Long memory constitutes a unified mesoscopic mechanism consistent with nonextensive statistical mechanics

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Abstract. - We unify two paradigmatic mesoscopic mechanisms for the emergence of nonextensive statistics, namely the multiplicative noise mechanism leading to a *linear* Fokker-Planck (FP) equation with *inhomogenous* diffusion coefficient, and the non-Markovian process leading to the *nonlinear* FP equation with *homogeneous* diffusion coefficient. More precisely, we consider the equation $\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}[F(x)p(x,t)] + \frac{1}{2}D\frac{\partial^2}{\partial x^2}[\phi(x,p)p(x,t)]$, where $D \in \mathcal{R}$ and $F(x) = -\partial V(x)/\partial x$, $V(x)$ being the potential under which diffusion occurs. Our aim is to find whether $\phi(x,p)$ exists such that the inhomogeneous linear and the homogeneous nonlinear FP equations become unified in such a way that the (ubiquitously observed) q-exponentials remain as stationary solutions. It turns out that such solutions indeed exist for a wide class of systems, namely when $\phi(x,p) = [A + BV(x)]^\theta [p(x,t)]^\eta$, where A, B, θ and η are (real) constants. Our main result can be sumarized as follows: For $\theta \neq 1$ and arbitrary confining potential $V(x)$, $p(x,\infty) \propto \{1-\beta(1-q)V(x)\}^{1/(1-q)} \equiv e_q^{-\beta V(x)}$, where $q = 1 + \eta/(\theta-1)$. The present approach unifies into a single mechanism, essentially *long memory*, results currently discussed and applied in the literature.

One of the cornerstones of statistical mechanics is the functional connection of the thermodynamic entropy with the set of probabilities $\{p_i\}$ of microscopic configurations. For the celebrated Boltzmann-Gibbs (BG) theory, this central functional is given by $S_{BG} = -k\sum_{i=1}^{W} p_i \ln p_i$, where W is the total number of microscopic states which are compatible with the information that we have about the system. This powerful connection is in principle applicable to a vast class of relevant systems, including (classical) dynamical ones whose maximal Lyapunov exponent is positive, thus generically warranting strong chaos, hence mixing in phase space, hence ergodicity (in some sense, Boltzmann embodied all these features in his insightful *molecular chaos hypothesis*). Within this theory, it ubiquitously emerges the Gaussian distribution $p_G \propto e^{-\beta x^2}$ $(\beta > 0)$. Indeed, this important probabilistic form (i) maximizes the (continuous version of the) entropy $S_{BG} = -k\int dx\, p(x)\ln[p(x)]$ under the basic constraints of normalizability and finite width; (ii) constitutes the exact solution, for all values of space and time, of the simplest form of the (linear and homogeneous) Fokker-Planck equation, in turn based on the simplest form of the Langevin equation (which includes *additive noise*); (iii) is the $N \to \infty$ attractor of the (appropriately centered and scaled) sum of N *independent* (or weakly correlated in an appropriate sense) discrete or continuous random variables whose second moment is finite (Central Limit Theorem, CLT); (iv) is the velocity distribution (Maxwell distribution) of any classical many-body Hamiltonian system whose canonical (thermal equilibrium with a thermostat) partition function is finite, i.e., if the interactions between its elements are sufficiently short-ranged, or inexistent. The simplest probabilistic model which realizes these paradigmatic features is a set of N independent equal binary random variables (each of them taking say the values 0 and 1 with probability 1/2). The probability of having, for fixed N, n 1's is given by $\frac{N!}{n!\,(N-n)!}\,2^{-N}$. Its limiting distribution is, after centering and scaling, a Gaussian (as first proved by de Moivre and Laplace), and its (extensive) entropy is the BG one, since $S_{BG}(N) = Nk\ln 2$.

