DIRAC'S EQUATION IN A BIANCHI-TYPE I UNIVERSE(*)

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1. Some Useful Formulae(**)

The basic 1-forms θ^A are constructed by means of the tetrads $e^A_{(\alpha)}$ (***)

(1.1) $$\theta^A = e^A_{(\alpha)}\, dx^\alpha$$

It corresponds to a decomposition of the line element

$$ds^2 = \eta_{AB}\, \theta^A \theta^B = (\theta^0)^2 - (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2$$

which defines in each point of the manifold a Cartan moving frame of reference.

Cartan's first structure equation gives

(1.2) $$d\theta^A + \omega^A{}_B \wedge \theta^B = 0$$

in which the symbol $\wedge$ means Grassman product and d is the exterior differentiation. The Ricci coefficients $\gamma^A{}_{BC}$ relate the 1-forms $\omega^A{}_B$ to the fundamental θ's through the relation

(1.3) $$\omega^A{}_B = \gamma^A{}_{BC}\, \theta^C$$

The ω-forms are set anti-symmetric by imposing

(1.4) $$d\eta_{AB} = \omega_{AB} + \omega_{BA} = 0$$

(*) Extracted from a forthcoming review paper on the gravitational interaction of neutrinos.

(**) This section is included here only for completeness.

(***) Capital Latin indices run 0 to 3; they are raised and Lowered with Minkowski metric η_{AB}, η^{AB} = diag(+1,-1,-1,-1)

The second Cartan structure equation relates the exterior derivative of $\omega^A{}_B$ with the 2-form curvature $\Omega^A{}_B$:

(1.5) $$\Omega^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B$$

Its relationship with Riemann curvature tensor in tetrad basis is given by

(1.6) $$\Omega^A{}_B = -\frac{1}{2} R^A{}_{BCD}\ \theta^C \wedge \theta^D$$

The covariant derivative for a spinor ψ (thus minimally coupled with gravitation) is given by

(1.7) $$D_\mu \psi = \partial_\mu \psi - \Gamma_\mu \psi$$

in which the internal connection Γ_μ has the form

(1.8) $$\Gamma_A = -\frac{1}{4} \gamma_{MNA}\ \gamma^M \gamma^N$$

in Cartan moving frame, where γ^M is a (constant) Dirac matrices.

2. Dirac's Equation in Kasner-Type Universes

In a Kasner-type Universe the fundamental length is given by

(2.1) $$ds^2 = dt^2 - a^2(t)\,dx^2 - b^2(t)\,dy^2 - c^2(t)\,dz^2$$

The fundamental 1-forms are given by

(2.2)
$$\begin{aligned}
\theta^0 &= dt\\
\theta^1 &= a(t)\,dx\\
\theta^2 &= b(t)\,dy\\
\theta^3 &= c(t)\,dz
\end{aligned}$$

The Ricci coefficients can be calculated by using definitions (1.2), (1.3), and the non-vanishing ones are

$$\gamma_{011} = \frac{\dot{a}}{a} \qquad ; \quad \gamma_{033} = \frac{\dot{c}}{c} \tag{2.3}$$
$$\gamma_{022} = \frac{\dot{b}}{b}$$

Using (1.8) we can evaluate the internal connections:

$$\tau_0 = 0 \tag{2.4}$$
$$\tau_1 = -\frac{1}{2}\frac{\dot{a}}{a}\gamma^0\gamma^1 \qquad \tau_3 = -\frac{1}{2}\frac{\dot{c}}{c}\gamma^0\gamma^3$$
$$\tau_2 = -\frac{1}{2}\frac{\dot{b}}{b}\gamma^0\gamma^2$$

These Dirac's equation

$$\gamma^A D_A \psi = \gamma^A \left(e_A^{(\alpha)} \partial_\alpha \psi - \tau_A \psi \right) = 0 \tag{2.5}$$

takes the form

$$\dot{\psi} + \frac{1}{2}\frac{\dot{V}}{V}\psi = 0 \tag{2.6}$$

in which we have assumed $\psi = \psi(t)$ and defined $V = abc$ as the volume of the Universe.

