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Multidimensional Cosmology and Gravitation

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Contents

Chapter 1

Classical Solutions in Multidimensional Cosmology

1. Multidimensional Cosmology with Multicomponent Perfect Fluid

1.1. Introduction. The model

Multidimensional gravitation and cosmology (see, for example [1-21] and references therein) is a very interesting object of investigations both from physical and mathematical points of view. Here we continue the study of such models started in [21].

Last decade the interest in multidimensional cosmology was stimulated mainly by the Kaluza-Klein and superstring paradigmas [22,23]. The "realistic" multidimensional cosmological models appeared mainly in a context of some unifications theories. Certainly, it is quite natural to believe that the Entire Universe is multidimensional and we live in a some sort of a (3+1)- dimensional layer, that is Our Universe. Of course, at first stage we should try to understand the structure of our 3-dimensional crude (dense) matter and the formation of Our Universe. But it seems to be very likely that at some stage of our development it will be just impossible to describe our (3+1)-dimensional layer (Our Universe) out of touch with other (multidimensional) layers and domains.

A large variety of multidimensional cosmological models is described by pseudo-Euclidean Toda-like systems [19] (see formula (1.1.10) below). These systems are not well studied yet. We note, that the Euclidean Toda-like systems are more or less well studied [24-28] (at least for certain sets of parameters, associated with finite-dimensional Lie algebras or affine Lie algebras). There is also a criterion of integrability by quadrature (algebraic integrability) for these (Euclidean) systems established by Adler and van Moerbeke [28]. Nevertheless, there are some indications that cosmological models may contain rather rich mathematical structures. For example, a self-dual reduction of the Bianchi-IX cosmology [29] leads us to the Halphen system of ordinary differential equations [30]. This system may be integrated in terms of modular forms [31] and is connected with a certain integrable reduction of the self-dual Yang-Mills equation [32] (with the infinite-dimensional group $SDiffSU(2)$). Another example is connected with the Kaluza-Klein dyon solution from [33]. The field equations for a spherically-symmetric Kaluza-Klein dyon in 5-dimensions were reduced in [33] to an open (Euclidean) Toda lattice with three points. Certainly, this problem may be formulated in terms of an appropriate cosmological model described by a pseudo-Euclidean Toda- like Lagrangian. So, we are led to an interesting nontrivial example of an integrable cosmological model.

In this lectures we consider a cosmological model describing the evolution of n Einstein spaces in the presence of m-component perfect-fluid matter. The metric of the model

$$g = -exp[2\gamma(t)]dt \otimes dt + \sum_{i=1}^{n} exp[2x^i(t)]g^{(i)}, \tag{1.1}$$

is defined on the manifold

$$M = R \times M_1 \times \ldots \times M_n, \tag{1.2}$$

where the manifold M_i with the metric $g^{(i)}$ is an Einstein space of dimension N_i, i.e.

$$R_{m_i n_i}[g^{(i)}] = \lambda^i g^{(i)}_{m_i n_i}, \tag{1.3}$$

$i = 1, \ldots, n$; $n \geq 2$. The energy-momentum tensor is adopted in the following form

$$T^M_N = \sum_{\alpha=1}^{m} T^{M(\alpha)}_N, \tag{1.4}$$

$$(T^{M(\alpha)}_N) = diag(-\rho^{(\alpha)}(t), p^{(\alpha)}_1(t)\delta^{m_1}_{k_1}, \ldots, p^{(\alpha)}_n(t)\delta^{m_n}_{k_n}). \tag{1.5}$$

$\alpha = 1, \ldots, m$, with the conservation law constraints imposed:

$$\bigtriangledown_M T^{M(\alpha)}_N = 0 \tag{1.6}$$

$\alpha = 1, \ldots, m-1$. The Einstein equations

$$R^M_N - \frac{1}{2}\delta^M_N R = \kappa^2 T^M_N \tag{1.7}$$

(κ^2 is gravitational constant) imply $\bigtriangledown_M T^M_N = 0$ and consequently $\bigtriangledown_M T^{M(m)}_N = 0$.

We suppose that for any α-th component of matter the pressures in all spaces are proportional to the density

$$p^{(\alpha)}_i(t) = (1 - h^{(\alpha)}_i(x(t)))\rho^{(\alpha)}(t), \tag{1.8}$$

where

$$h^{(\alpha)}_i(x) = \frac{1}{N_i}\frac{\partial}{\partial x^i}\Phi^{(\alpha)}(x), \tag{1.9}$$

$i = 1, \ldots, n$, where $\Phi^{(\alpha)}(x)$ is a smooth function on R^n, $\alpha = 1, \ldots, m$. So, the total model is anisotropic.

In Sec. 1.2 the Einstein equations (1.1.7) for the model are reduced to the equations of motion for some Lagrange system with the energy constraint $E = 0$ imposed. When $m = 1$ and all spaces are Ricci-flat ($\lambda^i = 0$ in (1.1.3), $i = 1, \ldots, n$) such reduction was performed previously in [9].

In Sec. 1.3 we consider the Einstein equations, when all spaces are Ricci-flat and $h_i^{(\alpha)} = const$, $i = 1,\ldots,n$, $\alpha = 1,\ldots,m$. In this case we deal with pseudo-Euclidean Toda-like system with the Lagrangian

$$L_A = \frac{1}{2}G_{ij}\dot{x}^i\dot{x}^j - \sum_{\alpha=1}^{m}\kappa^2 A^{(\alpha)}\exp(u_i^{(\alpha)}x^i), \qquad (1.10)$$

where $sign(G_{ij}) = (-,+,\ldots,+)$ [14,15], $u_i^{(\alpha)} = N_i h_i^{(\alpha)}$ and $A^{(\alpha)} = const$ $i = 1,\ldots,n$, $\alpha = 1,\ldots,m$. The Einstein equations are integrated in the following cases: 1) $m = 1$; 2) $n = 2$, $m \geq 2$, $A^{(\alpha)} \neq 0$, $u^{(\alpha)} - u^{(1)} = b^{(\alpha)}u$, $\alpha = 1,\ldots,m$, where $u^2 = G^{ij}u_iu_j = 0$, $u \neq 0$; 3) $u^{(\alpha)} = b^{(\alpha)}u$, $u^2 < 0$, $A^{(\alpha)} > 0$, $\alpha = 1,\ldots,m$.

1.2. The equations of motion

The non-zero components of the Ricci-tensor for the metric (1.1.1) are following

$$R_{00} = -\sum_{i=1}^{n} N_i[\ddot{x}^i - \dot{\gamma}\dot{x}^i + (\dot{x}^i)^2], \qquad (1.2.1)$$

$$R_{m_in_i} = g^{(i)}_{m_in_i}[\lambda^i + \exp(2x^i - 2\gamma)(\ddot{x}^i + \dot{x}^i(\sum_{i=1}^{n} N_i\dot{x}^i - \dot{\gamma}))], \qquad (1.2.2)$$

$i = 1,\ldots,n$.

We put

$$\gamma = \gamma_0 \equiv \sum_{i=1}^{n} N_i x^i \qquad (1.2.3)$$

in (1.1.1) (the harmonic time is used). Then it follows from (1.2.1) and (1.2.2) that the Einstein equations (1.1.7) for the metric (1.1.1) with γ from (1.2.3) and the energy-momentum tensor from (1.1.4), (1.1.5) are equivalent to the following set of equations

$$\frac{1}{2}G_{ij}\dot{x}^i\dot{x}^j + V_c + \kappa^2\sum_{\alpha=1}^{m}\rho^{(\alpha)}\exp(2\gamma_0) = 0, \qquad (1.2.4)$$

$$\lambda^i + \ddot{x}^i\exp(2x^i - 2\gamma_0) = \kappa^2\exp(2x^i)\sum_{\alpha=1}^{m}[p_i^{(\alpha)} + (D-2)^{-1}(\rho^{(\alpha)} - \sum_{j=1}^{n} N_j p_j^{(\alpha)})], \qquad (1.2.5)$$

$i = 1,\ldots,n$. Here

$$G_{ij} = N_i\delta_{ij} - N_iN_j \qquad (1.2.6)$$

are the components of the minisuperspace metric,

$$V_c = -\frac{1}{2}\sum_{i=1}^{n}\lambda^i N_i exp(-2x^i + 2\gamma_0) \qquad (1.2.7)$$

is the potential and $D \equiv dim M = 1 + \sum_{i=1}^{n} N_i$.

The conservation law constraint (1.1.6) for $\alpha \in \{1,...,m\}$ reads

$$\dot{\rho}^{(\alpha)} + \sum_{i=1}^{n} N_i \dot{x}^i (\rho^{(\alpha)} + p_i^{(\alpha)}) = 0. \tag{1.2.8}$$

We impose the conditions of state in the form (1.1.8), (1.1.9). Then eq. (1.2.8) gives

$$\rho^{(\alpha)}(t) = A^{(\alpha)} exp[-2N_i x^i(t) + \Phi^{(\alpha)}(x(t))], \tag{1.2.9}$$

where $A^{(\alpha)} = const$ and eqs. (1.2.4), (1.2.5) may be written in the following manner

$$\frac{1}{2} G_{ij} \dot{x}^i \dot{x}^j + V_c + \kappa^2 \sum_{\alpha=1}^{m} A^{(\alpha)} \exp \Phi^{(\alpha)} = 0, \tag{1.2.10}$$

$$\lambda^i + \ddot{x}^i \exp(2x^i - 2\gamma_0) = -\kappa^2 \sum_{\alpha=1}^{m} u^i_{(\alpha)} A^{(\alpha)} \exp(2x^i - 2\gamma_0 + \Phi^{(\alpha)}), \tag{1.2.11}$$

$i = 1, \ldots, n$. In (1.2.11) we denote

$$u_i^{(\alpha)} \equiv N_i h_i^{(\alpha)} = \partial_i \Phi^{(\alpha)}, \qquad u^i_{(\alpha)} = G^{ij} u_j^{(\alpha)}, \tag{1.2.12}$$

where [15]

$$G^{ij} = \frac{\delta^{ij}}{N_i} + \frac{1}{2-D} \tag{1.2.13}$$

are the components of the matrix inverse to the matrix (G_{ij}) (1.2.6).

It is not difficult to verify that equations (1.2.11) are equivalent to the Lagrange equations for the Lagrangian

$$L = \frac{1}{2} G_{ij} \dot{x}^i \dot{x}^j - V \tag{1.2.14}$$

where

$$V = V(x) = V_c(x) + \sum_{\alpha=1}^{m} \kappa^2 A^{(\alpha)} \exp[\Phi^{(\alpha)}(x)]. \tag{1.2.15}$$

Eq. (1.2.10) is the zero-energy constraint

$$E = \frac{1}{2} G_{ij} \dot{x}^i \dot{x}^j + V = 0. \tag{1.2.16}$$

Remark 1. In terms of 1-forms $u^{(\alpha)} = u_i^{(\alpha)} dx^i$, the relations (1.1.9) read: $u^{(\alpha)} = d\Phi^{(\alpha)}$, $\alpha = 1, \ldots, m$. In this case

$$du^{(\alpha)} = 0, \tag{1.2.17}$$

$\alpha = 1, \ldots, m$. The set of eqs. (1.2.17) (on R^n) is equivalent to (1.1.9). An open problem is to generalize the considered here formalism for the following cases: a) $du^{(\alpha)} \neq 0$ for some $\alpha \in \{1, \ldots, m\}$; b) $du^{(\alpha)} = 0$ for all $\alpha = 1, \ldots, m$, but $u^{(\alpha)}$ are defined on an open submanifold $\Omega \in R^n$ with the non-trivial cohomology group $H^1(\Omega, R) \neq 0$.

Using eqs. (1.2.1) and (1.2.2), it is not difficult to verify that the Einstein equations (1.1.7) for the metric (1.1.1) and the energy-momentum tensor from (1.1.4), (1.1.5), (1.1.8), (1.1.9) are equivalent to the Lagrange equations for the following degenerate Lagrangian (see also [15])

$$L = \frac{1}{2}\exp(-\gamma + \gamma_0(x))G_{ij}\dot{x}^i\dot{x}^j - \exp(\gamma - \gamma_0(x))V(x) \tag{1.2.18}$$

$(L = L(\gamma, x, \dot{x}))$. Fixing the gauge

$$\gamma = \gamma_0(x) - 2f(x), \tag{1.2.19}$$

where $f = f(x)$ is a smooth function on R^n, we get the Lagrangian

$$L_f = \frac{1}{2}\exp(2f(x))G_{ij}\dot{x}^i\dot{x}^j - \exp(-2f(x))V(x). \tag{1.2.20}$$

For $f = 0$ we have the harmonic-time gauge (1.2.3). The set of Lagrange equations for the Lagrangian (1.2.18) (or equivalently the set of the Einstein equations) with γ from (1.2.19) is equivalent to the set of Lagrange equations for the Lagrangian (1.2.20) with the energy constraint imposed

$$E_f = \frac{1}{2}\exp(2f(x))G_{ij}\dot{x}^i\dot{x}^j + \exp(-2f(x))V(x) = 0. \tag{1.2.21}$$

Remark 2. We remind that the action of the relativistic particle of mass m, moving in the pseudo-Euclidean background space with the metric $\hat{G}_{ij}(x)$ has the following form

$$S = \int d\tau[\hat{G}_{ij}(x(\tau))\frac{\dot{x}^i\dot{x}^j}{2e(\tau)} - \frac{m^2}{2}e(\tau)], \tag{1.2.22}$$

where $e = e(\tau)$ is 1-bein. Comparing (1.2.18) and (1.2.22), we find that for $V(x) > 0$ the cosmological model (1.2.18) is equivalent to the model of relativistic particle with the mass $m = 1$, moving in the conformally-flat (pseudo-Euclidean) space with the metric $\hat{G}_{ij}(x) = 2V(x)G_{ij}$). In this case $e = 2V(x)\exp(\gamma - \gamma_0(x))$. For $V(x) < 0$ we have a tachyon. The problem may be also reformulated in terms of a geodesic-flow problem for conformally-flat metric (this follows from (1.2.22) or from a more general scheme).

1.3. Classical solutions

Now, we consider the following case: $\lambda^i = 0$ (all spaces are Ricci-flat), $u_i^{(\alpha)} = N_i h_i^{(\alpha)} = const$, $i = 1, \ldots, n$. Then $V_c = 0$ and we put $\Phi^{(\alpha)} = u_i^{(\alpha)}x^i$ in (1.2.15). In this case the Lagrangian (1.2.14) has the form (1.1.10).

Remark 3. The curvature induced term V_c (1.2.7) may be generated in the framework of the model with the Ricci-flat spaces M_i by the addition of n new components of the perfect fluid with $u_i^{(k)} = 2N_i - 2\delta_i^k$ and $\kappa^2 A^{(k)} = -\lambda^k N_k/2$, $i, k = 1, \ldots, n$. The introduction of the cosmological constant Λ into the model is equaivalent to the addition of a new component with $u_i^{(n+1)} = 2N_i$ and $\kappa^2 A^{(n+1)} = \Lambda$.

One-component matter

We consider the case $m = 1$, $A^{(1)} = A \neq 0$. We denote $h_i^{(1)} = h_i$, $u_i^{(1)} = u_i = N_i h_i$.

We remind [14, 15] that the minisuperspace metric

$$G = G_{ij} dx^i \otimes dx^i \tag{1.3.1}$$

has pseudo-Euclidean signature $(-, +, \ldots, +)$, i.e. there exist a linear transformation

$$z^a = V_i^a x^i, \tag{1.3.2}$$

diagonalizing the minisuperpace metric (1.3.1)

$$G = \eta_{ab} dz^a \otimes dz^b = -dz^0 \otimes dz^0 + \sum_{i=1}^{n-1} dz^i \otimes dz^i, \tag{1.3.3}$$

where

$$(\eta_{ab}) = (\eta^{ab}) \equiv diag(-1, +1, \ldots, +1), \tag{1.3.4}$$

$a, b = 0, \ldots, n-1$.

Proposition 1. For any $u = (u_i) \in R^n$, $u \neq 0$, there exists a (nondegenegate) $n \times n$ matrix (V_i^a) such that

$$\eta_{ab} V_i^a V_j^b = G_{ij} \tag{1.3.5}$$

and a) $V_i^0 = u_i/\sqrt{-u^2}$, for $u^2 < 0$; b) $V_i^1 = u_i/\sqrt{u^2}$, for $u^2 > 0$; c) $V_i^0 + V_i^1 = u_i$, for $u^2 = 0$;

Here and below $(u = (u_i) = (N_i h_i))$

$$u^2 \equiv u_i u^i = G^{ij} u_i u_j = \sum_{i=1}^{n} N_i (h_i)^2 + \frac{1}{2-D} (\sum_{i=1}^{n} N_i h_i)^2. \tag{1.3.6}$$

(We note that in notations of [14] $u^2 = \acute{\Delta}(h)/(2-D)$.)

This proposition follows from the fact that $< u, v > \equiv G^{ij} u_i v_j$ is bilinear symmetric 2-form of signature $(-, +, \ldots, +)$ and the following quite obvious.

Proposition 2. Let $v \in E = R^n$, $n \geq 2$, and $< .,. >$: $E \times E \longrightarrow R$ is a bilinear symmetric 2-form of signature $(-, +, \ldots, +)$. Then there exists a basis $v^0, \ldots, v^{n-1}$ in E, such that $< v^a, v^b >= \eta^{ab}$ and a) $v = v^0$, b) $v = v^1$, c) $v = v^0 + v^1$, in the cases: a) $v^2 \equiv < v, v > = -1$, b) $v^2 = 1$, c) $v^2 = 0$ respectively.

Let $u \neq 0$. In $z = (z^a)$-coordinates (1.3.2) with the matrix (V_i^a) from the Proposition 1 the Lagrangian (1.2.14) has the following form

$$L_A = \frac{1}{2} \eta_{ab} \dot{z}^a \dot{z}^b - V_A = -\frac{1}{2} (\dot{z}^0)^2 + \sum_{i=1}^{n-1} \frac{1}{2} (\dot{z}^i)^2 - V_A, \tag{1.3.7}$$

where

$$V_A = \kappa^2 A \exp(2qz^0), \qquad u^2 < 0, \tag{1.3.8}$$

$$= \kappa^2 A \exp(2qz^1), \qquad u^2 > 0, \tag{1.3.9}$$

$$= \kappa^2 A \exp(z^0 + z^1), \qquad u^2 = 0, \tag{1.3.10}$$

is the potential (1.2.15). Here we denote

$$2q \equiv \sqrt{|u^2|}. \tag{1.3.11}$$

The Lagrange equations for the Lagrangian (1.3.7)

$$\ddot{z}^a = -\eta^{ab}\partial_b V_A \tag{1.3.12}$$

with the energy constraint (1.2.16)

$$E_A = \frac{1}{2}\eta_{ab}\dot{z}^a\dot{z}^b + V_A = 0, \tag{1.3.13}$$

can be easily solved. We present the solutions.

a) For $u^2 < 0$

$$z^i = p^i t + q^i, \qquad i = 1, \ldots, n-1, \tag{1.3.14}$$

$$2qz^0 = y(t), \tag{1.3.15}$$

where p^i and q^i are constants and

$$y(t) = \ln[C/D \sinh^2(\frac{1}{2}\sqrt{C}(t-t_0))], C \neq 0, D > 0, \tag{1.3.16}$$

$$= \ln[4/D(t-t_0)^2], \qquad C = 0, D > 0, \tag{1.3.17}$$

$$= \ln[-C/D \cosh^2(\frac{1}{2}\sqrt{C}(t-t_0))], C > 0, D < 0, \tag{1.3.18}$$

Here t_0 is an arbitrary constant, $D = -2u^2\kappa^2 A$, $C = -u^2(\vec{p})^2$ and $(\vec{p})^2 = \sum_{i=1}^{n-1}(p^i)^2$.

b) For $u^2 > 0$ we have

$$z^i = p^i t + q^i, \qquad i = 0, 2, \ldots, n-1, \tag{1.3.19}$$

$$2qz^1 = y(t), \tag{1.3.20}$$

with $(\vec{p})^2 = (p^0)^2 - \sum_{i=2}^{n-1}(p^i)^2$ in (1.3.15)-(1.3.18).

c) $u^2 = 0$, $u \neq 0$. In this case

$$z^i = p^i t + q^i, \qquad i = 2, \ldots, n-1, \tag{1.3.21}$$

$$z^+ = z^0 + z^1 = p^+ t + q^+, \tag{1.3.22}$$

$$z^- = z^0 - z^1 = p^- t + q^- + \kappa^2 A z(t), \tag{1.3.23}$$

where for $p^+ \neq 0$

$$z(t) = 2(p^+)^{-2}\exp(p^+ t + q^+), \qquad p^+ p^- = (\vec{p})^2 \tag{1.3.24}$$

($p^- = 0$ for $n = 2$) and for $p^+ = 0$

$$z(t) = t^2 \exp q^+, \qquad (\vec{p})^2 + 2\kappa^2 A \exp q^+ = 0. \tag{1.3.25}$$

Here $(\vec{p})^2 = \sum_{i=2}^{n-1}(p^i)^2$.

For $u = 0$ we have

$$z^a = p^a t + q^a, \qquad a = 0, \ldots, n-1, \tag{1.3.26}$$
$$\frac{1}{2}\eta_{ab} p^a p^b + \kappa^2 A = 0. \tag{1.3.27}$$

Kasner-like parametrization. Here we consider the case $u^2 < 0$, $A \neq 0$. For $C = -u^2(\vec{p})^2 > 0$ we reparametrize the time variable

$$\tau = \frac{T}{\sqrt{\varepsilon}} \ln \frac{\exp(\sqrt{C}(t-t_0)) + \sqrt{\varepsilon}}{\exp(\sqrt{C}(t-t_0)) - \sqrt{\varepsilon}}, \tag{1.3.28}$$

where

$$\varepsilon \equiv A/|A| = \pm 1, \qquad T \equiv (2/\kappa^2 |A||u^2|)^{1/2}. \tag{1.3.29}$$

We introduce new (Kasner-like) parameters

$$\alpha^i \equiv -2V^i_s p^s / \sqrt{-u^2(\vec{p})^2}, \tag{1.3.30}$$

where $(V^i_a) = (V^a_i)^{-1}$ and the summation parameter s runs: $s = 1, \ldots, n-1$. Then, due to relations (1.3.2), (1.3.5), (1.3.14)-(1.3.16), (1.3.18) and Proposition 1 we get the following expression for the metric (1.1.1) [40]

$$g = -(\prod_{i=1}^{n} (a_i(\tau))^{2N_i - u_i}) d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau) g^{(i)}, \tag{1.3.31}$$

where

$$a_i(\tau) = A_i [\frac{\sinh(\tau\sqrt{\varepsilon}/T)}{\sqrt{\varepsilon}}]^{2u^i/u^2} [\frac{\tanh(\tau\sqrt{\varepsilon}/2T)}{\sqrt{\varepsilon}}]^{\alpha^i}, \tag{1.3.32}$$

$i = 1, \ldots, n$; $A_i > 0$ are constants and the parameters α^i satisfy the relations

$$u_i \alpha^i = 0, \tag{1.3.33}$$
$$G_{ij} \alpha^i \alpha^j = -4/u^2 \tag{1.3.34}$$

(see Proposition 1 and (3.30)). For the density (2.15) we have

$$\rho(\tau) = A \prod_{i=1}^{n} (a_i(\tau))^{u_i - 2N_i}. \tag{1.3.35}$$

We note, that $(\vec{p})^2 = 2\kappa^2 |A| \prod_{i=1}^{n} A_i^{u_i}$

For $A > 0$ we have an exceptional solution (1.3.31), (1.3.33), (1.3.34) with the scale factors

$$a_i(\tau) = \bar{A}_i \exp(\pm 2u^i \tau / u^2 T), \tag{1.3.36}$$

$\bar{A}_i > 0$, $i = 1, \ldots, n$. This solution correspond to $C = 0$ case (1.3.17).

Remark 4. In [19] the Einstein equations (1.2.10), (1.2.11) were solved for $A^{(\alpha)} = 0$, $\alpha = 1, \ldots, m$, $\lambda^1 \neq 0$, $\lambda^i = 0$, $i > 1$. The solutions [19] may be also obtained from the formulas (1.3.31)-(1.3.34). We note that the spherically-symmetric analogue of the solution [19] was considered in [36] (the case $d = 2$ was considered previously in [35]). There exists an interesting special case of the solutions [35, 36]. It is the n-time generalization of the Schwarzschild solution

$g = -[(1-\frac{L}{R})^A]_{ab} dt^a \otimes dt^b + (1-\frac{L}{R})^{-spA} dR \otimes dR + (1-\frac{L}{R})^{1-spA} R^2 d\Omega^2$, where $L \neq 0$ and $A = (A_{ab})$ is symmetric $n \times n$ matrix, satisfying the relation $sp(A^2) + (spA)^2 = 2$.

We consider this solution in a separate publication.

Two spaces with m-component matter

Here we consider the following case: $n = 2,\ m \geq 2,\ A^{(\alpha)} \neq 0$,

$$u^{(\alpha)} - u^{(1)} = b^{(\alpha)} u \tag{1.3.37}$$

$\alpha = 1, \ldots, m$, where $u^2 = 0,\ u \neq 0$ and $b^{(\alpha)}$ are constants.

In z-coordinates (1.3.2), where the matrix (V_i^a) satisfies the Proposition 1 (see the case c) $u^2 = 0$) we have

$$z^+ = z^0 + z^1 = (V_i^0 + V_i^1) x^i = u_i x^i, \tag{1.3.38}$$

$$\Phi^{(1)} = u_i^{(1)} x^i = \alpha_+ z^+ + \alpha_- z^-, \tag{1.3.39}$$

where $2\alpha_+ = -< u^1, u^* >$, $2\alpha_- = -< u^1, u >$, and $u^* = (u_i^*)$ is defined by the relation : $u_i^* x^i = z^-$ (or equivalently $< u^*, u^* >= 0,\ < u^*, u >= -2$).

Due to (1.3.37)-(1.3.39) the potential in (1.1.10) is factorized

$$V = V_+(z^+) V_-(z^-), \tag{1.3.40}$$

where

$$V_+(z^+) = \exp(\alpha_+ z^+)(\kappa^2 A^{(1)} + \sum_{k=2}^{m} \kappa^2 A^{(\alpha)} \exp(b^{(\alpha)} z^+), \tag{1.3.41}$$

$$V_-(z^-) = \exp(\alpha_- z^-). \tag{1.3.42}$$

Let $A^{(\alpha)} > 0,\ \alpha = 1, \ldots, m,$. We consider the f-gauge (1.2.19) with

$$F = e^{2f} = V. \tag{1.3.43}$$

In this gauge the Lagrangian (1.2.20) reads

$$L_f = -\frac{1}{2} V_+(z^+) \dot{z}^+ V_-(z^-) \dot{z}^- - 1. \tag{1.3.44}$$

In the variables

$$w^\pm = w^\pm(z^\pm) = \int_{z_0}^{z^\pm} dx V_\pm(x) \tag{1.3.45}$$

the Lagrangian (1.3.44) has a rather simple form

$$L_f = -\frac{1}{2} \dot{w}^+ \dot{w}^- - 1. \tag{1.3.46}$$

The equations of motion for (1.3.46) give

$$w^{\pm}(t) = p^{\pm}t + q^{\pm}. \tag{1.3.47}$$

The parameters $p^{\pm}$ satisfy the energy constraint

$$2E_f = -p^{+}p^{-} + 2 = 0. \tag{1.3.48}$$

Remark 5. It is interesting to note that the so-called D-dimensional Schwarzschild-deSitter solution [44,45] may be obtained from the considered here cosmological solution with $n = m = 2$ and $N_1 = 1$, $N_2 = D - 2$.

n spaces with m component matter

Now we consider the simplest case of the multicomponent matter. We put in (1.1.10) $n \geq 2$, $A^{(\alpha)} > 0$, $u^{(\alpha)} = b^{(\alpha)}u$, $u^2 < 0$, where $b^{(\alpha)}$ are constants, $\alpha = 1, \ldots, m$.

In z-coordinates (1.3.2), corresponding to the case a) from the Proposition 1, the Lagrangian (1.1.10) has the form (1.3.7) with the potential

$$V_A = V_A(z^0) = \sum_{i=1}^{m} \kappa^2 A^{(\alpha)} \exp(2qb^{(\alpha)}z^0), \tag{1.3.49}$$

where q is defined in (1.3.11) ($A = (A^{(\alpha)})$). The solutions of the equations (1.3.12) and (1.3.13) are expressed by the formula (1.3.14) and the following relation

$$\int_{c_0}^{z^0} dx[2\mathcal{E} + 2V_A(x)]^{-1/2} = \pm(t - t_0), \tag{1.3.50}$$

where $2\mathcal{E} = \sum_{i=1}^{n-1}(p^i)^2$, and c_0, t_0 are constants.

1.4. Concluding remarks

In this section we investigated the multidimensional cosmological model with n ($n > 1$) Ricci-flat spaces, filled by m-component perfect fluid. In some sense, this model may be considered as "universal" cosmological model: a lot of cosmological models may be obtained from it under a suitable choice of parameters. This fact may be used for "Toda-like" classification of known exact cosmological (and spherically-symmetric) solutions of the Einstein equations. (We note, that the Bianchi-IX cosmological model is described by the "Toda-like" Lagrangian (1.10) with $n = 3$ and $m = 6$.)

Here we integrated the Einstein equations for some sets of parameters. But an open problem is the problem of integrability of the considered here model (at classical and quantum levels) for arbitrary values of the parameters m, n, N_i and $u_i^{(\alpha)}$. We hope to continue the investigation of this problem in forthcoming publications.

2. Multidimensional Cosmology with Multicomponent Perfect Fluid and Toda Lattices

2.1. Introduction

We consider dynamical systems with $n \geq 2$ degrees of freedom described by the Lagrangian

$$L = \frac{1}{2}\sum_{i,j=1}^{n} \eta_{ij}\dot{x}^i\dot{x}^j - \sum_{s=1}^{m} a^{(s)} \exp[\sum_{i=1}^{n} b_i^s x^i], \quad m \geq 2. \tag{2.1.1}$$

A lot of systems in gravitation [33,44] and as well in multidimensional cosmology [1-21,45-48] reduce to the systems with such a Lagrangian.

Without loss of generality it can be assumed that matrix (η_{ij}) is diagonalized and $\eta_{ii} = \pm 1$ for $i = 1,\ldots,m$. Such system is an algebraic generalization of a well-known Toda lattice [24,39] suggested by Bogoyavlensky [25,40]. We say that it is an Euclidean Toda-like system, if bilinear form of kinetic energy is positively definite, i.e. $\eta_{ij} = \delta_{ij}$. Nearly nothing is known about Euclidean Toda-like systems with arbitrary sets of vectors $b_1,\ldots,b_m$, where $b_s = (b_1^s,\ldots,b_n^s)$ for $s = 1,\ldots,m$. But, if they form the set of admissible roots of a simple Lie algebra, then the system is completely integrable and possesses a Lax representation. Remind, that the set of roots $\alpha_1,\ldots,\alpha_m$ is called admissible [25,40], provided vectors $\alpha_r - \alpha_s$ are not roots for all $r,s = 1,\ldots,m$. Each simple Lie algebra possesses the following set of admissible roots

$$\omega_1,\ldots,\omega_n,-\Omega \tag{2.1.2}$$

where $\omega_1,\ldots,\omega_n$ are simple roots and Ω is the maximal root [41] (usually $\Omega = \omega_1+\ldots+\omega_n$). Any subset of the set (2.1.2) is also admissible.

If the maximal root holds in the set (2.1.2), then generalized periodic Toda lattices arise. The different $L-A$ pairs for them were found by Bogoyavlensky [25,40]. There were also presented the Hamiltonians of this systems connected with simple Lie algebras.

The further progress in this field was attained by a number of authors (see, for example, [26-28,42,43] and refs. therein). In ref. [28] Adler and van Moerbeke established a criterion of algebraic complete integrability for Euclidean Toda-like systems. (This criterion was formally applied to multidimensional vacuum cosmology with n Einstein spaces in [19].) The explicit integration of the equations of motion for the generalized open Toda lattices (in this case the maximal root is thrown away) was developed by Olshanetsky and Perelomov [27] and Kostant [26]. (See also [42].)

Here we are interested in the problem of integrability of the Toda-like systems with the indefinite bilinear form of the kinetic energy. Let us call such systems pseudo-Euclidean Toda-like systems. To our knowledge, this problem has not been discussed intensively in the mathematical literature before. The reason, as it seems to us, consists in the following. If one try to connect a pseudo-Euclidean Toda-like system by the known manner with simple Lie algebra it reduces to an Euclidean system for the part of coordinates (see Sect. 2.4). Nevertheless, integrable pseudo-Euclidean Toda-like systems and search for their solutions in explicit form evoke a special interest, because such systems arise in cosmology. For instance, 4-dimensional vacuum homogeneous cosmological model of Bianchi IX-type is described by the Lagrangian (2.1.1) with $(\eta_{ij}) = \mathrm{diag}(-1,+1,+1)$ [25,40]. (In [31] it was shown, that this model has a rather rich mathematical structure.)

So, in this section we study integrable pseudo-Euclidean Toda-like systems appearing in multidimensional cosmology. This trend in the modern theoretical physics has appeared within the new paradigm based on unified theories and hypothesis of additional space-time dimensions. According to this hypothesis the physical space-time manifold has the topology $M^4 \times B$, where M^4 is a 4-dimensional manifold, and B is a so-called internal space (or spaces). Nonobservability of additional dimensions is attained in multidimensional cosmology by spontaneous or dynamical compactification of internal spaces to the Planck scale (10^{-33} cm.). Integrable cosmological models are of great interest, because the exact solutions allow to study dynamical properties of the model, in particular compactification of internal spaces, in detail.

In the Sect. 2.2, as in [37], we consider the cosmological model where multidimensional space-time manifold M is a direct product of the time axis R and of the n Einstein spaces $M_1, \ldots, M_n$. We remind, that any manifold of constant curvature is the Einstein one. It is shown that Einstein equations for the scale factors with a source in the multicomponent perfect fluid form correspond to the Lagrangian (2.1.1) with $(\eta_{ij}) = \mathrm{diag}(-1,+1,\ldots,+1)$. We develop the integration procedure to the case of an orthogonal set of vectors $b_1,\ldots,b_m$ in Sect. 2.3. Sect. 2.4 is devoted to the reduction of pseudo-Euclidean Toda-like system to the Euclidean one for a part of coordinates. This reduction allows us to obtain the class of the exact solutions for some nonorthogonal sets of the vectors $b_1,\ldots,b_m$. We present the exact solution in the simplest case, when the reducible pseudo-Euclidean system is connected with the Lie algebra A_2. Discussion of results is presented is Sect. 2.5. We single out some interesting solutions, in particular, Euclidean wormholes.

We denote by n the number of Einstein spaces and by m the number of the matter components. Indices i and j run from 1 to n. Index s runs from 1 to m.

2.2. The model

Here we consider a cosmological model describing the evolution of $n \geq 2$ Einstein spaces in the presence of m-component perfect-fluid matter [37] as in Sect. 1.1 with $h_i(x) = const$ (see (1.1.9)). Then

$$\rho^{(\alpha)}(t) = A^{(\alpha)} \exp[-2\gamma_0 + \sum_{i=1}^{n} u_i^{(\alpha)} x^i]. \tag{2.2.1}$$

where $A^{(\alpha)} = \mathrm{const}$ and

$$u_i^{(\alpha)} = N_i h_i^{(\alpha)}. \tag{2.2.2}$$

The Einstein eqs. (1.1.7) may be written in the following manner

$$\frac{1}{2} \sum_{i,j=1}^{n} G_{ij} \dot{x}^i \dot{x}^j + V = 0, \tag{2.2.3}$$

$$\lambda^i + \ddot{x}^i \exp[2x^i - 2\gamma_0] = -\kappa^2 \sum_{\alpha=1}^{m} u^i_{(\alpha)} A^{(\alpha)} \exp[2x^i - 2\gamma_0 + \sum_{j=1}^{n} u_j^{(\alpha)} x^j]. \tag{2.2.4}$$

Here

$$G_{ij} = N_i\delta_{ij} - N_i N j \tag{2.2.5}$$

are the components of the minisuperspace metric,

$$V = -\frac{1}{2}\sum_{i=1}^{n}\lambda^i N_i \exp[-2x^i + 2\gamma_0] + \kappa^2 \sum_{\alpha=1}^{m} A^{(\alpha)} \exp[\sum_{i=1}^{n} u_i^{(\alpha)} x^i]. \tag{2.2.6}$$

We denote

$$u^i_{(\alpha)} = \sum_{j=1}^{n} G^{ij} u_j^{(\alpha)}, \tag{2.2.7}$$

where

$$G^{ij} = \frac{\delta^{ij}}{N_i} + \frac{1}{2-D} \tag{2.2.8}$$

are the components of the matrix inverse to (G_{ij}) [15].

It is not difficult to verify that eqs. (2.2.14) are equivalent to the Lagrange-Euler eqs. for the Lagrangian

$$L = \frac{1}{2}\sum_{i,j=1}^{n} G_{ij}\dot{x}^i\dot{x}^j - V. \tag{2.2.9}$$

Eq. (2.2.3) is the zero-energy constraint.

We note, that in the framework of our model the curvature induced terms in the potential (2.2.6) may be considered as additional components of the perfect fluid. The introduction of the cosmological constant Λ into the model is equivalent also to the addition of a new component with $u_i = 2N_i$ and $\kappa^2 A = \Lambda$.

Finally, we present the potential (2.2.6) modified by introduction of Λ-term in the following form

$$\begin{aligned} V = \quad & \sum_{k=1}^{n}(-\frac{1}{2}\lambda^k N_k)\exp[\sum_{i,j=1}^{n} G_{ij} v^i_{(k)} x^j] + \\ & \sum_{\alpha=1}^{m} \kappa^2 A^{(\alpha)} \exp[\sum_{i,j=1}^{n} G_{ij} u^i_{(\alpha)} x^j] + \Lambda \exp[\sum_{i,j=1}^{n} G_{ij} u^i x^j], \end{aligned} \tag{2.2.10}$$

where we denote:

$$v^i_{(k)} = \sum_{j=1}^{n} G^{ij} v_j^{(k)} = -2\frac{\delta^i_k}{N_k}, \quad v_j^{(k)} \equiv 2(N_j - \delta^k_j), \tag{2.2.11}$$

$$u^i = \sum_{j=1}^{n} G^{ij} u_j. \tag{2.2.12}$$

Let $<.,.>$ be a symmetrical bilinear form defined on n-dimensional real vector space R^n with the components $G_{ij} = <e_i, e_j>$ in the canonical basis $e_1, \dots e_n$. $(e_1 =$

$(1,0,\ldots,0)$ etc.) It was shown [14,15], that the bilinear form $<.,.>$ is pseudo-Euclidean one with the signature $(-,+,...,+)$. Then the Lagrangian (2.2.9) may be written as:

$$L = \frac{1}{2} < \dot{x}, \dot{x} > - \sum_{\alpha=1}^{m} a^{(\alpha)} \exp[< b_\alpha, x >]. \tag{2.2.13}$$

$(x = x^1 e_1 + \ldots + x^n e_n, \;\; x \in R^n)$. Here we denoted by m the total number of components, including curvature, perfect fluid and the cosmological term. We note, that for $m = 1$ the Lagrangian system (2.2.13) is always integrable. The exact solutions were obtained in [37]. (Some special cases were considered in [20,48].) In the present paper we consider multicomponent case: $m \geq 2$.

We say that a vector $y \in R^n$ is called time-like, space-like or isotropic, if $< y, y >$ has negative, positive or null values correspondingly. Vectors y and z are called orthogonal if $< y, z >= 0$.

2.3. Exact solutions for orthogonal sets of vectors

Let vectors $b_1, \ldots, b_m$ satisfy the conditions: 1. They are linear independent;
2. $< b_\alpha, b_\beta >= 0$ for all $\alpha \neq \beta$, i.e. the set of vectors is orthogonal.

Then $m \leq n$. It is not difficult to prove

Proposition 1. The set of vectors $b_1, \ldots, b_n$ may contain at most one isotropic vector.

Proposition 2. The set of vectors $b_1, \ldots, b_n$ may contain at most one time-like vector and, if it holds the other vectors must be space-like.