What happens with the above properties when the correlations between say the elements of a probabilistic model are strong enough (in the sense that they spread over all elements of the system)? There is in principle no reason for expecting the relevant limiting distribution to be a Gaussian, and the entropy which is extensive (i.e., $S(N) \propto N$ for $N \gg 1$) to be S_{BG}. The purpose of the present paper is to focus on such and related questions for a class of systems which are ubiquitous in natural, artificial and even social systems, namely those which are *scale-invariant* in a probabilistic sense which we shall define below. Let us now discuss the frequent emergence of q-exponentials, defined as

$$P_q(x) = N_q[1-(1-q)\beta V(x)]^{1/(1-q)} = N_q\, e_q^{-\beta V(x)} \quad , \quad (1)$$

where N_q is a normalization factor; $P_1(x) = N_1\, e^{-\beta V(x)}$ is the standard BG case; for $V(x) \propto x^2$, $P_q(x)$ is a q-Gaussian, which displays asymptotic power-laws and can be seen as a natural generalization of the Gaussian ($q = 1$).

At this point let us make a few remarks. (i) q-Gaussians appear as the exact solutions of paradigmatic non-Markovian Langevin processes and their associated Fokker-Planck equations . Langevin equations with both additive and multiplicative noise [1], or Langevin equations with long-range-memory [2], lead respectively to inhomogeneous linear [3], or homogeneous nonlinear [4, 5] Fokker-Planck equations (see also [6–10]). (ii) q-CLT attractors are q-Gaussians [11]. (iii) The extremization of the entropy S_q with norm and finite width constraints yields q-Gaussians, where S_q is a generalization of BG entropy, namely [12, 13]

$$S_q = k\frac{1-\int dx\,[p(x)]^q}{q-1} \qquad (q \in R;\, S_1 = S_{BG}) \qquad (2)$$

This entropy is, for $q \neq 1$, *nonadditive* (see [14] for the current definition of additivity), i.e., for arbitrary probabilistically independent systems A and B, the equality $S(A+B) = S(A) + S(B)$ is not satisfied. However, for many systems a value of q, denoted by $q_{\rm ent}$, exists for which $S_{q_{\rm ent}}$ is *extensive*, i.e., $S_{q_{\rm ent}}(N) \propto N \quad (N \gg 1)$. As is well known, for all standard short-range-interacting many-body Hamiltonian systems, we have $q_{\rm ent} = 1$. However, some systems exist for which $q_{\rm ent} < 1$ [15–18]. (iv) Numerical indications [19] for the distributions of velocities in quasistationary states of long-range Hamiltonians [20] suggest q-Gaussians. Further, experimental and observational evidence for q-Gaussians exists for the motion of biological cells [21,22], defect turbulence [23], solar wind [24,25], cold atoms in dissipative optical lattices [26], dusty plasma [27], among others (see also [28]). Numerical indications are also available at the edge of chaos of unimodal maps [29].

A domain where the nonadditive entropy S_q can be naturally incorporated is for describing anomalous diffusion like-phenomena. From modified Langevin equations, inhomogeneous linear or homogeneous nonlinear Fokker-Planck equations have been derived and used in order to obtain the mesoscopic dynamic evolution of systems where such diffusion occurs.

Let us consider here the nonlinear Fokker-Planck (FP) equation given by

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}[F(x)p(x,t)] + \frac{1}{2}D\frac{\partial^2}{\partial x^2}[\phi(x,p)p(x,t)] \qquad (3)$$

where $D \in \mathcal{R}$ is the coeficient of diffusion, $F = -\frac{\partial V(x)}{\partial x}$ the drift term, and V a confining potential. We shall further assume the following wide connection:

$$\phi(x,p) = [g(V)]^\theta [p(x,t)]^\eta \,, \qquad (4)$$

with $g(V) = [A+BV]$, A, B, θ and η being real constants.

It should be noted that the present equation encompasses also the general inhomogeneous ($\theta \neq 0$) nonlinear ($\eta \neq 0$) case. Our aim is to find whether a q-exponential $P_q(x)$ exists as a stationary solution (i.e., $\lim_{t\to\infty} p(x,t) = P_q(x)$) of this FP equation such that the *inhomogeneous linear* and the *homogeneous nonlinear* FP equations become unified.

