MONOGRAFIAS DE FÍSICA

IX

LECTURES ON RELATIVISTIC WAVE EQUATIONS*

by

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CHAPTER 1

REPRESENTATION OF THE THREE DIMENSIONAL ROTATION GROUP

1. SPINOR SPACE.

Consider the two-dimensional complex vector space S_2 and the set $\{A\}$ of all 2×2, non-singular matrices A which transform S_2 into itself:

$$\psi' = A\psi , \quad \text{where} \quad \psi, \psi' \in S_2, \tag{1.1}$$

or

$$\psi'_i = a_{ik} \psi_k, \quad (i, k = 1, 2)$$

with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} .$$

We further impose the condition

$$\det A = 1.$$

The set of these A's forms a group which by virtue of the last condition is called the unimodular group. The two-component vectors ψ , transforming according to the unimodular group are known as (first rank) spinors.

Second-rank spinors are defined by the transformation:

$$\psi'_{k\ell} = a_{km} a_{\ell n} \psi_{mn} \, (k, \ell, m, n = 1, 2) .$$

Higher rank spinors are defined in a similar way.

Consider two spinors ψ and η :

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad , \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \quad .$$

Then

$$\begin{pmatrix} \psi_1' & \eta_1' \\ \psi_2' & \eta_2' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \psi_1 & \eta_1 \\ \psi_2 & \eta_2 \end{pmatrix}$$

from which it follows that

$$\psi_1' \; \eta_2' - \eta_1' \; \psi_2' = (\det A) \, (\psi_1 \eta_2 - \eta_1 \psi_2).$$

Hence, when $\det A = 1$:

$$\psi_1 \; \eta_2 - \eta_1 \; \psi_2 = \text{invariant.} \qquad (1.2)$$

Let us define a new spinor η^i:

$$\eta^1 \equiv \eta_2 \, , \quad \eta^2 \equiv - \eta_1 \, ,$$

then

$$\psi_1 \; \eta^1 + \psi_2 \; \eta^2 = \text{invariant.}$$

Introduce:

$$\epsilon^{ij} \equiv - \epsilon_{ij} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; ; \; \epsilon^{ij} \, \epsilon_{jk} = + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad .$$

We see that:

$$\epsilon^2 = \epsilon^{ij}\epsilon^{jk} = \epsilon_{ij}\epsilon_{jk} = -1, \quad \epsilon^{-1} = -\epsilon .$$

Hence:

$$\eta^i = \epsilon^{ij} \eta_j, \quad \eta_i = \epsilon_{ij} \eta^j .$$

Therefore:

$$\psi_i \eta^i = \epsilon_{ij} \psi^j \epsilon^{ik} \eta_k = -\psi^i \eta_i$$

$$\psi_i \psi^i = 0.$$

Similarly we can prove for all odd-rank spinors:

$$\psi^{ijk\,\cdots} \psi_{ijk\,\cdots} = 0.$$

By convention we call spinors with lower indices covariant spinors, and spinors with upper indices contravariant spinors.

One can easily verify by direct substitution that the following relation holds for any non-singular A:

$$\epsilon A \epsilon^{-1} = (\det A)\,(A^{-1})^T \;; \quad A^T \equiv \text{ transpose of A.} \tag{1.3}$$

Therefore,

$$\psi'^T \epsilon \psi' = (\psi^T A^T)\epsilon\,(A\psi) = \psi^T A^T \epsilon A \epsilon^{-1} \epsilon \psi = (\det A)\,\psi^T \epsilon \psi . \tag{1.4}$$

We will use this relation later.

2. REPRESENTATION OF 3-D ROTATION GROUP.

We define

$$\vec{x} = \tfrac{1}{2}\psi^T \epsilon \vec{\sigma} \varphi , \tag{1.5}$$

where:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices. We find

$$\begin{cases} x_1 = \tfrac{1}{2}(\psi_1 \ \psi_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \tfrac{1}{2}(\psi_1\varphi_1 - \psi_2\varphi_2) , \\ x_2 = \tfrac{i}{2}(\psi_1\varphi_1 + \psi_2\varphi_2) \\ x_3 = -\tfrac{1}{2}(\psi_1\varphi_2 + \psi_2\varphi_1) . \end{cases} \tag{1.6}$$

We now impose the further restriction

$$A^\dagger A = 1, \quad A^\dagger = \text{Hermitian conj. of } A,$$

i.e. we only consider the <u>unitary unimodular</u> group. This implies the following form for A:

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} , \quad |\alpha|^2 + |\beta|^2 = 1.$$

Then we have

$$\begin{cases} \psi_1' = \alpha\psi_1 + \beta\psi_2 , \\ \psi_2' = -\beta^*\psi_1 + \alpha^*\psi_2 . \end{cases} \tag{1.7}$$

It follows that

$$\begin{cases} \psi_2'^* = \alpha\psi_2^* + \beta(-\psi_1^*) \\ -\psi_1'^* = -\beta^* \psi_2^* + \alpha^* (-\psi_1^*). \end{cases}$$

Hence $\begin{pmatrix}\psi_2^* \\ -\psi_1^*\end{pmatrix}$ transforms like $\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$. If we now set

$\varphi_1 = \psi_2^*$, $\varphi_2 = -\psi_1^*$ we have

$$\begin{cases} x_1 = \frac{1}{2}\left(\psi_1 \psi_2^* + \psi_1^* \psi_2\right) = \text{real}, \\ x_2 = \frac{1}{2}\left(\psi_1 \psi_2^* - \psi_1^* \psi_2\right) = \text{real}, \\ x_3 = \frac{1}{2}\left(\psi_1 \psi_1^* - \psi_2 \psi_2^*\right) = \text{real}. \end{cases} \qquad (1.8)$$

Consider now

$$x^2 = x_1^2 + x_2^2 + x_3^2 = \frac{1}{4}\left(\psi_1 \varphi_2 - \psi_2 \varphi_1\right)^2 = \frac{1}{4}\left(\psi_1 \psi_1^* + \psi_2 \psi_2^*\right)^2 .$$

We see from (1.2) that x^2 is an invariant. Under the unitary unimodular group the vector $\vec{x}$ therefore transforms like a real three-dimensional vector under three-dimensional rotations:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = R(\alpha, \beta) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} .$$

From (1.6), (1.7) we find the explicit form for $R(\alpha, \beta)$

$$R(\alpha,\beta) = \tfrac{1}{2}\begin{pmatrix} (\alpha^2-\beta^2+\alpha^{*2}-\beta^{*2}), & i\,(-\alpha^2-\beta^2+\alpha^{*2}+\beta^{*2}), & 2(-\alpha\beta-\alpha^*\beta^*) \\ i(\alpha^2-\beta^2-\alpha^{*2}+\beta^{*2}), & (\alpha^2+\beta^2+\alpha^{*2}+\beta^{*2}), & -2i(\alpha\beta-\alpha^*\beta^*) \\ 2(\alpha\beta^*+\alpha^*\beta), & 2i\,(-\alpha\beta^*+\alpha^*\beta), & 2(\alpha\alpha^*-\beta\beta^*) \end{pmatrix} = \text{real.} \tag{1.9}$$

It is easily seen that the 3-dimensional rotation group $\{R\}$ is homomorphic to the unimodular, unitary group $\{A\}$. Thus $\{A\}$ is a 2-dimensional (spinor) representation of the rotation group. However, since the elements of $R(\alpha,\beta)$ depend on the squares of α and β, a change of sign of α and β does not change R so that to $R(\alpha,\beta)$ there correspond two elements, A and $-A$ of $\{A\}$. Hence $\{A\}$ is not a faithful representation of $\{R\}$.

In general, we see that a vector of $v + 1$ dimensions

$$\psi = \left(\psi^{(o)},\ \psi^{(1)},\ \ldots\ldots\ \psi^{(v)}\right)$$

which transforms like

$$\left(\psi_1^{v}\ ,\ \psi_1^{v-1}\ \psi_2\ ,\ldots\ldots\ \psi_2^{v}\right)$$

undergoes a linear transformation when ψ transforms under A. In other words if we form a vector with components proportional to ψ_1^v, $\psi_1^{v-1}\psi_2$ ...... etc., which we label $\psi^{(o)}, \psi^{(1)}$ ...... etc. then when the ψ_1 and ψ_2 transform according to (1-1), the vector $\left(\psi^{(o)}, \psi^{(1)}, \ldots\ldots\ \psi^{(v)}\right)$ undergoes a linear transformation. The set of matrices which act in this space of $v + 1 = 2J + 1$ dimensions constitutes a representation D_J of the unimodular unitary group. The group $\{A\}$ itself in this new notation is $D_{\frac{1}{2}}$.

We now wish to show that the <u>transformation matrices of D_j are unitary</u>.

If A is unitary

$$\psi'^{+} \psi' = \psi^{+} \psi$$

then

$$(\psi'^{+} \psi')^{v} = (\psi^{+}\psi)^{v}$$

or

$$\left(\psi_1'^{*} \psi_1' + \psi_2'^{*} \psi_2'\right)^{v} = \left(\psi_1^{*} \psi_1 + \psi_2^{*} \psi_2\right)^{v} ,$$

$$\sum_{r=0}^{v} \binom{v}{r} \psi_1'^{*v-r} \psi_2'^{*r} \psi_1'^{v-r} \psi_2'^{r} = \sum_{r=0}^{v} \binom{v}{r} \psi_1^{*v-r} \psi_2^{*r} \psi_1^{v-r} \psi_2^{r} ,$$

so we must have

$$\sum_{r=0}^{v} \binom{v}{r} \psi'^{*(r)} \psi'^{(r)} = \sum_{r=0}^{v} \binom{v}{r} \psi^{*(r)} \psi^{(r)} .$$

If we define

$$\varphi(r) \equiv \binom{v}{r}^{\frac{1}{2}} \frac{\psi^{(r)}}{\sqrt{v!}} = \frac{\psi^{(r)}}{\sqrt{r!(v-r)!}} , \qquad (1.10)$$

then the matrices of D_j leave the following form invariant:

$$\sum_{r=0}^{v} \varphi'^{*(r)} \varphi'^{(r)} = \sum_{r=0}^{v} \varphi^{*(r)} \varphi^{(r)}$$

and thus the matrices of D_j are unitary.

We put

$$J = \tfrac{1}{2} v \qquad M = r - J$$

and hence the components of $\varphi^{(M)}$ transform like:

$$\varphi^{(M)} \equiv \varphi^{(J+M)} \equiv \frac{\psi^{(J+M)}}{\sqrt{(J+M)!(J-M)!}} \sim \frac{\psi_1^{J-M}\,\psi_2^{J+M}}{\sqrt{(J+M)!(J-M)!}}\,, \quad M = -J,\ldots+J.$$

For the various values of J we have:

J = 0 : the identity,

J = ½ : the unitary group,

J = 1 : the rotation group (the representation of the group by itself, or principal representation).

And so on. In general J = integral are known as <u>tensor representations</u>, whereas J = half-integral are known as <u>spinor representations</u>. However, one can speak of both as spinor representations.

3. INFINITESIMAL ROTATIONS.

Consider the rotation

$$x_i' = x_i + \epsilon_{ij}\,x_j \;,$$

$$(\epsilon\text{ -infinitesimal}) .$$

From the condition

$$(\vec{x})^2 = \text{invariant} ,$$

it follows that:

$$\epsilon_{ij} = -\epsilon_{ji} \quad \text{(3 independent parameters)}$$

Consider now the spinor space. We have:

$$\psi' = D(\epsilon)\,\psi,$$

where

$$D(\epsilon)\,D(\epsilon') = D(\epsilon + \epsilon') \qquad (1.11)$$

and:

$$D(0) = 1.$$

then:

$$\psi' = D(\epsilon)\,\psi = \psi + \tfrac{1}{2}\,(\partial\psi/\partial\epsilon_{ij})_{\epsilon = 0} \times \epsilon_{ij} \qquad (1.12)$$

(the factor $\frac{1}{2}$ is needed because we sum over all i, j but only 3 ϵ's are independent).

Define

$$I_{ij}\,\psi \equiv \left(\frac{\partial\psi'}{\partial\epsilon_{ij}}\right)_{\epsilon = 0};$$

then

$$\psi' = \psi + \tfrac{1}{2}\, I_{ij}\,\psi\,\epsilon_{ij}$$

and:

$$D(\epsilon) = I + \tfrac{1}{2}\, I_{ij}\;\epsilon_{ij}\,.$$

Now, from (1.11) is follows that

$$\frac{d}{dt}\, D\,(t\epsilon) = D\,(t\epsilon)\,\tfrac{1}{2}\, I_{ij}\;\epsilon_{ij}\,.$$

Integrating and setting t = 1 we obtain

$$D(\epsilon) = e^{\frac{1}{2}\, I_{ij}\;\epsilon_{ij}}$$

If we set

$$L_{ij} \equiv - iI_{ij}$$

then

$$D(\epsilon) = e^{\frac{1}{2} i L_{ij} \epsilon_{ij}} . \qquad (1.13)$$

The I_{ij}'s have to satisfy the following commutation relations which are required for the integrability of the differential equations which determine the representation:

$$\left[I^{\nu\xi}, I^{\nu'\xi'}\right] = I^{\nu\xi'} \delta_{\xi\nu'} + I^{\xi\nu'}\delta_{\nu\xi'} - I^{\nu\nu'}\delta_{\xi\xi'} - I^{\xi\xi'}\delta_{\nu\nu'} . \qquad (1.14)$$

These reduce to the usual conditions,

$$\begin{cases} \left[L_1, L_2\right] = i L_3 \\ \left[L_2, L_3\right] = i L_1 \\ \left[L_3, L_1\right] = i L_2 , \end{cases} \qquad (1.15)$$

if we set

$$I_{23} \equiv I_1 \equiv i L_1$$

$$I_{31} \equiv I_2 \equiv i L_2$$

$$I_{12} \equiv I_3 \equiv i L_3 .$$

Consider a rotation $\varphi = \epsilon_{21}$ about the x_3 axis. The equations of transformation for $\vec{x}$ written in the infinitesimal form are:

$$\begin{cases} x_1' = x_1 - \varphi x_2 \\ x_2' = x_2 + \varphi x_1 \\ x_3' = x_3 . \end{cases} \qquad (1.16)$$

The R matrix then has the form:

$$R(\varphi) = \begin{pmatrix} 1 & -\varphi & 0 \\ \varphi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (1.17)$$

Alternatively R can be written as:

$$R(\varphi) = I - iL_3\varphi$$

where

$$L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (1.18)$$

For finite values of φ one has from (1.13):

$$R_3(\varphi) = e^{-iL_3\varphi}$$

Expanding in series and keeping in mind the fact that:

$$L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_3^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ etc.,}$$

we have:

$$R_3(\varphi) = e^{-iL_3\varphi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \varphi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\varphi^2}{2!} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \ldots$$

Collecting the terms one gets

$$R_3(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (1.19)$$

which is of course the expected result. Similarly for finite rotations $R_2(\psi)$ and $R_1(\theta)$ about the x_2 and x_1 axes, respectively, we have:

$$R_2(\psi) = e^{-iL_2\psi} = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix} \qquad (1.20)$$

and

$$R_1(\theta) = e^{-iL_1\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}. \qquad (1.21)$$

L_1 and L_2 have the following form:

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad ; \quad L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}. \qquad (1.22)$$

The A matrices corresponding to the various rotations R_1, R_2 and R_3 can be found by comparing (1.19), (1.20) and (1.21) with (1.9) and by taking account of the unimodularity condition. Straight forward calculations yield:

$$A_1 = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \quad \text{(rotation about } x_1\text{);}$$

$$A_2 = \begin{pmatrix} \cos\frac{\psi}{2} & -\sin\frac{\psi}{2} \\ \sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{pmatrix} \quad \text{(rotation about } x_2\text{);}$$

$$A_3 = \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix} \quad \text{(rotation about } x_3\text{).}$$

The form of the A's for infinitesimal rotations is:

$$A_j = I - i S_j \varphi_j$$

where $\varphi_j = \theta, \psi, \varphi$ for $j = 1, 2, 3$ respectively, and the S_j's turn out to be Pauli's spin matrices:

$$\begin{cases} S_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ +\frac{1}{2} & 0 \end{pmatrix} = \frac{1}{2}\sigma_1 \\ S_2 = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \frac{1}{2}\sigma_2 \\ S_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = \frac{1}{2}\sigma_3 . \end{cases} \qquad (1.23)$$

We must note that the correspondence between $D_{\frac{1}{2}}$ (A) and D_1 (R' S) <u>is not one-to-one</u>. The identity in D_1 leaves x_1, x_2, x_3 unchanged. To that there correspond in $D_{\frac{1}{2}}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus to <u>every</u> rotation there correspond two elements of $D_{\frac{1}{2}}$, A and -A. $D_{\frac{1}{2}}$ <u>is a two valued representation of the rotation group</u>.

The representations with half integer index (spinor representations) are two-valued. The representations with J = integer (tensor representations) are one-valued or faithful.

We saw that the problem of finding all the representations of the rotation group is reduced to the problem of finding all possible infinitesimal operators L_1, L_2, L_3 which satisfy the commutation relations (1.15). For the principal representation the

L's are given by (1.18) and (1.22). For the unitary group the L's are nothing but the Pauli spin matrices.

In general the space of the representation of an arbitrary number of dimensions can be built up in the following manner: let

$$L_+ \equiv L_1 + iL_2$$

$$L_- \equiv L_1 - iL_2$$

then:

$$[L_+, L_3] = - L_+$$

$$[L_-, L_3] = + L_-$$

$$[L_+, L_-] = 2 L_3 .$$

To find L_+, L_- and L_3 we take as the basis the eigenvectors of L_3:

$$L_3 \mathcal{V}_M = M \mathcal{V}_M .$$

The operators L_+ and L_- have the property that:

$$L_+ \mathcal{V}_M = \varkappa_{M+1} \mathcal{V}_{M+1}$$

$$L_- \mathcal{V}_M = \varkappa_{M-1} \mathcal{V}_{M-1}$$

$$\varkappa_M = \sqrt{J(J+1) - M (M-1)}$$

with

$$|M| \leqslant J .$$

The vectors $\mathcal{V}_J \ldots\ldots \mathcal{V}_{-J}$ transform among themselves under L_+, L_-, L_3 so that the R_{2J+1}, space transforms into itself under all transformations of the representation. Furthermore, R_{2J+1} is irreducible,that is to say, it contains no subspace other than itself and zero,which is invariant under D_J.

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CHAPTER 2

THE LORENTZ GROUP AND ITS SPINOR REPRESENTATIONS

1. REPRESENTATIONS OF THE LORENTZ GROUP.

Consider the space of 4-vectors $x^\mu = (x^0 = t, x^1, x^2, x^3)$ with the metric

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}.$$

The group of real transformations which transforms this space into itself while leaving the length of the 4-vector invariant,

$$x^2 \equiv g_{\mu\nu} x^\mu x^\nu = t^2 - (\bar{x})^2 = \text{invariant},$$

is called the (homogeneous) Lorentz group.