Remark 1. The additional term $a^{(0)} \exp[< b_0, x >]$ with zero-vector $b_0 = 0$ does not change the equations of motion, but changes the energy constraint (2.2.3)

$$\frac{1}{2} < \dot{x}, \dot{x} > + \sum_{\alpha=1}^{m} a^{(\alpha)} \exp[< b_\alpha, x >] + a^{(0)} = 0. \tag{2.3.1}$$

It corresponds to the perfect fluid with $h_i^{(0)} = 0$ for all $i = 1, \ldots, n$. Such a perfect fluid is called the stiff or Zeldovich matter [49]. It may be considered also as minimally coupled real scalar field [50]. We take into account this additional component by modification of the energy constraint

$$\frac{1}{2} < \dot{x}, \dot{x} > + \sum_{\alpha=1}^{m} a^{(\alpha)} \exp[< b_\alpha, x >] = E_0. \tag{2.3.2}$$

These propositions allow to split the class of exact solutions under consideration into following subclasses:

A. There are one time-like vector and at most $(n-1)$ space-like vectors.

B. There are at most $(n-1)$ space-like vectors.

C. There are one isotropic vector and at most $(n-2)$ space-like vectors (this subclass arises for $n \geq 3$).

To integrate eqs. of motion in all subclasses we consider an orthonormal basis $e'_1, \ldots, e'_n$. These vectors are such that

$$< e'_i, e'_j >= \eta_{ij}, \tag{2.3.3}$$

where we denote by η_{ij} the components of the matrix

$$(\eta_{ij}) = \mathrm{diag}(-1, +1, \ldots, +1). \tag{2.3.4}$$

Let us define coordinates of the vectors in this basis by

$$x = X^1 e'_1 + \ldots + X^n e'_n. \tag{2.3.5}$$

For these new coordinates we have

$$X^i = \eta_{ii} < e'_i, x >, \quad x^i = \sum_{k=1}^{n} t^i_k X^k, \tag{2.3.6}$$

where we denoted by t^i_k the components of a non-degenerate matrix defined by

$$e'_k = \sum_{i=1}^{n} t^i_k e_i. \tag{2.3.7}$$

Components t^i_k satisfy the relations:

$$\sum_{k,l=1}^{n} G_{kl} t^k_i t^l_j = \eta_{ij}. \tag{2.3.8}$$

Let us try to find exact solutions for subclasses A, B and C.

A. Let b_1 be a time-like vector. Then $< b_r, b_r >> 0$ for $r = 2, \ldots, m$ (in this case $m \leq n$). We choose the orthonormal basis $e'_1, \ldots, e'_n$ as

$$e'_s = b_s / | < b_s, b_s > |^{1/2}, \quad s = 1, \ldots, m. \tag{2.3.9}$$

Then we have:

$$< b_s, x >= \eta_{ss} | < b_s, b_s > |^{1/2} X^s. \tag{2.3.10}$$

The Lagrangian (2.2.13) and the energy constraint (2.3.2) for the coordinates $X^1, \ldots, X^n$ have the form

$$L = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j - \sum_{s=1}^{m} a^{(s)} \exp[\eta_{ss} | < b_s, b_s > |^{1/2} X^s], \tag{2.3.11}$$

$$E_0 = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j + \sum_{s=1}^{m} a^{(s)} \exp[\eta_{ss} | < b_s, b_s > |^{1/2} X^s]. \tag{2.3.12}$$

Lagrangian (2.3.11) leads to the set of eqs.

$$\ddot{X}^s = -| < b_s, b_s > |^{1/2} a^{(s)} \exp[\eta_{ss} | < b_s, b_s > |^{1/2} X^s], \tag{2.3.13}$$

$$\ddot{X}^{m+1} = \ldots = \ddot{X}^n = 0, \tag{2.3.14}$$

which is easily integrable. We get

$$X^s = -\eta_{ss} | < b_s, b_s > |^{-1/2} \ln[F_s^2(t - t_{0s})], \tag{2.3.15}$$

$$X^{m+1} = p^{m+1} t + q^{m+1}, \ldots, X^n = p^n t + q^n, \tag{2.3.16}$$

where we denoted

$$\begin{aligned} F_s(t-t_{0s}) \quad &= \sqrt{|a^{(s)}/E_s|}\cosh[\sqrt{|E_s < b_s, b_s > |/2}(t-t_{0s})], \quad \text{if } \eta_{ss}a^s > 0,\ \eta_{ss}E_s > 0, \\ &= \sqrt{|a^{(s)}/E_s|}\sin[\sqrt{|E_s < b_s, b_s > |/2}(t-t_{0s})], \quad \text{if } \eta_{ss}a^s < 0,\ \eta_{ss}E_s < 0, \\ &= \sqrt{|a^{(s)}/E_s|}\sinh[\sqrt{|E_s < b_s, b_s > |/2}(t-t_{0s})], \quad \text{if } \eta_{ss}a^s < 0,\ \eta_{ss}E_s > 0, \\ &= \sqrt{| < b_s, b_s > a^{(s)}|/2}(t-t_{0s}), \quad \text{if } \eta_{ss}a^s < 0,\ E_s = 0. \end{aligned} \tag{2.3.17}$$

By t_{0s}, E_{0s} $(s = 1,\ldots,m)$, $p^{m+1},\ldots,p^n$, $q^{m+1},\ldots,q^n$ we denoted the integration constants. Some of them are not arbitrary and connected by the relation

$$E_1 + \ldots + E_m + \frac{1}{2}(p^{m+1})^2 + \ldots + \frac{1}{2}(p^n)^2 = E_0. \tag{2.3.18}$$

We have for components t^i_k

$$t^i_s = b^i_s/| < b_s, b_s > |^{1/2}. \tag{2.3.19}$$

It is convenient to present the exact solution in a Kasner-like form. Kasner-like parameters are defined by

$$\alpha^i = t^i_{m+1}p^{m+1} + \ldots + t^i_n p^n, \tag{2.3.20}$$
$$\beta^i = t^i_{m+1}q^{m+1} + \ldots + t^i_n q^n. \tag{2.3.21}$$

Then for the scale factors of the spaces M_i (see (2.3.6)) we get

$$\exp[x^i] = \prod_{s=1}^{m}[F^2_s(t-t_{0s})]^{-b^i_s/<b_s,b_s>}\exp[\alpha^i t + \beta^i]. \tag{2.3.22}$$

Vectors α, $\beta \in R^n$, are defined by

$$\alpha \quad = \alpha^1 e_1 + \ldots + \alpha^n e_n, \quad \beta \quad = \beta^1 e_1 + \ldots + \beta^n e_n \tag{2.3.23}$$

satisfy the relations

$$< \alpha, \alpha >= 2(E_0 - E_1 - \ldots - E_m) \geq 0, \tag{2.3.24}$$
$$< \alpha, b_s >=< \beta, b_s >= 0, \quad s = 1,\ldots,m. \tag{2.3.25}$$

We remind that $< \alpha, \beta >= \sum_{i,j=1}^n G_{ij}\alpha^i\beta^j$.

Remark 2. If $m = n$ then $\alpha = \beta = 0$.

Remark 3. The set of constants E_0, E_s, t_{0s}, α^i and β^i is the final set. Only $2n$ constants from them are independent.

Remark 4. The subclass of the solutions may be easily enlarged. It is clear, that the addition of new component inducing a vector collinear to one of $b_1,\ldots,b_m$ leads to the integrable by quadrature model. Let us take into account the following additional terms in the Lagrangian (2.2.13)

$$-\sum_{\alpha=1}^{m(\sigma)} a^{(\sigma\alpha)}\exp[b_{(\sigma\alpha)} < b_\sigma, x >], \tag{2.3.26}$$

where $b_{(\sigma\alpha)}$ =const$\neq 0$ for $\alpha = 1, \ldots, m(\sigma)$, $1 \leq \sigma \leq m$. It is not difficult to show, that the modification of the exact solution (2.3.22) only consists in the replacement of the function $F_\sigma(t - t_{0\sigma})$ by one $F(t - t_{0\sigma})$, satisfying the quadrature

$$\int dF/\sqrt{E_\sigma F^2 - a^{(\sigma)} - \sum_{\alpha=1}^{m(\sigma)} a^{(\sigma\alpha)} F^{2(1-b_{(\sigma\alpha)})}} =< b_\sigma, b_\sigma > (t - t_{0\sigma}). \tag{2.3.27}$$

The additional components with other numbers σ may be taken into account by the same manner.

B. We have the set of m space-like vectors $b_1, \ldots, b_m$ $(m \leq n-1)$ and the orthonormal basis defined by

$$e'_{s+1} = b_s/\sqrt{< b_s, b_s >}, \quad s = 1, \ldots, m. \tag{2.3.28}$$

The Lagrangian (2.2.13) and the energy constraint (2.3.2) in terms of X-coordinates have the form

$$L = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j - \sum_{s=1}^{m} a^{(s)} \exp[\sqrt{< b_s, b_s >} X^{s+1}], \tag{2.3.29}$$

$$E_0 = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j + \sum_{s=1}^{m} a^{(s)} \exp[\sqrt{< b_s, b_s >} X^{s+1}]. \tag{2.3.30}$$

The corresponding eqs. of motion

$$\ddot{X}^1 = \ddot{X}^{m+2} = \ldots = \ddot{X}^n = 0, \tag{2.3.31}$$

$$\ddot{X}^{s+1} = -\sqrt{< b_s, b_s >} a^{(s)} \exp[\sqrt{< b_s, b_s >} X^{s+1}] \tag{2.3.32}$$

lead to the solution

$$X^1 = p^1 t + q^1, \tag{2.3.33}$$

$$X^{s+1} = \frac{-1}{\sqrt{< b_s, b_s >}} \ln[F_s^2(t - t_{0s})], \tag{2.3.34}$$

$$X^{m+2} = p^{m+2} t + q^{m+2}, \ldots, X^n = p^n t + q^n, \tag{2.3.35}$$

where functions $F_s(t - t_{0s})$ are defined by (2.3.17) (in this case all $\eta_{ss} = 1$). Some of integration constants in (2.3.33)-(2.3.35) satisfy the relation

$$E_1 + \ldots + E_m - \frac{1}{2}(p^1)^2 + \frac{1}{2}(p^{m+2})^2 + \ldots + \frac{1}{2}(p^n)^2 = E_0. \tag{2.3.36}$$

To present the scale factors in a Kasner-like form we define the parameters:

$$\alpha^i = t_1^i p^1 + t_{m+2}^i p^{m+2} + \ldots + t_n^i p^n, \tag{2.3.37}$$

$$\beta^i = t_1^i q^1 + t_{m+1}^i q^{m+2} + \ldots + t_n^i q^n. \tag{2.3.38}$$

Then from (2.3.6) we obtain the same formula:

$$\exp[x^i] = \prod_{s=1}^{m} [F_s^2(t - t_{0s})]^{-b_s^i/<b_s,b_s>} \exp[\alpha^i t + \beta^i]. \tag{2.3.39}$$

The relations (2.3.8) lead to the following constraints for the Kasner-like parameters α^i and β^i :

$$< \alpha, \alpha >= 2(E_0 - E_1 - \ldots - E_m), \tag{2.3.40}$$

$$< \alpha, b_s >=< \beta, b_s >= 0, \quad s = 1, \ldots, m. \tag{2.3.41}$$

Remark 5. If $m = n - 1$, then either $< \alpha, \alpha >< 0$ or $\alpha = 0$; and β has the same properties.

Remark 6. We may also consider the enlargement of this subclass by the manner described in Remark 4. If we add to the Lagrangian (2.2.13) the terms (2.3.26) for some $\sigma \leq m$, we should replace the function $F_\sigma(t - t_{0\sigma})$ in eq. (2.3.39) by the function $F(t - t_{0\sigma})$, satisfying (2.3.27).

C. Let b_1 be an isotropic vector. Then $< b_r, b_r >> 0$ for $r = 2, \ldots, m$ (in this case $m \leq n - 1$). We choose the orthonormal basis $e'_1, \ldots, e'_n$ by

$$e'_r = b_r / \sqrt{< b_r, b_r >}, \quad b_1 = e'_1 + e'_{m+1}. \tag{2.3.42}$$

Then we get

$$L = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j - a^{(1)} \exp[-X^1 + X^{m+1}] -$$

$$\sum_{r=2}^{m} a^{(r)} exp[\sqrt{< b_r, b_r >} X^r], \tag{2.3.43}$$

$$E_0 = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j + a^{(1)} \exp[-X^1 + X^{m+1}] +$$

$$\sum_{r=2}^{m} a^{(r)} \exp[\sqrt{< b_r, b_r >} X^r], \tag{2.3.44}$$

The corresponding eqs. of motion have the form

$$\ddot{X}^1 = -a^{(1)} \exp[-X^1 + X^{m+1}], \tag{2.3.45}$$

$$\ddot{X}^{m+1} = -a^{(1)} \exp[-X^1 + X^{m+1}], \tag{2.3.46}$$

$$\ddot{X}^r = -\sqrt{< b_r, b_r >} a^{(r)} \exp[\sqrt{< b_r, b_r >} X^r], \tag{2.3.47}$$

$$\ddot{X}^{m+2} = \ldots = \ddot{X}^n = 0. \tag{2.3.48}$$

To integrate (2.3.45), (2.3.46) it is useful to consider the eqs. of motion for $X^+ = X^1 + X^{m+1}$ and $X^- = -X^1 + X^{m+1}$. Then we get the solution

$$X^1 = \frac{1}{2}(p^+ - p^-)t + \frac{1}{2}(q^+ - q^-) - 2 \ln[f(t)], \tag{2.3.49}$$

$$X^{m+1} = \frac{1}{2}(p^+ + p^-)t + \frac{1}{2}(q^+ + q^-) - 2 \ln[f(t)], \tag{2.3.50}$$

$$X^r = \frac{-1}{\sqrt{< b_r, b_r >}} \ln[F_r^2(t - t_{0r})], \tag{2.3.51}$$

$$X^{m+2} = p^{m+2} t + q^{m+2}, \ldots, X^n = p^n t + q^n, \tag{2.3.52}$$

Here by $f(t)$ we denoted the function

$$f(t) \quad = \exp[\frac{a^{(1)}}{2(p^-)^2}\exp[p^- t + q^-]], \quad p^- \neq 0, \tag{2.3.53}$$

$$= \exp[\frac{a^{(1)}}{4}\exp[q^-]t^2], \quad p^- = 0. \tag{2.3.54}$$

The integration constants satisfy the relations

$$\frac{1}{2}p^+ p^- + E_2 + \ldots + E_m + \frac{1}{2}(p^{m+2})^2 + \ldots + \frac{1}{2}(p^n)^2 = E_0, \quad p^- \neq 0, \tag{2.3.55}$$

$$a^{(1)}\exp[q^-] + E_2 + \ldots + E_m + \frac{1}{2}(p^{m+2})^2 + \ldots + \frac{1}{2}(p^n)^2 = E_0, \quad p^- = 0. \tag{2.3.56}$$

The Kasner-like parameters are defined by

$$\alpha^i = \frac{1}{2}t_1^i(p^+ - p^-) + \frac{1}{2}t_{m+1}^i(p^+ + p^-) + t_{m+2}^i p^{m+2} + \ldots + t_n^i p^n, \tag{2.3.57}$$

$$\beta^i = \frac{1}{2}t_1^i(q^+ - q^-) + \frac{1}{2}t_{m+1}^i(q^+ + q^-) + t_{m+2}^i q^{m+2} + \ldots + t_n^i q^n. \tag{2.3.58}$$

Then from (2.3.6) we obtain the scale factors in a Kasner-like form:

$$\exp[x^i] = [f(t)]^{-b_1^i}\prod_{r=2}^{m}[F_r^2(t - t_{0r})]^{-b_r^i/<b_r,b_r>}\exp[\alpha^i t + \beta i]. \tag{2.3.59}$$

The Kasner-like parameters satisfy

$$< \alpha, \alpha > \quad = 2(E_0 - E_2 - \ldots - E_m), \quad < \alpha, b_1 > \neq 0 \tag{2.3.60}$$

$$= (E_0 - a^{(1)}\exp[< \beta, b_1 >] - E_2 - \ldots - E_m), \quad < \alpha, b_1 >= 0, \tag{2.3.61}$$

$$< \alpha, b_r >=< \beta, b_r >= 0, \quad r = 2, \ldots, m. \tag{2.3.62}$$

Remark 7. For the parameters p^- and q^- we get:

$$p^- =< \alpha, b_1 >, \quad q^- =< \beta, b_1 > . \tag{2.3.63}$$

Remark 8. If $m = n - 1$ and $< \alpha, b_1 >= 0$, then $< \alpha, \alpha >= 0$, i.e. $\alpha = p^+ b_1$. If $m < n - 1$ and $< \alpha, b_1 >= 0$, then $< \alpha, \alpha >\geq 0$.

Remark 9. Let us consider the enlargement of this subclass by the addition of the terms (2.3.26) to the Lagrangian. The modification of the exact solution (2.3.59) for each $\sigma = 2, \ldots, m$ is described in the Remark 6. Let us take into account the additional components, induced by isotropic vectors collinear to b_1 . It is not difficult to show that in this case (for $\sigma = 1$) the additional terms (2.3.26) leads to the following modification of the function $f(t)$

$$f(t) \quad = \exp\{\frac{a^{(1)}}{2(p^-)^2}\exp[p^- t + q^-] + \sum_{\alpha=1}^{m(1)}\frac{a^{(1\alpha)}}{2b_{(1\alpha)}(p^-)^2}\exp[b_{(1\alpha)}(p^- t + q^-)]\}, \ p^- \neq 0,$$

$$= \exp\{(a^{(1)}\exp[q^-] + \sum_{\alpha=1}^{m(1)} a^{(1\alpha)}\exp[b_{(1\alpha)}q^-])\frac{t^2}{4}\}, \ p^- = 0. \tag{2.3.64}$$

In (2.3.56) and (2.3.61) the additional terms appear

$$\sum_{\alpha=1}^{m(1)} a^{(1\alpha)}\exp[b_{(1\alpha)}q^-]. \tag{2.3.65}$$

These are all modifications in this case.

2.4. Reduction of pseudo-Euclidean Toda-like system to Euclidean one

Now we consider the case, when the set of vectors $b_1, \dots, b_m$ is not orthogonal. It is easily shown that eqs. of motion of our system with the Lagrangian

$$L = \frac{1}{2} < \dot{x}, \dot{x} > - \sum_{\alpha=1}^{m} a^{(\alpha)} \exp[< b_\alpha, x >]. \tag{2.4.1}$$

for the new variables

$$p = \dot{x} \in R^n, \tag{2.4.2}$$
$$l_\alpha = a^{(\alpha)} \exp[< b_\alpha, x >] \tag{2.4.3}$$

have the following form

$$\dot{p} = - \sum_{\alpha=1}^{m} l_\alpha b_\alpha, \tag{2.4.4}$$
$$\dot{l}_\alpha = l_\alpha < b_\alpha, p > . \tag{2.4.5}$$

Note that this representation is valid for non-degenerate bilinear form $< .,. >$ with arbitrary signature.

Let us consider a simple complex Lie algebra G. Let H be a Cartan subalgebra, and h_i, e_{ω_γ} be a Weyl-Cartan basis in G [41]. We denote by $h_1, \dots, h_n$ some basis in H and by $\omega_1, \dots, \omega_N$ the set of roots ($\omega_\gamma \in H$, $\gamma = 1, \dots, N$). If the roots $\omega_1, \dots, \omega_m$ are admissible, then we have [25,40]

$$[h, e_{\omega_\alpha}] = (\omega_\alpha, h) e_{\omega_\alpha}, \quad h \in H \tag{2.4.6}$$
$$[e_{\omega_\alpha}, e_{-\omega_\beta}] = \delta_{\alpha\beta} \omega_\alpha, \quad \alpha, \beta = 1, \dots, m, \tag{2.4.7}$$

where we denote by $(.,.)$ the Killing-Cartan form. Let us define in the algebra G the vectors ($L - A$ pair) [25,40]

$$L(t) = \sum_{\alpha=1}^{m} f_\alpha(t) e_{-\omega_\alpha} + C \sum_{i=1}^{n} h^i(t) h_i + C^2 \sum_{\alpha=1}^{m} e_{\omega_\alpha}, \tag{2.4.8}$$
$$A(t) = -\frac{1}{C} \sum_{\alpha=1}^{m} f_\alpha(t) e_{-\omega_\alpha}, \tag{2.4.9}$$

where C is arbitrary constant. Using (2.4.6-2.4.7), it can be easily checked that eq.

$$\dot{L}(t) = [L(t), A(t)] \tag{2.4.10}$$

is equivalent to the following set of eqs. for the variables $f_\alpha(t), h^i(t)$

$$\dot{h} = - \sum_{\alpha=1}^{m} f_\alpha \omega_\alpha, \tag{2.4.11}$$
$$\dot{f}_\alpha = f_\alpha (\omega_\alpha, h), \tag{2.4.12}$$

where we denoted $h = h^1(t) h_1 + \dots + h^n(t) h_n$, $h \in H$.

Consider the real linear subspace of dimension n $H' \in H$ such that the Killing-Cartan form $(.,.)$ on H' is a real non-degenerate bilinear form with the signature $(-,+,\ldots,+)$, i.e. $< .,. >$. It is evident, that the sets of eqs. (2.4.4-2.4.5) and (2.4.11-2.4.12) are identical, if $h, \omega_1, \ldots, \omega_m \in H'$. Thus, if the set of vectors $b_1, \ldots, b_m \in R^n$ equipped with the bilinear form $< .,. >$ may be identified with a set of admissible roots $\omega_1 \ldots, \omega_m \in H'$, then pseudo-Euclidean Toda-like system with the Lagrangian (2.4.1) possesses the Lax representation. If such identification is possible, then the system is called to be connected with the simple complex Lie algebra.

Proposition 3. Let a pseudo-Euclidean Toda-like system is connected with a simple complex Lie algebra. Then it is reducible to an Euclidean Toda-like system for a part of coordinates.

Proof. We get in an arbitrary orthonormal basis $e'_1, \ldots, e'_n$

$$L = \frac{1}{2}\sum_{i,j=1}^{n} \eta_{ij}\dot{X}^i\dot{X}^j - \sum_{s=1}^{m} a^{(s)} \exp[\sum_{i=1}^{n} B_i^s X^i], \tag{2.4.13}$$

where we denoted

$$B_i^s = \sum_{j=1}^{n} \eta_{ij} B_s^j. \tag{2.4.14}$$

We remind, that $b_s = B_s^1 e'_1 + \ldots + B_s^n e'_n$.

It is known [41] that the Killing-Cartan form defined on the real linear span of roots of a simple (or semi-simple) complex Lie algebra is positively definite. But we have the indefinite bilinear form $< .,. >$. Then the first components of the vectors $b_1, \ldots, b_m$ must be zero in a suitably chosen orthonormal basis, i.e. $B_1^s = 0$ for $s = 1, \ldots, m$. Then, in this basis Lagrangian (2.4.1) has the form

$$L = \frac{1}{2}\sum_{i,j=1}^{n} \eta_{ij}\dot{X}^i\dot{X}^j - \sum_{s=1}^{m} a^{(s)} \exp[\sum_{k=2}^{m} B_k^s X^k]. \tag{2.4.15}$$

Coordinate X^1 satisfies the eq.: $\ddot{X}^1 = 0$. Eqs. of motion for $X^2, \ldots, X^n$ are followed from the Euclidean Toda-like Lagrangian

$$L_E = \frac{1}{2}\sum_{k,l=2}^{n} \delta_{kl}\dot{X}^k\dot{X}^l - \sum_{s=1}^{m} a^{(s)} \exp[\sum_{k=2}^{m} B_k^s X^k]. \tag{2.4.16}$$

Thus, we obtained the reduction of a pseudo-Euclidean Toda-like system to the Euclidean one.

Integrating the eqs. of an Euclidean Toda-like system by known methods [26,27,42], we obtain the class of exact solutions for some nonorthogonal set of vectors $b_1, \ldots, b_m$. Here we consider this procedure for the simplest 2-component case ($n \geq 3$), when Toda lattice is connected with Lie algebra A_2.

Suppose, that the vectors b_1 and b_2, induced by two components in the Lagrangian

$$L = \frac{1}{2} < \dot{x}, \dot{x} > -a^{(1)} \exp[< b_1, x >] - a^{(2)} \exp[< b_2, x >], \tag{2.4.17}$$

satisfy the following conditions

$$< b_1, b_2 >= -\frac{1}{2} < b_1, b_1 >= -\frac{1}{2} < b_2, b_2 >< 0. \tag{2.4.18}$$

Then, we have two space-like vectors with the same lengths. The angle between them is equal to $120°$. We denote

$$\sqrt{< b_1, b_1 >} = \sqrt{< b_2, b_2 >} = b. \tag{2.4.19}$$

Let us define the orthonormal basis $e'_1, \ldots, e'_n$ in R^n by

$$b_1 = be'_2, \tag{2.4.20}$$

$$b_2 = b(-\frac{1}{2}e'_2 + \frac{\sqrt{3}}{2}e'_3). \tag{2.4.21}$$

In this basis the Lagrangian (2.4.17) and corresponding energy constraint have the form

$$L = \frac{1}{2}\sum_{i,j=1}^{n} \eta_{ij}\dot{X}^i\dot{X}^j - a^{(1)}\exp[bX^2] - a^{(2)}\exp[b(-\frac{1}{2}X^2 + \frac{\sqrt{3}}{2}X^3)], \tag{2.4.22}$$

$$E_0 = \frac{1}{2}\sum_{i,j=1}^{n} \eta_{ij}\dot{X}^i\dot{X}^j + a^{(1)}\exp[bX^2] + a^{(2)}\exp[b(-\frac{1}{2}X^2 + \frac{\sqrt{3}}{2}X^3)] \tag{2.4.23}$$

For the coordinates $X^1,\ X^4, \ldots, X^n$ we get the following eqs. of motion:

$$\ddot{X}^1 = \ddot{X}^4 = \ldots = \ddot{X}^n. \tag{2.4.24}$$

Therefore

$$X^1 = p^1t + q^1,\ X^4 = p^4t + q^4, \ldots, X^n = p^nt + q^n, \tag{2.4.25}$$

where $p^1, p^4, \ldots, p^n, q^1, q^4, \ldots, q^n$ are arbitrary integration constants. The eqs. of motion for the coordinates X^2 and X^3 follow from the Lagrangian

$$L_E = \frac{1}{2}((\dot{X}^2)^2 + (\dot{X}^3)^2) - a^{(1)}\exp[bX^2] - a^{(2)}\exp[b(-\frac{1}{2}X^2 + \frac{\sqrt{3}}{2}X^3)]. \tag{2.4.26}$$

Let us introduce new coordinates y^1 and y^2 as

$$y_1 = \frac{b}{2\sqrt{2}}X^2, \quad y_2 = \frac{b}{2\sqrt{2}}X^3. \tag{2.4.27}$$

We obtain the Lagrangian of the open Toda lattice connected with the Lie algebra $A_2 = SL(3, C)$

$$L_T = \frac{1}{2}((\dot{y}^1)^2 + (\dot{y}^2)^2) - \epsilon g_1^2\exp[2\sqrt{2}y_1] - \epsilon g_2^2\exp[-\sqrt{2}y_1 + \sqrt{6}y_2], \tag{2.4.28}$$

where we denoted

$$b^2a^{(1)}/8 = \epsilon g_1^2, \quad b^2a^{(2)}/8 = \epsilon g_2^2, \tag{2.4.29}$$

$$\epsilon = \mathrm{sgn}[a^{(1)}] = \mathrm{sgn}[a^{(2)}] = \pm 1. \tag{2.4.30}$$

To study the open Toda lattice it is useful to add the additional coordinate y_3:

$$L_T = \frac{1}{2}((\dot{y}^1)^2 + (\dot{y}^2)^2 + (\dot{y}^3)^2) - \epsilon g_1^2 \exp[2\sqrt{2}y_1] - \epsilon g_2^2 \exp[-\sqrt{2}y_1 + \sqrt{6}y_2], \tag{2.4.31}$$

After the orthogonal linear transformation

$$q_1 = \frac{1}{\sqrt{6}}(\sqrt{3}y_1 + y_2 + \sqrt{2}y_3),$$

$$q_2 = \frac{1}{\sqrt{6}}(-\sqrt{3}y_1 + y_2 + \sqrt{2}y_3), \tag{2.4.32}$$

$$q_3 = -2y_2 + \sqrt{2}y_3 \tag{2.4.33}$$

the Lagrangian (2.4.31) takes the well-known form [24,26-28,42,43]

$$L_T = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - \epsilon g_1^2 \exp[2(q_1 - q_2)] - \epsilon g_2^2 \exp[2(q_2 - q_3)]. \tag{2.4.34}$$

In this representation the additional degree of freedom corresponds to the free motion of the center of mass ($\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3 = 0$). The integrating of the eqs. of motion for this system leads to the result [26,27,42]

$$g_1^2 \exp[2(q_1 - q_2)] = \frac{F_+}{F_-^2}, \quad g_2^2 \exp[2(q_2 - q_3)] = \frac{F_-}{F_+^2}, \tag{2.4.35}$$

where

$$F_\pm = \frac{4}{9A_1A_2(A_1 + A_2)}\{A_1 \exp[\pm(A_1 + 2A_2)t \pm B_1] + \tag{2.4.36}$$
$$\epsilon(A_1 + A_2)\exp[\pm(A_1 - A_2)t \mp (B_1 - B_2)] + A_2 \exp[\mp(2A_1 + A_2)t \mp B_2]\}.$$

The integration constants B_1, B_2 are arbitrary and A_1, A_2 satisfy the condition: $A_1A_2 > 0$. For the energy of the system with Lagrangian (2.4.24) we have

$$\frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) + \epsilon g_1^2 \exp[2(q_1 - q_2)] + \epsilon g_2^2 \exp[2(q_2 - q_3)] = \frac{3}{4}(A_1^2 + A_1A_2 + A_2^2). \tag{2.4.37}$$

Doing the inverse linear transformation

$$y_1 = \frac{1}{\sqrt{2}}[q_1 - q_2],$$

$$y_2 = \frac{1}{\sqrt{6}}([q_1 - q_2] + 2[q_2 - q_3]), \tag{2.4.38}$$

$$y_3 = \frac{1}{\sqrt{3}}(q_1 + q_2 + q_3),$$

for the system with Lagrangian (2.4.22) we get the solution

$$X^2 = \frac{1}{b}\ln[\frac{8}{b^2|a^{(1)}|}\frac{F_+}{F_-^2}], \tag{2.4.39}$$

$$X^3 = \frac{\sqrt{3}}{b}\ln[\frac{8}{b^2(|a^{(1)}|(a^{(2)})^2)^{1/3}}\frac{1}{F_+}], \tag{2.4.40}$$

and the following energy constraint

$$E_0 = -\frac{1}{2}(p^1)^2 + \frac{1}{2}(p^4)^2 + \ldots + \frac{1}{2}(p^n)^2 + \frac{6}{b^2}(A_1^2 + A_1A_2 + A_2^2). \tag{2.4.41}$$

To present the scale factors in the Kasner-like form let us introduce the Kasner-like parameters

$$\alpha^i = t_1^i p^1 + t_4^i p^4 + \ldots + t_n^i p^n, \tag{2.4.42}$$
$$\beta^i = t_1^i q^1 + t_4^i q^4 + \ldots + t_n^i q^n, \tag{2.4.43}$$

where components t_k^i are determined by (2.3.7). In this case they satisfy the relations

$$t_2^i = \frac{1}{b}b_1^i, \quad t_3^i = \frac{2}{\sqrt{3}}(\frac{1}{b}b_2^i + \frac{1}{2b}b_1^i). \tag{2.4.44}$$

From (2.3.6) we obtain the coordinates x^i and finally present the exact solution in the form

$$\exp[x^i] = [\tilde{F}_-^2]^{-b_1^i/<b_1,b_1>}[\tilde{F}_+^2]^{-b_2^i/<b_2,b_2>}\exp[\alpha^i t + \beta^i], \tag{2.4.45}$$

where

$$\tilde{F}_- = \frac{1}{8}b^2\{(a^{(1)})^2|a^{(2)}|\}^{\frac{1}{3}}F_-, \tag{2.4.46}$$
$$\tilde{F}_+ = \frac{1}{8}b^2\{(a^{(2)})^2|a^{(1)}|\}^{\frac{1}{3}}F_+. \tag{2.4.47}$$

The vectors α and β defined by (2.3.23) satisfy the relations

$$< \alpha, \alpha >= 2(E_0 - \frac{6}{b^2}(A_1^2 + A_1A_2 + A_2^2)), \tag{2.4.48}$$
$$< \alpha, b_r >=< \beta, b_r >= 0, \quad r = 1, 2. \tag{2.4.49}$$

Remark 10. If $n = 3$, then $< \alpha, \alpha >\leq 0$ and $< \beta, \beta >\leq 0$.

2.5. Discussion

Let us consider some cosmological models corresponding to the introduced in the Sect. 2.3 integrable subclasses of pseudo-Euclidean Toda-like systems. For this purpose in Table I we present values of the bilinear form $< .,. >$ (see Sect. 2.2) for the vectors

$$v_i \equiv v_{(i)}^1 e_1 + \ldots + v_{(i)}^n e_n, \quad v_{(i)}^j = -2\frac{\delta_i^j}{N_i}, \tag{2.5.1}$$

$$u_\alpha \equiv u_{(\alpha)}^1 e_1 + \ldots + u_{(\alpha)}^n e_n, \quad u_\alpha^j = h_j^{(\alpha)} + \frac{1}{2-D}\sum_{i=1}^n N_i h_i^{(\alpha)}, \tag{2.5.2}$$

$$u \equiv u^1 e_1 + \ldots + u^n e_n, \quad u^j = \frac{2}{2-D}, \tag{2.5.3}$$

induced by curvature, perfect fluid and Λ-term correspondingly.

Within the subclass A we are able to construct the model with one Einstein space of non-zero curvature. Let $(n-1)$ Einstein spaces are Ricci-flat and one, for instance M_1 , have a non-zero Ricci tensor. Then we put $b_1 \equiv v_1$. To get the orthogonality with b_1 for at most $(n-1)$ available components of the perfect fluid ($b_{(\alpha+1)} \equiv u_{(\alpha)}$ for $\alpha \leq n-1$) we put: $h_1^{(\alpha)} = 0$ (see Table I). Then, these components appeared to be in the manifold M_1 in the Zeldovich matter form (see Remark 1). The model of such a type was integrated in [47]. In the same way the model with all Ricci-flat spaces and Λ-term arises . In this case we put $b_1 = u$. The condition of the orthogonality reads: $\sum_{i=1}^{n} h_i^{(\alpha)} N_i = 0$ for all $\alpha \leq n-1$. Then we get the negative values for the some $h_i^{(\alpha)}$. It means that for such perfect fluids $p > \rho$ in some spaces (see (1.1.8)).

The vectors v_i and u induced by curvature and Λ-term correspondingly are time-like, therefore subclasses B and C correspond to the Ricci-flat models without Λ-term for some multicomponent perfect fluid source. These vectors can not be roots of any simple complex Lie algebra. Therefore, the models with more than one non-zero curvature space and the models with curvature and Λ-term are not trivially reducible to the Euclidean Toda lattices. Some possibilities of integration of these models were studied in [18,46].

In conclusion we discuss the existence of the Euclidean wormholes [51-54] within the class of the obtained exact solutions. We consider the simple model within subclass A with the manifold $R \times M_1 \times M_2$, when M_1 has a nonzero Ricci tensor with $\lambda_1 > 0$ (see 1.1.3) and M_2 is Ricci-flat. The integrable model arises in the presence of the perfect fluid in the Zeldovich matter form for the space M_1. It means $h_1 = 0$ and the other parameter in the equation of state for M_2 (see 1.1.8)) may be arbitrary positive constant h. If we demand the positiveness of the mass-energy density for the perfect fluid ($A > 0$), then from (2.3.22) we get for the scales factors of the M_1 and M_2

$$\exp[x^1] = \{F_1^2(t-t_{01})\}^{-\frac{1}{2(N_1-1)}} \{F_2^2(t-t_{02})\}^{\frac{1}{h(N_1-1)}}, \tag{2.5.4}$$

$$\exp[x^2] = \{F_2^2(t-t_{02})\}^{-\frac{1}{hN_2}}, \tag{2.5.5}$$

where

$$F_1(t-t_{01}) = \sqrt{\frac{1}{2}\lambda_1 N_1/|E_1|}\cosh[\sqrt{2|E_1|(N_1-1)/N_1}(t-t_{01})], \tag{2.5.6}$$

$$F_2(t-t_{01}) = \sqrt{\kappa^2 A/E_2}\cosh[h\sqrt{\frac{1}{2}(N_1-1)N_2|E_2/(N_1+N_2-1)}(t-t_{02})]. \tag{2.5.7}$$

In this case $E_1 < 0$ and $E_2 > 0$. The energy constraint (2.3.24) leads to the condition: $-E_1 = E_2 \equiv E$.

We may suppose that M_1 is 3-dimensional sphere S^3 and M_2 is d-dimensional torus T^d. Then formulas (2.5.4-2.5.7) present the multidimensional generalization of closed Friedmann model. This model may be relevant in the theory of the Early Universe, because the Zeldovich matter equation of state: $p = \rho$ is valid on the earlier stage of its evolution [49].

To prove the existence of the Euclidean wormholes we use the transformation $t \to it$. Then for the case $t_{01} = t_{02} = 0$ we obtain

$$\exp[x^1] = \{\frac{\kappa^2 A}{E}\cos^2[\sqrt{\frac{Ed}{d+2}}ht]\}^{1/(2h)}\{\frac{3\lambda_1}{2E}\cos^2[\sqrt{\frac{4E}{3}}t]\}^{-1/4}, \tag{2.5.8}$$

$$\exp[x^2] = \{\frac{\kappa^2 A}{E}\cos^2[\sqrt{\frac{Ed}{d+2}}ht]\}^{-1/(hd)}. \tag{2.5.9}$$

It is easy to see that when $\frac{d}{d+2}h^2 > \frac{4}{3}$ one has wormhole with respect to the internal space T^d. The case $\frac{d}{d+2}h^2 < \frac{4}{3}$ corresponds to the wormhole for the external space S^3. Note, that for $h = 2$ and $d = 1$ the wormhole for the internal space is accompanied by the static external space. It is not difficult to show that wormhole with respect to the whole space for this model arises in the presence of the additional component in the form of minimally coupled scalar field.

	v_j	u_β	u
v_i	$4(\frac{\delta_{ij}}{N_i}-1)$	$-2h_i^{(\beta)}$	-4
u_α	$-2h_j^{(\alpha)}$	$\sum_{i=1}^n h_i^{(\alpha)}h_i^{(\beta)}N_i+$ $\frac{1}{2-D}[\sum_{i=1}^n h_i^{(\alpha)}N_i][\sum_{j=1}^n h_j^{(\beta)}N_j]$	$\frac{2}{2-D}\sum_{i=1}^n h_i^{(\alpha)}N_i$
u	-4	$\frac{2}{2-D}\sum_{i=1}^n h_i^{(\beta)}N_i$	$-4\frac{D-1}{D-2}$

TABLE I

3. Billiard Representation for Multidimensional Cosmology with Multicomponent Perfect Fluid near the Singularity

3.1. Introduction

A lot of interesting topics in multidimensional cosmology were considered: exact solutions and the problem of integrbility, superstring cosmology and the problem of compactification, variation of constants, classical and quantum wormholes, chaotic behaviour near the singularity, etc.

In the present section we deal with a stochastic behavior in multidimensional cosmological models [53-55,18]. This direction in higher-dimensional gravity was stimulated by well-known results for "mixmaster" model [56-59]. We note, that there is also an elegant explanation for stochastic behavior of scale factors of Bianchi-IX model suggested by Chitre [58-59] and recently considered in [60-62]. (For "history" of the problem see also [63].) In the Chitre's approach the Bianchi-IX cosmology near the singularity is reduced to a billiard on the Lobachevsky space H^2 (see Fig. 4 below). The volume of this billiard is finite. This fact together with the well-known behavior (exponential divergences) of geodesics on the spaces of negative curvature leads to a stochastic behavior of the dynamical system in the considered regime [64,65].

Chitre's approach [58] may also be used in the multidimensional case [55]. It allows us to obtain a more evident picture for the origin of the oscillatory behaviour near the singularity using the formation of billiard walls. The present section is devoted to a construction of the "billiard representation" for the multidimensional cosmological model describing the evolution of n Einstein spaces in the presence if $(m+1)$-component perfect fluid [37] (see section 3.2). One of these components corresponds to the cosmological constant term [66]. In some sense the model [37] may be considered as "universal" cosmological model: a lot of cosmological models (not obviously multidimensional) may be embedded in this model.

We impose certain restrictions on the parameters of the model [37] and reduce its dynamics near the singularity to a billiard on the $(n-1)$-dimensional Lobachevsky space H^{n-1} (Sec. 3.3). The geometrical criterion for the finiteness of the billiard volume and its compactness is suggested. This criterion reduces the considered problem to the geometrical (or topological) problem of illumination of $(n-2)$-dimensional unit sphere S^{n-2} by $m_+ \leq n$ point-like sources located outside the sphere [68-69]. These sources correspond to the components with $(u^{(\alpha)})^2 > 0$ (Sec. 3.3). When these sources illuminate the sphere then, and only then, the billiard has a finite volume and the cosmological model possesses a stochastic behavior near the singularity. (We note, that, for cosmological and curvature terms $(u^{(\alpha)})^2 < 0$ and these terms may be neglected near the singularity). For the case of an infinite billiard volume the cosmological model has a Kasner-like behavior near the singularity. When the minimally coupled massless scalar field is added into consideration, the evolution in time is bounded: $t > t_0$ and the limit $t \to t_0$ corresponds to the approach to the singularity. In this case the stochastic behavior near the singularity is absent.

In Sec. 3.4 we illustrate the suggested approach on an example of the Bianchi-IX cosmology.

