Francesco Toppan

CBPF - CCP
Rua Dr. Xavier Sigaud 150, cep 22290-180
Rio de Janeiro (RJ), Brazil

Abstract

The basic principles of topology are introduced. The Euler formula is discussed. It is applied to derive some mathematical classification theorems, including the classification of regular polyhedra (platonic solids) and of some more general geometrical figures (such as the "fullerens" or "soccerballs") of interest e.g. for chemistry. Further pedagogical examples and applications are derived. Their relevance to physics is mentioned.

CBPF-MO-004/02

E-mail: toppan@cbpf.br

The Euler formula

The Euler formula is the first example in mathematics of a topological invariant. It was first discovered by Descartes as a formula involving solid angles. However, Descartes did not realize its topological character, due to Euler, namely the relation between the number of faces $\mathbf{F}$, vertices $\mathbf{V}$ and edges $\mathbf{E}$ of any triangularized solid figure.

The derivation of the Euler formula is sketched in figure **1**. It should be noticed that the original Euler derivation was slightly incorrect.

In the following we use the Euler formula to derive some mathematical classification theorems and also to rederive some of the most celebrated results of greek mathematics, i.e. the classification of platonic solids.

Different Euler formulas

i) The "planar" formula.

$$\mathbf{V} + \mathbf{F} - \mathbf{E} = 1. \quad (1)$$

ii) The original Euler formula for triangularized convex solids.

$$\mathbf{V} + \mathbf{F} - \mathbf{E} = 2. \quad (2)$$

iii) The generalized Euler formula for generic triangularized compact boundaryless (Riemann) surfaces.

$$\mathbf{V} + \mathbf{F} - \mathbf{E} = 2 - 2h. \quad (3)$$

Here h, the number of "handles", is a topological invariant, also known as the *genus* of the surface. In string theory, e.g., it characterizes the order of the perturbation expansion.

Application of the planar Euler formula: the tessellation of the plane with regular polygons.

Let f polygons of p sides meeting at a single vertex. Then

$$E \approx \frac{Vf}{2}, \tag{4}$$

$$F \approx \frac{Vf}{p}. \tag{5}$$

Remark: For a fixed bounded region inside a circle of size R, the formula is approximated. It is recovered in the limit $R \to \infty$. In such a limit $V \to \infty$. We get

$$Vf(\frac{1}{f} + \frac{1}{p} - \frac{1}{2}) = 1, \tag{6}$$

that is in the $V \to \infty$ limit

$$\frac{1}{f} + \frac{1}{p} = \frac{1}{2}. \tag{7}$$

Solutions of the above equation for integer values (≥ 3) of f and p:

$$
\begin{array}{llll}
a) & f = 3 & , & p = 6; \\
b) & f = 4 & , & p = 4; \\
c) & f = 6 & , & p = 3.
\end{array}
\tag{8}
$$

Remark: solutions a and c are dual in the $f \leftrightarrow p$ exchange (solution b is self-dual).

Remark: The formula (7) is usually derived from the property that the internal sum of the angles in a triangle is equal to π:

$$
f(1 - \frac{2}{p})\pi \;=\; 2\pi. \tag{9}
$$

Other application: tessellation of the plane with 2 polygons of p sides and 1 polygon of q sides meeting at each vertex.

We look for $p \geq 3$ and $q \geq 3$ solutions of the diophantine equation

$$\frac{2}{p} + \frac{1}{q} \;=\; \frac{1}{2}. \tag{10}$$

Setting

$$p = \tfrac{4q}{q-2} = 4 + \tfrac{8}{q-2} \tag{11}$$

we get the complete set of solutions

$$\begin{aligned} q = 3 \quad &, \quad p = 12; \\ q = 4 \quad &, \quad p = 8; \\ q = 6 \quad &, \quad p = 6; \\ q = 10 \quad &, \quad p = 5. \end{aligned} \tag{12}$$

The first and most famous problem of graph theory, solved with topological methods by Euler: the Königsberg's bridges problem.

Is it possible to cross all bridges of Königsberg (given by figure 2) exactly once?

Remark on the existence of a necessary condition: on each intermediate vertex the number of incoming lines must coincides with the number of outgoing lines

$$n_{int} = n_{out}. \tag{13}$$

A physical application of the planar Euler's formula.

It controls (see figure 3) the $\hbar$ order in the perturbation expansion of Feynman graphs.