Thus

$$x'^\mu = \ell^\mu_\nu x^\nu, \quad \text{or} \quad x' = L\,x, \tag{2.1}$$

$$g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} x^\mu x^\nu ,$$

or

$$g_{\mu\nu} \ell^\mu_\lambda \ell^\nu_\tau = g_{\lambda\tau} . \tag{2.2}$$

This is a condition on the transformation matrix $L = (\ell^\mu_\nu)$. It follows from (2.2) that

$$\det |\ell^\mu_\nu| = \pm 1$$

For $\lambda = \tau = 0$, (2.2) gives:

$$g_{\mu\nu} \, l^{\mu}_{o} \, l^{\nu}_{o} = 1,$$

or

$$(l^{o}_{o})^2 - \sum_{i=1}^{3} (l^{i}_{o})^2 = 1 .$$

Since all l^{μ}_{ν} are real,

$$(l_{oo})^2 = (l^{o}_{o})^2 \geqslant 1 .$$

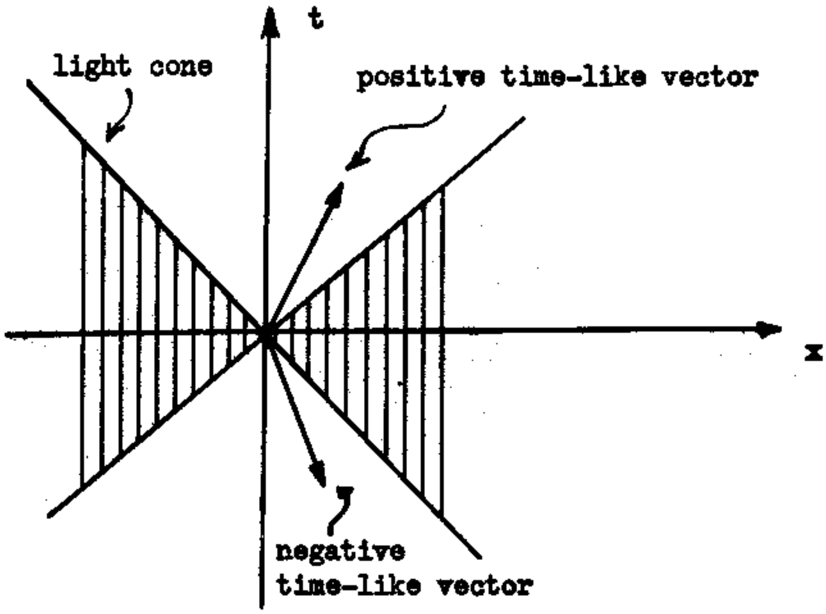
We distinguish between the following four parts of the homogeneous Lorentz group:

$$\det |l^{\mu}_{\nu}| = 1, \quad \begin{cases} l_{oo} \geqslant 1, & \text{(I)} \\ l_{oo} \leqslant -1, & \text{(II)} \end{cases}$$

$$\det |l^{\mu}_{\nu}| = -1, \quad \begin{cases} l_{oo} \geqslant 1, & \text{(III)} \\ l_{oo} \leqslant -1. & \text{(IV)} \end{cases}$$

Only the first part (I) is a subgroup of the Lorentz group, because only it contains the identity. It is called the proper orthochronous Lorentz group or continuous Lorentz group; it transforms a positive time-like vector ($(x^o)^2 - \sum_{i=1}^{3} (x^i)^2 > 0$, $x_o > 0$) into another positive time-like vector. The continuous Lorentz group together with any one of the other three parts again form a group, which then includes space and time reflections (discontinuous operations).

Just as the three-dimensional rotation group has a two-valued representation by the two-dimensional unitary unimodular group, so the continuous Lorentz group has a two-valued representation by the two-dimensional unimodular group (not unitary!).



Consider the group of unimodular, 2 × 2 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} ,$$

$$\det A = \alpha\delta - \gamma\beta = 1 .$$

A has 6 parameters corresponding to the 6 parameters specifying a general Lorentz transformation.

The contravariant spinor $\psi \equiv \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$ transforms like

$$\psi' = A\psi \qquad \text{or} \qquad \psi'^{\mu} = a^{\mu\lambda} \psi^{\lambda}$$

We also define a dotted contravariant spinor $\dot{\varphi} \equiv \begin{pmatrix} \varphi^{\dot{1}} \\ \varphi^{\dot{2}} \end{pmatrix}$ by the transformation:

$$\dot{\varphi}' = (A^{\dagger})^{T} \dot{\varphi} \qquad \text{or} \qquad \varphi'^{\dot{\mu}} = a^{*\dot{\mu}\dot{\lambda}} \varphi^{\dot{\lambda}}$$

We see that a dotted spinor transforms like the complex conjugate of an undotted spinor. Tensors with dotted and undotted indices transform like the corresponding products of dotted and undotted spinors.

In particular,

$$\Psi'^{\lambda\dot{\mu}} = a^{\lambda\lambda'} \quad a^{*\dot{\mu}\dot{\mu}'} \quad \Psi^{\lambda'\dot{\mu}'} \qquad (2.3)$$

Regarding $\Psi^{\lambda\dot{\mu}}$ as a 2×2 matrix, this can be written

$$\Psi' = A \Psi A^{\dagger} \qquad (A^{\dagger} \equiv (A^{*})^{T}).$$

It is obvious that, since det A = 1, det Ψ' = det Ψ, or

$$\det \Psi = \begin{vmatrix} \psi^{1\dot{1}} & \psi^{1\dot{2}} \\ \psi^{2\dot{1}} & \psi^{2\dot{2}} \end{vmatrix} = \text{invariant.}$$

Now we have seen that the dotted indices transform "like the complex conjugates" of the correponding undotted indices. Therefore we choose $\psi^{1\dot{1}}$, $\psi^{2\dot{2}}$ real, and $\psi^{1\dot{2}} = \left(\psi^{2\dot{1}}\right)^{*}$. We can then introduce the real vector

$$x = (x^{o}, \vec{x}) = (t, x, y, z)$$

by:

$$\begin{cases} \psi^{1\dot{1}} = z + t, \\ \psi^{2\dot{2}} = -z + t, \\ \psi^{1\dot{2}} = x + iy, \\ \psi^{2\dot{1}} = x - iy. \end{cases} \qquad (2.4)$$

Then

$$\det \begin{vmatrix} \Psi^{1\dot{1}} & \Psi^{1\dot{2}} \\ \Psi^{2\dot{1}} & \Psi^{2\dot{2}} \end{vmatrix} = \det \begin{vmatrix} z + t & x + iy \\ x - iy & -z + t \end{vmatrix} =$$

$$= t^2 - x^2 - y^2 - z^2 = \text{invariant.}$$

Under the unimodular group, the vector x defined by (2.4) therefore undergoes a linear transformation L such that $t^2 - \vec{x}^2 = $ = inv:

$$x' = L\,x\,, \qquad t^2 - \vec{x}^2 = \text{inv.}$$

By using (2.3) and (2.4), we find the explicit expression for L in terms of the parameters $\alpha, \beta, \gamma, \delta$ of the unimodular group:

$$\left(\begin{array}{c|c|c|c}
\mathrm{Re}(\beta\gamma^* + \delta\alpha^*) & -\mathrm{Im}(\alpha^*\delta + \gamma^*\beta) & \mathrm{Re}(\alpha\gamma^* - \beta\delta^*) & -\mathrm{Re}(\alpha\gamma^* + \beta\delta^*) \\
\hline
-\mathrm{Im}(\beta\gamma^* + \alpha\delta^*) & \mathrm{Re}(\alpha\delta^* - \gamma\beta^*) & \mathrm{Im}(\alpha\gamma^* - \beta\delta^*) & \mathrm{Im}(\alpha\gamma^* + \beta\delta^*) \\
\hline
\mathrm{Re}(\beta\alpha^* - \delta\gamma^*) & -\mathrm{Im}(\beta\alpha^* - \delta\gamma^*) & \begin{array}{c}\alpha\alpha^* - \beta\beta^* - \\ -\gamma\gamma^* + \delta\delta^*\end{array} & \begin{array}{c}\alpha\alpha^* + \beta\beta^* - \\ -\gamma\gamma^* - \delta\delta^*\end{array} \\
\hline
-\mathrm{Re}(\beta\alpha^* + \delta\gamma^*) & \mathrm{Im}(\beta\alpha^* + \delta\gamma^*) & \begin{array}{c}\alpha\alpha^* - \beta\beta^* \\ +\gamma\gamma^* - \delta\delta^*\end{array} & \begin{array}{c}\alpha\alpha^* + \beta\beta^* \\ +\gamma\gamma^* + \delta\delta^*\end{array}
\end{array}\right)$$

where Re = real part,

Im = imaginary part.

We also see that

$$\ell_{oo} = \tfrac{1}{2}\,(\alpha\alpha^* + \beta\beta^* + \gamma\gamma^* + \delta\delta^*) \geqslant 0$$

and

$$\det L = +1,$$

so we have indeed found a representation of the proper orthochronous (continuous) Lorentz group.

As in the case of the three-dimensional rotation group, it can be shown that to every general continuous Lorentz transformation there correspond just <u>two</u> matrices of the unimodular group, A and -A; thus the unimodular group is a two-valued representation of the continuous Lorentz group. This is also obvious from the explicit expression for L (2.5), which is unchanged if A is replaced by -A. If we specialize to the case A = unitary, then

$$\gamma = -\beta^* , \quad \delta = \alpha^* , \quad \alpha\delta - \beta\gamma = 1,$$

and

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \boxed{\diagup\diagup\diagup} & \\ 0 & & & \end{pmatrix} ;$$

these are the three-dimensional rotations which leave $\vec{x}^2$ invariant.

It is clear that the set of

$$\psi_{MM'} = \frac{(\psi^1)^{J+M}\,(\psi^2)^{J-M}}{\sqrt{(J+M)!\,(J-M)!}} \cdot \frac{(\varphi^{\dot{1}})^{J'+M'}\,(\varphi^{\dot{2}})^{J'-M'}}{\sqrt{(J'+M')!\,(J'-M')!}} \qquad (2.6)$$

form a space R(J,J') of (2J + 1) (2J'+ 1) dimensions which transforms into itself under continuous transformations. A linear combination of the $\psi_{MM'}$, is expressed by

$$C_{\lambda\mu\ldots\nu\dot{\rho}\dot{\sigma}\ldots\dot{\tau}}\,\psi^\lambda\psi^\mu\ldots\psi^\nu\varphi^{\dot{\rho}}\varphi^{\dot{\sigma}}\ldots\varphi^{\dot{\tau}},$$

where C is separately symmetric in the dotted and undotted indices.

2. CERTAIN INVARIANT QUANTITIES AND THE COVARIANT DIRAC EQUATION.

Earlier we introduced the invariant spinor quantities

$$-\epsilon_{\lambda\mu} = \epsilon^{\lambda\mu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which serve to raise and lower undotted spinor indices.

In the same spirit we now introduce the invariant dotted spinor quantities

$$-\epsilon_{\dot{\lambda}\dot{\mu}} = \epsilon^{\dot{\lambda}\dot{\mu}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

to raise and lower dotted spinor indices.

The metric

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$ is an example

of an invariant tensor quantity in the four-dimensional vector space. It serves to raise and lower the four-valued space-time indices.

We further introduce the invariant mixed spinor-tensor quantities

$$(\sigma_k)^{\lambda\dot{\mu}} \quad (k = 0, 1, 2, 3; \lambda, \dot{\mu} = 1, 2):$$

$$\begin{cases} (\sigma_0)^{\lambda\dot{\mu}} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (\sigma^0)_{\lambda\dot{\mu}} , \\ (\sigma_1)^{\lambda\dot{\mu}} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\sigma^1)_{\lambda\dot{\mu}} , \\ (\sigma_2)^{\lambda\dot{\mu}} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -(\sigma^2)_{\lambda\dot{\mu}} , \\ (\sigma_3)^{\lambda\dot{\mu}} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (\sigma^3)_{\lambda\dot{\mu}} . \end{cases} \tag{2.7}$$

The last equations follow from the first equations by lowering $\lambda, \dot{\mu}$ and raising k, e. g.:

$$(\sigma^2)_{\lambda\dot{\mu}} = g^{2k} \epsilon_{\lambda\lambda'} \epsilon_{\dot{\mu}\dot{\mu}'} (\sigma_k)^{\lambda'\dot{\mu}'}$$

$$= -(\epsilon_{\lambda\lambda'}) (\sigma_2)^{\lambda'\dot{\mu}'} (\epsilon_{\dot{\mu}'\dot{\mu}})^T$$

$$= - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Using (2.7) explicitly, we prove

$$(\sigma_k)^{\lambda\dot{\mu}} (\sigma^{k'})_{\lambda\dot{\mu}} = 2 \delta_k{}^{k'} . \qquad (2.8)$$

Also

$$(\sigma^k)^{\lambda\dot{\nu}} (\sigma_{k'})_{\lambda\dot{\nu}'} + (\sigma_{k'})^{\lambda\dot{\nu}} (\sigma^k)_{\lambda\dot{\nu}'} = 2 \delta_{k'}{}^k \delta_{\dot{\nu}'}{}^{\dot{\nu}} ,$$

or after contraction with $g^{\ell k'}$:

$$(\sigma^k)^{\lambda\dot{\nu}} (\sigma^\ell)_{\lambda\dot{\nu}'} + (\sigma^\ell)^{\lambda\dot{\nu}} (\sigma^k)_{\lambda\dot{\nu}'} = 2 g^{k\ell} \delta_{\dot{\nu}'}{}^{\dot{\nu}} . \qquad (2.9)$$

Similarly,

$$(\sigma^k)^{\lambda\dot{\mu}} (\sigma^\ell)_{\lambda'\dot{\mu}} + (\sigma^\ell)^{\lambda\dot{\mu}} (\sigma^k)_{\lambda'\dot{\mu}} = 2 g^{k\ell} \delta_{\lambda'}{}^{\lambda} .$$

With the help of $(\sigma_k)^{\lambda\dot{\mu}}$ we see that (2.[illegible]) can also be written as

$$\psi^{\lambda\dot{\mu}} = x^k (\sigma_k)^{\lambda\dot{\mu}} \left(= \begin{pmatrix} z+t & x+iy \\ x-iy & -z+t \end{pmatrix} \right).$$

Multiply this by $(\sigma^\ell)_{\lambda\dot{\mu}}$, and contract over $\lambda\dot{\mu}$. Then (2.8) gives

$$x^\ell = \tfrac{1}{2} (\sigma^\ell)_{\lambda\dot{\mu}} \psi^{\lambda\dot{\mu}} ,$$

which is the exact generalization of (1.5) to the four-dimensional space.

An invariant connection between an undotted spinor ψ^λ and a dotted spinor $\varphi_{\dot\mu}$ is clearly given by

$$\psi^\lambda = \psi^{\lambda\dot\mu}\,\varphi_{\dot\mu} = x^k\,(\sigma_k)^{\lambda\dot\mu}\,\varphi_{\dot\mu}\ .$$

Multiply this by $x^\ell\,(\sigma_\ell)_{\lambda\dot\nu}$ and contract over λ.

Then (2.9) gives

$$\varphi_{\dot\nu} = \frac{1}{x^2}\,x^\ell\,(\sigma_\ell)_{\lambda\dot\nu}\,\varphi^\lambda\ , \text{ where } x^2 \equiv x^\mu x_\mu\ .$$

An invariant connection of this type is given by the set of Dirac equations:

$$\begin{cases} p^k\,(\sigma_k)_{\lambda\dot\mu}\,\chi^{\dot\mu} = m\,\psi_\lambda \\ \psi_\lambda\,p^k\,(\sigma_k)^{\lambda\dot\mu} = m\,\chi^{\dot\mu}. \end{cases} \tag{2.10}$$

With the help of (2.9) we obtain

$$(p_k\,p^k - m^2)\begin{Bmatrix}\psi \\ \dot\chi\end{Bmatrix} = 0\ .$$

We write the Dirac equations in a more familiar notation:

$$p^\mu \equiv (p^0, \vec p),$$

$$(\sigma^k)_{\lambda\dot\mu} \equiv (1, \vec\sigma),$$

$$\psi \equiv \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}\ ;\ \dot\chi \equiv \begin{pmatrix}\chi^{\dot 1} \\ \chi^{\dot 2}\end{pmatrix}$$ → ($\vec\sigma$ are the familiar Pauli matrices).

Then (2.10) can be written

$$\begin{cases} (p^o \ \sigma^o - \vec{p}.\vec{\sigma})\dot{\chi} = m\,\psi \ , \\ \psi^T\,(p^o\sigma^o - \vec{p}\,.\,\epsilon\vec{\sigma}\epsilon^T) = m\dot{\chi}^T \ , \end{cases}$$

or since

$$\epsilon\vec{\sigma}\epsilon^T = -\vec{\sigma}^T \ :$$

$$\begin{cases} (p^o - \vec{p}.\vec{\sigma})\,\dot{\chi} = m\,\psi \\ (p^o + \vec{p}.\vec{\sigma})\,\psi = m\,\dot{\chi} \ . \end{cases} \tag{2.11}$$

By summing and subtracting the equations (2.11) we find also

$$\begin{cases} p^o \ \psi_\ell - (\vec{p}.\vec{\sigma})\,\psi_o = m\,\psi_\ell \\ p^o \ \psi_o - (\vec{p}.\vec{\sigma})\,\psi_\ell = -m\,\psi_o, \end{cases} \tag{2.12}$$

where

$$\begin{cases} \psi_\ell = \dot{\chi} + \psi \\ \psi_o = \dot{\chi} - \psi \ . \end{cases}$$

Both (2.11) and (2.12) can be written in the form

$$\left\{ \gamma^o \ p^o - \vec{\gamma}\cdot\vec{p} \right\} \Psi = m\Psi \tag{2.13}$$

or

$$\gamma^\mu p_\mu \Psi = m\,\Psi \ ,$$

where the γ's satisfy the familiar commutation relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\,g^{\mu\nu} \ .$$

In (2.13) we first choose the representation

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \text{ and } \Psi = \begin{pmatrix} \psi \\ \dot{\chi} \end{pmatrix},$$

and obtain (2.11).

Then we use the representation

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \vec{\gamma} \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \text{ and } \Psi = \begin{pmatrix} \psi_\ell \\ \psi_o \end{pmatrix},$$

in (2.13) and obtain (2.12).

Both representations are of course related by a similarity transformation:

$$\gamma'^\mu = S\,\gamma^\mu\,S^{-1},$$

where:

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix}$$

3. <u>INFINITESIMAL LORENTZ TRANSFORMATIONS. SPINOR REPRESENTATIONS.</u>

Under infinitesimal Lorentz transformations,

$$x'^\mu = (\delta^\mu_\nu + \epsilon^\mu_\nu)\,x^\nu, \quad \epsilon_{\mu\nu} = -\epsilon_{\nu\mu},$$

any representation space is transformed into itself according to:

$$\psi' = D(\epsilon)\,\psi,$$

where

$$D(\epsilon) = I - \frac{1}{2}\,\epsilon_{\mu\nu}\,M^{\mu\nu}; \quad (M^{\mu\nu} = -M^{\nu\mu}).$$

The $M^{\mu\nu}$ are the infinitesimal transformation operators of the representation of the Lorentz group. It can be shown by requiring that the differential equations which determine the representation be integrable, that the $M^{\mu\nu}$ must satisfy the commutation rules

$$\left[M^{\alpha\beta}, M^{\alpha'\beta'}\right] = i\left\{M^{\alpha\beta'} g^{\beta\alpha'} + M^{\beta\alpha'} g^{\alpha\beta'} - M^{\alpha\alpha'} g^{\beta\beta'} - M^{\beta\beta'} g^{\alpha\alpha'}\right\}. \tag{2.14}$$

Introduce

$$M^{ij} \equiv M^k \text{ (Cyclic)}; \quad M^{oi} \equiv N^i \ (i, j, k = 1, 2, 3).$$

Furthermore define

$$K_j \equiv \tfrac{1}{2}(M^j - iN^j); \qquad L_j \equiv \tfrac{1}{2}(M^j + iN^j) \ ;$$

$$K_{\pm} \equiv K_1 \pm iK_2 \ ; \qquad L_{\pm} \equiv L_1 \pm iL_2.$$

Then we find from (2.14) that all K's commute with all L's and:

$$\begin{cases} [K_3, K_{\pm}] = \pm K_{\pm} \\ [K_+, K_-] = 2K_3 \end{cases} \qquad \begin{cases} [L_3, L_{\pm}] = \pm L_{\pm} \\ [L_+, L_-] = 2L_3. \end{cases}$$

We see that the K's and L's operate in two independent spaces and satisfy separately the commutation rules of the three-dimensional angular momenta. For these it is known how the vector spaces for K and L are to be constructed. As a result we obtain a finite-dimensional vector space $\{v_{MM'}\}$ formed of eigenvectors of K^2, K_3, L^2, L_3, with

$$K_+ v_{MM'} = \sqrt{(J-M)(J+M+1)}\ v_{M+1,M'} ;$$
$$(M = -J, \ldots, J)$$
$$K_- v_{MM'} = \sqrt{(J+M)(J-M+1)}\ v_{M-1,M'} ;$$

$$K_3 v_{MM'} = M\, v_{MM'} ;$$

$$L_+ v_{MM'} = \sqrt{(J'-M')(J'+M'+1)}\ v_{M,M'+1} ;$$
$$(M'=-J',\ldots,J')$$
$$L_- v_{MM'} = \sqrt{(J'+M')(J'-M'+1)}\ v_{M,M'-1} ;$$

$$L_3 v_{MM'} = M'\, v_{MM'} .$$

This determines a (2J+1)(2J'+1) dimensional representation of the Lorentz group which can be easily shown to be irreducible. The corresponding space is actually identical with the representation space of D(J, J'), given by the vectors $\psi_{MM'}$ of (2.6).

