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QUANTUM SCALING IN MANY-BODY SYSTEMS

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The theory of quantum critical phenomena is used and extended to study some current many-body problems in condensed matter physics. Renormalization group concepts are applied to strongly correlated electronic materials which are close to a zero temperature instability. These systems have enhanced effective masses and susceptibility. Scaling ideas are developed to obtain the exponents which govern the critical behavior of these quantities in terms of the usual critical exponents associated with a zero temperature phase transition. We show the existence of a new energy scale, related to the quantum nature of the many-body instability, which can be generally associated with the setting in of Fermi-liquid behavior with decreasing temperature in strongly interacting electronic systems. We use these ideas to investigate the Kondo lattice problem which provides a model to describe heavy fermion systems. The usefulness of the scaling approach is illustrated by applying it to a well studied heavy fermion. We introduce a scaling theory of the Mott transition with special emphasis on charge fluctuation effects. It leads us to distinguish between the thermal mass, as obtained from the linear term of the temperature dependent specific heat, from the optical mass describing the effect of interactions on the conductivity and which signals the localization transition. Finally we discuss briefly how disorder can be included in this approach and how it should affect our results.

1.1 QUANTUM PHASE TRANSITIONS

1.1.1 INTRODUCTION

Quantum phase transitions distinctively from temperature-driven critical phenomena occur due to a competition between different parameters describing the basic interactions of the system[1,2]. They have as a specific feature the quantum character of the critical fluctuations. Besides at zero temperature time plays a crucial and fundamental role, the static properties being coupled to the dynamics. In this report we will be mainly interested in quantum phase transitions which occur in strongly correlated electronic materials and how scaling concepts can be useful to understand the properties of these many-body systems[1]. A similar approach has recently been applied to the case of interacting bosons[3]. However the fermionic problem has its own idiosyncrasies and difficulties. For example there is no natural order parameter associated with the localization transition in the electronic case while bosons at zero temperature are either localized or superfluid and the superfluid order parameter can then be used to distinguish between both phases. We shall be specifically concerned here with heavy-fermion materials[4,5] and the highly correlated electron gas[6,7,8]. The former are electronic systems with unstable f-shell elements which are close to a magnetic instability and have a huge thermal effective mass. The latter are characterized by their proximity to a metal-insulator transition induced by electronic correlations. In section 1 we introduce the main ideas and the scaling concepts which are used in this work. For this purpose we consider the simplest model exhibiting a quantum phase transition. In section 2 the heavy fermion problem is introduced and a scaling theory bearing in the competition between magnetism and Kondo effect is developed. The theory is applied to a well studied heavy fermion material allowing to fully appreciate it's usefulness and also limitations. Finally in section 3 the scaling approach is used to describe the Mott transition under the assumption that at zero temperature this is a continuous transition. Different approximations to the Hubbard model are examined from the point of view of the theory of quantum critical phenomena resulting in a better understanding of their nature. The exact solution of the Hubbard model in one dimension combines with scaling to provide a new perspective to the problem of correlated eletronic systems.

The renormalization group (RG) provides the appropriate framework to describe quantum critical phenomena[9,10]. The concepts of crossover, unstable fixed points, attractors, flow in parameter space and relevant or irrelevant fields turn out to be extremely useful also to describe the physical behavior of strongly correlated many body systems as we will show here. The renormalization group is particularly useful in cases where the phase transition is not clearly associated with an order parameter like in the case of the metal-insulator transition due to correlations[6] (Mott transition). It turns out that the flow of the RG equations to the different attractors is sometimes sufficient to characterize the metallic or insulating phases with no need of considering explicitly an order parameter[11].

Although the notion of crossover had an early application in the single impurity Kondo problem, which dealt with the formation of a local magnetic moment in a metal, we show it turns out to be extremely relevant also for the Kondo lattice in spite of the lattice translation invariance of this system. In this case it is associated with a new energy scale and with the concept of coherence which marks the onset of the Fermi liquid regime in dense Kondo systems. The scaling approach provides a simple and clear interpretation of the concept of coherence. This notion underlies most of the physics of heavy fermions and is very difficult to capture within usual many-body treatments which are generally restricted to a mean-field level.

The scaling properties of a system close to a quantum phase transition, can be derived considering an expansion of the RG equations near the unstable zero temperature fixed point governing this transition. The set of critical exponents associated with this fixed point characterize the universality class of the transition. Besides specifying the divergence of the correlation length, susceptibility, the critical slowing down, etc., for a many-body system these exponents also characterize the critical behavior of the compressibility[12], the conductivity mass[13] and the enhancement of the thermal mass[12] close to the zero temperature instability. In spite of the wide application of scaling ideas in condensed matter, only recently they have been systematically used in the study of strongly correlated electronic materials[3,4,8] like heavy fermions and Mott insulators. However before we describe how these ideas can be useful to understand the physics of these systems, we shall give a brief introduction to the scaling theory at a zero temperature phase transition using the simplest model borrowed from the theory of localized magnetism.

1.2 RENORMALIZATION GROUP AND SCALING RELATIONS

Let us consider the simplest model which exhibits a quantum phase transition, namely the one-dimensional Ising model in a transverse magnetic field[9,10]:

$$\mathcal{H} = - J \sum_i S_i^z S_{i+1}^z - h \sum_i S_i^x - H \sum_i S_i^z \qquad (1.1)$$

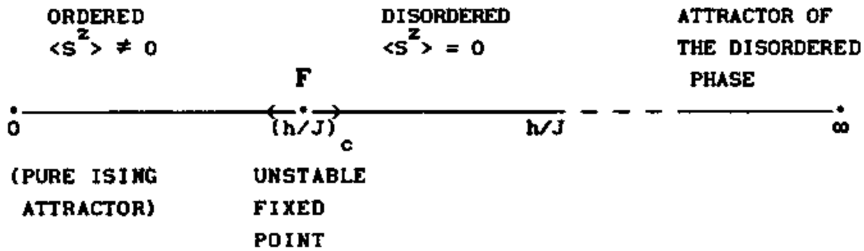
where J is the nearest-neighbor coupling, h the transverse field and H the uniform magnetic field in the z-direction conjugate to the order parameter $\langle S^z \rangle$. Let us consider initially the case H = 0. Physically we expect that:

i) T = 0, h = 0. There is long range magnetic order with an order parameter $\langle S^z \rangle \neq 0$.

ii) T = 0, h = ∞. The transverse field destroys the long range magnetic order and $\langle S^z \rangle = 0$.

Then we expect that at a critical value of the ratio (h/J) there is a zero temperature phase transition from an ordered state with $\langle S^z \rangle \neq 0$ to a disordered state, i.e. to a state with a vanishing order parameter,

$\langle S^z \rangle = 0$. From the RG point of view this phase transition at $(h/J) = (h/J)_c$ is associated with an unstable zero temperature fixed point at $(h/J)_c$. Schematically we have:



Associated with the unstable fixed point F we have a set of critical exponents which characterize the universality class of the transition. The stable fixed points at $(h/J) = 0$ and $(h/J) = \infty$, are the attractors of the ordered and disordered phases respectively.

1.3 SCALING PROPERTIES CLOSE TO A ZERO TEMPERATURE FIXED POINT

Let us consider how the parameters of the Hamiltonian (1) scale, under a length scale transformation by a factor b, close to the zero temperature unstable fixed point[14]. Let us define $g = \left| \left(\frac{h}{J}\right) - \left(\frac{h}{J}\right)_c \right|$ which measures the distance to the critical point in parameter space. We have:

$$\begin{cases} J' = b^{-y}J \qquad \text{or} \qquad h' = b^{-y}h \\ g' = b^{a}g \\ H' = b^{x}H \end{cases} \tag{1.2}$$

since $(h/J)' = (h/J)$ at the fixed point and where we have introduced three exponents y, a and x. The prime refers to the quantities renormalized under the length scale transformation. The ground state energy density (singular part) can be written as:

$$E_s = Jf(g,H/J) \tag{1.3}$$

where $f(x,y)$ is a scaling function. The new energy density and correlation length are:

$$\begin{cases} E'_s = b^{d}E_s = J'f(g',H'/J') \\ \xi' = \xi(g',H'/J') = b^{-1}\xi(g,H/J) \end{cases} \tag{1.4}$$

Using the relations (1.2) in (1.4) we get:

$$\begin{cases} \dfrac{E_s(g,H/J)}{J} = b^{-(d+y)}\, f(b^{a}g,b^{(x+y)}H/J) \\ \xi(g,H/J) = b\xi(b^{a}g,b^{(x+y)}H/J) \end{cases} \tag{1.5}$$

Now since b is arbitrary we make

$$b^a g = 1$$

or

$$b = g^{-1/a}$$

to obtain

$$\begin{cases} \dfrac{E_s}{J} = g^{(d+y)/a} f\left(1, \dfrac{H/J}{g^{(x+y)/a}}\right) \\ \xi = g^{-1/a} \xi\left(1, \dfrac{H/J}{g^{(x+y)/a}}\right) \end{cases} \tag{1.6}$$

from where we identify the correlation length exponent $\nu = 1/a$ and the exponent $\Delta = (x+y)/a = \nu(x+y)$. We can define an exponent α through the singularity of the ground state energy density, i.e. the relation $\frac{E_s}{J} = g^{2-\alpha}$, to get:

$$2 - \alpha = \nu(d + y) \tag{1.7}$$

This is a modified hyperscaling relation which relates the critical exponents ν and α to the dimensionality of the system d and to the exponent y which renormalizes the coupling J at the fixed point. It differs from the usual hyperscaling relation of finite temperature critical phenomena in a very fundamental way since d is replaced by d + y. We shall return to this important point later.

The magnetization $m = \langle S^z \rangle$ is defined by:

$$m = -\left(\partial E_s/\partial H\right)_{H=0} \propto |g|^{\beta}$$

which defines the exponent β.

Deriving Eq. (1.6) we get:

$$m = -\left(\frac{\partial E_s}{\partial H}\right)_{H=0} g^{(d+y)/a} g^{-(x+y)/a} f'(1,0) \propto |g|^{\nu(d-x)}$$

consequently

$$\beta = \nu(d-x) \tag{1.8}$$

The susceptibility is defined by

$$\chi = -\left(\frac{\partial^2 E_s}{\partial H^2}\right)_{H=0} \propto |g|^{-\gamma}$$

which in turn defines the exponent γ.

Taking the second derivative of Eq. (1.6) and comparing it with the equation above defining γ, we find

$$\gamma = \nu(2x + y - d) \qquad (9)$$

Equation (1.7), (1.8) and (1.9) yield the standard scaling relation[15]

$$\alpha + 2\beta + \gamma = 2 \qquad (1.10)$$

Finally defining the exponent δ through in $m \propto H^{1/\delta}$ at $g = 0$ we get:

$$\delta = (x + y)/(d - x)$$

and the scaling law

$$\delta = \Delta/\beta \qquad (1.11)$$

with $\Delta = \beta + \gamma$.

1.4 THE SPECIAL ROLE OF TIME AND THE DYNAMIC EXPONENT

Close to the zero temperature fixed point time τ scales as:

$$\tau' = b^z \tau$$

which defines the dynamic exponent z. The quantum character of the critical fluctuations allows us to relate z to the exponent y governing the scaling of the coupling constant at the unstable fixed point as we now show.

We can write for the critical fluctuations in J and τ

$$\Delta J' = b^{-y} \Delta J$$

$$\Delta\tau' = b^z \Delta\tau$$

Now we want the uncertainty relation $\Delta J \Delta\tau \geq \hbar$ to be a scaling invariant, i.e.:

$$\Delta J' \Delta\tau' = b^{(z-y)} \Delta J \Delta\tau \geq \hbar$$

and consequently we must have[4]:

$$y = z$$

and the modified hyperscaling relation

$$2 - \alpha = \nu(d + z) \qquad (1.12)$$

Note that the dimension d in this relation is replaced by $d_{eff} = d + z$ which plays the role of an *effective dimensionality*. This shift in the dimensionality has important consequences:

i) It implies that the exponents of the quantum system are the same of the corresponding classical one in $d_{eff} = d + z$ dimensions[1]. For example the d = 1 Ising model in transverse field has critical exponents[8], associated with the zero temperature fixed point F, with values: $\beta = 1/8$, $\alpha = 0$, $\nu = 1$, $\gamma = 1.75$. We can immediately identify these exponents with the exact results obtained by Onsager for the classical Ising model in two dimensions[8]. So in this case we expect to find z = 1 which is indeed confirmed by exact calculations[16].

ii) Since d_{eff} is increased it may reach the upper critical dimension in which case the exponents associated with the T = 0 fixed point assume classical (mean field) values. This is actually what happens for some of the phase transitions studied in this report.

1.5 THE CORRELATION FUNCTION AT T = 0

The fluctuation-dissipation[17] theorem gives us the following relation between the wave-vector dependent static susceptibility $\chi(q)$ and the dynamic q-dependent correlation function $S(q,\omega)$ at T = 0.

$$\chi(q) = \int \frac{d\omega}{2\pi} \frac{S(q,\omega)}{\omega} \tag{1.13}$$

The singularity of $\chi(q = 0) \propto |g|^{-\gamma}$ defines the exponent γ. On the other hand the static correlation function is given by:

$$S(q) = \int d\omega S(q,\omega) \tag{1.14}$$

and $S(q = 0) \propto |g|^{-\gamma_s}$ which defines γ_S.

For *finite temperatures* T we have $\chi.T = S(q = 0)$ and $\gamma = \gamma_s$. This is not the case however for $T = 0$. Let us consider the scaling ansatz for $S(q,\omega)$ which defines the exponent η:

$$S(q,\omega) = \xi^{2-\eta} D(q\xi, \omega\xi^{z}) \tag{1.15}$$

taking this scaling expression in Eqs. (1.13) and (1.14) we get[17] the relations

$$\gamma_s = (2 - z - \eta)\nu = \gamma - \nu z \tag{1.16}$$

and $2\beta = \nu(d+z-2+\eta)$ at zero temperature while for $T \neq 0$ we have $\gamma_s = \gamma = (2-\eta)\nu$. Note that Eq. (1.15) implies $G(r) = \frac{1}{r^{d+z-2+\eta}} g(r/\xi)$ since $S(q=0) = \int d\vec{r} G(r)$. Comparing with the finite temperature case we see again that d + z plays the role of an effective dimension in the quantum transitions.

1.5 EXTENSION TO FINITE TEMPERATURES

We would like to extend this approach for small but finite temperatures. Since temperature is a parameter it is renormalized by the characteristic energy or coupling constant at the zero temperature fixed point. We then have

$$\left(\frac{T}{J}\right)' = b^{y}\left(\frac{T}{J}\right) \tag{1.17}$$

Formally we may think of the renormalized temperature as a "field" which renormalizes under a scale transformation according to Eq. (1.7). It is interesting here to consider three possibilities[4] depending on the value assumed by the exponent y.

i) $y > 0$

In this case the flow of the renormalization group equations is away from the zero temperature fixed point. In the renormalization group language we say temperature is a "relevant field". This is the case for the Ising model in a transverse field ($y = z = 1$) and also for the many-body problems we are concerned here.

ii) $y = 0$

This is the marginal case generally associated with the collapse of a finite temperature fixed point and the one at zero temperature. Such a situation occurs for example in the anisotropic Heisenberg ferromagnet in two dimensions at the isotropic fixed point[18].

iii) $y < 0$

This is a peculiar situation which may occur in random magnetic systems. It leads to dimensional reduction ($d_{eff} < d$) and gives rise to anomalous critical slowing down for the finite temperature transition controlled by the $T = 0$ fixed point. This is the case of the Ising ferromagnet in a random field[19,20] for $d > 2$.