We have indeed

$$L = I - V + 1, \quad (14)$$

where L is the number of loops and I the number of internal lines (propagators).

The formula (14) coincides with the planar formula (7), with

$$\begin{aligned} L &\equiv F, \\ I &\equiv E. \end{aligned} \quad (15)$$

A pedagogical problem: let us connect n generic points in the circumference of a circle with straight lines (see figure 4). How many regions are individuated?

Explicit check:

$$\begin{aligned} n &= 1 \quad F = 1, \\ n &= 2 \quad F = 2, \\ n &= 3 \quad F = 4, \\ n &= 4 \quad F = 8, \\ n &= 5 \quad F = 16, \\ n &= 6 \quad F = ?, \\ n &= \ldots \quad \ldots \end{aligned} \tag{16}$$

Hints for a solution: planar Euler's formula and induction.

Results of the previous problem.

External vertices $V_{ext} \equiv n$.

Internal vertices $V_{int}(n)$, edges $E(n)$, faces $F(n)$.

$$
\begin{aligned}
E(n) &= 2V_{int}(n) + \frac{n(n+1)}{2}, \\
F(n) &= E(n) + 1 - n - V_{int}(n).
\end{aligned}
$$

$$V_{int}(n) = \frac{1}{24}(n^4 - 6n^3 + 11n^2 - 6n).$$

$$F(n) = \frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24).$$

Comment on figure 4.

$$V_{int}(n+1) = V_{int}(n) + \frac{1}{2}\sum_{j=1}^{n}(n-j)(n-j+1).$$

Since

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2} \quad , \quad \sum_{j=1}^{n} j^2 = \frac{n}{6}(2n^2+3n+1),$$

then

$$V_{int}(n+1) = V_{int}(n) + \frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{n}{3}.$$

Due to induction

$$V_{int}(n) = An^4 + Bn^3 + Cn^2 + Dn + E,$$

with

$$A = \tfrac{1}{24},\ B = -\tfrac{1}{4},\ C = \tfrac{11}{24},\ D = -\tfrac{1}{4},\ E = 0.$$

Higher-dimensional generalization of Euler formula (e.g. for Kaluza-Klein motivated theories).

The alternate sum alt for hypercubes in d dimensions (i.e. $(x_1, x_2, \ldots, x_d)$ with $0 \leq x_i \leq 1$ for $i = 1, \ldots, d$).

number of k-faces: $2^k \left(\begin{array}{c} d \\ k \end{array} \right)$.

$$alt = \sum_{k=1}^{d} (-1)^{k+1} 2^k \left(\begin{array}{c} d \\ k \end{array} \right). \tag{17}$$

Due to the binomial formula

$$1 = (2-1)^d = \sum_{j=0}^{d} (-1)^{d-j} 2^j \left(\begin{array}{c} d \\ j \end{array} \right),$$

the result of alt is 2 in d odd-dimensional spaces and 0 in even-dimensional spaces.

Application of the Euler formula to the classification of the regular polyhedra (platonic solids).

(Regular polyhedra: f polygons of p sides meet at each vertex.)

$$(\frac{1}{f} + \frac{1}{p} - \frac{1}{2}) = \frac{1}{E}. \qquad (18)$$

Full table of integral solutions with $p \geq 3$, $f \geq 3$.

	f	p	E	V	F
a	3	3	6	4	4
b	3	4	12	8	6
c	3	5	30	20	12
d	4	3	12	6	8
e	5	3	30	12	20

with a tetrahedron, b cube, c dodecahedron, d octahedron, e icosahedron.

Due to the $f \leftrightarrow p$ duality, we have 5 platonic solids but only 3 groups of symmetry.

Comment:

In the previous table the values F, E, V can be used to determine how many ingredients are necessary to construct a specific polyhedrum.

For instance F determines the number of p-sides paper polygons that have to be glued together to produce the corresponding regular polyhedrum.

Alternatively, V and E determine the number of magnetized spheres and magnetized bars respectively, necessary to construct the given polyhedrum with, let's say, Geomag (see figure **5**).

Further classification. Necessary conditions for the existence of polyhedra with more polygonal faces (with application to fullerens and soccerballs.)

Polyhedra made with two kind of polygons of p_a and p_b sides. f_a and respectively f_b such polygons meet at each vertex.