For different values J, J' we have:

J	J'	dimension of space	representation space
0	0	1	ψ (scalar)
1/2	0	2	ψ^{λ}(spinor)
0	1/2	2	$\psi^{\dot\lambda}$(dotted spinor)
1/2	1/2	4	$\psi^{\mu\dot\lambda}$(vector; principal representation)
1	0	3	$\psi^{\lambda\mu}$(self-dual second rank skew tensor)
...	...	...	....

If J, J' are integers, the representation is faithful.

Otherwise it is two-valued.

These are all the finite-dimensional representations of the Lorentz group. In contrast to the finite-dimensional representations of the three-dimensional rotation group, these are not unitary. In fact, it can be shown that only infinite-dimensional representations of the Lorentz group can be unitary. We indicate the proof in the following section.

4. FINITE - AND INFINITE-DIMENSIONAL REPRESENTATIONS OF THE LORENTZ GROUP.

The consideration of the operators K and L led us only to the finite-dimensional representations of the Lorentz group. For the general discussion we consider instead the operators:

$$\begin{cases} M_{\pm} = M^{1}{}_{\pm}\ i\, M^{2}\,, \\ M_{3} \end{cases} \quad \text{and} \quad \begin{cases} N_{\pm} = N^{1}{}_{\pm}\ i\, N^{2}\,, \\ N_{3}\,. \end{cases}$$

They satisfy the commutation rules:

$$\begin{cases} \left[M_3, M_{\pm}\right] = \pm M_{\pm}\ ; \quad \left[N_3, N_{\pm}\right] = \pm M_{\pm}\ ; \\ \left[M_+, M_-\right] = 2\, M_3\ ; \quad \left[N_+, N_-\right] = -2\, M_3\ ; \\ \left[M_+, M_+\right] = \left[M_-, N_-\right] = \left[M_3, N_3\right] = 0\ ; \\ \left[N_3, M_{\pm}\right] = \pm N_{\pm}\ ; \left[M_3, N_{\pm}\right] = \pm N_{\pm}\ ; \\ \left[N_{\mp}, M_{\pm}\right] = \mp 2\, N_3\,. \end{cases} \tag{2.15}$$

A representation of the proper orthochronous Lorentz group is also a representation of the three-dimensional rotation subgroup. The M's are infinitesimal operators of this subgroup representation. The representation space generated by these operators is determined in the same way as in the three-dimensional case. The treatment of the N's is different, since they do not obey the same commutation rules as the M's. With the help of the commutation rules (2.15) the following <u>theorem</u> can be proven:

The irreducible representations of the proper orthochronous Lorentz group are determined by a pair of numbers (k_o, C) where k_o is an integral or half-odd integral non-negative number and C is a complex number. The irreducible representation corresponding to a given (k_o, C) is given by the following relations for a suitable basis f^k_μ:

$$M_+ \; f^k_\mu = \sqrt{(k+\mu+1)(k-\mu)} \; f^k_{\mu+1} \; ;$$

$$M_- \; f^k_\mu = \sqrt{(k+\mu)(k-\mu+1)} \; f^k_{\mu-1} \; ;$$

$$M_3 \; f^k_\mu = \mu \; f^k_\mu \; ;$$

These are the familiar relations for the three-dimensional angular momenta.

$$N_+ \; f^k_\mu = \sqrt{(k-\mu)(k-\mu-1)} \; C_k \; f^{k-1}_{\mu+1} - \sqrt{(k-\mu)(k+\mu+1)} \; A_k \; f^k_{\mu+1}$$

$$+ \sqrt{(k+\mu+1)(k+\mu+2)} \; C_{k+1} \; f^{k+1}_{\mu+1} \; ;$$

$$N_- f^k_\mu = - \sqrt{(k+\mu)(k+\mu-1)}\; C_k\, f^{k-1}_{\mu-1} - \sqrt{(k+\mu)(k-\mu+1)}\; A_k\, f^k_{\mu-1}$$

$$- \sqrt{(k-\mu+1)(k-\mu+2)}\; C_{k+1}\, f^{k+1}_{\mu-1} \;;$$

$$N_3 f^k_\mu \quad \sqrt{(k-\mu)(k+\mu)}\; C_k\, f^{k-1}_\mu - \mu A_k\, f^k_\mu - \sqrt{(k+\mu+1)(k-\mu+1)}\; \cdot$$

$$C_{k+1}\, f^{k+1}_\mu \;; \tag{2.16}$$

where

$$A_k = \frac{i\, k_o\, C}{k(k+1)} \;, \quad C_k = \frac{i}{k}\sqrt{\frac{(k^2-k_o^2)(k^2-C^2)}{4\,k^2-1}}$$

a) If $C^2 = (k_o + n)^2$ for some integral n, the representation is finite-dimensional and then

$$\mu = -k, -k+1, \ldots k-1, k; \; k = k_o, k_o+1, \ldots, k_o+n$$

b) If $C^2 \neq (k_o+n)^2$ for any integral n, then the representation is infinite-dimensional and

$$\mu = -k, \ldots, k; \quad k = k_o, k_o+1, \ldots$$

In case a) the coefficients C_k are all zero for $k \geqslant k_o + n$. We see from (2.16) that the operators N generate a representation space of n dimensions in the index K. This space is equivalent to the previously mentioned spinor representation space of $D(J, J')$. In case b) the C_k's will not become zero for any possible value $k > k_o$, so that the operators N generate an infinite-dimensional space in the index K.

5. UNITARY REPRESENTATIONS OF THE LORENTZ GROUP.

We saw that (for ϵ infinitesimal):

$$D(\epsilon) = I - \frac{i}{2} M^{\lambda\mu} \epsilon_{\lambda\mu} .$$

If we now require that the representation $D(\epsilon)$ be unitary, $D^{+} = D^{-1}$, we find

$$M^{\lambda\mu\dagger} = M^{\lambda\mu}, \quad \text{or} \quad M^{i\dagger} = M^{i}, \; N^{i\dagger} = N^{i}.$$

Hence

$$\left(N_3 f, g\right) = \left(f, N_3 g\right) .$$

Apply this first to the basis vectors:

$$\left(N_3 \; f^k_\mu , f^k_\mu\right) = \left(f^k_\mu , N_3 \; f^k_\mu\right) .$$

Using the orthonormality of the f^k_μ and (2.16), we find

$$A_k = A_k^* .$$

Now

$$A_k = \frac{i \, k_o \, C}{k(k+1)} \quad \text{, hence}$$

either 1) C is imaginary

or 2) $K_o = 0$.

Now consider

$$\left(N_3 \; f^k_\mu \; , f^{k-1}_\mu\right) = \left(f^k_\mu , \; N_3 \; f^{k-1}_\mu\right) .$$

This leads to

$$C_k = - C_k^* .$$

Since

$$C_k = \frac{i}{k} \sqrt{\frac{(k^2 - k_o^2)(k^2 - C^2)}{4k^2 - 1}} , \quad k^2 \geqslant k_o^2 , \; 4k^2 - 1 > 0$$

we have $k^2 - C^2 > 0$, i.e., C is either real or imaginary, and

$$k^2 - C^2 > 0 .$$

If now

1) C imaginary, we always have $k^2 - C^2 > 0$.

2) C real, then $k_o = 0$, and $C_k = i\sqrt{\frac{k^2 - C^2}{4k^2 - 1}}$,

the condition $k^2 - C^2 \geqslant 0$ for k = 1, 2 ....
leads to the condition $C^2 \leqslant 1$.

Hence we have the theorem:

If the representation of the proper Lorentz group is unitary, then (k_o, C) satisfies one of the following conditions:

1) C is pure imaginary, k_o is an arbitrary non-negative integer or half-odd integer.

2) $C^2 \leqslant 1$, $k_o = 0$.

Since in neither case the condition $C^2 = (k_o + n)^2$ can be fulfilled, we see that unitary representations can only be infinite-dimensional. Later we shall see that the states describing the elementary particles as vectors of a Hilbert space must form the basis of an infinite-dimensional unitary representation of the proper Lorentz group. On the other hand, the representations of this group whose elements act on the tensor or spinor indices of the field variables are finite and non-unitary.

6. IMPROPER LORENTZ TRANSFORMATIONS AND THEIR REPRESENTATIONS IN SPINOR SPACE.

We shall call the improper Lorentz group the group which consists of the proper Lorentz group plus those Lorentz transformations which have determinant (-1) (improper Lorentz transformation) and $\ell_0^0 \geqslant 1$. The space reflection $\vec{x}' = -\vec{x}$, $x'^0 = x^0$ is such an improper Lorentz transformation. Any improper Lorentz transformation can be regarded as a space reflection followed by a proper Lorentz transformation.

The transformation matrix $\mathcal{S}$ for space reflection,

$$\mathcal{S} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

does not commute with the general Lorentz transformation

$$\mathcal{S} L = \begin{pmatrix} \ell_0^0 \; \ell_1^0 \; \cdots\cdots \\ \ell_0^1 \; \cdots\cdots\cdots \end{pmatrix} \neq \begin{pmatrix} \ell_0^0 \; \cdots\cdots \\ \ell_0^1 \; \cdots\cdots \end{pmatrix} = L \mathcal{S}$$

Therefore, if L is represented by a 2×2 matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, i.e. by the unimodular group, and $\mathcal{S}$ by a 2×2 matrix S, we must also have

$$AS \neq SA \tag{2.17}$$

But in the special case of the unitary unimodular group U, which represents space rotations R, we have

$$US = SU$$

because

$$\mathcal{S}R = R\mathcal{S} \qquad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\ddots} \\ 0 & & \end{pmatrix} .$$

Since U is irreducible, S must be a multiple of the unit matrix. This contradicts (2.17).

Thus the improper Lorentz group cannot be represented by a two-dimensional representation. We now show how the improper Lorentz group can be represented by a four-dimensional representation.

Earlier we introduced the contravariant two component spinor ψ^{λ}, which transforms according to

$$\psi^{\lambda'} = a^{\lambda\mu} \psi^{\mu} .$$

We also defined the dotted contravariant spinor $\psi^{\dot\lambda}$ by the transformation

$$\psi^{\dot\lambda'} = a^{\dot\lambda\dot\mu *} \psi^{\dot\mu} .$$

The spinor indices can be lowered with the help of the ϵ matrix. Thus

$$\psi'_{\dot\lambda} = \epsilon_{\dot\lambda\dot\nu} \psi^{\dot\nu'} = \epsilon_{\dot\lambda\dot\nu} a^{\dot\nu\dot\mu *} \epsilon^{-1}{}_{\dot\mu\dot\sigma} \psi_{\dot\sigma} ,$$

$$\begin{pmatrix} \psi'_{\dot 1} \\ \psi'_{\dot 2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi_{\dot 1} \\ \psi_{\dot 2} \end{pmatrix} ,$$

or

$$\begin{pmatrix} \psi'_{\dot 1} \\ \psi'_{\dot 2} \end{pmatrix} = \begin{pmatrix} \delta^* & -\gamma^* \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} \psi_{\dot 1} \\ \psi_{\dot 2} \end{pmatrix} , \quad \psi'_{\dot\mu} = (A^+)^{-1}{}_{\dot\mu\dot\lambda}\, \psi_{\dot\lambda} \; .$$

In the special case of three-dimensional rotations $A^+ A = 1$, and it follows that $\psi_{\dot\mu}$ transforms like ψ^μ under three-dimensional rotations.

We therefore define a "four-component" spinor Ψ as follows:

$$\Psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi_{\dot 1} \\ \psi_{\dot 2} \end{pmatrix}$$

and the matrix

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & (A^\dagger)^{-1} \end{pmatrix} .$$

The set of these $\mathcal{A}$ is manifestly reducible under the proper Lorentz group. But now we include space reflections such as to extend the group to the improper Lorentz group, and take as the four-dimensional representation for space reflection the matrix:

$$S = i \left(\begin{array}{cc|cc} & 0 & 1 & 0 \\ & & 0 & 1 \\ \hline 1 & 0 & & \\ 0 & 1 & & 0 \end{array}\right) , \quad S^2 = -1 \; . \quad (2.19)$$

The operation with S on Ψ takes an upper undotted index into a lower dotted index and viceversa. The set of $\mathcal{A}$'s together with S is now an irreducible representation of the improper Lorentz group. S does not commute with the $\mathcal{A}$'s. Only in the special case of three-dimensional rotations, where $\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, does S commute with the $\mathcal{A}$, as was to be expected.

We still have to show that S indeed corresponds to a space reflection. Consider

$$\begin{aligned} \psi^{1}_{\dot 2} &= \psi^{1\dot 1} = z+t & -\psi^{1}_{\dot 1} &= \psi^{1\dot 2} = x-iy \\ -\psi^{2}_{\dot 1} &= \psi^{2\dot 2} = -z+t & \psi^{2}_{\dot 2} &= \psi^{2\dot 1} = x+iy \,. \end{aligned} \tag{2.20}$$

Now

$$\begin{aligned} S\psi^{1\dot 1} &= S\psi^{1}_{\dot 2} = -\psi^{\dot 2}_{1} = +\psi^{\dot 2 2} \\ S\psi^{2\dot 2} &= -S\psi^{2}_{\dot 1} = +\psi^{\dot 1}_{2} = +\psi^{\dot 2 1} \\ S\psi^{1\dot 2} &= -S\psi^{1}_{\dot 1} = +\psi^{\dot 1}_{1} = -\psi^{\dot 2 1} \\ S\psi^{2\dot 1} &= S\psi^{2}_{\dot 2} = -\psi^{\dot 2}_{2} = -\psi^{\dot 1 2} \,. \end{aligned} \tag{2.21}$$

In operating with S on $\psi^{\mu}_{\dot\lambda}$, we not only raise or lower the two indices, but also introduce a minus sign on account of the factor i in (2.19).

Under the operation S therefore

$$\begin{aligned} \psi^{1\dot 1} &\rightarrow \psi^{\dot 2 2} = \psi^{2\dot 2} \;; & \psi^{1\dot 2} &\rightarrow -\psi^{\dot 2 1} = -\psi^{1\dot 2} \\ \psi^{2\dot 2} &\rightarrow \psi^{\dot 1 1} = \psi^{1\dot 1} \;; & \psi^{2\dot 1} &\rightarrow -\psi^{\dot 1 2} = -\psi^{2\dot 1} \,. \end{aligned}$$

Hence, according to (2.20),

$$t \rightarrow t \;; \quad x, y, z \rightarrow -x, -y, -z, \qquad \text{Q.E.D.}$$

7. SPACE REFLECTIONS IN SPINOR SPACE

We saw in the first chapter that for any 2×2 matrix A the following relation holds:

$$A^T \epsilon = D \epsilon A^{-1} .$$

The transformation $\psi' = A\psi$ in spinor space then induces the following transformation for the 3-vector $\vec{x}$, as defined in the first chapter:

$$\vec{x}' = \psi'^T \epsilon \vec{\sigma} \psi' = \psi^T A^T \epsilon \vec{\sigma} A \psi \qquad (2.22)$$

$$= D \psi^T \epsilon A^{-1} \vec{\sigma} A \psi .$$

The explicit expressions for the matrices A for general 3-D rotations were given in chapter 1.

In this case we have $D = 1$, and

$$x_i' = \psi^T \epsilon A^{-1} \sigma_i A \psi ;$$

on the other hand:

$$x_i' = a_{ij} x_j = \psi^T \epsilon a_{ij} \sigma_j \psi .$$

Hence:

$$A^{-1} \sigma_i A = a_{ij} \sigma_j . \qquad (2.23)$$

We now define the reflection with respect to the $x-y$ plane by:

$$\psi' = \sigma_3 \psi , \text{ i.e. } A_R = \sigma_3, \; D_R = -1 .$$

Substituting this in (2.22), we indeed obtain:

$$x_1' = x_1$$
$$x_2' = x_2$$
$$x_3' = -x_3$$

as required.

For the reflections with respect to the other two planes we obtain similar results, which are summarized in the following table:

Reflection w.r.t.	Representation A_R
yz - plane	σ_1
zx - plane	σ_2
xy - plane	σ_3
origin (the product of the three preceding reflections)	$\sigma_1 \sigma_2 \sigma_3 = i I$
plane with normal $\vec{\alpha}$	$\vec{\alpha} \cdot \vec{\sigma}$

Since for all reflections $D_R = -1$, we have that the expression

$$\psi^T \epsilon \psi$$

mentioned in the first chapter, is a <u>pseudoscalar</u>:

$$\psi^{T'} \epsilon \psi' = \psi^T A_R{}^T \epsilon A_R \psi = D_R \psi^T \epsilon \psi = -\psi^T \epsilon \psi .$$

We further define

$$\vec{l} = \psi^+ \vec{\sigma} \psi .$$

Under 3-D rotations $\vec{l}$ behaves like a vector ($A^{+} = A^{-1}$):

$$\vec{l}' = \psi^{+} A^{+} \vec{\sigma} A \psi = \psi^{+} A^{-1} \vec{\sigma} A \psi$$

or, according to (2.23):

$$l_i' = \psi^{+} A^{-1} \sigma_i A \psi = a_{ij} \psi^{+} \sigma_j \psi = a_{ij} l_j .$$

However, under reflections, l_i transforms in the opposite way to x_i; e.g., for reflection w.r. to the xy-plane ($A_R = \sigma_3$) we have

$$l_1' = \psi^{\dagger} \sigma_3 \sigma_1 \sigma_3 \psi = -l_1, \; l_2' = -l_2, \; l_3' = l_3 .$$

Thus $\vec{l}$ is a <u>pseudovector</u>.

Finally, it is easily seen that

$$\psi^{+} \psi$$

is a <u>scalar</u>.

The ψ's we have been dealing with are undotted contravariant spinors $\psi \equiv \psi^{\lambda}$. We saw earlier that the dotted covariant spinor $\chi \equiv \chi_{\dot{\lambda}}$ transforms like ψ^{λ} under 3-D rotations. However, under any one of the reflections listed in the previous table, $\chi_{\dot{\lambda}}$ transforms <u>oppositely to ψ^{λ}</u>. We have, in an obvious notation:

$$\psi = A_R \psi, \text{ where } \begin{cases} A_R^{*} = A_R^{T} \\ A_R^{2} = 1 \\ D_R = -1 \end{cases} .$$

Now

$$\chi_{\cdot} = \epsilon_{\cdot\cdot} \psi^{\cdot} ,$$

and

$$\psi_{\cdot}{}' = A_R^* \psi^{\cdot} .$$

Hence

$$\chi_{\cdot}' = \epsilon_{\cdot\cdot} A_R^* \psi^{\cdot} = \epsilon_{\cdot\cdot} A_R^T \psi^{\cdot} .$$

According to (1.4):

$$\chi_{\cdot}' = D_R A_R^{-1} \epsilon_{\cdot\cdot} \psi' = D_R A_R \chi.$$

Thus

$$\chi' = -A_R \chi. \qquad \text{Q.E.D.}$$

χ is called a <u>spinor of the second kind</u> by D'Espagnat and Prentki (Ref. 21). It was already introduced by Cartan (Ref. 22). There is no linear connection between ψ and χ.