By varying the metric on the Lagrangian of the spinor field, we can obtain (see, for instante Brill et al., 1957), the energy-momentum tensor of the field

$$T_{AB} = i \left\{ \bar{\psi}\gamma_A D_B \psi + \bar{\psi}\gamma_B D_A \psi - D_A \bar{\psi}\gamma_B \psi - D_B \bar{\psi}\gamma_A \psi \right\} \tag{2.7}$$

A straight forward calculation gives:

$$T_{00} = 2i \left(\bar{\psi}\gamma^0 \dot{\psi} - \dot{\bar{\psi}}\gamma^0 \psi \right) \tag{2.8}$$
$$T_{01} = i \left(\dot{\bar{\psi}}\gamma^1 \psi - \bar{\psi}\gamma^1 \dot{\psi} \right)$$
$$T_{02} = i \left(\dot{\bar{\psi}}\gamma^2 \psi - \bar{\psi}\gamma^2 \dot{\psi} \right)$$

$$T_{03} = i(\dot{\bar{\psi}}\gamma^3\psi - \bar{\psi}\gamma^3\dot{\psi})$$

$$T_{12} = i\left(\frac{\dot{b}}{b} - \frac{\dot{a}}{a}\right)\bar{\psi}\gamma^0\gamma^1\gamma^2\psi$$

$$T_{13} = i\left(\frac{\dot{c}}{c} - \frac{\dot{a}}{a}\right)\bar{\psi}\gamma^0\gamma^1\gamma^3\psi \quad , \quad T_{23} = i\left(\frac{\dot{c}}{c} - \frac{\dot{b}}{b}\right)\bar{\psi}\gamma^0\gamma^2\gamma^3\psi$$

All other components are null. Now from Dirac's equation we can easily show that

(2.9) $\qquad T_{00} = T_{01} = T_{02} = T_{03} = 0$

identically.

Some special cases.

(a) Friedmann-type:

Set $a = b = c = R(t)$. Then by (2.8) the energy-momentum tensor vanishes identically,

(2.10) $\qquad T_{\mu\nu}[\psi] \equiv 0 \qquad ;$

such class of neutrino, satisfying Dirac's equation

$$\dot{\psi} + \frac{3}{2}\frac{\dot{R}}{R}\psi = 0$$

does not create curvature, but it reacts to the gravitational field. Such type of behaviour will be called passon-like (from Novello's paper "A new model of gravitational interaction"). It is also known as ghost-neutrino, by Ray and Davies who first obtained a solution of this type.

(b) General Kasner-type

Let us set ... in

(2.11) $\qquad \psi = f(t)\,\chi^{(0)}$

where $\chi^{(0)}$ is an arbitrary constant spinor. Dirac's equation gives

$$\dot{f} + \frac{3}{2} \frac{\dot{R}}{R} f = 0 \tag{2.12}$$

Now, let us impose a $\chi^{(0)}$ of the form

$$\chi^{(0)} = \begin{pmatrix} \varphi^{(0)} \\ \sigma^{3} \varphi^{(0)} \end{pmatrix} \tag{2.13}$$

in which σ^3 is the Pauli 2x2 matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\varphi^{(0)}$ is an arbitrary constant 2-spinor. Then, we find

$$\bar{\psi}\gamma^0\gamma^1\gamma^2\psi = -2i\, \varphi^{(0)+}\sigma^3\varphi^{(0)}$$
$$\bar{\psi}\gamma^0\gamma^1\gamma^1\psi = \bar{\psi}\gamma^0\gamma^2\gamma^3\psi = 0 \tag{2.14}$$

If we impose further

$$\varphi^{(0)+}\sigma^3\varphi^{(0)} = 0 \tag{2.15}$$

then, for any function a,b and c we have

$$T_{\mu\nu}[\psi] = 0$$

It seems worthwhile to remark that in both cases (a) and (b) neutrino current $j^\mu = \bar{\psi}\gamma^\mu\psi$ is not *null* . Indeed, for the latter, the density $\rho = j^0$ is given by

$$\rho = 2|f|^2\, \varphi^{(0)+}\varphi^{(0)} \tag{2.16}$$

and $\rho \neq 0$ unless $\psi \equiv 0$.

(c) Massive fermions in a general Kasner-type Universe

For a massive fermion ψ , Dirac's equation assumes the form

$$\gamma^0 \left(\dot{\psi} + \frac{1}{2}\frac{\dot{V}}{V}\psi\right) + im\psi = 0 \tag{2.17}$$

A direct calculation gives, for the non-null components of the energy-momentum tensor:

$$T_{00} = 4m\,\bar{\psi}\psi$$

$$T_{12} = i\left(\frac{\dot b}{b} - \frac{\dot a}{a}\right)\psi^{+}\begin{pmatrix} -i\sigma^{3} & 0 \\ 0 & -i\sigma^{3}\end{pmatrix}\psi \tag{2.18}$$

$$T_{13} = i\left(\frac{\dot c}{c} - \frac{\dot a}{a}\right)\psi^{+}\begin{pmatrix} -i\sigma^{2} & 0 \\ 0 & -i\sigma^{2}\end{pmatrix}\psi\,,\quad T_{23} = i\left(\frac{\dot c}{c} - \frac{\dot b}{b}\right)\psi^{+}\begin{pmatrix} -i\sigma^{1} & 0 \\ 0 & -i\sigma^{1}\end{pmatrix}\psi$$