The edges are given by

$$E = \frac{V}{2}(f_a + f_b),$$

the faces are respectively given by

$$F_a = V\frac{f_a}{p_a} \qquad , \qquad F_b = V\frac{f_b}{p_b}.$$

Let $x \equiv \frac{f_a}{f_a+f_b}$, $f \equiv f_a + f_b$, $p \equiv p_a$, $q \equiv p_b$.

Then the Euler formula reads as follows

$$\frac{1}{f} + \frac{x}{p} + \frac{(1-x)}{q} = \frac{1}{2} + \frac{1}{E}.$$

Classification of the following case. 2 polygons of p sides and a single polygon of q sides meet at each vertex.

Besides respecting the Euler formula, the values V, E, F_p, F_q, p and q must all be integrals. Moreover $p, q \geq 3$, $V \geq 4$ and $E \geq 6$. The following constraint must be satisfied

$$V = \frac{2}{3}E, \quad F_p = \frac{2V}{p}, \quad F_q = \frac{V}{q}, \qquad (19)$$

as well as

$$V \;\; > \;\; p, q. \qquad (20)$$

The Euler formula reads now as follows:

$$E \;\; = \;\; \frac{6pq}{4q + (2-q)p}. \qquad (21)$$

Remark: for $p = 3$, no other solution respecting the constraint, besides $q = 3$ (the already known case of the tetrahedron) is found.
E.g. for $p = 3$, $q = 6$ the constraint (20) is violated.

Comment. For $p = 4$ the Euler formula is degenerate ($E = 3q$) and admits solution for any value of $q = 3, 4, \ldots$. The corresponding polyhedra are illustrated in figure 6.

The non-trivial solution to the classification problem are therefore found for $p \geq 5$.

It is convenient to organize the classification in terms of increasing values of $q = 3, 4, \ldots$. It is easily checked that no solution is found for $q \geq 10$.

The complete set of solutions is given by the table below.

Table with the solution of the problem.

	q	p	E	V	F_p	F_q
a	3	6	18	12	4	4
b	3	9	54	36	8	12
c	3	10	90	60	12	20
d	3	11	198	132	24	44
e	4	6	36	24	8	6
f	4	7	84	56	16	14
g	5	6	90	60	20	12
h	6	5	45	30	12	5
i	8	5	120	80	32	10
j	9	5	270	180	72	20

The cases a and e are explicitly constructed in figure **7** and figure **8** respectively. The case g is concretely realized by the C60 molecule of carbonium (the fulleren), or by the soccer balls. Remark:the Euler formula provides a necessary condition for the existence of the corresponding polyhedra. The explicit construction of a given polyhedrum requires specifying how to glue the polygons together, whenever is possible. This information however is not furnished by the Euler formula.

[CR] **R. Courant and H. Robbins**, *What is Mathematics. An Elementary Approach to Ideas and Methods*, Oxford Un. Press, New York 1941.

[Wey] **H. Weyl**, *Symmetry*, Princeton Univ. Press, 1952.

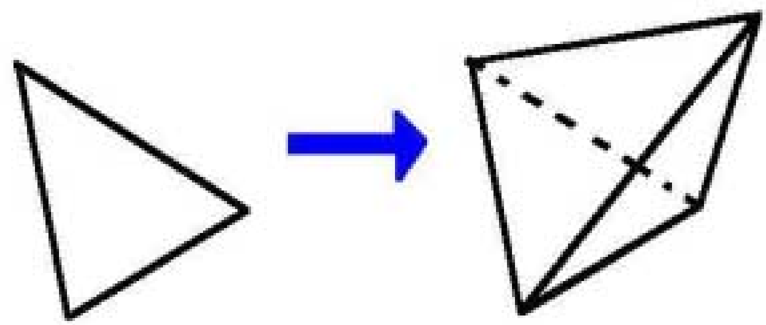
[Cox] **H.S.M. Coxeter**, *Regular Polytopes*, Dover, New York, reprint 1973.

[Hir] **G. Hirsch**, *Topologie*, pages 380-415 in *J. Dieudonné*, *Abregé d'histoire des mathématiques*, Hermann, Paris 1978.

[Des]**Descartes**, *Oeuvres inédites de Descartes par le comte Foucher de Careil*, t. 2 1859, ed. de Jonquières, Paris t. 110, pages 169-173, 261-266, 677-680, 1890.