It is now clear that we can build up the following table of covariants with the help of the spinors ψ and χ:

Scalars	$\psi^\dagger \psi$	$\chi^\dagger \chi$	$\chi^T \epsilon \psi$
Pseudoscalars	$\psi^T \epsilon \psi$	$\chi^T \epsilon \chi$	$\chi^\dagger \psi$
Vectors	$\psi^T \epsilon \vec{\sigma} \psi$	$\chi^T \epsilon \vec{\sigma} \chi$	$\chi^\dagger \vec{\sigma} \psi$
Pseudovectors	$\psi^\dagger \vec{\sigma} \psi$	$\chi^\dagger \vec{\sigma} \chi$	$\chi^T \epsilon \vec{\sigma} \psi$

CHAPTER 3

INHOMOGENEOUS LORENTZ GROUP

The inhomogeneous Lorentz group is defined by:

$$x'^{\mu} = a^{\mu} + \alpha^{\mu}_{\nu} x^{\nu} .$$

The product of two inhomogeneous Lorentz transformations is again an inhomogeneous Lorentz transformation:

If,

$$x'^{\mu} = a^{\mu} + \alpha^{\mu}_{\nu} x^{\nu}$$

$$x^{\nu} = b^{\nu} + \beta^{\nu}_{\lambda} x^{\lambda}_{o}$$

then:

$$x'^{\mu} = c^{\mu} + \gamma^{\mu}_{\nu} x^{\nu}_{o}$$

where

$$\begin{cases} c^{\mu} = a^{\mu} + \alpha^{\mu}_{\nu} \; b^{\nu} \\ \gamma^{\mu}_{\nu} = \alpha^{\mu}_{\lambda} \beta^{\lambda}_{\nu} \quad . \end{cases} \qquad (3.1)$$

Symbolically:

$$(a, \alpha) (b, \beta) = (c, \gamma) . \qquad (3.2)$$

For the case of an infinitesimal transformation we have:

$$x'^{\mu} = a^{\mu} + (\delta^{\mu}_{\nu} + \epsilon^{\mu}_{\nu}) x^{\nu}$$

where a^{μ} and ϵ^{μ}_{ν} are infinitesimals of first order. A representation of the inhomogeneous Lorentz group $D(a, \alpha)$ is written:

$$D(a, \alpha) = T(a)\, D(\alpha) \quad \text{(translation follows rotation)}.$$

It follows from (3.1) and (3.2) that

$$T(a)\, T(b) = T(a+b) \quad y(a,b)$$

$$D(\alpha)\, D(\beta) = D(\alpha\beta) \quad y(\alpha,\beta)$$

$$D(\alpha)\, T(a) = T(\alpha a) \quad D(\alpha) \quad y(\alpha a),$$

where the y's are factors of modulus one. The representation expressed in terms of infinitesimal displacements and rotations now is:

$$D(a, \alpha) = (I - i a_\mu p^\mu)(I - \tfrac{i}{2} \epsilon_{\mu\nu} M^{\mu\nu})$$

where p^μ are the operators corresponding to displacements and are defined by:

$$p^\mu \psi = \left(\frac{\partial \psi}{\partial a_\mu}\right)_{\substack{a = 0 \\ \epsilon = 0}} \quad , \quad \psi' = D(a, \alpha)\, \psi.$$

In order that the differential equations determining the representation be integrable the p^μ and the $M^{\mu\nu}$ have to satisfy the following commutation relations:

$$[p^\mu , p^\nu] = 0$$
$$[M^{\mu\nu}, p^\lambda] = i(g^{\lambda\nu} p^\mu - g^{\lambda\mu} p^\nu)$$
$$[M^{\nu\xi}, M^{\nu'\xi'}] = i(M^{\nu\xi'} g^{\xi\nu'} + M^{\xi\nu'} g^{\nu\xi'} - M^{\nu\nu'} g^{\xi\xi'} - M^{\xi\xi'} g^{\nu\nu'}).$$

With the p's and the M's we can form the following operators:

$$\Gamma_\sigma^{*} = \tfrac{1}{2} p^\mu M^{\nu\lambda} \epsilon_{\mu\nu\lambda\sigma}.$$

We have:

$$\Gamma_1 = p^0 M^{23} + p^3 M^{02} - p^2 M^{03}$$

$$\Gamma_2 = p^0 M^{31} + p^1 M^{03} - p^3 M^{01}$$

$$\Gamma_3 = p^0 M^{12} + p^2 M^{01} - p^1 M^{02}$$

$$\Gamma_0 = - p^1 M^{23} - p^2 M^{31} - p^3 M^{12} .$$

We see that for $\vec{p} = 0$ (rest system) we have $\Gamma_0 = 0$, $\Gamma_i = p^0 M^i$ so that Γ_i / p_0 is the <u>intrinsic spin of the physical system</u>. Note that:

$$\Gamma_\mu \; p^\mu = 0 .$$

Also:

$$\left[M^{\mu\nu} , \Gamma^\sigma \right] = i (g^{\nu\sigma} \Gamma^\mu - g^{\mu\sigma} \Gamma^\nu)$$

$$\left[p^\mu , \Gamma^\sigma \right] = 0 ; \quad \left[\Gamma_\mu , \Gamma_\sigma \right] \neq 0 .$$

It is easy to see that the inhomogeneous Lorentz group has two invariant operators:

$$P \equiv p^\nu \; p_\nu$$

and

$$W \equiv - \Gamma_\sigma \Gamma^\sigma = \tfrac{1}{2} M^{\mu\lambda} M_{\mu\lambda} p_\nu p^\nu - M^{\lambda\mu} M_{\lambda\nu} \; p_\mu \; p^\nu$$

which commute with all the matrices of the representation.

Indeed:

$$\left[p^\mu , p^\nu p_\nu \right] = 0 \quad \text{because} \quad \left[p^\mu , p^\nu \right] = 0 ;$$

$$\left[M^{\nu'\xi'}, p^{\nu} p_{\nu}\right] = \left[M^{\nu'\xi'}, p^{\nu}\right] p_{\nu} + p^{\nu} \left[M^{\nu'\xi'}, p_{\nu}\right]$$

$$= i(p^{\xi'} p^{\nu'} - p^{\nu'} p^{\xi'}) + i(p^{\xi'} p^{\nu'} - p^{\nu'} p^{\xi'}) = 0;$$

$$\left[p^{\mu}, W\right] = 0 \quad \text{because} \quad \left[p^{\mu}, \Gamma^{\sigma}\right] = 0;$$

$$\left[M^{\mu\nu}, W\right] = 0 \quad \text{because} \quad \left[p^{\mu}, \Gamma^{\sigma}\right] = 0;$$

$$\left[M^{\mu\nu}, W\right] = \left[M^{\mu\nu}, \Gamma_{\sigma}\right] \Gamma^{\sigma} + \Gamma_{\sigma} \left[M^{\mu\nu}, \Gamma^{\sigma}\right]$$

$$= i\left[\Gamma^{\mu}, \Gamma^{\nu}\right] + i\left[\Gamma^{\nu}, \Gamma^{\mu}\right] = 0 .$$

Therefore P and W commute with the infinitesimal operators of the inhomogeneous Lorentz group, then they commute with all the representation matrices of this group. Now, if the representation is irreducible, W and P, by Schur's Lemma, have to be multiples of the unit operator. The classification of all irreducible representations is reduced therefore to the determination of the Spectra of the two invariants P and W. Indeed, if ψ is an element of the representation space we have:

$$D(a,\alpha)\, W \psi = W D(a,\alpha) \psi .$$

But

$$W = w\, I$$

where w is a number

$$W\psi = w\psi$$

so: $W\, D\,(a,\alpha)\psi = w D\,(a,\alpha)\psi$

which means that $D(a,\alpha)\psi$ spans all the eigenfunctions of W belonging to w. So, $D(a,\alpha)\psi$ are the eigenfunctions of W with eigenvalue w. In this way D is characterized by the spectra of W and P, $D_{w,p}$

or $$D(a, \alpha) = D_{sm}(a, \alpha) ,$$

where we have set:

$$P = m^2 I$$
$$W = m^2 s (s + 1) I .$$

:*:*:*:*:*:

CHAPTER 4

WAVE EQUATIONS OF FREE ELEMENTARY PARTICLES

1. INTRODUCTION.

From the study of the Lorentz group representations given in the preceding chapter, it is possible to obtain the relativistic wave equations of free particles. We shall, however, follow closely in this chapters, sections 1 and 2, the elegant discussion as given by Bargmann and Wigner.

The wave functions which describe physical states of relativistic particles form a linear vector space. Call ψ_ℓ and $\psi_{\ell'}$ the wave functions of the same state of a free particle in two Lorentz frames ℓ and ℓ'. Then there is a linear connection between ψ_ℓ and $\psi_{\ell'}$:

$$\psi_{\ell'} = D(\ell', \ell)\, \psi_\ell$$

The requirement that the frames ℓ and ℓ' are physically equivalent implies that the $D(L) \equiv D(\ell', \ell)$ form a representation of the inhomogeneous Lorentz group.

Since all Lorentz frames are equivalent for the description of the particle, it follows that together with ψ, $D(L)\psi$ is also a possible state viewed from the original frame ℓ. With ψ, the vector space contains all transforms $D(L)\psi$. This means that the representation $D(L)$ may replace the wave equations of the system. To every relativistically invariant system of equations there corresponds a representation of the inhomogeneous Lorentz group.

So a classification of all possible relativistic wave equations is given by a classification of the irreducible representations of the inhomogeneous Lorentz group, i.e. by a study of the eigenvalue spectra of the two invariant operators,

$$W \equiv - \Gamma_\sigma \Gamma^\sigma \quad \text{and} \quad P = p^\nu p_\nu \, .$$

Since the translation operators p_ν commute among themselves and with all the Γ_σ, we can choose the wave functions to be eigenfunctions of the p_ν. In addition, they can be chosen to be eigenfunctions of one of the components Γ_σ, say Γ_3 (with eigenvalues ξ). So the wave functions will depend on the four numbers p_ν, which vary over the manifold $p^\nu p_\nu = P =$ const, and the parameter ξ, which may assume a finite number of values:

$$\psi = \psi(p, \xi).$$

The inhomogeneous Lorentz transformation

$$x'^\mu = a^\mu + l^\mu_\nu x^\nu$$

induces the transformation

$$D(L)\, \psi(p,\xi) = e^{-ia_\nu p^\nu} \quad Q(p^\mu, l^\mu_\nu)\, \psi(L^{-1} p, \xi)$$

in representation space. The operator Q may depend on p_ν, but affects only the variables ξ (the "spin" variables).

The infinitesimal operators of the homogeneous Lorentz group, $M^{\mu\nu}$, divide into a part $L^{\mu\nu}$ which affects only the p_ν, and a part $S^{\mu\nu}$ which affects only the ξ :

$$M^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} .$$

We find $L^{\mu\nu}$ by studying the effect of $M^{\mu\nu}$ on a wave function with $\xi = 0$ (no "spin effects"). In this case $a_\nu = 0$, $Q = 1$, therefore

$$D(L)\psi(p) = \psi(L^{-1} p) = \psi(p^\mu - \epsilon^\mu{}_\nu p^\nu)$$

$$= \psi(p) - \frac{\partial\psi}{\partial p^\mu} \epsilon^\mu{}_\nu p^\nu .$$

On the other hand:

$$D(L)\ \psi(p) = (I - \frac{i}{2} \epsilon_{\mu\nu} M^{\mu\nu})\ \psi(p) .$$

Hence

$$\frac{i}{2} \epsilon_{\mu\nu} M^{\mu\nu} = g^{\lambda\mu} \epsilon_{\mu\nu} p^\nu \frac{\partial}{\partial p^\lambda} .$$

Since this is an identity in $\epsilon_{\mu\nu}$, we have

$$M^{\mu\nu} = -2 i p^\nu g^{\mu\lambda} \frac{\partial}{\partial p^\lambda} ,$$

of which only the antisymmetric part $\frac{1}{2}(M^{\mu\nu} - M^{\nu\mu})$ is of interest. We therefore have for the part $L^{\mu\nu}$ of $M^{\mu\nu}$:

$$L^{\mu\nu} = i(p^\mu g^{\nu\lambda} - p^\nu g^{\mu\lambda}) \frac{\partial}{\partial p^\lambda} .$$

We have

(4.1)

$$p^\rho L^{\mu\nu} \epsilon_{\rho\mu\nu\sigma} = 0 , \qquad (4.1)$$

because

$$\underbrace{p^\rho\ p^\mu}\ g^{\nu\lambda}\ \frac{\partial}{\partial p^\lambda}\ \underbrace{\epsilon_{\rho\mu\nu\sigma}} = 0$$

symmetric antisymmetric

$$\overbrace{p^\rho\ p^\nu}\ g^{\mu\lambda}\ \frac{\partial}{\partial p^\lambda}\ \overbrace{\epsilon_{\rho\mu\nu\sigma}} = 0\ .$$

Also

$$\left[S^{\mu\nu},\ p^\lambda \right] = 0\ ,$$

because the $S^{\mu\nu}$ act only on the ξ.

Owing to (4.1), we can write

$$\Gamma_\sigma = \tfrac{1}{2}\ p^\mu\ S^{\nu\lambda}\ \epsilon_{\mu\nu\lambda\sigma}\ . \qquad (4.2)$$

Call

$$\vec{S} \equiv (S^{23}, S^{31}, S^{12})\ ; \qquad \vec{S}' \equiv (S^{01}, S^{02}, S^{03})$$

$$\vec{p} \equiv (p^1\ , p^2\ , p^3\ )\ .$$

Then

$$\Gamma_0 = -\vec{p}\cdot\vec{S}$$

$$\vec{\Gamma} = p^0\ \vec{S} - \left[\vec{p}\times\vec{S}'\right]\ . \text{ (polarization) } (4.3)$$

Also

$$W \equiv -\Gamma_\sigma\,\Gamma^\sigma = \tfrac{1}{2}\ S^{\mu\lambda}\ S_{\mu\lambda}\ p^\nu\ p_\nu - S^{\lambda\mu}\ S_{\lambda\nu}\ p_\mu\ p^\nu\ .$$

We see from (4.3) that the spin of a massive particle is given by $\vec{S} = \vec{\Gamma}/p^0$ in the rest system. For a particle with zero mass there is no rest system, but it is possible to find a convenient special system. We therefore give the general definition of spin by means of the so-called little group.

The subgroup of the homogeneous Lorentz transformation which keep a certain chosen momentum vector p_c^μ unchanged is called the "little group". It is the group of transformations within the three-dimensional "plane" perpendicular to the fixed four-vector p_c^μ. In the special case of the rest system ($\vec{p}_c = 0$), the little group is the three-dimensional rotation group. This group is obviously generated by the operators $S^{\mu\nu}$. If the wave functions belonging to the irreducible representations of this group have $2s + 1$ components, the corresponding spin is s.

More explicity, the situation is as follows. In order to define the spin of a particle with rest mass, we study it in the rest system (otherwise we measure the total angular momentum and there is no invariant way to separate the proper from the orbital angular momentum). In this system some of the components of the wave function may vanish. The remaining ones must transform among themselves under a Lorentz transformation which conserves $\vec{p} = 0$. If the number of these is $2s + 1$, the spin is s.

However, to find the irreducible representations and the spin of the particle we could have chosen any arbitrary little group corresponding to some arbitrary fixed momentum p_c^μ. This is important for the case of zero mass particles, where it is not possible to find a rest system for p_c^μ.

The infinitesimal generators of the little group corresponding to an arbitrary fixed momentum p_c^μ are the Γ_μ's (which of course commute with all the components p_c^{μ}). Only three Γ's are independent on account of

$$p^{\mu} \Gamma_{\mu} = 0$$

($p^{\sigma} \Gamma_{\sigma} = \frac{1}{2} p^{\sigma} p^{\mu} M^{\nu\lambda} \epsilon_{\mu\nu\lambda\sigma} = 0$ because we contract the symmetric tensor $p^{\sigma} p^{\mu}$ with the antisymmetric tensor $\epsilon_{\mu\nu\lambda\sigma}$).

2. CLASSIFICATION OF RELATIVISTIC WAVE EQUATIONS.

We set

$$P \equiv p^{\nu} p_{\nu} = m^2 I,$$

and obtain four different classes of representations:

I. Class P_m (or P_s). $m^2 > 0$; p^{μ} is timelike.

II. Class P_o (or D_s). $m = 0$; p^{μ} lies on the momentum light cone.

III. Class P_o. $p^{\mu} = 0$ for $\mu = 0, 1, 2, 3$.

IV. Class P_{π}. $m^2 < 0$; p^{μ} is spacelike.

Only Classes I and II have a known physical meaning.

I. Class P_m (or P_s): Particles of finite mass and spin s.

Since p_{μ} is time-like ($p^{\mu} p_{\mu} = m^2 > 0$), the sign of p_o is invariant under the orthochronous group. So the irreducible representations will fall into two groups according to the sign of the energy sgn $p_o = p_o/|p_o|$ (or of the mass $m = (\text{sgn}\ p_o)(p_\mu p^\mu)^{\frac{1}{2}}$).

We go to the rest system and find

$$\Gamma_o = 0, \quad \Gamma_i = m\, S_i.$$

From the commutation relations for the Γ's,

$$\left[\Gamma_i , \Gamma_j\right] = im\, \epsilon_{ijk}\, \Gamma_k ,$$

we find

$$\left[S_i , S_j\right] = i\ \epsilon_{ijk}\, S_k ,$$

in agreement with the fact that S_i is the spin vector. Still in the rest system, we have

$$W = m^2\, \vec{S}^2 = m^2\, s\,(s+1)\, I;\ s = 0, \tfrac{1}{2}, 1, \ldots$$

The possible representations are listed in the table:

	s
$m > 0$	0, ½, 1, 3/2, 2, ...
$m < 0$	0, ½, 1, 3/2, 2, ...

We now turn to the determination of the wave equations of the class P_m.

Spin $s = 0$. ψ depends only on p, because $\xi = 0$. The wave equations is

$$p^\mu\ p_\mu \psi(p) = m^2 \psi(p);\ \ W\psi = 0.$$

It is the familiar Klein-Gordon equation for scalar or pseudoscalar particles. In coordinate space the wave equation is

$$(\Box + m^2)\ \Psi(x) = 0\ ; \qquad \Box = \frac{\partial^2}{\partial x_0^2} - \vec{\nabla}^2 .$$

<u>Spin $s = \frac{1}{2} N (N = 1, 2, 3, \ldots)$.</u> For N = 1 we have the case of the Dirac equation for spin ½ particles, which we discussed in Chapter 2. There we introduced the 4 × 4 matrices γ_μ and the four-row spinor

$\Psi = \Psi(p, \xi)$, where $\xi = 1, 2, 3, 4$. The Dirac

equation corresponds to a representation of the orthochronous Lorentz group with space reflections, with $m \gtrless 0$ and $s = \frac{1}{2}$.

The derivation of the wave equations for a general $s = \frac{1}{2} N$ follows the same pattern as in the case of the Dirac equation.

As wave functions we choose

$$\Psi = \Psi\,(p; \xi_1, \xi_2, \ldots, \xi_N),$$

which we require to be <u>symmetric in the ξ-variables</u>. Each ξ_i can take on four values: $\xi_i = 1, 2, 3, 4$. $(i = 1, 2, \ldots, N)$.

We further introduce the 4 × 4 matrices $\gamma^\nu_{(i)}$ such that:

$$\gamma^\mu_{(i)}\ \gamma^\nu_{(i)} + \gamma^\nu_{(i)}\ \gamma^\mu_{(i)} = 2\,g^{\mu\nu} ,$$

$$\gamma^\mu_{(i)}\ \gamma^\nu_{(j)} = \gamma^\nu_{(j)}\ \gamma^\mu_{(i)} \quad \text{for } i \neq j .$$

$\gamma^\mu_{(i)}$ acts on the variable ξ_i of Ψ.

