\sigma_{10}(s')}{s'\ s'-s}\, ds' + \Delta C'_{oo} + P_{stu}. \qquad (5.10)$$

Consider the channel for which t is the energy and let $t \to \infty$ with the momentum transfer S being held constant and negative. The boundedness condition (5.7) requires then, that for $p \geqslant 2$:

$$\int s'^{p-1} \frac{\Delta\sigma_{1p}(s')}{s' - s}\, ds' = 0 \qquad (s < 0) \qquad (5.11)$$

and by analytic continuation it follows that $\sigma_{1p}(s) = 0$. Therefore the representation for the difference of the two amplitudes is of the form:

$$\Delta f(s,t,u) = \frac{1}{\pi}\int \frac{s}{s'} \frac{\Delta\sigma_{10}(s')}{s' - s} + \frac{t}{\pi}\int \frac{\Delta\sigma_{11}(s')}{s' - s}\, ds' + P_{s,t,u} + \Delta C'_{oo} \qquad (5.12)$$

Since the single spectral functions $\sigma_{10}(s)$ and $\sigma_{11}(s)$ are determined by the s and p waves in the s-energy channel, one concludes that within the framework of the Mandelstam representation one can arbitrarily introduce particles with spin zero and one but not of higher spins. This limitation corresponds and is closely related to the criterium of renormalizability in field theory.

We shall next discuss the so called Gribov Paradox. [20]

One important feature of Pameranchuk's model is that the width of the difraction peak remains finite as the energy tends to infinity. The asymptotic behaviour, predicted by the model, for the amplitude f(s,t) is:

$$f(s,t) \simeq s\, f(t) \qquad (5.13)$$

This behaviour was shown by Gribov to be inconsistent with the Mandelstam representation. For simplicity we shall consider the scattering of isoscalar particles without spin. According to the Mandelstam representation, unitarity in the elastic region of the t-channel, gives for the spectral function the following expression:

$$\rho_{13}(s,t) = \frac{1}{16\pi^2 \sqrt{t}} \iint \frac{A_1(S_1,\, t)\, A_1(s_2,\, t)^*}{K(t;\, s,\, s_1,\, s_2)^{\frac{1}{2}}}\, ds_1\, ds_2 \qquad (5.14)$$

where the integration is extended over the region:

$$s_1 \geqslant 4\mu^2; \qquad s_2 \geqslant 4\mu^2; \qquad K \geqslant 0 \qquad (5.15)$$

with

$$K(t;s,s_1,s_2) = (t-4\mu^2)(s^2+s_1^2+s_2^2-2ss_1-2ss_2-2s_1s_2)-4ss_1s_2 \qquad (5.16)$$

and $A_1(s,t)$ is the absorptive part of f(s,t), for the reaction in which s is the energy, analytically continued in the variable t from the physical region $t < 0$, onto the spectral region $t \geqslant 4\mu^2$. The asymptotic behaviour of the absorption amplitude in the physical region $t < 0$ is given by:

$$A_1(s,t) \simeq s \operatorname{Im} f(t). \tag{5.17}$$

Let us assume that, for $t > 0$, the asymptotic behaviour of $A_1(s,t)$ is of the same form (5.17), $A_1(s,t) \simeq s\, a(t)$, where for $t > 4\mu^2$ $a(t)$ is complex. Since for fixed t, $A_1(s,t)$ is: an increasing function of s_1 in the asymptotic limit, then for large s, most of the contribution to the integrals in (5.15) comes from the region near the boundary $K = 0$, where s_1 and s_2 are both large, behaving like $\sqrt{s}$. Then for large s, $\rho(s,t)$ will be given by the following asymptotic expression:

$$\rho(s,t) = \frac{1}{16\pi^2 \sqrt{t}} \iint |a(t)|^2 \, K(t:s,s_1,s_2)^{-\frac{1}{2}} \, s_1 s_2 ds_1 ds_2 \tag{5.18}$$