The non-null components of the contracted Riemann tensor are

$$\frac{1}{2}R_{00} = \frac{\ddot a}{a} + \frac{\ddot b}{b} + \frac{\ddot c}{c}$$

$$-\frac{1}{2}R_{11} = \frac{\ddot a}{a} + \frac{\dot a}{a}\left(\frac{\dot b}{b} + \frac{\dot c}{c}\right) \tag{2.19}$$

$$-\frac{1}{2}R_{22} = \frac{\ddot b}{b} + \frac{\dot b}{b}\left(\frac{\dot a}{a} + \frac{\dot c}{c}\right)$$

$$-\frac{1}{2}R_{33} = \frac{\ddot c}{c} + \frac{\dot c}{c}\left(\frac{\dot a}{a} + \frac{\dot b}{b}\right)$$

By imposing Einstein's equation

$$R_{AB} = -T_{AB} \tag{2.20}$$

we must have

$$R_{11} = R_{22} = R_{33} = 0$$

$$2m\,\bar{\psi}\psi = \frac{\ddot a}{a} + \frac{\ddot b}{b} + \frac{\ddot c}{c} \tag{2.21}$$

$$\dot\psi + \frac{1}{2}\frac{\dot V}{V}\psi + im\gamma^{0}\psi = 0$$

either $a = b = c$

or $\varphi^{+}\sigma^{k}\varphi + \eta^{+}\sigma^{k}\eta = 0$ for $\psi = \begin{pmatrix}\varphi \\ \eta\end{pmatrix}$

From these results we can make the following table:

We used the following conventions

$$\gamma^{k} = \begin{pmatrix} 0 & \sigma^{k} \\ -\sigma^{k} & 0 \end{pmatrix} \qquad \gamma^{0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma_{5} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^{\mu +} = \gamma^{0}\gamma^{\mu}\gamma^{0} \qquad \bar{\psi} = \psi^{+}\gamma^{0}$$

Finally it is important to remark that we have a strong suggestion of a symmetry of passon-like fermions solutions in Bianchi type I (including the flat-space case) models. That is, there should be a group acting on $\{\psi\}$ which preserves the condition

$$T_{\mu\nu}[\psi] = 0$$

or, equivalently, given a passon-lihe fermion solution ψ, there exists a transformation

$$\psi \rightarrow \tilde{\psi} = S\psi$$

which generates other PL fermion solutions.

PASSON-LIKE FERMIONS

I Metric Coefficients	II Spinor ψ	$\bar\psi\psi$ / m	$T_{AB}[\psi]=0$ due to I	 due to II	 due to Dirac's equation	Additional Information	Current j^A
$a=b\neq c$	• $\psi=\begin{pmatrix}\varphi\\ \sigma^1\varphi\end{pmatrix}$	0 / 0	T_{12}	T_{13}, T_{23}	T_{01}, T_{02}, T_{03}	$T_{00}\sim\bar\psi\psi$	$(\rho,0,0,\rho)$
$a=c\neq b$	• $\psi=\begin{pmatrix}\varphi\\ \sigma^2\varphi\end{pmatrix}$	0 / 0	T_{13}	T_{12}, T_{23}	T_{01}, T_{02}, T_{03}		$(\rho,0,\rho,0)$
$b=c\neq a$	• $\psi=\begin{pmatrix}\varphi\\ \sigma^3\varphi\end{pmatrix}$	0 / 0	T_{23}	T_{12}, T_{13}	T_{01}, T_{02}, T_{03}		$(\rho,\rho,0,0)$
$a=b=c$	$\psi=\begin{pmatrix}\varphi\\ \eta\end{pmatrix}$ $\varphi^\dagger\varphi=\eta^\dagger\eta$	0 / m	T_{12}, T_{13}, T_{23}		T_{01}, T_{02}, T_{03}	$T_{11}=T_{22}=T_{33}=0$ $T_{AB}\neq 0$, other wise it is flat	$(\rho,\varphi^\dagger\vec\sigma\eta+\eta^\dagger\vec\sigma\varphi)$
arbitrary	$\psi=\begin{pmatrix}\varphi\\ \eta\end{pmatrix}$ $\eta^T=\varphi^\dagger$ $\varphi^\dagger\sigma^2\varphi+\varphi^T\sigma^2\varphi^*=0$	0 / m		T_{12}, T_{13}, T_{23}	T_{01}, T_{02}, T_{03}		$(\rho,\varphi^\dagger\vec\sigma\eta+\eta^\dagger\vec\sigma\varphi)$

Obs.: • Incompatible with Dirac's equation if $m\neq 0$

$\rho=2\varphi^\dagger\varphi$ if $\varphi=\eta$.

$\vec\sigma=(\sigma^1,\sigma^2,\sigma^3)$