3.2. The model

Here we start also from the cosmological model describing the evolution of n Einstein spaces in the presence of $(m+1)$-component perfect-fluid matter (see section 1.2). The metric of the model

$$g = -\exp[2\gamma(t)]dt \otimes dt + \sum_{i=1}^{n} \exp[2x^i(t)]g^{(i)}, \tag{3.2.1}$$

is defined on the manifold

$$M = R \times M_1 \times \ldots \times M_n, \tag{3.2.2}$$

where the manifold M_i with the metric $g^{(i)}$ is an Einstein space of dimension N_i, i.e.

$$R_{m_i n_i}[g^{(i)}] = \lambda^i g^{(i)}_{m_i n_i}, \tag{3.2.3}$$

$i = 1, \ldots, n$; $n \geq 2$. The energy-momentum tensor is adopted in the following form

$$T^M_N = \sum_{\alpha=0}^{m} T^{M(\alpha)}_N, \tag{3.2.4}$$

$$(T^{M(\alpha)}_N) = diag(-\rho^{(\alpha)}(t), p^{(\alpha)}_1(t)\delta^{m_1}_{k_1}, \ldots, p^{(\alpha)}_n(t)\delta^{m_n}_{k_n}). \tag{3.2.5}$$

$\alpha = 0, \ldots, m$, with the conservation law constraints imposed:

$$\nabla_M T^{M(\alpha)}_N = 0, \tag{3.2.6}$$

$\alpha = 0, \ldots, m-1$. The Einstein equations

$$R^M_N - \frac{1}{2}\delta^M_N R = \kappa^2 T^M_N \tag{3.2.7}$$

(κ^2 is gravitational constant) imply $\nabla_M T^M_N = 0$ and consequently $\nabla_M T^{M(m)}_N = 0$.

We suppose that for any α-th component of matter the pressures in all spaces are proportional to the density

$$p^{(\alpha)}_i(t) = (1 - \frac{u^{(\alpha)}_i}{N_i})\rho^{(\alpha)}(t), \tag{3.2.8}$$

where $u^{(\alpha)}_i = const$, $i = 1, \ldots, n$; $\alpha = 0, \ldots, m$.

Non-zero components of the Ricci-tensor for the metric (3.2.1) are the following

$$R_{00} = -\sum_{i=1}^{n} N_i[\ddot{x}^i - \dot{\gamma}\dot{x}^i + (\dot{x}^i)^2], \tag{3.2.9}$$

$$R_{m_i n_i} = g^{(i)}_{m_i n_i}[\lambda^i + \exp(2x^i - 2\gamma)(\ddot{x}^i + \dot{x}^i(\sum_{i=1}^{n} N_i\dot{x}^i - \dot{\gamma}))], \tag{3.2.10}$$

$i = 1, \ldots, n$.

The conservation law constraint (3.2.6) for $\alpha \in \{0,...,m\}$ reads

$$\dot{\rho}^{(\alpha)} + \sum_{i=0}^{n} N_i \dot{x}^i (\rho^{(\alpha)} + p_i^{(\alpha)}) = 0. \tag{3.2.11}$$

From eqs. (3.2.8), (3.2.11) we get

$$\rho^{(\alpha)}(t) = A^{(\alpha)} \exp[-2N_i x^i(t) + u_i^{(\alpha)} x^i(t)], \tag{3.2.12}$$

where $A^{(\alpha)} = const$. Here and below the summation over repeated indices is understood.

We define

$$\gamma_0 \equiv \sum_{i=1}^{n} N_i x^i \tag{3.2.13}$$

in (3.2.1).

Using relations (3.2.8), (3.2.9), (3.2.10), (3.2.12) it is not difficult to verify that the Einstein equations (3.2.7) for the metric (3.2.1) and the energy-momentum tensor from (3.2.4), (3.2.5) are equivalent to the Lagrange equations for the Lagrangian

$$L = \frac{1}{2} \exp(-\gamma + \gamma_0(x)) G_{ij} \dot{x}^i \dot{x}^j - \exp(\gamma - \gamma_0(x)) V(x). \tag{3.2.14}$$

Here

$$G_{ij} = N_i \delta_{ij} - N_i N_j \tag{3.2.15}$$

are the components of the minisuperspace metric,

$$V = V(x) = -\frac{1}{2} \sum_{i=1}^{n} \lambda^i N_i \exp(-2x^i + 2\gamma_0(x)) + \sum_{\alpha=0}^{m} \kappa^2 A^{(\alpha)} \exp(u_i^{(\alpha)} x^i). \tag{3.2.16}$$

is the potential. This relation may be also presented in the form

$$V = \sum_{\alpha=0}^{\bar{m}} A_\alpha \exp(u_i^{(\alpha)} x^i), \tag{3.2.17}$$

where $\bar{m} = m + n$; $A_\alpha = \kappa^2 A^{(\alpha)}$, $\alpha = 0, \ldots, m$; $A_{m+i} = -\frac{1}{2}\lambda^i N_i$ and

$$u_j^{(m+i)} = 2(-\delta_j^i + N_j), \tag{3.2.18}$$

$i, j = 1, \ldots, n$. We also put $A_0 = \Lambda$ and

$$u_j^{(0)} = 2N_j, \tag{3.2.19}$$

$j = 1, \ldots, n$. Thus the zero component of the matter describe a cosmological constant term (Λ-term).

Diagonalization. We remind [14,15] that the minisuperspace metric

$$G = G_{ij} dx^i \otimes dx^i \tag{3.2.20}$$

has a pseudo-Euclidean signature $(-,+,\ldots,+)$, i.e. there exist a linear transformation

$$z^a = e^a_i x^i, \tag{3.2.21}$$

diagonalizing the minisuperspace metric (3.2.20)

$$G = \eta_{ab} dz^a \otimes dz^b = -dz^0 \otimes dz^0 + \sum_{i=1}^{n-1} dz^i \otimes dz^i, \tag{3.2.22}$$

where

$$(\eta_{ab}) = (\eta^{ab}) \equiv diag(-1,+1,\ldots,+1), \tag{3.2.23}$$

$a,b = 0,\ldots,n-1$. The matrix of the linear transformation (e^a_i) satisfies the relation

$$\eta_{ab} e^a_i e^b_j = G_{ij} \tag{3.2.24}$$

or equivalently

$$\eta^{ab} = e^a_i G^{ij} e^b_j = < e^a, e^b > . \tag{3.2.25}$$

Here

$$G^{ij} = \frac{\delta^{ij}}{N_i} + \frac{1}{2-D} \tag{3.2.26}$$

are components of the matrix inverse to the matrix (3.2.15) [15], $D = 1 + \sum_{i=1}^n N_i$ is the dimension of the manifold M (3.2.2) and

$$< u, v > \equiv G^{ij} u_i v_j \tag{3.2.27}$$

defines a bilinear form on R^n $(u = (u_i)$, $v = (v_i))$. Inverting the map (3.2.21) we get

$$x^i = e^i_a z^a, \tag{3.2.28}$$

where for the components of the inverse matrix $(e^i_a) = (e^a_i)^{-1}$ we obtain from (3.2.25)

$$e^i_a = G^{ij} e^b_j \eta_{ba}. \tag{3.2.29}$$

Like in [15,21] we put

$$z^0 = e^0_i x^i = q^{-1} N_i x^i, \qquad q = [(D-1)/(D-2)]^{1/2}. \tag{3.2.30}$$

In this case the 00-component of eq. (3.2.25) is satisfied and the set $(e^a, a = 1,\ldots,n-1)$ is defined up to $O(n-1)$-transformation. A special example of the diagonalization with the relations (3.2.30) and

$$z^a = e^a_i x^i = [N_a/(\sum_{j=a}^n N_j)(\sum_{j=a+1}^n N_j)]^{1/2} \sum_{j=a+1}^n N_j (x^j - x^i), \tag{3.2.31}$$

$a = 1,\ldots,n-1$, was considered in [14,15].

In z-coordinates (3.2.21) with z^0 from (3.2.30) the Lagrangian (3.2.14) reads

$$L = L(z^a, \dot{z}^a, \mathcal{N}) = \frac{1}{2}\mathcal{N}^{-1}\eta_{ab}\dot{z}^a\dot{z}^b - \mathcal{N}V(z), \tag{3.2.32}$$

where

$$\mathcal{N} = \exp(\gamma - \gamma_0(x)) > 0 \tag{3.2.33}$$

is the Lagrange multiplier (modified lapse function) and

$$V(z) = \sum_{\alpha=0}^{\bar{m}} A_\alpha \exp(u^\alpha_a z^a) \tag{3.2.34}$$

is the potential. Here we denote

$$u^\alpha_a = e^i_a u^{(\alpha)}_i =< u^{(\alpha)}, e^b > \eta_{ba}, \tag{3.2.35}$$

$a = 0, \ldots, n-1$, (see (3.2.27) and (3.2.29)). From (3.2.35) we get (see (3.2.26), (3.2.27) and (3.2.30))

$$u^\alpha_0 = - < u^{(\alpha)}, e^0 >= (\sum_{i=1}^{n} u^{(\alpha)}_i)/q(D-2). \tag{3.2.36}$$

For Λ-term and curvature components (see (3.2.19) and (3.2.18)) we have

$$u^0_0 = 2q > 0, \qquad u^{m+j}_0 = 2/q > 0, \tag{3.2.37}$$

$j = 1, \ldots, n$. The calculation of

$$(u^\alpha)^2 = \eta^{ab}u^\alpha_a u^\alpha_b =< u^{(\alpha)}, u^{(\alpha)} >= (u^{(\alpha)})^2, \tag{3.2.38}$$

for these components gives

$$(u^0)^2 = 4(D-1)/(2-D) < 0, \qquad (u^{m+j})^2 = 4(\frac{1}{N_j} - 1) < 0, \tag{3.2.39}$$

for $N_j > 1$, $j = 1, \ldots, n$. For $N_j = 1$ we have $\lambda^j = A_{m+j} = 0$.

3.3. Billiard representation

Here we consider the behavior of the dynamical system, described by the Lagrangian (3.2.32) for $n \geq 3$ in the limit

$$z^0 \to -\infty, \qquad z = (z^0, \vec{z}) \in \mathcal{V}_-, \tag{3.3.1}$$

where $\mathcal{V}_- \equiv \{(z^0, \vec{z}) \in R^n | z^0 < -|\vec{z}|\}$ is the lower light cone. For the volume scale factor

$$v = \exp(\sum_{i=1}^{n} N_i x^i) = \exp(q z^0) \tag{3.3.2}$$

(see (3.2.30)) we have in this limit $v \to 0$. Under certain additional assumptions the limit (3.3.1) describes the approaching to the singularity. We impose the following restrictions on the parameters u^α in the potential (3.2.34) for components with $A_\alpha \neq 0$:

$$1) A_\alpha > 0 \; if \; (u^\alpha)^2 = -(u_0^\alpha)^2 + (\vec{u}^\alpha)^2 > 0; \tag{3.3.3}$$

$$2) u_0^\alpha > 0 \; for \; all \; \alpha. \tag{3.3.4}$$

We note that due to (3.2.37) the second condition is always satisfied for Λ-term and curvature components (i.e. for $\alpha = 0, m+1, \ldots, m+n = \bar{m}$).

We restrict the Lagrange system (3.2.32) on $\mathcal{V}_-$, i.e. we consider the Lagrangian

$$L_- \equiv L|_{TM_-}, \qquad M_- = \mathcal{V}_- \times R_+, \tag{3.3.5}$$

where TM_- is tangent vector bundle over M_- and $R_+ \equiv \{\mathcal{N} > 0\}$. (Here $F|_A$ means the restriction of function F on A.) Introducing an analogue of the Misner-Chitre coordinates in $\mathcal{V}_-$ [58-59]

$$z^0 = -\exp(-y^0)\frac{1+\vec{y}^2}{1-\vec{y}^2}, \tag{3.3.6}$$

$$\vec{z} = -2\exp(-y^0)\frac{\vec{y}}{1-\vec{y}^2}, \tag{3.3.7}$$

$|\vec{y}| < 1$, we get for the Lagrangian (3.2.32)

$$L_- = \frac{1}{2}\mathcal{N}^{-1}e^{-2y^0}[-(\dot{y}^0)^2 + h_{ij}(\vec{y})\dot{y}^i\dot{y}^j] - \mathcal{N}V. \tag{3.3.8}$$

Here

$$h_{ij}(\vec{y}) = 4\delta_{ij}(1-\vec{y}^2)^{-2}, \tag{3.3.9}$$

$i, j = 1, \ldots, n-1$, and

$$V = V(y) = \sum_{\alpha=0}^{\bar{m}} A_\alpha \exp\bar{\Phi}(y, u^\alpha), \tag{3.3.10}$$

where

$$\bar{\Phi}(y,u) \equiv -e^{-y^0}(1-\vec{y}^2)^{-1}[u_0(1+\vec{y}^2) + 2\vec{u}\vec{y}], \tag{3.3.11}$$

We note that the $(n-1)$-dimensional open unit disk (ball)

$$D^{n-1} \equiv \{\vec{y} = (y^1, \ldots, y^n) | |\vec{y}| < 1\} \subset R^{n-1} \tag{3.3.12}$$

with the metric $h = h_{ij}(\vec{y})dy^i \otimes dy^j$ is one of the realization of the $(n-1)$-dimensional Lobachevsky space H^{n-1}.

We fix the gauge

$$\mathcal{N} = \exp(-2y^0) = -z^2. \tag{3.3.13}$$

Then, it is not difficult to verify that the Lagrange equations for the Lagrangian (3.3.8) with the gauge fixing (3.3.13) are equivalent to the Lagrange equations for the Lagrangian

$$L_* = -\frac{1}{2}(\dot{y}^0)^2 + \frac{1}{2}h_{ij}(\vec{y})\dot{y}^i\dot{y}^j - V_* \tag{3.3.14}$$

with the energy constraint imposed

$$E_* = -\frac{1}{2}(\dot{y}^0)^2 + \frac{1}{2}h_{ij}(\vec{y})\dot{y}^i\dot{y}^j + V_* = 0. \tag{3.3.15}$$

Here

$$V_* = e^{-2y^0}V = \sum_{\alpha=0}^{\bar{m}} A_\alpha \exp(\Phi(y, u^\alpha)), \tag{3.3.16}$$

where

$$\Phi(y,u) = -2y^0 + \bar{\Phi}(y,u). \tag{3.3.17}$$

Now we are interested in the behavior of the dynamical system in the limit $y^0 \to -\infty$ (or, equivalently, in the limit $z^2 = -(z^0)^2 + (\vec{z})^2 \to -\infty$, $z^0 < 0$) implying (3.3.1). Using the relations ($u_0 \neq 0$)

$$\Phi(y,u) = -u_0 \exp(-y^0)\frac{A(\vec{y}, -\vec{u}/u_0)}{1-\vec{y}^2} - 2y^0, \tag{3.3.18}$$

$$A(\vec{y},\vec{v}) \equiv (\vec{y}-\vec{v})^2 - \vec{v}^2 + 1, \tag{3.3.19}$$

we get

$$\lim_{y^0 \to -\infty} \exp\Phi(y,u) = 0 \tag{3.3.20}$$

for $u^2 = -u_0^2 + (\vec{u})^2 \leq 0$, $u_0 > 0$ and

$$\lim_{y^0 \to -\infty} \exp\Phi(y,u) = \theta_\infty(-A(\vec{y}, -\vec{u}/u_0)) \tag{3.3.21}$$

for $u^2 > 0$, $u_0 > 0$. In (3.3.21) we denote

$$\theta_\infty(x) \equiv \begin{matrix} +\infty, & x \geq 0, \\ 0, & x < 0. \end{matrix} \tag{3.3.22}$$

Using restrictions (3.3.3), (3.3.4) and relations (3.3.16), (3.3.20), (3.3.21) we obtain

$$V_\infty(\vec{y}) \equiv \lim_{y^0 \to -\infty} V_*(y^0, \vec{y}) = \sum_{\alpha \in \Delta_+} \theta_\infty(-A(\vec{y}, -\vec{u^\alpha}/u_0^\alpha)). \tag{3.3.23}$$

Here we denote

$$\Delta_+ \equiv \{\alpha | (u^\alpha)^2 > 0\}. \tag{3.3.24}$$

We note that due to (3.2.39) Λ-term and curvature components do not contribute to V_∞ (i.e. they may be neglected in the vicinity of the singularity).

The potential V_∞ may be also written as following

$$V_\infty(\vec{y}) = V(\vec{y}, B) \equiv \begin{array}{ll} 0, & \vec{y} \in B, \\ +\infty, & \vec{y} \in D^{n-1} \setminus B, \end{array} \tag{3.3.25}$$

where

$$B = \bigcap_{\alpha \in \Delta_+} B(u^\alpha) \subset D^{n-1}, \tag{3.3.26}$$

$$B(u^\alpha) = \{\vec{y} \in D^{n-1} || \vec{y} + \frac{\vec{u}^\alpha}{u_0^\alpha}| > \sqrt{(\frac{\vec{u}^\alpha}{u_0^\alpha})^2 - 1}\}, \tag{3.3.27}$$

$\alpha \in \Delta_+$. B is an open domain. Its boundary $\partial B = \bar{B} \setminus B$ is formed by certain parts of $m_+ = |\Delta_+|$ (m_+ is the number of elements in Δ_+) of $(n-2)$-dimensional spheres with the centers in the points

$$\vec{v}^\alpha = -\vec{u}^\alpha / u_0^\alpha, \qquad \alpha \in \Delta_+, \tag{3.3.28}$$

($|\vec{v^\alpha}| > 1$) and radii

$$r_\alpha = \sqrt{(\vec{v}^\alpha)^2 - 1} \tag{3.3.29}$$

respectively (for $n = 3$, $m_+ = 1$, see Fig. 1).

Fig. 1

So, in the limit $y^0 \to -\infty$ we are led to the dynamical system

$$L_\infty = -\tfrac{1}{2}(\dot{y}^0)^2 + \tfrac{1}{2} h_{ij}(\vec{y}) \dot{y}^i \dot{y}^j - V_\infty(\vec{y}), \tag{3.3.30}$$

$$E_\infty = -\tfrac{1}{2}(\dot{y}^0)^2 + \tfrac{1}{2} h_{ij}(\vec{y}) \dot{y}^i \dot{y}^j + V_\infty(\vec{y}) = 0, \tag{3.3.31}$$

which after the separating of y^0 variable

$$y^0 = \omega(t - t_0), \tag{3.3.32}$$

($\omega \neq 0$, t_0 are constants) is reduced to the Lagrange system with the Lagrangian

$$L_B = \frac{1}{2} h_{ij}(\vec{y}) \dot{y}^i \dot{y}^j - V(\vec{y}, B). \tag{3.3.33}$$

Due to (3.3.32)

$$E_B = \frac{1}{2} h_{ij}(\vec{y}) \dot{y}^i \dot{y}^j + V(\vec{y}, B) = \frac{\omega^2}{2}. \tag{3.3.34}$$

We put $\omega > 0$, then the limit $t \to -\infty$ describes the approach to the singularity. When the set (3.3.24) is empty ($\Delta_+ = \emptyset$) we have $B = D^{n-1}$ and the Lagrangian (3.3.33) describes the geodesic flow on the Lobachevsky space $H^{n-1} = (D^{n-1}, h_{ij} dy^i \otimes dy^j)$. In this case there are two families of non-trivial geodesic solutions (i.e. $y(t) \neq const$):

1. $$\vec{y}(t) = \vec{n}_1[\sqrt{v^2 - 1} \cos\varphi(\bar{t}) - v] + \vec{n}_2 \sqrt{v^2 - 1} \sin\varphi(\bar{t}), \tag{3.3.35}$$

$$\varphi(\bar{t}) = 2 \arctan[(v - \sqrt{v^2 - 1}) \tanh(\omega \bar{t})], \tag{3.3.36}$$

2. $$\vec{y}(t) = \vec{n} \tanh(\omega \bar{t}). \tag{3.3.37}$$

Here $\vec{n}^2 = \vec{n}_1^2 = \vec{n}_2^2 = 1$, $\vec{n}_1\vec{n}_2 = 0$, $v > 1$, $\omega > 0$, $\bar{t} = t - t_0$, $t_0 = const$.

Graphically the first solution corresponds to the arc of the circle with the center at point $(-v\vec{n}_1)$ and the radius $\sqrt{v^2 - 1}$. This circle belongs to the plane spanned by vectors $\vec{n}_1$ and $\vec{n}_2$ (the centers of the circle and the ball D^{n-1} also belong to this plane). We note, that the solution (3.3.35)-(3.3.36) in the limit $v \to \infty$ coincides with the solution (3.3.37).

We note, that the boundary of the billiard ∂B is formed by geodesics. For some billiards this fact may be used for "gluing" certain parts of boundaries.

When $\Delta_+ \neq \emptyset$ the Lagrangian (3.3.33) describes the motion of the particle of unit mass, moving in the $(n-1)$-dimensional billiard $B \subset D^{n-1}$ (see (3.3.26)). The geodesic motion in B (3.3.35)-(3.3.37) corresponds to a "Kasner epoch" and the reflection from the boundary corresponds to the change of Kasner epochs. For $n = 3$ some examples of (2-dimensional) billiards are depicted in Figs. 2-4.

Figs. 2-4

The billiard B in Fig. 2. has an infinite volume: $volB = +\infty$. In this case there are three open zones at the infinite circle $|\vec{y}| = 1$. After a finite number of reflections from the boundary the particle moves toward one of these open zones. For corresponding cosmological model we get the "Kasner-like" behavior in the limit $t \to -\infty$ [19].

For billiards depicted in Figs. 3 and 4 we have $volB < +\infty$. In the first case (Fig. 3) the closure of the billiard $\bar{B}$ is compact (in the topology of D^{n-1}) and in the second case (Fig. 4) $\bar{B}$ is non-compact. In these two cases the motion of the particle is stochastic.

Analogous arguments may be applied to the case $n > 3$. So, we are interested in the configurations with finite volume of B. We propose a simple geometric criterion for the finiteness of the volume of B and compactness of $\bar{B}$ in terms of the positions of the points (3.3.28) with respect to the $(n-2)$-dimensional unit sphere S^{n-2} $(n \geq 3)$. We say that the point $\vec{y} \in S^{n-2}$ is (geometrically) illuminated by the point-like source located at the point $\vec{v}$, $|\vec{v}| > 1$, if and only if $|\vec{y} - \vec{v}| \leq \sqrt{|\vec{v}|^2 - 1}$. In Fig. 1 the source P illuminates the closed arc $[P_1, P_2]$. We also say that the point $\vec{y} \in S^{n-2}$ is strongly illuminated by the point-like source located at the point $\vec{v}$, $|\vec{v}| > 1$, if and only if $|\vec{y} - \vec{v}| < \sqrt{|\vec{v}|^2 - 1}$. In Fig. 1 the source P strongly illuminates the open arc (P_1, P_2). The subset $N \subset S^{n-2}$ is called (strongly) illuminated by point-like sources at $\{\vec{v}^\alpha, \alpha \in \Delta_+\}$ if and only if any point from N is (strongly) illuminated by some source at $\vec{v}^\alpha$ $(\alpha \in \Delta_+)$.

Proposition 1. The billiard B (3.3.26) has a finite volume if and only if the point-like sources of light located at the points $\vec{v}^\alpha$ (3.3.28) illuminate the unit sphere S^{n-2}. The closure of the billiard $\bar{B}$ is compact (in the topology of $D^{n-1} \simeq H^{n-1}$) if and only if the sources at points (3.3.28) strongly illuminate S^{n-2}.

Proof. We consider the set $\partial^c B \equiv B^c \setminus \bar{B}$, where B^c is the completion of B (or, equivalently, the closure of B in the topology of R^{n-1}). We remind that $\bar{B}$ is the closure of B in the topology of D^{n-1}. Clearly, that $\partial^c B$ is a closed subset of S^{n-2}, consisting of all those points that are not strongly illuminated by sources (3.3.28). There are three possibilities: i) $\partial^c B$ is empty; ii) $\partial^c B$ contains some interior point (i.e. the point belonging to $\partial^c B$ with some open neighborhood); iii) $\partial^c B$ is non-empty finite set, i.e. $\partial^c B = \{\vec{y}_1, \dots \vec{y}_l\}$. The first case i) takes place if and only if $\bar{B}$ is compact in the topology of

D^{n-1}. Only in this case the sphere S^{n-2} is strongly illuminated by the sources (3.3.28). Thus the second part of proposition is proved. In the case i) $volB$ is finite. For the volume we have

$$volB = \int_B d^{n-1}\vec{y}\sqrt{h} = \int_0^1 dr(1-r^2)^{1-n}S_r. \tag{3.3.38}$$

The "area" $S_r \to C > 0$ as $r \to 1$ in the case ii) and, hence, the integral (3.38) is divergent. In the case iii)

$$S_r \sim C_1(1-r)^{2(n-2)} \ as \ r \to 1 \tag{3.3.39}$$

($C_1 > 0$) and, so, the integral (3.3.38) is convergent. Indeed, in the case iii), when $r \to 1$, the "area" S_r is the sum of l terms. Each of these terms is the $(n-2)$-dimensional "area" of a transverse side of a deformed pyramid with a top at some point $\vec{y}_k$, $k = 1,\ldots,l$. This multidimensional pyramid is formed by certain parts of spheres orthogonal to S^{n-2} in the point of their intersection $\vec{y}_k$. Hence, all lengths of the transverse section $r = const$ of the "pyramid" behaves like $(1-r)^2$, when $r \to 1$, that justifies (3.3.39). But the unit sphere S^{n-2} is illuminated by the sources (3.3.28) only in the cases i) and iii). This completes the proof.

The problem of illumination of convex body in multidimensional vector space by point-like sources for the first time was considered in [68,69]. For the case of S^{n-2} this problem is equivalent to the problem of covering the spheres with spheres [70,71]. There exist a topological bound on the number of point-like sources m_+ illuminating the sphere S^{n-2} [69]:

$$m_+ \geq n. \tag{3.3.40}$$

Thus, we are led to the following.
Proposition 2: When $m_+ < n$, i.e. the number of the components with $(u^\alpha)^2 > 0$ is less than the minisuperspace dimension, the billiard B (3.3.26) has infinite volume: vol $B = +\infty$.

In this case there exist an open zone on the sphere S^{n-2} and the stochastic behaviour near the singularity is absent (we get a Kasner-like behaviour for $t \to -\infty$).

Remark 1. Let the points (3.3.28) form an open convex polyhedron $P \subset R^{n-1}$. Then the sources at (3.3.28) illuminate S^{n-2}, if $D^{n-1} \subset P$, and strongly illuminate S^{n-2}, if $\overline{D^{n-1}} \subset P$.

Scalar field generalization. Let us assume that an additional $(m+1)$-th component with the equation of state $p_i^{(m+1)} = \rho^{(m+1)}$ is considered, $i = 1,\ldots,n$. This component describes Zeldovich matter [49] in all spaces and is equivalent to homogeneous massless free minimally coupled scalar field [50]. In this case $u_i^{(m+1)} = 0$, $i = 1,\ldots,n$ and the potential (3.2.17) is modified by the addition of constant $A_{m+1} > 0$. Then the potential V_* (3.3.16) is modified by the addition of the following term

$$\Delta V = A_{m+1}\exp(-2y^0). \tag{3.3.41}$$

This do not prevent from the formation of the billiard walls but change the time dependence of y^0-variable:

$$\exp(2y^0) = 2A_{m+1}\sinh^2[\omega(t-t_0)]/\omega^2, \tag{3.3.42}$$

($\omega > 0$) instead of (3.3.32). In the limit $t \to t_0+0$ we have $y^0 \to -\infty$ and $\vec{y}(t) \to \vec{y}_0 \in B$. So, the stochastic behavior near the singularity is absent in this case.

3.4. Bianchi-IX cosmology

Here we consider the well-known mixmaster model [56,57] with the metric

$$g = -\exp[2\gamma(t)] dt \otimes dt + \sum_{i=1}^{3} \exp[2x^i(t)] e^i \otimes e^i, \tag{3.4.1}$$

where 1-forms $e^i = e^i_\nu(\zeta) d\zeta^\nu$ satisfy the relations

$$de^i = \frac{1}{2}\varepsilon_{ijk} e^j \wedge e^k, \tag{3.4.2}$$

$i, j, k = 1, 2, 3$. The Einstein equations for the metric (3.4.1) lead to the Lagrange system (3.2.14)-(3.2.17) with (see, for example, [57]) $n = 3$, $N_1 = N_2 = N_3 = 1$, $m = 6$, $A_1 = A_2 = A_3 = 1/4$, $A_4 = A_5 = A_6 = -1/2$, $A_0 = A_7 = A_8 = A_9 = 0$, and

$$u_i^{(\alpha)} = 4\delta_i^\alpha, \qquad u_i^{(3+\alpha)} = 2(1 - \delta_i^\alpha), \tag{3.4.3}$$

$\alpha = 1, 2, 3$. In this case $\gamma_0 = \sum_{i=1}^n x^i$, the minisuperspace metric (3.2.14) is $G_{ij} = \delta_{ij} - 1$ and the potential (3.2.17) reads

$$V = V_{mix} \equiv \frac{1}{4}(e^{4x^1} + e^{4x^2} + e^{4x^3} - 2e^{2x^1+2x^2} - 2e^{2x^2+2x^3} - 2e^{2x^1+2x^3}). \tag{3.4.4}$$

In the z-coordinates (3.2.30), (3.2.31) we have for 3-vectors (3.2.35)

$$u^1 = \frac{4}{\sqrt{6}}(1, 1, -\sqrt{3}), \ u^2 = \frac{4}{\sqrt{6}}(1, 1, +\sqrt{3}), \ u^3 = \frac{4}{\sqrt{6}}(1, -2, 0), \tag{3.4.5}$$

$$u^4 = \frac{1}{2}(u^1 + u^2), \ u^5 = \frac{1}{2}(u^1 + u^3), \ u^6 = \frac{1}{2}(u^2 + u^3), \tag{3.4.6}$$

and, consequently,

$$(u^\alpha)^2 = 8, \qquad (u^{3+\alpha})^2 = 0, \tag{3.4.7}$$

$\alpha = 1, 2, 3$. Thus the conditions (3.3.3), (3.3.4) are satisfied. The components with $\alpha = 4, 5, 6$ do not survive in the approaching to the singularity . For the vectors (3.3.28) we have

$$\vec{v}^1 = (1, -\sqrt{3}), \ \vec{v}^2 = (1, +\sqrt{3}), \ \vec{v}^3 = (-2, 0), \tag{3.4.8}$$

i.e. a triangle from Fig. 4 (see also [60]). In this case the circle S^1 is illuminated by sources at points $\vec{v}^i$, $i = 1, 2, 3$, but not strongly illuminated. In agreement with Proposition the billiard B has finite volume, but $\bar{B}$ is not compact.

3.5. Discussions

We have obtained the "billiard representation" for the asymptotic cosmological model [37] and proved the geometrical criterion for the finiteness of the billiard volume and the compactness of the billiard (Proposition 1, Sec. 3.3). This criterion may be used as a rather effective (and universal) tool for the selection of the cosmological models with a stochastic behavior near the singularity.

For an "isotropic" component: $p_i^{(\alpha)} = (1-h)\rho^{(\alpha)}$, $i = 1, \ldots, n$, with $h \neq 0$ we have $(u^{(\alpha)})^2 = h^2(D-1)/(2-D) < 0$ and, hence, this component may be neglected near the singularity. Only "anisotropic" components with $(u^{(\alpha)})^2 > 0$ take part in the formation of billiard walls near the singularity. According to the topological bound (3.3.40) [69] the stochastic behavior near the singularity in the considered model may occur only if the number of components with $(u^{(\alpha)})^2 > 0$ is not less than the minisuperspace dimension.

We also note that here, like in the Bianchi-IX case [58,59], the considered reduction scheme uses a special time gauge (or parametrization of time). As it was pointed in [60] one should be careful in the interpretations of the results of computer experiments for other choices of time.

Restrictions on parameters. Here we discuss the physical sense of the restrictions on parameters of the model (3.3.3) and (3.3.4). The condition (3.3.3) means that the densities of the "anisotropic" components with $(u^{(\alpha)})^2 > 0$ should be positive. Using (3.2.8) and (3.2.36) we rewrite the restriction (3.3.4) in the equivalent form

$$\sum_{i=1}^{n} N_i \frac{\rho^{(\alpha)} - p_i^{(\alpha)}}{\rho^{(\alpha)}} > 0, \tag{3.5.1}$$

$(\rho^{(\alpha)} \neq 0)$ $\alpha = 1, \ldots, m$ (for curvature and Λ-terms (3.3.4) is satisfied). For

$$\rho^{(\alpha)} > 0, \qquad p_i^{(\alpha)} < \rho^{(\alpha)}, \tag{3.5.2}$$

$\alpha = 1, \ldots, m$, $i = 1, \ldots, n$, (3.5.1) is satisfied identically.

Remark 2. It may be shown that the condition (3.3.4) may be weakened by the following one

$$u_0^\alpha > 0, \ if \ (u^\alpha)^2 \leq 0. \tag{3.5.3}$$

In this case there exists a certain generalization of the set $B(u^\alpha)$ from (3.3.27) for arbitrary u_0^α $((u^\alpha)^2 > 0)$. The Proposition 1 (Sec. 3.3) should be modified by including into consideration the sources at infinity (for $u_0^\alpha = 0$) and "anti-sources" (for $u_0^\alpha < 0$). For "anti-source" the shadowed domain coincides with the illuminated domain for the usual source (with $u_0^\alpha > 0$). In this case we deal with the kinematics of tachyons. (We may also consider a covariant and slightly more general condition instead of (3.5.3)

$$\mathrm{sign} u_0^\alpha = \varepsilon, \ for \ all \ (u^\alpha)^2 \leq 0, \ \varepsilon = \pm 1.) \tag{3.5.4}$$

We note that for the component $\nu \in \Delta_+$ with $u_0^\nu < 0$ or, equivalently, $\sum_{i=1}^n u_i^{(\nu)} < 0$, the relation (3.4.37) should be substituted by

$$\sum_{i=1}^{n} u_i^{(\nu)} \alpha^1 < 0 \tag{3.5.5}$$

4. Dynamics of Inhomogeneities of Metric in the Vicinity of a Singularity in Multidimensional Cosmology

4.1. Introduction

As is well known a number of unified theories predict that dimension of the Universe exceeds that of we normally experience at a macroscopic level [23]. It is assumed that presently additional dimensions are hidden, for they are compactified to the Planckian size, and they do not display themselves in macroscopic and even in microscopic processes. However, the situation must be changed as we come back with time to the very beginning of the evolution of our Universe. Standard cosmological models predict the existence of a singular point at the very beginning and, therefore, the universe size could approach to the Planckian scale. Thus, in the early universe the additional dimensions, if exist, must not be different from ordinary dimensions and should be taken into account. Moreover, one could expect that the existence of additional dimensions may drastically change properties of the singularity and even remove it. The main aim of this section is to construct a general solution of multidimensional Einstein equations near a singularity and to investigate properties of inhomogeneities.

The way to construct a general solution with singularity was indicated first by Belinsky et al.in Ref. [57] for $D = 4$, where D is the dimension of a spacetime. Dynamics of metric at a particular point of space was shown to resemble the behaviour of the well studied "mixmaster" (or of the type-IX) homogeneous model and the last one has a complex stochastic nature [57,74]. Subsequent utilizing of that construction has been done in Ref.[53] where the so-called scalar-vector-tensor theory (or the case $D = 5$) was considered and the main feature of the mixmaster model, i.e. the complex oscillatory regime was shown to be also present in the 5-dimensional case.

An investigation of inhomogeneities of metric based on the general solutions has been considered first in Ref. [75]. The case of the scalar-tensor theory (or $D = 4+$ scalar fields) was considered and it turned out that the oscillatory regime leads to the fractioning of the coordinate scale λ of the inhomogeneities of Kasner exponents ($\lambda \approx \lambda_0 2^{-N}$, where N is the number of elapsed Kasner epochs and λ_0 is the initial scale of inhomogeneities). However, the methods by means of which the properties and statistics of the inhomogeneities were investigated turned out to be unapplicable for general case (i.e. for the absence of scalar fields as well as for the expanding universe). This problem has been solved recently in Ref. [61]. In this paper we generalize the results obtained in Ref. [61] to the case of arbitrary number of dimensions D.

As it was mentioned above the main features of the dynamics of an inhomogeneous gravitational field nearby the singularity in 4-dimensional case may be summarized as follows:

1. Locally dynamics of metric functions resembles the behaviour of the most general homogeneous "mixmaster" model [57], which has stochastic behaviour [74]. Just the stochastic behaviour leads to a monotonic decrease of the coordinate scale of the metric inhomogeneities [61,75].
2. In the vicinity of a singularity a scalar field is the only kind of matter effecting the dynamics of metric [53].

These facts may be simply understood under the following qualitative estimates (that is confirmed by subsequent consideration). As is well known in cosmology the horizon size l_h is a natural scale measuring a distance from the singularity. Therefore, inhomogeneities may be divided into the large-scale ($l_i \overline{g} l_h$) and small-scale ($l_i \ll l_h$) ones. The horison size varies with time as $l_h \sim t$ (where t is the time in synchronous reference system) whereas the characteristic spatial dimension of the inhomogeneity may be estimated as $l_i \sim t^\alpha$ (as $t \to 0$). In a linear theory for an isotropic background the exponent α may be expressed via the state equation of matter as $\alpha = \frac{2\epsilon}{3(p+\epsilon)}$ and what is important $\alpha < 1$. Thus, it is clear that an arbitrary inhomogeneous field becomes large-scale in the sufficient closeness to the singularity. Since the inhomogeneities are large-scale there are no effects connected with propagating of gravitational waves etc, and this would mean that inhomogeneities become passive. Consequently, dynamics of the field may be approximately described by the most general homogeneous model depending parametrically upon the spatial coordinates. Note, however, that the homogeneous model would appear to be in a general non-diagonal form.

The second fact may be understood in the same way. As it was shown in Ref. [76] the gravitational part of the Einstein equations at the singular point varies with time, in the leading order, as $R^\alpha_\beta \sim t^{-2}$ whereas the matter has the order $T^\alpha_\beta \sim t^{-2k}$, where k depends upon the state equation as $k = \frac{\epsilon+p}{2\epsilon}$. Thus, one can see that for the equation of state satisfying the inequality $p < \epsilon$ we have $k < 1$ and only for the limiting case $p = \epsilon$ ($k = 1$) the both sides turn out to be of the same order. We note that in the vicinity of a singularity scalar fields give just this equation of state.

As it is well known (see for example Ref.[21,37,53,77,78]) additional dimensions may be treated in ordinary gravity as a set of nonminimally coupled scalar and vector fields. Therefore, one could expect that the main contribution to dynamics in the vicinity of a singularity would be given by those dynamical functions which are connected with scalar fields, whereas other functions would play a passive role.

Thus, one could expect that in multidimensional cosmology local behaviour of the metric functions (at a particular point of space) will be described by a most general homogeneous model. Here, it is necessary to recall the important property of the mixmaster universe that is the stochastic behaviour. The problem of stochasticity of homogeneous multidimensional cosmological models has been investigated in a number of papers [18,54]. In particular, in Refs. [54] the result was obtained that chaos is absent in the spaces whose dimension $D \geq 11$, since in this case the last stage of a cosmological collapse is described by a minimally-coupled scalar field [53]. Therefore it seems to be sufficient to consider the Einstein equation of $D < 10$ dimensions with the scalar field matter source.

Thus, here we consider the D-dimensional Einstein equations with the matter source given by a minimally-coupled scalar field. Using generalized Kasner variables we divide the dynamical functions connected with physical degrees of freedom into two parts. One part has a simple behaviour while the other is described by a billiard on an appropriate Lobachevsky space. In dimensions $D < 11$ the billiard has a finite volume and shows stochastic properties. This stochasticity causes the degree of inhomogeneity of the part of dynamical functions and leads to the formation of spatial chaos. The presence of a scalar field results in the fact that lengths of trajectories on the billiard take finite values. This destroys the chaotic properties which, however, are restored in the limit when the

ADM energy density for the scalar field turns out to be small as compared with that of the gravitational variables.