The wave equations then are

$$\gamma^{\mu}_{(i)}\, p_\mu \psi = m\, \psi \ (i = 1, 2, \ldots, N = 2s), \qquad (4.4)$$

which give rise to the invariant

$$\gamma^{\mu}_{(i)}\, p_\mu\, \gamma^{\nu}_{(i)}\, p_\nu \psi = \tfrac{1}{2}\,(\gamma^{\mu}_{(i)}\, \gamma^{\nu}_{(i)} + \gamma^{\nu}_{(i)}\, \gamma^{\mu}_{(i)})\, p_\mu\, p_\nu \psi =$$

$$= g^{\mu\nu} p_\mu\, p_\nu \psi = p^\mu\, p_\mu\, \psi = m^2\, \psi .$$

We proceed to find the infinitesimal operators $M^{\mu\nu}$ of the homogeneous Lorentz group from the invariance requirement on the wave equations. From

$$(\gamma^{\mu}_{(i)}\, p'_\mu - m)\, \psi'\, (p', \xi) = 0 \ ; \quad p'_\mu = \ell_\mu^{\ \nu}\, p_\nu ,$$

and

$$\psi'\, (p', \xi) = D\, (L)\, \psi(p, \xi)$$

we find

$$(\gamma^{\mu}_{(i)}\, \ell_\mu^{\ \nu}\, p_\nu - m)\, D\, (L)\, \psi(p, \xi).$$

Multiply by D^{-1} on the left:

$$(D^{-1}\, \gamma^{\mu}_{(i)}\, \ell_\mu^{\ \nu} D\, p_\nu - m)\, \psi\, (p, \xi) = 0 .$$

Hence we recover the wave equation in the unprimed system, if

$$D^{-1}\, (L)\, \gamma^{\mu}\, (i)\ \ell_\mu^{\ \nu} D\, (L) = \gamma^{\nu}_{(i)} \qquad (4.5)$$

or

$$D\, (L)\, \gamma^{\nu}_{(i)}\, D^{-1}\, (L) = \gamma^{\mu}_{(i)}\, \ell_\mu^{\ \nu} .$$

For an infinitesimal Lorentz transformation

$$\begin{cases} \ell_\mu^{\ \nu} = \delta_\mu^{\ \nu} + \epsilon_\mu^{\ \nu} \\ D(\ell) = I - \sum_j \tfrac{i}{2} \epsilon_{\mu\nu} M^{\mu\nu}_{(j)} \ . \end{cases}$$

Substituting this into (4.5) and regarding the resulting equation as an identity in $\epsilon_{\mu\nu}$, we find

$$i \sum_j \left[M^{\mu\nu}_{(j)}, \gamma^\lambda_{(i)} \right] = \gamma^\nu_{(i)} g^{\mu\lambda} - \gamma^\mu_{(i)} g^{\nu\lambda}.$$

We split this relation up into

$$i \left[M^{\mu\nu}_{(i)}, \gamma^\lambda_{(i)} \right] = \gamma^\nu_{(i)} g^{\mu\lambda} - \gamma^\mu_{(i)} g^{\nu\lambda}$$

and

$$\left[M^{\mu\nu}_{(j)}, \gamma^\lambda_{(i)} \right] = 0 \quad \text{for} \quad i \neq j \ .$$

Now we set

$$\sum_i M^{\mu\nu}_{(i)} = L^{\mu\nu} + \sum_i S^{\mu\nu}_{(i)} \quad \text{with} \quad L^{\mu\nu} = i\left(p^\mu \frac{\partial}{\partial p_\nu} - p^\nu \frac{\partial}{\partial p_\mu} \right) .$$

Since $L^{\mu\nu}$ commutes with the γ's, we have

$$\begin{cases} i \left[S^{\mu\nu}_{(i)}, \gamma^\lambda_{(i)} \right] = \gamma^\nu_{(i)} g^{\mu\lambda} - \gamma^\mu_{(i)} g^{\nu\lambda} \ , \\ \left[S^{\mu\nu}_{(j)}, \gamma^\lambda_{(i)} \right] = 0 \quad \text{for} \quad i \neq j \ . \end{cases}$$

This is satisfied by

$$S^{\mu\nu}_{(i)} = \frac{i}{4} \left(\gamma^\mu_{(i)} \gamma^\nu_{(i)} - \gamma^\nu_{(i)} \gamma^\mu_{(i)} \right).$$

The total spin is

$$S^{\mu\nu} = \sum_i S^{\mu\nu}_{(i)} \ .$$

With the special choice $\vec{\gamma}_{(i)}$ = antihermitian, $\gamma^{o}_{(i)}$ = hermitian, we find that

$$\begin{cases} (S^{i\ell})^{\dagger} = S^{i\ell} \\ (S^{o\ell})^{\dagger} = -\,S^{o\ell} \end{cases} \qquad i,\ \ell = 1,\ 2,\ 3,$$

a result which is of course independent of the special representation of the γ's.

Now we show that the representation of the little group given by $\psi(p;\xi_1,\ldots,\xi_{2s})$ is D_s, corresponding to the spin s.

Since $m \neq 0$, we choose the rest system

$$\vec{p}_c = 0, \quad p^o_c = m,$$

to define our little group (here the 3-D rotation group). We saw earlier that

$$\left[\gamma^{o}_{(j)},\ S^{i\ell}_{(j)}\right] = 0 \qquad i,\ell = 1,\ 2,\ 3.$$

We therefore choose $\gamma^{o}_{(j)}$ diagonal. Since it must be different from the identity matrix, with eigenvalues ± 1 and zero trace, we set

$$\gamma^{o}_{(i)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Then in the rest system:

$$\gamma^{o}_{(i)}\, p_o\, \psi = m\, \psi\ .$$

For $p_o = +\,m$, we have

$$\gamma^{o}_{(i)}\, \psi = \psi\ ;$$

i.e., all components of ψ which correspond to the third and fourth rows of the $\gamma_{(i)}$'s are zero. For $p_o = -m$, $\gamma^o_{(i)} \psi = -\psi$, and the components of ψ corresponding to the first and second rows of the $\gamma_{(i)}$'s are zero.

We shall consider only the case $p_o = + m$. Then we write

$$\psi \equiv \begin{pmatrix} & \xi_1 \; \xi_2 \; \xi_3 \; \cdots \; \xi_N \\ \psi & (p;\; 1 \;\; 1 \;\; 1 \; \ldots \; 1) \\ \psi & (p;\; 2 \;\; 1 \;\; 1 \; \ldots \; 1) \\ 0 & \\ 0 & \\ \psi & (p;\; 1 \;\; 1 \;\; 1 \; \ldots \; 1) \\ \psi & (p;\; 1 \;\; 2 \;\; 1 \; \ldots \; 1) \\ 0 & \\ 0 & \\ \cdot & \\ \cdot & \end{pmatrix}$$

Of the 4^N components therefore only 2^N are $\neq 0$. Furthermore, we require that ψ be symmetric in the ξ_i's. Hence the components of ψ in which the same number k of the N indices correspond to the first row of the γ's(i.e., k indices have the value 1), the N - k other indices corresponding to the second row of the γ's (i.e., having the value 2), are identical. Since k goes from 0 to N, there are N + 1 independent components of ψ. Example N = 3:

$$\begin{cases} \psi(p;111) \\ \psi(p;211) = \psi(p;121) = \psi(p;112) \\ \psi(p;221) = \psi(p;212) = \psi(p;122) \\ \psi(p;222)\ . \end{cases}$$

We now study transformation properties of these N + 1 components under the little group, i.e., the three-dimensional rotation group. The corresponding infinitesimal operators are the $M^{i\ell} = S^{i\ell}$, where $\vec{p}_c = 1, 2, 3$. (The $L^{i\ell}$ give zero if applied to a ψ with $\vec{p}_c = 0$). In particular, $S^{12}{}_{(j)} = \frac{1}{2}\gamma^1{}_{(j)}\gamma^2{}_{(j)}$ can be chosen diagonal. Since it is hermitian, has the trace zero, and satisfies $\left[S^{12}{}_{(j)}\right]^2 = \frac{1}{4}$, we can set it equal to

$$S^{12}{}_{(j)} = \begin{pmatrix} \frac{1}{2} & & & 0 \\ & -\frac{1}{2} & & \\ & & \frac{1}{2} & \\ 0 & & & -\frac{1}{2} \end{pmatrix} .$$

Now for a ψ with k of the ξ_i corresponding to the first row of the γ's and N - k to the second row, we have

$$S^{12}\,\psi = \sum_j S^{12}{}_{(j)}\,\psi = \left[\tfrac{1}{2}\,k - \tfrac{1}{2}\,(N-k)\right]\psi = (k-s)\,\psi\ .$$

For a given s, $m_s \equiv k-s$ takes on the $2s+1$ values $-s, \ldots, +s$. Therefore the $2s+1$ independent components of $\psi(p;\xi_1, \ldots, \xi_{2s})$ <u>belong to the representation Ds of the little group. The particle [redacted]ribed by this set of components has spin s</u>.

We have used 2s variables ξi to describe particles with spin s. This is a consequence of the symmetry of the wave functions in the ξ_i . Any other type of symmetry of the wave functions would require a larger number of variables and the description would be more complicated.

II. Class P_o (or 0_s): Particles of Zero Rest Mass and Spin s.

This class is defined by

$$p^\mu p_\mu = 0, \qquad p_\mu \neq 0.$$

The wave functions again depend on p_μ and the spin variables ξ_i. We now have

$$W \equiv - \Gamma_\sigma \Gamma^\sigma = - M^{\lambda\mu} M_{\lambda\nu} p_\mu p^\nu .$$

W will not be very useful in the classification of the wave equations for zero rest mass, however, for we shall find that $W\Psi = 0$ for any spin. Instead we consider the Γ_σ directly.

The little group now cannot be defined by a p_c^μ in the rest system, for now $g_{\mu\nu} p_c^\mu p_c^\nu = 0$, so that at least two components of p_c must be $\neq 0$ (we exclude the case where all $p_c^\mu = 0$). So let us choose

$$p_c^\mu = (1, 0, 0, 1).$$

Since under this special little group the vector $p^\mu = (1, 0, 0, 1)$ is left invariant, we may set $p^0 = p^3 = 1$, $p^1 = p^2 = 0$ in (4.2) and obtain

$$\begin{cases} \Gamma_0 = -\,S^{12} \\ \Gamma_1 = S^{23} + S^{02} \\ \Gamma_2 = S^{31} - S^{01} \\ \Gamma_3 = S^{12} \quad . \end{cases}$$

As wave equations we use

$$\gamma^{\mu}_{(i)}\; p_{\mu}\; \psi = 0, \qquad i = 1, 2, \ldots, N = 2s \quad . \tag{4.6}$$

We define

$$\gamma^5{}_{(j)} \equiv i\,\epsilon_{\mu\nu\lambda\sigma}\; \gamma^{\mu}_{(j)}\; \gamma^{\nu}_{(j)}\; \gamma^{\lambda}_{(j)}\; \gamma^{\sigma}_{(j)} = i\gamma^0\gamma^1\gamma^2\gamma^3 \; .$$

Then

$$\gamma^5{}_{(j)}\; \gamma^{\mu}_{(j)} + \gamma^{\mu}_{(j)}\gamma^5{}_{(j)} = 0 \;,\; (\gamma^5{}_{(j)})^2 = 1,\; (\gamma^5{}_{(j)})^{\dagger} =$$

$$\gamma^5{}_{(j)},\; (\gamma^{\ell})^2 = -1;\; (\gamma^0)^2 = 1;\; (\vec{\gamma})^{+} = -\vec{\gamma};\; (\gamma^0)^{\dagger} = \gamma^0 \; .$$

It is clear that if ψ is a solution of the wave equation (4.6), then $\gamma^5{}_{(j)}\psi$ is also a solution (This is of course not true for the case $m \neq 0$, see formula (4.4). It is therefore convenient to choose $\gamma^5{}_{(j)}$ to be diagonal. In our special system $p^1 = p^2 = 0$, $p^3 = p^0 = 1$ we have then from (4.6):

$$(-\gamma^0{}_{(i)} + \gamma^3{}_{(i)})\,\psi = 0,$$

or

$$\gamma^3{}_{(i)}\gamma^0{}_{(i)}\,\psi = -\psi\,, \quad i = 1, 2, \ldots, N$$

We choose the representation

$$\gamma^0_{(i)} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \vec{\gamma}_{(i)} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \quad \gamma^5_{(i)} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix},$$

$$\vec{\alpha}_{(i)} = \gamma^0_{(i)} \vec{\gamma}_{(i)} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}.$$

Then we have

$$\alpha^3_{(i)} \psi = \psi \qquad \text{with} \quad \alpha^3_{(i)} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}.$$

This means that ψ has only components corresponding to the second and third rows of the γ's.

Now, since the wave equation (4.6) is invariant under any one of the operators $\gamma^5_{(i)}$, we can decompose the manifold of wave equations into two invariant manifolds by imposing restrictions on $\gamma^5_{(i)} \psi$. In particular, we choose

$$\gamma^5_{(i)} \psi = -\psi . \tag{4.7}$$

Then only the components of ψ corresponding to the second row of the γ's are $\neq 0$. There is only one such component:

$$\psi(p;\, 222 \, \ldots \, 2) .$$

This <u>one</u> independent component describes a <u>left circular polarized particle</u>. If instead of (4.7) we impose

$$\gamma^5_{(i)} \psi = \psi ,$$

we obtain a ψ whose only non-zero independent component (corresponding to the third row of the γ's) describes a <u>right circular</u>

polarized particle.

We see that for a given p, there are only two independent components of the wave function describing a massless particle, regardless of the number $N = 2s$ of the variables ξ_i(i.e. the spins).

We now show that the spin is $S = \frac{1}{2}N$. First of all we have $\Gamma_1\psi = \Gamma_2\psi = 0$.

Multiply

$$\gamma^3_{(i)}\,\gamma^0_{(i)}\,\psi = -\psi$$

on the left by $\gamma^0_{(i)}\gamma^2_{(i)}$, and find:

$$\gamma^2_{(i)}\gamma^3_{(i)}\,\psi = -\gamma^0_{(i)}\gamma^2_{(i)}\,\psi\,,$$

or

$$\Gamma_1\psi = \sum_i\left(S^{23}_{(i)} + S^{02}_{(i)}\right)\psi = \frac{1}{2}\sum_i\left(\gamma^2_{(i)}\gamma^3_{(i)} + \gamma^0_{(i)}\gamma^2_{(i)}\right)\psi = 0\,.$$

The proof for $\Gamma_2\psi$ is similar. For $\Gamma_3\psi$ $(= -\Gamma_0\psi = S^{12}\psi)$ we find the following.

Multiply

$$\gamma^3_{(i)}\,\gamma^0_{(i)}\,\psi = -\psi$$

on the left by $\gamma^1_{(i)}\,\gamma^2_{(i)}$, and find

$$\gamma^0_{(i)}\,\gamma^1_{(i)}\gamma^2_{(i)}\gamma^3_{(i)}\,\psi = \gamma^1_{(i)}\,\gamma^2_{(i)}\,\psi\,.$$

Hence

$$\tfrac{1}{2}\gamma^5{}_{(i)}\psi = \tfrac{1}{2}\,\gamma^1{}_{(i)}\gamma^2{}_{(i)}\psi = S^{12}{}_{(i)}\psi \;,$$

$$S^{12}\psi = \tfrac{1}{2}\sum_{i=1}^{N(=2s)} \gamma^5{}_{(i)}\psi \;.$$

So for

$$\gamma^5{}_{(i)}\psi = -\psi\ (\text{left}) : \qquad S^{12}\psi = -s\psi \;,$$

and for

$$\gamma^5{}_{(i)}\psi = \psi\ (\text{right}) : \quad S^{12}\psi = s\psi \;.$$

Thus the spin of the two states of the particle is indeed s.

We also see that

$$W\psi = -\left(\Gamma_o\,\Gamma^o + \Gamma_3\,\Gamma^3\right)\psi = 0 \;.$$

So we see that W cannot serve to characterize the spin of the particle. Instead we define the two manifolds corresponding to $p^\mu\,p_\mu = 0$ and spin S by:

$$\begin{aligned} \Gamma_\mu &= s\,p_\mu &\quad \text{for} \quad \gamma^5{}_{(i)}\psi &= \psi \;, \\ \Gamma_\mu &= -s\,p_\mu &\quad \text{for} \quad \gamma^5{}_{(i)}\psi &= -\psi \;. \end{aligned} \tag{4.8}$$

These equations are not invariant under space reflections. Note that these equations are an illustration of the theorem: two four-vectors Γ_μ, p_μ of zero length $\left(\Gamma_\mu\,\Gamma^\mu = 0,\ p_\mu\,p^\mu = 0\right)$ and orthogonal to each other $\left(p_\mu\,\Gamma^\mu = 0\right)$, are parallel to each other: $\Gamma_\mu = \text{const.}\;\; p_\mu$.

3. WAVE EQUATIONS FOR PARTICLES WITH SPIN 1. PROCA'S EQUATIONS.

3a. CONNECTION WITH THE DUFFIN-KEMMER THEORY.

The particles with physical interest up to now are those with spin 0, $\frac{1}{2}$, and 1. For spin 0 we have the Klein-Gordon equation, for spin $\frac{1}{2}$, the Dirac equation. We now discuss in more detail the case of spin 1.

For the wave function we use the notation:

$$\psi \equiv \psi\,(p;\xi_1\,,\xi_2) \equiv \Psi_{\xi_1\,\xi_2} \equiv \Psi_{ij}\;.$$

Ψ is symmetric in the indices i, j(i, j = 1, 2, 3, 4) so it has 10 independent components.

The wave equations are:

$$\begin{cases} \gamma^{\mu}{}_{ij}\,\delta_{i'j'}\;p_{\mu}\;\Psi_{jj'} = m\,\Psi_{ii'} \\ \gamma^{\mu}_{i'j'}\;\delta_{ij}\;p_{\mu}\;\Psi_{jj'} = m\,\Psi_{ii'}\;, \end{cases} \tag{4.9}$$

or

$$\begin{cases} (\gamma^{\mu} \times I')\;p_{\mu}\Psi = m\Psi \\ (I \times \gamma'^{\mu})\;p_{\mu}\;\Psi = m\Psi\;. \end{cases}$$

Therefore, with

$$\beta^{\mu} = \tfrac{1}{2}\left\{\gamma^{\mu} \times I' + I \times \gamma'^{\mu}\right\}\;,$$

we have

$$\beta^{\mu}\;p_{\mu}\;\Psi = m\,\Psi\;. \tag{4.10}$$

From the commutation rules of the γ^μ and γ'^μ we find that

$$\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = \beta^\mu g^{\nu\lambda} + \beta^\lambda g^{\nu\mu} . \qquad (4.11)$$

Equation (4.10), together with (4.11), are the basic Duffin-Kemmer equations. They lead again to the Klein-Gordon equation:

Multiply (4.10) on the left by $p_\lambda \beta^\lambda \beta^\nu$:

$$p_\lambda \beta^\lambda \beta^\nu \beta^\mu p_\mu \psi = m \; p_\lambda \beta^\lambda \beta^\nu \psi .$$

Hence

$$\tfrac{1}{2} p_\lambda p_\mu \; (\beta^\lambda \beta^\nu \beta^\mu + \beta^\mu \beta^\nu \beta^\lambda) \psi = m \, p_\lambda \beta^\lambda \beta^\nu \psi ,$$

or

$$p^\nu p_\mu \beta^\mu \psi = m \, p_\lambda \beta^\lambda \beta^\nu \psi .$$

The left hand side gives $m p^\nu \psi$, so we have

$$p^\nu \psi = p_\lambda \beta^\lambda \beta^\nu \psi .$$

Multiply by p_ν , and get:

$$p_\nu p^\nu \psi = p_\lambda \beta^\lambda \; p_\nu \beta^\nu \psi = m^2 \psi .$$

The β's of the Duffin-Kemmer theory together with the symmetric wave function ψ lead to a description of particles with spin 1 (Proca's equations). We indicate here how this works out, leaving the more exact treatment for the next subsection.