[Eul] **Euler**, travail présenté le 26 août 1735 à l'Académie de Saint-Petersbourg, *Opera Omnia*, t. 7, pages 1-10.

FIGURE 1



V (=3) --> V'= V+1

F=1 --> F'= V

E (=3) --> E'= E+V

V+F-E = V'+F'-E'

FIGURE 1

surfaces with holes

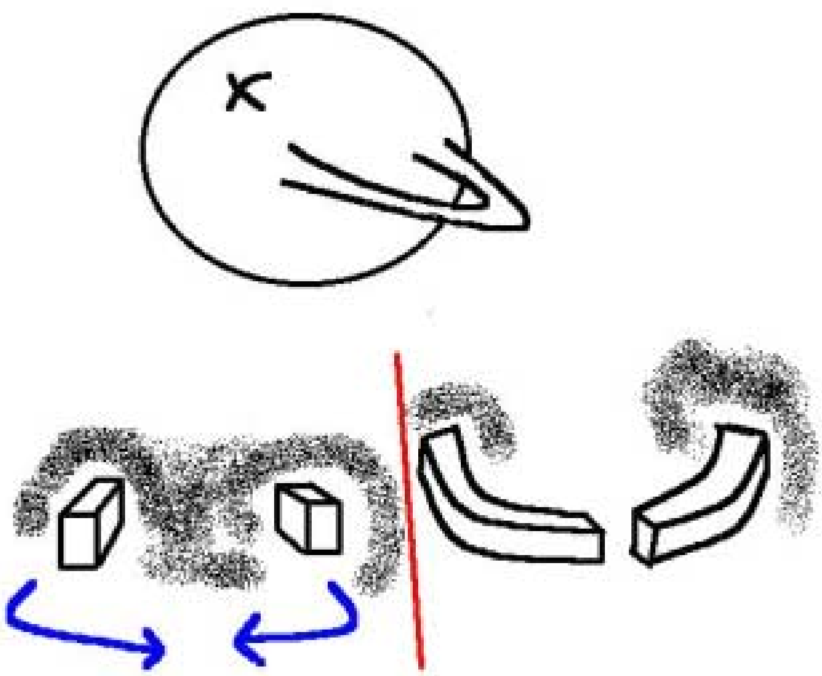


FIGURE 1

surfaces with holes

-2 in the sum of V+F-E
when gluing together

V+F-E = 2 - 2 h

(h number of holes)