4.2. Generalized Kasner Solution, Generalized Kasner Variables

We consider the theory in canonical formulation. Basic variables are the Riemann metric components $g_{\alpha\beta}$ with signature $(+,-,...,-)$ and a scalar field ϕ specified on the n-manifold S, and its conjugate momentum $\Pi^{\alpha\beta} = \sqrt{g}(K^{\alpha\beta} - g^{\alpha\beta}K)$ and Π_ϕ, where $\alpha = 1,...,n$ and $K^{\alpha\beta}$ is the extrinsic curvature of S. For the sake of simplicity we shall consider S to be compact i.e. $\partial S = 0$. The action has in Planck units the following form

$$I = \int_S (\Pi^{ij}\frac{\partial g_{ij}}{\partial t} + \Pi_\phi \frac{\partial \phi}{\partial t} - NH^0 - N_\alpha H^\alpha) d^n x dt, \tag{4.2.1}$$

where

$$H^0 = \frac{1}{\sqrt{g}}\left\{ \Pi^\alpha_\beta \Pi^\beta_\alpha - \tfrac{1}{n-1}(\Pi^\alpha_\alpha)^2 + \tfrac{1}{2}\Pi^2_\phi + g(W(\phi) - R) \right\}, \tag{4.2.2}$$

$$H^\alpha = -2\Pi^{\alpha\beta}_{|\beta} + g^{\alpha\beta}\partial_\beta\phi\Pi_\phi, \tag{4.2.3}$$

here

$$W(\phi) = \frac{1}{2}\left\{ g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi) \right\}. \tag{4.2.4}$$

A generalized Kasner solution is realized under the following assumption

$$\sqrt{g}T \sim (\Pi^\alpha_\beta, \Pi_\phi)\overline{g}V = g(W - R), \tag{4.2.5}$$

where $\sqrt{g}T$ denotes the first three terms in (4.2.2). Then, using (4.2.1) one can find the following solution of the multidimensional Einstein equations

$$ds^2 = dt^2 - \sum_{a=0}^{n-1} t^{s_a} l^a_\alpha \, , l^a_\beta dx^\alpha dx^\beta \tag{4.2.6}$$

where l^a_α, s_a are functions of space coordinates. Kasner exponents s_a satisfy the identities $\sum s_a = \sum s^2_a + q^2 = 1$, and run the domain $-\frac{n-2}{n} \le s_a \le 1$ (here $q^2 = \frac{(n-1)^2}{2}\frac{\Pi^2_\phi}{(\Pi^\alpha_\alpha)^2}$). Since, as it was shown in Ref.[53,57] the generalized Kasner solution takes a substantial portion of the evolution of metric it is convenient to introduce a Kasner-like parametrization of the dynamical variables [61]. We consider the following representation for metric components and their conjugate momenta

$$g_{\alpha\beta} = \sum_a \exp\{q^a\}\, l^a_\alpha l^a_\beta \, , \tag{4.2.7}$$

$$\Pi^\alpha_\beta = \sum_a p_a L^\alpha_a l^a_\beta \, , \tag{4.2.8}$$

here $L^\alpha_a l^b_\alpha = \delta^b_a$ $(a, b = 0, ..., (n-1))$, and the vectors l^a_α contain only $n(n-1)$ arbitrary functions of spatial coordinates. Further parametrization may be taken in the following form

$$l^a_\alpha = U^a_b S^b_\alpha, \; U^a_b \in SO(n), \; S^a_\alpha = \delta^a_\alpha + R^a_\alpha \tag{4.2.9}$$

where R^a_α denotes a triangle matrix ($R^a_\alpha = 0$ as $a \leq \alpha$). Substituting (4.2.7) - (4.2.9) into (4.2.1) one gets the following expression for the action functional

$$I = \int_S (p_a \frac{\partial q^a}{\partial t} + T^\alpha_a \frac{\partial R^a_\alpha}{\partial t} + \Pi_\phi \frac{\partial \phi}{\partial t} - NH^0 - N_\alpha H^\alpha) d^n x dt, \tag{4.2.10}$$

here $T^\alpha_a = 2\sum_b p_b L^\alpha_b U^b_a$ and the Hamiltonian constraint takes the form

$$H^0 = \frac{1}{\sqrt{g}} \{ \sum p_a^2 - \tfrac{1}{n-1} (\sum p_a)^2 + \tfrac{1}{2}\Pi^2_\phi + V \} . \tag{4.2.11}$$

In the case of $n = 3$ the functions R^a_α are connected purely with transformations of a coordinate system and may be removed by solving momentum constraints $H^\alpha = 0$. In the multidimensional case the functions R^a_α contain $\frac{n(n-3)}{2}$ dynamical functions as well. Now it is easy to see that the choice of Kasner-like parametrization simplifies the procedure of the constructing of the generalized Kasner solution. Indeed, if we now neglect the potential term in (4.2.10) and put $N^\alpha = 0$ we find that Hamiltonian does not depend on the scale functions and other dynamical variables contained in Kasner vectors introduced by expressions (4.2.7) (4.2.8).

4.3. The asymptotic model in the vicinity of a cosmological singularity

As it is well known, [53], [57], the Kasner regime (4.2.6) turns out to be unstable in a general case. This happens due to the violation of the condition (4.2.5) because the potential V contains increasing terms which lead to replacement of Kasner regimes. To find out the law of replacement it is more convenient to use an asymptotic expression for the potential [61], [55]. For this aim we put the potential in the following form

$$V = \sum_{A=1}^{k} \lambda_A g^{u_A}, \tag{4.3.1}$$

here λ_A is a set of functions of all dynamical variables and of their derivatives and u_A are linear functions of the anisotropy parameters $Q_a = \frac{q^a}{\sum q}$ ($u_A = u_A(Q)$). Assuming the finiteness of the functions λ and considering the limit $g \to 0$ we find that the potential V may be modeled by potential walls

$$g^{u_A} \to \theta_\infty[u_A(Q)] = \begin{cases} +\infty, & u_A < 0, \\ 0, & u_A > 0 \end{cases} \tag{4.3.2}$$

Thus, putting $N^\alpha = 0$ we can remove the passive dynamical function T^α_a, R^a_α from the action (4.2.10) and get the reduced dynamical system

$$I = \int_S \left\{ p_a \tfrac{\partial q^a}{\partial t} + \Pi_\phi \tfrac{\partial \phi}{\partial t} - \lambda \left\{ \sum p^2 - \tfrac{1}{n-1} (\sum p)^2 + \tfrac{1}{2}\Pi^2_\phi + U(Q) \right\} \right\} d^n x dt, \tag{4.3.3}$$

here λ is expressed via the lapse function as $\lambda = \frac{N}{\sqrt{g}}$. In harmonic variables the action (4.3.3) takes the form formally coincided with the action for a relativistic particle

$$I = \int_S \{ P_r \tfrac{\partial z^r}{\partial t} - \lambda' (P_i^2 + U - P_0^2) \} d^n x dt, \tag{4.3.4}$$

here $r = 0, ..., n$, $i = 1, ..., n$, $q^a = A^a_j z^j + z^0$ $(j = 1, ..., n-1)$, $z^n = \sqrt{\frac{n(n-1)}{2}}\phi$ and the constant matrix A^a_j obeys the following conditions

$$\sum_a A^a_j = 0, \; \sum_a A^a_j A^a_k = n(n-1)\delta_{jk} \tag{4.3.5}$$

and can be expressed in the following form

$$A^a_j = \sqrt{\frac{n(n-1)}{j(j-1)}}(\theta^a_j - j\delta^a_j),$$

where $\theta^a_j = \left\{ \begin{array}{l} 1, j > a \\ 0, j \leq a \end{array} \right.$.

Since the timelike variable z^0 varies during the evolution as $z^0 \sim \ln g$ the positions of potential walls turn out to be moving. It is more convenient to fix the positions of walls. This may be done by using the so-called Misner-Chitre like variables [55] ($\vec{y} = y^j$)

$$z^0 = -e^{-\tau}\frac{1+y^2}{1-y^2}, \vec{z} = -2e^{-\tau}\frac{\vec{y}}{1-y^2}, \qquad y = \mid \vec{y} \mid < 1. \tag{4.3.6}$$

Using these variables one can find the following expressions for the anisotropy parameters

$$Q_a(y) = \frac{1}{n}\left\{ 1 + \frac{2A^a_j y^j}{1+y^2} \right\}, \tag{4.3.7}$$

which are now independent of timelike variable τ. From (4.3.7) one can find the range of the anisotropy functions $-\frac{n-2}{n} \leq Q_a \leq 1$.

Choosing as a time variable the quantity τ (i.e. in the gauge $N = \frac{n(n-1)}{2}\sqrt{g}\exp(-2\tau)/P^0$) we put the action (4.3.4) into the ADM form

$$I = \int_S \{ \vec{P}\frac{\partial}{\partial\tau}\vec{y} + P^n\frac{\partial}{\partial\tau}z^n - P^0(P, y) \} d^n x d\tau, \tag{4.3.8}$$

where the quantity

$$P^0(P, y) = (\epsilon^2(\vec{y}, \vec{P}) + V[y] + (P^n)^2 e^{-2\tau})^{1/2}, \tag{4.3.9}$$

plays the role of the ADM Hamiltonian density and

$$\epsilon^2 = \frac{1}{4}(1-y^2)^2\vec{P}^2. \tag{4.3.10}$$

The part of the configuration space connected with the variables $\vec{y}$ is a realization of the $(n-1)$ -dimensional Lobachevsky space [64] and the potential V cuts a part of it. Thus, locally (at a particular point of S) the action (4.3.9) describes a billiard on the Lobachevsky space. The positions of walls which form the boundary of the billiard are determined, due to (4.3.1) by the inequalities

$$\sigma_{abc} = 1 + Q_a - Q_b - Q_c \geq 0, \; a \neq b \neq c \tag{4.3.11}$$

and the total number of walls is $\frac{n(n-1)(n-2)}{2}$. Using the matrix (4.3.5) one can find that the walls are formed by spheres determined by the equations

$$\sigma_{abc} = \frac{n-1}{n(1+y^2)} \{ (\vec{y} + \vec{B}_{abc})^2 + 1 - B^2_{abc} \} , \; \vec{B}_{abc} = \frac{1}{n-1}(\vec{A}^a - \vec{A}^b - \vec{A}^c), \tag{4.3.12}$$

here for arbitrary a, b, c we have $B^2 = 1 + \frac{2n}{n-1}$. In a general case n points of the billiard having the coordinates $\vec{P}_a = \frac{1}{n-1}\vec{A}^a$ lie on the absolute (at infinity of the Lobachevsky space). The trajectories which end with these points correspond to the set of Kasner exponent $(0, \cdots 0, 1)$. When $n = 9$ there appear additional isolated points S_{ab} lying on the absolute. The coordinates of these points are given by the vectors $\vec{S}_{abc} = \frac{1}{12}(\vec{A}_a + \vec{A}_b + \vec{A}_c)$, $a \neq b \neq c$ (see appendix). In the case of $n \geq 10$ in addition to the points P_a and S_{abc} there appear open accessible domains on the absolute (see appendix of Refs. [54] where has been used another approach) and the volume of the billiard becomes infinite. If on the contrary $n < 10$, the volume of the billiard is finite and the billiard turns out to be a mixing one. We give two simplest examples for illustration of the billiards on fig.5. The case $n = 3$ on fig.5a coincides with the well-known "mixmaster" model and on fig.5b we illustrate the case of $n = 4$ considered in Ref.[53].

4.4. Dynamics of inhomogeneities

The system (4.3.8) has the form of the direct product of "homogeneous" local systems. Each local system in (4.3.8) has two variables ϵ and P^n as integrals of motion. The solution of this local system for remaining functions represents a geodesic flow on a manifold with negative curvature. As it is well known the geodesic flow on a manifold with negative curvature is characterized by exponential instability [64]. This means that during the motion along a geodesic the normal deviations grow no slower than the exponential of the traversed path $\xi \simeq \xi_0 e^s$), where the traversed path is determined by the expression

$$s = \int_{\tau_0}^{\tau} dl = \int_{\tau_0}^{\tau} \frac{2 \mid \frac{\partial y}{\partial \tau} \mid}{(1-y^2)} d\tau = \frac{1}{2} \ln \mid \frac{P^0 - \epsilon}{P^0 + \epsilon} \mid_{\tau_0}^{\tau} . \tag{4.4.1}$$

This instability leads to the stochastic nature of the corresponding geodesic flow. The system possesses the mixing property [65] and an invariant measure induced by the Liouviulle one

$$d\mu(y, P) = const\delta(E - \epsilon) d^{n-1}y d^{n-1}P, \tag{4.4.2}$$

where E is a constant. Integrating this expression over ϵ we find

$$d\mu(y, s) = const \frac{d^{n-1}y d^{n-2}s}{(1-y^2)^n}, \tag{4.4.3}$$

where $\vec{s} = \frac{\vec{P}}{\epsilon}$, $|s| = 1$.

Since the inhomogeneous system (4.3.8) is the direct product of "homogeneous" systems one can simply describe its behaviour as in ref [61]. In particular, the scale of the inhomogeneity decreases as

$$\lambda_i \sim (\frac{\partial y}{\partial x})^{-1} \sim \lambda_i^0 \exp(-s) \tag{4.4.4}$$

and after sufficiently large time ($s(\tau) \to \infty$) the dynamical functions $\vec{y}(x)$, $\vec{P}(x)$ become a random functions of the spatial coordinates. In order to calculate different mean values one can use the following n-point distribution functions [61]

$$\rho_{x_1 \cdots, x_n}(y_1, \cdots, y_n, m_1 \cdots, m_n) =< \prod_{i=1}^{n} \delta(y_i - y(x_i))\delta(m_i - m(x_i)) >, \tag{4.4.5}$$

where the angular brackets can denote the averaging out either over an initial distribution or over a certain coordinate volume $\Delta V >> (\lambda_i^0)^3$. The mixing results in the relaxation of initial functions (4.4.5) to the limiting ones which have the form of the direct product of measures (4.4.3): $d\mu = \prod_i d\mu_i$. Thus, asymptotic expressions for averages and correlating functions have the form

$$< y(\vec{x}) >=< P(\vec{x}) >= 0, < y_k(x), y_l(x') >=< y_k, y_l > \delta(x, x'), \tag{4.4.6}$$

for $|x - x'| \overline{g} \lambda_i^0 \exp(-s)$.

Here it is necessary to point out a role of the scalar field in dynamics and statistical properties of inhomogeneities. As may be easily seen from (4.4.1) in the absence of a scalar field (i.e. $P^n = 0$) the transversed path coincides with the duration of motion (we have $s = \Delta\tau = \tau - \tau_0$ instead of (4.4.1)). Thus, the effect of scalar fields is displayed in the replacement of the dependence for transversed path of time variable and, therefore, in the replacement of the rate of increasing of the inhomogeneities. This replacement does not change qualitatively the evolution of the universe in the case of cosmological expansion. But in the case of the contracting universe the situation changes drastically. Indeed, in the limit $\tau \to -\infty$ from (4.4.1) we find that the transversed path s takes a limited value s_0 and therefore the increasing of inhomogeneities turns out to be finite. One of consequences of such behaviour is the fact that at the singularity the functions $\vec{y}$ and $\vec{P}$ take constant values. In other words in the presence of scalar fields a cosmological collapse ends with a stable Kasner-like regime (4.2.6). This fact may be seen in the other way. Indeed, in the limit $\tau \to -\infty$ the scalar field gives the leading contribution in ADM Hamiltonian (4.3.9) and P^0 does not depend on gravitational variables at all.

The finiteness of the transversed path $s(\tau)$ leads, generally speaking, to the destruction of the mixing properties [65], since for establishment of the invariant measure it is necessary to satisfy the condition $s_0 \to \infty$. Evidently, this condition requires the smallness of the energy density for scalar field as compared with the ADM energy of gravitational field (the last term in (4.3.9) in comparison with the first ones). Indeed, in this case s_0 is determined by the expression $s_0 = -\ln \frac{P^n e^{\tau_0}}{2\epsilon}$, which follows from (4.4.1), and as $P^n \to 0$ one get $s_0 \to \infty$ (i.e. s can have arbitrary large values).

Thus, in the case of cosmological contraction one may speak of the mixing and, therefore, of establishment of the invariant statistical distribution just only for those spatial domains which have suficiently small energy density of the scalar field.

4.5. Estimates and concluding remarks

In this manner the large-scale structure of the space in the vicinity of singularity acquires a quasi-isotropic nature. A distribution of inhomogeneities is determined by the set of functions of spatial coordinates $\epsilon(x)$, $\Pi_\phi(x)$ and R^a_α which conserve during the evolution a primordial degree of inhomogeneity of the space. The scale of inhomogeneity of other functions grows as $\lambda \approx \lambda_0 e^{-s(\tau)}$. In this section we give some estimates clarifying the behaviour of the inhomogeneities. For simplicity we consider the case when the scalar field is absent.

To find the estimate for the inhomogeneity growth in a synchronous time t ($dt = Nd\tau$) we put $y = 0$. Then for variation of the variable τ one may find the following estimate $\sqrt{g} \sim \exp(-\frac{n}{2}e^{-\tau}) \sim P^0 t$, (here the point $t = 0$ corresponds to the singularity). According to (4.4.4) the dependence of the coordinate scale of inhomogeneity upon the time t takes the form

$$\lambda \approx \lambda_0 \ln(1/g_0)/\ln(1/g)$$

in the case of contracting ($g \to 0$) and

$$\lambda \approx \lambda_0 \ln(1/g)/\ln(1/g_0)$$

in the case of the expanding universe.

A rapid generation of the more and more small scales leads to the formation of spatial chaos in metric functions and so the large-scale structure acquires a quasi-isotropic nature. Speeds of the scale growing (Hubble constants) for different directions turn out to be equal after averaging over a spatial domains having the size $\approx \lambda_0$. Indeed, using (4.3.7) one may find the expressions for averages $< Q_a >= 1/n$.

Besides, it is necessary to mention one more characteristic feature of the oscillatory regime in the inhomogeneous case. This is the formation of a cellular structure in the scale functions Q_a during the evolution which demonstrate explicitely the stochastic process of development of inhomogeneities. Indeed, let us consider some region of coordinate space ΔV. Two functions $y(\vec{x})$ define the map of that region on some square $\Sigma \in K$ (see fig.1c). During the evolution the size of the square Σ grows $\simeq e^{s(\tau)}$ and Σ covers the domain of the billiard K many times. Each covering determines its own preimage in ΔV. In this manner the initial coordinate volume is splitted up in "cells" $\Delta V = \bigcup_i \Delta V_i$. In the every cell the vector $y(\vec{x})$ takes almost all admissible values $\vec{y} \in K$ and that of the functions $Q_a(Q_a \in [Q_{min}, 1]$ where $Q_{min} = -\frac{(n-1)^2-(n+1)}{n(n+1)}$. To illustrate this process let us consider the case $n = 3$. In this case it is convenient to use the Poincaré model of the Lobachevsky plane on the upper complex half-plane $H = \{W = U + iV, V \geq 0\}$ (see fig.5c). The line $V = 0$ is called the absolute and its points lie at infinity. Geodesics in H are given by semi-circles with centers on the absolute, or by rays perpendicular to the absolute. The billiard constitutes the region $K \in H$, bounded by geodesics triangle $\partial K = [|W| = 1, U = \pm 1]$. The area of the billiard is equal to π. The motion can be continued to the whole plane H. For this aim one needs to reflect the domain of the billiard with respect to one of the boundary walls and make iteration of such procedure. In this way the Lobachevsky plane will be covered by a set of domains K^n each of which is connected with the region of the billiard K by a one-to-one mapping. During the evolution an arbitrary initial square Σ^0 begins to grow and covers the more and more number of the domains K^n (see fig.5c). Such cellular structure turns out to be depending on time and the number of cells increases as $N \approx N_0 e^{s(\tau)}$. However, the situation will be changed if we consider a contracting space filled with a scalar field. Then the evolution of this structure in the limit $g \to 0$ ends, because the functions Q_a become independent of time, and on the final stage of the collapse one would have a real cellular structure [75].

In spite of the isotropic nature of the spatial distribution of the field the large local anisotropy displays itself in the anomalous dependence of spatial lengths upon time variable for vectors and curves. Indeed, a moment of scale function $< g^{MQ_a} >$ (where $M > 0$) decreases in the asymptotic $g \to 0$ as the Laplace integral $\int_{Q_{min}}^{1} g^{MQ_a} \rho(Q_a) dQ_a$, where $\rho(Q_a)$ is the distribution which follows from (4.4.3). The main contribution in this integral is given by the point $Q = Q_{min}$ and $Q = Q_{min}$ and in the case of $n > 3$ in the limit $(Q - Q_{min}) \to 0$ one can find $\rho(Q) \approx C(Q - Q_{min})^{n-1}$, where C is a constant and we obtain the estimate

$$< g^{MQ_a} > \approx \frac{g^{MQ_{min}}}{(M \ln 1/g)^{n-1}}. \tag{4.5.1}$$

This expression shows that for $n > 3$ average lengths even increase while approaching the singularity. The case $n = 3$ must be considered separately. In this case we have $Q_{min} = 0$ and the explicit form of the distribution function $\rho(Q_a)$, as it follows from (4.4.3), is

$$\rho(Q) = \frac{2}{\pi}(Q(1 - Q))^{-1/2}(1 + 3Q)^{-1}. \tag{4.5.2}$$

As $Q \ll 1$ one has $\rho(Q_a) \approx \frac{2}{\pi}(Q_a)^{-1/2}$ and, thus, in the limit $g \to 0$ we get the estimate

$$< g^{MQ_a} > \approx (M \ln(1/g))^{-1/2}. \tag{4.5.3}$$

In conclusion we briefly repeat the main results. The general ihomogeneous solution of D-dimensional Einstein equations with any matter sources satisfying the inequality $\epsilon \geq p$ near the cosmological singularity is constructed. It is shown that near the singularity a local behavior of metric functions (at a particular point of the coordinate space) is described by a billiard on the $(D-1)$-dimensional Lobachevsky space. In the case of $D < 11$ the billiard has a finite volume and consequently a mixing one. The rate of growth of inhomogeneities of metric is obtained. Statistical properties of inhomogeneities are described by the invariant measure. It is shown that a minimally-coupled scalar field leads, in general, to the distruction of stochastic properties of the inhomogeneous model.

Appendix

Here we show that the billiards in the dimensions exceeding $n = 9$ become infinite. Let us introduce a new set of variables connected with the old ones as $\vec{x} = \frac{2\vec{y}}{1+y^2}$. Within these variables the absolute of the Lobachevsky space keeps the old position $|x^2| = 1$ and the walls become planes (see (4.3.7), (4.3.12)). Furthermore, it will be more convenient to select a region on the Lobachevsky space on which the anisotropy parameters are in the increasing order $Q_0 \leq Q_1 \leq \cdots \leq Q_{n-2} \leq Q_{n-1}$ and which is restricted by the only wall (see (4.3.11)) $\sigma(\vec{x}) = \sigma_{1,n-2,n-1}$. This region is formed by the vectors of the type $\vec{x} = \Sigma_{i=1}^{n-1} u^i \vec{e}_i$, where the parameters $0 \leq u^i \leq 1$ and the set of basic vectors is given by: $\vec{e}_i = \frac{1}{n+1} \sum_{a=i}^{n-1} \vec{A}^a$ for $i \leq n-2$, $\vec{e}_{n-2} = \frac{1}{2(n-1)}(\vec{A}^{n-2} + \vec{A}^{n-1})$ and $\vec{e}_{n-1} = \frac{1}{n-1}\vec{A}^{n-1}$. They are normalized so that $\sigma(\vec{e}_i) = 0$. It is easy to find that the wall causes the restrictions on the parameters $u^i : \sum u^i \leq 1$. The Euclidian norms of the basic vectors are $e_i^2 = \frac{i(n-i)(n-1)}{(n+i)^2}$ for $i \leq n-2, e_{n-2}^2 = \frac{n-2}{2(n-1)}$ and $|e_{n-1}| = 1$ (here we used the following property of $\vec{A}^a : \sum_{k=1}^{n-1} A_k^a A_k^b = n(n-1)\delta^{ab} - (n-1)$). Now, it is easy to find that for $n < 9$ all basic vectors except $\vec{e}_{n-1}$ have norms less than unity and we have $|\vec{x}| \leq 1$ (equality is achieved only when $\vec{x} = \vec{e}_{n-1}$). In the case $n = 9$ we get $e_3^2 = e_8^2 = 1$, all the other vectors have norms less than unity and we have the similar situation as above (i.e., $|x| = 1$ only when $\vec{x} = \vec{e}_3$ and $\vec{x} = \vec{e}_8$). In the case $n > 9$ a number of basic vectors have norms exceeding unity, e.g., $\vec{e}_i$ for $i = \left[\frac{n}{3}\right] + 1$ or $i = \left[\frac{n}{3}\right] + 1$, where $\left[\frac{n}{3}\right]$ denotes the entire part of the number $\frac{n}{3}$. This means that the wall in these directions lies outside the absolute of the Lobachevsky space and there appears an open accessible domain. In other words, the trajectories do not meet any obstacle in these directions and run to the infinity. This proves the statement made in Sec. 4.3.

5. Multidimensional Cosmology and the Time Variation of G: a Dynamical System Approach [79]

5.1. Introduction

Multidimensional cosmology has since long ago attracted the attention of cosmologists, who were stimulated initially mainly by the Kaluza-Klein theory [80-81] and more recently by superstrings models [23]. The idea that the Universe we live in can be represented as a 4-dimensional hypersurface imbedded in a (4+n)-spacetime manifold has actually different versions. In particular, we could mention the one put forward by Wesson, who has developed an embedding scheme in which the Friedmann-Robertson-Walker-Lemaitre cosmology can be entirely obtained in a rather simple and elegant way from (4+1)-dimensional Ricci-flat spacetimes [82-83]. Further generalization of this theory to arbitrary dimensionality with applications to multidimensional cosmology and lower dimensional gravity was later carried out by Rippl et al [84]. General multidimensional and multicomponent schemes were studied in [21] (see also refs. therein).

In addition to the role multidimensional theories might play in providing a theoretical framework in which the most fundamental laws of physics appear to be unified, another motivation may come from a conjecture - originally proposed by Dirac [85] - regarding the time variation of the Newtonian gravitational constant G. Indeed, this idea, which was to be taken seriously by superstrings theory and recent inflationary models, is also present in the context of multidimensional cosmological models where G is considered not as a fundamental constant of Nature, but as a cosmological function depending on the geometry of an 'internal space' [12,50,86].

Among the several attempts to construct gravity theories with varying G is Brans-Dicke theory, where the strength of the gravitational force is determined by a scalar field [87,88]. Here we find again the same idea underlying the connection between higher dimensions and time variation of G, as it can be shown that n-dimensional Kaluza-Klein models reduce to Brans-Dicke vacuum models for $w = 0$. Other theories with scalar field (especially conformal) see in [50].

In this section we consider, as in [21], a (4+n)-spacetime manifold defined by the topological product $M^{4+n} = R \times M_k^3 \times K^n$, where M_k^3 is a 3-dimensional space of constant curvature (i.e.,$M_k^3 = S^3, R^3, L^3$ according to $k = +1, 0, -1$, respectively), and K^n is a n-dimensional Ricci-flat manifold. We assume also that this spacetime is generated by a (4+n)-dimensional multicomponent perfect fluid.

Now, it turns out that the field equations for the special case $k = 0$ may be reduced to an autonomous homogeneous system of the second order. This system contains some free parameters, one of them being n (the dimensionality of the internal space) and the others come from the equations of state of the multicomponent-fluid. However, by restricting ourselves to 'dust-like' matter, we are left with n as the only parameter of the system. Then, we construct the phase diagram of the system to obtain a general picture of the solutions. As a by-product of the analysis we also obtain analytical solutions of the equations for arbitrary values of n (see also [14]).

5.2. The field equations

The gravitational field equations in a (4+n)-dimensional gravity are postulated to be

$$ {}^{(4+n)}R_{\mu\nu} = \kappa^2 \left({}^{(4+n)}T_{\mu\nu} - g_{\mu\nu}\frac{T}{(n+2)} \right), \tag{5.2.1} $$

where all the geometric quantities are defined in $(4+n)$ dimensions and κ^2 is the generalized Einstein constant [21]. We take the metric tensor to be given by the line element

$$ ds^2 = dt^2 - R^2(t) {}^{(3)}g_{ij}(x^k)dx^i dx^j - b^2(t) {}^{(n)}g_{pq}(y^r)dy^p dy^q, \tag{5.2.2} $$

where $i,j,k = 1,2,3; p,q,r = 4,...,n+3$; ${}^{(3)}g_{ij}$, ${}^{(n)}g_{pq}$, $R(t)$ and $b(t)$ are, respectively, the metrics and scale factors for ${}^{(3)}M_k$ and K^n. The $(4+n)-$dimensional energy-momentum tensor for a multicomponent perfect fluid is taken to be

$$ T^{\mu}_{\nu} = diag(\varrho(t), -p_3(t)\delta^i_j, -p_n(t)\delta^m_n) \tag{5.2.3} $$

From (5.2.2) and (5.2.3) the Einstein equations become:

$$ 3\frac{\ddot{R}}{R} + n\frac{\ddot{b}}{b} = \frac{\kappa^2}{n+2}\left(-(n+1)\varrho - 3p_3 - np_n\right), \tag{5.2.4} $$

$$ \frac{2k}{R^2} + \frac{\ddot{R}}{R} + n\frac{\dot{b}}{b}\frac{\dot{R}}{R} + 2\frac{\dot{R}^2}{R^2} = \frac{\kappa^2}{n+2}\left(\varrho + (n-1)p_3 - np_n\right) \tag{5.2.5} $$

$$ \frac{\ddot{b}}{b} + (n-1)\frac{\dot{b}^2}{b^2} + 3\frac{\dot{R}}{R}\frac{\dot{b}}{b} = \frac{\kappa^2}{n+2}\left(\varrho - 3p_3 + 2p_n\right) \tag{5.2.6} $$

At this point it is worthwhile mentioning the way by which higher dimensional gravity theories of this type can be naturally related to their 4-dimensional counterparts with varying G [21]. This is simply done by integrating the $(4+n)$-dimensional energy density over the K^n compact space and equating the result to ${}^{(4)}\varrho(t)$, thereby defining the energy density in 4-dimensional spacetime:

$$ {}^{(4)}\varrho(t) = \int_{K^n} dy^n \sqrt{{}^{(n)}g}\, b^n(t)\varrho(t) = \varrho(t)b^n(t), \tag{5.2.7} $$

where $\sqrt{{}^{(n)}g}$ is the determinant of ${}^{(n)}g_{pq}$. It is convenient to 'normalize' the scale factor $b(t)$ by imposing the condition $\int_{K^n} \sqrt{{}^{(n)}g}dy^n = 1$. Thus, in order to get the equations of the 4-dimensional gravity we put

$$ 8\pi G(t)\left[{}^{(4)}\varrho(t)\right] = \kappa^2\varrho(t). \tag{5.2.8} $$

This procedure leads us to the definition of an effective gravitational 'constant' $G(t)$ given by $8\pi G(t) = \kappa^2 b^{-n}(t)$. In this way the time variation of G is directly related to the time variation of the internal space scale factor $b(t)$ by

$$ \frac{\dot{G}}{G} = -n\frac{\dot{b}}{b} \tag{5.2.9} $$

Clearly for $n = 0$ the Friedmann cosmology in ordinary 4-dimensional spacetime is recovered.

5.3. The dynamical system and the phase portraits

In this section we let $M_k^3 = R^3$ and assume that the multicomponent fluid satisfies the equations of state $p_3 = p_n = 0$, i.e., we assume that matter behaves as a $(n+4)$-dimensional 'dust'. Then, letting $x = \frac{3\dot{R}}{R}$ and $y = \frac{\dot{b}}{b}$ the equations (5.2.4-6) become

$$\dot{x} + \frac{x^2}{3} + n\dot{y} + \dot{y}^2 = -\frac{n+1}{n+2}\kappa^2\varrho \tag{5.3.1}$$

$$\dot{x} + x^2 + Nxy = \frac{3\kappa^2\varrho}{n+2} \tag{5.3.2}$$

and

$$\dot{y} + ny^2 + xy = \frac{\kappa^2\varrho}{n+2}. \tag{5.3.3}$$

Eliminating ϱ from these equations results in

$$\dot{x} = \frac{1}{2(n+2)}\left[-2(n+1)x^2 + 2n(1-n)xy + 3n(n-1)y^2\right] \tag{5.3.4}$$

and

$$\dot{y} = \frac{1}{2(n+2)}\left[\frac{2x^2}{3} - 4xy - n(n+5)y^2\right] \tag{5.3.5}$$

Defined[1] in this way x can be interpreted as a measure of the usual cosmological expansion of the 4-dimensional observable Universe, while y is a measure of the time variation of the gravitational constant G or, equivalently, the expansion of the compact space K^n (see eq.(5.2.9)). The above system of equations represents a homogeneous autonomous dynamical system of the second-order. To carry out an analysis of this system we first note that, as the system is homogeneous, the origin of the phase space $x = y = 0$ corresponds to an equilibrium point (in fact , an isolated equilibrium point) [89]. Physically, this point represents nothing else but the flat Minkowski spacetime of General Relativity, with $\varrho = 0$.

In order to construct the phase diagram of a homogeneous dynamical system we first determine the *invariant rays* of the system [89] by introducing the polar coordinates in the phase plane: $x = r\cos\theta, y = r\sin\theta$. In these coordinates a general homogeneous dynamical system of order m of the form

$$\dot{x} = X_m(x,y), \dot{y} = Y_m(x,y)$$

is transformed into

$$\dot{r} = r^m Z(\theta), \dot{\theta} = r^{m-1}N(\theta),$$

where the functions $Z(\theta)$ and $N(\theta)$ are given by

$$Z(\theta) = Y_m(\cos\theta, \sin\theta)\sin\theta + X_m(\cos\theta, \sin\theta)\cos\theta \tag{5.3.6}$$

[1]It is possible, of course, to absorb the factor $\frac{1}{2(n+2)}$ defining a new time $d\tau = 2(n+2)dt$. However, nothing is gained by this in terms of simplicity.

$$N(\theta) = Y_m(\cos\theta, \sin\theta)\cos\theta - X_m(\cos\theta, \sin\theta)\sin\theta. \tag{5.3.7}$$

Then, the invariant rays of the system are obtainded by solving the equation $N(\theta) = 0$. Clearly, in the phase plane they will be depicted as straight semi-lines starting from the origin and it is not difficult to see that if they do exist then they are automatically solutions of the dynamical system [89]. In our case $m = 2$ and a straightforward calculation leads to

$$\begin{aligned} Z(\theta) &= \frac{1}{2(n+2)}\Big[-n(n+5)\sin^3\theta + (3n^2 - 3n - 4)\sin^2\theta\cos\theta \\ &+ (2n - 2n^2 + \frac{2}{3})\sin\theta\cos^2\theta - 2(n+1)\cos^3\theta\Big] \end{aligned} \tag{5.3.8}$$

$$\begin{aligned} N(\theta) &= \frac{1}{2(n+2)}\Big[-3n(n-1)\sin^3\theta + n(n-7)\sin^2\theta\cos\theta \\ &+ 2(n-1)\cos^2\theta\sin\theta + \frac{2}{3}\cos^3\theta\Big]. \end{aligned} \tag{5.3.9}$$

Here let us make some comments. First, we should point out that the dynamical system (5.3.4-5) is not defined for $n = 0$, since in this case we would not have equation (5.2.6). If $n = 1$, then the solutions of the equation $N(\theta) = 0$ yield six invariant rays which correspond to the angles $\theta_i = \pm\frac{\pi}{2}$ and $\arctan(\pm\frac{1}{3})$, with $i = 1, ..., 6$. For an arbitrary $n > 1$ we can put the equation (5.3.9) in the following factorized form:

$$N(\theta) = \frac{\cos^3\theta}{2n+4}\left\{(\frac{1}{3} - a)[3n(n-1)a^2 + 6na + 2]\right\} \tag{5.3.10}$$

where we have defined $a = \tan\theta$. Then, for $n > 1$ we have again six invariant rays, now corresponding to the angles $\theta_i = \arctan a_i$, with

$$a_0 = \frac{1}{3}, a_\pm = \frac{1}{n-1}\left(-1 \pm \sqrt{\frac{1}{3}(1 + \frac{2}{n})}\right)$$

See figs. 6 and 7. The knowledge of the invariant rays as well as the analytic expressions for the functions $N(\theta)$ and $Z(\theta)$ allow us to draw separately the following phase diagrams for the two cases $n = 1$ and $n > 1$ (for details see appendix). These diagrams show the behaviour of all solutions of the equations (5.3.4-5) which make up our dynamical system. Each curve corresponds to a specific cosmological model satisfying the field equations (5.3.4-5), the origin representing the Minkowski spacetime M. In order to know the behaviour of the solutions at the infinity we employed a method due to Poincare', consisting of projecting the phase plane onto a plane circle [93]. In this compactified phase plane the points at infinity correspond to points located in the border of the circle. The directions of the invariant rays are not affected by the transformation (see appendix 5.8).

5.4. The physical picture

Let us begin our analysis considering $n > 1$, and leave the comments on the case $n = 1$ to the end of this section. In figure 6 we have a typical diagram for arbitrary $n > 1$. First we note that the invariant rays divide up the phase plane in six topologically distinct regions (or sectors) A,B,...,F. Each of these regions contains an infinite number of solutions which represent cosmological models with different physical properties. The arrows in the curves are to be interpreted as the time evolution of the corresponding models.
Since there is no closed curve in the phase plane we can conclude that all models are singular (the expansion parameter x tends to infinity either in the past or in the future), some of them starting from a big-bang ($x \rightarrow +\infty$)while others collapsing to a big-crunch ($x \rightarrow -\infty$). In this sense the solutions represented by the invariant rays exhibit the same behaviour. It would be rather tedious to describe exhaustively the time evolution of the models corresponding to all the curves of the phase diagram. So, we will pick up some illustrative cases, although the complete informations about all solutions are provided by the phase portrait.

To begin with let us consider the solution represented by the invariant ray depicted in figure 61 as the semi-line I^+. This curve clearly describes a universe starting from a big-bang ($x = +\infty$) and evolving towards the Minkowski spacetime (depicted in the diagram as the fixed point M located at the origin) . Since $y > 0$ along this trajectory we see that as time goes by the gravitational constant G decreases. This is in agreement with the known hypothesis formulated by Dirac who, postulated, inspired on a different reasoning (*the large numbers conjecture*), that Newtonian gravitational constant should decrease as the Universe expands [85].

Analogously, the same analysis shows us that the invariant ray II^+ corresponds to an expanding universe starting from a big-bang and tending to Minkowski spacetime. Since y is negative in this *anti-Dirac* universe the gravitational constant G increases with the cosmic time.

The invariant rays I^+ and II^+ encloses an infinite class of solutions all lying within the region A. A typical solution of this class describes an expanding and singular universe undergoing a transition from an increasing G (*anti-Dirac*) to an decreasing G era (*Dirac phase*).

A quite different situation arises when one examines the solution corresponding to the invariant ray III^+. Here we observe an initially static universe ($x = 0$) entering an expansion regime during which the gravitational constant increases with time.
At this point it is interesting to note that one might look alternatively at the dynamics of the models corresponding to II^+ and III^+ as describing the usual cosmic expansion taking place in ordinary 4-dimensionality (here expressed by the variable x) followed by a contraction of the internal n-dimensional space (represented here by y). The sector B, which is delimited by II^+ and III^+, contains only solutions which do not approach Minkowski spacetime, neither in the future nor in the past. On the other hand, the solutions lying in sector F all tend to M and start their trajectories as contracting universes, slowing down before enter an expanding era. In this class of models the gravitational constant is an ever decreasing function of the cosmic time.

We shall not carry out a detailed analysis of the solutions lying in sectors D and

E as these describe only contracting universes, *ipso facto* not being physically relevant. (As we shall see later, in section 5.6, sector E as well as sector B both represent classes of solutions with negative energy density.) In sector C a typical universe comes from Minkowski spacetime in the past and has a contracting era followed by further expansion.

In the case $n = 1$ (see figure 7) the physical picture is very similar. However, now as two of the invariant rays, namely III^+ and III^- lie exactly on the y-axis they represent vacuum flat solutions with a time-varying G.(In fact, an identical configuration has been already found in the context of Brans-Dicke theory by Romero-Barros [90]). An alternative way to look at these solutions is to consider them as a topological product of a static Minkowski spacetime by a time-dependent (expanding or contracting) compact internal space.

5.5. Exact solutions of the field equations

Often the knowledge of the invariant rays present in a homogeneous dynamical system is helpful in obtaining exact analytical solutions of the system. In that case the problem of finding the solutions corresponding to the invariant rays reduces to solving an algebraic equation of one order higher as the system itself. In our particular case we will have to solve a cubic polynomial equation, the roots of which are nothing more than the already known tangents a_i of the arcs defined by the invariant rays. Let us express the equations of the invariant rays simply by $y = ax$, where clearly a generically denotes a_i. Now, putting this into the equations (5.3.4-5) we get

$$\dot{x} = \frac{x^2}{2(n+2)}\left[-2(n+1) + 2n(1-n)a + 3n(n-1)a^2\right] \tag{5.5.1}$$

$$\dot{y} = a\dot{x} = \frac{x^2}{2(n+2)}\left[\frac{2}{3} - 4a - n(n+5)a^2\right] \tag{5.5.2}$$

The condition for (5.5.1) and (5.5.2) to be consistent is the algebraic equation

$$3n(n-1)a^3 + n(7-n)a^2 + 2(1-n)a - \frac{2}{3} = 0 \tag{5.5.3}$$

which is, in fact, equivalent to eq.(5.3.9). Again, we have to consider the two cases a) $n > 1$ and b) $n = 1$:

a) If $n > 1$ then the roots of (5.5.3) are given by

$$a_0 = 1/3, a_\pm = \frac{1}{n-1}\left[-1 \pm \sqrt{\frac{1}{3}(1+\frac{2}{n})}\right]$$

Now, going back to equation (5.3.4) and putting $y = ax$, with $a = a_0, a_\pm$, we get respectively:

$$\dot{x} = \gamma x^2 \tag{5.5.4}$$

where $\gamma = \gamma_0, \gamma_\pm$ and

$$\gamma_0 = -\frac{(n+3)}{6} \tag{5.5.5}$$

$$\gamma_\pm = -(1 + na_\pm) \tag{5.5.6}$$

These last equations can be immediately integrated to give $R(t)$ and $b(t)$.Then, corresponding to the three values of $a = a_0, a_\pm$ we have respectively (after suitable coordinate transformations):

$$R(t) \sim t^{-\frac{1}{3\gamma_0}} = R_0 t^{\frac{2}{n+3}} \tag{5.5.7}$$

$$b(t) \sim [R(t)]^{3a_0} = b_0 t^{\frac{2}{n+3}} \tag{5.5.8}$$

$$R(t) \sim t^{\frac{-1}{3\gamma_\pm}} = R_0 t^{\frac{-1}{3(1+na_\pm)}} \tag{5.5.9}$$

$$b(t) \sim [R(t)]^{3a_\pm} = b_0 t^{\frac{a_\pm}{1+na_\pm}} \tag{5.5.10}$$

where R_0 and b_0 are constants (see also 12).
b) If $n = 1$ then the equation (5.5.3) has two solutions, namely, $a = \pm\frac{1}{3}$. Naturally, these solutions correspond to the invariant rays defined by $\theta_i = \arctan \pm\frac{1}{3}$ in section 5.3. The third solution, corresponding to the other invariant rays, $\theta_i = \pm\frac{\pi}{2}$ can be obtained directly from the dynamical system (eqs.(5.3.4-5)) just putting $n = 1$ and $x = 0$. This procedure leads us back to the static solution referred earlier in section 5.4:

$$R(t) = constant, \tag{5.5.11}$$

$$b(t) = b_0 t \tag{5.5.12}$$

The other solutions are:

$$R(t) = R_0 t^{\frac{1}{3}} \tag{5.5.13}$$

$$b(t) = b_0 t^{\frac{1}{3}} \tag{5.5.14}$$

$$R(t) = R_0 t^{\frac{1}{3}} \tag{5.5.15}$$

$$b(t) = b_0 t^{-\frac{1}{3}}. \tag{5.5.16}$$

We conclude this section by noting that equations (5.5.7-16) actually represent six distinct pair of solutions $R(t)$, $b(t)$, each being singular at $t = 0$. Indeed, after integrating (5.5.4) we obtain (apart from a constant of integration which can be further eliminated by a coordinate transformation)

$$x = -\frac{1}{\gamma t}, \tag{5.5.17}$$

which,in fact, has to be understood as representing different solutions (for the same γ) according to t $\in (-\infty, 0)$ or $t \in (0, +\infty)$. In the phase diagrams these *twofold degeneracy* is reflected by the presence of distinct solutions (including the equilibrium point M) all lying on the same line $y = ax$. Finally, we should mention that if $n = 0$ in (5.5.7) we recover Friedmann's solution for a dust filled universe.