We study the behaviour of the second order spinor $\psi = \psi_{jj'}$ under space reflections. We choose a representation in which the reflection operator is

$$S_{ij} = S_{i'j'} = i \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} .$$

Operating on Ψ_{ij} with S, we find:

$$\begin{cases} \Psi_{11} \longrightarrow -\Psi_{11} \\ \Psi_{22} \longrightarrow -\Psi_{22} \\ \Psi_{33} \longrightarrow -\Psi_{33} \\ \Psi_{44} \longrightarrow -\Psi_{44} \\ \Psi_{12} \longrightarrow -\Psi_{12} \\ \Psi_{34} \longrightarrow -\Psi_{34} \end{cases} \qquad \begin{cases} \Psi_{13} \longrightarrow \Psi_{13} \\ \Psi_{14} \longrightarrow \Psi_{14} \\ \Psi_{23} \longrightarrow \Psi_{23} \\ \Psi_{24} \longrightarrow \Psi_{24} \end{cases} .$$

We can, therefore, separate the Ψ components into a group forming a four-vector, and a group forming the six components of an antisymmetric tensor:

$$\phi^{\mu} \begin{cases} \phi^{\ell} \longrightarrow -\phi^{\ell} & \text{3 components} \\ \phi^{0} \longrightarrow \phi^{0} & \text{1 component} \end{cases}$$

$$F \begin{cases} F^{0\ell} \longrightarrow -F^{0\ell} & \text{3 components} \\ F^{i\ell} \longrightarrow F^{i\ell} & \text{3 components} \end{cases} .$$

So we choose

$$\Psi = (\phi^{0}, \phi^{1}, \phi^{2}, \phi^{3}, F^{01}, F^{02}, F^{03}, F^{23}, F^{31}, F^{12}) ,$$

and find that (4.10) leads to the Proca equations:

$$\begin{cases} p_\nu \ F^{\mu\nu} = i m \ \phi^\mu \\ i m \ F^{\mu\nu} = p^\mu \ \phi^\nu - p^\nu \ \phi^\mu \ . \end{cases}$$

The corresponding β matrices are 10 × 10 matrices with the explicit form:

$$\beta_0 = \left[\begin{array}{c|c|c|c} 0 & \begin{smallmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 1 \end{smallmatrix} & 0 & 0 \\ \hline \begin{smallmatrix} 0 & & 0 \\ & -1 & \\ 0 & & -1 \end{smallmatrix} & 0 & \begin{smallmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 0 \end{smallmatrix} & 0 \\ \hline 0 & \begin{smallmatrix} -1 & & 0 \\ & 0 & \\ 0 & & 0 \end{smallmatrix} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] , \quad \beta_1 = \left[\begin{array}{c|c|c|c} 0 & \begin{smallmatrix} 0 & -1 & 0 \\ & 0 & \end{smallmatrix} & 0 & \begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix} \\ \hline \begin{smallmatrix} 0 & \\ 1 & 0 \\ 0 & \end{smallmatrix} & 0 & \begin{smallmatrix} 0 & 0 & -1 \\ & 0 & \end{smallmatrix} & 0 \\ \hline 0 & \begin{smallmatrix} 0 & \\ 0 & 0 \\ 1 & \end{smallmatrix} & 0 & 0 \\ \hline 0 \ 0 \ {-1} & 0 & 0 & 0 \end{array}\right]$$

$$\beta_2 = \left[\begin{array}{c|c|c|c} 0 & \begin{smallmatrix} 0 & 0 & -1 \\ & 0 & \end{smallmatrix} & 0 & \begin{smallmatrix} 0 \\ -1 \\ 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} 0 & \\ 0 & 0 \\ 1 & \end{smallmatrix} & 0 & \begin{smallmatrix} 0 & 1 & 0 \\ & 0 & \end{smallmatrix} & 0 \\ \hline 0 & \begin{smallmatrix} 0 & \\ -1 & 0 \\ 0 & \end{smallmatrix} & 0 & 0 \\ \hline 0 \ 1 \ 0 & 0 & 0 & 0 \end{array}\right] , \quad \beta_3 = \left[\begin{array}{c|c|c|c} 0 & 0 & \begin{smallmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{smallmatrix} & \begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix} \\ \hline 0 & 0 & 0 & 0 \\ \hline \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{smallmatrix} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

3b. RIGOROUS DERIVATION OF PROCA'S EQUATIONS.

Since $\Psi_{ij} = \Psi_{ji}$, the two equations (4.9) are actually identical, and can be written simply

$$\gamma^\mu \, p_\mu \psi = m \psi \, , \tag{4.12}$$

where γ^μ acts either on the first or on the second index of $\psi = \Psi_{ij}$. Ψ_{ij}, regarded as a 4×4 matrix, can be written as a linear combination of the 16 independent matrix γ_A, where

$$\gamma_A = 1 \, , \gamma^\mu \, , \tfrac{1}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu), \gamma^5 \gamma^\mu, \gamma^5 \, .$$

Thus

$$\Psi = \sum_{A=1}^{16} \varphi_A \gamma_A \, . \tag{4.13}$$

We want Ψ to be symmetric, so we investigate the symmetry properties of the γ_A. Since $(\gamma^\mu)^T$ satisfies the same commutation rules as γ^μ, the two are connected by a similarity transformation:

$$(\gamma^\mu)^T = B \gamma^\mu B^{-1} \quad ,$$

Then

$$\gamma^\mu = (B^{-1})^T (\gamma^\mu)^T B^T = (B^{-1})^T B \gamma^\mu B^{-1} B^T =$$

$$= (B^{-1} B^T)^{-1} \gamma^\mu (B^{-1} B^T) \, .$$

Hence $B^{-1} B^T$ commutes with all the γ^μ, and therefore with all the γ_A, which form an irreducible set of matrices. Therefore, according to Schur's lemma,

$$B^{-1} B^T = k \, I \, .$$

Then

$$B^T = k \, B \, .$$

$$B = k\,B^T = k^2\,B\,; \qquad \text{hence} \quad k = \pm 1\,.$$

Now

$$(B^{-1})^T = k\,B^{-1},$$
$$(\gamma^\mu B^{-1})^T = k\gamma^\mu B^{-1}\,,$$
$$(\gamma^5\gamma^\mu\gamma^\nu B^{-1})^T = -k\,\gamma^5\gamma^\mu\gamma^\nu B^{-1}\,,$$
$$(\gamma^\mu\gamma^5\,B^{-1})^T = -k\,\gamma^\mu\gamma^5\,B^{-1}\,,$$
$$(\gamma^5\,B^{-1})^T = k\,\gamma^5\,B^{-1}\,.$$

If we set $k = +1$, then there are 10 antisymmetric matrices $\gamma^\mu\gamma^5\,B^{-1}$ and $\gamma^5\gamma^\mu\gamma^\nu B^{-1}$, which have to be linearly independent, since $\gamma^\mu\gamma^5$ and $\gamma^5\gamma^\mu\gamma^\nu$ are linearly independent. But this is a contradiction, since there can be only six linearly independent antisymmetric 4×4 matrices.

Therefore

$$k = -1\,.$$

Then we can write for the general symmetric wave function Ψ:

$$\Psi(p) = \phi_\mu\gamma^\mu\gamma^5\,B^{-1} + \frac{1}{2}\,F_{\mu\nu}\gamma^\mu\gamma^\nu\gamma^5\,B^{-1}\,.$$

Now we obtain from (4.12):

$$\left\{p_\mu\gamma^\mu(\phi_\lambda\gamma^\lambda + \frac{1}{2}F_{\lambda\nu}\gamma^\lambda\gamma^\nu) - m\,(\phi_\lambda\gamma^\lambda + \frac{1}{2}F_{\mu\lambda}\gamma^\mu\gamma^\lambda)\right\} \times \gamma^5\,B^{-1} = 0,$$

or

$$\frac{1}{2}\,(p_\mu\gamma^\mu\,\phi_\lambda\gamma^\lambda + p_\lambda\gamma^\lambda\phi_\mu\gamma^\mu - im\,F_{\mu\lambda}\gamma^\mu\gamma^\lambda) =$$

$$= m\,\phi_\lambda\,\gamma^\lambda - \frac{i}{2}\,p_\mu\,F_{\lambda\nu}\,\gamma^\mu\,\gamma^\lambda\,\gamma^\nu$$

Now, since

$$\gamma^\mu\,\gamma^\lambda + \gamma^\lambda\,\gamma^\mu = 2\,g^{\lambda\mu}\,,$$

$$\begin{aligned}&\tfrac{1}{2}\,(p_\mu\,\phi_\lambda - p_\lambda\,\phi_\mu - im\,F_{\mu\lambda})\,\gamma^\mu\,\gamma^\lambda = \\ &= -g^{\lambda\mu}\,p_\lambda\,\phi_\mu + m\,\phi_\lambda\gamma^\lambda - p_\mu\,F_{\lambda\nu}\,\gamma^\mu\,\gamma^\lambda\,\gamma^\nu\,.\end{aligned} \tag{4.14}$$

We take the trace and obtain:

$$\left(\mathrm{tr}\;\gamma^\lambda = \mathrm{tr}\left\{\gamma^\mu\,\gamma^\lambda\,\gamma^\nu\right\} = 0;\quad \mathrm{tr}\left\{\gamma^\mu\,\gamma^\lambda\right\} = 4\,g^{\mu\lambda}\right)$$

$$p^\mu\,\phi_\mu = 0\,. \tag{4.15}$$

Now multiply (4.14) by γ^α and take the trace, considering that

$$\mathrm{tr}\,\left\{\gamma^\mu\,\gamma^\lambda\,\gamma^\nu\,\gamma^\alpha\right\} = 4\,\left(g^{\mu\alpha}\,g^{\lambda\nu} + g^{\mu\lambda}\,g^{\nu\alpha} - g^{\mu\nu}\,g^{\lambda\alpha}\right),$$

we find

$$m\,\phi^\alpha - \frac{i}{2}\left(p^\lambda\,F_\lambda^{\ \alpha} - p^\nu\,F^\alpha_{\ \nu}\right) = 0\,,$$

or

$$p_\mu\,F^{\mu\alpha} = -\,im\,\phi^\alpha\,. \tag{4.16}$$

Finally, multiply (4.14) by $\gamma^\alpha\gamma^\beta$ and take the trace. The result is (using (4.15)):

$$p_\mu\,\phi_\nu - p_\nu\,\phi_\mu = im\,F_{\mu\nu}\,. \tag{4.17}$$

Equations (4.16), (4.17) are Proca's equations.

4. THE SPINOR FIELD ($s = \frac{1}{2}$).

We now study in more detail the Dirac equation:

$$(\gamma^\mu p_\mu - m)\,\psi(p) = 0\,. \qquad (4.18)$$

The γ^μ's satisfy

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\, g^{\mu\nu}\,.$$

A representation is:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}.$$

A solution of (4.18) is also a solution of the Klein-Gordon equation:

$$(\gamma^\nu p_\nu + m)(\gamma^\mu p_\mu - m)\psi = (\gamma^\nu \gamma^\mu p_\nu p_\mu - m^2)\psi =$$

$$= (p_\mu p^\mu - m^2)\,\psi = 0\,.$$

Introduce

$$\beta = \gamma^0\,, \quad \vec{\alpha} = \gamma^0 \vec{\gamma}\,,$$

and find from (4.18):

$$H\,\psi(p) = p^0\,\psi(p)\,, \qquad (4.19)$$

where

$$H \equiv \vec{\alpha}\cdot\vec{p} + \beta m\,.$$

In coordinate space we find, from

$$\psi(p) = \frac{1}{(2\pi)^{3/2}} \int \psi(x)\, e^{ipx}\, d_x^4 ,$$

that

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \psi(x) = 0 .$$

For the hermitian conjugate, $\psi^\dagger$, or rather, the Dirac adjoint, $\bar{\psi} = \psi^+ \gamma^o$, we find the equations

$$\bar{\psi}(p) \quad (\gamma^\mu p_\mu - m) = 0$$

and

$$\bar{\psi}(x) \left(i\gamma^\mu \frac{\overleftarrow{\partial}}{\partial x^\mu} + m \right) = 0 .$$

The Dirac operator $\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right)$ is invariant under translations, so the field $\psi(x)$ must be a scalar under translations, in order to keep the Dirac equation invariant.

For homogeneous Lorentz transformations we have seen that

$$\psi'(p') = D\,(L)\, \psi\,(p) . \tag{4.20}$$

For the infinitesimal transformation operator we obtain

$$M^{\mu\nu} = \frac{i}{4} \left(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \right) .$$

For the special case of a rotation by the angle φ in the i,ℓ-plane we have

$$D_{(i\ell)}(\varphi) = e^{\frac{1}{2}\gamma^i \gamma^\ell \varphi} = \cos \varphi/2 + \gamma^i \gamma^\ell \sin \varphi/2 ,$$

and for rotations in the o,ℓ -plane:

$$D_{(o\ell)}(\varphi) = e^{-\frac{1}{2}\gamma^o\gamma^\ell\varphi} = \cosh \varphi/2 - \gamma^o\gamma^\ell \sinh \varphi/2$$

We see again that the spinor representation is not one-valued, because a rotation about the angle 2π takes ψ into $-\psi$.

We now derive the transformation law for the Dirac adjoint. From (4.20) we find

$$\overline{\psi'(p')} = \psi'^\dagger(p')\gamma^o = \psi^\dagger(p)\,D^\dagger\gamma^o = \overline{\psi_{(p)}}\,\gamma^o D^\dagger \gamma^o .$$

Now D is a function of $\gamma^\mu\gamma^\nu(\mu \neq \nu)$,

$$D = D\ (\gamma^\mu \gamma^\nu)\ ,$$

hence

$$\gamma^o D^+ \gamma^o = \gamma^o D\ (\gamma^{\nu\dagger}\gamma^{\mu\dagger})\,\gamma^o = D\ (\gamma^o\gamma^{\nu\dagger}\gamma^o\gamma^o\gamma^{\mu\dagger}\gamma^o)$$

$$= D\ (\gamma^\nu\gamma^\mu) = D\ (-\gamma^\mu\gamma^\nu) = D^{-1} .$$

Therefore

$$\overline{\psi'(p')} = \overline{\psi(p)}\ D^{-1} .$$

It follows that quadratic forms in $\overline{\psi}$ and ψ transform according to the tensor representations of the Lorentz group:

$$\overline{\psi'(p')}\ \gamma_A\ \psi'(p') = \overline{\psi_{(p)}}\,D^{-1}\ \gamma_A\ D\ \psi_{(p)}\ ,$$

where γ_A are the 16 independent γ matrices mentioned earlier. We saw previously that under Lorentz transformations:

$$D^{-1}\gamma^\mu D = \ell^\mu_{\ \nu}\,\gamma^\nu .$$

Hence we find:

$$
\begin{array}{lll}
\text{for} \quad \gamma_A = 1, & \bar{\psi}\gamma_A\psi & \text{is a scalar,} \\
\gamma_A = \gamma^\mu & \text{"} & \text{is a vector} \\
\gamma_A = \tfrac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) & \text{"} & \text{is an antisymmetric second rank tensor} \\
\gamma_A = \gamma^\mu\gamma^5 & \text{"} & \text{is a pseudovector} \\
\gamma_A = \gamma^5 & \text{"} & \text{is a pseudoscalar.}
\end{array}
$$

The justification of the last two lines will become clearer after discussion of the parity operation for spinors (See J. Leite Lopes, Inversion Operations in Quantum Field Theory, Notas de Física, vol. VI, Nº 2, 1960).

5. THE NEUTRINO FIELD.

The neutrino is a spin ½ particle with zero mass. It satisfies the Dirac equation with $m = 0$. However, let us first make some remarks on the Dirac equation with $m \neq 0$.

We have

$$\vec{\alpha} = \gamma^0 \vec{\gamma}, \ \beta = \gamma^0, \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

and

$$\sigma^1 = i\gamma^2\gamma^3 \ \text{(cyclic)}$$

Therefore

$$\vec{\alpha} = \gamma^5 \vec{\sigma}.$$

Now define

$$\psi_L = \tfrac{1}{2}(1 - \gamma^5)\psi \quad ,$$

$$\psi_R = \tfrac{1}{2}(1 + \gamma^5)\psi .$$

Then we find from the Dirac equation (4.19):

$$p^0 \psi = (\gamma^5 \vec{\sigma} . \vec{p} + m \beta)\psi ,$$

$$\gamma^5 p^0 \psi = (\vec{\sigma} . \vec{p} - m \beta \gamma^5)\psi ,$$

so that

$$p^0 \psi_L = - \vec{\sigma} . \vec{p} \, \psi_L + m \beta \psi_R ,$$

$$p^0 \psi_R = \vec{\sigma} . \vec{p} \, \psi_R + m \beta \psi_L .$$

We define the helicity $\mathcal{H}$ as:

$$\mathcal{H} \equiv \frac{\vec{\sigma} . \vec{p}}{|\vec{p}|} .$$

Then

$$\begin{cases} p^0 \psi_L = - |\vec{p}| \mathcal{H} \psi_L + m \beta \psi_R \\ p^0 \psi_R = |\vec{p}| \mathcal{H} \psi_R + m \beta \psi_L . \end{cases}$$

Now we set $m = 0$:

$$\begin{cases} p^0 \psi_L = - |\vec{p}| \mathcal{H} \psi_L \\ p^0 \psi_R = |\vec{p}| \mathcal{H} \psi_R . \end{cases}$$

For a positive energy neutrino $(p^0 = |\vec{p}|)$, we therefore find:

$$\begin{cases} \mathcal{H} \psi_L = - \psi_L \\ \mathcal{H} \psi_R = \psi_R . \end{cases}$$

This justifies the name left-handed neutrino for ψ_L, and right-handed neutrino for ψ_R. A particle with $m \neq 0$ is in general a superposition of right and left-handed states, which are coupled through the mass term in the equation of motion. For very high energy, $|\vec{p}| \gg m$, the mass term becomes less important, and one has a situation of near polarization.

In the case of $m = 0$, it is convenient to change our usual representation for the γ's to the following one:

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

In this new representation

$$\psi_L = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_L \\ 0 \end{pmatrix}, \quad \psi_R = \begin{pmatrix} 0 \\ 0 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi_R \end{pmatrix},$$

where we have introduced the two-component spinors φ_L and φ_R. ψ_L satisfies the equations for the left handed neutrino:

$$\gamma^\mu p_\mu \; \psi_L = 0 \; ; \quad \gamma_5 \psi_L = - \psi_L \; .$$

The equations for the right-handed neutrino, ψ_R, are

$$\gamma^\mu p_\mu \psi_R = 0 \; ; \quad \gamma_5 \psi_R = \psi_R \; .$$

:*:*:*:

CHAPTER 5

LAGRANGIAN FORMALISM

The wave functions and the wave equations which we have obtained can be regarded as fields and field equations respectively to be derived from a Lagrangian function. The basic requirement on this function is, therefore, invariance with respect to the inhomogeneous Lorentz group.

The Lagrangian is defined as a real function of the field functions and their first derivatives. Second and higher order derivatives are excluded because we want the field equations to be of at most second order:

$$\mathcal{L}(x) \equiv \mathcal{L}\left(U_i(x), \frac{\partial U_i}{\partial x^\mu}\right) ,$$

where $U_i(x)$ is the Fourier transform of $\psi_i(p)$ and $\partial U_i/\partial x^\mu$, the first derivative of $U_i(x)$. $\mathcal{L}(x)$ is required not to depend explicitly on the coordinates $x \equiv (x^0, x^1, x^2, x^3)$ as it has to be invariant under the inhomogeneous Lorentz group.