For large s the boundary curve $K = 0$ is approximately given by the hyperbola:

$$4\, s_1\, s_2 = (t - 4\mu^2)\, s, \tag{5.19}$$

along any direction not parallel to the axis s_1 and s_2. Let us introduce new variables u and defined by:

$$4\, s_1\, s_2 = (t - 4\mu^2)\, s\, u$$

$$\frac{s_2}{s_1} = v \tag{5.20}$$

The domain of integration of these new variables taking the hyperbola (5.19) as a boundary curve is:

$$(u_o/u) \leqslant v \leqslant (u/u_o); \quad u_o \leqslant u \leqslant 1 \tag{5.21}$$

where $u_o = 4(4\mu^2)^2/s(t - 4\mu^2)$. Therefore:

$$\rho_{13}(s,t) \simeq \frac{1}{16\pi^2 \sqrt{t}} \, |a(t)|^2 \, (t - 4\mu^2)^{3/2} \, \frac{s}{32} \int_{u_o}^{1} \frac{u\, d\, u}{\sqrt{1-u}} \int_{u_o/u}^{u/u_o} \frac{dv}{v} \tag{5.22}$$

whence, performing the integration, one obtains the following asymptotic expression for $\rho(s,t)$:

$$\rho_{13}(s,t) \simeq \frac{|a(t)|^2}{16\pi^2\sqrt{t}}(t-4\mu^2)^{3/2}\,\frac{s}{24}\left(\ln\frac{1}{u_o}+O(1)\right) \simeq C(t)s\ \ln s \tag{5.23}$$

But, Mandelstam's representation gives:

$$\rho_{13}(s,t) = \text{Im}\ A_1(s,t) \simeq s\ \text{Im}\ a\ (t) \tag{5.24}$$

which is inconsistent with the expression (5.23). Gribov shows that the paradox still remains should the asymptotic behaviour be of the form $f(s,t) \simeq S^{\alpha}\ln^{\beta} s\ f(t)$, with $\text{Re}\,\alpha \geqslant 1$ and $\text{Re}\ \beta \geqslant -1$. He proposed an asymptotic behaviour with $\alpha = 1$ and $\beta < -1$ but this implies a cross section vanishing at infinity.

Although Gribov's paradox strongly suggests that Pomeranchuk's model for high energy scattering is inadequate, if the amplitude is to satisfy Mandelstam's representation, it is not altogether conclusive since $f(s,t)$ could, conceivably, have an asymptotic behaviour for $t \geqslant 4\mu^2$ different from that for $t < 0$.

* * *

APPENDIX A - Asymptotic behaviour of the amplitudes in nucleon-nucleon and nucleon-antinucleon scattering.

In this appendix we shall discuss the asymptotic behaviour of nucleon-nucleon and nucleon-antinucleon covariant amplitudes within the framework of Pomeranchuk's model. Let us consider the process:

$$\text{I)} \quad N_1 + N_2 \longrightarrow N_1' + N_2'$$

$$\text{II)} \quad N_1 + \bar{N}_2' \longrightarrow N_1' + \bar{N}_2$$

$$\text{III)} \quad N_1 + \bar{N}_1' \longrightarrow \bar{N}_1 + N_2'$$

where N stands for a nucleon and $\bar{N}$ for an anti-nucleon. Let p_1, p_2 and p_1', p_2', be, respectively, the initial and final momenta for process I. Conservation of total momentum gives:

$$p_1 + p_2 = p_1' + p_2' \tag{A.1}$$

The amplitude for this process depend on two scalar variables. We shall use the Mandelstam invariants defined by:

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ t &= (p_1' - p_1)^2 \\ u &= (p_1' - p_2)^2 \end{aligned} \tag{A.2}$$

where, in the center of mass system, s is the square of the total energy, $-t$ and $-u$ the squares of the momentum transfers for the pairs (1,1) and (1,2) respectively. They satisfy the relation

$$s + t + u = 4M^2 \tag{A.3}$$

The momenta for processes II and III will be denoted by the same letters, but with the sign reversed for antinucleons. The scattering matrix for each process may be expressed in terms of a set of ten covariant amplitudes, five for each isotopic state $I = 0$ or $I = 1$. These amplitudes may be chosen, as in ref. 4), by analogy with -decay four-fermion interactions, by writing the Feynman amplitude for a given isotopic spin transition, in the following way:

$$\mathcal{F}^I = \sum_{i=1}^{5} F_i^I \left(O_i + (-1)^i \tilde{O}_i\right) \qquad \text{(A.4)}$$

where the upper index is the isotopic spin $I = 0$ or $I = 1$, and

$$\begin{aligned} O_i &= \bar{u}(p_2') A_i u(p_2) \bar{u}(p_1') A_i u(p_1) \\ \tilde{O}_i &= \bar{u}(p_1') A_i u(p_2) \bar{u}(p_2') A_i u(p_1) \end{aligned} \qquad \text{(A.5)}$$

where the A's are the five sets of covariant Dirac matrices in the order (S, T, A, V, P). The F's are functions of the variables s, t, u and satisfy the Mandelstam representation. The helicity amplitudes (2.10) are related to the F's by (Ref. 4, 4.17):

$$(\varphi_2 + \varphi_1) = \frac{1}{\pi}\left[M^2(F_1 + (F_2 + F_4)\cos\theta) - (4E^2 - M^2)F_3\right]$$

$$(\varphi_2 - \varphi_1) = \frac{1}{\pi}\left[-E^2 F_1 + \left(\left[2E^2 - M^2\right]F_2 + M^2F_4\right)\cos\theta + 3M^2F_3 - p^2F_5\right]$$

$$(\varphi_4 + \varphi_3) = \frac{1}{\pi}\left[2M^2F_2 + 2E^2F_4 + p^2(-F_1 + 2F_3 + F_5)\right]\cos^2\frac{\theta}{2} \qquad \text{(A.6)}$$

$$(\varphi_4 - \varphi_3) = \frac{1}{\pi}\left[2M^2F_2 + 2E^2F_4 - p^2(-F_1 + 2F_3 + F_5)\right]\sin^2\frac{\theta}{2}$$

$$\varphi = \frac{-1}{\pi} M(F_2 + F_4) E \sin\theta \,.$$

where, here, E and p are the energy and momentum of a nucleon in the center of mass system. From the boundedness of the helicity amplitudes one can easily deduce that in the physical region all the F's are bounded when $E \to \infty$. In addition, the relations $\varphi_i \approx \varphi_j (i, j = 1 \ldots 4)$, $\varphi_5 \ll \varphi_i$, at fixed momentum transfer t, mean that:

$$\varphi_i - \varphi_j = 0(s); \qquad \varphi_5 = 0(s) \tag{A.7}$$

They imply in the following restrictions on the covariant amplitudes in the asymptotic region of large s:

$$\varphi_2 - \varphi_1 = 0(s) \longrightarrow -F_1 + 2F_2 - F_5 = 0(1) \tag{A.8a}$$

$$\varphi_4 - \varphi_3 = 0(s) \longrightarrow F_1 - 2F_3 + 2F_4 - F_5 = 0(1) \tag{A.8b}$$

$$\varphi_3 - \varphi_1 = 0(s) \longrightarrow -F_1 + 6F_3 + 2F_4 + F_5 = 0(1) \tag{A.8c}$$

$$\varphi_5 = 0(s) \longrightarrow F_2 + F_4 = 0(s^{\frac{1}{2}}) \tag{A.8d}$$

Similarly, isotopic spin independence in the asymptotic region gives:

$$F_j^0 - F_j^1 = 0(1) \tag{A.9}$$

Now, the amplitudes $F_j(s,u,t)$ for process I are related to the amplitudes $\bar{F}(u,s,t)$ and $\bar{F}(t,s,u)$ for processes II and III by the so called "crossing relations" or "crossing symetries". One has (Ref. 4, 2.19):

$$F_j^I(s,u,t) = B^{II'} \Gamma_{jk} F_k^{I'}(u,s,t). \tag{A.10}$$

where the first variable in the argument of this functions F and $\bar{F}$ stands for the energy in the corresponding process, and:

$$B = \frac{1}{2} \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \tag{A.11}$$

$$\Gamma = \frac{1}{4}\begin{bmatrix} -1 & 6 & -4 & +4 & -1 \\ 1 & 2 & 0 & 0 & 1 \\ -1 & 0 & 2 & 2 & 1 \\ 1 & 0 & 2 & 2 & -1 \\ -1 & 6 & 4 & -4 & -1 \end{bmatrix} \tag{A.12}$$

Since Pomeranchuk's conjecture applies to nucleon-antinucleon scattering as well, the $\bar{F}$'s must also satisfy the asymptotic relations (A.8) and (A.9). One can verify that these relations are, indeed, invariant under the transformation (A.10). On the other hand, one can obtain from (A.10), similar relations between $F_j^I(s,u,t)$ and $\bar{F}_j^I(t,s,u)$ by making use of the Pauli principle which requires that:

$$F_j^I(s,u,t) = (-1)^{I+j}\, F_j^I(s,t,u) \tag{A.13}$$

Therefore, corresponding to (A.8) and (A.9) one obtain the following asymptotic relations for the $\bar{F}_j^I(t,s,u)$ at fixed t and large s:

(A.8a) $\longrightarrow \bar{F}_2 = O(1)$ (A.14a)

(A.8b) $\longrightarrow \bar{F}_3 = O(1)$ (A.14b)

(A.8c) $\longrightarrow -\bar{F}_1 + 2\bar{F}_3 + \bar{F}_5 = O(1)$ (A.14c)

(A.8d) $\longrightarrow \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4 = O(s^{\frac{1}{2}})$ (A.14d)

and

(A.9) $\longrightarrow \bar{F}_j^1 = O(1)$ (A.15)

The helicity amplitudes $\bar{\varphi}$ for process III will then have a pole as the momentum transfer u(or s) $\longrightarrow \infty$ and the residues of the pole for $t = 0$ and taking into account (A.14) will be given by:

$$\text{Res.}\ \bar{\varphi}_1 = \text{Res.}\ \bar{\varphi}_2 = \frac{1}{2} \lim_{S \to \infty} \bar{F}_4(0,s,u)$$
$$\text{Res.}\ \bar{\varphi}_3 = \text{Res.}\ \bar{\varphi}_4 = \text{Res.}\ \bar{\varphi}_5 = 0 \qquad (A.16)$$

and

$$\text{Res}\ \bar{\varphi}^1_j = 0 \qquad (A.17)$$

As anticipated in Sec. 2, (A.8b) and (A.8d) do not imply in strong restrictions on the bahaviour of $\bar{\varphi}(t,s,u)$ when $t \to 0$, $s \to \infty$, as the conditions (A.16) on the residues of the pole result, exclusively from (A.14a) and (A.14c). This behaviour of the helicity amplitudes is formally the same as if there were a pole in the energy at $t = 0$, in the state of angular momentum $J = 1$ and G-parity even. Since, by (A.17) the pole occurs only in the isotopic state $I = 0$, these results coincide with the predictions of a model in which the asymptotic behaviour of the scattering amplitude is dominate by a Regge pole which moves along a trajectory with the quantum numbers of the vacuum, the so called Pomeranchuk trajectory, passing through $J = 1$ at $t = 0$. Therefore, both models predict the same relations for the asymptotic amplitudes in nucleon-nucleon and nucleon-antinucleon forward scattering as well as for the total cross sections.