CBPF-MO-004/02

FIGURE 2

the Koenisgberg´s bridges

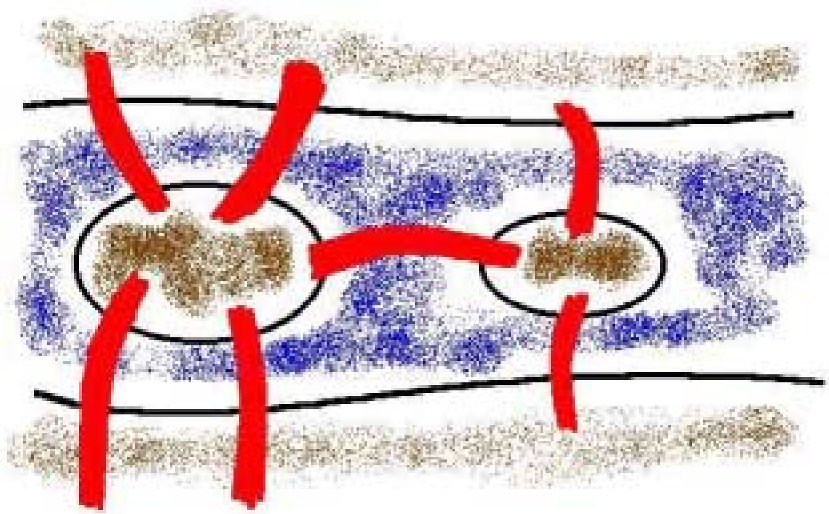


FIGURE 2

the Koenisgberg´s bridges

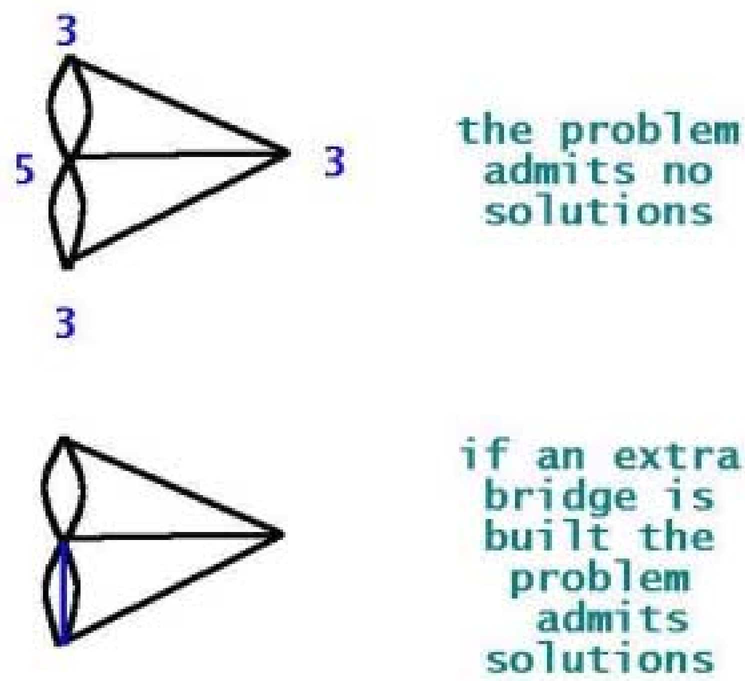