Let us now obtain how temperature will appear in the scaling functions. Returning to Eq. (1.2) and with the additional renormalization equation for temperature Eq. (1.17) we get for example for the temperature dependent correlation length:

$$\xi = |g|^{-\nu} f\left(\frac{H/J}{|g|^{\Delta}} , \frac{T/J}{|g|^{\nu y}}\right)$$

where $f(x,y)$ is a scaling function.

For the free energy density we find:

$$f = |g|^{2-\alpha} f_E \left[\frac{H/J}{|g|^{\Delta}} , \frac{T/J}{|g|^{\nu y}} \right] \tag{1.18}$$

where $\Delta = \beta + \gamma$. We will find out convenient to define a crossover exponent $\phi = \nu y = \nu z$.

1.6 THE CROSSOVER LINE

Let us consider the one dimensional Ising model in a transverse field and make an expansion of the RG equations close to the zero temperature unstable fixed point. We get

$$K_{n+1} = b^{a}(K_n - K_c) + K_c$$

where $K_0 = (h/J)$ and $K_c = (h/J)_c$ is the fixed point. For finite temperatures this equation can be generalized to lowest order in (T/J) and is given by:

$$\begin{cases} K_{n+1} = K_c + b^{a}(K_n - K_c) - T_n \\ T_{n+1} = b^{y} T_n \end{cases} \tag{1.19}$$

where $T = (T/J)$. These equations can be iterated in the following way[21]

$$\begin{cases} K_1 = K_c + b^a(K_0 - K_c) - T_0 \\ T_1 = b^y T_0 \end{cases} \tag{1.20}$$

and

$$\begin{cases} K_2 = K_c + b^a(K_1 - K_c) - T_1 \\ T_2 = b^y T_1 \end{cases} \tag{1.21}$$

Using Eq. (1.20) the equation for K_2 can be rewritten as:

$$K_2 = K_c + b^a\left[K_c + b^a(K_0 - K_c) - T_0 - K_c\right] - b^y T_0$$

$$K_2 = K_c + b^{2a}(K_0 - K_c) - b^a T_0 - b^y T_0$$

$$K_2 = K_c + b^{2a}(K_0 - K_c) - \frac{(b^a+b^y)(b^a-b^y)}{(b^a-b^y)} T_0$$

$$K_2 = K_c + b^{2a}\left(K_0 - K_c - \frac{1}{b^a-b^y} T_0\right) + \frac{1}{b^a-b^y} b^{2y} T_0$$

repeating the iteration n times we get:

$$K_n = K_c + b^{na}(K_0 - K_c - a_0 T_0) + a_0 b^{ny} T_0$$

where $a_0 = 1/(b^a - b^y)$, $K_0 = h/J$ and $T_0 = T/J$. Taking $\ell = b^n$ we finally obtain:

$$K_\ell = K_c + \ell^a(K_0 - K_c - a_0 T_0) + a_0 \ell^y T_0 \tag{1.22}$$

Since ℓ is arbitrary we repeat the scaling procedure until

$$\ell^a(K_0 - K_c - a_0 T_0) = 1$$

and this length scale defines a correlation length

$$\ell = \xi = \frac{1}{(K_0 - K_c - a_0 T_0)^{1/a}} \propto \frac{1}{[\,h - h_c(T)\,]^\nu}$$

with the correlation length exponent $\nu = 1/a$ and $h_c(T) = h_c + a_0 T$.

Substituting the expression for ξ into Eq. (1.22) we find that at length scale ξ,

$$K_\xi = K_c + 1 + a_0 \frac{T}{[h - h_c(T)]^{\nu y}}$$

Notice that if $\dfrac{T}{[h-h_c]^{\nu y}} \ll 1$, K_ξ is essentially a constant independent of temperature. On the other for $\dfrac{T}{[h-h_c]^{\nu y}} \gg 1$, K_ξ acquires a temperature dependence. The same holds for a scaling function $f(K_\xi)$ whenever K_ξ appears as an argument.

Consequently the line

$$T_c = (h - h_c)^{\phi} \quad \text{with} \quad \phi = \nu y = \nu z \tag{1.23}$$

represents a crossover line which separates two different regimes for the behavior of K_ξ or of a scaling function $f(K_\xi)$, in the non-critical part of the phase diagram, i.e for $(h/J) > (h/J)_c$. Furthermore if there is a critical line in the h X T phase diagram, as in the two-dimensional Ising model in a transverse field for $(h/J) < (h/J)_c$, *this critical line is governed by the same exponent ϕ of the crossover line*. This assertion is the content of the so called **generalized scaling hypothesis**, which plays a fundamental role in the analysis of the problems discussed in this paper[22]. This hypothesis is well stablished in experiments and holds also for disordered systems[23].

The existence of a crossover line, associated with a zero temperature fixed point, in the non-critical region of the phase diagram, is a general feature in quantum systems. What is the physical significance of this crossover line, since there is no phase transition occurring along it? Consider a physical quantity X written in the scale invariant form as $X = |g|^{-x} f(T/T_c)$ where $T_c = |g|^{\nu y}$ and $g = 0$ defines the critical point. Deriving with respect to temperature yields $\partial X/\partial T = |g|^{-x-\nu y} f'(t)$ where $f'(t)$ is the derivative of the scaling function $f(t)$ with respect to $t = T/T_c$.

If we equate $\partial X/\partial T$ to zero to find the extrema of the quantity X, we get $f'(t) = 0$, beside trivial roots. This equation will have a solution let's say for $t_M = w$, where w is a constant. This yields $T_M = w\ T_c = w|g|^{\nu y}$ implying that the extrema of the quantity X occur along the crossover line. The same holds for inflexion points. Consequently any anomaly on physical quantities, like maxima for example, in the non-critical region of the phase diagram will occur along the crossover line making this line accessible experimentally. Although the constant w may depend on the particular physical quantity, as expected for a crossover effect, the relevant, universal information is contained in the crossover exponent $\phi = \nu y$ which is determined by the universality class of the transition.

An interesting feature of the scaling approach is that it allows to determine the singular behavior of the physical quantities of interest, as a function of temperature for example, just at criticality. Let us consider the uniform susceptibility of the Ising model in a transverse field as it approaches the critical point at $(h/J)_c$ or $|g| = 0$, $T = 0$, from finite temperatures. The general scaling form for the susceptibility is $\chi = |g|^{-\gamma} f(T/T_c)$ with $T_c \propto |g|^{\nu z}$. In order to have a non trivial result for χ at the critical point, we require that the dependence on $|g|$ cancels out. For this purpose the scaling function $f(T/T_c)$ is expanded as $f(T/T_c) \approx (T/T_c)^x$ such that $\chi = |g|^{-\gamma}[T/T_c]^x = |g|^{-\gamma}[T/|g|^{\nu z}]^x$ or $\chi = |g|^{-(\gamma+\nu z x)}T^x$. Finally from the condition that the dependence on $|g|$ cancels we determine the exponent x. This condition is, $\gamma + \nu z x = 0$, which yields $x = -\gamma/\nu z$, implying that the susceptibility diverges as $\chi = T^{-\gamma/\nu z}$, at the critical point

$|g| = 0$, with decreasing temperature.

In the next sections we use the ideas introduced above to study strongly correlated electronic materials. In spite of the strong interactions it turns out, as evidenced by experiments, that at sufficiently low temperatures and above some lower critical dimension these systems behave as Fermi liquids with renormalized parameters. More specifically this Fermi liquid regime is characterized by enhanced effective mass and susceptibility due to the proximity of a zero temperature phase transition. We use scaling arguments to obtain how the exponents, which govern the enhancement of these quantities, are related to the usual exponents associated with a zero temperature fixed point and which were introduced above. As expected the dynamic exponent z also plays a special role in these many-body systems and shows up explicitly in some important thermodynamic quantities. We study in section 2 the Kondo lattice problem which is a useful model to describe heavy fermion materials. We illustrate the usefulness of the scaling theory by applying it to the well studied heavy fermion material $CeRu_2Si_2$. Next, in section 3, we start discussing a scaling theory for the metal-insulator transition due to correlations (Mott transition) giving special attention to charge fluctuation effects. Finally we briefly discuss how the presence of disorder will modify the results we have obtained.

2.1 HEAVY FERMIONS

Heavy fermions are metallic systems containing elements with unstable f-shells like Ce, Yb and U. They are characterized by a huge linear temperature dependent term in the specific heat which is attributed to quasiparticles with large effective masses[24]. Heavy fermions can attain different ground states: magnetic, quite generally with long range antiferromagnetic order, superconductor or a Fermi liquid with renormalized parameters[25]. Recent experiments suggest still another possibility where the system reaches a non-magnetic, insulating ground state[26]. From the magnetic point of view heavy fermions present at sufficiently high temperatures ($T >> T_c$) a Curie-Weiss susceptibility indicating the existence of interacting local moments on the f-shells. For $T << T_c$ the susceptibility becomes Pauli-like, i.e., temperature independent but with an enhanced value comparable to that of the thermal mass[25]. The characteristic temperature T_c is much lower than the Fermi temperature expected from band-structure calculations.

A naive interpretation of the behavior of heavy-fermions in terms of a large density of states, associated with the f-electrons at the Fermi surface, is insufficient to explain the properties of these systems. This is dramatically illustrated by relaxation experiments which, within this simple model, yield values for the density states which are inconsistent with those obtained by thermodynamic measurements[27]. In fact the physics of the heavy fermions is dominated by many-body effects as evidenced for example by the proximity of these materials to a magnetic instability[5].

The characteristic temperature T_c associated, for example, with the crossover in the magnetic susceptibility provides a clear evidence for the importance of many-body effects as will become clear below. This crossover from a high temperature local moment behavior, as indicated by the Curie-Weiss temperature dependence of the susceptibility, to a Fermi-liquid, i.e. temperature independent behavior at low temperatures, is reminiscent of the Kondo effect in dilute alloys. This has led to some confusion on the nature of the energy scale which is contained in T_c resulting in a wrong identification of this characteristic temperature with the single-ion Kondo temperature T_K (see Fig.1). If one looks for other physical properties of heavy fermions, like the resistivity as a function temperature, it becomes clear that a new low energy scale, related to the interaction between the f-electrons, is coming into action and is intrinsically distinct from that of the single ion Kondo effect[5,25]. This is illustrated in the resistivity versus temperature curve of a non-magnetic heavy fermion, like $CeCu_6$, shown in Fig.2. At high temperatures the resistivity is metallic increasing with temperature. As the temperature is lowered the resistivity rises with a logarithmic behavior which can be described as a Kondo effect due to the f-moments scattering the conduction electrons incoherently. A further decrease in temperature does not lead to a saturation of the resistivity as in the dilute alloy Kondo effect. On the contrary the resistivity drops due to the fact that the interacting f moments are displayed on a translation invariant lattice resulting in a coherent scattering of the conduction electrons. The resistivity goes to zero with a T^2 behavior characteristic of an interacting Fermi-liquid. The drop in the resistivity at low temperatures is inextricably linked with the concept of *coherence*. It marks the onset of the interactions between the f-electrons, periodically displayed on a lattice, giving rise to a physical behavior which is completely distinct from that which occurs in the single-impurity Kondo case where translation invariance is lost. Concomitantly to the decrease of the resistivity and the appearance of a T^2 term, one observes the crossover in the susceptibility which we described above. From the magnetic point of view coherence manifests as a passage from local moment to Pauli-like behavior as the system enters a renormalized Fermi-liquid regime. The characteristic or *coherence temperature* T_c provides the energy scale in which this crossover occurs and is a truly many-body effect[5]. If we neglect the occurrence of superconductivity the physics of heavy fermions, contained in the energy scale given by T_c , may be understood as a direct consequence of the competition between the Kondo effect, which acts to reinforce a non-magnetic ground state and the indirect interaction between the f-moments, mediated by the conduction electrons, which aims to establish long range magnetic order[28].

The essence of this competition is contained in the so-called Kondo lattice hamiltonian which provides a useful model to describe *the magnetic degrees of freedom* of the heavy fermion problem[28]. This hamiltonian can be written as:

$$\mathcal{H} = \sum_{k,\sigma} E_k c^{+}_{k\sigma} c_{k\sigma} + J \sum_{i} \vec{S}_i . \vec{\sigma}_i \tag{2.1}$$

where the first term describes a band of conduction electrons of width W and the second the coupling between the conduction electrons of spin $\vec{\sigma}$ and the local moments $\vec{S}_i$ associated with the f-electrons which are displayed on the sites i of a lattice.

This hamiltonian has been studied by different methods[28,29,10]. The general picture which emerges is that at zero temperature, depending on the ratio (J/W), we may have a non-magnetic ground state ($(J/W) > (J/W)_c$) or a magnetic ground state ($(J/W < (J/W)_c$). Here $(J/W)_c$ is the critical value of the ratio between the coupling J and the bandwidth W at which the zero temperature transition occurs[28]. Within the renormalization group approach[9,10] this T=0 transition is associated with an unstable fixed point occurring at $(J/W)_c$ (see Fig.1).

We can use the scaling concepts which were developed in the last section to obtain the scaling expressions of the relevant physical quantities close to the zero temperature unstable fixed point at $(J/W)_c$. We get[5]

$$f \propto |j|^{2-\alpha} f_F[T/T_c, H/H_c]$$

$$\chi_S \propto |j|^{-\gamma} f_s[T/T_c, H/H_c]$$

$$m_T \propto \gamma \propto C/T \propto |j|^{2-\alpha-2\nu z} f_c[T/T_c, H/H_c] \qquad (2.2)$$

$$\tau \propto |j|^{-\nu z} f_t[T/T_c, H/H_c]$$

$$\xi \propto |j|^{-\nu} f_L[T/T_c, H/H_c]$$

where f, χ_S, m_T stand for the singular part of the free energy density, the order parameter susceptibility and the thermal mass obtained from the linear term of the specific heat respectively. τ is the characteristic relaxation time which governs the critical slowing down and ξ the correlation length measured by neutron scattering. Also $T_c = |j|^{\nu z}$, $H_c=|j|^{\Delta}$ and $j = (J/W)-(J/W)_c$ measures the distance to the critical point in parameter space. Since heavy fermions are generally close to an antiferromagnetic instability[25], the order parameter is the staggered magnetization m_s which at zero temperature scales as $m_s \propto |j|^{\beta}$. Consequently in the equations above, H is a staggered field conjugated to the order parameter and χ_s a staggered susceptibility. The exponents α, $\Delta = \beta + \gamma$, ν and z are associated with the zero temperature fixed point and obey standard scaling relations like $\alpha + 2\beta + \gamma = 2$. However, as shown in the previous section, because we are dealing with a zero temperature instability the hyperscaling relation is modified and is given by $2-\alpha = \nu(d+z)$.