APPENDIX B - General crossing symmetry for isospin amplitudes.

Let us consider the transition matrix in spin and isospin space, for process Ia (Sec. 3), defined by:

$$\mathcal{T}(p_1p_2;\ p_3p_4)(2\pi)^4\ \delta(p_1 + p_2 - p_3 - p_4) = i \int d^4x\ d^4y\ d^4z\ d^4w$$

$$\times\ \exp.i(p_4\cdot w + p_3\cdot z - p_2\cdot y - p_1\cdot x)\langle \vec{S}_4(w)\vec{S}_3(z)T\left[\phi_4(w)\phi_3(z)\bar{\phi}_1(x)\bar{\phi}_2(y)\right]$$

$$\bar{\overleftarrow{S}}_1(x)\ \bar{\overleftarrow{S}}_2(y)\rangle \qquad (B.1)$$

For particles of spin zero $\vec{S}(x) = \vec{\Box}_x + m^2$, and for spin $\frac{1}{2}$ particles $\vec{S}(x) = -i\not{\partial}_x + m$. The matrix element of the $\mathcal{T}$-matrix between free particle spin and isospin states, give the Feynman amplitude for a transition in these states.

Now it follows from $\mathcal{C}$-invariance of strong interactions that the Heisenberg field operators for particle and antiparticle are related by a unitary transformation $\mathcal{C}$ defined by:

$$\mathcal{C}^{-1}\ \phi(x)\mathcal{C} = \phi^c(x) = C\bar{\phi}(x)^T \qquad (B.2)$$

$$\mathcal{C}^{-1}\ \bar{\phi}(x)\mathcal{C} = \bar{\phi}^c(x) = \mp\ \phi(x)^T\ C^{-1} \qquad (B.3)$$

where the sign + applies to boson fields and the sign - to fermion fields in the last equation. This operation of charge conjugation is the same as that defined by Lee and Yang. The unitary matrix C acts in both spin and isospin space and has the following properties:

$$C^{-1}\ SC = \bar{S}^T \qquad (B.4)$$

$$C^{-1}SC = -\vec{S}^T; \qquad C^{-1}\vec{I}C = -\vec{I}^T \qquad (B.5)$$

$\bar{S}$ and $\bar{\tau}$ are the spin and isospin matrices, respectively. Applying this transformation to the fields ϕ_2 and ϕ_4 in (B.1), the right hand side becomes:

$$i\int d^4x\ d^4y\ d^4z\ d^4w\ \exp.i(p_4w + p_3z - p_2y - p_1x)\ \times$$

$$\times\left\{C_2^{-1}\langle\vec{S}_2(y)\vec{S}_3(z)T\left[\phi_2^c(y)\phi_3(z)\bar{\phi}_1(x)\bar{\phi}_4^c(w)\right]\bar{\overleftarrow{S}}_1(x)\bar{\overleftarrow{S}}_4(w)\rangle C_4\right\}^{T_{24}} \qquad (B.6)$$

where T_{24} means transposed in the space of the variables of particles 2 and 4. The minus sign in (B.3) for fermion fields is compensated by a charge in sign of the T-product when the fields ϕ_2 and ϕ_4 are interchanged. If one express (B.6) in terms of the transition matrix $\bar{\mathcal{T}}$ for process IIa, then, comparing with (B.1) one obtains:

$$\mathcal{T}(p_1 p_2; p_3 p_4) = \left\{ C_2^{-1} \bar{\mathcal{T}}(p_1 - p_4; p_3 - p_2) C_4 \right\}^{T_{24}} \tag{B.7}$$

Under the substitution $p_2 \longleftrightarrow -p_4$, the invariants defined by (A.2) are transformed in the following way: $s \leftrightarrow u$, $t \longleftrightarrow t$, so that (B.7) relates the amplitudes for process Ia with energy S to the amplitudes for process IIa with energy u, and the same value of the momentum transfer.