5.6. The energy density

So far we have not been concerned with the energy density predicted by the models. A brief look into the field equations shows us that ϱ must be given by

$$\varrho = \frac{1}{6\kappa^2}\left[2x^2 + 3n(n-1)y^2 + 6nxy\right]. \tag{5.6.1}$$

If $n > 1$ the above equation however can be put into the factorized form :

$$\varrho = \frac{1}{6\kappa^2}(y - a_+ x)(y - a_- x), \tag{5.6.2}$$

with $a_\pm$ as defined in section 5.5. This last equation allows us to draw the following conclusions:

i) For $n > 1$ we verify that the solutions lying on the invariant rays corresponding to $a_\pm$ are vacuum solutions.

ii) All solutions lying on the sector B and F are non-physical (in the sense that they have negative energy, which classically is forbidden). Incidentally, these are the only solutions which never tend to Minkowski spacetime neither in the past nor in the future.

iii) Solutions lying on the invariant ray corresponding to a_0 have positive energy density for arbitrary value of $n > 1$. This can be easily verified by computing ϱ for this case as we have $\varrho = \frac{x^2}{36\kappa^2}\left[2n^2 + n + 12\right]$.

All the properties mentioned above are ilustrated in figure 8. [2]

For $n = 1$ the same procedure leads to the picture displayed by fig. 9.

5.7. Conclusions

The idea that the Newtonian constant of gravitation G could indeed vary with time on a cosmic scale, which seems to have ocurred first to Dirac, in 1938, is far from being supported by current experimental data. Recent results [91] based on solar-system experiments tend to indicate an upper limit given by $\left|\dot{G}/G\right| < 10^{-12}$ to any possible variation of G. Yet even this rather stringent condition has not prevented cosmologists to speculate and investigate what theoretical consequences would such hypothesis lead to (for a list of references on past and recent works see [12,21,50,86,92]). Among other attempts to insert G in gravity theories as a scalar field (e.g. , Brans-Dicke-Jordan theories), is the multi-dimensional cosmology approach [21] which was described in section 5.2. The fact that in this scheme the field equations plus some symmetry assumptions may be tractable by mathematical techniques of dynamical system theory led us to obtain a whole spectrum

[2] One could argue that it is not exactly ϱ , but ${}^{(4)}\varrho$ the physical quantity which would be actually measured. However, from equation (5.2.7) we see that all that has been said in this section of ϱ is also true for ${}^{(4)}\varrho$.

of cosmic configurations where the matter of the Universe is regarded as a multicomponent perfect fluid in higher dimensions. It turns out that in this scheme some solutions exhibit a non-physical behaviour (at least from a classical standpoint). However, other solutions seem not to be in contradiction with generally accepted and standard models of the Universe, as they manifest properties such as cosmic expansion and the existence of an initial singularity. Also, in some of these expanding solutions the gravitational constant G decreases with time, a property which may justify calling them *Dirac universes* (we detect the presence of *anti-Dirac* models as well). Evidently, it was not our aim here to provide a quantitative discussion of the solutions, even of the more physically relevant ones, trying to square them in the context of present observational and experimental data. Rather, our interest in this paper was actually to call the attention of theorists for the extremely rich scenario which arises when one allows for higher dimensionality and the varying gravitational constant hypothesis.

5.8. Appendix

In order to construct the phase diagrams corresponding to the figures 6 and 7 all we need is to calculate the values of the functions $N^l(\theta)$, and $Z(\theta)$ at $\theta = \theta_i$, where θ_i is an invariant ray and the superscript l refers to the first non-vanishing derivative evaluated at θ_i [89]. Since the system is quadratic the phase portraits are symmetric by plane reflections ($x \to -x$, $y \to -y$), although the time orientation of the curves must be reversed in this operation. Such property means we only need carrying out our analysis in the neighbourhood of just three of the six invariant rays. Then, let us summarize the results which come from straightforward calculations.

For both cases $n > 1$ and $n = 1$, we obtain the following:

$l = 1$, $N^1(\theta_1) < 0$, $N^1(\theta_2) < 0$, $N^1(\theta_3) > 0$, $Z(\theta_1) < 0$, $Z(\theta_2) < 0$, and $Z(\theta_3) > 0$; where for the case $n > 1$ the invariant rays are: $\theta_1 = \arctan\frac{1}{3}$, $\theta_2 = \arctan a_+$, $\theta_3 = \arctan a_-$, whereas for the case $n = 1$, $\theta_1 = \arctan +\frac{1}{3}$, $\theta_2 = \arctan -\frac{1}{3}$ and $\theta_3 = -\frac{\pi}{2}$. With these results we can classify for arbitrary values of n the invariant rays θ_1 and θ_2 as being of type (β), while θ_3 is of type (α) [89]. From this classification we are led to the diagrams displayed in figs. 6 and 7.

To carry out the Poincare' compactification of phase plane we perform the transformations of variables $u = \frac{y}{x}$ and $z = \frac{1}{x}$. Then, starting from the equations (5.5.1) and (5.5.2), we end up with the dynamical system:

$$\frac{du}{d\tau} = \frac{1}{2(n+2)}\left[(\frac{1}{3} - u)\left(3n(n-1)u^2 + 6nu + 2\right)\right] \tag{5.8.1}$$

$$\frac{du}{d\tau} = \frac{z}{2(n+2)}\left[2(n+1) + 2n(n-1)u + 3n(1-n)u^2\right], \tag{5.8.2}$$

where $zd\tau = dt$. The equilibrium points of the dynamical system in the plane uz are: $(1/3, 0), (u_\pm, 0)$, with $u_\pm = a_\pm$. A simple analysis of the topological character of these points reveals that they correspond to a saddle-point and two nodes (unstable and stable), respectively [93].

6. Bulk Viscosity and Entropy Production in Multidimensional Integrable Cosmology

6.1. Introduction

Up till now we studied different properties of multidimensional cosmology using the matter source of multidimensional Einstein equations in the form of the perfect fluid [37-38]. But, of course, more realistic may be the model which incorporates some viscosity effects. Within 4-dimensional cosmology the viscous Universe was considered by a number of authors from quite different points of view. Without carrying of a detailed review of the subject (extensive review was given by Gron [94]), we mention some main trens in cosmology with viscous fluid as a source.

First, Misner [95] considered neutrino viscosity as a mechanism for reducing the anisotropy in the Early Universe. Stewart [96] and Collins and Stewart [97] proved that it is possible only if initial anisotropies are small enough. Another series of papers was started by Weinberg [98] which concerns the production of entropy in the viscous Universe. Both isotropization and production of entropy during lepton era in models of Bianchi types I,V were considered by Klimek [99]. Caderni and Fabbri [100] calculated coefficients of shear and bulk viscosity in plasma and lepton eras within the model of Bianchi type I. The next trend is connected with obtaining of singularity free viscous solutions. The first non-singular solution was obtained by Murphy [101] within flat Friedman-Robertson-Walker model with fluid possessing a bulk viscosity. Murphy supposed that the coefficient of a bulk viscosity is proportional to the density of a fluid. However, Belinsky and Khalatnikov [102,103] showed that this solution corresponds to the very peculiar choice of parameters and is unstable with respect to the anisotropy perturbations. Other nonsingular solutions with bulk viscosity were obtained by Novello and Araújo [104], Romero [105], Oliveira and Salim [106].

In this section we study the multidimensional cosmological model with a chain of Ricci-flat spaces for the source in the form of a fluid possessing bulk viscosity. In section 6.2 we describe the model and get basic equations. For their integration we develop some vector formalism proposed in our previous papers. In section 6.3 we summarize thermodynamics in multidimensional cosmology and obtain the formula for the rate of change of entropy. In section 6.4 we integrate equations of motion for special set of parameters in the first and second equations of state. Exact solutions are presented in the Kasner-like form and their properties are studied.

6.2. The model

As in previous sections we consider here a multidimensional cosmological model with the metric (1.1.1) defined on the D-dimensional manifold (1.1.2). We consider only Ricci-flat spaces $M_1, \ldots, M_n$, i.e.

$$R_{n_i l_i}[g^{(i)}] = 0, \quad n_i, l_i = 1, \ldots, N_i. \tag{6.2.1}$$

It is easy to obtain in the usual way the following non-zero components of the Ricci-tensor for the metric (1.1.1)

$$R_0^0 = \mathrm{e}^{-2\gamma(t)} \left(\sum_{i=1}^{n} N_i (\dot{x}^i)^2 + \ddot{\gamma}_0 - \dot{\gamma}\dot{\gamma}_0 \right), \tag{6.2.2}$$

$$R_{k_i}^{m_i} = \mathrm{e}^{-2\gamma(t)} \left(\ddot{x}^i + (\dot{\gamma}_0 - \dot{\gamma})\dot{x}^i \right) \delta_{k_i}^{m_i}, \tag{6.2.3}$$

where we denoted $\gamma_0 = \sum_{i=1}^{n} N_i x^i$. Indices m_i and k_i run over from $D - \sum_{j=i}^{n} N_j$ to $D - \sum_{j=i}^{n} N_j + N_i$ for $i = 1, \ldots, n$ ($D = 1 + \sum_{i=1}^{n} N_i = \mathrm{dim M}$).

We take the energy-momentum tensor for a viscous fluid in the standard form (without shear)

$$T_B^A = \rho u^A u_B + (p - \zeta\theta) P_B^A, \tag{6.2.4}$$

where ρ and p are the fluid density and the pressure, respectively, ζ is the bulk viscosity coefficient. Vector u^A is the D-dimensional velocity of a fluid and $P_B^A = \delta_B^A + u^A u_B$ is the projector on the $(D-1)$-dimensional space orthogonal to u^A. By θ we denote the scalar expansion $\theta = u^A{}_{;A}$.

We impose the comoving observer condition for the D-dimensional velocity: $u^A = \delta_0^A \mathrm{e}^{-\gamma(t)}$. Then

$$(u^A u_B) = \mathrm{diag}(-1, 0, \ldots, 0), \tag{6.2.5}$$

$$(P_B^A) = \mathrm{diag}(0, 1, \ldots, 1), \tag{6.2.6}$$

$$\theta = \dot{\gamma}_0 \mathrm{e}^{-\gamma(t)}. \tag{6.2.7}$$

Let us remark that the function $\gamma(t)$ in (1.1.1) determines a time gauge for the comoving observer. We have the harmonic time gauge for $\gamma(t) = \gamma_0$ and the proper time gauge for $\gamma(t) = 0$. Harmonic time t and proper time τ are connected by $d\tau = \exp[\gamma_0] dt$.

We admit that the pressure and the bulk viscosity term in (6.2.4) are anisotropic with respect to the whole space $M_1 \times \ldots \times M_n$. Such an admission leads to the following generalization of the expression (6.2.4)

$$(T_B^A) = \mathrm{diag}(-\rho, (p_1 - \theta\zeta_1)\delta_{k_1}^{m_1}, \ldots, (p_n - \theta\zeta_n)\delta_{k_n}^{m_n}), \tag{6.2.8}$$

where p_i and ζ_i are the pressure and the bulk viscosity coefficient in the space M_i. Furthermore, we suppose that the barotropic equations of state holds

$$p_i = (1 - h_i)\rho(t), \tag{6.2.9}$$

where h_i=const for $i = 1, \ldots, n$.

It is easy to show that the equation of motion $\nabla_M T_0^M = 0$ for the viscous fluid with the tensor (6.2.8) looks as follows

$$\dot{\rho} + \sum_{i=1}^{n} N_i \dot{x}^i (\rho + p_i - \zeta_i \theta) = 0. \tag{6.2.10}$$

The Einstein equations $R^A_B - \frac{1}{2}\delta^A_B R = \kappa^2 T^A_B$ (κ^2 is gravitational constant) may be written as $R^A_B = \kappa^2(T^A_B - \frac{T}{D-2}\delta^A_B)$. Further, we employ the equation $R^0_0 - \frac{1}{2}\delta^0_0 R = \kappa^2 T^0_0$ and the equations $R^{m_i}_{k_i} = \kappa^2(T^{m_i}_{k_i} - \frac{T}{D-2}\delta^{m_i}_{k_i})$. Using (6.2.2), (6.2.3) and (6.2.8) we get

$$\sum_{i=1}^{n} N_i(\dot{x}^i)^2 - \dot{\gamma}_0^2 = -2\kappa^2 \mathrm{e}^{2\gamma}\rho, \tag{6.2.11}$$

$$\ddot{x}^i + (\dot{\gamma}_0 - \dot{\gamma})\dot{x}^i = \kappa^2 \left[\left(-h_i + \frac{\sum_{k=1}^n N_k h_k}{D-2} \right) \rho \mathrm{e}^{2\gamma} + \left(-\zeta_i + \frac{\sum_{k=1}^n N_k \zeta_k}{D-2} \right) \dot{\gamma}_0 \mathrm{e}^{\gamma} \right]. \tag{6.2.12}$$

To develop the integration procedure for the equations of motion (6.2.11),(6.2.12) we introduce the n-dimensional real vector space R^n. By $e_1, \ldots, e_n$ we denote the canonical basis in R^n, i.e. $e_1 = (1, 0, \ldots, 0)$ etc.

Let $< .,. >$ be a symmetric bilinear form defined on R^n, such that

$$< e_i, e_j >= \delta_{ij} N_j - N_i N_j \equiv G_{ij}. \tag{6.2.13}$$

In our previous papers this form was introduced as a minisuperspace metric for the cosmological models. It was shown that it is a nongenerate form with the pseudo-Euclidean signature $(-, +, \ldots, +)$. So, for vectors $a = a^1 e_1 + \ldots + a^n e_n$ and $b = b^1 e_1 + \ldots + b^n e_n$ we have

$$< a, b >= \sum_{i,j=1}^{n} G_{ij} a^i b^j. \tag{6.2.14}$$

The form $< a, b >$ may be also written as

$$< a, b >= \sum_{i=1}^{n} a_i b^i = \sum_{i=1}^{n} a^i b_i = \sum_{i,j=1}^{n} G^{ij} a_i b_j, \tag{6.2.15}$$

if we introduce the covariant components of vectors by

$$a_i = \sum_{j=1}^{n} G_{ij} a^j. \tag{6.2.16}$$

By $G^{ij} = \delta^{ij}/N_i + 1/(2-D)$ we denote components of a matrix inverse to (G_{ij}).

We call a vector $y \in R^n$ time-like, space-like or isotropic, if $< y, y >$ takes negative, positive or null values, respectively. Vectors y and z are called orthogonal if $< y, z >= 0$.

In our model the following vectors are used

$$x = x^1 e_1 + \ldots + x^n e_n, \tag{6.2.17}$$

$$u = u^1 e_1 + \ldots + u^n e_n, \quad u^i = h_i - \frac{\sum_{k=1}^n N_k h_k}{D-2}, \quad u_i = N_i h_i \tag{6.2.18}$$

$$\xi = \xi^1 e_1 + \ldots + \xi^n e_n, \quad \xi^i = \zeta_i - \frac{\sum_{k=1}^n N_k \zeta_k}{D-2}, \quad \xi_i = N_i \zeta_i. \tag{6.2.19}$$

If $h_i = 1$ for $i = 1, \ldots, n$, we have dust in the whole space ($p_i = 0$, see (6.2.9)). The vector (6.2.18) corresponding to dust in the whole space is denoted by u_d. We note that

$$(u_d)_i = N_i, \quad u_d^i = \frac{-1}{D-2}, \quad < u_d, u_d >= -\frac{D-1}{D-2}, \quad < u_d, x >= \gamma_0. \tag{6.2.20}$$

Thus, using (6.2.14), (6.2.17)-(6.2.19) we obtain the Einstein equations in the form

$$< \dot{x}, \dot{x} >= -2\kappa^2 \mathrm{e}^{2\gamma} \rho, \tag{6.2.21}$$

$$\ddot{x} + (< u_d, \dot{x} > -\dot{\gamma})\dot{x} = -\kappa^2 \left(\rho \mathrm{e}^{2\gamma} u + < u_d, \dot{x} > \mathrm{e}^{\gamma} \xi \right). \tag{6.2.22}$$

The equation of motion (6.2.10) can be written as

$$\dot{\rho} + \rho < 2u_d - u, \dot{x} > -\mathrm{e}^{-\gamma} < u_d, \dot{x} >< \xi, \dot{x} >= 0. \tag{6.2.23}$$

Excluding the density ρ from (6.2.22) by (6.2.21) we get the following equation

$$\ddot{x} + (< u_d, \dot{x} > -\dot{\gamma})\dot{x} = \frac{1}{2} < \dot{x}, \dot{x} > u - \kappa^2 < u_d, \dot{x} > \mathrm{e}^{\gamma} \xi. \tag{6.2.24}$$

To integrate (6.2.24) we need a second equation of state for the bulk viscosity coefficients ζ_i. To obtain an exact solution in a 4-dimensional flat Friedman-Robertson-Walker model with bulk viscosity Murphy [101] used the second equation of state of the form $\zeta =$const$\,\rho$. Belinsky and Khalatnikov [107] studied the qualitative behavior of this model with a more general equation: $\zeta = \alpha \rho^{\nu}$, where α, ν=const. It is easy to show that for this model on manifold $R \times M_1^3$ for $\gamma(t) = 0$ the set of equations (6.2.23),(6.2.24) may be written as

$$3H^2 = \kappa^2 \rho, \tag{6.2.25}$$

$$\dot{H} = \frac{\alpha}{2} 3^{\nu+1} H^{2\nu+1} + \frac{3}{2}(h-2)H^2, \tag{6.2.26}$$

where H is the Hubble parameter of the 3-dimensional Ricci-flat manifold M_1^3, i.e. $H = \dot{x}^1$. The set of equations (6.2.25)-(6.2.26) coincides with the one obtained by Belinsky and Khalatnikov [107]. It is easy to see that equation (6.2.26) for H is always integrable by quadrature. In the simplest case with $\nu = 1$ we get the exact solution obtained by Murphy [101]. Other solutions for special parameters ν and h and a solution for arbitrary ν and h were also obtained (see [94] for details).

For multidimensional cosmological model with manifold $M = R \times M_1 \times \ldots \times M_n$ the set of equations (6.2.21)-(6.2.22) is more complicated. Obviously , we have the set of nonlinear differential equations (6.2.24) for scale factors $\exp[x^i]$ of the spaces $M_1, \ldots, M_n$. If we adopt Belinsky and Khalatnikov's condition: $\zeta \sim \rho^{\nu}$, then rather complicated equations arise. In particular, for $\nu = 1$ Appel and Ricatti equations appear. Chakraborty and Nandy [108] within a 5-dimensional model with manifold $R \times M_1^3 \times S_2^1$ avoided this difficulty by imposing an additional constraint for the scale factors: $\exp[x^2] = \mu \exp[\omega x^1]$, $\mu, \nu =$const.

Here, with no loss of generality, we consider an integration of the set of equations (6.2.22) for another second equation of state.We suppose that the bulk viscosity coefficient ζ_i corresponding to the space M_i is proportional to $\exp[-\gamma_0]$, i.e.

$$\zeta_i \sim [\text{scale factor of } \mathrm{M}_1]^{-\mathrm{dim M}_1} \cdot \ldots \cdot [\text{scale factor of } \mathrm{M}_\mathrm{n}]^{-\mathrm{dim M}_\mathrm{n}}. \tag{6.2.27}$$

Physically, the assumption (6.2.27) means that the expansion of the spaces $M_1, \ldots, M_n$ is accompanied by a decreasing of the bulk viscosity effect.

Let us notice that the metric dependence of the bulk viscosity coefficient was also considered by other authors. Lukacs [109] integrated the homogeneous and isotropic 4-dimensional model with a viscous dust for such second equation of state: $\zeta = \text{const}[\text{scale factor}]^{-1}$. Curvature-dependent bulk viscosity was studied in a multidimensional cosmology by Wolf [110]. Recently Motta and Tomimura [111] studied a 4-dimensional inhomogeneous cosmology with some metric dependence of the bulk viscosity coefficient.

6.3. Thermodynamics of viscous fluid in multidimensional Universe

We first summarize thermodynamics in multidimensional cosmology on the manifold $M = R \times M_1 \times \ldots \times M_n$ following papers [112,113]. The first law of thermodynamics can be written as follows

$$TdS = d(\rho V) + V \sum_{i=1}^{n} p_i \frac{dV_i}{V_i}, \tag{6.3.1}$$

where V_i is any fluid volume in the space M_i, V is a fluid volume in the whole space: $V = V_1 \cdot \ldots \cdot V_n$ and S is an entropy in the volume V. We suppose the conservation law for the baryon particle number N_B in volume V. Then, for entropy per baryon $s = S/N_b$ and baryon number density $n = N_b/V$ we obtain from (6.3.1)

$$nT\dot{s} = \dot{\rho} + \rho \sum_{i=1}^{n} N_i \dot{x}^i + \sum_{i=1}^{n} p_i N_i \dot{x}^i, \tag{6.3.2}$$

We remind that $\exp[x^i]$ is the scale factor of the space M_i of the dimension N_i.

For the perfect fluid ($\zeta_i = 0$) comparing (6.3.2) and equation of motion (6.2.10) we get the conservation of entropy :$s =$const and by the barotropic equations of state (6.2.9) the integral of motion

$$\rho \exp[\sum_{i=1}^{n} (2 - h_i) N_i x^i] = \text{const}. \tag{6.3.3}$$

Temperature of the perfect fluid can be obtained in such a way [113]. From (6.3.2) we have

$$(\frac{d\rho}{dx^i})_{s,x^j} = -\rho N_i - p_i N_i = (h_i - 2) N_i \rho, \quad j \neq i. \tag{6.3.4}$$

Then

$$\rho = K(s) \exp[\sum_{i=1}^{n} (h_i - 2) N_i x^i], \tag{6.3.5}$$

where $K(s)$ is an unknown function of the entropy s. By inverting (6.3.5) we get

$$s = s(\rho \exp[\sum_{i=1}^{n}(2-h_i)N_i x^i]). \tag{6.3.6}$$

Substituting (6.3.6) for s in (6.3.2), we obtain

$$nT \frac{ds}{d(\rho \exp[(2-h_i)N_i x^i])} = \exp[\sum_{i=1}^{n}(h_i - 2)N_i x^i]. \tag{6.3.7}$$

For the perfect fluid we have $ds/d(\rho \exp[2-h_i)N_i x^i]) = B = \text{const}$ (see (6.3.3)), then

$$nT = \frac{1}{B}\exp[\sum_{i=1}^{n}(h_i - 2)N_i x^i] = \frac{1}{B}\exp[< u - 2u_d, x >]. \tag{6.3.8}$$

Now we consider the fluid with a bulk viscosity. Comparing (6.2.10) and (6.3.2) we obtain

$$nT\dot{s} = \theta \sum_{i=1}^{n} N_i \zeta_i \dot{x}^i. \tag{6.3.9}$$

Using (6.2.7),(6.2.14),(6.2.17)and (6.2.19) we get

$$\dot{s} = \frac{\dot{\gamma}_0 \mathrm{e}^{-\gamma}}{nT}\sum_{i=1}^{n} N_i \zeta_i \dot{x}^i = \frac{\mathrm{e}^{-\gamma}}{nT} < u_d, \dot{x} >< \xi, \dot{x} > . \tag{6.3.10}$$

This formula gives the rate of change of entropy per baryon in multidimensional cosmology on the manifold $M = R \times M_1 \times \ldots \times M_n$ with anisotropic bulk viscosity. The production of entropy in the model can be calculated if the temperature of a fluid is known. Further, we suppose that the temperature is given by the perfect fluid formula (6.3.8). Then we get

$$\dot{s} = B \exp[< 2u_d - u, x > -\gamma] < u_d, \dot{x} >< \xi, \dot{x} > . \tag{6.3.11}$$

6.4. Exact solutions

In this section we consider only the model with identical pressures and identical bulk viscosity coefficients in each space M_i, i.e.

$$p_i = (1-h)\rho \quad \text{or} \quad u = hu_d, \tag{6.4.1}$$

$$\zeta_i = \frac{\zeta_0}{\kappa^2}\mathrm{e}^{-\gamma_0} \quad \text{or} \quad \xi = \frac{\zeta_0}{\kappa^2}\mathrm{e}^{-\gamma_0} u_d, \quad i = 1, \ldots, n, \tag{6.4.2}$$

where ζ_0 and h are constants. Here we suppose that

$$h > 0, \qquad \zeta_0 > 0. \tag{6.4.3}$$

Then, the set of equations (6.2.24) in the harmonic time gauge ($\gamma = \gamma_0$) looks as follows

$$\ddot{x} = \frac{h}{2} < \dot{x}, \dot{x} > u_d - \zeta_0 < u_d, \dot{x} > u_d. \tag{6.4.4}$$

(We remind that $\gamma_0 =< u_d, x >$.) To integrate (6.4.4) we use the following decomposition of the vector x

$$x =< u_d, x > \frac{u_d}{< u_d, u_d >} + \sum_{i=2}^{n} < e'_i, x > e'_i. \tag{6.4.5}$$

The vectors $u_d, e'_2, \dots, e'_n$ form an orthogonal basis in R^n, i.e.

$$< u_d, e'_i >= 0, \quad < e'_i, e'_j >= \delta_{ij}, \quad i, j = 2, \dots, n. \tag{6.4.6}$$

We notice that in this basis any vector e_i can not be time-like or isotropic because the vector u_d is time-like. The set of equations (6.4.4) may be written as

$$< u_d, \ddot{x} >=< u_d, u_d > \left[\frac{h}{2} \left(\frac{< u_d, \dot{x} >^2}{< u_d, u_d >} + \sum_{i=2}^{n} < e'_i, \dot{x} >^2 \right) - \zeta_0 < u_d, \dot{x} > \right], \tag{6.4.7}$$

$$< e'_i, \ddot{x} >= 0, \quad i = 2, \dots, n. \tag{6.4.8}$$

Integration of (6.4.8) leads to the results

$$< e'_i, x >= p^i t + q^i, \quad i = 2, \dots, n, \tag{6.4.9}$$

where p^i and q^i are arbitrary constants. To present the scale factors $\exp[x^i]$ in a Kasner-like form, we introduce the vectors $\alpha, \beta \in R^n$

$$\alpha = p^2 e'_2 + \dots + p^n e'_n \equiv \alpha^1 e_1 + \dots + \alpha^n e_n, \tag{6.4.10}$$

$$\beta = q^2 e'_2 + \dots + q^n e'_n \equiv \beta^1 e_1 + \dots + \beta^n e_n. \tag{6.4.11}$$

We remind that the vectors $e_1, \dots, e_n$ form the canonical basis in R^n. The coordinates α^i and β^i are the Kasner-like parameters. Integration of (6.4.7) results in

$$< u_d, x >= -\frac{1}{h} \ln[C f^2] + \frac{\zeta_0}{h} < u_d, u_d > t, \tag{6.4.12}$$

where $C > 0$ is an integration constant.

Using (6.4.5),(6.4.9)-(6.4.12) we obtain the exact solution in the Kasner-like form

$$\mathrm{e}^{x^i} = (C f^2)^{-\frac{1}{h(D-1)}} \exp[(\alpha^i - \frac{\zeta_0}{h(D-2)}) t + \beta_i]. \tag{6.4.13}$$

The Kasner-like parameters obey the relations

$$< \alpha, u_d >= \sum_{i=1}^{n} \alpha^i N_i = 0, \quad < \beta, u_d >= \sum_{i=1}^{n} \beta_i N_i = 0, \tag{6.4.14}$$

$$< \alpha, \alpha >= \sum_{i=1}^{n} (\alpha^i)^2 N_i = \sum_{j=2}^{n} (p^j)^2. \tag{6.4.15}$$

Using (6.2.21) we obtain the density

$$\rho = \frac{a^2 + < \alpha, \alpha >}{2\kappa^2} (C f^2)^{\frac{2}{h}} \exp[\frac{2a^2 h}{\zeta_0} t] \left(F + \frac{a - \sqrt{< \alpha, \alpha >}}{\sqrt{a^2 + < \alpha, \alpha >}} \right) \left(F + \frac{a + \sqrt{< \alpha, \alpha >}}{\sqrt{a^2 + < \alpha, \alpha >}} \right). \tag{6.4.16}$$

For the functions f and F in (6.4.12),(6.4.13) and (6.4.16) we have the following variants

$$f = \sinh[Ah(t-t_0)/2], \quad F = \coth[Ah(t-t_0)/2], \quad C > 0, \tag{6.4.17}$$

$$f = \cosh[Ah(t-t_0)/2], \quad F = \tanh[Ah(t-t_0)/2], \quad C > 0, \tag{6.4.18}$$

$$f = \exp[Ah(t-t_0)/2], \quad F = 1, \quad C = \exp[-\frac{\zeta_0(D-1)}{D-2}t_0], \tag{6.4.19}$$

$$f = \exp[-Ah(t-t_0)/2], \quad F = -1, \quad C = \exp[-\frac{\zeta_0(D-1)}{D-2}t_0]. \tag{6.4.20}$$

Constants A and a are such that

$$a = \frac{\zeta_0}{h}\sqrt{\frac{D-1}{D-2}}, \quad A = \frac{D-1}{D-2}\sqrt{\frac{\zeta_0^2}{h^2} + \frac{D-2}{D-1} < \alpha, \alpha >} \tag{6.4.21}$$

Using (6.3.11) we obtain the rate of change of entropy per baryon in this model

$$\dot{s} = \frac{B}{\kappa^2}\zeta_0 C f^2 \exp[\zeta_0 \frac{D-1}{D-2}t]\left(\frac{\zeta_0}{h}\frac{D-1}{D-2} + AF\right). \tag{6.4.22}$$

Let us consider the properties of this model. Further we consider only solutions with

$$< \alpha, \alpha >> 0. \tag{6.4.23}$$

Condition $< \alpha, \alpha >= 0$ means that all Kasner-like parameters are zero, then the identical dynamics follows for all spaces $M_1, \ldots, M_n$. Such solutions in the framework of multi-dimensional cosmology are out of interest. Indeed, the observable distinction between external and internal dimensions demands the stage of various dynamics for the external and internal spaces. In this connection the solutions with expansion of the 3-dimensional external space and simultaneous contraction of the internal space (or spaces) are mostly attractive.

Also we suppose the weak energy condition for the solutions obtained, i.e. $\rho(\tau) \geq 0$ for any proper time τ. It is not hard to prove that only solutions with $f = \exp[Ah(t-t_0)/2]$ and $f = \sinh[Ah(t-t_0)/2]$ satisfy the weak energy condition under the condition (6.4.23).

We first consider the properties of the solution with $f = \exp[Ah(t-t_0)/2]$. In the proper time τ it can be written as follows

$$\mathrm{e}^{x^i(\tau)} = \mathrm{e}^{\tilde{\beta}^i}\left(\frac{\tau_0 - \tau}{T_0}\right)^{1/(D-1) - T_0\alpha^i}, \quad \tau < \tau_0, \tag{6.4.24}$$

$$\rho(\tau) = \frac{\zeta_0 T_0}{\kappa^2 h}\frac{1}{(\tau_0 - \tau)^2}, \tag{6.4.25}$$

where τ_0 is arbitrary constant and parameters $\tilde{\beta}^i$ obey the relations (6.4.14). For constant T_0 we have

$$\frac{1}{T_0} = \frac{D-1}{D-2}(\frac{\zeta_0}{h} + \sqrt{\frac{\zeta_0^2}{h^2} + \frac{D-2}{D-1} < \alpha, \alpha >}). \tag{6.4.26}$$

The formula (6.4.22) for the rate of change of entropy per baryon is easily integrable in this case

$$s(\tau) = s(-\infty) + \frac{B\zeta_0}{\kappa^2 T_0 h}\left(\frac{T_0}{\tau_0 - \tau}\right)^h. \tag{6.4.27}$$

It is evident from (6.4.25) that this solution is singular at the final point of evolution $\tau = \tau_0$, because $\rho(\tau) \to +\infty$ as $\tau \to \tau_0 - 0$. We also notice that $\rho(\tau) \to 0$ as $\tau \to -\infty$, so this solution can be interpreted as that describing creation of matter in the Universe.

The entropy per baryon $s(\tau)$ under the conditions (6.4.3) is monotonically increasing to infinity function on the interval $(-\infty, \tau_0)$. Existence of the solutions with similar unbounded production of entropy at the final stage of evolution within 4-dimensional viscous models of Bianchi types I,IX with the second equation of state $\zeta = \alpha\rho^\nu$ was proved by Belinsky and Khalatnikov [107]. Such solutions can be considered in connection with the problem of extremely large entropy per baryon in the present Universe. Indeed, it is evident that such solutions (multidimensional or not) are applicable up to some proper time τ_c. From the time τ_c other equations of state are valid, then the evolution of the Universe is described by another model. However, it is possible that on reaching the time τ_c the entropy per baryon (6.4.27) is large enough (see fig.10).

It is also worth noticing, that this solution describes contraction of at least one space of $M_1, \ldots, M_n$. Indeed, due to the relations (6.4.14) at least one of the Kasner-like parameters is nonpositive, so the corresponding scale factor monotonically decreases on the interval $(-\infty, \tau_0)$.This process can be interpreted as contraction of the internal space (or spaces) to the Planck scale ($10^{-33} cm.$). In fact the unbounded production of entropy arises due to the necessary contraction of part of the spaces, which we interpret as internal. Moreover, it can be shown that for some set of Kasner-like parameters the solution describes expansion of one part of spaces and simultaneous contraction of the other part.

Let us consider this property for a simplest model on the manifold $R \times R^3 \times T^d$, where R^3 is a 3-dimensional flat external space and T^d is an internal space having the shape of d-dimensional torus. The exact solution (6.4.24) gives

$$e^{x^1(\tau)} = e^{\tilde{\beta}^1} \left(\frac{\tau_0 - \tau}{T_0}\right)^{1/(d+3) - T_0\alpha^1}, \tag{6.4.28}$$

$$e^{x^2(\tau)} = \exp[-\frac{3}{d}\tilde{\beta}^1] \left(\frac{\tau_0 - \tau}{T_0}\right)^{1/(d+3) + \frac{3}{d}T_0\alpha^1}, \tag{6.4.29}$$

where

$$\frac{1}{T_0} = \frac{d+3}{d+2}\left(\frac{\zeta_0}{h} + \sqrt{\frac{\zeta_0^2}{h^2} + 3\frac{d+2}{d}(\alpha^1)^2}\right), \tag{6.4.30}$$

τ_0, $\tilde{\beta}^1$ and α^1 are arbitrary constants. If $\alpha^1 > 0$ then the internal space monotonically contracts. It is not difficult to show that under the condition

$$\frac{(d+3)(d-1)}{d}\alpha^1 > 2\frac{\zeta_0}{h} \tag{6.4.31}$$

we obtain the monotonic expansion of the external space on the interval $(-\infty, \tau_0)$ (see fig. 11). This condition can be satisfied for $d \geq 2$.

Let us suppose that the solution (6.4.28),(6.4.29) describes the evolution of the multidimensional Universe on the time interval $(\tau_0 - T_0, \tau_c)$. Also we put $s(-\infty) = 0$ in (6.4.27). Then under the condition of expansion of the external space (6.4.31) we obtain

$$\left(\frac{\exp[x^2(\tau_0 - T_0)]}{\exp[x^2(\tau_c)]}\right)^{hd} > \frac{s(\tau_c)}{s(\tau_0 - T_0)}, \tag{6.4.32}$$

i.e. if the internal space T^d contracts on the time interval $(\tau_0 - T_0, \tau_c)$ in K times then the entropy per baryon increases on this interval less then in K^{hd} times. Thus, there exists the upper limit for the production of entropy provided the expansion of the external space. This limit depends on the final sizes of the internal space T^d and can be removed to infinity as $d \to +\infty$.

The exact solution (6.4.13),(6.4.16) with $f = \sinh[Ah(t - t_0)]$ under the condition (6.4.23) satisfies the weak energy condition for any $t \in (t_0, +\infty)$ and this interval corresponds to the proper time interval $(-\infty, \tau_0)$. It follows from (6.4.13),(6.4.16) that this solution and that with $f = \exp[Ah(t - t_0)/2]$ have identical behavior near the singularity point $\tau = \tau_0$. So, they have the same main properties. We only note, that for $2/h > 1$ we have $\rho(\tau) \to 0$ as $\tau \to -\infty$, then this solution also can be interpreted as that describing creation of matter.

7. Inflationary Solutions in Multidimensional Cosmology with Perfect Fluid

7.1. The model

It is of interest to study also inflationary solutions in multidimensional cosmology which [119-120]. We consider a cosmological model describing the evolution of n Ricci-flat spaces in the presence of the 1-component perfect-fluid matter [37] and a homogeneous massless minimally coupled scalar field. The metric of the model and the manifold are taken as (1.1.1-2)

We take the field equations in the following form:

$$R^M_N - \frac{1}{2}\delta^M_N R = \kappa^2 T^M_N, \tag{7.1.1}$$

$$\Box\varphi = 0, \tag{7.1.2}$$

where κ^2 is the gravitational constant, $\varphi = \varphi(t)$ is scalar field, $\Box$ is the d'Alembert operator for the metric (1.1.1) and the energy-momentum tensor is adopted in the following form

$$T^M_N = T^{M(pf)}_N + T^{M(\phi)}_N, \tag{7.1.3}$$

$$(T^{M(pf)}_N) = diag(-\rho, p_1\delta^{m_1}_{k_1}, \dots, p_n\delta^{m_n}_{k_n}), \tag{7.1.4}$$

$$T^{M(\phi)}_N = \partial^M\varphi\partial_N\varphi - \frac{1}{2}\delta^M_N(\partial\varphi)^2. \tag{7.1.5}$$

We put pressures of the perfect fluid in all spaces to be proportional to the density

$$p_i(t) = (1 - \frac{u_i}{N_i})\rho(t), \tag{7.1.6}$$

where $u_i = const$, $i = 1, \dots, n$.

We impose also the following restriction on the vector $u = (u_i) \in R^n$

$$< u, u >_* < 0. \tag{7.1.7}$$

Here bilinear form $< .,. >_*$: $R^n \times R^n \to R$ is defined by the relation

$$< u, v >_* = G^{ij}u_i v_j, \tag{7.1.8}$$

$u, v \in R^n$, where

$$G^{ij} = \frac{\delta^{ij}}{N_i} + \frac{1}{2-D} \tag{7.1.9}$$

are components of the matrix inverse to the matrix of the minisuperspace metric [8,9]

$$G_{ij} = N_i\delta_{ij} - N_i N_j. \tag{7.1.10}$$

In (7.1.9) $D = 1 + \sum_{i=1}^n N_i$ is the dimension of the manifold M (1.1.2).

7.2. Classical solutions

We get the following non-exceptional solutions of the field equations (7.1.1-2) [121]

$$g = -(\prod_{i=1}^{n}(a_i(\tau))^{2N_i-u_i})d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau)g^{(i)}, \tag{7.2.1}$$
$$a_i(\tau) = A_i[\sinh(r\tau/T)/r]^{2u^i/<u,u>_*}[\tanh(r\tau/2T)/r]^{\beta^i}, \tag{7.2.2}$$
$$\exp(\kappa\varphi(\tau)) = A_\varphi[\tanh(r\tau/2T)/r]^{\beta_\varphi}, \tag{7.2.3}$$
$$\kappa^2\rho(\tau) = A\prod_{i=1}^{n}(a_i(\tau))^{u_i-2N_i}, \tag{7.2.4}$$

$i = 1,\ldots,n$; where $r = \sqrt{A/|A|}$, $T = (\frac{1}{2}|A < u,u >_* |)^{-1/2}$. $A_i, A_\varphi > 0$ are constants and the parameters β^i, β_φ satisfy the relations

$$\sum_{i=1}^{n} u_i\beta^i = 0, \qquad \sum_{i,j=1}^{n} G_{ij}\beta^i\beta^j + (\beta_\varphi)^2 = -4/ < u,u >_* . \tag{7.2.5}$$

Here $\tau > 0$ for $A > 0$ and $0 < \tau < \pi T$ for $A < 0$.