FIGURE 3

simply connected graphs

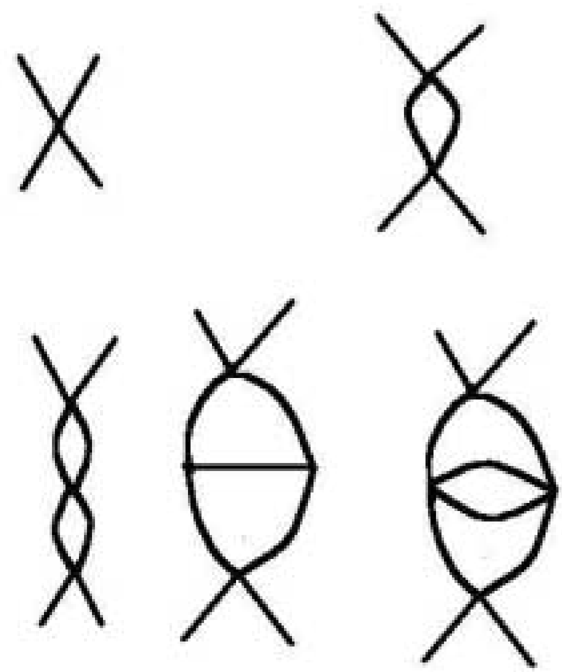


FIGURE 4

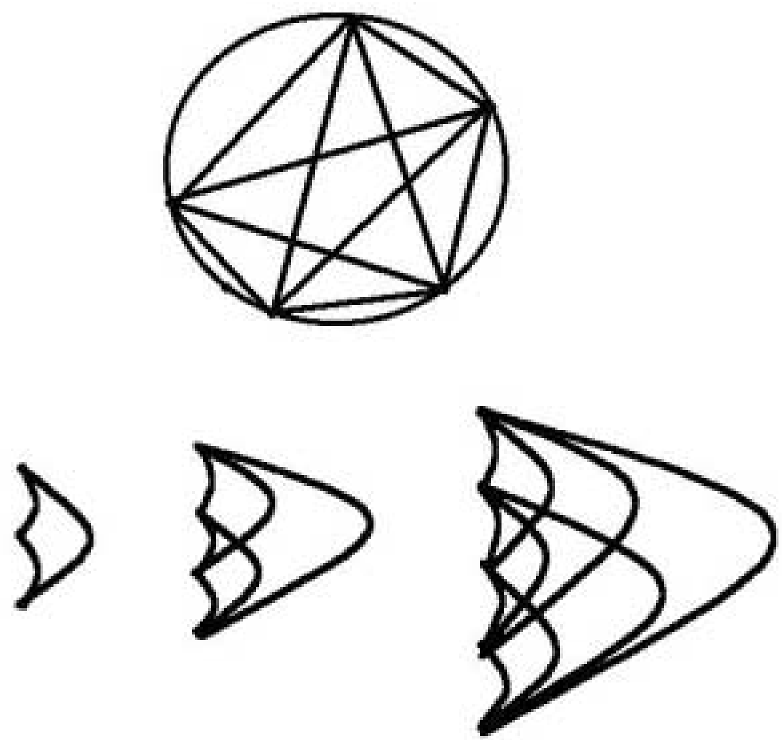
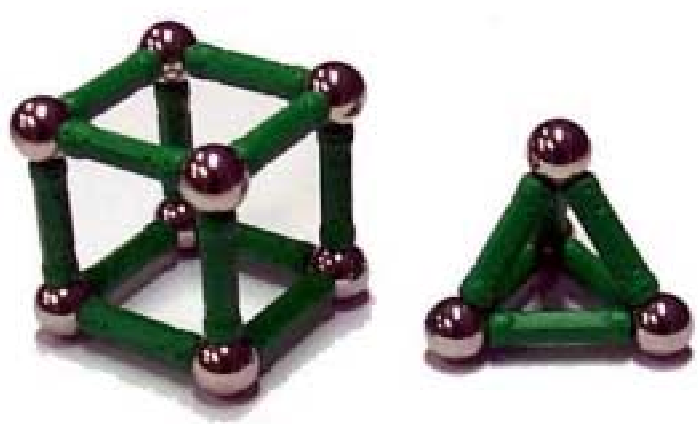