We are now interested on the transformation of isospin indices under the crossing symmetry (B.7). So we shall for the moment igonore the transformation properties of the states in ordinary space. If $|\mu\rangle$ is an eigenstate of I_z with eigenvalue μ, then it follows from (B.5) that $C|\mu\rangle^*$ is the eigenstate $|-\mu\rangle$, but for a phase. In order to determine the phase let us consider the representation with I_z diagonal and the usual phase conventions. Since I_x is then symmetric and I_y antisymmetric one obtains from (B.5):

$$C\, I_- = -I_+ C \tag{B.8}$$

Therefore:

$$C|\mu\rangle = \frac{1}{N} C\, I_-^{I-\mu}|I\rangle = (-1)^{I-\mu} \frac{1}{N} I_+^{I-\mu} C|I\rangle = (-1)^{I-\mu} \frac{1}{N} I_+^{I-\mu} \eta |-I\rangle =$$

$$= \eta (-1)^{I-\mu} |-\mu\rangle \tag{B.9}$$

The arbitrary phase η is usualy taken igual to one. With this choice one has:

$$C^2 = (-1)^{2I} = \begin{cases} +1 & \text{for isobosons} \\ -1 & \text{for isofermions} \end{cases} \qquad (B.10)$$

As an example the nucleon isospinor $\binom{p}{n}$ transforms under this operation into $\binom{\bar{n}}{-\bar{p}}$.

In order to get the crossing relations between isospin amplitudes for processes Ia and IIa we take matrix elements of B(7) between states $|\mu_i \mu_j\rangle$. The result is

$$f_{\mu_1\mu_2;\mu_3\mu_4}(p_1, p_2; p_3, p_4) =$$

$$= (-1)^{I_2+I_4-\mu_2-\mu_4} \bar{f}_{\mu_1-\mu_4;\mu_3-\mu_2}(p_1, -p_4; p_3, -p_2) \qquad (B.11)$$

where f and $\bar{f}$ are invariant matrices in spin space only. It can be shown that in the special system where $\vec{p}_1 + \vec{p}_3 = 0$ and if $M_1 = M_3$ and $M_2 = M_4$, helicity amplitudes transform in a likewise manner but there is no change in sign of the helicities.

The amplitudes for transitions in eigen states of the total isotopic spin may be writen as linear combinations of the amplitudes f_i with Clebsh-Gordon coefficients. One can then write.

$$f_\mu^I(s,u,t) = \sum \mathcal{C}_I(I_1\mu_1, I_2\mu_2)\, \mathcal{C}_{I\mu}(I_3\mu_3, I_4\mu_4) f_{\mu_1\mu_2;\mu_3\mu_4}(s,u,t)$$

$$= \sum_{\mu_j} \mathcal{C}_{I\mu}(I_1\mu_1;I_2\mu_2)\mathcal{C}_{I\mu}(I_3\mu_3;I_4\mu_4)\bar{f}_{\mu_1,-\mu_4;\mu_3,-\mu_2}(u,s,t) \times$$
$$\times (-1)^{I_2+I_4+\mu_2+\mu_4}$$

$$\sum_{\mu_j} \mathcal{C}_I(I_1\mu_1,I_2\mu_2)\mathcal{C}_{I\mu}(I_3\mu_3;I_4\mu_4)\sum_{I} \mathcal{C}_{I'\mu'}(I_1\mu_1,I_4-\mu_4)\mathcal{C}_{I'\mu'}(I_3\mu_3;I_2-\mu_2)\times$$
$$\times\ \bar{f}^{I'}_{\mu}(u,s,t)(-1)^{I_2+I_4+\mu_2+\mu_4} \qquad (B.9)$$