The condition $\frac{\partial p(x,t)}{\partial t} = 0$ implies

$$\frac{\partial F(x)p(x,\infty)}{\partial x} = \frac{D}{2}\frac{\partial^2}{\partial x^2}[\phi(x,p)p(x,\infty)]\,. \qquad (5)$$

By assuming appropriate boundary conditions (basically $p(\pm\infty,\infty) = 0$), changing variables and developing, we obtain

$$\frac{\partial\left[g(V)^\theta p(V)^{1+\eta}\right]}{\partial V} = -\frac{2p(V)}{D}\,, \qquad (6)$$

where $p(V) \equiv p(V(x),\infty)$. It follows that

$$\frac{\partial p(V)}{\partial V} = -\frac{2g(V)^{-\theta}p(V)^{1-\eta} + \theta BDg(V)^{-1}p(V)}{D(1+\eta)}\,. \qquad (7)$$

Now, we will first consider the $\theta \neq 1$ case. Let us propose a solution of Eq.(7) satisfying

$$g(V)^{1-\theta}p(V)^{-\eta} = C \qquad (8)$$

with $\partial C/\partial V = 0$. By direct substitution in Eq. (7), we can easily show that $p(V) = P_q(V)$, as given by Eq. (1), with

$$q = 1 + \frac{\eta}{\theta - 1}\,, \qquad (9)$$

and

$$\beta = \frac{2C + \theta BD}{AD(1+\eta)}\,, \qquad (10)$$

where

$$C = \left[A N_q^{q-1}\right]^{1-\theta} . \quad (11)$$

These results reproduce, for $\theta = 0$, the homogeneous nonlinear case discussed in [2, 4, 5], i.e., $q = 1 - \eta$ and $N_q^{1-q}\beta = 2/D(1+\eta)$, which leads to normal diffusion for $\eta = 0$.

Eq. (9) shows that, for $\eta \neq 0$, the limit $\theta \to 1$ corresponds to a singularity in the $|q|$ value. However, in this limit, we still find q-exponentials as stationary solutions of the above FP equation, in two different situations:

a) For η considered as a function of θ with the leading term given by $\eta \sim \alpha(\theta - 1)^\delta$, the analytical extension of the solution given by Eqs. (8-11) results in q-exponentials presenting the indices $q = 1$ for $\delta > 1$ and $q = 1 + \alpha$ for $\delta = 1$, with $\beta = (2 + BD)/AD$ in both cases.

b) An isolated solution (which is not a particular case of the previous ones) can also be found by setting $\eta = 0$ and $\theta = 1$ in Eq. (7). By so doing we obtain

$$\frac{\partial p(V)}{\partial V} = -\frac{[2 + BD]\, g(V)^{-1} p(V)}{D} . \quad (12)$$

Now, imposing the relation

$$g(V) p(V)^{q-1} = \bar{C} . \quad (13)$$

where $\bar{C}$ is independent of the potential V. Following the same procedure used to obtain Eqs. (9) and (10), we finally have $p(V) = P_q(V)$, with

$$q = \frac{(1 + BD)}{(1 + BD/2)} , \quad (14)$$

and

$$\beta = \frac{(2 + BD)}{AD} . \quad (15)$$

It should be noticed that this case ($(\eta, \theta) = (0, 1)$) recovers previous results already obtained in [1, 3].

As final remarks, we note that:

(i) For $A \neq 0$, we may take $A = 1$ without loss of generality, if the (B, D) parameters are properly rescaled;

(ii) It is known [13], that the q-Gaussians, which emerge for $V(x) \propto x^2$ are normalized only for $q < 3$, which implies the following restrictions to the values of the parameters of the FP equation: (a) For $\theta \neq 1$, $\frac{\eta}{\theta - 1} < 2$, (b) For $\theta \to 1$, $\alpha < 2$ if $\delta = 1$ and $BD > -2$ or $BD < -4$. For each V(x), an analysis of integrability of P_q must be performed, in order to establish the accepted range of values of the parameters (η, θ, B, D, q) (e.g., if $V(x) \propto |x|^\rho$, in the limit $|x| \to \infty$, then it must be $q < 1 + \rho$);

(iii) We stress that the Eq. (3) used in this paper corresponds to the Itô form of a generalized FP equation (a Stratonovich approach has already been examined [10], and the existence of q-exponential distributions as stationary solutions has also been proved).