In the argument of the scaling functions temperature appears scaled by the characteristic temperature $T_c = |j|^{\nu z}$. Since the scaling functions have in general different asymptotic behavior for $(T/T_c) \gg 1$ and $(T/T_c) \ll 1$, the line $T_c = |j|^{\nu z}$, in the non-critical part of the (T/W) versus (J/W) phase diagram, represents a crossover line between different regimes. The arguments summarized in the beginning of this section concerning the physics of heavy fermions and in particular the behavior of the susceptibility and the resistivity have led us to identify this crossover temperature or line with the coherence temperature associated with the onset with decreasing temperature of the Fermi liquid or dense Kondo regime in these materials. This crossover temperature clearly represents a new energy scale much lower than the single-ion Kondo temperature which is also shown in the phase diagram of Fig.1. Besides the collective nature of this crossover or coherence line is clearly indicated by the fact that, according to the generalized scaling hypothesis[22], the critical Neel line in the critical region of the phase diagram ($J < J_c$) is governed by the same exponent of the crossover line , i.e. $T_N \propto |j|^{\nu z}$ (see Fig.1). The identification of the crossover line with the coherence transition showing unambiguously the existence of a new low energy scale T_c , different from that of the single impurity problem T_K , represents an achievement of the scaling approach. The critical Neel line in Fig.1 gives the Neel temperature T_N for a given ratio (J/W) smaller than the critical one. Since temperature is a relevant field the exponents determining the singularities at the finite temperature antiferromagnetic instability along the Neel line are different from those associated with the zero temperature fixed point at $(J/W)_c$

The discussions above imply that in establishing the Fermi-liquid regime below T_c interactions between the moments play an important role. Consequently the nature of the screening, which leads to the observed Fermi liquid behavior for $T \ll T_c$, is in the lattice problem quite distinct from that which occurs in the single impurity Kondo effect[30].

Since for $T << T_c$, i.e., below the coherence line, the system attains a Fermi liquid regime as evidenced by experiments, the scaling functions $f(T/T_c)$ in Eqs. (2.2) have Sommerfeld-like expansions in this region of the phase diagram, i.e. $f(T/T_c << 1) \cong 1 + a(T/T_c)^2 + b(T/T_c)^4 + \ldots$. Using such an expansion for the free energy we have obtained the scaling expression of the thermal mass m_T , defined as the coefficient of the linear term of the specific heat, which is given in Eq. (2a). This thermal mass is meaningful only for $T << T_c$ when the system has reached a truly Fermi-liquid regime. Notice that if $2-\alpha-2\nu z < 0$, the thermal mass will be enhanced as a consequence of the proximity to the magnetic instability. A similar enhancement occurs for the limiting Pauli-like uniform susceptibility as we discuss below.

Further insight into the significance of the crossover line and the asymptotic limits of the scaling functions can be obtained examining the uniform magnetic susceptibility. For this purpose we have to introduce a new exponent σ which controls the renormalization of the uniform magnetic field h close to the zero temperature fixed point. We have

$$(h/J)' = b^{\sigma+z}(h/J) \tag{2.3}$$

and the scaling form of the magnetic field dependent free energy density is given by:

$$f \propto |j|^{2-\alpha} f_h[\ (h/J)/|j|^{\phi_h}\ ] \tag{2.4}$$

where $\phi_h = \nu(\sigma+z)$. The uniform susceptibility χ_h is given by $\chi_h = -\partial^2 f/\partial h^2 \propto |j|^{2-\alpha-2\phi_h} f_c[T/T_c, h/h_c]$ with $h_c \propto |j|^{\phi_h}$. The temperature independent or Pauli-like behavior of the uniform low field susceptibility χ_0 for $T \ll T_c$ implies that in the equation above $f_c(\ T/T_c \ll 1\ ) = \text{constant}$ such that $\chi_0 \propto |j|^{2-\alpha-2\phi_h}$. For $2-\alpha-2\phi_h < 0$ this uniform susceptibility is enhanced as indeed is found experimentally[25]. On the other hand local moment or Curie-Weiss behavior for $T \gg T_c$ implies the following asymptotic behavior for the susceptibility scaling function: $f_c(\ T/T_c \gg 1\ ) \sim (T/T_c)^{-1}$ such that in this limit $\chi_0 \propto \frac{|J|^{2-\alpha-2\phi_h+\nu z}}{T}$. Since the uniform susceptibility in this high temperature regime ($T \gg T_c$) is related to the magnitude of the local moments μ through $\chi_0 \propto \mu^2/T$ we conclude, by comparing with the previous expression for χ_0 , that even in this regime the moments are renormalized by the interactions. A useful expression for the uniform susceptibility, which interpolates between the two regimes, is given by $\chi_0 \propto \frac{\mu^2}{T+T_c}$ with $\mu^2 \propto |j|^{2-\alpha-2\phi_h+\nu z}$ and $T_c = |j|^{\nu z}$.

2.2 THE KONDO LATTICE PHASE DIAGRAM AND THE INFLUENCE OF PRESSURE

How can we explore the phase diagram of the Kondo lattice? Can we verify the scaling predictions and obtain the critical exponents associated with the zero temperature unstable fixed point at $(J/W)_c$? It is clear that a given physical system corresponds to a fixed value of the ratio (J/W). Fortunately this ratio depends also on the volume of the system and can be varied by applying external pressure or chemical pressure what is achieved by doping conveniently. This allows to move along the phase diagram of the Kondo lattice and provides a possibility of extracting the critical exponents from experimental data. The volume dependence of the ratio (J/W) is in general modeled by an exponential volume dependence $(J/W) = (J/W)_0 \exp[-q(V-V_0)/V_0]$ where V_0 and $(J/W)_0$ are the equilibrium values of the volume and of the ratio (J/W) respectively and q a parameter[31].

Let us consider a physical quantity X whose critical behavior is characterized by the exponent x, i.e. $X \propto |j|^{-x}$. We can define a Gruneisen parameter associated with the physical quantity X through[5]

$$\Gamma_x = \frac{dLnX}{dLnV} = \frac{-qx}{1-(J/W)_c/(J/W)_o} \qquad (2.5)$$

This equation shows that if the system is close to the magnetic instability such that $(J/W)_o \simeq (J/W)_c$ the Gruneisen parameter will become very large. This is indeed the case in heavy fermions where the large Gruneisen parameters lead to an extreme sensitivity of these systems to volume changes or external applied pressures[32]. Introducing the compressibility $\kappa = (-1/V)\partial V/\partial P$ we can express the Gruneisen parameters explicitly in terms of variation with respect to external pressure P. We get

$$\Gamma_x = (-1/\kappa)\,\frac{\partial LnX}{\partial P}$$

It turns out to be convenient to consider the expansion of Ln X(P) for small external pressures:

$$LnX(P) = LnX(0) + \left.\frac{\partial LnX}{\partial P}\right|_{V=V_o} .P + O(P^2) \qquad (2.6)$$

from which we find

$$Ln[X(P)/X(0)] \simeq -x\,(\kappa_o\Gamma)\,P \qquad (2.7)$$

where κ_o is the equilibrium compressibility and Γ a property independent Gruneisen parameter defined by $\Gamma = |\partial Lnj/\partial LnV|$ such that $\Gamma_x = -x\,\Gamma$ ($x > 0$).

The analysis above shows that by comparing the Gruneisen parameters of different physical quantities we can obtain relations between the exponents governing their critical behavior. Furthermore for small applied pressures the logarithm of a given quantity, normalized by it's equilibrium (zero pressure) value varies linearly with pressure, with a coefficient that is directly related to the exponent characterizing it's critical behavior[5] (Eq. 2.7).

2.3 QUANTUM SCALING IN $CeRu_2Si_2$

The system $CeRu_2Si_2$ is a moderate heavy fermion[33] with a γ value given by $\gamma \cong 360mJ/mole\ K^2$. This material has been extensively studied by neutron scattering[34]. Magnetic[33] and resistivity[33,35] measurements were done for different pressures and external magnetic fields. The neutron scattering measurements[34] indicate that in $CeRu_2Si_2$ the magnetic correlation lengths (for the ab and c directions) increase with decreasing temperature before saturating at a constant value for temperatures smaller than a characteristic temperature. These results show that this system does not order magnetically down to the lowest temperatures. The susceptibility at high temperatures has a temperature dependence characteristic of interacting local moments i.e. a Curie-Weiss behavior. With decreasing temperature it presents a maximum at a temperature T_M before reaching a temperature independent Pauli-like behavior at low temperatures[33] (Fig.3).

Doping $CeRu_2Si_2$ with Lanthanum (La) is equivalent to applying negative pressure in this system[36]. In fact $Ce_{1-x}La_xRu_2Si_2$ at a critical concentration x_c= 7% becomes antiferromagnetic[36]. This is a nice illustration of the Kondo lattice mechanism in operation since magnetic ions (Ce) are being substituted by non-magnetic ions (La) and magnetism is being induced in the system.

In Figure 4 we show the logarithm of several normalized physical quantities measured in the system $CeRu_2Si_2$ as a function of pressure for small pressures[5] ($P \le 8$ kbars). The expected general linear behavior given by Eq.2.7 is observed in every case. Furthermore all the straight lines have the same inclination implying that the physical quantities shown there are governed by exponents which assume the same numerical values within experimental accuracy. The quantities whose pressure variations are shown in this figure are:

i) $Ln[T_c(P)/T_c(0)]$ where T_c is the coherence temperature defined by the maximum of the low field uniform susceptibility[33] (see Fig.3). In fact using the scaling expression for χ_0 it is easy to show (see section 1) that maxima in this quantity will occur along the coherence line $T_c = |j|^{\nu z}$.

ii) $Ln\{[A(P)/A(0)]^{1/2}\}$ where A is the coefficient of the T^2 term of the low temperature resistivity ($T << T_c$) and defined by $\rho = \rho_0 + AT^2$ where ρ_0 is the residual resistivity[33,35]. In the Fermi liquid regime at $T << T_c$, i.e. below the coherence line, we expect $A \propto T_c^{-2}$.

iii) $Ln[h_c(P)/h_c(0)]$ where h_c is the characteristic uniform magnetic field at which the uniform differential susceptibility $\chi_h = -\partial^2 f/\partial h^2 = dm/dh$ has a maximum for a fixed temperature $T << T_c$. This maximum at h_c defines a metamagnetic-like transition[32] which shows the existence of strong antiferromagnet correlations in the Fermi liquid.

iv) $Ln[\chi_h(h=h_c, T\cong 0, P=0)/\chi_h(h=h_c, T\cong 0, P)]$ where $\chi_h(h=h_c, T\cong 0)$ is the

value of the uniform differential susceptibility[32] $\chi_h = dm/dh$, at the characteristic field h_c for $T << T_c$.
v) $\mathrm{Ln}[\chi_0(h\cong 0,\ T=T_c,\ P=0)/\chi_0(h\cong 0,\ T=T_c,\ P)]$ where $\chi_0(h\cong 0,\ T=T_c)$ is the value of the uniform low field susceptibility at the coherence temperature[33].

Let us see which information on the exponents we can obtain from these experimental results. The fact that the data of i) and ii) fall on the same line just confirms the Fermi liquid nature of the state attained below the coherence line since in this case $A \propto T_c^{-2}$ as expected from the scaling form of the resistivity $\rho = \rho(T/T_c)$. The data of i) and iii) falling on the same line imply the important exponent equality $\phi_h = \nu z$ since the characteristic field h_c and the coherence temperature T_c shift with pressure at the same rate.

On the other hand the scaling form of the uniform susceptibility $\chi_h \propto |J|^{2-\alpha-2\phi_h} f(T/T_c, h/h_c)$, the results of iv), v) together with the data of iii) and i) falling on a unique line, imply the exponents relations $2 - \alpha = \nu z$ with $\nu z = \phi_h$.

The equality $2 - \alpha = \nu z$ shows that the exponents associated with the zero temperature unstable fixed point of the Kondo lattice violate the modified hyperscaling relation $2-\alpha = \nu(d+z)$. This in turn suggests that the magnetic phase transition associated with this fixed point occurs above the upper critical dimension d_c for this transition.

We can extract some consequences of the empirical relations $2 - \alpha = \nu z = \phi_h$, implied by the results contained in Fig.4, which are independent of the particular value of these exponents. These are:

i) First the Wilson ratio $\chi_0(T << T_c, h \cong 0)/m_T$ turns out to be a constant independent of pressure (this is due to the equality $\phi_h = \nu z$). Unfortunately the specific heat of $CeRu_2Si_2$ has not been measured yet as a function of pressure to confirm this prediction.

ii) Second the empirical Kadowaki-Woods type of relation[37] $A \alpha \gamma^2$ or A/γ^2= constant (see Fig.5), where A is the coefficient of the T^2 term of the resistivity and $\gamma \propto m_T$ the coefficient of the linear term of the specific heat, is directly obtained from the Fermi liquid relation $A \alpha T_c^{-2}$ and the equality $2 - \alpha = \nu z$.

iii) Thirdly consider the scaling expression for the uniform magnetization $m \alpha \partial f/\partial h \alpha |j|^{2-\alpha-\phi_h} f_m(T/T_c,h/h_c)$. The equality $2 - \alpha = \phi_h$ implies that the metamagnetic-like transition at $h = h_c$ and $T << T_c$ occurs always at the same fixed value of the magnetization independent of pressure i.e. $m = f_m(h/h_C)$ = constant for $h = h_C(P)$. This has indeed been verified experimentally[33].

iv) The amplitude of the magnetic moment obtained from the Curie-Weiss form of the high temperature susceptibility ($T >> T_c$) scales as $\mu^2 = |j|^{2-\alpha-2\phi_h+\nu z}$ as we have shown before. Consequently the relation $2 - \alpha = \nu z = \phi_h$ implies that μ does not renormalize as $|j|$ varies. In the table below we show the limiting Pauli susceptibility and the value of the effective moment extracted from the high temperature susceptibility for two non-magnetic heavy fermions. Notice the small relative renormalization of the moments μ compared to that of the susceptibilities χ_0.

System	$\mu(\mu_B)$	$\chi_0(10^{-3}$ emu mol$^{-1})$
$CeCu_6$	2.69	34
$CeRu_2Si_2$	2.44	14

Table 1. The effective moment obtained from the high temperature susceptibility ($\chi = \mu^2/3k_B(T-\theta)$) and the limiting Pauli susceptibility $\chi_0(T\rightarrow 0)$, for different heavy fermions. The effective moment for a free trivalent Ce ion is 2.54 μ_B (from A.Lacerda, PhD Thesis, Universite J.Fourier, Grenoble, 1990).

The experiments mentioned above either confirm the exponents relations we have found analyzing the data shown on Fig.4 or remain to be checked. In any case the actual values of these critical exponents α, ν, z etc., cannot be unambiguously determined from the experimental data given in this figure. However considering the Ising nature of the antiferromagnetic state of doped $Ce_{1-x}La_xRu_2Si_2$ and the fact that the uniform magnetic field acts as an additional relevant field besides the staggered field, as evidenced by the behavior of the uniform susceptibility, has led us to suggest that the exponents associated with the zero temperature Kondo lattice unstable fixed point can be identified with those of a classical tricritical point[5]. In fact for a tricritical point the upper critical dimension d_c = 3. Since $d_{eff} = d + z$, we get $d_{eff} \geq d_c$ even for the marginal case z = 0 in three dimensions, supporting the idea that we are dealing here with a classical tricritical transition with mean field exponents. Also the tricritical mean field value for the crossover exponent, $\phi_t = \phi_h/\nu z = 1$, is in agreement with the experimental results. The classical tricritical exponents[38] $\alpha = 1/2$, $\nu = 1/2$ together with the empirical relation $2 - \alpha = \nu z$ imply that the dynamic exponent z assumes the value $z = 3$. This value for the dynamic critical exponent is generally associated with ferromagnetic spin fluctuations. These fluctuations are plausible to occur at a metamagnetic transition which for $(J/W) = (J/W)_c$ occurs at arbitrarily low magnetic fields. Another possibility to have $z = 3$ is to take into account the long range nature of the RKKY interaction between the local moments which show a $1/r^3$ dependence, z being related to the exponent of this power law.