Making use of the symmetry properties of Clebsh-Gordon coefficients one finds after some manipulation:

$$f^I(s,u,t) = \sum_{I'}(2I'+1)(-1)^{2(I_1+I_2)}\begin{Bmatrix} I_1 & I_2 & I \\ I_3 & I_4 & I' \end{Bmatrix}\bar{f}^{I'}(u,s,t) \qquad (B.10)$$

where $\begin{Bmatrix} I_1 & I_2 & I \\ I_3 & I_4 & I' \end{Bmatrix}$ is the 6 - j symbol. The crossing matrix:

$$O_{II'} = (2I'+1)(-1)^{2(I_1+I_2)}\begin{Bmatrix} I_1 & I_2 & I \\ I_3 & I_4 & I \end{Bmatrix} \qquad (B.11)$$

satisfies the orthogonality condition:

$$\sum(2I+1)(2I'+1)(-1)^{2I_2+2I_4}\begin{Bmatrix} I_1 & I_4 & I'' \\ I_3 & I_2 & I \end{Bmatrix}\begin{Bmatrix} I_1 & I_2 & I \\ I_3 & I_4 & I' \end{Bmatrix} = (-1)^{2I_2+2I_4}\delta_{II'} \qquad (B.12)$$

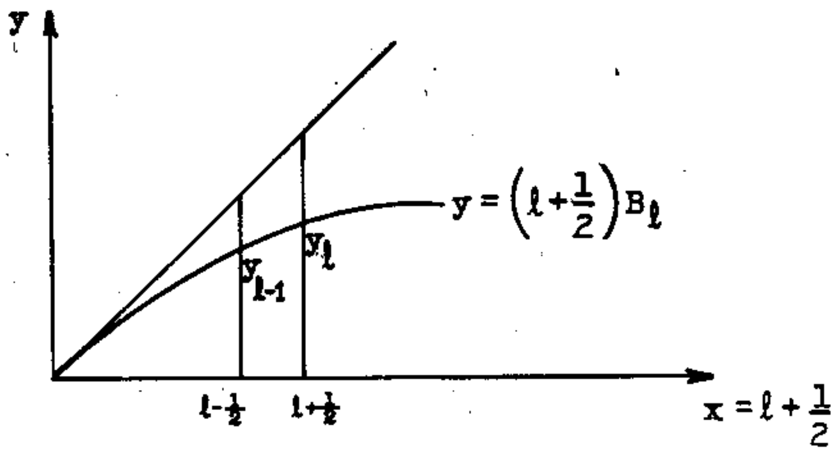
The reason for the factor $(-1)^{2I_2+2I_4}$ is that for isofermions $C^2 = -1$ according to the usual convention.

APPENDIX C

We want to calculate an upper bound for the sum:

$$\sum_{0}^{L-1}(2\ell+1)\,B_\ell(\cos\theta) + a_L(2L+1)B_L(\cos\theta) \qquad (C.1)$$

We take first $a_L = 0$ and evaluate the sum by the method of integration. The function $y_\ell = (\ell+\frac{1}{2})B_\ell$ has its concavity turned downwards for $\ell+\frac{1}{2} = x > 0$ (See Fig. 1)



Then:

$$\frac{1}{2}\left(y_\ell + y_{\ell-1}\right) < \int_{\ell-\frac{1}{2}}^{\ell+\frac{1}{2}} y \, dx \tag{C.2}$$

and

$$2 \sum_{0}^{L-1} y_\ell < 2 \int_{\frac{1}{2}}^{L-\frac{1}{2}} y \, dx + \left(y_{L-1} + y_o\right) = I + \left(y_{L-1} + y_o\right) \tag{C.3}$$

where

$$I_{L-1} = \frac{4}{3}\left\{\left[1 + (L-1)L \operatorname{sen}^2\theta\right]^{3/4} - 1\right\} / \operatorname{sen}^2\theta \tag{C.4}$$

Then

$$R(s,t) < \frac{I_{L-1} + \left(y_{L-1} + y_o\right)}{L^2} = \frac{I_{L-1}}{(L-1)L} \; \frac{1 + \left(y_{L-1} + y_o\right)/I_{L-1}}{1 + (1/L-1)} \tag{C.5}$$