For positive energy density ($A > 0$), see (7.2.4), we have a family of exceptional solutions with the constant real scalar field [37]

$$g = -(\prod_{i=1}^{n}(a_i(\tau))^{2N_i-u_i})d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau)g^{(i)}, \tag{7.2.6}$$
$$a_i(\tau) = \bar{A}_i\exp[\pm 2u^i\tau/(T < u,u >_*)], \tag{7.2.7}$$
$$\varphi(\tau) = const, \tag{7.2.8}$$

and $\rho(\tau)$ is defined by (7.2.4). Here $\bar{A}_i > 0$ ($i = 1,\ldots,n$) are constants, and T is defined as in (7.2.1-4).

We note that for $A > 0$ the solution (7.2.7) with the sign "+" is an attractor for the solutions (7.2.2).

Inflationary solutions. First we consider the case

$$< u^{(\Lambda)} - u, u >_* \neq 0, \tag{7.2.9}$$

where $u_i^{(\Lambda)} = 2N_i$ correspond to the cosmological term. The solution (7.2.6), (7.2.7) in synchronous time parametruization reads as

$$g = -dt_s \otimes dt_s + \sum_{i=1}^{n} a_i(t_s)g^{(i)}, \tag{7.2.10}$$
$$a_i(t_s) = A_i t_s^{\nu^i}, \tag{7.2.11}$$
$$\kappa^2\rho = \frac{-2<u,u>_*}{<u^{(\Lambda)}-u,u>_*^2 t_s^2}. \tag{7.2.12}$$

where

$$\nu^i = 2u^i/ < u^{(\Lambda)} - u, u >_* . \tag{7.2.13}$$

$i = 1,\ldots,n$. Thus, formulas (7.2.10)-(7.2.13) and $\varphi = const$ describe exceptional solutions for the case (7.2.9). We call these solutions as the power-law inflationary solutions.

The solution is a self-similar one.

Now we consider the case

$$< u^{(\Lambda)} - u, u >_* = 0. \tag{7.2.14}$$

In this case

$$\kappa^2 \rho = const \tag{7.2.15}$$

and

$$a_i(t_s) = \bar{A}_i \exp[\mp \frac{u^i}{\sqrt{- < u, u >_*}} \frac{t_s}{T_0}], \tag{7.2.16}$$

where

$$T_0 = (2\kappa^2 \rho)^{-1/2}. \tag{7.2.17}$$

The relations (7.2.10), (7.2.15)-(7.2.17) and $\varphi = const$ describe the exponential-type inflation for the case (7.2.14). In the special case $u = u^{(\Lambda)}$ (cosmological constant case) this solution was considered in [48].

The corresponding quantum solutions were considered in [121]. Applying the arguments considered in [67] one may show that the ground state wave function

$$\Psi_0^{(HH)} = I_0 \left(\frac{\sqrt{2|A|}}{q} \exp(qz^0) \right), \qquad A < 0, \tag{7.2.18}$$

$$J_0 \left(\frac{\sqrt{2A}}{q} \exp(qz^0) \right), \qquad A > 0, \tag{7.2.19}$$

satisfies the Hartle-Hawking boundary condition. Here $2q = \sqrt{- < u, u >_*}$ and $\exp(qz^0) = \prod_{i=1}^{n} a_i^{u_i/2}$ is quasivolume.

7.3. Some Examples

Let us consider the isotropic case when pressures in all spaces are equal. Then

$$u_i = hN_i = \frac{h}{2} u_i^{(\Lambda)} , \tag{7.3..1}$$

$$p_i = (1 - h)\rho = p \tag{7.3..2}$$

For this case

$$< u, u >_* = -h \frac{D-1}{D-2} < 0 \tag{7.3..3}$$

if $h \neq 0$ or $p \neq \rho$.

The cosmological constant corresponds to $h = 2$, and the dust-like matter to $h = 1$. Then,

$$u^i = G^{ij} u_j = h/(2 - D) , \tag{7.3..4}$$

$$\nu^i = 2/h(D-1) = \nu$$

We see that for $h > 0$ (or $p < \rho$) we have according to (7.2.11) the isotropic expansion and for $h < 0$ ($p > \rho$) the isotropic contraction. We may calculate also for this isotropic case

$$< u^{(\Lambda)} - u, u >_* = \frac{1}{4}\,(2-h) < u^{(\Lambda)}, u^{(\Lambda)} >_* \, , \tag{7.3..5}$$

which for $h = 2$ is equal to zero.

Accordingly, we have the power-law (in general) and the exponential law ($h = 2$) inflations here as well.

8. Integrable Weyl Geometry in Multidimensional Cosmology. Numerical Investigation [122]

8.1. Introduction

The multidimensional gravitation theories are very attractive in the context of the unification of fundamental interactions. Moreover, several modern theories require space-time to have more than four dimensions [23,123-128]. The nonobservability of additional dimensions in such theories needs an explanation. Among different possible ways of such explanation the hypothesis about dynamical contraction of internal manifold during expansion of the universe is very popular. This idea is realized in many exact cosmological solution of multidimensional Einstein's equations [21,129-139]. As a rule such models require additional fields and do not avoid initial big bang singularity. The introduction of additional fields in multidimensional gravitation theories destroy their pure geometrical character and require an additional motivation [126]. Such motivation may be done in the framework of some generalizations of Riemannian geometry. In four dimensional case such generalization in several cases leads to removing of cosmological big bang singularity [140-142]. That is why the unification of generalized geometric structures and multidimensional gravity seems to be very attractive. Unfortunately, only in several papers the multidimensional gravitation theory and cosmology are considered in the scope of some generalization of Riemannian geometry [143-145].

One of the simplest generalization of the Riemannian geometry is the integrable Weyl geometry with the connection components

$$\Gamma^{\alpha}_{\beta\gamma} = \widetilde{\Gamma}^{\alpha}_{\beta\gamma} - \frac{1}{2}\left(\omega_\beta \delta^\alpha_\gamma + \omega_\gamma \delta^\alpha_\beta - g_{\beta\gamma}\omega^\alpha\right), \tag{8.1.1}$$

where $\widetilde{\Gamma}^{\alpha}_{\beta\gamma}$ are the Christoffel symbols, $\omega_\alpha = \omega_{,\alpha}$, ω is a scalar field, δ^α_β are the Kroneker symbols, $g_{\alpha\beta}$ is a metric tensor; the small Greek indices take values from 0 to $n-1$, n is a dimension of space-time. The Ricci tensor and the curvature scalar of the connection (1) are equal to

$$R_{\mu\nu} = \widetilde{R}_{\mu\nu} + \frac{n-2}{2}\omega_{\mu||\nu} + \frac{1}{2}g_{\mu\nu}\Box\omega + \frac{n-2}{4}\left(\omega_\mu\omega_\nu - g_{\mu\nu}\omega^\lambda\omega_\lambda\right), \tag{8.1.2}$$

$$R = \widetilde{R} + (n-1)\Box\omega - \frac{(n-1)(n-2)}{4}\omega^\lambda\omega_\lambda, \tag{8.1.3}$$

where the tildes denote the quantities calculated in the connection $\widetilde{\Gamma}^{\alpha}_{\beta\gamma}$, two parallel vertical bars and $\Box$ denote the covariant derivative and the d'Alembert operator of this connection. It is necessary to note that the integrable Weyl space-time is also conformally-Riemannian, since there is a conformal transformation of metric tensor $g_{\alpha\beta}$ which maps the Riemannian space-time into integrable Weyl space-time. As the integrable Weyl space-time is defined by the pair $(g_{\alpha\beta}, \omega)$ the gravitation theory in this space-time does not coincide with Einsteinian general relativity because the field ω must be contained in the Lagrangian independently from $g_{\alpha\beta}$ and cannot be excluded by the conformal transformation.

Some features of the Einsteinian cosmological models with scalar fields were recently considered by several authors [21,129,131,137,139,145-150] both in 4-dimensional and in (4+d)-dimensional space-times. The cosmological models in four-dimensional Weyl-integrable space-time were recently considered by Novello et al. in [140], where the existence of nonsingular open cosmological models was shown. The appearance of Weyl geometry in multidimensional cosmology was discussed also in [144].

In this paper we consider the influence of Weyl geometry on the evolution of Friedman-Robertson-Walker (FRW) cosmological models in multidimensional gravitation theory. As usually the space-time is assumed to have the structure of direct product $M^4 \times V^d$ of four-dimensional FRW space-time M^4 and d-dimensional interior space V^d that is supposed to be d-sphere S^d or d-torus T^d. The metric of space-time is supposed to be block-diagonal

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) - \tilde{g}_{ab} du^a du^b, \qquad (8.1.4)$$

where $k = +1,\ 0,\ -1$ for closed, plane and open models, $d\Omega^2$ is a line element on two-sphere, u^a, $a = 1, ..., d$, and $\tilde{g}_{\alpha\beta}$ are the coordinates and metric tensor of the interior space V^d. Once we consider only spatially homogeneous FRW cosmologies, it is natural to make the Weyl scalar field ω to be a function of cosmic time t only: $\omega = \omega(t)$. We consider both vacuum case and non vacuum case with the additional scalar field φ with non minimal coupling. The 4-dimensional case will be briefly considered also for completeness. The existence of the conformal map between Riemannian and integrable Weyl space-times may be used for generation of exact solutions from the known solutions of general relativity. Such approach admits obtaining only the particular solutions. Therefore to demonstrate general qualitative behavior of the models we solve the system of cosmological equations numerically with initial values given at $t = 0$ and satisfying the constraint equation. For that purpose we use adaptive numerical methods with automatic choice of integration step and with the stiffness checking. The geometrical units where $G = c = 1$ are used in what follows.

8.2. Integrable Weyl cosmology in vacuum

Following [140] we shall consider the vacuum cosmological models in the gravitation theory with the Lagrangian

$$L = R + \xi \omega_\alpha \omega^\alpha \qquad (8.2.1)$$

where R is defined by (8.1.3) and $\xi = const$. After excluding the total derivatives of the scalar field Lagrangian (8.2.1) takes the form

$$L = \tilde{R} - \frac{(n-1)(n-2) - 4\xi}{4} \omega^\alpha \omega_\alpha \qquad (8.2.2)$$

So, the theory differs from the Einstein theory with the massless scalar field by the coefficient before the square of the scalar field gradient and has different geodesic lines. Note also that due to the definition of the Weyl connection (8.1.1) the scalar field ω cannot be renormalized and hence the coefficient ξ before $\omega^\alpha \omega_\alpha$ cannot be put to ± 1 as it may be done in the pure Einstein theory with massless scalar field. Variation of (8.2.2) with respect to the pair $(g_{\alpha\beta}, \omega)$ of independent variables yields the equations

$$\widetilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \widetilde{R} - \frac{(n-1)(n-2) - 4\xi}{4} \left(\omega_\mu \omega_\nu - \frac{1}{2} g_{\mu\nu} \omega^\alpha \omega_\alpha \right) = 0, \tag{8.2.3}$$

and

$$\Box \omega = 0 \tag{8.2.4}$$

The equations (8.2.3-4), coincide with the Einstein equations for the massless scalar field, whose solutions for the FRW cosmological models were investigated both in four-dimensional [148] and multidimensional cases [78]. By this reason here we only summarize briefly the main results.

8.2.1 Four-dimensional case. As the scalar field ω is a function on t only, equation (8.2.4) yields the first integral

$$\dot{\omega} = \frac{\gamma}{a^3} \tag{8.2.5}$$

where overdot denotes time differentiation and $\gamma = const$ is the integration constant. Due to (8.2.5) equations (8.2.3) take the form

$$\dot{a}^2 + k - \frac{\lambda \gamma^2}{12 a^4} = 0 \tag{8.2.6}$$

and

$$2a\ddot{a} + \dot{a}^2 + k + \frac{\lambda \gamma^2}{4 a^4} = 0 \tag{8.2.7}$$

where $\lambda = (3 - 2\xi)$. As it is easy to see from (8.2.6), only singular and static solution of equations (8.2.6-7) exist if $\lambda > 0$. For negative values of λ solution exists only for the open models. In this case $a(t) \geq a_0 = (\xi - 3)\gamma^2/12$ and so the cosmological singularity is absent. The qualitative behavior of scale factor $a(t)$ for negative λ is shown in figure 12 and its features are discussed in detail in [140].

8.2.2. Multidimensional case. In the multidimensional case the behavior of the model depends not only on the parameter ξ, as in the previous case, but on the structure of the interior space also. For simplicity only 5- and 6-dimensional models will be considered in the following. We consider these two cases separately. The main qualitative features of models in general n-dimensional ($n > 6$) case are the same as in 5- and 6-dimensions.

8.2.2.1. 5-dimensional models. In 5-dimensions space-time interval (8.1.4) reads

$$ds^2 = dt^2 - a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) - s^2(t) du^2 \tag{8.2.8}$$

where $u \in S^1$ is the interior space coordinate. Assuming, as above, a scalar field ω to be a function of the cosmological time only, the first integral of equation (8.2.4) takes the form

$$\dot{\omega} = \frac{\gamma_1}{a^3(t) s(t)}, \tag{8.2.9}$$

where $\gamma_1 = const$. Due to (8.2.8-9) equations (8.2.3) become after simplification

$$3\frac{\dot{a}}{a}\frac{\dot{s}}{s} + 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3k}{a^2} - \frac{\gamma_1^2(3-\xi)}{2a^6 s^2} = 0, \tag{8.2.10}$$

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}\frac{\dot{s}}{s} + \frac{k}{a^2} = 0, \tag{8.2.11}$$

and

$$\frac{\ddot{s}}{s} + 3\frac{\dot{a}}{a}\frac{\dot{s}}{s} = 0 \tag{8.2.12}$$

The last equation has the first integral

$$\dot{s} = \frac{\gamma_2}{a^3} \tag{8.2.13}$$

where $\gamma_2 = const$. It is easy to see that analogous to the four-dimensional case the nonsingular solutions of equations (15), (17) exist only for the open models ($k = -1$). In this case for $t < 0$ the scale factor of 3-space $a(t)$ decreases monotonically from infinity to its minimal value a_0 and then grows to infinity at $t > 0$, while the Weyl field $\omega(t)$ and the scale factor of interior space evolve monotonous from $\omega_- = \lim_{t\to-\infty}\omega(t)$ and $s_- = \lim_{t\to-\infty} s(t)$ to $s_+ = \lim_{t\to\infty} s(t)$, where a_0, $\omega_\pm$ and $s_\pm$ are defined by the integration constants and may have arbitrary values. Note that if $\gamma_2 < 0$ than the constants s_- and s_+ satisfy the condition $s_- > s_+$ and so the standard dimensional reduction scenario is realized. The typical shape of the functions $a(t)$ and $s(t)$ are shown in the figures (13.a,b).

The figure (13a) shows that unlike the 4-dimensional case the evolution of 3-space in 5-dimensional model is time-asymmetric. This asymmetry appears because the equation (8.2.11) depends not only on $\dot{s}(t)$ but also on the time-asymmetric interior space scale factor $s(t)$.

8.2.2.2. 6-dimensional models. In 6-dimensional case we consider two types of topological structures for the interior space: the 2-sphere S^2 and 2-dimensional torus T^2. Therefore, the space-time metric (8.1.4) may have one of two forms

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) - s^2(t)\left(\frac{du^2}{1-u^2} + u^2 dv^2\right) \tag{8.2.14}$$

or

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) - s_1^2(t)du^2 - s_2^2(t)dv^2, \tag{8.2.15}$$

where $\{u, v\}$ are the coordinates on S^2 or T^2 respectively. First integrals of equation (8.2.4) take the form

$$\dot{\omega} = \frac{q_1}{a^3 s^2}, \tag{8.2.16}$$

for metric (8.2.14) and

$$\dot{\omega} = \frac{q_2}{a^3 s_1 s2}, \tag{8.2.17}$$

for metric (8.2.15).

Equations (8.2.3) for the metric (8.2.14) after simplification take the form

$$\frac{\ddot a}{a} + 2\left(\frac{\dot a}{a}\right)^2 + 2\frac{\dot a}{a}\frac{\dot s}{s} + \frac{2k}{a^2} = 0, \tag{8.2.18}$$

$$\frac{\ddot s}{s} + \left(\frac{\dot s}{s}\right)^2 + 3\frac{\dot a}{a}\frac{\dot s}{s} + \frac{1}{s^2} = 0, \tag{8.2.19}$$

and the constraint equation

$$3\frac{\dot a}{a}\left(\frac{\dot a}{a} + 2\frac{\dot s}{s}\right) + \frac{3k}{a^2} + \frac{1}{s^2} - \frac{q_1\,(5-\xi)}{2a^6 s^4} = 0. \tag{8.2.20}$$

The first two equations are dynamical and the last is the constraint.

It is easy to see that only singular solutions of equations (8.2.18-20) exist: the scale factor $s(t)$ of the interior space evolves from zero at $t = t_0$ to its maximal value s_{max} and return to zero at $t = t_1 > t_0$. The behavior of $a(t)$ depends on the sign of k. Namely, if $k = +1$ then the qualitative evolution of $a(t)$ is the same as the evolution of $s(t)$. If $k = 0$ than $a(t)$ increase from zero at $t = t_0$ to infinity at $t = t_1$ or decrease from infinity to zero; the unstable solutions with $a(t) = const$ are also exist. Finally, if $k = -1$ then $a(t)$ evolves from infinity at $t = t_0$ to its minimum a_{min} and then grows to infinity at $t = t_1$.

For the metric (8.2.15) equations (8.2.3) after simplification read

$$\frac{\ddot a}{a} + \frac{\dot a}{a}\left(\frac{\dot s_1}{s_1} + \frac{\dot s_2}{s_2}\right) + 2\left(\frac{\dot a}{a}\right)^2 + \frac{2k}{a^2} = 0, \tag{8.2.21}$$

$$\frac{\ddot s_1}{s_1} + 3\frac{\dot a}{a}\frac{\dot s_1}{s_1} + \frac{\dot s_1}{s_1}\frac{\dot s_2}{s_2} = 0, \tag{8.2.22}$$

$$\frac{\ddot s_2}{s_2} + 3\frac{\dot a}{a}\frac{\dot s_2}{s_2} + \frac{\dot s_1}{s_1}\frac{\dot s_2}{s_2} = 0, \tag{8.2.23}$$

and the constraint equation

$$3\frac{\dot a}{a}\left(\frac{\dot a}{a} + \frac{\dot s_1}{s_1} + \frac{\dot s_2}{s_2}\right) + \frac{\dot s_1}{s_1}\frac{\dot s_2}{s_2} + \frac{3k}{a^2} - \frac{q_2^2\,(5-\xi)}{2a^6 s_1^2 s_2^2} = 0. \tag{8.2.24}$$

As in 4- and 5-dimensional cases the nonsingular solutions of the equations (8.2.21-24) exist only for the open models ($k = -1$). Analogously to 5-dimensional case the scale factor of 3-space $a(t)$ in these models decreases monotonously from infinity to its minimal value a_0 and then grows to infinity at $t \to +\infty$, while the scale factors $s_i(t)$, $i = 1,\ 2$, of interior space changes monotonously from $s_{i-} = \lim_{t\to-\infty} s_i(t)$ to $s_{i+} = \lim_{t\to\infty} s_i(t)$. The necessary condition for the realization of the dimensional reduction scenario in this case are defined by the following inequalities

$$\frac{3}{a_0^2} + \frac{q_2^2(5-\xi)}{2a_0^6 s_{10} s_{20}} > 0, \tag{8.2.25}$$

and

$$\dot s_1(0) < 0, \dot s_2(0) < 0 \tag{8.2.26}$$

It is necessary to note that inequality (8.2.25) is the necessary condition for $\dot s_1$ and $\dot s_2$ to be of the same sign. The time behavior of scale factors $a(t)$, $s_1(t)$ and $s_2(t)$ in this case is qualitatively the same as in 5-dimensional case (Figure 13).

8.3. Integrable Weyl cosmology in theory with non minimal scalar field

In this section we consider cosmological models in gravitation theories with Lagrangian

$$L = R\left(1 + \frac{1}{2(n-1)}\varphi^2\right) + \xi\omega^\alpha\omega_\alpha + \eta\varphi^\alpha\varphi_\alpha, \tag{8.3.1}$$

where R is defined by (8.1.3), φ is a real scalar field, $\eta = \pm 1$ and $\xi = const$ as above. In the limiting case $\varphi = const$ Lagrangian (8.3.1) coincides with (8.2.1) while in another limiting case $\omega = const$ it coincides with the Lagrangian for the conformal-invariant scalar field.

The substitution of (8.1.3) into (8.3.1) gives after simplification

$$L = \tilde{R}\left(1 + \frac{\varphi^2}{2(n-1)}\right) - \varphi\varphi^\alpha\omega_\alpha - \frac{(n-1)(n-2) - 4\xi}{4}\omega^\alpha\omega_\alpha - \frac{(n-2)}{8}\varphi^2\omega^\alpha\omega_\alpha + \eta\varphi^\alpha\varphi_\alpha, \tag{8.3.2}$$

where the total derivatives of the scalar fields are omitted.

Variation of (8.3.2) with respect to independent variables $g_{\mu\nu}$, ω and φ yields the equations

$$\begin{gathered}\left(\tilde{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{R}\right)\left(1 + \frac{\varphi^2}{2(n-1)}\right) - \frac{(n-1)(n-2) - 4\xi}{4}\left(\omega_\mu\omega_\nu - \frac{1}{2}g_{\mu\nu}\omega^\alpha\omega_\alpha\right) - \\ \frac{1}{2}\varphi\left(\varphi,_\mu\omega,_\nu + \varphi,_\nu\omega,_\mu\right) - \frac{n-2}{8}\varphi^2\left(\omega,_\mu\omega,_\nu - \frac{1}{2}g_{\mu\nu}\omega^\alpha\omega_\alpha\right) + \frac{\varphi}{n-1}\left(g_{\mu\nu}\Box\varphi - \varphi,_{\mu||\nu}\right) + \\ \varphi,_\mu\varphi,_\nu\left(\eta - \frac{1}{n-1}\right) + g_{\mu\nu}\varphi^\alpha\varphi_\alpha\left(\frac{1}{n-1} - \frac{\eta}{2}\right) = 0,\end{gathered} \tag{8.3.3}$$

$$\left(\frac{(n-1)(n-2) - 4\xi}{2} + \frac{n-2}{4}\varphi^2\right)\Box\omega - \varphi\Box\varphi - \varphi,_\nu\varphi,^{nu} = 0, \tag{8.3.4}$$

and

$$\eta\Box\varphi - \left(\Box\omega + \frac{1}{n-1}\tilde{R} - \frac{n-2}{4}\omega,^\nu\omega,_\nu\right)\varphi = 0, \tag{8.3.5}$$

Equation (8.3.5) shows that non-Riemannian nature of space-time geometry in the considered model leads to the effective mass generation for the scalar field φ.

8.3.1. Four-dimensional models. In four-dimensional case the equations (8.3.3)-(8.3.5) consist of the constraint equation

$$\left(\frac{\dot{a}}{a} + \frac{k}{a^2}\right)^2\left(3 + \frac{\varphi^2}{2}\right) + \frac{\dot{a}}{a}\varphi\dot{\varphi} + \frac{\eta}{2}\dot{\varphi}^2 - \frac{1}{2}\varphi\dot{\varphi}\dot{\omega} - \frac{1}{8}\varphi^2\dot{\omega}^2 - \frac{3-2\xi}{4}\dot{\omega}^2 = 0, \tag{8.3.6}$$

and three dynamical equations

$$\left(2 + \frac{\varphi^2}{3}\right)\frac{\ddot{a}}{a} + \frac{1}{3}\varphi\ddot{\varphi} + 2\left(\frac{\dot{a}}{a}\right)^2\left(2 + \frac{\varphi^2}{3}\right) + \frac{5}{3}\frac{\dot{a}}{a}\varphi\dot{\varphi} + \frac{1}{3}\dot{\varphi}^2 + \left(2 + \frac{\varphi^2}{3}\right)\frac{2k}{a^2} = 0, \tag{8.3.7}$$

$$\left(\frac{\varphi^2}{2} - 2\xi + 3\right)\ddot{\omega} - \varphi\ddot{\varphi} + \left(9 - 6\xi + \frac{3}{2}\varphi^2\right)\frac{\dot{a}}{a}\dot{\omega} - 3\frac{\dot{a}}{a}\varphi\dot{\varphi} - \dot{\varphi}^2 = 0, \tag{8.3.8}$$

and

$$\eta\ddot{\varphi} - \varphi\ddot{\omega} + 2\varphi\frac{\ddot{a}}{a} + 3\frac{\dot{a}}{a}\left(\eta\dot{\varphi} - \varphi\dot{\omega}\right) + \frac{1}{2}\varphi\dot{\omega}^2 + 2\varphi\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2}\varphi = 0. \qquad (8.3.9)$$

The coefficients before $\ddot{a}/a$, $\ddot{\omega}$ and $\ddot{\varphi}$ in the equations (8.3.7)-(8.3.9) depend both on the parameters ξ, η and on the scalar field φ. The determinant of the matrix of coefficients before $\ddot{a}/a$, $\ddot{\omega}$ and $\ddot{\varphi}$ is equal to

$$d = \left(\tfrac{\eta}{2} - \tfrac{2}{3}\right)\varphi^4 + \left(2\eta + \tfrac{4\xi}{3} - \tfrac{2\eta\xi}{3} - 4\right)\varphi^2 + 6\eta - 4\eta\xi\,.$$

The points where $d = 0$ are the singular points of the system (8.3.7)-(8.3.9). These points are not described by the system (8.3.6)-(8.3.9) because for fixed η and ξ equation $d = const$ defines not more than four fixed values of φ and the system (8.3.6)-(8.3.9) reduces to the first order system. Therefore the initial value of the field φ must be from the open set $d \neq 0$.

For η equation $d = 0$ divide the half-plane (ξ, φ^2), in three regions that will be denoted as A, B and C, while for $\eta = -1$ there are only two regions A and B (figure 14a,b). The behavior of the model depends on the region where the point (ξ, φ_0^2) is situated.

Numerical investigation of equations (8.3.1)-(8.3.3) shows that for the closed ($k = 1$) and flat ($k = 0$) cosmological models only singular solutions exist for any initial conditions. For the open models ($k = -1$) if the pair (ξ, φ_0^2) defines the point in the region B (both for $\eta = 1$ and $\eta = -1$) or C (for $\eta = 1$) than only singular solutions of the equations (8.3.7)-(8.3.9) exist. If the pair (ξ, φ_0^2) defines the point in the region A then solutions may be both regular and singular. The numerical investigation does not permit to find the exact conditions of regularity, but it shows that both regular and singular solutions are stable against finite perturbations of the initial conditions. The typical qualitative behavior of the universe scale factor $a(t)$, Weyl field ω and the matter scalar field φ are shown in figure 15a-c.

The universe scale factor $a(t)$ in the typical nonsingular solution evolves from infinity at $t = -\infty$ to its minimal value $a_0 = a(0)$ and then grows to infinity at $t \to \infty$ (figure 15a). Both scalar fields, the Weyl field ω and the field φ evolves between two limiting values: from $\omega_- = \lim_{t\to-\infty}\omega(t)$ and $\varphi_- = \lim_{t\to-\infty}\varphi$ to $\omega_+ = \lim_{t\to\infty}\omega(t)$ and $\varphi_+ = \lim_{t\to\infty}\varphi(t)$. The difference in the evolution of these fields is that the field ω evolves monotonously (figure 15b) while the field φ near $t = 0$ (i. e. near the minimum of $a(t)$) may have several intermediate extrema with one absolute maximum if $\eta = 1$ (figure 15c) or absolute minimum if $\eta = -1$. As $\varphi(t)$ for big $|t|$ tends asymptotically to constants, the model evolves asymptotically as an empty Weyl cosmological model that is considered in section 8.2.1. It is necessary to note also that the evolution of the universe scale factor $a(t)$ has a small time-asymmetry in comparison with the case of the empty space. This asymmetry is a result of non symmetrical evolution of the matter field φ because the field equations (8.3.7)-(8.3.9) contain both φ and $\dot{\varphi}$.

8.3.2. 5-dimensional models. In 5-dimensional case equations (8.3.3)-(8.3.5) after simplification become

$$3\left\{\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a} + \frac{\dot{s}}{s}\right) + \frac{k}{a^2}\right\}\left\{1 + \frac{\varphi^2}{8}\right\} + \frac{\varphi\dot{\varphi}}{4}\left(\frac{3\dot{a}}{a} + \frac{\dot{s}}{s}\right) + \frac{\eta\dot{\varphi}^2}{2} - \frac{\varphi\dot{\varphi}\dot{\omega}}{2} +$$
$$+\frac{\dot{\omega}^2}{2}\left(\xi - 3 - \frac{3\varphi^2}{8}\right) = 0, \qquad (8.3.10)$$

$$\left\{\frac{\ddot{a}}{a}-\frac{\ddot{s}}{s}+2\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}-\frac{\dot{s}}{s}\right)+\frac{2k}{a^2}\right\}\left\{1+\frac{\varphi^2}{8}\right\}+\frac{\varphi\dot{\varphi}}{4}\left(\frac{\dot{a}}{a}-\frac{\dot{s}}{s}\right)=0, \tag{8.3.11}$$

$$3\left\{1+\frac{\varphi^2}{8}\right\}\left\{\frac{\ddot{a}}{a}+\frac{\dot{a}}{a}\left(2\frac{\dot{a}}{a}+\frac{\dot{s}}{s}\right)+\frac{2k}{a^2}\right\}+\frac{\varphi\ddot{\varphi}}{4}+\frac{\varphi\dot{\varphi}}{2}\left(3\frac{\dot{a}}{a}+\frac{\dot{s}}{2s}+\frac{\dot{\varphi}}{2\varphi}\right)=0, \tag{8.3.12}$$

$$\ddot{\omega}\left(\frac{3\varphi^2}{4}-2\xi+6\right)-\varphi\ddot{\varphi}+\dot{\omega}\left(3\frac{\dot{a}}{a}+\frac{\dot{s}}{s}\right)\left(3\frac{\varphi^2}{4}-2\xi+6\right)-\varphi\dot{\varphi}\left(3\frac{\dot{a}}{a}+\frac{\dot{s}}{s}+\frac{\dot{\varphi}}{\varphi}\right)=0, \tag{8.3.13}$$

and

$$\frac{\varphi}{2}\left(\frac{3\ddot{a}}{a}+\frac{\ddot{s}}{s}\right)+\eta\ddot{\varphi}-\varphi\ddot{\omega}+\varphi\dot{\omega}\left(\frac{3\dot{\omega}}{4}-3\frac{\dot{a}}{a}-\frac{\dot{s}}{s}\right)+\eta\dot{\varphi}\left(3\frac{\dot{a}}{a}+\frac{\dot{s}}{s}\right)+\frac{3\varphi\dot{a}}{2a}\left(\frac{\dot{a}}{a}+\frac{\dot{s}}{s}\right)$$
$$+\frac{3k\varphi}{2a^2}=0. \tag{8.3.14}$$

The equation (8.3.1) is the constraint that must be satisfied by the initial conditions and the equations (8.3.11-14) are the dynamical. The determinant of the matrix of the coefficients before $\ddot{a}/a$, $\ddot{s}/s$, $\ddot{\omega}$ and $\ddot{\varphi}$ in the dynamical equations (8.3.11-14) is equal to $d=\left(\frac{9}{256}\eta-\frac{3}{32}\right)\varphi^6+\left(\frac{27}{32}\eta+\frac{1}{8}\xi-\frac{3}{32}\eta\xi-\frac{3}{2}\right)\varphi^4+\left(\frac{27}{4}\eta+\xi-\frac{3}{2}\eta\xi-6\right)\varphi^2+18\eta-6\eta\xi$.

The qualitative features of function $d(\xi,\eta,\varphi)$ are the same as in 4-dimensional case: for $\eta=1$ equation $d=0$ divides the half-plane $(\xi,\varphi^2>0)$ in three regions that are denoted as A, B and C, while for $\eta=-1$ there are only two regions A and B (figure 16a,b). The behavior of the model depends on the region where the point (ξ,φ_0^2) is situated.

Numerical investigation of equations (8.3.10-14) shows that as well as in the previous 4-dimensional case only singular solutions exist at any initial conditions for the closed $(k=1)$ and flat $(k=0)$ cosmological models. For the open models $(k=-1)$ if the pair (ξ,φ_0^2) defines the point in the region B (both for $\eta=1$ and $\eta=-1$) or C (for $\eta=1$) than only singular solutions of the equations (8.3.10-14) exist, while if the pair (ξ,φ_0^2) defines the point in the region A than the solution may be both regular and singular. The regularity of solutions depends on the constants of integration that may be considered as the initial conditions at $t=0$. It was found that the regularity of solutions depends mainly on the signs of $\dot{s}(0)$, $\dot{\omega}(0)$ and $\dot{\varphi}(0)$. Their possible combinations that give nonsingular solutions of equations (8.3.11-14) are represented in table 2. The last column of this table shows the general direction of the interior space evolution by means of the signs of the difference $\Delta=s_+-s_-$, where $s_\pm=\lim_{t\to\pm\infty}s(t)$.

Table 2.
Conditions of the solutions regularity and the direction of $s(t)$ evolution

sign$\dot{s}(0)$	sign$\dot{\omega}(0)$	sign$\dot{\varphi}(0)$	sign$(s_+ - s_-)$
-1	-1	0	-1
-1	+1	0	-1
-1	-1	+1	-1
-1	+1	-1	-1
0	+1	0	-1
0	-1	0	+1
+1	+1	-1	+1
+1	-1	0	+1
+1	+1	0	+1
+1	-1	+1	+1

The typical behavior of the nonsingular solution of the equations (8.3.11-14) for $\eta = 1$ is shown at the figures 17a-d for the case $\Delta \leq 0$, i. e. for the contracting interior space.

In general nonsingular solution the radius of the universe changes monotonously from infinity at $t = -\infty$ to minimal value a_0 and then grows to infinity (figure 17a), while the radius of the internal space starts from $s_- = \lim_{t\to-\infty} s(t)$, passes through several (one or two) intermediate extrema, that are situated near minimum of $a(t)$ and may be absent in some cases, and then changes to $s_+ = \lim_{t\to\infty} s(t)$ (figure 17b). Note that s_+ and s_- may be of the same or different order. The field φ evolves analogously to 4-dimensional case (figure 17c). Note that the extremal points of the functions $a(t)$, $s(t)$ and $\varphi(t)$ do not coincide with each other in general case and the function $a(t)$ is time asymmetrical especially near its minimum. Finally the Weyl field ω changes monotonously between two limiting values: $\omega_- = \lim_{t\to-\infty} \omega(t)$ and $\omega_+ = \lim_{t\to\infty} \omega(t)$ (figure 17d). In the case $\eta = -1$ the model evolves as above but the extremal points of the field φ change type: minimum becomes maximum and vice versa.

8.4. Concluding remarks

We have considered the qualitative evolution of multidimensional cosmological models based on the integrable Weyl geometry both in vacuum space-time and in the presence of nonminimal scalar field. The existence of nonsingular solutions of field equations for open cosmological models that realized the dimensional reduction scenario was demonstrated. It was shown that in multidimensional case the evolution of the scale factor of the universe $a(t)$ becomes time-asymmetric unlike the four-dimensional case. We have shown also that all nonsingular cosmological models considered above have some common features. In particular the evolution of the universe scale factor (radius) $a(t)$ for big $|t|$ is asymptotically linear. Further in all nonsingular models Weyl scalar field $\omega(t)$ as well as the matter field $\varphi(t)$ in the models with nonminimal coupling tend asymptotically to constants. So the models tend to the pure Einsteinian models of the corresponding dimensions and the change of the collapse era into expansion one may be considered as a cosmological phase transition induced by the transition of scalar fields $\omega(t)$ and $\varphi(t)$

from one stationary state $\omega = \omega_-$ and $\varphi = \varphi_-$ into another stationary state $\omega = \omega_+$ and $\varphi = \varphi_+$. At the late stages of the universe evolution the fields $\omega(t)$ and $\varphi(t)$ are unobservable.

There are several qualitative differences between the vacuum models and the models with nonminimal scalar field. First of all in vacuum models the existence of cosmological singularity depends only on the parameters of the theory while in the case of nonminimal scalar field it depends on the initial conditions also. Secondly, in the models with nonminimal scalar field the evolution of the internal space scale factor $s(t)$ may be non-monotonous. In the typical scenario one of the limiting values of $s(t)$ at $t = \pm\infty$ is much smaller than another but in several models both limiting values of internal radius $s(t)$ may be arbitrary small and it become finite only near minimum of the universe scale factor $a(t)$.

We have discussed here only the models with the one- or two- dimensional interior space because if interior space has dimension $d \geq 3$ and direct product topology of torus on several spheres then the models have the same qualitative features as considered above. In particular, the nonsingular solutions exist only for toroidal interior space topology.

The models considered above show that the real geometrical structure of space-time may have a non-Riemaniann nature but the universe may evolve in such a way that its non-Riemaniann nature is essential only near $t = 0$ and become unobservable at late stages of the evolution. Therefore, the consideration of generalized geometrical structures in multidimensional cosmology may be of a considerable interest. In particular, the models considered above may be generalized in the following manner. First of all, both Weyl scalar field $\omega(t)$ and matter field $\varphi(t)$ may be massive and have nonlinear potential. Secondly, the possible influence of the cosmological term Λ must be considered also. At last, the term $R\varphi^2/2(n-1)$ in the lagrangian (8.3.1) may have negative sign. One may suppose that in this case nonsingular solutions of the field equations may be obtained not only for open models, but for closed and flat models also. These possibilities will be considered elsewhere.

9. Exact Solutions in Integrable Weyl Geometry in Multidimensional Cosmology [152-153]

9.1. Introduction

Here we continue to study multidimensional models in integrable Weyl geometry started in section 8. We stress that the gravitational field in a Weyl-integrable space-time (WIST) is determined by the tensor g_{AB} and the scalar ω, just as in scalar-tensor theories (STT) of gravity. The difference between these two cases is determined by Eq. (8.1.1). Namely, both in STT and in WIST there is a conformal gauge in which test particles move along geodesics; however, in WIST, unlike STT, even in this frame the motion in general depends on both the metric and the scalar field. Thus, a gravitation theory on the basis of WIST is in general not a special case of STT due to a nonminimal coupling between the matter and the scalar field.

However, field equations in STT and WIST-based theories in many cases coincide, in particular, for all vacuum space-times.

The description of cosmological models in STT is often reduced to that of Einsteinian cosmologies with scalar fields. The latter were considered by many authors [21,50] in both 4-dimensional and (4+d)-dimensional space-times.

In this section we consider the evolution of multidimensional cosmological models based on integrable Weyl geometry with finding exact solutions for some simplest cases of empty spaces. The main characteristic features of the solutions are illustrated graphically. Keeping in mind the possible applications of the results to the description of quantum stages of the universe evolution we also consider WIST with the Euclidean signature.

9.2. Model

As is the case with STT, the gravitational field Lagrangian may in general contain various invariant combinations of g_{AB} and ω. Let us restrict ourselves to Lagrangians which are (a) linear in the scalar curvature and (b) quadratic in ω_A. Then the general form of the Lagrangian satisfying (a) and (b) is

$$L = A(\omega)R + B(\omega)\omega^A\omega_A - 2\Lambda(\omega) + L_m \tag{9.2.1}$$

where R is the Weyl scalar curvature corresponding to the connection (8.1.1), A, B and Λ are arbitrary functions and L_m is the nongravitational matter Lagrangian.

Using the expression (8.1.2) for R in terms of the Riemannian curvature $\widetilde{R}$ corresponding to the metric g_{AB}, the conformal mapping well-known in STT [154], modified for D dimensions [78,152]:

$$g_{MN} = A^{-2/(D-2)}\overline{g}_{MN}. \tag{9.2.2}$$

and omitting a total divergence, we obtain the following form of the Lagrangian:

$$\overline{L} = A(\omega)\overline{R} + F(\omega)\overline{g}^{AB}\omega_A\omega_B + A^{-D/(D-2)}[-2\Lambda(\omega) + L_m] \tag{9.2.3}$$

where

$$F(\omega) = \frac{1}{A(\omega)^2}\Big[A(\omega)B(\omega) - (D-1)A(\omega)\Big(A_\omega + \frac{D-2}{4}\Big) + \frac{D-1}{D-2}A_\omega^2\Big]. \tag{9.2.4}$$

Let us consider vacuum cosmological models with the following structure of the space-time W_D:

$$W_D = \mathrm{R} \times M_1 \times \ldots \times M_n; \qquad \dim M_i = N_i; \tag{9.2.5}$$

where the subspaces M_i are assumed to be maximally symmetric. The component R corresponds to the time τ; besides, we assume $\omega = \omega(\tau)$. Thus, the effective Riemannian metric is written in the form

$$d\overline{s}^2 = \overline{g}_{AB} dx^A dx^B = \mathrm{e}^{2\gamma(\tau)} d\tau^2 - \sum_{i=1}^{n} \mathrm{e}^{2\beta_i(\tau)} ds_i^2 \tag{9.2.6}$$

where ds_i^2 are τ-independent metrics of the N_i-dimensional spaces of constant curvatures K_i; with no loss of generality one can put $K_i = 0, \ \pm 1$.