The <u>action</u> is defined as:

$$I \equiv \int \mathcal{L}(x)\, d^4x , \qquad d^4x = dx^0\, dx^1\, dx^2\, dx^3 .$$

The variational principle, which we assume to hold here, states that the action must be stationary:

$$\delta I = 0 ,$$

for variations $\delta U_i(x)$ of the field variables. In addition it is assumed that the δU_i's vanish at the boundary of the integration domain. One then obtains the Lagrange-Euler equations:

$$\frac{\delta I}{\delta U_i(x)} = \frac{\partial \mathcal{L}}{\partial U_i(x)} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial U_i}{\partial x^\mu}\right)} = 0$$

which are the field equations.

1. NOETHER'S THEOREM:

For every continuous coordinate transformation which makes $\delta I = 0$ and for which we know the law of transformation of the fields, we can construct a combination of the U_i and their first derivatives, which is covariant and independent of time.

In order to prove the theorem let us consider the group of infinitesimal inhomogeneous Lorentz transformations:

$$x^{\mu'} = x^\mu + \alpha^\mu_\nu x^\nu + a^\mu$$

where a^μ and α^μ_ν are infinitesimals of the first order - these transformations are of the general type:

$$x'^\mu = x^\mu + \delta x^\mu$$

where for example for the case of translations we have:

$$\delta x^\mu = \sum X^\mu_\nu \delta\omega^\nu ,$$

$$X^\mu_\nu = \delta^\mu_\nu ; \delta_\omega{}^\nu = a^\nu ,$$

and

$$\delta x^{\mu} = \sum_{\alpha<\beta} X^{\mu}_{\alpha\beta}\, \delta\omega^{\alpha\beta}$$

$$X^{\mu}_{\alpha\beta} = \delta^{\mu}_{\alpha}\, g_{\beta\nu}\, x^{\nu} - \delta^{\mu}_{\beta}\, g_{\alpha\nu}\, x^{\nu};\ \delta\omega^{\alpha\beta} = \alpha^{\alpha\beta}\ ,$$

for the case of Lorentz rotations.

In the general case we can then write

$$x^{\mu\prime} = x^{\mu} + \delta x^{\mu}\ ;\ \delta x^{\mu} = \sum X^{\mu}_{\nu}\, \delta\omega^{\nu}.$$

The transformed field functions are written:

$$\begin{aligned} U_i'\ (x') &= U_i(x) + \delta U_i(x) \\ &= U_i(x) + \sum \Omega_{ij}(x)\, \delta\omega^{j} \\ \delta U_i(x) &= \sum \Omega_{ij}(x)\, \delta\omega^{j}\ . \end{aligned}$$

$\delta U_i(x)$ is the variation of the field function due to both the change of its form and the change of its argument.

The variation of the form only is:

$$\begin{aligned} \bar{\delta} U_i(x) &= U_i'(x) - U_i(x) \\ &= U_i'(x') - U_i(x) - \left(U_i'(x') - U_i'(x)\right) \\ &= \sum_j \Omega_{ij}(x)\, \delta\omega^{j} - \left(U_i'(x+\delta x) - U_i(x)\right) \\ &= \sum_j \Omega_{ij}(x)\, \delta\omega^{j} - \delta x^{K} \frac{\partial U_i'}{\partial x^{K}} \\ &= \sum_j \left(\Omega_{ij} - \sum_K \frac{\partial U_i}{\partial x_K}\, X^{K}_{j}\right) \delta\omega^{j}\ , \end{aligned}$$

the action is:

$$I = \int \mathcal{L}(x)\, d^4x \quad .$$

Its variation $\delta \int \mathcal{L}(x)\, d^4x$ is made up of the variation of the integrand and the variation of the domain of integration. The variation of the limits $\delta(b-a) = \delta b - \delta a$ of the one-dimensional integral $\int_a^b \delta(dt)$ can be written symbolically:

$$\int_a^b \delta(dt) = \int_a^b dt\, \frac{\partial(\delta t)}{\partial t} \quad .$$

Similarly the variation of the 4-dimensional volume of integration can be written:

$$\delta(d^4x) = d^4x \sum_\nu \frac{\partial(\delta x^\nu)}{\partial x^\nu} \quad .$$

Now

$$\delta I \equiv \int \mathcal{L}(x)\, d^4x = \int \delta\mathcal{L}(x)\, d^4x + \int \mathcal{L}(x)\, \delta(d^4x) \quad .$$

But:

$$\delta\mathcal{L} = \mathcal{L}'(x') - \mathcal{L}(x) = \mathcal{L}'(x') - \mathcal{L}(x') + \Big(\mathcal{L}(x') - \mathcal{L}(x)\Big)$$

$$= \overline{\delta\mathcal{L}} + \sum_\nu \frac{\partial\mathcal{L}}{\partial x^\nu}\, \delta x^\nu \quad .$$

where

$$\overline{\delta\mathcal{L}} = \mathcal{L}'(x') - \mathcal{L}(x') = \sum_i \frac{\partial\mathcal{L}}{\partial U_i}\, \bar{\delta} U_i + \sum_{i,\nu} \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial U_i}{\partial x^\nu}\right)} \bar{\delta}\left(\frac{\partial U_i}{\partial x^\nu}\right) \quad .$$

However, from the field equations we have:

$$\frac{\partial\mathcal{L}}{\partial U_i} = \sum_\nu \frac{\partial}{\partial x^\nu} \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial U_i}{\partial x^\nu}\right)} \quad .$$

Now

$$\bar{\delta}\mathcal{L} = \sum_{i,\nu} \frac{\partial}{\partial x^\nu} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial U_i}{\partial x^\nu}\right)} \bar{\delta} U_i + \sum_{i,\nu} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial U_i}{\partial x^\nu}\right)} \frac{\partial}{\partial x^\nu} (\bar{\delta} U_i)$$

$$= \sum_{i,\nu} \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial U_i}{\partial x^\nu}\right)} \bar{\delta} U_i \right).$$

So:

$$\delta I = \int \delta \mathcal{L}(x)\, d^4x + \int \mathcal{L}(x)\, \delta(d^4x)$$

$$= \int \left[\bar{\delta}\mathcal{L} + \sum_\nu \frac{\partial \mathcal{L}}{\partial x^\nu} \delta x^\nu\right] d^4x + \int \mathcal{L}(x)\, d^4x \sum_\nu \frac{\partial(\delta x^\nu)}{\partial x^\nu}$$

$$= \int d^4x \sum_\nu \frac{\partial}{\partial x^\nu} \left\{ \sum_i \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial U_i}{\partial x^\nu}\right)} \bar{\delta} U_i + \mathcal{L}\delta x^\nu \right\}$$

$$= -\int d^4x \sum_{\nu,j} \frac{\partial \Theta_j^\nu}{\partial x^\nu} \delta\omega^j ,$$

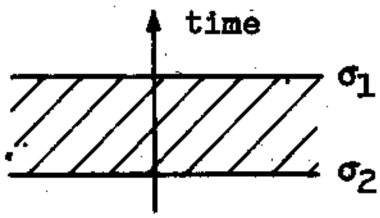
where

$$\Theta_j^\nu = \sum_i \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial U_i}{\partial x^\nu}\right)} \left(\sum_\lambda \frac{\partial U_i}{\partial x^\lambda} X_j^\lambda - \Omega_{ij} \right) - \mathcal{L} X_j^\nu \;;$$

the postulate $\delta I = 0$ therefore gives rise to:

$$\frac{\delta I}{\delta\omega^j} = -\sum_\nu \frac{\partial \Theta_j^\nu}{\partial x^\nu} = 0 .$$

This theorem permits us to obtain from this equation the conservation laws of the corresponding surface integrals.



Integrating over a region bounded in time by space like surfaces σ_1 and σ_2 and extending to infinity in space like directions and assuming that the field vanishes at infinity we get:

$$\sum_{\nu} \left(\int_{\sigma_2} d\,\sigma_\nu\ \Theta_j^\nu - \int_{\sigma_1} d\,\sigma_\nu\ \Theta_j^\nu \right) = 0 \ ,$$

where $d\sigma_\nu$ is the projection of the surface element $d\sigma$ into the hyperplane perpendicular to the x^ν axis.

Thus:

$$\sum_{\nu} \int_{\sigma} d\,\sigma_\nu\ \Theta_j^\nu = S_j(\sigma)$$

are independent of the surface σ. For σ taken as hyperplane, $x^o =$ = constant:

$$S_j(x^o) = \int d^3 x\ \ \Theta_j^o$$

are independent of the time. This proves <u>Noether's theorem</u>, the covariants independent of time being the S_j.

The quantities Θ_j^ν are not unique. Since they enter the integral in the form

$$\delta I = - \int d^4x \ \ \sum_{\nu, j} \frac{\partial \Theta_j^\nu}{\partial x^\nu}\ \delta\omega^j$$

it is clear that we can add to Θ_j^ν an expression like

$$\sum_{\lambda} \frac{\partial f_j^{\lambda\nu}}{\partial x^\lambda}$$

where

$$f_j^{\lambda\nu} = - f_j^{\nu\lambda} \ ,$$

one can use this property to symmetrize the Θ_j^{ν} .

Energy-momentum tensor and energy-momentum vector.

Consider translations alone:

$$x^{\mu'} = x^{\mu} + \delta x^{\mu}$$

$$\delta x^{\mu} = X_{\nu}^{\mu} \, \delta\omega^{\nu}$$

$$X_{\nu}^{\mu} = \delta_{\nu}^{\mu} , \delta\omega^{\nu} = a^{\nu}$$

and as:

$$U_i'(x') = U_i(x)$$

then:

$$\Omega_{ij} = 0$$

so

$$\Theta_j^{\nu} \equiv T_j^{\nu} = \sum_i \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial U_i}{\partial x^{\nu}}\right)} \frac{\partial U_i}{\partial x^j} - \mathcal{L}\delta_j^{\nu} \; ; (\nu, j = 0, 1, 2, 3)$$

or

$$T^{\nu\lambda} = \sum_i g^{j\lambda} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial U_i}{\partial x^{\nu}}\right)} \frac{\partial U_i}{\partial x^j} - \mathcal{L} g^{\nu\lambda}$$

this is the energy-momentum tensor. The surface integral S_j is here:

$$\int T^{\nu\lambda} \, d\sigma\nu = P^{\lambda}$$

thus

$$\int T^{0\lambda} \, d^3x = P^{\lambda}$$

is the energy-momentum vector, which is conserved in time.

<u>Angular-momentum tensor and spin tensor</u>.

Now consider infinitesimal 4-rotations:

$$x'^{\mu} = x^{\mu} + \delta x^{\mu}$$

$$\delta x^{\mu} = \sum_{\lambda<\nu} X^{\mu}_{\lambda\nu}\ \delta\omega^{\lambda\nu}$$

$$X^{\mu}_{\lambda\nu} = \left(\delta^{\mu}_{\lambda} g_{\nu\alpha} - \delta^{\mu}_{\nu} g_{\lambda\alpha}\right) x^{\alpha}\ ;\ \delta\omega^{\lambda\nu} = \alpha^{\lambda\nu}\ .$$

We shall write

$$U'_i(x') = U_i(x) + \sum_{j;\lambda<\nu} M^{j}_{i;\lambda\nu} U_j(x)\ \delta\omega^{\lambda\nu}$$

where we have put:

$$\Omega_{ij} = M_{ij;\lambda\nu} U_j(x)$$

so that:

$$\delta U_i(x) = \sum M^{j}_{i;\lambda\nu} U_j(x)\ \delta\omega^{\lambda\nu}\ .$$

For a scalar field:

$$M^{j}_{i;\lambda\nu} = 0$$

For a vector field:

$$M^{j}_{i;\lambda\nu} = \delta^{j}_{\nu} g_{i\lambda} - \delta^{j}_{\lambda} g_{i\nu}\ .$$

For a spinor field:

$$M^{j}_{i;\lambda\nu} = \tfrac{1}{2} (\gamma_{\lambda} \gamma_{\nu})_{ij}\ .$$

Because the parameters are $\delta\omega^{\lambda\nu}$ instead of $\delta\omega^{j}$ we see that the index j in Θ^{ν}_{j} is replaced by a pair ℓm:

$$\Theta^{\nu}_{\ell m} = \sum_{i} \frac{\partial \mathcal{L}}{\partial(\partial U_i/\partial x^{\nu})} \left(\sum_{\lambda} \frac{\partial U_i}{\partial x^{\lambda}}\ X^{\lambda}_{\ell m} - \Omega_{ij\,\ell m} \right) - \mathcal{L}\, X^{\nu}_{\ell m}$$

Replacing $X^{\nu}_{\ell m}$ and $\Omega_{i\ell m}$ by their appropriate expression we get:

$$M^{\nu}_{\ell m} = \sum_i \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial U_i}{\partial x^{\nu}}\right)} \left\{ \sum_{\lambda,\alpha} \frac{\partial U_i}{\partial x^{\lambda}} \left(\delta^{\lambda}_{\ell} g_{m\alpha} - \delta^{\lambda}_{m} g_{\ell\alpha}\right) x^{\alpha} - \sum_j M^j_{i;\ell m} U_j \right\} -$$

$$-\mathcal{L}\left(\delta^{\nu}_{\ell} g_{m\alpha} - \delta^{\nu}_{m} g_{\ell\alpha}\right) x^{\alpha}$$

$$= T^{\nu}_{\ell} x_m - T^{\nu}_{m} x_{\ell} - \sum_{ij} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial U_i}{\partial x^{\nu}}\right)} M^j_{i;\ell m} U_j(x)$$

which is the angular momentum density tensor. For a scalar field we already saw that $M^j_{i;\ell m} = 0$

hence for this field

$$M^{\nu;\ell m} \longrightarrow L^{\nu;\ell m} = T^{\nu\ell} x^m - T^{\nu m} x^{\ell}$$

which is the <u>orbital</u> angular momentum density of the wave field. This is conserved:

$$\frac{\partial M_o^{\nu;\ell m}}{\partial x^{\nu}} = 0$$

if

$$T^{m\ell} = T^{\ell m} .$$

For a field which is not scalar, the term

$$S^{\nu}_{\ell m} = - \sum_{ij} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial U_i}{\partial x^{\nu}}\right)} M^j_{i;\ell m} U_j(x)$$

defines the <u>spin tensor</u> of the field.

The 3-dimensional densities of L and S are:

$$L^{oj\ell m} = X^m T^{o\ell} - X^\ell T^{om}$$

$$S^{oj\ell m} = - g^{\ell\ell'} g^{mm'} \sum_{ij} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial U_i}{\partial x^o}\right)} M^j_{i;\ell' m'} U_j (x)$$

the tensor of orbital and spin angular momenta are the integrals of these densities:

$$L^{\ell m} = \int L^{o;\ell m} d^3x$$

$$S^{\ell m} = \int S^{o;\ell m} d^3x$$

Charge and current vector

Assume that the wave fields are complex. The Lagrangian must be real (hermitian in the quantized version of the theory) so it can only depend on expression like $U^* U$ and will be invariant if the field is multiplied by an arbitrary phase factor (gauge transformations of the first kind):

$$U_i' = e^{i\alpha} U_i, \; U_i'^* = e^{-i\alpha} U_i^* \quad ,$$

then

$$\mathcal{L} \left(U', \frac{\partial U'}{\partial x}\right) = \mathcal{L} \left(U, \frac{\partial U}{\partial x}\right).$$

For α infinitesimal we write:

$$U_i' = U_i + i\alpha \; U_i \; ;$$

$$U_i'^* = U_i^* - i\alpha \; U_i^* \; .$$

Clearly:

$$X^\mu_\nu = 0, \quad \delta \omega^j = \alpha, \quad \Omega_{ij} = iU_i \; ,$$

the corresponding Θ^{ν} (omitting the irrelevant index j) are:

$$\Theta^{\nu} = J^{\nu} = i \sum_{i} \left\{ \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial U_i^*}{\partial x^{\nu}} \right)} U_i^* (\mathbf{x}) - \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial U_i}{\partial x^{\nu}} \right)} U_i (\mathbf{x}) \right\} ,$$

J^{ν} is the current four-vector and

$$Q = \int J^{0}(\mathbf{x}) \, d^3\mathbf{x} = \text{const.}$$

is the charge of the field (electric charge, nucleon (mesic) charge, etc. ...).

Summarizing, we have:

Invariance under		Conservation of
<u>Space-time displacements</u>	⟶	<u>energy-momentum vector</u>
<u>Homogeneous-Lorentz transformations</u>	⟶	<u>angular momentum tensor</u>
<u>Gauge transformation of the first kind</u>	⟶	<u>current 4-vector</u>

2. EXAMPLES OF FIELDS

We next give a table of some of the most important fields encountered, together with the corresponding "conserved" quantities.

2a. REAL SCALAR FIELD

This describes neutral spinless particles.

<u>Lagrangian</u>:

$$\mathcal{L} = -\tfrac{1}{2} \left(m^2 \phi^2 - \frac{\partial \phi}{\partial x_{\mu}} \frac{\partial \phi}{\partial x^{\mu}} \right) .$$

Field equation:

$$(\Box + m^2)\,\phi = 0 \;.$$

Energy momentum tensor:

$$T^{\mu\lambda} = g^{\lambda\lambda'} \frac{\partial\phi}{\partial x_\mu} \frac{\phi}{\partial x^{\lambda'}} - \mathcal{L}\, g^{\mu\lambda} ,$$

also

$$T^{\mu}_{\lambda} = \frac{\partial\phi}{\partial x_\mu} \frac{\partial\phi}{\partial x^{\lambda}} - \mathcal{L}\,\delta^{\mu}_{\lambda} \,.$$

Energy density:

$$T^{oo} = \tfrac{1}{2}\left[\sum_\mu \left(\frac{\partial\phi}{\partial x^\mu}\right)^2 + m^2 \phi^2\right] .$$

Momentum density vector:

$$T^{oi} = - \frac{\partial\phi}{\partial x_o} \frac{\partial\phi}{\partial x^i} , \quad i = 1, 2, 3.$$

Angular momentum density:

$$M^{\nu}_{\ell m} \equiv L^{\nu}_{\ell m} = \frac{\partial\phi}{\partial x_\nu}\left(x_m \frac{\partial\phi}{\partial x^\ell} - x_\ell \frac{\partial\phi}{\partial x^m}\right) + \mathcal{L}\left(x_\ell\ \delta^{\nu}_{m} - x_m \delta^{\nu}_{\ell}\right) .$$

3-dimensional density of the angular-momentum tensor:

$$M^{o}_{\ell m} \equiv L^{o}_{\ell m} = \frac{\partial\phi}{\partial x_o}\left(x_m \frac{\partial\phi}{\partial x^\ell} - x_\ell \frac{\partial\phi}{\partial x^m}\right) + \mathcal{L}\left(x_\ell \delta^{o}_{m} - x_m\ \delta^{o}_{\ell}\right)$$

$$S^{\nu}_{\ell m} = 0 \;.$$

Current 4-vector:

$$J^{\nu} = 0 \;.$$

Momentum representation of real scalar field:

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^4 K \, e^{-iKx} \, \phi(K); \; Kx = x^0 K^0 - \vec{K}.\vec{x}.$$

The reality condition $\phi^*(x) = \phi(x)$ leads to: $\phi^*(K) = \phi(-K)$.

The equation of motion is:

$$(K_\mu K^\mu - m^2) \; \phi(K) = 0 .$$

Clearly the solution is of the form:

$$\phi(K) = \delta(K^2 - m^2) \; \varphi(K),$$

so:

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^4 K \, \delta(K^2 - m^2) \, e^{-iKx} \varphi(K) .$$

Now write the above equation in the form:

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^4 K \, \delta(K^2 - m^2) \left\{ e^{-iKx} \varphi^{(+)}(K) + e^{iKx} \varphi^{(-)}(K) \right\}$$

where

$$\varphi^{(+)}(K) = \tfrac{1}{2} (1 + \mathrm{sgn}\, K^0) \, \varphi(K)$$

$$\varphi^{(-)}(K) = \tfrac{1}{2} (1 + \mathrm{sgn}\, K^0) \, \varphi(-K) .$$

Note that:

$$\left(\varphi^{(+)}(K) \right)^+ = \varphi^{(-)}(K) .$$

So:

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x),$$

$$\phi^{(+)}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3K}{2K^0} \; e^{-iKx} \varphi^{(+)}(K) ,$$

$$\phi^{(-)}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 K}{2K^0} e^{+iKx} \varphi^{(-)}(K) ,$$

where the integration over K_0 $\left(= +\sqrt{\vec{K}^2 + m^2} \right)$ has been carried out.