This bound is slightly better than (4.35) for one can shown that

$$y_{L-1} + y_o \leqslant \frac{I_{L-1}}{L-1} \tag{C.6}$$

The equality holds in the limit $L \to 1$. Indeed (C.6) is equivalent to

$$\left\{\frac{3}{4}(L-1)\left(L-\frac{1}{2}\right)\left[1+(L-1)L \operatorname{sen}^2\theta\right]^{-\frac{1}{4}} + \frac{3}{8}(L-1)\right\} \operatorname{sen}^2\theta \quad \left[1+(L-1)L \operatorname{sen}^2\theta\right]^{\frac{3}{4}} - 1$$

or

$$-\frac{1}{4}(L-1)L\ \mathrm{sen}^2\theta+\left(1+\frac{3}{8}(L-1)\ \mathrm{sen}^2\theta\right)\left[(1+(L-1)L\ \mathrm{sen}^2\theta)^{\frac{1}{4}}-1\right]<0$$

or writing $x = (L-1)L\ \mathrm{sen}^2\theta$;

$$-\frac{1}{4}x+\left(1+\frac{3}{8}\frac{x}{L}\right)\left[(1+x)^{\frac{1}{4}}-1\right]<0 \qquad (C.7)$$

But

$$L=\frac{1}{2}\left[1+\sqrt{1+x/\mathrm{sen}^2\theta}\right]\therefore\ \frac{1}{2L}=\frac{1}{1+\sqrt{1+x/\mathrm{sen}^2\theta}}<\frac{1}{1+\sqrt{1+x}}=\frac{\sqrt{1+x}-1}{x}$$

Therefore the left hand side of (C.7) is less than:

$$-\frac{1}{4}x+\left(\frac{1}{4}+\frac{3}{4}\sqrt{1+x}\right)\left[(1+x)^{1/4}-1\right]=$$

$$=-\frac{1}{4}\left\{(x+1)-3(x+1)^{3/4}+3(x+1)^{2/4}-(x+1)^{1/4}\right\}=$$

$$=-\frac{1}{4}(x+1)^{1/4}\left[(x+1)^{1/4}-1\right]^3<0$$

Then (C.7) holds, hence also (C.6).

When $a_L \neq 0$ we write $\bar{L}-L=\delta$ which is related to a_L by

$$\bar{L}^2-L^2=(\bar{L}+L)\delta=a_L(2L+1) \qquad (C.8)$$

Then:

$$a_L(2L+1)\left[1+L(L+1)\mathrm{sen}^2\theta\right]^{-\frac{1}{4}}=\delta(\bar{L}+L)\left[1+L(L+1)\mathrm{sen}^2\theta\right]^{-\frac{1}{4}}$$

$$<\frac{1}{2}\delta(y_{\bar{L}-1}+y_L) \qquad (C.9)$$

and

$$2\sum_0^{L-1}y+2a_Ly_L<2\int_{\frac{1}{2}}^{L-\frac{1}{2}}y\,dx+(y_{L-1}+y_o)-\delta(y_{L-1}+y_{\bar{L}-1})+$$

$$+\delta(y_{\bar{L}-1}+y_L)=2\int_{\frac{1}{2}}^{\bar{L}-\frac{1}{2}}y\,dx+\left[(1-\delta)y_{L-1}+\delta y_L\right]+y_o$$

or

$$2\sum_{0}^{L} y_\ell + 2\, a_L y_L \leqslant 2\int_{\frac{1}{2}}^{\bar{L}-\frac{1}{2}} y\, dx + (y_{\bar{L}-1} + y_o) = I_{\bar{L}-1} + (y_{\bar{L}-1} + y_o) \qquad (C.11)$$

Hence the results (C.5) and (4.35) for L an integer hold true for $\bar{L}$ non-integer as well.

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