An unambiguous confirmation, or refutation, of these values for the Kondo lattice critical exponents could be obtained from measurements of the correlation length at very low temperatures ($T \ll T_c$) for different applied pressures. Also NMR or EPR on dissolved magnetic impurities under applied pressure could provide information on these exponents confirming or not the tricritical values we have suggested. A similar analysis carried on for $CeAl_3$ for pressures above 3 kbars and which include EPR data at different pressures[39] confirm the classical tricritical nature of the zero temperature Kondo lattice exponents and show that this system is in the same universality class of $CeRu_2Si_2$.

As we have mentioned before $Ce_{1-x}La_xRu_2Si_2$ above a critical concentration $x_c = 7\%$ becomes antiferromagnetic[36]. The system at the critical concentration x_c i.e with $(J/W)=(J/W)_c$ is particularly interesting. This system does not cross the coherence line and consequently does not enter the Fermi liquid regime. Besides it has a phase transition just at zero temperature. Consequently we have here an interesting possibility of having a metallic system with weak disorder which does not behave as a Fermi-liquid. The thermodynamic properties of this system at the critical concentration can be easily obtained from the scaling theory. Requiring that the dependence on $|j|$ cancels out in the thermodynamic quantities, we get for the specific heat at criticality ($|j| = 0$): $C/T = m_T = T^x$ where $x = (2 - \alpha - 2\nu z)/\nu z$ or using the modified hyperscaling relation, $x = (d - z)/z$. Similarly we obtain for the uniform susceptibility at criticality: $\chi_0 = T^p$ where $p = (2 - \alpha - 2\phi_h)/\nu z$. For small uniform magnetic fields χ_h diverges at small fields like $\chi_h = h^q$ with $q = (2-\alpha-2\phi_h)/\phi_h$. So depending on the value of the critical exponents we may find that the system just at criticality does not behave as a Fermi-liquid as characterized by the behavior of the specific heat and susceptibility.

The empirical relations between the critical exponents $2-\alpha = \nu z = \phi_h$ that we have found analyzing the data of $CeRu_2Si_2$ implies for the system at x_c, i.e. ($|j| = 0$), the following behavior: C = constant, $\chi_0 = T^{-1}$ and $\chi_h = h^{-1}$. The temperature independent anomalous behavior of the specific heat indicates that we have in fact to look for the expression of the free energy density at criticality. We then find a contribution linear in temperature at $|j| = 0$ for this quantity which in turn gives rise to a finite residual entropy. Recalling the behavior of the uniform low field susceptibility, $\chi_0 \propto T^{-1}$, we can give a simple interpretation for these results. That is the system with $x = x_c$ behaves as a collection of non-interacting local moments, decoupled from the electron gas, with a Curie type of susceptibility down to the lowest temperatures and a finite residual entropy due to the degeneracy on the orientation of these local magnetic moments. From the point of view of the antiferromagnetic side of the phase diagram ($(J/W) < (J/W)_c$), this is consistent with the fact that the Neel temperature goes to zero at x_c or $|j| = 0$. In the real system however residual interactions may mask this simple behavior but the result looks sufficiently interesting to be worthwhile doing experiments at this particular concentration.

The positive and large exponent of the specific heat, $\alpha = 1/2$, that our analysis of the experiments on $CeRu_2Si_2$ under pressure suggests, implies a great sensitivity of these materials to disorder. This a consequence of the Harris criterion[40] which for $\alpha > 0$ implies that disorder is a relevant "field" in the renormalization group sense. In this case dilution and the disorder inherent to it, may give rise to new physical behavior leading to a different universality class and cannot be viewed simply as a negative pressure effect. In fact we have failed to analyze recent experiments[41] on UYPd with the exponents obtained for $CeRu_2Si_2$. These systems are strongly disordered and non-Fermi liquid behavior in this case can be attributed to disorder and not necessarily to the effect discussed above namely the system being on the edge of an instability. The best system to study scaling properties would be an antiferromagnetic system with a small Neel temperature. Pressure in this case would lead to destruction of the long range order and eventually would drive the system along the coherence line allowing to check the scaling predictions and in particular the generalized scaling hypothesis.

3.1 SCALING APPROACH TO THE MOTT TRANSITION

3.1.1 THE HUBBARD MODEL

The Hubbard Hamiltonian was originally proposed to describe the magnetism of transition metals where the electrons involved have a large degree of itinerancy[42]. Hubbard made some approximations concerning the Coulomb interaction arguing that it was sufficient to consider correlations between electrons on the same site. Even within this drastic approximation the resulting hamiltonian still represents a very difficult many-body problem for which only in one-dimension an exact solution can be found[43]. The Hubbard hamiltonian is given by:

$$\mathcal{H} = \sum_{i,j,\sigma} t_{ij} c^{+}_{i\sigma} c_{j\sigma} + U \sum_{i} n_{i\uparrow} n_{i\downarrow} \tag{3.1}$$

where the first term allows the electrons to hop from site to site and gives rise to a narrow band of width W. The second term represents the Coulomb interaction between two electrons on the same site. Many techniques have been devised to deal with this hamiltonian. A variety of solutions have been found besides that describing itinerant ferromagnetism for which it was originally conceived[44].

For n = 1, where n is the number of electrons per atom, it is generally accepted[45] that this model yields at zero temperature, above a lower critical dimension, a metal-insulator transition occurring as a function of the ratio of the parameters U/W. For $n \neq 1$ one finds magnetic transitions occurring in the metallic state, as the ferromagnetic instability which has been used to model the ferromagnetism of the transition metals and still more complicated types of long range magnetic order[44,46].

The Hubbard model gives also an adequate description of the excitations and physical properties of nearly ferromagnetic systems, like Palladium[47] which is close to a ferromagnetic instability. On the other hand in the half-filled band case (n = 1), where the metal-insulator transition occurs accompanied by the appearance of antiferromagnetic order (Mott transition), this model provides the basis for understanding the behavior of systems in the metallic side of the localization transition like doped V_2O_3 or V_2O_3 under pressure[48]. Mott in his classical book "Metal-insulator transitions" uses the expression "highly correlated electron gas" to designate these "nearly localized" electronic systems[49]. The recent discovery of superconductivity on doped antiferromagnetic insulating compounds of transition metals has led Anderson[50] to suggest that superconductivity in these materials is just another manifestation of the "Mott phenomenon" which is essentially the physics of highly correlated electronic systems contained in the Hubbard hamiltonian.

3.2 THE HIGHLY CORRELATED ELECTRON GAS AND THE NEARLY FERROMAGNETIC METAL

The metallic phase of the materials which are close to a Mott transition that is the "highly correlated electron gas" resembles that of "nearly ferromagnetic" metals in many aspects. Both kind of systems have an enhanced Pauli susceptibility at low temperatures and a large electronic contribution to the low temperature specific heat. An interesting and instructive illustration of the applicability of these two concepts[46] is provided by the Fermi liquid system ^3He. Initially it was suggested that liquid ^3He is nearly ferromagnetic in order to understand the enhancement of it's thermal mass , obtained from the linear term of the specific heat and it's susceptibility. Also a logarithmic temperature dependence in these quantities provided evidence for the existence of paramagnons which are excitations characteristic of nearly ferromagnetic systems. In the paramagnon approximation for a nearly magnetic metal, as described by the Hubbard model, the mass m is enhanced as[47]:

$$m \propto \mathrm{Ln}\,(u - u_c) \tag{3.2}$$

and the uniform susceptibility as:

$$\chi \propto (u-u_c)^{-1}$$

where $u = (U/W)$ and u_c is the critical value of the ratio between the Coulomb repulsion U and the bandwidth W at which the ferromagnetic instability occurs. Notice that in this case the ratio χ/m will depend on $(u-u_c)$ and consequently on pressure which is the external parameter used to vary the ratio U/W. On the other hand in the Brinkman and Rice[45] approach to the highly correlated electron gas, using Gutzwiller's approximation[51] (BRG), they have obtained:

$$m \;\alpha\; (u-u_c)^{-1}$$

and (3.3)

$$\chi \;\alpha\; (u-u_c)^{-1}$$

for a metallic system close to localization. In this case the ratio χ/m turns out to be independent of pressure. The mass m is generally calculated from the residue of the pole of the one-electron propagator **Z** ($m \propto 1/Z$). In principle then the pressure dependence of the ratio χ/m allows to distinguish a nearly ferromagnetic system from a nearly localized one. The experiments on ^{3}He favor the latter approach since the ratio χ/m experimentally varies little with pressure, although the compressibility decreases, suggesting that ^{3}He is nearly localized it's physics being more close to that of the highly correlated electron gas than to that of a nearly ferromagnetic metal[46]. Notice that localization in this case may be interpreted as a tendency to solidification. In the experiments the mass m is obtained from the linear term of the specific heat and the result above relies in that this mass is the same as that calculated from the quantity Z, an assumption that the scaling approach will lead us to question.

An interesting aspect of the Brinkman-Rice-Gutzwiller (BRG) approach to the highly correlated electron gas is that the divergence of the uniform susceptibility at U_c in this case is not related to the appearance of long range magnetic order but to the formation of independent local moments at zero temperature[45]. In fact the Curie susceptibility of free moments $\chi_0 \alpha \mu^2/T$ is infinite at $T = 0$. The BRG approach describes a pure localization transition missing the long range magnetic order which is expected to occur at the Mott transition[45]. Although magnetic fluctuations are included in this method to a certain extent[46] they are not the dominant excitations which in this case are due to charge fluctuations.

The discussion above shows the importance of determining the exponents which govern the critical behavior of the susceptibility, the compressibility and in particular the enhancement of the thermal mass close to an instability of the interacting electron gas. These exponents determine how these quantities vary with pressure and reflect the nature of the incipient instability.

3.3 THE CONDUCTIVITY AND THE CHARGE STIFFNESS

The most direct way of constructing a scaling theory of the Mott transition is to consider the situation where both the localization transition and the magnetic one occur at the same value of the ratio (U/W). In this case the staggered magnetization associated with the long range antiferromagnetic order accompanying the localization transition can be used as the order parameter[8]. Since however magnetic fluctuations have already been considered in detail in the heavy fermion problem we shall emphasize here the pure localization aspects of the Mott transition. We shall look at this transition from the point of view of the scaling theory of quantum critical phenomena which was developed in the previous sections[12]. Although this localization transition can generally be associated with a zero temperature unstable fixed point at $(U/W)_c$, the scaling approach to this problem differs in some fundamentals aspects to the previous one developed for heavy fermions. The reason, as we mentioned before, is that in this case there is no identifiable order parameter associated with the localization transition. In spite of the fact that the flow of the renormalization group equations to the metallic attractor ($(U/W) = 0$) or the insulating one ($(U/W) = \infty$) allows to distinguish the nature of the phases, it is essential to find the relevant physical quantity which can be unambiguously associated with the localization phenomenon. This quantity turns out to be the conductivity effective mass[52]

m^*, obtained from the frequency dependent conductivity $\sigma(\omega)$ or alternatively the charge stiffness[53] $D_c \propto 1/m^*$. For a perfect conductor, which is the case of the lattice translation invariant systems at T = 0 that we are considering, the frequency dependent conductivity $\sigma^{xx}(\omega)$ has a delta function at zero frequency due to free acceleration of the carriers. The weight of this delta function defines the charge stiffness D_c. We have:

$$\sigma^{xx}(\omega) = (e^2/\hbar)D_c\delta(\hbar\omega) + \sigma_r^{xx}(\omega) \tag{3.4}$$

The free acceleration term in the real part of the frequency dependent conductivity gives rise, through the Kramers-Kronig relation, to an imaginary part which is inversely proportional to frequency[52] i.e.

$$\mathrm{Im}\sigma^{xx}(\omega) = (2e^2/\hbar^2\omega)D_c + \mathrm{Im}\sigma_r^{xx}(\omega) \tag{3.5}$$

These results can be made more concrete by considering the simplest approximation for the frequency dependent conductivity namely the Drude equation[54]:

$$\sigma(\omega) = \frac{ne^2}{m^*}\frac{\tau}{1 - i\omega\tau} \tag{3.6}$$

with a real part

$$\mathrm{Re}\sigma(\omega) = \frac{ne^2}{m^*}\frac{\tau}{1 + \omega^2\tau^2} \tag{3.7}$$

and an imaginary part given by:

$$\mathrm{Im}\sigma(\omega) = \frac{ne^2}{m^*}\frac{\omega\tau^2}{\omega^2\tau^2+1} \tag{3.8}$$

For a perfect conductor, i.e. in the limit that $\tau \to \infty$, we obtain

$$\begin{cases} \mathrm{Re}\sigma(\omega) = \dfrac{ne^2\pi}{m^*}\,\delta(\omega) \\ \mathrm{Im}\sigma(\omega) = \dfrac{ne^2}{m^*\omega} \end{cases} \tag{3.9}$$

the last equation leads to

$$\lim_{\omega\to 0} \omega\mathrm{Im}\sigma(\omega) = \frac{ne^2}{m^*} \tag{3.10}$$

comparing with the previous equation for the charge stiffness, Eqs.3.4 and 3.5, we find the relation between this quantity and the effective mass, $D_c = n\pi h^2/m^*$. Equation 3.10 above was used by Kohn[52] to discuss the metal-insulator transition due to correlations. The basic idea is that for an interacting electronic system the vanishing of D_c , or the divergence of m^* , should signal the metal-insulator transition. There is however a basic difficulty in implementing this idea for Fermi liquids since for an interacting Fermi-liquid the conductivity or optical mass m^* never gets renormalized by the interactions due to Galilean invariance[55]. This is not the case however for *interacting lattice systems* as noticed by Shastry and Sutherland[53]. For these lattice systems a divergence of the optical mass is expected to occur at the localization transition.

The authors[53] above also pointed out an important connection between the charge stiffness and the sensitivity of the electronic system to a change in the boundary conditions. They considered the difference in ground state energy of a d-dimensional interacting fermionic lattice system of finite size L under a twist ϕ in the boundary conditions. The difference in the total ground state energy density of the twisted and untwisted system can be written as[53] (to order L^{-2}):

$$\frac{E(\phi) - E(0)}{L^d} = \frac{\Delta E(\phi)}{L^d} = \frac{D_c}{L^2}\phi^2 \qquad (3.11)$$

where D_c is the charge stiffness appearing in the frequency dependent conductivity. Long ago Byers and Yang[56] have shown that a change in boundary conditions in a finite system of size L is formally equivalent to imagine such system, in the shape of a ring, threaded by a flux Φ in the so-called Bohm-Aharonov conditions where the electrons are not in direct contact with the magnetic field. The flux Φ/Φ_0 where $\Phi_0 = (hc/e)$ is related to the twist ϕ by $\Phi/\Phi_0 = \phi$. Notice that from this point of view an insulator, for which $D_c = 0$ or $m^* = \infty$, has an additional symmetry (see Eq. 3.11), namely gauge invariance, as compared with the perfect conductor for which the energy of the individual states and in principle the total ground state energy density depends on the flux through the ring at least to order L^{-2}.