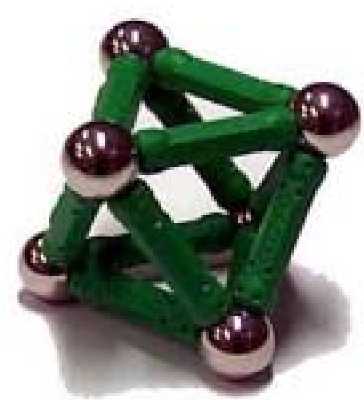
FIGURE 5

Polyhedra with Geomag



the cube and the tetrahedron

FIGURE 5
Polyhedra with Geomag



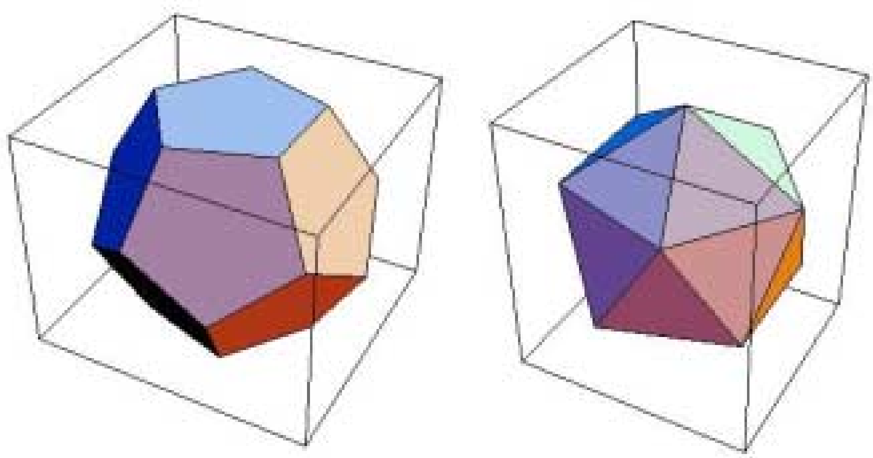
the octahedron

FIGURE 5

Polyhedra with Geomag

4 hexagons and 4 triangles

FIGURE 5



dodecahedron and icosahedron

FIGURE 6

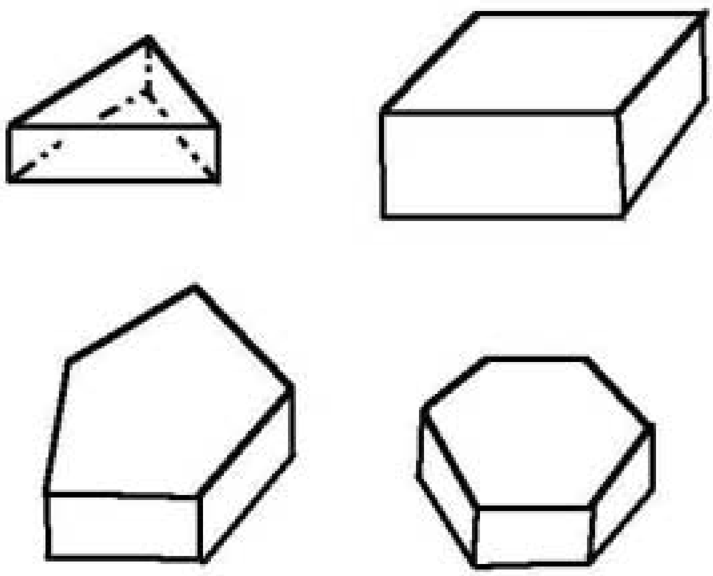


FIGURE 7

4 hexagons + 4 triangles

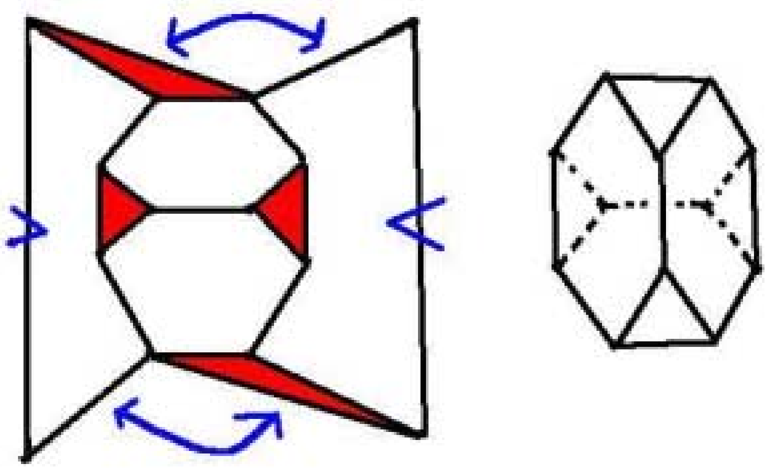


FIGURE 8

8 hexagons and 6 squares

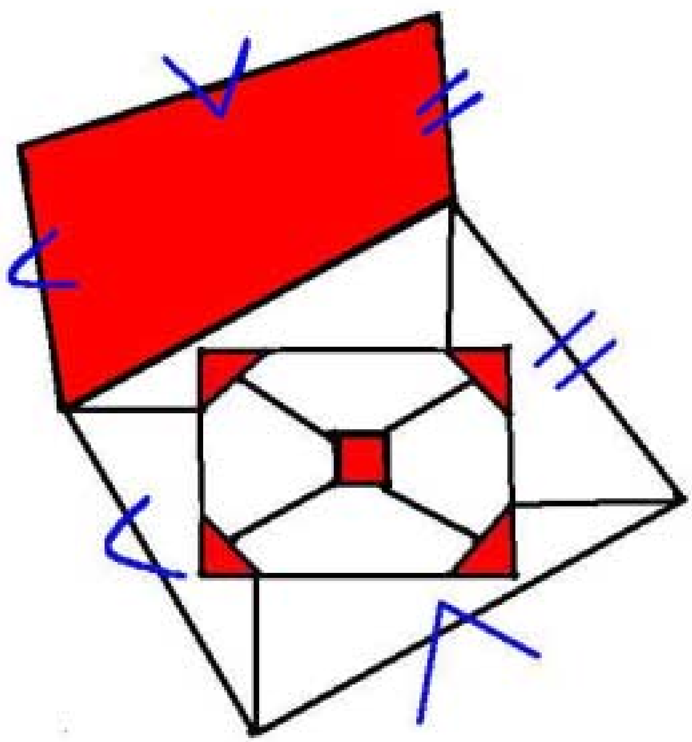


FIGURE 9