Let us emphasize that Eq. (9), here presented for the first time, enables us to analyze in an unique way the present general inhomogeneous nonlinear case, which contains, as particular cases, several FP equations used to describe a large class of nongaussian natural systems. This unification constitutes the main result of this paper.

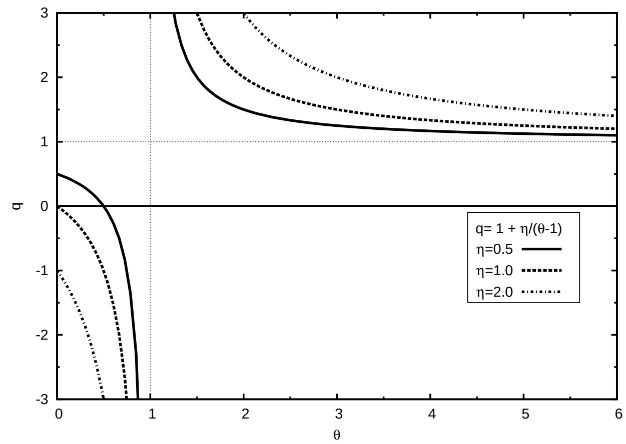

Fig. 1: $q = 1 + \frac{\eta}{\theta - 1}$ as a function of θ for selected values of η (the upper bound $q = 3$ herein shown corresponds to a potential $V(x) \propto x^2$, i.e., $\rho = 2$). For the special point ($\theta \to 1$, $\eta \to 0$), there are three different solutions for q, namely $q = 1$, $q = 1 + \alpha$, and $q = (1 + BD)/(1 + BD/2)$ (see the text).

To summarize, the *nonlinear inhomogeneous* FP process given by Eq. (3) has, as a stationary solution for any confining potential $V(x)$, a probability distribution given by the q-exponential $P_q(V)$, with finite values of q and β (see Fig. 1 and Table 1). These results exhibit that we may retain long memory (basically, the probability distribution longstandingly maintains memory of its form at $t = 0$) as a unified mesoscopic mechanism consistent with nonextensive statistical mechanics.

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REFERENCES

[1] C. Anteneodo and C. Tsallis, J. Math. Phys. **44**, 5194 (2003).
[2] M.A. Fuentes and M.O. Caceres, Phys. Lett. A **372**, 1236 (2008). To compare with the present paper, it must be done $q \to 2 - q$.
[3] L. Borland, Phys. Lett. A **245**, 67 (1998).
[4] A.R. Plastino and A. Plastino, Physica A **222**, 347 (1995).
[5] C. Tsallis and D.J. Bukman, Phys. Rev. E **54**, R2197 (1996).
[6] V. Schwammle, E.M.F. Curado and F.D. Nobre, Eur. Phys. J. B **58**, 159-165 (2007).
[7] V. Schwammle, F.D. Nobre and E.M.F. Curado, Phys. Rev. E **76**, 041123 (2007).
[8] B.C.C. dos Santos and C. Tsallis, Phys. Rev. E **82**, 061119 (2010).
[9] J. S. Andrade Jr., G.F.T. da Silva, A.A. Moreira, F.D. Nobre and E.M.F. Curado, Phys. Rev. Lett. **105**, 260601 (2010).

[10] L. Borland, F. Pennini, A. R. Plastino and A. Plastino European Phys. Journal B **12**, 285 (1999).

[11] S. Umarov, C. Tsallis and S. Steinberg, Milan J. Math. **76**, 307 (2008); for a more pedagogical version, see S.M.D. Queiros and C. Tsallis, AIP Conference Proceedings **965**, 21 (New York, 2007). See also S. Umarov, C. Tsallis, M. Gell-Mann and S. Steinberg, J. Math. Phys. **51**, 033502 (2010); C. Vignat and A. Plastino, J. Phys. A **40**, F969 (2007); M.G. Hahn, X.X. Jiang and S. Umarov, J. Phys. A **43**, 165208 (2010).