3.4 PERFECT CONDUCTOR VERSUS SUPERCONDUCTOR

The assertions above must be taken with extreme care and their content constitute the central point of an interesting discussion[57]. In the same paper[56] referred above, Byers and Yang introduced a criterion to distinguish a superconductor from a perfect conductor ring which is essentially based on the sensitivity of the system to a flux threading it. According to them for a ring of superconducting material the total ground state energy depends on the enclosed flux (in fact is a periodic function of Φ with period $\Phi_0/2$). This is not the case however for a perfect conductor where, although the individual energy levels depend on the flux, the total energy does not. This cancellation is a macroscopic effect and is illustrated in Fig.6 for systems of different sizes. For small, mesoscopic systems, flux dependence does occur leading to interesting and important behavior[58]. How can we reconcile the Byers and Yang criterion for superconductivity with the results of Shastry and Sutherland relating the charge stiffness to the flux or boundary condition dependence of the ground state energy of a *perfect conductor* as given by Eq. 3.11 ? The point is that the result given in this equation is valid only for small values of ϕ that is for $\phi < 1/L^{d-1}$. For larger values level crossing occurs giving rise in the limit of large systems to the cancellation effects mentioned above and which form the basis of the Byers and Yang criterion to distinguish perfect conductivity from superconductivity.

3.5 SCALING APPROACH TO THE MOTT TRANSITION

3.5.1 THE CONDUCTIVITY MASS

The scaling properties of the charge stiffness D_c or of the effective mass m^* close to a Mott transition can be obtained[13] either from finite size scaling theory together with Eq. 3.11 or more directly from the scaling form of the frequency dependent conductivity $\sigma(\omega)$. In fact just on dimensional grounds we can write $\sigma(\omega) = (e^2/\hbar)\xi^{2-d}f(\omega\tau_\xi)$ where $\tau_\xi = \xi^z$. In both cases we find $D_c \propto |g|^{2-\alpha-2\nu}$ or alternatively $D_c \propto \xi^{-(d+z-2)}$ where we used the modified hyperscaling relation. The quantity g measures the distance in parameter space to the transition ($g = 0$ defines the critical point) and $\xi = |g|^{-\nu}$ is the correlation length in the metallic phase which can be identified with a characteristic screening length[59]. The characteristic time $\tau_\xi = \xi^z$ sets the timescale for the critical slowing down close to the transition. In the derivation of the critical behavior of the charge stiffness using the scaling form of the conductivity, we make use of the asymptotic properties of the scaling function $f(\omega\tau_\xi)$ in $\sigma(\omega)$ at certain limits. Specifically for a perfect conductor $\mathrm{Im}f(\omega\tau_\xi \to 0) \propto 1/\omega\tau_\xi$ (see Eq.3.9) which together with the expression $\mathrm{Im}\sigma(\omega) = ne^2/m^*\omega$ yields the above result for the scaling behavior of $D_c \propto 1/m^*$ close to the metal-insulator transition. The critical exponents α, ν and the dynamical exponent z are associated with the unstable zero temperature fixed point controlling the transition.

It is interesting to point out out that the charge stiffness scales as the superfluid density close to a $T = 0$ superfluid-insulator transition[3]. Also for one dimensional Lorentz invariant systems such that

$d + z = 1 + 1 = 2$, the charge stiffness reduces to a constant amplitude due to conformal invariance[13].

Our previous study of the heavy fermion problem led us to recognize the existence of still another mass for the highly correlated electron gas namely the thermal mass m_T obtained from the linear term of the specific heat. This mass is meaningful below a crossover line T_c in the non-critical part of the phase diagram of (T/U) versus (U/U_c) (Fig.7) where a Fermi liquid regime is attained. In this case for $T \ll T_c$ the scaling function for the free energy has a Sommerfeld-like expansion and allows to define a thermal mass from the linear term of the specific heat as a function of temperature. As in the previous section we find that the thermodynamic mass m_T scales as $m_T \propto |g|^{2-\alpha-2\nu z}$ close to the transition and consequently with a different exponent of the conductivity mass m^*.

3.5.2 THE HUBBARD MODEL AND THE MOTT TRANSITION

A metal-insulator transition associated with a zero temperature fixed point has been obtained for the Hubbard model with one electron per atom ($n = 1$) using a renormalization group approach[60] for dimensions $d \geq 2$. The unstable fixed point at the critical value of the ratio $u_c = (U/W)_c$ separates two regions in parameter space. For $u > u_c$ the flow of the renormalization group equations is towards the strong repulsion attractor which characterizes an insulating antiferromagnetic phase. For $u < u_c$ the flow is towards $U/W = 0$, i.e. the Fermi liquid fixed point, which is identified as the attractor of the metallic phase. This approach yields values for the zero temperature critical exponents which are not rigorous due to the approximations involved in the renormalization procedure. It considers the renormalization of the relevant parameters of the Hubbard hamiltonian under scale transformations on blocks of finite size. Since the fluctuations in the number of particles are of the order of the total number in these finite cells this method has difficulties when dealing for example with the Mott transition which occurs as the number of electrons per site is varied to reach the critical value $n_c = 1$. Besides the compressibility of the phases involved are not given correctly for the same reasons. Since it will turn out very important for our scaling analysis to distinguish between these two situations namely, the Mott transition occurring at fixed density n = 1 but varying the ratio U/W, from that at fixed $u > u_c$ but varying the electron density, we shall take as the basis of our discussion of the localization transition the Brinkman and Rice approach to the Hubbard model based on the Gutzwiller approximation (BRG). This method as mentioned before describes a Mott transition occurring as a function of U for fixed n = 1. However, as shown by Nozieres[61], it can be generalized to obtain a phase diagram as a function of the chemical potential[12]. We shall give a fresh look at the BRG solution from the point of view of the scaling theory of quantum critical phenomena developed in previous sections. We identify the critical exponents associated with the BRG approximation and discuss the mean-field nature of this approach.

In Fig.8 we show the zero temperature phase diagram of the Hubbard model, in the reduced chemical potential versus U/W phase space, as described by the BRG approximation extended by Nozieres. The dashed line gives the trajectory of the fixed density n = 1 transition which occurs by varying the ratio U/W between the Coulomb repulsion U and the bandwidth W. Alternatively the Mott insulating phase can be reached by varying the chemical potential μ or the electron density. The equation for the phase boundary[61,12] is

$\frac{\mu-\mu_c}{\mu_c} = \left(\frac{U-U_c}{U}\right)^{1/2}$ where μ_c = U/2 is the chemical potential for the half-filled band (n = 1) and $U_c = (U/W)_c W$ the critical value of the Coulomb repulsion at μ_c.

In order to identify the critical exponents associated with the fixed density metal-insulator transition as described the BRG approach we have to deduce first how the relevant physical quantities like the compressibility and susceptibility scale close to the Mott transition. The behavior of the optical and thermal masses have been obtained before. For this purpose we consider the renormalization of the chemical potential μ close to the critical point of the fixed density transition at $U = U_c$, $\mu = U/2$. Since this is just a parameter it will acquire a renormalization when properly normalized by the relevant interactions. Within our previous notation we take the exponent y to describe the scaling of U or W at the zero temperature unstable fixed point i.e. $U' = b^{-y}U$ or $W' = b^{-y}W$ at $(U/W)_c$, $\mu = U/2$. Consequently we have for the renormalized chemical potential

$$(\delta\mu/U)' = b^{y}(\delta\mu/U) \qquad (3.12)$$

where $\delta\mu = \mu - U/2$ and which allows to obtain the scaling form of the free energy density in terms of the chemical potential close to U_c. We get

$$f \propto (U - U_c)^{2-\alpha}\, f\left[\frac{(\delta\mu/U)}{(u-u_c)^{\nu z}}\right] \qquad (3.13)$$

where we identified the exponent y with the dynamic exponent z using the same arguments as before. We can now deduce quite generally the scaling form for the compressibility or charge susceptibility κ which behaves as

$\kappa = \dfrac{\partial^2 f}{\partial \mu^2} \propto (U-U_c)^{2-\alpha-2\nu z}$ close to the critical value U_c for the fixed density, n = 1, Mott transition independent of a particular approximation.

The scaling of the susceptibility close to the localization transition described by the BRG approximation can be obtained considering the effect of a small external magnetic field in the chemical potential[12], we find $\chi \propto |g|^{-(2-\alpha-2\nu z)}$. Notice that there is no new exponent introduced here for the susceptibility since in the BRG treatment localization occurs without the appearance of long range magnetic order.

It is a well known result[46] that in the BRG approximation, the singular part of the ground state energy density varies as $E_s \propto (U-U_c)^2$ for n = 1 close to U_c. From this result we identify the BRG exponent $\alpha = 0$. Similarly a characteristic screening length can be obtained within this approach[46] which diverges at the localization transition as $\xi \propto (U-U_c)^{-1/2}$. This length plays the role of the correlation length in the metallic phase and consequently we identify the critical exponent $\nu = 1/2$ within the BRG theory.

From the equation of the critical line limiting the incompressible Mott insulating phase in the (μ/U) versus (U/U_c) phase diagram, $\dfrac{\mu-\mu_c}{\mu_c} = \left(\dfrac{U-U_c}{U}\right)^{1/2}$, see Fig.8, the scaling form of the free energy (Eq.3.13) and using the generalized scaling hypothesis[22] we can identify the exponent $\nu z = 1/2$ within the BRG approach. Since we found previously that $\nu = 1/2$ we get $z = 1$. This value of z corresponds to what is known as the Lorentz invariant case since time enters on the same footing as the space coordinates. Now that we have obtained the basic exponents associated with the BRG approximation, namely $\alpha = 0$, $\nu = 1/2$ and $z = 1$, we can substitute in the scaling expression for the compressibility to find that it vanishes linearly at the localization transition i.e. $\kappa \propto (U-U_c)$. This well known result[46] represents the behavior of the full compressibility and not just the singular part. This is due to the fact that the Mott phase is incompressible ($\kappa \propto \frac{\partial n}{\partial \mu} = 0$, since n is fixed and equal to 1 in this phase) and consequently there can be no regular or non-critical contribution for the compressibility at the transition.

For the optical mass, we introduced before, the BRG exponents give rise to a critical behavior $m^* \propto (U-U_c)^{-1}$ and this divergence signals the appearance of localization. This optical mass is not necessarily equal to that obtained from the residue of the pole of the one-electron propagator Z , which is connected to the discontinuity of the average occupation number at the Fermi level, $m_Z \propto 1/Z$ and was found to diverge as $m_Z \propto (U-U_c)^{-1}$ in the original paper by Brinkman and Rice[45]. This is clearly illustrated by the case of Luttinger liquids[62] for which Z = 0 although $D_c \neq 0$ in general and m^* is finite in the metallic phase[53]. It turns out then from these one-dimensional Luttinger liquids that Z is not the most appropriate quantity to signal the metal-insulator transition. On the other hand the effective mass $m^* \propto (1/D_c)$ is sensitive to this transition even for one-dimensional interacting electronic systems[53] (see below).

It would be desirable to obtain the scaling properties of the mass defined from the quantity Z. Since Z is obtained from the renormalized one-particle Green's function, differently from the optical mass or compressibility which are related to two-particle propagators, it is more appropriate to consider this quantity as an order parameter for the localization transition[63]. In fact we expect that Z scales as $Z \propto |g|^{2\beta}$ where β is the usual exponent associated with the order parameter (see below for the reason of the factor 2). The situation for the Luttinger liquids, i.e. for d = 1, resembles that of the XY model in d = 2. In both cases the order parameter is always zero however the susceptibility (the optical mass for the localization transition) diverges at the transition.

Since the quantity Z is related to the self-energy $\Sigma(\omega)$ (neglecting k-dependence) by[64] $Z = (1-\partial\Sigma/\partial\omega)^{-1}$, this implies, assuming a scaling ansatz for $\Sigma(\omega)$, that this quantity scales as $\Sigma(\omega) = \xi^{d-2+\eta}f(\omega\tau_\xi)$ where we used the modified scaling relation $2\beta = \nu(d+z-2+\eta)$ and $\tau_\xi = \xi^z$. Alternatively we get $\Sigma(\omega) = \omega^{(2-d-\eta)/z}f(\omega\tau_\xi)$. Then if $d = 2-\eta$ is the lower critical dimension for some instability of the electron gas we expect to find logarithmic corrections for the frequency dependence of the self energy at this instability. This behavior of the self-energy characterizes the marginal Fermi liquid of Varma et al.[65]. Originally this argument was used by Fisher, Grinstein and Girvin[3] to predict a zero temperature universal conductance at the superconductor-insulator transition in two dimensions. Starting from the scaling expression for the conductivity $\sigma(\omega) \propto \xi^{2-d}f(T/T_c)$ they obtained, using the procedure described in section 1, that at the transition $\sigma(T) \propto T^{(d-2)/z}$. For d = 2 the conductivity attains a constant, universal value at T = 0. For this argument to apply it is essential that d = 2 is neither the lower nor the upper critical dimension for this transition in which case logarithmic corrections invalidate this result. In our case this is just the situation of interest since it generates this type of corrections to the self-energy. Notice that the same expression for $\sigma(T)$ at criticality holds for the metal-insulator transition considered here.

As we have shown previously there is still another mass in the problem namely the thermal mass m_T which scales as $m_T \propto (u-u_c)^{2-\alpha-2\nu z}$. If we use the BRG exponents we find the surprising result that $m_T \propto (U-U_c)$ and consequently vanishes at the localization transition.

The quantum scaling approach to the interacting electron gas shows that near an instability, independent of any particular approximation, one has to distinguish between different masses which scale differently close to the transition. In particular the different critical behavior of the optical and thermal masses close to a zero temperature instability implies that in general the excitations which transport heat may be different in nature from those responsible for electric transport. In this case the Wiedmann-Franz ratio close to the transition will not necessarily be a universal constant. Considering the next term in the Sommerfeld expansion for the free energy, the BRG exponents yield a T^3 contribution to the specific heat with a coefficient that is independent of $(U-U_c)$. Sufficiently close to the transition this contribution may be dominant signalizing a change in the character of the elementary excitations as the localization transition is approached. All these results point to a breakdown of Fermi-liquid theory. Indeed for a Fermi-liquid one expects that the masses m_Z, m^* and m_T . For classical, mean field exponents this occurs whenever the dynamic exponent z takes the value z = 3 in which case $m^* = m_T = m_Z = m \propto (U-U_c)^{-1}$. For classical tricritical exponents this happens for z = 2 and $m \propto (U-U_c)^{-1/2}$, this assuming that $Z \propto |g|^{2\beta}$ in both cases. The essential point to recognize is that the validity of Fermi liquid theory imposes restrictions on the value of the dynamic exponent z at a metal-insulator transition. We recall that in Hubbard's approach to the the Mott transition[42] he obtained the gap exponent $\nu z = 3/2$ differently from BRG where $\nu z = 1/2$.

Another important consequence of the existence of different masses concerns the ^{3}He problem summarized earlier. We recall that one of the evidences that this system was nearly localized was obtained on the basis of a comparison between the predictions of the BRG and paramagnon models for the pressure dependence of the ratio χ/m_T. A main argument supporting that liquid ^{3}He is nearly localized as opposed to nearly ferromagnetic is that the mass, *obtained from the low temperature specific heat*, has an enhancement and varies with pressure at the same rate as the uniform susceptibility χ. The BRG method yields $\chi \propto (U-U_c)^{-1}$ as can be easily seen from the expression for χ given before and the BRG exponents. However the scaling analysis shows that in this approach the thermal mass is reduced as the system approaches localization in spite that the mass obtained from the residue of the one-particle Green's function, $m_z \propto (1/Z)$ diverges. The enhanced thermal mass observed experimentally[46] in He^3 may be due to magnetic correlations which are not fully taken into account in the BRG approach. On the other hand these results suggest that charge fluctuations anticipating a localization transition do not cause an enhancement of the thermal mass.