Set

$$a(\vec{K}) = \varphi^{(+)}(\vec{K}) = \frac{\varphi^{(+)}(K)}{\sqrt{2 K_0}} ,$$

$$a^+(\vec{K}) = \varphi^{(-)}(\vec{K}) = \frac{\varphi^{(-)}(K)}{\sqrt{2 K_0}} ,$$

then:

$$\phi^{(+)}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 K}{\sqrt{2K_0}} e^{-iKx} a(\vec{K}) ,$$

$$\phi^{(-)}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 K}{\sqrt{2K_0}} e^{+iKx} a^+(\vec{K}) .$$

Now:

$$P^0 = \int T^{00} d^3x = \tfrac{1}{2} \int \left[\sum_\mu \left(\frac{\partial \phi}{\partial x^\mu} \right)^2 + m^2 \phi^2 \right] d^3x$$

$$= \int d^3x \left\{ \sum_\mu \frac{\partial \phi^{(+)}}{\partial x^\mu} \frac{\partial \phi^{(-)}}{\partial x^\mu} + m^2 \phi^{(+)} \phi^{(-)} \right\} .$$

In the momentum representation:

$$P^0 = \int d^3 K \; K^0 a^+ (\vec{K}) \; a(\vec{K}) ,$$

$$P^\alpha = \int d^3 K \; K^\alpha a^+ (\vec{K}) \; a(\vec{K}) ,$$

2b. COMPLEX SCALAR FIELD.

This describes charged spinless particles.

Lagrangian:

$$\mathcal{L} = g_{\mu\nu} \frac{\partial \phi^*}{\partial x_\mu} \frac{\partial \phi}{\partial x_\nu} - m^2 \phi^* \phi .$$

Field equations:

$$(\Box + m^2)\, \phi = 0, \quad (\Box + m^2)\, \phi^* = 0 .$$

Energy-momentum tensor:

$$T^{\mu\lambda} = \frac{\partial \phi}{\partial x_\mu} \frac{\partial \phi}{\partial x_\lambda} + \frac{\partial \phi^*}{\partial x_\lambda} \frac{\partial \phi}{\partial x_\mu} - \mathcal{L}\, g^{\mu\lambda}$$

$$T^{oo} = \sum_\alpha \frac{\partial \phi^*}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\alpha} + m^2 \phi^* \phi ,$$

$$T^{o\alpha} = -\left(\frac{\partial \phi^*}{\partial x_o} \frac{\partial \phi}{\partial x_\alpha} + m^2 \frac{\partial \phi^*}{\partial x_\alpha} \frac{\partial \phi}{\partial x_o} \right) .$$

Current:

$$J^\nu = i g^{\nu\nu'} \left(\phi^* \frac{\partial \phi}{\partial x^{\nu'}} - \frac{\partial \phi^*}{\partial x^{\nu'}} \phi \right) .$$

Momentum representation:

$$\phi = \phi^{(+)} + \phi^{(-)}; \quad \phi^* = \phi^{*(+)} + \phi^{*(-)}$$

$$\phi^{(\pm)}(x) = \frac{1}{(2\pi)^{3/2}} \int d^4 K \;\delta(K^2 - m^2)\, e^{\mp iKx} \varphi^{(\pm)} \quad (K)$$

$$= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 K}{\sqrt{2 K^o}} \; e^{\mp iKx} \varphi^{(\pm)}(\vec{K}) ,$$

$$\phi^{*(\pm)} = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 K}{\sqrt{2 K^o}} \; e^{\mp iKx} \varphi^{*(\pm)}(\vec{K}) .$$

Let

$$\varphi^{(+)}(\vec{K}) = a(\vec{K}), \quad \varphi^{+(+)}(\vec{K}) = b(\vec{K}) ,$$

$$\varphi^{(-)}(\vec{K}) = b(\vec{K}), \quad \varphi^{+(-)}(\vec{K}) = a^{+}(\vec{K}),$$

$$P^{\ell} = \int d^3 K\, K^{\ell} \left\{a^{+}(\vec{K})\; a(\vec{K}) + b(\vec{K})\; b^{+}(\vec{K})\right\}$$

$$\ell = 0, 1, 2, 3, \qquad K^{0} = + \sqrt{K^2 + m^2},$$

$$Q = \int d^3 K \left\{a^{+}(\vec{K})\; a(\vec{K}) - b(\vec{K})\; b^{+}(\vec{K})\right\}.$$

2c. SPINOR FIELD

Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left(\bar{\psi}(x)\gamma^{\mu}\frac{\partial\psi}{\partial x^{\mu}} - \frac{\partial\bar{\psi}}{\partial x^{\mu}}\gamma^{\mu}\psi(x)\right) - m\bar{\psi}(x)\psi(x) .$$

Energy-momentum tensor:

$$T^{\lambda\nu} = g^{j\lambda}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x^{\nu}}\right)}\;\frac{\partial\psi}{\partial x^{j}} + \frac{\partial\bar{\psi}}{\partial x^{j}}\;\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\bar{\psi}}{\partial x^{\nu}}\right)}\right) - \mathcal{L}g^{\nu\lambda}$$

$$= \frac{1}{2}\; g^{j\lambda}\left(\bar{\psi}(x)\gamma^{\nu}\frac{\partial\psi}{\partial x^{j}} - \frac{\partial\bar{\psi}}{\partial x^{j}}\gamma^{\nu}\;\psi(x)\right) .$$

because $\mathcal{L} = 0$ for the solutions of the Dirac's equation.

Current vector:

$$J^{\mu}(x) = \bar{\psi}\;\gamma^{\mu}\;\psi .$$

Spin tensor:

$$S^{\nu;\alpha\beta} = -\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x^{\nu}}\right)}\; M^{\alpha\beta}{}_{ij}\,\psi_j - \bar{\psi}_i\; M^{\alpha\beta}{}_{ij}\,\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\bar{\psi}}{\partial x^{\nu}}{}_{j}\right)} .$$

The transformation laws for spinors is:

$$\psi'(x') = \psi(x) + \frac{1}{4}\left(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha\right)\psi(x)\,\delta\omega_{\alpha\beta},$$

so that

$$M^{\alpha\beta}\,\psi = \frac{1}{4}\left(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha\right)\psi,$$

$$\bar{\psi}\,M^{\alpha\beta} = -\frac{1}{4}\bar{\psi}\left(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha\right).$$

Hence:

$$S^{\nu;\alpha\beta} = \frac{1}{4}\,\bar{\psi}(x)\left\{\gamma^\nu\sigma^{\alpha\beta} + \sigma^{\alpha\beta}\gamma^\nu\right\}\psi(x) ,$$

where

$$\sigma^{\alpha\beta} = \frac{1}{2i}\left(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha\right).$$

<u>Momentum representation</u>

$$\psi(x) = \frac{1}{(2\pi)^{3/2}}\int \psi(p)\, e^{-ipx}\, d^4 p ,$$

Set

$$\psi(p) = \varphi(p)\;\delta(p^2 - m^2) ,$$

so

$$\psi(x) = \psi^{(+)}(x) + \psi^{(-)}(x) ,$$

$$\psi^{(\pm)}(x) = \frac{1}{(2\pi)^{3/2}}\int d^3 p\, e^{\mp ipx}\,\varphi^{(\pm)}(\vec{p}) ,$$

$$\varphi^{(\pm)}(\vec{p}) = \frac{1}{2p_o}\,\frac{1}{2}\,(1 + \mathrm{sgn}\, p_o)\;\varphi(\pm p) ,$$

$$p_o = +\sqrt{\vec{p}^{\,2} + m^2}$$

$$\text{sgn } p_o = \begin{cases} +1 & p_o > 0 \\ -1 & p_o < 0 \, . \end{cases}$$

The $\varphi^{(\pm)}(\vec{p})$'s are solutions of the equation:

$$(\gamma^K p_K \mp m)\, \varphi^{(\pm)}(\vec{p}) = 0 \, .$$

In the rest system ($\vec{p} = 0$) these have the form:

$$\varphi^{(+)}(\vec{0}) = \begin{pmatrix} c_1 \\ c_2 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad \varphi^{(-)}(\vec{0}) = \begin{pmatrix} 0 \\ 0 \\ c_3 \\ c_4 \end{pmatrix}$$

where we have used the representation $\gamma^o = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$.

So:

$$\varphi = \begin{pmatrix} \varphi(+) \\ \varphi(-) \end{pmatrix}, \quad \text{or} \quad \psi = \begin{pmatrix} \psi^{(+)} \\ \psi^{(-)} \end{pmatrix}$$

where now $\varphi^{(+)}$, $\varphi^{(-)}$, $\psi^{(+)}$, $\psi^{(-)}$ are two component spinors.

In general one can write:

$$\varphi_\alpha^{\pm}(\vec{p}) = \sum_{\nu = 1,2} a_\nu^{(\pm)}(\vec{p})\, \vartheta_\alpha^{\nu(\pm)}(\vec{p})$$

$$\bar{\varphi}_\alpha^{\pm}(\vec{p}) = \sum_{\nu = 1,2} a_\nu^{+(\pm)}(\vec{p})\, \bar{\vartheta}_\alpha^{\nu(\pm)}(\vec{p}) \, .$$

Call

$$a_\nu^{(-)}(\vec{p}) = b_\nu^{+}(\vec{p}) \qquad \vartheta^{\nu(+)}(\vec{p}) = U_\nu(\vec{p})$$

$$a_\nu^{(+)}(\vec{p}) = a_\nu(\vec{p}) \qquad \vartheta^{\nu(-)}(\vec{p}) = \mathcal{V}_\nu(\vec{p})$$

then the momentum representation of $\psi(x)$ is:

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p_o}} \sum_{s=1,2} \left\{ a_s(\vec{p}) U_s(\vec{p}) e^{-ipx} + b_s^+(\vec{p}) \mathcal{V}_s(\vec{p}) e^{ipx} \right\}.$$

One then has for the energy-momentum

$$P^\mu = \int d^3x T^{o\mu} = \int d^3p\, p^\mu \sum_{\nu=1,2} \left(a_\nu^{+(+)}(\vec{p}) a_\nu^{(-)}(\vec{p}) - a_\nu^{+(-)}(\vec{p}) a_\nu^{(+)}(\vec{p}) \right)$$

but

$$a_\nu^{(+)\ +} = a_\nu^{+(-)} = b_\nu(\vec{p}),$$

$$a_\nu^{(-)\ +} = a_\nu^{+(+)} = a\ (\vec{p})$$

thus

$$P^\mu = \int d^3p\, p^\mu \sum_\nu (-b_\nu(\vec{p})\, b_\nu^+(\vec{p}) + a_\nu^+(\vec{p})\, a_\nu(\vec{p})).$$

This expression (in the classical spinor field theory) is not positive definite.

For the 3-dimensional spin tensor we get:

$$S^{o;iK} = \tfrac{1}{2} \int \psi^*(x)\, \sigma^{iK} \psi(x)\, d^3x, \qquad i, K = 1, 2, 3,$$

or

$$\vec{S} = \tfrac{1}{2} \int \psi^*(x)\, \vec{\sigma}\, \psi(x)\, d^3x \quad .$$

Since $T^{\mu\nu}$ is not symmetric, $\vec{S}$ is not conserved in time. If however $\psi, \bar{\psi}$ are independent of some coordinate like x_1, x_2 then there is conservation of S_3. In fact:

$$\frac{\partial S^{\nu;\alpha\beta}}{\partial x^\nu} = \frac{1}{4} \frac{\partial \bar{\psi}}{\partial x^\nu} \left\{ \gamma^\nu \sigma^{\alpha\beta} + \sigma^{\alpha\beta} \gamma^\nu \right\} \psi(x) +$$

$$+ \frac{1}{4} \bar{\psi} \left\{ \gamma^\nu \sigma^{\alpha\beta} + \sigma^{\alpha\beta} \gamma^\nu \right\} \frac{\partial \psi}{\partial x^\nu}$$

$$= \frac{1}{4} \frac{\partial \bar{\psi}}{\partial x^\nu} \left\{ \gamma^\nu \sigma^3 + \sigma^3 \gamma^\nu \right\} \psi(x) + \frac{1}{4} \bar{\psi} \left\{ \gamma^\nu \sigma^3 + \sigma^3 \gamma^\nu \right\} \frac{\partial \psi}{\partial x^\nu} ,$$

but if $\frac{\partial \psi}{\partial x^1} = \frac{\partial \psi}{\partial x^2} = 0$ then

$$\frac{\partial S^{\nu;12}}{\partial x^\nu} = \frac{1}{2} \frac{\partial \bar{\psi}}{\partial x^\nu} \gamma^\nu \sigma^3 \psi(x) + \frac{1}{2} \bar{\psi} \sigma^3 \gamma^\nu \frac{\partial \psi}{\partial x^\nu}$$

$$= \frac{1}{2} \left\{ - im\, \psi \sigma^3 \psi + im\, \bar{\psi} \sigma^3 \psi \right\} = 0 ,$$

$S_3 = \int d^3x\, S^{0;12}$ is conserved in time.

For the charge Q we find:

$$\begin{aligned} Q &= \int \psi^*(x)\ \psi(x)\ d^3x \\ &= \int d^3 p \sum_\nu \left(a_\nu^{+(+)}(\vec{p}) a_\nu^{(-)}(\vec{p}) + a_\nu^{+(-)}(\vec{p}) a_\nu^{(+)}(\vec{p}) \right) \\ &= \int d^3 p \sum_\nu \left(\left(a_\nu^{(-)}(\vec{p}) \right)^+ a_\nu^{(-)}(\vec{p}) + \left(a_\nu^{(+)}(\vec{p}) \right)^+ a_\nu^{(+)}(\vec{p}) \right) . \end{aligned}$$

The charge is positive definite.

2d. VECTOR FIELD

This type of field represents the electromagnetic field ($m = 0$) or the vector meson field ($m \neq 0$).

3. INTERACTIONS

So far we have studied classical free fields. Interaction is taken into account by adding a new term to the Lagrangian build with the interacting field variables, such that the total

Lagrangian be invariant with respect to the proper inhomogeneous Lorentz group. In addition, some terms may be invariant under the improper group, where as others (example: β -decay) we know not to be invariant.

One can obtain local interaction Lagrangian by simply multiplying a scalar formed with the first field functions by a scalar formed with the second field function, both at the same space-time point.

We write:

$$\mathcal{L} = \mathcal{L}_o + \mathcal{L}_{int} .$$

$\mathcal{L}_o$, $\mathcal{L}_{int}$ both invariant under the proper inhomogeneous Lorentz group. From the total $\mathcal{L}$ the invariant corresponding to translations and rotations, namely, energy-momentum and angular momentum are given by Noether's theorem.

As an example we give the interaction of the spinor field ψ with the real scalar field, the real pseudoscalar field and the real vector field

	Real Scalar $\phi(x)$	Real Pseudoscalar $\varphi(x)$	Real Vector $A_\mu(x)$
Spinor ψ	$g_1 \bar{\psi}\, \psi\, \phi$ $+ g_2 \bar{\psi} \gamma^\mu \psi \frac{\partial \phi}{\partial x^\mu}$	$g_1 \bar{\psi} \gamma^5\, \psi\, \varphi$ $+ g_2 \bar{\psi} \gamma^5 \gamma^\mu \psi \frac{\partial \varphi}{\partial x^\mu}$	$e_1 \bar{\psi} \gamma^\mu \psi A_\mu$ $- e_2 \bar{\psi} \sigma^{\mu\nu} \psi F_{\mu\nu}$

Another example is given by the possible interaction form between four fermions P, N, e, ν .

$$\mathcal{L}_{int} = (\bar{\psi}_e\ O_{(s)}\ \psi_N)\left(\psi_e\ O_{(s)}\left[C_s + C_s'\gamma^5\right]\psi_\nu\right) +$$

$$+\ (\bar{\psi}_e\ O_\mu^{(V)}\psi_N)\left(\bar{\psi}_e\ O^{(V)\mu}\left|C_V + C_V'\gamma^5\right|\psi_\nu\right) +$$

$$+\ (\bar{\psi}_e\ O^{(T)}_{\mu\lambda}\psi_N)\left(\bar{\psi}_e\ O^{(T)\mu\lambda}\left[C_T + C_T'\gamma^5\right]\psi_\nu\right) +$$

$$+\ (\bar{\psi}_e\ O^{(A)}_\mu\ \psi_N)\left(\psi_e\ O^{(A)\mu}\left[C_A + C_A\gamma^5\right]\psi_\nu\right) +$$

$$+\ (\bar{\psi}_e\ O^{(P)}\psi_N)\left(\psi_e\ O^{(P)}\left[C_P + C_P'\gamma^5\right]\psi_\nu\right) +$$

+ hermitian conjugate. (See page 75 where the operators O are given as γ_A).

This corresponds to the decay process:

$$N \longrightarrow P + e + \bar{\nu}$$

Recent experiments seem to indicate that only C_V, C_V', C_A, $C_A' \neq 0$.

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REFERENCES

The following will be useful to consult:

1) E. P. Wigner, Group Theory and its Applications to the Atomic Spectra, Academic Press (1959).

2. L. Van der Waerden, Gruppentheoretische Methoden in der Quantenmechanik, Julius Spinger, (1932).

3. W. Pauli, Lectures on Continuous Groups and Reflections in Quantum Mechanics. UCRL-8213.

4. P. T. Mattews, Relativistic Quantum theory of Elementary Particle Interactions, NYD-2097.

5. I. M. Gelfand and Z. Ys. Sapiro, Representations of the Group of Rotations of 3-dimensional Space and their Applications, Amer. Math. Soc. Translations Series 2, Vol. 2, (1956).

6) M. A. Naimark, Linear Representations of the Lorentz Group, Amer. Math. Soc. Translations Series 2, Vol. 6 (1951).

7) E. P. Wigner, Ann. Math. 40, 149 (1939).

8) V. Bargmann and E. P. Wigner, Proc. Nat. Acd. Sci. 34, 211, (1948).

9) I. M. Shirokov, A Group Theoretical consideration of the Basis of Relativistic Quantum Mechanics, JETP, 6, (33), 664 (I); 929 (III), (1958).

10) Cartan, Leçons sur la theorie des spineurs, Hermann & Cie, Paris.

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INTRODUCTION

Elementary particles are described by relativistic wave equations. The fields which represent them are assumed to possess certain properties which follow from an invariance of the wave equations under certain groups of transformations. These are suggested by experiment. Examples of such transformations are the displacements and rotations, and invariance of the laws under these groups (suggested by the homogeneity and isotropy of space) leads to the important principles of conservation of momentum and angular momentum.

The most important invariance principle in field theory is the principle of relativity. It states that the laws of nature must be independent of the position and the (constant) velocity of the observer. Mathematically the principle imposes that the equations of motion be invariant with respect to the transformations of the inhomogeneous proper Lorentz group. The basic character of this principle is then seen in the fact that the wave fields must form a representation space of the Lorentz group. Their geometrical nature is thus defined and the fields can only be scalars, spinors, vectors, and spinors or tensors of higher rank.

We shall give the representations of the three-dimensional rotation group and of the Lorentz group. The equations of motion of free elementary particles are obtained from the study of the invariants of the latter group. The mass - as an arbitrary real number which characterizes one of the two invariants of the group - and the spin are introduced in the theory and thus seen to be a consequence of relativity.

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