Making use of the freedom to choose the time coordinate τ, let us introduce the harmonic time by putting

$$\gamma = \sum_{i=1}^{n} N_i \beta_i. \tag{9.2.7}$$

Then the Ricci tensor for $\overline{g}_{AB}$ has the following nonzero components:

$$\begin{aligned} \overline{R}^{\tau}_{\tau} &= \mathrm{e}^{-2\gamma}\Big(\ddot{\gamma} - \dot{\gamma}^2 + \sum_{i=1}^{n} N_i \dot{\beta}_i^2\Big), \\ \overline{R}^{m_i}_{n_i} &= \delta^{m_i}_{n_i}\Big[\mathrm{e}^{-2\gamma}\ddot{\beta}_i + (N_i - 1) K_i \mathrm{e}^{-2\beta_i}\Big] \end{aligned} \tag{9.2.8}$$

where the indices $m_i, \ n_i$ belong to the subspace M_i.

The field equations take an especially simple form under the additional condition $\Lambda \equiv 0$:

$$\overline{R}_{MN} + F(\omega)\omega_M \omega_N = 0, \tag{9.2.9}$$

$$2\overline{\nabla}_M [F(\omega)\omega^M] - F_\omega \omega^M \omega_M = 0. \tag{9.2.10}$$

9.3. Solutions

They can be integrated completely under one of the above assumptions: (i) if all the subspaces M_i are Ricci-flat and (ii) if one of M_i (for instance, M_1) is a space of nonzero constant curvature (K_1). Indeed, putting $K_i = 0$ ($i > 1$), we obtain:

$$(F\dot{\omega}^2)^{\cdot} = 0 \quad \Rightarrow \quad F\dot{\omega}^2 = S = \mathrm{const}; \tag{9.3.1}$$

$$\ddot{\beta}_i = 0 \quad \Rightarrow \quad \beta_i = \beta_{i0} + h_i \tau, \qquad i > 1; \tag{9.3.2}$$

$$\ddot{\gamma} - \ddot{\beta}_1 = -K_1 d^2 \mathrm{e}^{2\gamma - 2\beta_1} \tag{9.3.3}$$

where $d{+}1 = N_1 = \dim M_1$. The equation (9.3.3) leads to different results for different K_1: for $K_1 = 0$ (case (i)) Eq. (9.3.2) may be regarded to include $i = 1$; for $K_1 \neq 0$ (case (ii)) we get:

$$\mathrm{e}^{\beta_1 - \gamma} = \frac{d}{k}\cosh k\tau, \qquad k > 0 \qquad (K_1 = +1), \tag{9.3.4}$$

$$\mathrm{e}^{\beta_1 - \gamma} = d \cdot s(k, \tau) \equiv \begin{cases} (d/k)\sinh k\tau, & k > 0, \\ d \cdot \tau, & k = 0, \\ (d/k)\sin k\tau, & k < 0, \end{cases} \qquad (K_1 = -1) \tag{9.3.5}$$

where $k =$ const and another integration constant is elimintaed by a particular choice of the origin of τ. Lastly, a combination of components of (9.2.9) representing the temporal component of the Einstein equations (the initial data equation) leads to the following relation among the integration constants:

$$\Big(\sum_{i=1}^{n} N_i h_i\Big)^2 - \sum_{i=1}^{n} N_i h_i^2 = S, \qquad K_1 = 0; \tag{9.3.6}$$

$$\frac{d{+}1}{d} k^2 \text{sign } k = \frac{1}{d}\Big(\sum_{i=2}^{n} N_i h_i\Big)^2 + \sum_{i=2}^{n} N_i h_i^2 + S, \qquad K_1 \neq 0. \tag{9.3.7}$$

Thus, the set of equations (9.2.9-10) has been integrated in quadratures.

As the original functions $A(\omega)$ and $B(\omega)$ and hence $F(\omega)$ are arbitrary, it is difficult to describe the physical properties of the models in a general form. Therefore, here we would like to restrict ourselves to some simple special cases.

Thus, we will assume $A \equiv 1$ while $B(\omega)$ remains arbitrary, so that the metrics $\overline{g}_{AB}$ and g_{AB} coincide.

9.4. Special Cases

As the first step consider 4-dimensional homogeneous isotropic cosmologies. For this purpose we must put $n = 1$, $d = 2$, $\beta_1 \equiv \beta(\tau)$. The condition that τ is a harmonic coordinate takes the form $\gamma = 3\beta$ and for the scale factor we get:

$$e^{2\beta} = a^2(\tau) = \begin{cases} 1/2s(k,\tau), & K_1 = 1, \\ e^{k\tau}, & K_1 = 0, \\ 1/2\cosh k\tau, & K_1 = -1, \end{cases} \tag{9.4.1}$$

where $s(k,\tau)$ is defined by (9.3.5) and the physical time is determined by the integral $t = \pm\int e^{\gamma(\tau)} d\tau$. The constant k is connected with the "scalar charge" S according to (9.3.6), (9.3.7) where one should substitute $h_i = 0$ $(i > 1)$ and $h_1 = k/2$:

$$2S = \begin{cases} 3k^2 \text{sign } k, & K_1 = \pm 1, \\ 3k^2, & K_1 = 0. \end{cases} \tag{9.4.2}$$

It is easy to obtain that in the case of a spherical world ($K_1 = 1$) the values $\tau = \pm\infty$ correspond to finite times t_1 and t_2 at which $a = 0$ (the initial and final singularities). For a flat world ($K_1 = 0$) at $k \neq 0$ and a hyperbolic one ($K_1 = -1$) at $k > 0$ an initial or final singularity is observed at infinite τ. In the special case $K_1 = -1$, $k = 0$ we obtain the Milne vacuum model which is known to describe a domain in flat space-time (in this case $S = 0$, so that the scalar field is trivial).

Lastly, in the case $K_1 = -1$, $k < 0$ we see that the limits $\tau \to 0$, $\pi/|k|$ correspond to $t \to \pm\infty$; the scale factor $a(t)$ decreases in an asymptotically linear manner in the remote past ($t \to -\infty$), reaches a minimum at $\tau = \pi/2|k|$ and grows in an asymptotically linear manner at $t \to \infty$ while the scalar field ω changes monotonically from one limiting value ω_- at $t \to -\infty$ to another limiting value ω_+ at $t \to +\infty$. The model is time-symmetric with respect to the maximum contraction instant. The tipical shape of the function $a(t)$ for this case is shown in Fig. 18.

By (9.4.2) a necessary condition for the existence of nonsingular solutions is the restriction $F<0$ on the function (9.2.4), (as in this case $S<0$), or, in terms of the initial function $B(\omega)$: $B<3/2$.

These results confirm those of Ref. [140].

Consider now the metric $\overline{g}_{AB}$ for $n=2$: let $a(t)\equiv \mathrm{e}^{\beta_1(\tau)}$ be the scale factor of the ordinary physical space ($N_1=3$), while $b(t)\equiv \mathrm{e}^{\beta_2(\tau)}$ that of the internal space ($N_2=N$).

In the case $K_1=0$ (spatially flat models) we obtain:

$$d\overline{s}^2=\mathrm{e}^{2(3h_1+Nh_2)\tau}d\tau^2-\mathrm{e}^{2h_1\tau}ds_1^2-\mathrm{e}^{2h_2\tau}ds_2^2 \tag{9.4.3}$$

where with no loss of generality the scales in M_1 and M_2 are chosen so that $\beta_{10}=\beta_{20}=0$. Herewith

$$6(h_1+Nh_2/2)^2=N(N+1/2)+S \tag{9.4.4}$$

In the special case $3h_1+Nh_2=0$ the time coordinate τ is synchronous, in other words, physical. The metric (9.4.3) is nonsingular at finite τ and describes an exponential expansion (inflation) of one of the spaces (e.g., the physical one, M_1) and a simultaneous exponential contraction of the other, M_2, since h_1 and h_2 have different signs. However, by (9.4.4) and (9.3.1)

$$S=F\dot{\omega}^2=-h_1^2(2N+1)/N<0. \tag{9.4.5}$$

So, a necessary condition for the existence of the special solution (9.4.3) is the restriction

$$B(\omega)<(D-1)(D-2)/4, \tag{9.4.6}$$

more general than $B<3/2$ for the 4-dimensional case.

In the more general case $3h_1+Nh_2=H\neq 0$ a transition to the physical time $dt=\mathrm{e}^{H\tau}d\tau$ leads to the metric

$$d\overline{s}^2=dt^2-t^{2h_1/H}ds_1^2-t^{2h_2/H}ds_2^2 \tag{9.4.7}$$

which is singular at $t=0$ if at least one of the constants h_1 or h_2 is nonzero. At $h_1=h_2=0$ the metric is static and (9.4.5) implies that either $\dot{\omega}=0$ (the solution is trivial), or $F\equiv 0$, a special choice of B such that $\omega(\tau)$ has no dynamics.

For a spherical world ($K_1=1$) the metric is

$$d\overline{s}^2=\frac{\mathrm{e}^{-Nh\tau}}{2\cosh k\tau}\left[\frac{d\tau^2}{4\cosh^2 k\tau}-ds_1^2\right]-\mathrm{e}^{2h\tau}ds_2^2 \tag{9.4.8}$$

where ds_1^2 is the line element on a unit sphere. A consideration like that as for $K_1=1$ leads to the following conclusions:

(a) The model behavior is classified by the values of the constant $h=h_2$ as compared with $k>0$. The physical time $t=\pm\int \mathrm{e}^{\gamma(\tau)}d\tau$ varies either within a finite segment $[t_1,\ t_2]$ (if $|Nh|<3k$), or within a semi-infinite range (if $|Nh|\geq 3k$).

(b) At any finite boundary of the range of t at least one of the scale factors $a(t)$ or $b(t)$ vanishes, i.e., a singularity takes place.

(c) At $t \to \pm\infty$ either $a \to 0,\ b \to \infty$, or conversely, $a \to \infty,\ b \to 0$.

The value $S = -F\dot{\omega}^2$ is determined at $K_1 = \pm 1$ from

$$3k^2 \mathrm{sign}\, k = N(N+2)h^2 + 2S. \tag{9.4.9}$$

For hyperbolic models ($K_1 = -1$) the metric has the form

$$d\overline{s}^2 = \frac{\mathrm{e}^{-Nh\tau}}{2s(k,\tau)}\left[\frac{d\tau^2}{4s^2(k,\tau)} - ds_1^2\right] - \mathrm{e}^{2h\tau} ds_2^2 \tag{9.4.10}$$

(the same as (9.4.8) but the function $\cosh k\tau$ is replaced by $s(k,\tau)$ defined in (9.3.5). Preserving generality, let us assume $\tau > 0$.

The model behavior may be briefly described as follows:

(a) At $k > 0,\ Nh \le -3k$ or $k = 0,\ h < 0$ the physical time $t = \pm\int \mathrm{e}^{\gamma(\tau)} d\tau$ ranges from $-\infty$ to $+\infty$. The factor $b(t) = \mathrm{e}^{h\tau}$ varies from a finite value at $\tau = 0$ ($t = -\infty$) to zero at $\tau \to \infty$ ($t \to \infty$). The factor $a(t)$ describes a power-law contraction from infinity (at $t \to -\infty$) to a regular minimum and an infinite (in general, power-law) expansion at $t \to \infty$. There is no singularity at finite t.

(b) At $k \ge 0,\ Nh > 3k$ the model is singular at finite t corresponding to $\tau \to \infty$. In the special case $h = k = 0$ we come again to the Milne model supplemented with the space M_2 with a constant scale factor.

(c) At $k < 0$ the time t ranges again from $-\infty$ to $+\infty$. The factor $a(t)$ behaves as it did in item (a), however, its variation at $t \to \pm\infty$ is linear (but in general with unequal slopes at the two asymptotics). The factor $b(t)$ changes monotonically between two finite boundary values. The typical time dependence of the scale factors $a(t)$ and $b(t)$ in this case is shown in Fig. 19.

It is necessary to note that, unlike the 4-dimensional models, the nonsingular multidimensional ones with $h \ne 0$ exhibit a time-asymmetric behavior of $a(t)$.

It is seen in a straightforward way that in all the nonsingular models the requirement (9.4.6) is imposed on $B(\omega)$, which, as it could be formulated in general relativity, means the negative scalar field energy density.

Some properties of the above models have been discovered in numerical calculations for a number of special cases with $D = 5$ and $D = 6$ ([125] and section 8).

9.5. Euclidean Solutions

Keeping in mind possible applications of our models to quantum stages of the universe evolution, let us continue them to the Euclidean sector. For this purpose let us replace the metric (9.2.6) by a slightly more general one

$$d\overline{s}^2 = \overline{g}_{AB} dx^A dx^B = \mathrm{e}^{2\gamma(\tau)} d\tau^2 + \sum_{i=1}^{n} \varepsilon_i \mathrm{e}^{2\beta_i(\tau)} ds_i^2 \tag{9.5.1}$$

where $\varepsilon_i = \pm 1$. Then in Eqs.(9.2.8) and consequently in the field equations the only change is that K_i are replaced by $\varepsilon_i K_i$. If we put, as before, $K_i = 0$ for $i \ge 2$, the equations depend only on $\varepsilon_1 K_1$. That means that the evolution of the Lorentzian open model ($K_1 = -1$, $\varepsilon_1 = -1$) coincides with that of the Euclidean closed model ($K_1 = 1$, $\varepsilon_1 = 1$) and vice versa, and the evolution of models with a flat 3-space ($K_1 = 0$) does not depend on the metric signature. In particular, the nonsingular Lorentzian model with an open 3-space, whose exterior and interior scale factors are shown in Figs. 19a,b, corresponds to the Euclidean four-dimensional wormhole $\mathrm{S}^3 \times \mathrm{R}^1$.

In conclusion, we have seen that many of the multidimensional Weyl cosmologies with flat additional spaces are nonsingular: there are special flat-space models with eternally increasing or decreasing scale factors (such models are absent in 4 dimensions) and there are more general hyperbolic models with a cosmological bounce (generalizing the 4-dimensional ones [140]) which realize the dimensional reduction scenario. It has been shown that in the multidimensional case the evolution of the scale factor of the universe $a(t)$ becomes time-asymmetric, unlike the 4-dimensional case. In particular the evolution of $a(t)$ for big $|t|$ is asymptotically linear.

Chapter 2

Multidimensional Gravity. Spherical and Axial Symmetry Cases

1. The Birkhoff Theorem in Multidimensional Gravity [155]

1.1. Introduction

The original Birkhoff theorem [156] states that in general relativity (GR) the spherically symmetric vacuum field is static and is thus reduced to the Schwarzschild solution. From a wider viewpoint, the theorem indicates a case when the field equations induce, under certain circumstances, an additional field system symmetry that was not postulated at the outset. The theorem is closely related to the quadrupole nature of the gravitational field in GR, more precisely, to the absence of monopole gravitational waves. Thus theorems of this sort are able not only to simplify the treatment of certain physically relevant situations but also to provide their better understanding.

After Birkhoff the theorem was extended to spherical systems with a nonzero cosmological constant Λ, the Maxwell or Born-Infeld electromagnetic fields [157-158] and others), scalar fields and $\Lambda \neq 0$ in GR [159] and some scalar-tensor theories of gravity [160]. In Ref. [161] the theorem was extended to planarly and pseudospherically symmetric Einstein-Maxwell fields.

Another approach was suggested in Refs. [162-163]: the study was aimed at finding out general conditions under which the staticity theorem could be proved. This allowed all the previously found cases of GR and scalar-tensor theories when the extended Birkhoff theorem is valid, to be included, along with many new ones. The theorem was generalized in two respects: to include more types of space-time symmetry (e.g., planar, cylindrical and pseudoplanar) and more kinds of matter (scalar fields, gauge fields, perfect fluid, etc.).

Here we would like to extend the approach of [162-163] to multidimensional gravity. One may recall that most modern unification theories incorporate more than four dimensions (e.g., that of superstrings [23]); on the other hand, some studies are undertaken in (2+1) and even (1+1) dimensions where certain hard problems simplify and admit a deeper insight. The low energy limit of many theories, actually embracing an enormous range of energy scales, is reduced to the multidimensional Einstein equations

$$G_A^B \equiv R_A^B - \delta_A^B R_C^C/2 = -T_A^B, \tag{1.1.1}$$

where R_A^B is the D-dimensional Ricci tensor and T_A^B is the matter energy-momentum tensor (EMT). We will assume the validity of (1.1.1) for some dimension D and some kind of matter and find certain general conditions under which these field equations make the system symmetry increase. The consideration essentially follows the lines of [162-163]. In Section 1.2 the extended Birkhoff theorem is proved for multidimensional GR. In Section 1.3 its different special cases are discussed and Section 1.4 contains some remarks, in particular, on situations excluded by the requirements of the theorem; its further extension to multidimensional scalar-tensor theories is presented.

1.2. THEOREM

Consider a D-dimensional Riemannian or pseudo-Riemannian space with the structure

$$V^D = M^2 \times V_1 \times V_2 \times \ldots \times V_n, \quad \dim V_i = N_i, \quad n = 1, 2, \ldots \tag{1.2.1}$$

where M^2 is an arbitrary two-dimensional subspace parametrized by the coordinates u and v and V_i are subspaces of arbitrary dimension (N_i) and signature whose metric depends on u and v only via conformal (scale) factors. Thus with no further loss of generality the D-dimensional metric may be written in the form

$$ds_D^2 = \eta_u \mathrm{e}^{2\alpha} du^2 + \eta_v \mathrm{e}^{2\gamma} dv^2 + \sum_{i=1}^{n} \mathrm{e}^{2\beta_i} ds_i^2 \tag{1.2.2}$$

where $\eta_u = \pm 1$, $\eta_v = \pm 1$; α, β_i and γ are functions of u and v and ds_i^2 are the u- and v-independent metrics of the subspaces. It is meant that M^2 along with one (two-dimensional) or two (one-dimensional) subspaces V_i form the conventional physical space-time while the rest V_i correspond to extra (internal) dimensions. For greater generality we would not like to fix the signs η_u and η_v.

Before formulating the theorem let us introduce the quantity

$$\rho(u,v) \equiv \sum_{i=1}^{n} N_i \beta_i(u,v). \tag{1.2.3}$$

and present the nonzero Ricci tensor components for the metric (1.2.2):

$$R_u^u = \Box_u \gamma + \Box_v \alpha + \eta_u \mathrm{e}^{-2\alpha}(\rho'' - \alpha'\rho' + \sum_{i=1}^{n} \beta_i'^2) + \eta_v \mathrm{e}^{-2\gamma} \dot\alpha \dot\rho; \tag{1.2.4}$$

$$R_v^v = \Box_u \gamma + \Box_v \alpha + \eta_u \mathrm{e}^{-2\alpha} \gamma'\rho' + \eta_v \mathrm{e}^{-2\gamma}(\ddot\rho - \dot\gamma\dot\rho + \sum_{i=1}^{n} \dot\beta_i^2); \tag{1.2.5}$$

$$R_{n_i}^{m_i} = \mathrm{e}^{-2\beta_i} \overline{R}_{n_i}^{m_i} + \delta_{n_i}^{m_i}[(\Box_u + \Box_v)\beta_i + \eta_u \mathrm{e}^{-2\alpha}\beta_i'\rho' + \eta_v \mathrm{e}^{-2\gamma}\dot\beta_i\dot\rho]; \tag{1.2.6}$$

$$R_{uv} = \dot\rho' - \gamma'\dot\rho - \dot\alpha\rho' + \sum_{i=1}^{n} \dot\beta_i \beta_i' \tag{1.2.7}$$

where primes and dots stand for partial derivatives ∂_u and ∂_v, respectively, and

$$\Box_u = \mathrm{e}^{-\alpha-\gamma}\partial_u(\mathrm{e}^{\gamma-\alpha}\partial_u), \quad \Box_v = \mathrm{e}^{-\alpha-\gamma}\partial_v(\mathrm{e}^{\alpha-\gamma}\partial_v), \tag{1.2.8}$$

The indices m_i and n_i belong to the subspace V_i; the Ricci tensors $R^{m_i}_{n_i}$ correspond to the metrics ds_i^2 and do not depend on u and v.

Theorem 1. *Let there be a Riemannian space V^D* (1.2.1) *with metric* (1.2.2) *obeying the Einstein equations* (1.1.1). *If*

(A) *there is a domain Δ in M^2 where*

$$\mathrm{sign}\,(\rho^{,A}\rho_{,A}) = \eta_u; \tag{1.2.9}$$

(B) *each $\beta_i(u,v)$ in Δ is functionally related to ρ* (certain relations $F_i(\rho,\beta_i)=0$ are valid);

(C) *in an orthogonal coordinate frame where $\rho=\rho(u)$* (its existence is guaranteed by (1.2.9)) *the EMT component $T_{uv}\equiv 0$ and there is a combination*

$$T^v_v + \mathrm{const}\cdot T^u_u \tag{1.2.10}$$

independent of v and γ,

then there is an orthogonal coordinate frame (u,v) in Δ such that the metric (1.2.2) *is v-independent.*

Proof. Let us choose an orthogonal coordinate frame where $\rho=\rho(u)$, which is possible by Condition A. Then by Condition B all $\dot\beta_i = 0$. By Condition C the mixed EMT component $T_{uv}=0$ (in the conventional case $\eta_u = -\eta_v$ that means that there is no energy flow in the frame of reference where $\rho=\rho(u)$), and the corresponding component of Eqs.(1.1.1) yields $\dot\alpha = 0$. Now only γ may depend on v. To make the last step and to obtain $\gamma=\gamma(u)$ it is sufficient to find a combination of the Einstein equations having the form $\gamma' = f(u)$, whence

$$\gamma = \gamma_1(u) + \gamma_2(v) \tag{1.2.11}$$

and γ_2 may be brought to zero by a coordinate transformation $v = v(\tilde v)$. Observing (1.2.4-6), one can see that any combination of the form $G^u_u + \mathrm{const}\cdot G^v_v$ of the left-hand sides of (1.1.1) does contain γ but only in the term $\mathrm{e}^{-2\alpha}\rho'\gamma'$. As $\rho\neq\mathrm{const}$, our problem is solved when the corresponding combination of T^B_A does not depend on γ and v, exactly what is required in Condition C. This completes the proof.

The theorem generalizes the results of [162-163] to arbitrary space-time dimension and signature, including multidimensional Kaluza-Klein-type models with a chain of internal spaces each with a scale factor of its own, such as considered in, e.g., [78,139,,165,166].

1.3. SPECIAL CASES

In the following examples, unless otherwise indicated, we will adhere to the conventional interpretation of the Birkhoff theorem, i.e., assume that v is time ($\eta_v = 1$ and u is a space variable ($\eta_u = -1$). Everything may be easily reformulated for coinciding η_u and η_v. No assumptions on the signatures of V_i are made since they do not affect the conclusions.

In general, the following matter sources satisfy the requirements C of Theorem 1 with no further restrictions on the structure of V^D:

(a) Linear or nonlinear, minimally coupled scalar fields with the Lagrangian $L = \varphi^{,A}\varphi_{,A} - V(\varphi)$ where $V(\varphi)$ is an arbitrary function, under the restriction $\varphi = \varphi(u)$:

$$2T^A_B = \delta^A_B V(\varphi) + \eta_u \mathrm{e}^{-2\alpha}\varphi'^2 \mathrm{diag}\ (1,\ -1,\ \ldots,\ -1); \tag{1.3.1}$$

here and henceforth positions in "diag " are ordered by the scheme $(u,\ v,\ \ldots)$.

(b) A massless, minimally coupled scalar field ($L = \varphi^{,A}\varphi_{,A}$) under the restriction $\varphi = \varphi(v)$: the EMT does not contain φ but only $\dot\varphi = \mathrm{const}$ (the so-called cosmological scalar field):

$$2T^A_B = \eta_v \mathrm{e}^{-2\gamma} \to \dot\varphi^2 \mathrm{diag}\ (-1,\ 1,\ -1,\ \ldots,\ -1). \tag{1.3.2}$$

(c) Abelian gauge fields ($L = -F^{AB}F_{AB},\ F_{AB} = \partial_A U_B - \partial_B U_A$) under the restriction that the vector potential U_A has a single nonzero component $U_K(u)$, with a fixed coordinate $K \neq v$, so that among F_{AB} only $F_{uK} = -F_{Ku} \neq 0$:

$$T^u_u = T^K_K = -F^{uK}F_{uK}; \quad \text{other}\ \ T^A_B = \delta^A_B F^{uK}F_{uK}. \tag{1.3.3}$$

(d) Nonlinear vector fields with Lagrangians of the form $\Phi(I),\ I = F^{AB}F_{AB}$, where Φ is an arbitrary function, under the same restriction as that in item (c) but with $K = v$ (an example is the Born-Infeld nonlinear electromagnetic field):

$$T^u_u = T^v_v = 2(d\Phi/dI)F^{uv}F_{uv} - \Phi/2; \quad \text{other}\ \ T^A_B = -\delta^A_B \Phi/2. \tag{1.3.4}$$

(e) Some kinds of interacting fields: for instance, the system of an Abelian gauge field and a scalar dilaton field ($L = \varphi^{,A}\varphi_{,A} - \mathrm{e}^{2\lambda\varphi}F^{AB}F_{AB},\ \lambda = \mathrm{const}$) under the constraints of items (a) and (c): the EMT structure combines those of (1.3.1) and (1.3.3). As (1.3.1) is v- and γ-independent, evidently the second condition C of Theorem 1 is satisfied by one of the two combinations $T^u_u \pm T^v_v$. This is just the interaction relevant for multidimensional dilatonic black holes [78,166,169].

The same is true if the expression $\mathrm{e}^{2\lambda\varphi}$ in the Lagrangian is replaced by any function of φ.

(f) The cosmological term $\Lambda\delta^A_B$ may be added to the left-hand side of (1.1.1) with no consequences.

One can easily find other forms of matter, as well as combinations of the above forms of matter and other ones, for which Theorem 1 holds.

As for the diversity of space-time structures to which the theorem applies, it is also very wide. In the 4-dimensional case it includes the symmetries mentioned in [162-163], namely: spherical, planar, pseudospherical, pseudoplanar, cylindrical, toroidal ($V^D = M^2\times S^2,\ \ M^2\times R^2,\ \ M^2\times L^2,\ \ M^2\times R^1\times R^1,\ \ M^2\times R^1\times S^1,\ \ M^2\times S^1\times S^1$, respectively, where L^2 is the Lobachevsky plane). It applies to both conventional (Lorentzian) GR and its "Euclidean" counterpart, as well as to Kaluza-Klein type models with a chain of internal spaces with u-dependent scale factors. Moreover, multidimensional extensions may incorporate generalized spherical and other symmetries in the spirit of Tangherlini [167], i.e., $S^m,\ R^m$ or L^m with an arbitrary $m>2$ instead of $S^2,\ R^2,\ L^2$.

For space-times with horizons, such as the black-hole ones, the theorem states the metric independence on different coordinates in different domains of M^2: thus, in the conventional Schwarzschild case it fixes the t-independence (staticity) in the R domain and r-independence (homogeneity) in the T domain. The same applies to multidimensional black holes considered in many papers (e.g., [21,78,165-169]).

Another point of interest is the existence of Abelian gauge fields of various directions which satisfy the theorem, see the above item (c). One may recall such evident examples as Coulomb-like fields for spherical and other similar symmetries, radial, longitudinal and azimuthal electric fields for conventional cylindrically symemtric space-times (and their magnetic counterparts); however, there are configurations with u-dependent vector potential components directed in extra dimensions, deserving a separate treatment.

1.4. COMMENTS

1.4.1. Condition A of Theorem 1 may be weakened if M^2 is a proper Riemannian space ($\eta_u=\eta_v$): instead of (1.2.9), it is sufficient to assume just $\rho\neq$ const. Indeed, in this case the orthogonal coordinates u and v may be always chosen so that $\rho=\rho(u)$, for instance, one may put just $u=\rho$.

If M^2 is pseudo-Riemannian ($\eta_u=-\eta_v$), then the gradient of $\rho(u,v)$ may be either u-like, or v-like, or null (sign $(\rho^{,A}\rho_{,A})=\eta_u,\ \eta_v,\ 0$, respectively). To extend the theorem to the case when it is v-like one may just change the notations of the coordinates, $u\leftrightarrow v$, irrespective of which of them is spacelike. So in both cases one can achieve $\rho=\rho(u)$.

1.4.2. Cancellation of Condition B of Theorem 1 (possible if $n>1$) leads to the existence of at least two functionally independent unknowns. The situation is most obviously exemplified by the Einstein-Rosen vacuum cylindrical gravitational waves [164] ($D=4,\ N_1=N_2=1,\ v=t$, i.e., time).

More generally, extra-dimension scale factors behave like minimally coupled scalar fields in 4 dimensions, so possible monopole waves may be eliminated just at the expense of the additional assumption B. When the latter is removed, these waves may manifest the instability of static configurations (as is the case for many non-black-hole spherically symmetric multidimensional space-times [21,78,166].

1.4.3. The possibility $\rho=$ const, excluded in the theorem, looks somewhat exotic in the spherically symmetric case but is quite natural for, say, planar symmetry. Let us show that it leads to wave solutions to Eqs. (1.1.1) taking as an example a 4-dimensional vacuum space-time with a cosmological constant ($T_B^A = \delta_B^A \Lambda$), possessing spherical or planar symmetry ($D = 4,\ n = 1,\ N_1 = 2;\ \overline{R}_2^2 = \overline{R}_3^3 = \epsilon = +1,\ 0$, respectively). In this case $\rho = 2\beta_1$ and Eqs. (1.1.1) give:

$$\epsilon \mathrm{e}^{-\rho} = \Lambda; \quad \mathrm{e}^{-2\alpha}(\eta_u \alpha'' + \eta_v \ddot{\alpha}) = \Lambda \tag{1.4.1}$$

where the coordinates are chosen so that the metric of M^2 is conformally flat ($\alpha = \gamma$). There are the following variants:

- $\epsilon = 1,\ \eta_u = -\eta_v;\ \Lambda > 0$. The space-time is formed by a congruence of spheres of equal radii and the only nontrivial metric coefficient α obeys a nonlinear wave equation.
- $\epsilon = 1,\ \eta_u = \eta_v;\ \Lambda > 0$. The same but the equation is nonlinear, elliptic type.
- $\epsilon = 0,\ \eta_u = -\eta_v;\ \Lambda = 0$. Linear waves in a planarly symmetric space-time: $\alpha = \gamma = f_1(u+v) + f_2(u-v)$.
- $\epsilon = 0,\ \eta_u = \eta_v;\ \Lambda = 0$. The only nontrivial metric coefficient $\alpha = \gamma$ is a harmonic function of u and v.

1.4.4. Another possibility rejected in Theorem 1 is that $\rho(u, v)$ has a null gradient in a pseudo-Riemannian M^2. The condition $\rho^{,A}\rho_{,A} = 0$ in the coordinates such that $\alpha = \gamma$ leads to $\dot{\rho} = \pm\rho'$. Let us choose the plus sign (re-defining $u \to -u$ if required), so that $\rho = \rho(\xi),\ \xi = u + v$.

Consider for instance vacuum planarly symmetric space-times of any dimension D, so that

$$V^D = M^2 \times R^{D-2}; \quad \rho = (D-2)\beta; \quad T_B^A = 0. \tag{1.4.2}$$

Substituting $\rho = \rho(\xi)$ and $\alpha = \gamma$ for (1.4.2) to Eqs.(1.1.1), we obtain:

$$\alpha = \alpha_1(\xi) + \alpha_2(\eta); \quad 2\alpha_1(\xi) = \ln|\beta'| + \beta, \quad \beta = \beta(\xi) \tag{1.4.3}$$

where $\eta = u - v$; $\beta(\xi)$ and $\alpha_2(\eta)$ are arbitrary functions. This is a planarly symmetric vacuum wave solution to Eqs. (1.1.1) for any dimension D.

The examples of items 1.4.3 and 1.4.4 show that in Theorem 1, establishing the sufficient conditions for staticity, no condition may be omitted or essentially weakened.

1.4.5. A case of interest is the one when some β_i are only u-dependent while others linearly depend on v, so that $\dot{\beta}_i =$ const. If still $\dot{\rho} = 0$ and the spaces V_i corresponding to $\dot{\beta}_i \neq 0$ are Ricci-flat, then the proof of Theorem 1 may be properly modified to conclude that all the remaining metric coefficients are v-independent.

Imagine, e.g., a 4-dimensional, static, spherically symmetric space-time ($u = r$, radial coordinate; $v = t$, time; $V_1 = S^2$, a 2-dimensional sphere) accompanied by internal Ricci-flat spaces of which some are static (in general, r-dependent), others exponentially expanding ($\dot{\beta}_i =$ const > 0) and still others exponentially contracting ($\dot{\beta}_i =$ const < 0). The specific forms of all functions are to be found from Eqs. (1.1.1) with a relevant choice of matter. However, as $\dot{\rho} = 0$, this class of solutions cannot contain those in which all extra dimensions are contracting. (This certainly does not mean that such solutions cannot exist at all: they are just not covered by the present treatment.)

1.4.6. The consideration of Section 1.2 rests on the geometric structure of the space V^D, including certain symmetry requirements. However, the theorem contains no requirements to internal symmetries of the subspaces V_i since no constraints on the dependence of $\overline{R}^{m_i}_{n_i}$ on the internal coordinates y^{n_i} have appeared. By Eqs. (1.1.1) this dependence is just the same as that in the EMT.

However, in most applications, as seen from the examples of Section 1.3, V_i are either Ricci-flat, or constant curvature spaces: as the EMT is independent of the internal coordinates, the same is true for V_i. Moreover, when all the diagonal components $T^{m_i}_{m_i}$ are equal to each other, Eqs.(1.1.1) force the components $\overline{R}^{m_i}_{m_i}$ to be equal as well, while the off-diagonal components are zero. Thus it is the EMT symmetry that forces V_i to be constant curvature spaces.

1.4.7. In [162-163] the generalized Birkhoff theorem was extended to a broad class of scalar-tensor theories of gravity in 4 dimensions. The same can be done in the multidimensional case. To do that let us consider in V^D with the metric g_{MN} a scalar-tensor theory described by the Lagrangian

$$\sqrt{g}L = \sqrt{g}[A(\phi)R + B(\phi)\phi^{,M}\phi_{,M} - 2\Lambda(\phi) + L_m] \tag{1.4.4}$$

where $g =\mid \det g_{MN} \mid$; $A > 0$, B and Λ are any smooth functions of ϕ and the matter Lagrangian L_m may depend on both g_{MN} and ϕ. The conformal mapping (suggested by Wagoner [154] for $D = 4$)

$$g_{MN} = A^{-2/(D-2)}\overline{g}_{MN} \tag{1.4.5}$$

brings (1.4.4) to the form (up to a divergence)

$$\sqrt{g}L = \sqrt{\overline{g}}\Big\{\overline{R}+\frac{1}{A^2}\Big[AB+\frac{D-1}{D-2}\left(\frac{dA}{d\phi}\right)^2\Big]\overline{g}^{MN}\phi_{,M}\phi_{,N}+A^{-D/(D-2)}[-2\Lambda(\phi)+L_m]\Big\} \tag{1.4.6}$$

where $\overline{g} =\mid \det \overline{g}_{MN} \mid$ and $\overline{R}$ is the scalar curvature corresponding to $\overline{g}_{MN}$. Variation of (4.6) with respect to $\overline{g}_{MN}$ yields the Einstein equations with an EMT containing the contribution of the (possibly nonlinear) scalar field ϕ and that of matter coupled to ϕ. For our purpose it is essential that the latter contribution $\overline{T}_{MN}$ coincides with the original EMT ($T_{MN} = (\delta/\delta g^{MN})(\sqrt{g}L_m)$ up to a ϕ-dependent factor. Consequently, if $\phi = \phi(u)$ and T_{MN} satisfies Condition C of Theorem 1, so does $\overline{T}_{MN}$ and Theorem 1 is applicable to the metric $\overline{g}_{MN}$. However, now it is $\overline{\rho} = \rho + \ln A$ that appears instead of ρ in the formulation of the theorem. Therefore Theorem 1 cannot be directly applied to g_{MN} and its formulation should be properly modified:

Theorem 2. *Consider a field system with the Lagrangian (1.4.4) in a Riemannian space V^D (1.2.1) with metric (1.2.2). Let there be a domain Δ in M^2 where*

(i) *all β_i and the field ϕ are functions of u;*

(ii) $\overline{\rho} = \rho + \ln A \neq \mathrm{const}$ *and*

(iii) *Conditions C of Theorem 1 are valid for the EMT derived from L_m.*

Then the coordinate v in Δ may be chosen so that all g_{MN} are v-independent.

1.4.8. It would be of interest to try to extend the theorem to multidimensional models with nonzero off-diagonal metric components such as g_{ui} with i from extra dimensions, as is the case in the original Kaluza-Klein model. This goes beyond the scope of this paper, although probably such a generalization does exist since the new effective vector fields are unlikely to create monopole waves.

2. Multitemporal generalization of Schwarzschild solution [170]

2.1. Introduction

In [171] the generalization of the Schwarzschild solution to the case of n internal Ricci-flat spaces was obtained.(The case $n = 1$ was considered earlier in [172].) In [36] this solution was generalized on $O(d+1)$-symmetrical (Tangherlini-like) case. (In [173] the special case of the solution [3] with $n = 2$ was considered).

This section is devoted to an interesting special case of the solution [171]. This is the n-time generalization of the Schwarzschild solution. We note, that the idea of considering of space-time manifolds with extra time dimensions was discussed earlier by different authors (see, for example, [174-181). Some revival of the interest in this direction was inspired recently by string models [178-181].

In sec. 2.2 the multitemporal generalization of the Schwarzschild formula is considered and corresponding geodesic equations are integrated. In sec. 2.3 the motion of the relativistic particle in the background of the solution is investigated and a multitemporal analogue of the Newton's formula is obtained. The sec. 2.4 is devoted to multitemporal generalization of Newton's mechanics and Newton's gravitational law for interacting objects described by the solution ("multitemporal hedgehogs").

2.2. The metric and geodesic equations

The metric generalizing the Schwarzschild solution to the multitemporal case reads

$$g = -\sum_{i=1}^{n} f^{a_i} dt^i \otimes dt^i + f^{-b} dR \otimes dR + f^{1-b} R^2 d\Omega^2, \tag{2.2.1}$$

where $f = 1 - (L/R)$, $L = const$, $d\Omega^2$ is the standard metric on 2-dimensional sphere and the parameters b, a_i satisfy the relations

$$b = \sum_{i=1}^{n} a_i, \qquad b^2 + \sum_{i=1}^{n} a_i^2 = 2. \tag{2.2.2}$$

The metric (2.1) satisfies the Einstein equations (or equivalently $R_{MN}[g] = 0$) and may be obtained as a special case of the solution [171] or more general solution [36].

The metrics $g(a, L)$ and $g(-a, -L)$ are equivalent for any set $a = (a_1, \ldots, a_n)$, satisfying (2.2.2). This may be easily verified using the following transformation of the radial variable: $R = R_* + L$. So, without loss of generality we restrict our consideration by the case $L > 0$ (the case $L = 0$ is trivial).

In the case

$$a_i = \delta_{ik}, \tag{2.2.3}$$

$k \in \{1, ..., n\}$, the metric (2.2.1) has the following form

$$g = g_{Sch}^{(k)} - \sum_{i \neq k} dt^i \otimes dt^i, \tag{2.2.4}$$

i.e. it is a trivial (cylindrical) extension of the Schwarzschild solution with the time t^k. It describes an extended (in times) membrane-like object. Any section of this object by hypersurface $t^i = t^i_0 = const$, $i \neq k$, is the 4-dimensional black hole, "living" in the time t^k. It may be proved that the solution (2.2.1) has a singularity at $R = L$ for all sets of parameters $(a_1, \ldots, a_n)$ except n Schwarzschild-like points (2.2.3) (for $n = 2$ this was proved in [182]).

We consider the geodesic equations for the metric (2.2.1)

$$\ddot{x}^M + \Gamma^M_{NP}[g]\dot{x}^N \dot{x}^P = 0, \tag{2.2.5}$$

where $x^M = x^M(\tau)$, $\dot{x}^M = dx^M/d\tau$ and τ is some parameter on a curve.

These equations are nothing more than the Lagrange equations for the Lagrangian

$$\begin{aligned} L_1 &= \frac{1}{2} g_{MN}(x)\dot{x}^M \dot{x}^N \\ &= \frac{1}{2}[f^{-b}(\dot{R})^2 + f^{1-b}R^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) - \sum_{i=1}^{n} f^{a_i}(\dot{t}^i)^2]. \end{aligned} \tag{2.2.6}$$

The complete set of integrals of motion for the Lagrange system (2.2.6) is following

$$f^{a_i}\dot{t}^i = \varepsilon^i, \tag{2.2.7}$$

$$f^{1-b}R^2\dot{\varphi} = j, \tag{2.2.8}$$

$$f^{-b}\dot{R}^2 + f^{1-b}R^2\dot{\varphi}^2 - \sum_{i=1}^{n} f^{a_i}(\dot{t}^i)^2 = 2E = 2L_1, \tag{2.2.9}$$

$i = 1, ..., n$. Without loss of generality we put here $\theta = \frac{\pi}{2}$.