The BRG exponents $\alpha = 0$, $\nu = 1/2$ and $z = 1$, are suggestively mean-field like[38]. This led us[12] to consider that this approximation provides the mean-field solution for the fixed density localization transition in correlated electronic systems described by the Hubbard model. If this is the case we can substitute the BRG exponents $\alpha = 0$, $\nu = 1/2$, $z = 1$ in the modified hyperscaling relation to find out the upper critical dimension $d_c = 3$ for this transition. We note that these exponents coincide with those for the classical fixed density Mott transition of interacting lattice bosons[3] ($d_c = 3$ also for bosons). We recall that taking $Z \propto |g|^{2\beta}$ with the mean field value $\beta = 1/2$ we reproduce the BRG result $m_z \propto 1/Z \propto (U-U_c)^{-1}$.

The Fermi liquid regime attained below the crossover line in the phase diagram shown in Fig.7 may be identified with the so called "highly correlated electron gas" as is usually referred in Mott's book a metallic system which is close to a Mott transition. The properties of this gas are not totally described by the BRG approach, as we have seen, since it does not take into account fully the magnetic correlations. These may lead to larger values for the dynamic exponent z and an enhanced thermal mass. In real systems the entrance into the renormalized Fermi-liquid regime is characterized, as in the case of heavy fermions, by a crossover in the magnetic susceptibility from a temperature dependent to a Pauli-like behavior. This behavior is illustrated for doped V_2O_3 as shown[66] in Fig.9 . Note that the "coherence temperature" in this case is much larger. The renormalization of the strong interactions, giving rise to a Fermi-liquid behavior below some characteristic temperature, seems to be a universal feature of the electronic systems considered in this report.

3.6 THE DENSITY-DRIVEN MOTT TRANSITION

The Mott insulating phase shown in Fig.8 can be reached either by varying the ratio (U/U_c) for fixed n = 1 or by varying the chemical potential or electron density for $U > U_c$. These transitions are not necessarily in the same universality class, that is they may be governed by different fixed points with different critical exponents. For the density-driven Mott transition, with the critical behavior expressed in terms of $\delta = \mu - \mu_c(U)$, the singular part of the free energy density defines a new exponent α' through $f \propto |\delta|^{2-\alpha'}$. Here $\mu_c(U)$ is the value of the chemical potential at the phase boundary of the Mott insulating phase for a given $U > U_c$. The singular part of the compressibility $\kappa_s \propto \partial^2 f/\partial\mu^2 \propto |\delta|^{-\alpha'}$ for this density-driven transition. The new exponents reflect the possibility that the fixed density and fixed U transitions are in different universality classes. Using a finite-size, finite-time scaling expression for the singular part of the free energy density $f = |\delta|^{\nu'(d+z')} f_s[\xi/L, \xi^{z'}/\beta]$ of a system of spatial size L and temporal extent β and assuming that the compressibility is the coefficient of the finite-time correction[3], we find alternatively for the total compressibility $\kappa \propto |\delta|^{\nu'(d-z')}$. The incompressible character of the Mott insulator, which rules out a finite contribution for the compressibility at the transition, allows to equate the expressions for the total and the singular part of the compressibility. We then get, $-\alpha' = \nu'(d-z')$ and using the modified hyperscaling relation $2-\alpha' = \nu'(d+z')$ we obtain the important relation $\nu'z' = 1$ for the density-driven transition. This equality is rigorous for the superfluid-Mott insulator transition in a lattice system of interacting bosons[3] as the number of particles is varied. In the fermion case the assumption above, that the compressibility is the coefficient for the finite time corrections, can not be rigorously justified. In the boson case it relies on the existence of an order parameter associated with the zero temperature superfluid-to-Mott-insulator transition and it's effective action. On the other hand since the compressibility determines the stiffness of the sound excitations of the Fermi liquid, like in the superfluid case we expect that, at least whenever charge and spin separate, the relation $\nu'z' = 1$ should hold for the metal-to-Mott-insulator density-driven transition. We will show below that this is indeed the case for d = 1.

Notice that a separation between spin and charge occurs for the localization transition in the Hubbard model described by the BRG approach in the sense that the charge degrees of freedom become critical at the transition without the appearance of long range magnetic order. For the density driven transition described by the BRG approximation, as extended by Nozieres, the equality $\nu' z' = 1$ is suggested from the linear shape of the boundary of the n = 1 Mott phase at $U_c/U = 0$ (see Fig.8 and the equation for the critical frontier). This leads to a linear relation between the chemical potential and the Coulomb repulsion U as U becomes very large.

A pure localization transition without the appearance of long range magnetic order as described in the BRG approach may not in fact occur in high dimensional fermionic systems. However for d = 1, where charge and spin degrees of freedom separate, a pure localization transition occurs *as a function of density or chemical potential* for the Hubbard model[53] (Fig.10). It can be shown[67] from the Bethe ansatz solution that the compressibility κ diverges as $\kappa \propto (n-n_c)^{-1}$ and that the charge stiffness D_c vanishes linearly[67] with the number of holes, i.e., $D_c \propto (n-n_c)$ at this transition with $n_c = 1$. The optical mass $m^* \propto (1/D_c) \propto (n-n_c)^{-1}$ and consequently diverges with the same numerical exponent of the compressibility[67]. The above exact result for the critical behavior of the compressibility together with a rigorous relation between the singular part of m^* and this quantity[67], namely $m^* \propto \kappa$, allow to determine unambiguously the critical exponents characterizing this density-driven transition when expressed in terms of $\delta = \mu - \mu_c(U)$, where $\mu_c(U)$ is the value of the chemical potential at the phase boundary shown in Fig.10. For this purpose we note that $n \propto \frac{\partial f}{\partial \mu} \propto |\delta|^{1-\alpha'}$. If $(1-\alpha') \geq 1$ i.e. $\alpha' \leq 0$, the relation between μ and n is regular, i.e. $(n-n_c) \propto \delta = [\mu-\mu_c(U)]$, otherwise[68] $(n-n_c) \propto |\delta|^{1-\alpha'}$. The compressibility $\kappa = \partial^2 f/\partial\mu^2 \propto |\delta|^{-\alpha'}$. Let us assume that the regular term dominates i.e. $1-\alpha' \geq 1$ ($\alpha' \leq 0$). Then $(n-n_c) \propto \delta$ and consequently $\kappa \propto (n-n_c)^{-\alpha'}$. Due to the exact result $\kappa \propto (n-n_c)^{-1}$ we should then have $\alpha' = 1$. This is in contradiction with the initial assumption that the relation between μ and n is regular, i.e. that $\alpha' \leq 0$. The alternative possibility yields $[\mu-\mu_c(u)] \propto (n-n_c)^{\frac{1}{1-\alpha'}}$ and $\kappa \propto (n-n_c)^{-(\alpha'/1-\alpha')}$. A comparison with the exact result $\kappa \propto (n-n_c)^{-1}$ determines $\alpha' = 1/2$. On the other hand the optical mass $m^* \propto |\mu-\mu_c(U)|^{-(2-\alpha'-2\nu')}$, as derived before, and since $m^* \propto \kappa$ we have $2-\alpha'-2\nu' = \alpha'$ or $\nu' = 1-\alpha' = 1/2$. Finally from the hyperscaling relation we get $z' = 2$ and consequently $\nu'z' = 1$. These exponents $\alpha' = 1/2$, $\nu' = 1/2$ and $z' = 2$ are the same which have been found for the density-driven superfluid-Mott-insulator transition for one-dimensional interacting bosons in a lattice[3,69]. It is possible, as the results above suggest, that whenever spin and charge separate the density-driven localization transition for fermions is in the same universality class of the corresponding transition for bosons. Furthermore $z' = 2$ is the dynamical critical exponent of the metal-insulator transition due to band filling for the non-interacting one-dimensional electron gas[9]. The gap exponent $\nu'z' = 1$ also in this case.

For the lattice boson problem[3] the upper critical dimension for the density-driven superfluid-localization transition is $d_c = 2$. For $d \geq 2$ the exponents assume the classical, mean-field values, $\alpha' = 0$, $\nu' = 1/2$, $z' = 2$ etc.. In the case of fermions, for $d = 2$, the density-driven localization transition is accompanied by long range magnetic order[70] at $T = 0$. In this case the analogy with bosons may cease and presumably $\nu'z' \neq 1$. However there is numerical evidence that the compressibility still diverges with the inverse of the hole concentration for a square lattice in the Hubbard model close to the density-driven transition[71]. If the analogy with the $d = 2$ boson case or with non-interacting electrons[71] was still valid we would expect a logarithmic divergence of κ. The numerical result suggests a similarity of the $d = 2$ density-driven transition with the one-dimensional case at least as concerns the behavior of the compressibility and the optical mass.

We have studied the metal-insulator transition described by the Hubbard model from the point of view of the theory of quantum critical phenomena. The n = 1 Mott insulating phase can be reached either at a fixed density or by varying the number of electrons. These transitions may be in different universality classes. They are quantum transitions governed by zero temperature fixed points. The exponents associated with these fixed points determine the critical behavior of the different quantities of physical interest. In particular they control the divergence of the optical mass which characterizes unambiguously the localization transition. Our analysis of the BRG approximation has led to new insight on the nature of this solution revealing it's mean field character. The scaling approach provides a critical view of the different approximations to the difficult many body problem posed by the Hubbard hamiltonian.

4.1 CONCLUSIONS and PERSPECTIVES

We have discussed the scaling theory of quantum critical phenomena and applied it to some current and important many-body problems. Although scaling ideas have been widely applied to magnetic systems, both pure[15] and disordered[19], only recently[4,5,8,12,13] they have been used to describe strongly interacting many-body electronic systems. This has coincided with extensive work on the scaling properties of interacting bosons on a lattice[3,69]. The instability of the interacting electrons has been associated with a zero temperature unstable fixed point. The exponents related to this fixed point determine the universality class of the transition and obey standard scaling relations. However because we are dealing with a zero temperature transition the hyperscaling relation is modified giving rise to a shifted dimensionality. We have emphasized that time plays a distinct role in quantum phase transitions. It is precisely the dynamic exponent z associated with the critical slowing down of the quantum fluctuations that appears in the modified hyperscaling relation.

The zero temperature exponents determine the critical behavior of the different physical quantities close to the phase transition. For an electronic system close to localization these exponents yield the critical behavior of the optical mass which characterizes unambiguously this transition, the enhancement of the thermal mass and of the compressibility. The scaling analysis led us to recognize the existence of distinct masses for the correlated electronic system as anticipated by Léggett[73]. Although we could not derive rigorously the scaling law for the mass $m_Z \propto 1/Z$ as we did for the thermal and the optical mass we argued, as previous authors[63], that the quantity Z scales as an order parameter. We have shown that the proportionality between m_T and m_Z , expected for a Fermi liquid, imposes constraints in the value the dynamic exponent z assumes at a metal-insulator transition. The concept of a thermodynamic mass m_T , defined from the linear term of the specific heat, is closely related to the existence of a crossover line in the non-critical part of the phase diagram. This line marks the onset of a renormalized Fermi-liquid regime which is evidenced by experimental observation on strongly correlated systems. In the heavy fermion problem this new energy scale, which is lower and distinct from the single ion Kondo temperature, has been associated with the notion of coherence. The scaling theory presented here is most useful above some lower critical dimension such that the zero temperature transition occurs at a finite critical ratio of the parameters. In this case a Fermi liquid regime for the interacting electronic system is always expected. The scaling analysis can however be extended for marginal cases where the relevant transition occurs for any finite value of the interactions[4] and also in one dimension. It relies only on the existence of a continuous transition at T = 0 associated with an unstable fixed point.

We have dealt here with pure systems with no disorder. If disorder is relevant, in the renormalization group sense, it will take the system to another fixed point and a different universality class. Close to this new fixed point a similar scaling analysis can in principle be carried out[72]. An important question concerns whether the crossover line associated with this new fixed point, as we have interpreted here, completely subsides implying that no Fermi-liquid regime can be found in the disordered case. It is clear that a large value of the dynamic exponent z will reduce the region in the phase diagram where Fermi liquid behavior is expected to occur specially close to the transition. Disorder may also raise the upper critical dimension such that mean field exponents are not expected to occur for the T = 0 phase transitions in these systems[3].

The application of simple scaling concepts to the many-body systems studied here leads to new insight into these difficult problems. These ideas play an important role in the understanding of the physics of highly correlated electronic materials and of the approximations implemented to solve the problems posed by these systems. The quantum scaling method should be viewed as a complementary approach to other techniques.

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FIGURE CAPTIONS

Figure 1. Phase diagram of the Kondo lattice (Ref.4). The critical Neel line is governed by the same exponent of the crossover or coherence line T_c. The renormalized Fermi liquid regime is attained for $T \ll T_c$. The figure shows unambiguously that the coherence temperature represents a new energy scale in the non-critical part of the phase diagram lower than the single ion Kondo temperature T_K.

Figure 2. The resistivity of the non-magnetic heavy fermion $CeCu_6$ as a function of temperature (Milliken et al., Ref.25). The drop on the resistivity at low temperatures is associated with the coherence effect.

Figure 3. The low field magnetic susceptibility of $CeRu_2Si_2$ as a function of temperature (Ref.33). Notice the crossover, passing through a maximum, from a temperature dependent behavior at high temperatures to a temperature independent, Pauli-like regime at low temperatures. The maximum defines the crossover or coherence temperature. This maximum shifts to higher temperatures as pressure increases and the system moves away from criticality tracing the coherence line.

Figure 4. Pressure dependence of: (*) $h_c(P)/h_c(0)$; (x) $\chi_h(h_c,P=0)/\chi_h(h_c,P)$ for $T \ll T_c$; (+) $[A(0)/A(P)]^{1/2}$; (*) $T_c(P)/T_c(0)$; (◊) $\chi_0(T_c,P=0)/\chi_0(T_c,P)$ for $h \ll h_c$ (Ref.5). For further details and references see text.

Figure 5. The relation between the the coefficient of the T^2 term of the resistivity (A) and the coefficient of the linear term of the specific heat γ for different heavy fermions (from Ref.37). The line represents the relation $A \propto \gamma^2$. The triangle is for $CeRu_2Si_2$ and the cross-dots are for $CeAl_3$ at different applied pressures (Phillips et al., Ref.25; J.Flouquet et.al., Ref.36).

Figure 6. The total energy of a two-dimensional free electron gas in a ring as a function of the flux through the ring. As the number N, of electrons and sites, increases the flux dependence of the total energy decreases approaching a flat line as N goes to infinity.

Figure 7. The phase diagram of the Hubbard model as described by the BRG approximation (Ref.12). The incompressible, n = 1, Mott insulating phase can be reached either at constant density (the dashed line trajectory) or by varying the chemical potential or electron density. These transitions are not necessarily in the same universality class.

Figure 8. The phase diagram of a Mott insulator (schematic) as a function of temperature. The continuous line is found in experiments to be first order. The highly correlated electron gas regime defined as a renormalized Fermi-liquid close to a Mott transition is attained below the crossover line (dashed line). The straight dashed line represents the Fermi energy of the non-interacting system.

Figure 9. The low field magnetic susceptibility of doped V_2O_3 as a function of temperature (Ref.66). This system does not become magnetic down to the lowest temperature. The susceptibility shows a crossover from a temperature dependent behavior to a Pauli-like one as the system enters the highly correlated electron gas regime with decreasing temperature.

Figure 10. The phase diagram of the one-dimensional Hubbard model. The critical frontier $\mu_c(U)$, separating the metallic from the insulating phase, is obtained from the exact expression for the gap (Refs. 43 and 67), namely $(\Delta/U) = (t/U)^2\int_1^\infty dz(z^2-1)^{1/2}\text{cosech}(\pi tz/2U)$. The insulating phase for $\mu = U/2$ ($\mu/U = 0.5$) extends to $(t/U) = \infty$, i.e for any value of $U > 0$.