[12] C. Tsallis, *J. Stat. Phys.* **52**, 479 (1988).

[13] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics - Approaching a Complex World* (Springer, New York, 2009).

[14] O. Penrose, *Foundations of Statistical Mechanics: A Deductive Treatment* (Pergamon, Oxford, 1970), p. 167.

[15] C. Tsallis, in *Nonextensive Entropy - Interdisciplinary Applications*, eds. M. Gell-Mann and C. Tsallis (Oxford University Press, New York, 2004).

[16] C. Tsallis, M. Gell-Mann and Y. Sato, Proc. Nat. Acad. Sci. **102**, 15377 (2005).

[17] F. Caruso and C. Tsallis, Phys. Rev. E **78**, 021101 (2008).

[18] A. Saguia and M.S. Sarandy, Phys. Lett. A **374**, 3384 (2010).

[19] A. Pluchino, A. Rapisarda and C. Tsallis, *Eur. Phys. Lett.* **80**, 26002 (2007), and Physica A **387**, 3121 (2008).

[20] C. Anteneodo and C. Tsallis, Phys. Rev. Lett. **80**, 5313 (1998).

[21] A. Upadhyaya, J.-P. Rieu, J.A. Glazier and Y. Sawada, Physica A **293**, 549 (2001).

[22] S. Thurner, N. Wick, R. Hanel, R. Sedivy and L.A. Huber, Physica A **320C**, 475 (2003).

[23] K.E. Daniels, C. Beck and E. Bodenschatz, Physica D **193**, 208 (2004).

[24] L.F. Burlaga and A.F.Vinas, Physica A **356**, 375 (2005).

[25] L.F. Burlaga, A.F. Vinas, N.F. Ness and M.H. Acuna, Astrophys. J. **644**, L83 (2006).

[26] P. Douglas, S. Bergamini and F. Renzoni, Phys. Rev. Lett. **96**, 110601 (2006).

[27] B. Liu and J. Goree, Phys. Rev. Lett. **100**, 055003 (2008).

[28] F.D. Nobre, M.A. Rego-Monteiro and C. Tsallis, Phys. Rev. Lett. **106**, 140601 (2011).

[29] U. Tirnakli, C. Beck and C. Tsallis, Phys. Rev. E **75**, 040106(R) (2007); U. Tirnakli, C. Tsallis and C. Beck, Phys. Rev. E **79**, 056209 (2009); C. Tsallis and U. Tirnakli, J. Phys. Conf. Series **201**, 012001 (2010).

FOKKER -PLANCK	**Linear** ($\eta \to 0$)	**Nonlinear** ($\eta \neq 0$)
Homogeneous ($\theta = 0$)	$q = 1$	$q = 1 - \eta$
Inhomogeneous ($\theta \neq 0$)	If $\eta \sim \alpha(\theta - 1)^\delta$ and $\theta \to 1$, $q = 1$ for $\delta > 1$ $q = 1 + \alpha$ for $\delta = 1$ If η is strictly 0 and $\theta \to 1$, $q = \frac{2(1+BD)}{2+BD}$	$q = 1 + \frac{\eta}{\theta - 1}$

Table 1: Nonlinear inhomogeneous Fokker-Planck equation with q-exponential stationary-state distribution. The present $\theta = 0$ result recovers that of Eqs. (2.8) and (2.9) of [4] with the notation correspondence (F, Q, D, K) in [4] $\leftrightarrow$ (P, D, N_q, F) here. It also recovers that of [5] by using there $\mu = 1$ and $\nu = 1 + \eta$. The result presented by the isolated solution at $(\theta, \eta) = (1, 0)$ recovers that cited (for the Itô approach) in [1] with the notation correspondence $(\frac{M}{\tau})$ in [1] $\leftrightarrow$ $(\frac{BD}{4})$ here. A comparison is also possible through Eq. (11) of [3] by using the notation correspondence $(K, D, U(x))$ in [3] $\leftrightarrow$ $(F, Dg(V)^\theta, V(x))$ here.