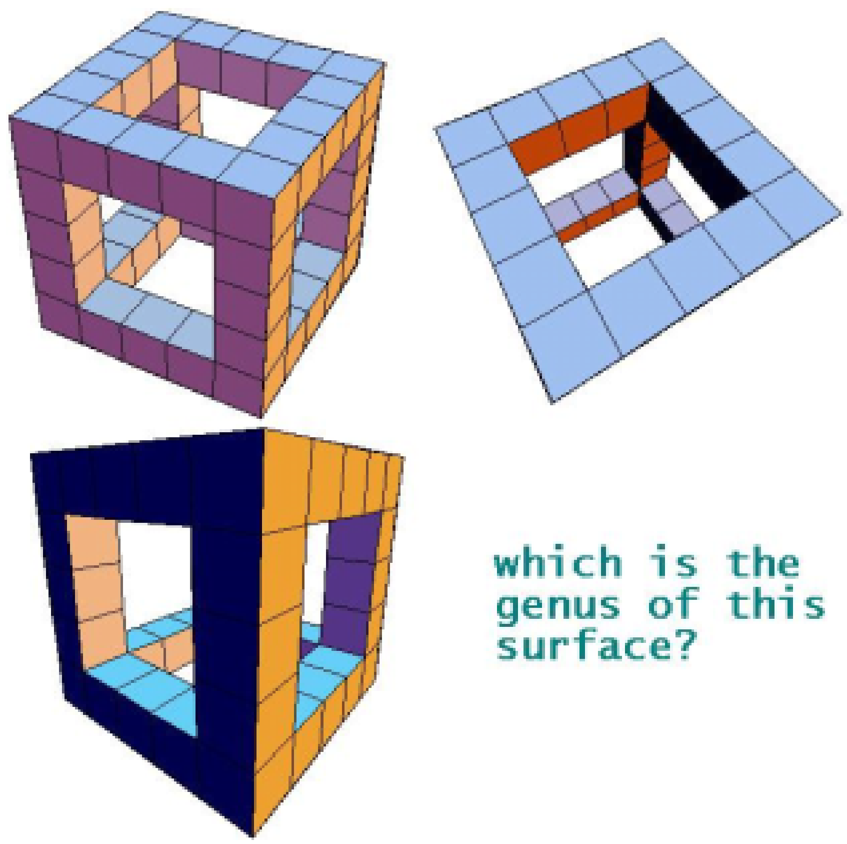
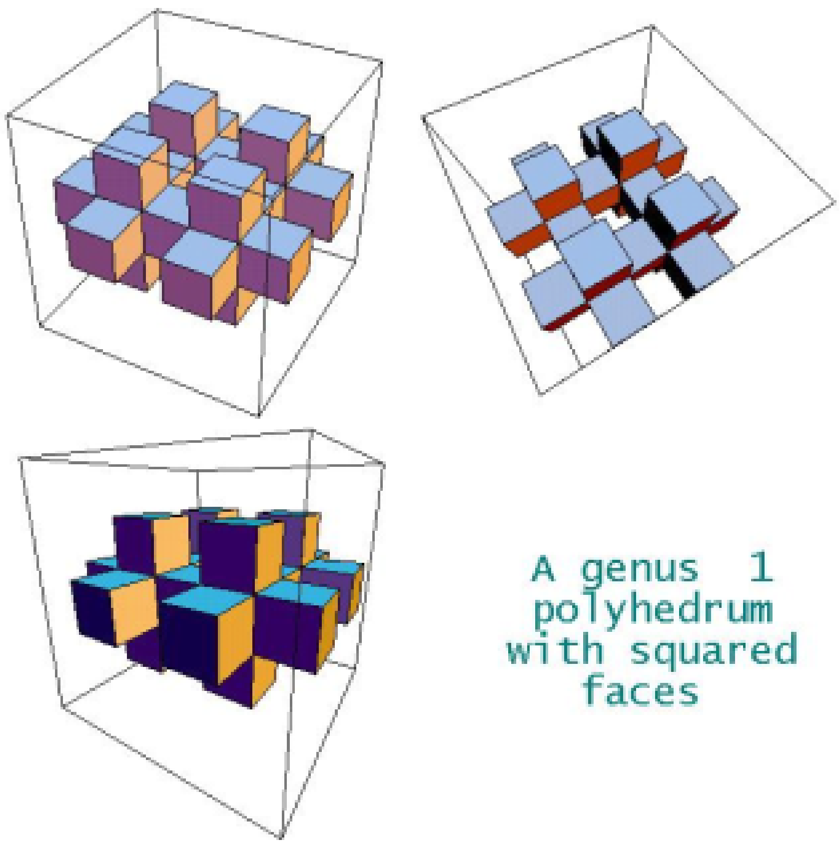


FIGURE 10



A genus 1 polyhedrum with squared faces

FURTHER PROBLEMS

check which of the cases labelled in table 18 indeed correspond to regular polyhedra.
How to construct them (i.e. how to glue the polygons together)?

generalize the result of table in page 18 to further cases. For instance Euler formula and regular polyhedra for genus $g>0$.

Notice: the genus 1 polyhedrum with squared faces can be trivially extended to higher genus.