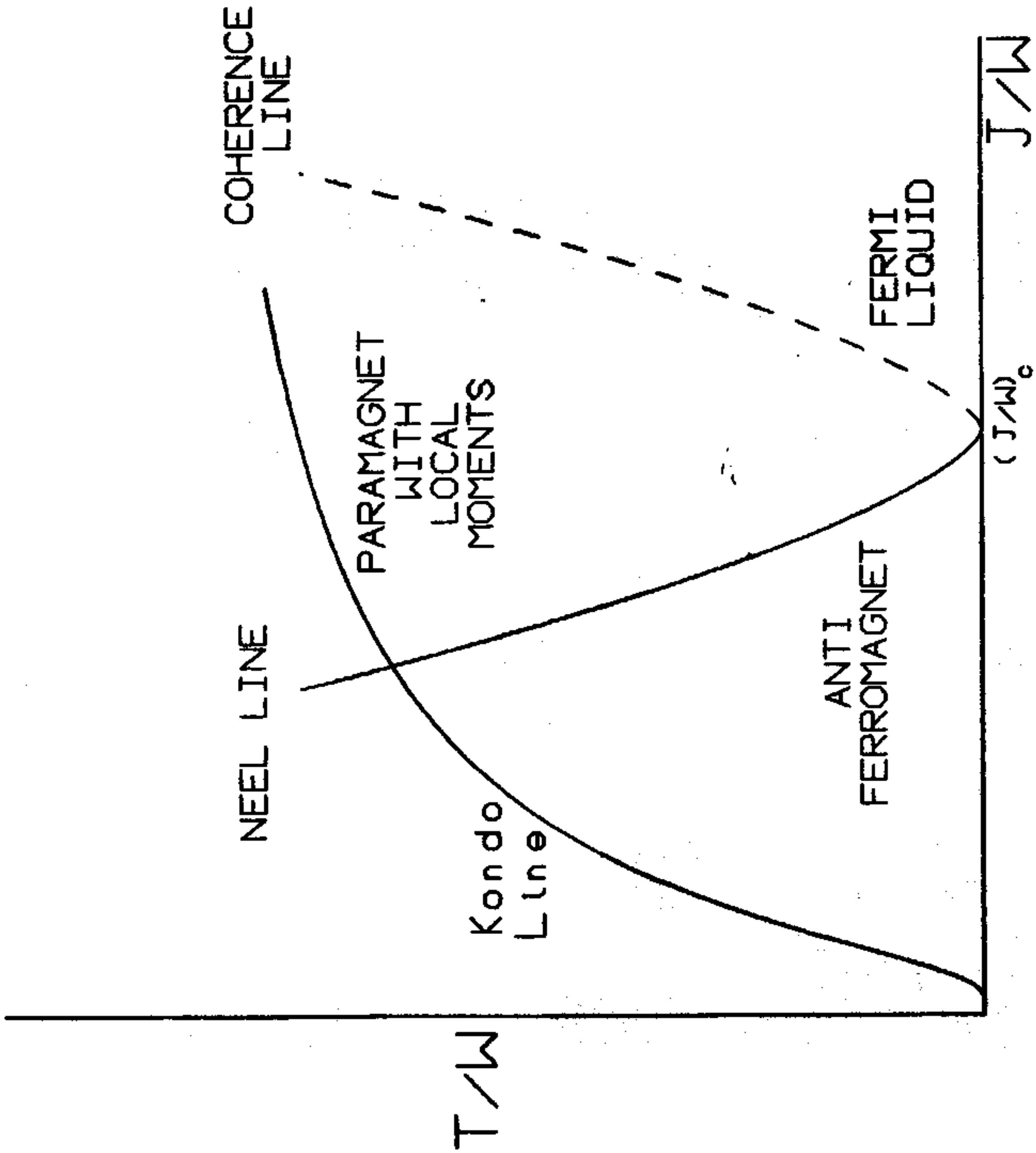
COHERENCE LINE
NEEL LINE
PARAMAGNET WITH LOCAL MOMENTS
FERMI LIQUID
ANTI FERROMAGNET
Kondo Line
T/W
J/W
(J/W)c

FIG. 1

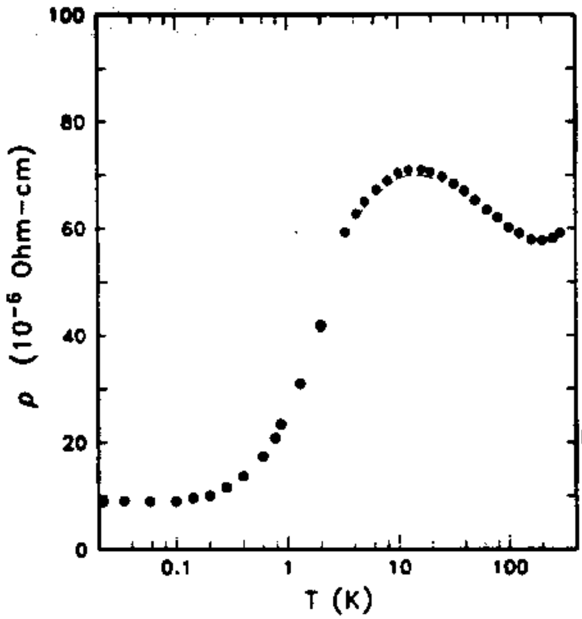

FIG. 2

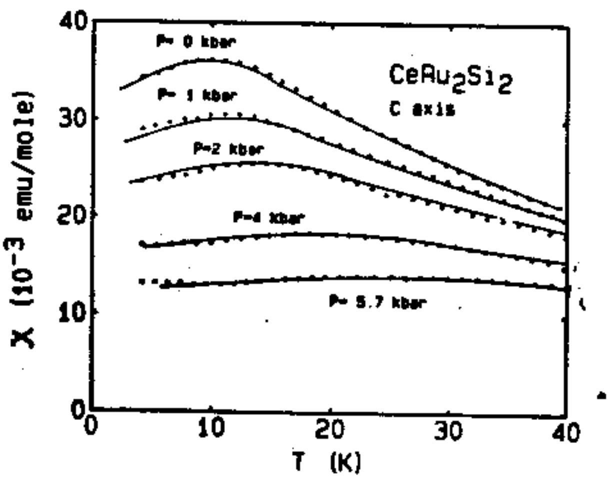
CeRu2Si2
C axis
P= 0 kbar
P= 1 kbar
P=2 kbar
P=4 kbar
P= 5.7 kbar
X (10-3 emu/mole)
T (K)
0
10
20
30
40

FIG. 3

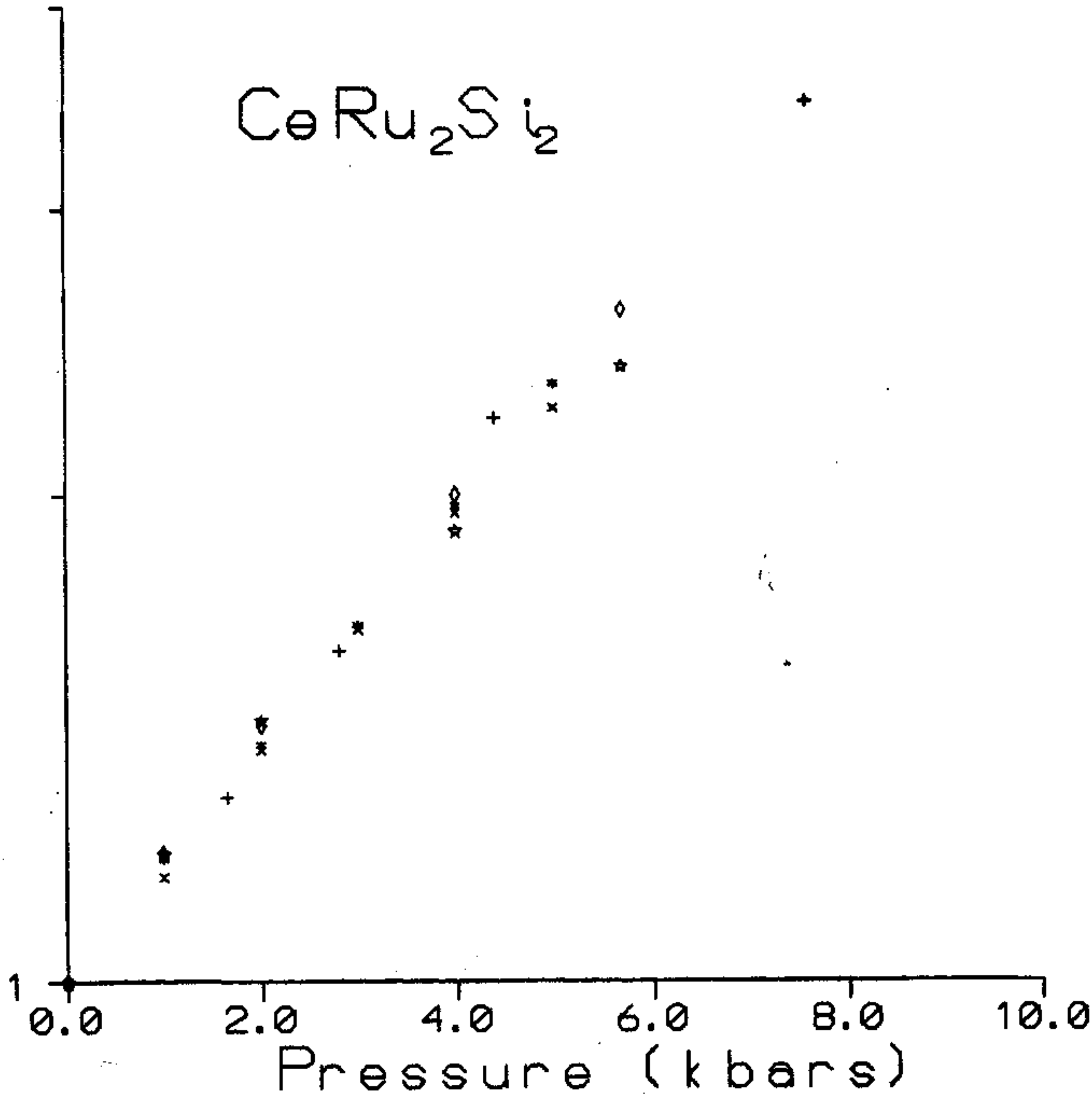

FIG. 4

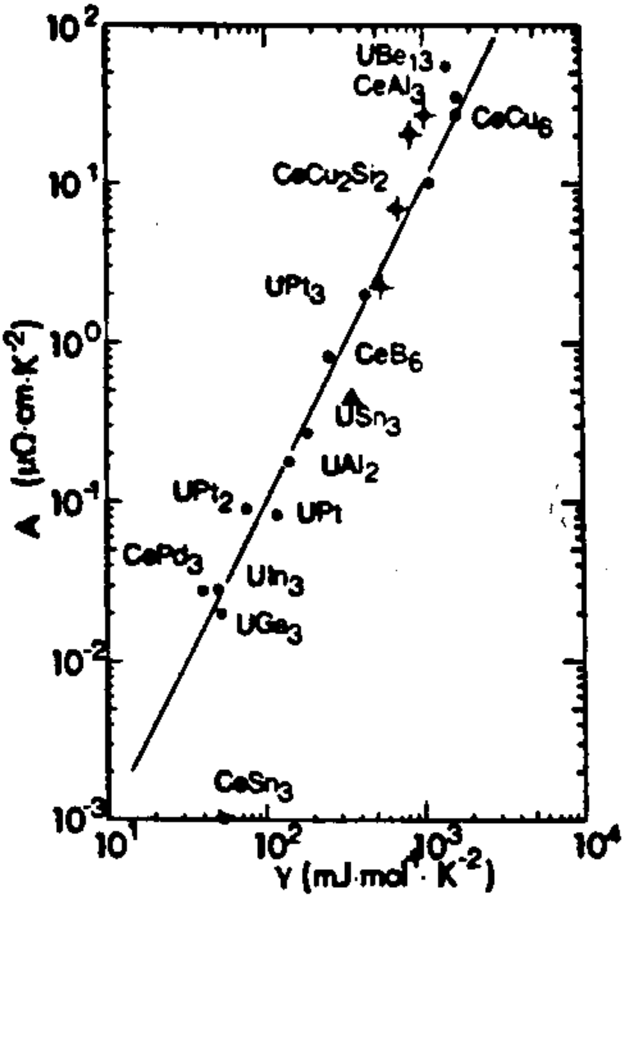
UBe13
CeAl3
CeCu6
CeCu2Si2
UPt3
CeB6
USn3
UAl2
UPt2
UPt
CePd3
UIn3
UGe3
CeSn3
A (μΩ·cm·K^-2)
γ (mJ·mol^-1·K^-2)
10^2
10^1
10^0
10^-1
10^-2
10^-3
10^1
10^2
10^3
10^4

FIG. 5

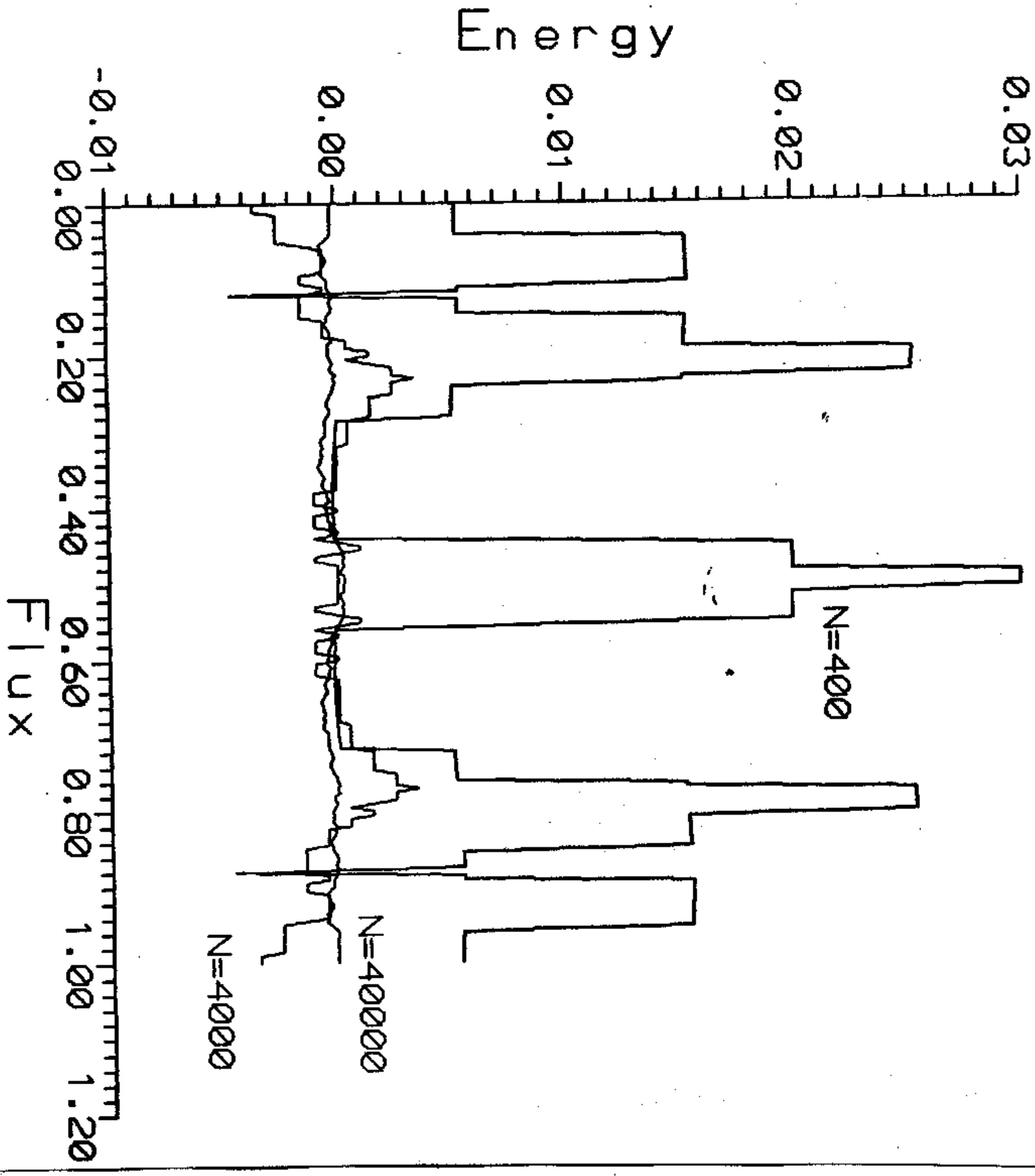
Energy
-0.01
0.00
0.01
0.02
0.03
0.00
0.20
0.40
0.60
0.80
1.00
1.20
Flux
N=400
N=4000
N=40000

FIG. 6

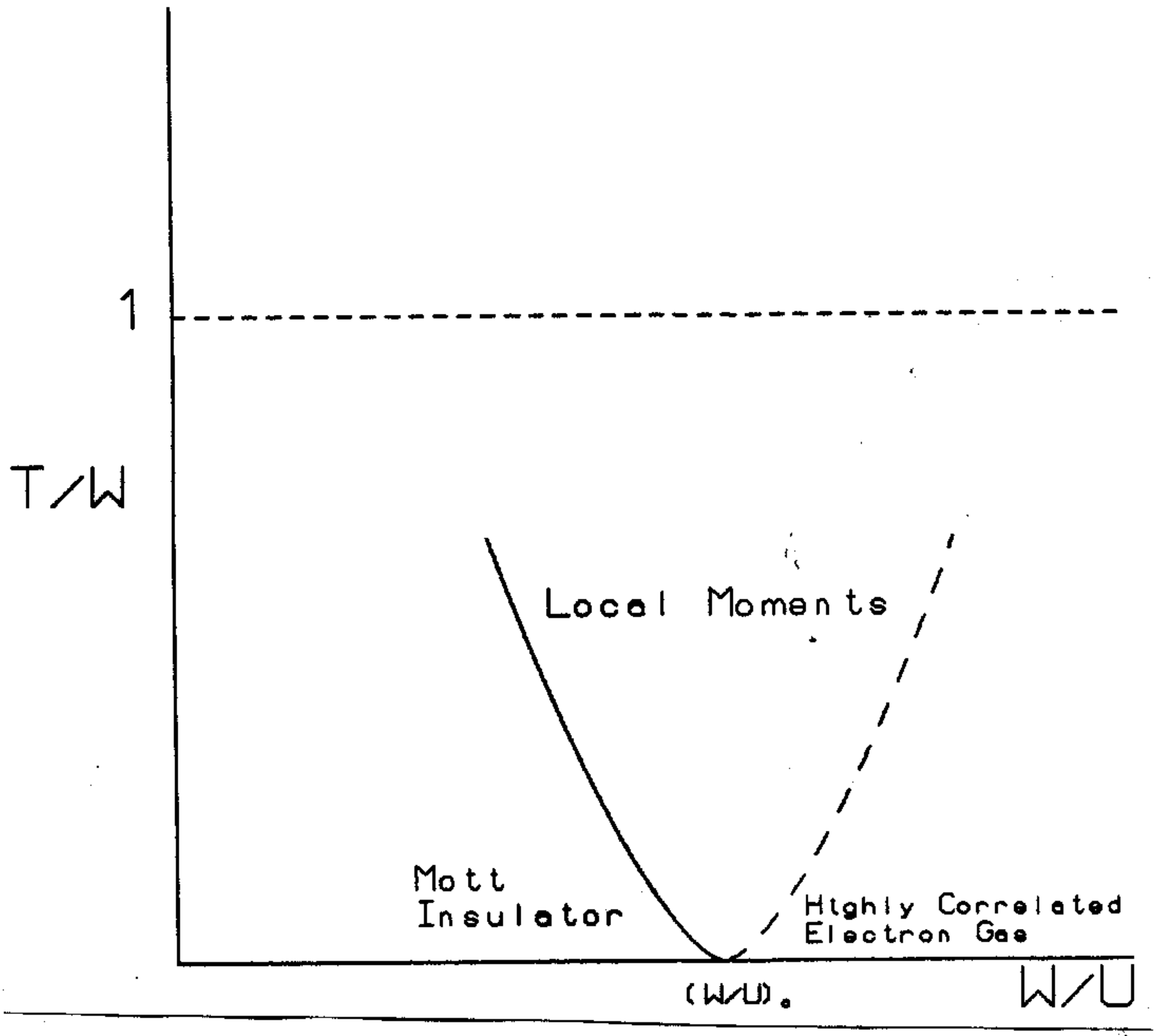
1
T/W
Local Moments
Mott
Insulator
Highly Correlated
Electron Gas
(W/U)$_c$
W/U

FIG. 7

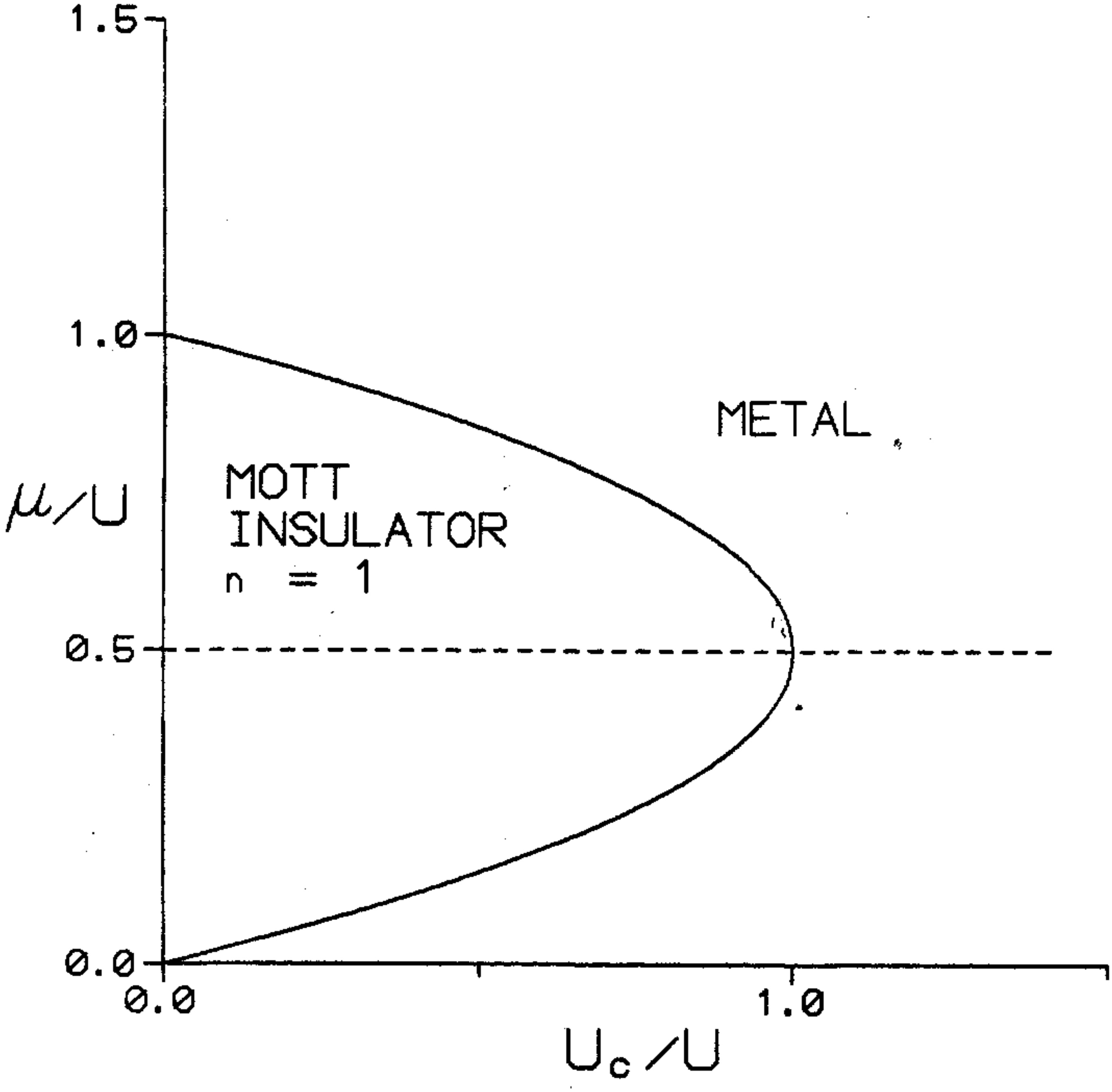

FIG. 8

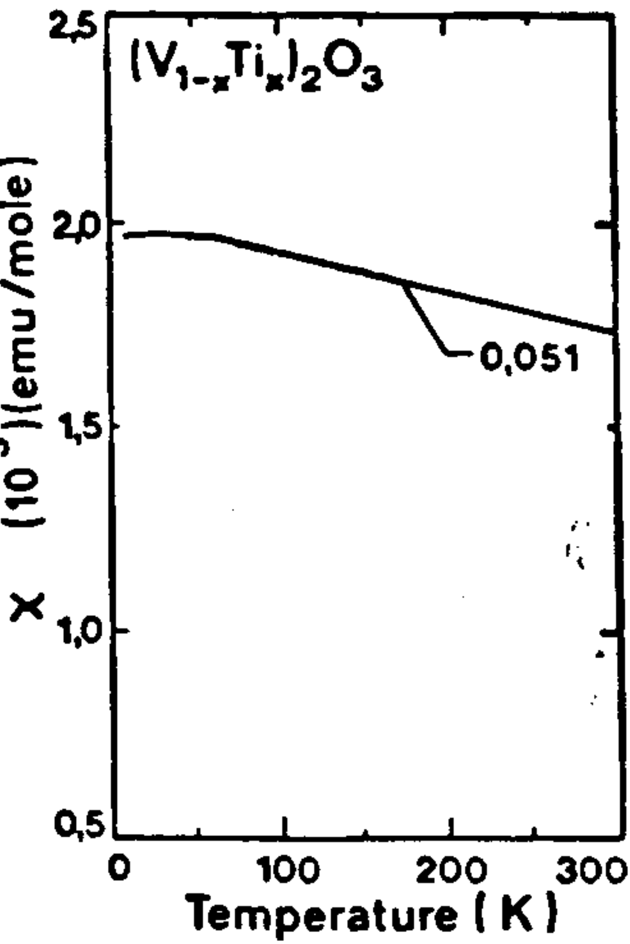
2,5
2,0
1,5
1,0
0,5
$(V_{1-x}Ti_x)_2O_3$
X (10^{-3})(emu/mole)
0,051
0
100
200
300
Temperature (K)

FIG. 9

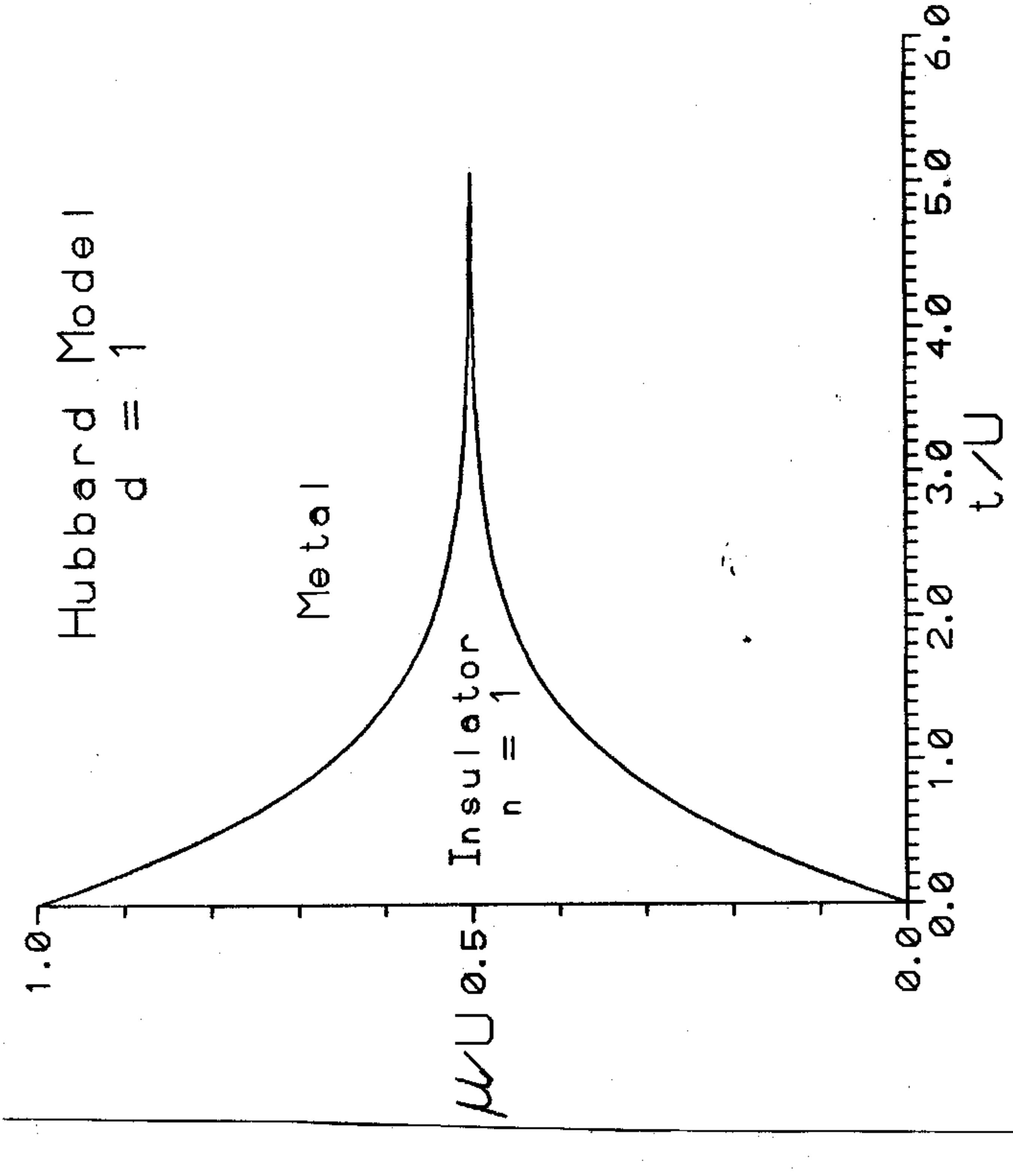
Hubbard Model
d = 1
Metal
Insulator
n = 1
t/U
μ/U
0.0
1.0
2.0
3.0
4.0
5.0
6.0
0.0
0.5
1.0

FIG. 10

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