Multitemporal horizon. Here we consider the null geodesics. Putting $E = 0$ in (2.2.9) we get for a light "moving" to the center

$$\dot{R} = -\sqrt{\sum_{i=1}^{n}(\varepsilon^i)^2 f^{b-a_i} - j^2 f^{-1+2b}R^{-2}} \tag{2.2.10}$$

and consequently

$$t^i - t^i_0 = -\int_{R_0}^{R} dx \frac{\varepsilon^i[f(x)]^{-a_i}}{\sqrt{\sum_{i=1}^{n}(\varepsilon^i)^2[f(x)]^{b-a_i} - j^2[f(x)]^{-1+2b}x^{-2}}}, \tag{2.2.11}$$

$i = 1, \ldots, n$.

Let $L > 0$, $\varepsilon = (\varepsilon^i) \neq 0$ and $a = (a_1, \ldots, a_n)$ satisfies (2.2.2). We say that the ε-horizon takes place for the metric (2.2.1) at $R = L$ if and only if

$$||t - t_0|| \equiv \sum_{i=1}^{n} |t^i - t^i_0| \to +\infty, \tag{2.2.12}$$

as $R \to L$ for all t_0 and j. It may be proved [182] that for $L > 0$ and for non-Schwarzschild set a the ε-horizon for the metric (2.2.1) at $R = L$ is absent for any $\varepsilon \neq 0$. For the Schwarzschild set of parameters (2.2.3) the ε-horizon takes place if $\varepsilon^k \neq 0$, i.e. light should "move" in t^k-direction.

2.3. Relativistic particle

Let us consider the motion of the relativistic particle in the gravitational field, corresponding to the metric (2.2.1). The Lagrangian of the particle is

$$L_2 = -m\sqrt{-g_{MN}(x)\dot{x}^M \dot{x}^N}, \tag{2.3.1}$$

where m is the mass of the particle. The Lagrange equations for (2.3.1) in the proper time gauge

$$g_{MN}(x)\dot{x}^M \dot{x}^N = -1 \tag{2.3.2}$$

coincide with the geodesic equations (2.2.5). In this case $(E^i) = (m\varepsilon^i)$ is the energy vector and $J = mj$ is the angular momentum. For fixed values of ε^i the 3-dimensional part of the equations of motion is generated by the Lagrangian

$$L_* = \frac{m}{2}[f^{1-b}\bar{g}_{Sch,\alpha\beta}(x)\dot{x}^\alpha \dot{x}^\beta + \sum_{i=1}^{n}(\varepsilon^i)^2 f^{-a_i}]. \tag{2.3.3}$$

where $\bar{g}_{Sch}$ is the space section of the Schwarzschild metric.

Now, we restrict our consideration by the non-relativistic motion at large distances: $R\overline{g}L$. In this approximation: $t^i = \varepsilon^i \tau$, $\quad \sum_{i=1}^{n}(\varepsilon^i)^2 = 1$. It follows from (2.3.3) that in the considered approximation we get a non-relativistic particle of mass m, moving in the potential

$$V = -\frac{m}{2}\sum_{i=1}^{n}(\varepsilon^i)^2\frac{a_i L}{R} = -G\frac{m(\varepsilon^i M_{ij}\varepsilon^j)}{R}, \tag{2.3.4}$$

where G is the gravitational constant and

$$M_{ij} = a_i\delta_{ij}L/2G, \tag{2.3.5}$$

are the components of the gravitational mass matrix.

We note, that the relation (2.3.4) may be rewritten as following

$$V = -G\frac{tr(MM_I)}{R} \tag{2.3.6}$$

where $M_I = (m\varepsilon_i\varepsilon_j)$ is the inertial mass matrix of the particle. For $n = 1$ the potential (2.3.6) coincides with the Newton's one.

Matrix form. The solution (2.2.1) may be also rewritten in the matrix form

$$\begin{aligned} g = \quad & -[(1-L/R)^A]_{ij} d\bar{t}^i \otimes d\bar{t}^j \\ & +(1-L/R)^{-trA} dR \otimes dR + (1-L/R)^{1-trA} R^2 d\Omega^2, \end{aligned} \tag{2.3.7}$$

where A is a real symmetrical $n \times n$-matrix satisfying the relation

$$(trA)^2 + tr(A^2) = 2. \tag{2.3.8}$$

Here $x^A \equiv \exp(A \ln x)$ for $x > 0$. The metric (2.3.7) can be reduced to the metric (2.2.1) by the diagonalization of the A-matrix: $A = S^T(a_i\delta_{ij})S$, $S^T S = 1_n$ and the reparametrization of the time variables: $S_i^j \bar{t}^i = t^j$. In this case the gravitational mass matrix is

$$(M_{ij}) = (A_{ij} L/2G). \tag{2.3.9}$$

We may also define the gravitational mass tensor as

$$\mathcal{M} = M_{ij} d\bar{t}^i \otimes d\bar{t}^j. \tag{2.3.10}$$

We call the extended (in time) object, corresponding to the solution (2.3.7)-(2.3.8) as multitemporal Schwarzschild hedgehog. At large distances $R \overline{g} L$ this object is described by the matrix analogue of the Newton's potential

$$\Phi_{ij} = -\frac{1}{2} L A_{ij}/R = -G M_{ij}/R. \tag{2.3.11}$$

Clearly, that this potential for the diagonal case (2.2.1) $A = a_i \delta_{ij}$ is a superposition of the potentials, corresponding to "pure states": Schwarzschild-like membranes (2.2.4). So, in the post-Newtonian approximation the Schwarzschild hedgehog is equivalent to the superposition of black hole membranes (2.2.4), corresponding to different times.

2.4. Multitemporal Newton laws

The solution (2.3.7), (2.3.8) may be also rewritten as following

$$\begin{aligned} g = \quad & -[(1 - ||L||/R)^{L/||L||}]_{ij} dt^i \otimes dt^j \\ & +(1 - ||L||/R)^{-trL/||L||} dR \otimes dR + (1 - ||L||/R)^{1-(trL/||L||)} R^2 d\Omega^2, \end{aligned} \tag{2.4.1}$$

where here $L = (L_{ij}) \neq 0$ is real symmetrical $n \times n$-matrix with the norm

$$||L|| \equiv \sqrt{\frac{1}{2}(trL)^2 + \frac{1}{2} tr(L^2)}. \tag{2.4.2}$$

We call matrix L as gravitational length matrix.

Now we consider the interaction between two multitemporal hedgehogs with gravitational length matrices $L_1 = (L_{1,ij})$ and $L_2 = (L_{2,ij})$ located at large distances from each other

$$|\vec{x}| \overline{g} ||L||_1, ||L||_2, \qquad \vec{x} \equiv \vec{x}_1 - \vec{x}_2. \tag{2.4.3}$$

We begin with the simplest case $n = 1$. In Newton's mechanics the equations of motion for two point-like masses $M_1 = L_1/2G$ and $M_2 = L_2/2G$ with world lines $\vec{x}_1 = \vec{x}_1(t)$ and $\vec{x}_2 = \vec{x}_2(t)$ respectively are well-known:

$$\frac{d^2 \vec{x}_1}{dt^2} = -L_2 \frac{\vec{x}}{2|\vec{x}|^3}, \tag{2.4.4}$$

$$\frac{d^2 \vec{x}_2}{dt^2} = L_1 \frac{\vec{x}}{2|\vec{x}|^3}, \tag{2.4.5}$$

where $\vec{x}$ is defined in (2.4.3). The equations (2.4.4), (2.4.5) may be obtained from the Einstein equations, when the solutions describing the post-Newtonian (2.4.3), non-relativistic motion

$$|\frac{d\vec{x}_a}{dt}| \ll 1, \tag{2.4.6}$$

$a = 1, 2$, of two black holes are considered.

Our hypothesis is that the generalization of this scheme to the multitemporal case should lead to the following equations of motion for two non-relativistic hedgehogs with gravitational length matrices L_1 and L_2 in the post-Newtonian approximation (2.4.3)

$$\frac{d^2\vec{x}_1}{dt^i dt^j} = -L_{2,ij}\frac{\vec{x}}{2|\vec{x}|^3}, \tag{2.4.7}$$

$$\frac{d^2\vec{x}_2}{dt^i dt^j} = L_{1,ij}\frac{\vec{x}}{2|\vec{x}|^3}. \tag{2.4.8}$$

The functions $\vec{x}_a = \vec{x}_a(t_1, \ldots, t_n)$, $a = 1, 2$, describe the world surfaces of two multitemporal objects in the considered approximation. The multitemporal analogue of the non-relativistic condition (2.4.6) reads

$$|\frac{d\vec{x}_a}{dt^i}| \ll 1, \tag{2.4.9}$$

$a = 1, 2$, $i = 1, \ldots, n$. Defining gravitational mass matrices

$$(M_{a,ij}) = (L_{a,ij}/2G), \tag{2.4.10}$$

and forces

$$\vec{F}_{a,ij} = M_{a,ij}\frac{d^2\vec{x}_a}{dt^i dt^j}, \tag{2.4.11}$$

$a = 1, 2$, we get

$$\vec{F}_{1,ij} = -GM_{1,ij}M_{2,ij}\frac{\vec{x}}{|\vec{x}|^3}, \tag{2.4.12}$$

$$\vec{F}_{1,ij} = -\vec{F}_{2,ij}, \tag{2.4.13}$$

$i, j = 1, \ldots, n$. Relations (2.4.11), (2.4.12) and (2.4.13) are multitemporal analogues of the Newton's laws , describing the multitemporal "motion" of two interacting non-relativistic hedgehogs in the post-Newtonian approximation. (The generalization to multi-hedgehog case is quite transparent.) We note, that for $\vec{F}_1 = tr(\vec{F}_{1,ij})$ we get the formula suggested previously in [183]

$$\vec{F}_1 = -Gtr(M_1M_2)\frac{\vec{x}}{|\vec{x}|^3}. \tag{2.4.14}$$

Scalar-vacuum generalization. The solution (2.2.1) can be easily generalized on a scalar-vacuum case. In this case the field equations corresponding to the action

$$S = \frac{1}{2}\int d^D x\sqrt{|g|}(R[g] - \partial_M\varphi\partial_N\varphi g^{MN}), \tag{2.4.15}$$

are satisfied for the metric (2.2.1) and the scalar field

$$\varphi = \frac{1}{2} q \ln(1 - \frac{L}{R}) + const, \tag{2.4.16}$$

with the parameters related as following

$$b = \sum_{i=1}^{n} a_i, \qquad b^2 + \sum_{i=1}^{n} a_i^2 + q^2 = 2. \tag{2.4.17}$$

This solution is a special case of the solution [184] or more general dilatonic-electro-vacuum solution [182,185-6].

Conclusion

In this paper we considered the multitemporal generalization of the Schwarzschild solution. We integrated the equations of geodesics for the metric and considered the motion of relativistic particle in the background , corresponding to the metric. We obtained the modification of Newton's law for interaction of massive non-relativistic particle with multitemporal hedgehog (i.e extended in time object, described by the solution). We also suggested multitemporal analogues of Newton's formulas for non-relativistic motion of interacting hedgehogs. We note, that the main difference of the multitemporal (n-time) case from the ordinary $n = 1$ case is following: in the space-time with n time coordinates the gravitational and inertial masses are $n \times n$ matrices, and the energy of a relativistic particle is the n-component vector.

3. Multitemporal generalization of the Tangherlini solution [182]

3.1. Introduction

In ref. [36] the Tangherlini solution [1,2] ($O(d+1)$-symmetric analogue of the Schwarzschild solution) was generalized on the case of $\bar{n}$ internal Ricci-flat spaces. The metric of this solution is defined on the manifold

$$M = M^{(2+d)} \times M_1 \times \ldots \times M_{\bar{n}}, \tag{3.1.1}$$

and has the following form

$$\begin{aligned} g = \quad & -f^a dt \otimes dt + f^{b-1} dR \otimes dR \\ & + f^b R^2 d\Omega_d^2 + \sum_{i=1}^{\bar{n}} f^{a_i} g^{(i)}, \end{aligned} \tag{3.1.2}$$

where $M^{(2+d)}$ is $(2+d)$-dimensional space-time ($d \geq 2$), $(M_i, g^{(i)})$ is Ricci-flat manifold ($g^{(i)}$ is metric on M_i), $dim M_i = N_i$, $i = 1, \ldots, \bar{n}$; $d\Omega_d^2$ is canonical metric on d-dimensional sphere S^d,

$$f = f(R) = 1 - BR^{1-d}, \tag{3.1.3}$$

$$b = (1 - a - \sum_{i=1}^{\bar{n}} a_i N_i)/(d-1), \tag{3.1.4}$$

$B = const$, and the parameters $a, a_1, \ldots, a_{\bar{n}}$ satisfy the relation

$$(a + \sum_{i=1}^{\bar{n}} a_i N_i)^2 + (d-1)(a^2 + \sum_{i=1}^{\bar{n}} a_i^2 N_i) = d. \tag{3.1.5}$$

(Here the notations slightly differ from those of ref. [36]). The metric (3.1.2) with the relations (3.1.3)-(3.1.5) imposed satisfies the Einstein equations or, equivalently,

$$R_{MN}[g] = 0. \tag{3.1.6}$$

Some special cases of the solution (3.1.2)-(3.1.5) were considered earlier in the following publications: [172,188,189] ($d = 2; \bar{n} = 1; N_1 = 1$), [125,190] ($d = 2; \bar{n} = 2, 3; N_1 = \ldots = N_{\bar{n}} = 1$), [172] ($d = 2; \bar{n} = 1$), [191] ($\bar{n} = 1$; $a = [(1 - (d + N_1)^{-1})/(1 - d^{-1})]^{1/2}$; $a_1 = -a/(d + N_1 - 1)$); [173] ($\bar{n} = 2$, $N_2 = 1$), [171] ($d = 2$; $\bar{n}$ is arbitrary).

It was shown in [36] that in the $(2+d)$-dimensional section of the metric (3.1.2) a horizon exists only in the trivial case: $a - 1 = a_1 = \ldots = a_n$ (this proposition was also suggested in [171]). We also note that the cosmological analogue of the solution [36] was presented in [19], where the tree-generalizations of the solution were considered. (Such tree-generalizations may be also constructed for spherically-symmetric case [36]).

In this section we consider an interesting special case of the solution [36]. This is the n-time generalization of the Tangherlini solution. We note that the idea of the existence of multidimensional domains with several times in the (multidimensional) Universe was suggested by Sakharov in [193].

The section is organized as following. In Sec. 3.2 the metric of the multitemporal solution is considered. The explicit expression for the Riemann tensor squared, corresponding to this solution is presented. The proposition concerning the singularity of the solution at $R = L > 0$ (L is the parameter) for any set of dimensionless parameters (a_i) except n "tangherlinian" sets is suggested. This proposition is proved for $n = 2$. In Sec. 3.3 the equations of the geodesics, corresponding to the solution are integrated. The notion of the multitemporal horizon is introduced. It is proved that for all non-exceptional sets of a_i-parameters the multitemporal horizon is absent. In Sec. 3.4 the motion of the relativistic particle is considered. The multitemporal $O(d+1)$- symmetric analogue of the Newton's formula for this case is obtained. In multitemporal case the inertial and gravitational masses are defined as matrices (or it may be defined also as tensors). In Sec. 3.5 the vacuum solution [36] is generalized on the electro-scalar- vacuum case for the model with exponential scalar-electro-magnetic coupling. Some infinite-dimensional generalizations of the solution (including infinite-temporal and Grassmann-Banach analogues) are presented.

3.2. The metric

We consider the special case of the solution (3.1.1)-(3.1.5) with $\bar{n} = n-1$ one-dimensional internal spaces: $M_i = R$, $g^{(i)} = -dt^i \otimes dt^i$, $i = 1, \ldots, n-1$. Denoting $t = t^n$ and $a = a_n$ we get from (3.1.2)

$$\begin{aligned} g = \quad & -\sum_{i=1}^{n}(1 - BR^{1-d})^{a_i} dt^i \otimes dt^i \\ & +(1 - BR^{1-d})^{b-1} dR \otimes dR + (1 - BR^{1-d})^b R^2 d\Omega_d^2, \end{aligned} \tag{3.2.1}$$

where

$$b = (1 - \sum_{i=1}^{n} a_i)/(d-1) \tag{3.2.2}$$

and the parameters $a_1, \ldots, a_n$ satisfy the relations

$$(\sum_{i=1}^{n} a_i)^2 + (d-1)\sum_{i=1}^{n} a_i^2 = d. \tag{3.2.3}$$

The metric (3.2.1) with the parameters satisfying (3.2.2) and (3.2.3) is the solution of the (n+d+1)-dimensional Einstein equations (or, equivalently, of the Ricci-flatness eqs. (3.1.6)).

The solution (3.2.1)-(3.2.3) is multitemporal (n-time) generalization of the Tangherlini solution [167].

Let $E(d,n)$ be the set of points $a = (a_1, \ldots, a_n) \in R^n$, satisfying the relation (3.2.3). Clearly, that $E(d,n)$ is ellipsoid. We denote the solution (3.2.1), corresponding to $a \in E(d,n)$, $B \in R$ by $g = g(a,B)$. An interesting fact is that the metrics $g(a,B)$ and $g(-a,-B)$ are equivalent (for any $a \in E(d,n)$, $B \in R$), i.e.

$$g(-a,-B) = \varphi^* g(a,B) \tag{3.2.4}$$

for some diffeomorphism φ. This diffeomorphism $\varphi = \varphi_B$ is defined by the relations

$$\varphi : (t^i, R_*, \theta^\alpha) \mapsto (t^i, R, \theta^\alpha), \qquad R^{d-1} = R_*^{d-1} + B. \tag{3.2.5}$$

Remark 1. An analogous equivalence takes place for the metric (3.1.2).

Due to relation (3.2.4) it is quite sufficient to restrict our consideration by the case $B > 0$ (the case $B = 0$ is trivial).

We introduce the following notations

$$T_1 \equiv (1, 0, \ldots, 0), \ldots, T_n \equiv (0, \ldots, 0, 1), \tag{3.2.6}$$

$$T \equiv \{T_1, \ldots, T_n\} \subset E(d, n). \tag{3.2.7}$$

We call the points (3.2.6) as Tangherlini points and the set (3.2.7) as Tangherlini set. The metric (3.2.1) for $a = T_k$ has a rather simple form

$$g(T_k, B) = g_T^{(k)} - \sum_{i \neq k} dt^i \otimes dt^i, \tag{3.2.8}$$

where $g_T^{(k)}$ is the Tangherlini solution with the time variable $t = t^k$, $k = 1, ..., n$. The metric (3.2.8) is a trivial (cylindrical) extension of the Tangherlini solution with the time t^k. It describes an extended membrane-like (string-like for $n = 2$) object. Any section of this object by hypersurface $t^i = t^i_0 = const$, $i \neq k$, is the $(2 + d)$-dimensional black hole [167,187], "living" in the time t^k.

Singularity. The Riemann tensor squared for the metric (3.2.1) has the following form

$$\begin{aligned}
I[g] \equiv \quad & R_{MNPQ} R^{MNPQ} = \bar{I}[g]/8 f^{2(b-1)}, && (3.2.9) \\
\bar{I}[g] = \quad & 16d(d-1)R^{-4} f^{-2} - 8d(d-1)R^{-2} f^{-1}(b\frac{f'}{f} + \frac{2}{R})^2 \\
& -d(b\frac{f'}{f} + \frac{2}{R})^4 + 2d[2b\frac{f''}{f} - b(\frac{f'}{f})^2 + \frac{2(b+1)}{R}\frac{f'}{f}]^2 \\
& + \sum_{i=1}^{n} \{-a_i^4 (\frac{f'}{f})^4 + 2a_i^2 [2\frac{f''}{f} + (a_i - 1 - b)(\frac{f'}{f})^2]^2\} \\
& + [d(b\frac{f'}{f} + \frac{2}{R})^2 + \sum_{i=1}^{n} a_i^2 (\frac{f'}{f})^2]^2, && (3.2.10)
\end{aligned}$$

where $f' \equiv df/dR$ and f is defined by (3.1.3). We denote $L = L_B \equiv B^{1/(d-1)}$ for $B > 0$. The relation (3.2.10) may be obtained from the formula presented in the Appendix.

Proposition 1. Let $B > 0$ and $a = (a_i) \in E(d, n) \setminus T$, i.e. the set of parameters a is non-tangherlinian. Then the quadratic invariant (3.2.9) for the metric (3.2.1) $g = g(a, B)$ is divergent: $I[g] \to \infty$, as $R \to L$.

Proof. Here we prove the proposition for the case $n = 2$. (The case $n > 2$ will be considered in a separate publication).

From eqs. (3.2.9)-(3.2.10) we get the following asymptotical formula (here n is arbitrary)

$$I[g] = \frac{A}{8}[f'(L)]^4 [f(R)]^{-2b-2}[1 + O(L - R)], \tag{3.2.11}$$

as $R \to L$, where

$$A = A(a) = \quad -db^4 + 2db^2 + (db^2 + \sum_{i=1}^{n} a_i^2)^2$$
$$+ \sum_{i=1}^{n} [-a_i^4 + 2a_i^2(a_i - 1 - b)^2]. \tag{3.2.12}$$

The formula (3.2.11) is valid, when $A \neq 0$. We note that

$$(1-r)/(d-1) \leq b \leq (1+r)/(d-1), \tag{3.2.13}$$

where $r \equiv \sqrt{dn/(d+n-1)}$ (see also Remark 2 below). It follows from (3.2.13) that

$$1 + b > 0 \tag{3.2.14}$$

and consequently (see (3.2.11)) $I[g] \to \infty$ as $R \to L$, when

$$A \neq 0 \tag{3.2.15}$$

Now, we prove the inequality (3.2.15) for $n = 2$ and $a \in E(d,n) \setminus T$. For $n = 2$ we have

$$A = \frac{1}{2} d(d+1) b^2 \bar{A}, \tag{3.2.16}$$

where

$$\bar{A} = -(d-1)(d+2)b^2 + 8b + 8. \tag{3.2.17}$$

Using inequalities (3.2.13) it is not difficult to verify that $\bar{A} = \bar{A}(b) > 0$ (see also Remark 3 below). On the other hand (in the case $n = 2$) $b = b(a) = 0$ only for the tangherlinian points $a = (0,1), (1,0)$. So, the inequality (3.2.15) takes place for all $a \in E(d,2) \setminus T$ ($n = 2$). The proposition 1 is proved for $n = 2$.

Remark 2. In the coordinates

$$\bar{a}_1 = (a_1 + \ldots + a_n)/\sqrt{n},$$
$$\bar{a}_2 = (a_1 - a_2)/\sqrt{2},$$
$$\bar{a}_3 = (a_1 + a_2 - 2a_3)/\sqrt{6}, \qquad (n > 2) \ldots$$
$$\bar{a}_n = (a_1 + \ldots + a_{n-1} - (n-1)a_n)/\sqrt{n(n-1)},$$

the ellipsoid equation (3.2.3) reads

$$(n + d - 1)\bar{a}_1^2 + (d-1) \sum_{i=2}^{n} \bar{a}_i^2 = d. \tag{3.2.18}$$

The inequalities (3.2.13) can be easily obtained from (3.2.18) and the relation

$$b = (1 - \sqrt{n}\bar{a}_1)/(d-1). \tag{3.2.19}$$

Remark 3. The inequality $\bar{A} > 0$ may be proved, using (3.2.13) and the following inequalities

$$1 + \sqrt{2d/(d+1)} < 5/2 < b_+(d-1),$$
$$1 - \sqrt{2d/(d+1)} > -1/2 > b_-(d-1),$$

where $b_\pm = [4 \pm \sqrt{8d(d+1)}]/(d-1)(d+2)$ are zeros of the quadratic polynomial $\bar{A}(b)$.

Thus for $n = 2$, $a \in E(d,2) \setminus T$, $B > 0$ the metric (3.2.1) $g = g(a, B)$ is singular at $R = L$.

In the case $a \in T$, $B > 0$ (n is arbitrary) the metric (3.2.1) $g = g(a, B)$ is regular for $R > 0$ and

$$I[g] = B^2 R^{-2-2d} d^2 (d^2 - 1). \tag{3.2.20}$$

Remark 4. In this case the metric has form (3.2.8). We remind that the regularity of the Tangherlini metric for $R > 0$ may be easily seen using the coordinates

$$\bar{t} = t + \int dx \varphi(x)(f(x))^{-1}, \qquad \bar{R} = R + \int dx(\varphi(x))^{-1}(f(x))^{-1}, \tag{3.2.21}$$

where $\varphi(x) = (L/x)^{(d-1)/2}$.

3.3. The geodesic equations

We consider the geodesic equations for the metric (3.2.1)

$$\ddot{x}^M + \Gamma^M_{NP}[g]\dot{x}^N \dot{x}^P = 0. \tag{3.3.1}$$

Here and below $x^M = x^M(\tau)$ and $\dot{x}^M = dx^M/d\tau$.

These equations are equivalent to the Lagrange equations for the Lagrangian

$$\begin{aligned} L &= \frac{1}{2} g_{MN}(x)\dot{x}^M \dot{x}^N \\ &= \frac{1}{2}[f^{b-1}(\dot{R})^2 + f^b R^2 \kappa_{ij}(\theta)\dot{\theta}^i\dot{\theta}^j - \sum_{i=1}^n f^{a_i}(\dot{t}^i)^2]. \end{aligned} \tag{3.3.2}$$

where the function $f = f(R)$ is defined in eq. (3.1.3) and

$$\kappa = d\theta^1 \otimes d\theta^1 + \sin^2\theta^1 d\theta^2 \otimes d\theta^2 + \ldots + \sin^2\theta^1 \ldots \sin^2\theta^{d-1} d\theta^d \otimes d\theta^d \tag{3.3.3}$$

is standard metric on S^d. Here $0 < \theta^1, \ldots, \theta^{d-1} < \pi$, $0 < \theta^d = \varphi < 2\pi$.

The complete set of integrals of motion for the Lagrange system (3.3.2) is following

$$f^{a_i}\dot{t}^i = \varepsilon^i, \tag{3.3.4}$$
$$f^b R^2 \dot{\varphi} = j, \tag{3.3.5}$$

$$f^{b-1}(\dot{R})^2 + j^2 f^{-b} R^{-2} - \sum_{i=1}^n (\varepsilon^i)^2 f^{-a_i} = 2E_L \tag{3.3.6}$$

$i = 1, ..., n$. We put here $\theta^1 = \ldots = \theta^{d-1} = \frac{\pi}{2}$. (This may be done for any trajectory by a suitable choice of coordinate system.) The radial equation

$$(f^{b-1}\dot{R})^{\cdot} + \frac{j^2}{2}(f^{-b}R^{-2})' - \frac{1}{2}(\dot{R})^2(f^{b-1})' - \frac{1}{2}\sum_{i=1}^{n}(\varepsilon^i)^2(f^{-a_i})' = 0 \tag{3.3.7}$$

(here $(.)' = d(.)/dR$) is generated by the Lagrangian

$$L_R = \frac{1}{2}[f^{b-1}(\dot{R})^2 - j^2 f^{-b}R^{-2} + \sum_{i=1}^{n}(\varepsilon^i)^2 f^{-a_i}]. \tag{3.3.8}$$

We note, that the case $2E_L = 2L > 0$ in (3.3.6) correspond to a tachion.

Multitemporal horizon. Here we consider the null geodesics. Putting $E_L = 0$ in (3.3.6) we get for a light "moving" to the center

$$\dot{R} = -\sqrt{\sum_{i=1}^{n}(\varepsilon^i)^2 f^{1-b-a_i} - j^2 f^{1-2b}R^{-2}} \tag{3.3.9}$$

and consequently

$$t^i - t_0^i = -\int_{R_0}^{R} dx \frac{\varepsilon^i[f(x)]^{-a_i}}{\sqrt{\sum_{i=1}^{n}(\varepsilon^i)^2[f(x)]^{1-b-a_i} - j^2[f(x)]^{1-2b}x^{-2}}}, \tag{3.3.10}$$

$i = 1, \ldots, n$.

Definition. Let $B > 0$, $\varepsilon = (\varepsilon^i) \neq 0$ and $a \in E = E(d,n)$. We say that the ε-horizon takes place for the metric $g(a,B)$ at $R = L \equiv B^{1/(d-1)}$ if and only if

$$||t - t_0|| \equiv \sum_{i=1}^{n}|t^i - t_0^i| \to +\infty, \tag{3.3.11}$$

as $R \to L$ for all t_0 and j.

Proposition 2. Let $B > 0$, and $a \in E = E(d,n) \setminus T$. Then the ε-horizon for the metric $g(a,B)$ at $R = L$ is absent for any $\varepsilon \neq 0$.

Proof. We put $j = 0$. It is sufficient to prove that all integrals in (3.3.10) are convergent, when $R \to L$. The integrals in (3.3.10) are convergent only if

$$s_i = -a_i - \frac{1}{2}min_\varepsilon(1 - b - a_i) > -1 \tag{3.3.12}$$

for all $i \in K_\varepsilon \equiv \{j|\varepsilon^j \neq 0\}$. Here

$$min_\varepsilon(u_i) \equiv min\{u_i|i \in K_\varepsilon\}. \tag{3.3.13}$$

Indeed, the integrand in the i-th integral in (3.3.10) behaves like $\varepsilon^i(L-x)^{s_i}$ as $x \to L$. The set of inequilities (3.3.12) may be rewritten as following

$$2a_i < max_\varepsilon(a_i) + 1 + b, \tag{3.3.14}$$

for all $i \in K_\varepsilon$, where max_ε in defined analogously to min_ε. It can be easily verified that the set of inequalities (3.3.14) is equivalent to the following inequality

$$max_\varepsilon(a_i) < 1 + b. \tag{3.3.15}$$

This inequlity follows from

$$max(a_i) < 1 + b. \tag{3.3.16}$$

Now we prove (3.3.16) for all $a \in E \setminus T$. Let us consider the tangent hypersurface to the ellipsoid E in the point $T_1 = (1, 0, \ldots, 0)$. The equation for this hypersurface has following form

$$d(a_1 - 1) + a_2 + \ldots + a_n = 0. \tag{3.3.17}$$

It is clear, that for all $a \in E \setminus T_1$

$$d(a_1 - 1) + a_2 + \ldots + a_n < 0, \tag{3.3.18}$$

or, equivalently,

$$a_1 < 1 + b. \tag{3.3.19}$$

In analogous manner it may be proved that

$$a_i < 1 + b. \tag{3.3.20}$$

for all $a \in E \setminus T_i$, $i = 1, ..., n$. The inequalities (3.3.20) imply (3.3.16). The proposition is proved.

Now we consider the case $a \in T$. Without loss of generality we put $a = T_1 = (1, 0, \ldots, 0)$. It is not difficult to verify that in this case the ε-horizon takes place only if $\varepsilon^1 \neq 0$.

3.4. Relativistic particle

Here we consider the motion of the relativistic particle in the gravitational field, corresponding to the metric (3.2.1). The Lagrangian of the particle is well-known

$$L_1 = -m\sqrt{-g_{MN}(x)\dot{x}^M \dot{x}^N}, \tag{3.4.1}$$

where m is the mass of the particle ($\dot{x}^M = dx^M/d\tau$).

The Lagrange equations for (3.4.1) in the proper time gauge

$$g_{MN}(x)\dot{x}^M \dot{x}^N = -1 \tag{3.4.2}$$

coincide with the geodesic equations (3.3.1). In this case $(E^i) = (m\varepsilon^i)$ is the energy vector and $J = mj$ is the angular momentum (see (3.3.4) and (3.3.5)). For fixed values of ε^i the (d+1)-dimensional part of the equations of motion is generated by the Lagrangian

$$L_* = \frac{m}{2}[f^b \bar{g}_{T,\alpha\beta}(x)\dot{x}^\alpha \dot{x}^\beta + \sum_{i=1}^{n}(\varepsilon^i)^2 f^{-a_i}], \tag{3.4.3}$$

where $\bar{g}_T$ is the space section of the Tangherlini metric.

Now, we restrict our consideration by the non-relativistic motion at large distances: $R\overline{g}L_B$. In this approximation: $t^i = \varepsilon^i \tau$, $\sum_{i=1}^n (\varepsilon^i)^2 = 1$. It follows from (3.4.3) that in this approximation we get a non-relativistic particle of mass m, moving in the potential

$$V = -\frac{m}{2} \sum_{i=1}^{n} (\varepsilon^i)^2 \frac{a_i B}{R^{d-1}} = -G \frac{m(\varepsilon^i M_{ij} \varepsilon^j)}{R^{d-1}}, \tag{3.4.4}$$

where G is the gravitational constant and

$$M_{ij} = a_i \delta_{ij} B/2G, \tag{3.4.5}$$

are the components of the gravitational mass matrix.

It is interesting to note that the relation (3.4.4) may be rewritten as following

$$V = -G \frac{tr(M M_I)}{R^{d-1}} \tag{3.4.6}$$

where $M_I = (m \varepsilon^i \varepsilon^j)$ is the inertial mass matrix of the particle.

The solution (3.2.1) may be also rewritten in the matrix form

$$\begin{aligned} g = \quad & -[(1 - BR^{1-d})^A]_{ij} d\bar{t}^i \otimes d\bar{t}^j \\ & + (1 - BR^{1-d})^{b-1} dR \otimes dR + (1 - BR^{1-d})^b R^2 d\Omega_d^2, \end{aligned} \tag{3.4.7}$$

where A is a real symmetric $n \times n$-matrix satisfying the relation

$$(trA)^2 + (d-1) tr(A^2) = d. \tag{3.4.8}$$

and

$$b = (1 - trA)/(d-1). \tag{3.4.9}$$

Here $x^A \equiv \exp(A \ln x)$ for $x > 0$. The metric (3.4.7) can be reduced to the metric (3.2.1) by the diagonalization of the A-matrix: $A = S^T (a_i \delta_{ij}) S$, $S^T S = 1_n$ and the reparametrization of the time variables: $\bar{t}^i = S_{ij} t^j$. In this case the gravitational mass matrix is

$$(M_{ij}) = (A_{ij} B/2G). \tag{3.4.10}$$

We may also define the gravitational mass tensor as

$$\mathcal{M} = M_{ij} d\bar{t}^i \otimes d\bar{t}^j. \tag{3.4.11}$$

We call the extended object, corresponding to the solution (3.4.7)-(3.4.9) as multi-temporal hedgehog. At large distances $R^{d-1}\overline{g}B$ this object is described by the matrix analogue of the Newton's potential

$$\Phi_{ij} = -\frac{1}{2} BR^{1-d} A_{ij} = -GR^{1-d} M_{ij}. \tag{3.4.12}$$

Clearly that this potential for the diagonal case (3.2.1) $A = a_i \delta_{ij}$ is a superposition of the potentials, corresponding to "pure" black hole states (3.2.8).

Remark 5. It is interesting to note that the formula

$$A = Q_i R^{1-d} dt^i \tag{3.4.13}$$

describe the multitemporal $O(d+1)$-analogue of the well-known electrostatic solution of the Maxwell equations. In this case the charge $Q = (Q_i)$ is a vector (or we may also define the charge as the 1-form $Q_i dt^i$).

Remark 6. Let us consider the solution (3.2.1) for $n = 2$ with $a_1 > 0$ and $a_2 < 0$. In this case under a suitable choice of the ε^i-parameters a point $R > L$, may be a libration point, i. e. the point of equilibrium. In this case

$$a_1(\varepsilon^1)^2 + a_2(\varepsilon^2)^2[f(R)]^{a_2 - a_1} = 0 \tag{3.4.14}$$

and $\varepsilon^2 \neq 0$. An analogous situation takes place for arbitrary n, when there exist positive and negative a_i-th parameters.

3.5. Some generalizations

Here we present some generalizations of the considered above solutions. First, we consider the model described by the following action

$$S = \int d^D x \sqrt{|g|} \{ \frac{1}{2\kappa^2} R[g] - \frac{1}{2\kappa^2} \partial_M \varphi \partial_N \varphi g^{MN} - \frac{1}{4} \exp(2\lambda\varphi) F_{MN} F^{MN} \}, \tag{3.5.1}$$

where $g = g_{MN} dx^M \otimes dx^N$ is the metric , $F = \frac{1}{2} F_{MN} dx^M \wedge dx^N = dA$ is the strength of the electromagnetic field and φ is the scalar field. Here λ is constant. The action (3.5.1) describes for certain values of parameters λ and D a lot of interesting physical models including standard Kaluza-Klein theory, dimensionally reduced Einstein-Maxwell theory, supergravity theories (see, for example [197]). We present the spherically-$O(d+1)$-symmetric solutions of the field equations corresponding to the action (3.5.1) with the topology (3.1.1). The solution is the following

$$\begin{aligned} g = \quad & -f_1^{(D-3)/A(\lambda)} f_\varphi^{2\lambda} dt \otimes dt \\ + \quad & f_1^{-1/A(\lambda)} (f_2^{-1} f_\varphi^{2\lambda} f^2)^{1/(1-d)} [f_2 du \otimes du + d\Omega_d^2] \\ + \quad & \sum_{i=1}^{\bar{n}} f_1^{-1/A(\lambda)} \exp(2A_i u + 2D_i) g^{(i)}, \end{aligned} \tag{3.5.2}$$

$$F = Q f_1 du \wedge dt, \tag{3.5.3}$$

$$\exp\varphi = f_1^{(2-D)\lambda/2A(\lambda)} f_\varphi. \tag{3.5.4}$$

In (3.5.2)-(3.5.4)

$$f_1 = f_1(u) = C_1(D-2)/\kappa^2 Q^2 A(\lambda) \sinh^2(\sqrt{C_1}(u - u_1)), \tag{3.5.5}$$

$$f_2 = f_2(u) = C_2/(d-1)^2 \sinh^2(\sqrt{C_2}(u - u_2)), \tag{3.5.6}$$

$$f_\varphi = f_\varphi(u) = \exp(Bu + D_\varphi), \tag{3.5.7}$$

$$f = f(u) = \exp[\sum_{i=1}^{\bar{n}} N_i(A_i u + D_i)], \tag{3.5.8}$$

$$A(\lambda) = D - 3 + \lambda^2(D-2), \tag{3.5.9}$$

and $Q \neq 0$, D_i, D_φ, u_1, u_2 are constants and the parameters C_1, C_2, B, A_i satisfy the the relation

$$\frac{C_2 d}{d-1} = \frac{C_1(D-2)}{D-3+\lambda^2(D-2)} + B^2(1+\lambda^2) + \frac{1}{d-1}(\lambda B + \sum_{i=1}^{\bar{n}} A_i N_i)^2 + \sum_{i=1}^{\bar{n}} A_i^2 N_i. \qquad (3.5.10)$$

The solution (3.5.2)-(3.5.10) generalizes the well-known Myers-Perry charged black hole solution [187] for the model (3.5.1) on the case of $\bar{n}$ internal Ricci-flat spaces. (We remind that $(M_i, g^{(i)})$ is Ricci-flat space of dimension N_i, $i = 1, \ldots, \bar{n}$.) The case $d = 2$ was considered previously in [165-166]. Some special cases of this solution were considered also in [196-197]. For $\bar{n} = n-1$, $M_i = R$, $g^{(i)} = -dt^i \otimes dt^i$, $i = 1, \ldots, n-1$, $t = t^n$ we get from (3.5.2) the multitemporal generalization of the solution [187] for the action (3.5.1).

In the zero charge case $F = 0$ we have

$$g = -\exp(2A_{-1}u + 2D_{-1}) dt \otimes dt + \exp[2(1-d)^{-1} \sum_\nu N_\nu (A_\nu u + D_\nu)] f_2^{1/(d-1)} [f_2 du \otimes du + d\Omega_d^2] + \sum_{i=1}^{\bar{n}} \exp(2A_i u + 2D_i) g^{(i)}, \qquad (3.5.11)$$

$$\varphi = Bu + D_\varphi. \qquad (3.5.12)$$

The integration constants satisfy the relation

$$\frac{C_2 d}{d-1} = \frac{1}{d-1} (\sum_\nu A_\nu N_\nu)^2 + \sum_\nu A_\nu^2 N_\nu + B^2. \qquad (3.5.13)$$

Hear $\nu = -1, 1, \ldots, n$; $N_{-1} = 1$.

The solution (3.2.1)-(3.2.3) may be also generalized on the infinite-time case: $n = \infty$. In this case the following restriction on the parameters a_i should be imposed (see also [19])

$$\sum_{i=1}^{\infty} |a_i| < +\infty. \qquad (3.5.14)$$

This relation implies

$$\sum_{i=1}^{\infty} |a_i|^2 < +\infty. \qquad (3.5.15)$$

In this case the metric (3.2.1) is correctly defined on a proper infinite-dimensional (Banach) manifold and satisfies the Einstein equations. We note that an infinite-dimensional version of the Einstein gravity was considered earlier by Kalitzin [192].

Remark 7. Another infinite-dimensional extension of the considered here solution may be obtained if the field of real numbers R is replaced by the even part G_0 of the infinite-dimensional Grassmann-Banach algebra $G = G_0 + G_1$ [198-199]. In this case all coordinates and the parameters of the solution (3.2.1) are elements of G_0. (The d-dimensional sphere with the metric on it should be replaced by its trivial G_0-extensions.)

3.6. Conclusion

In this paper the multitemporal analogue of the Tangherlini solution was considered. It was shown that in the case of two time directions the solution describes a naked singularity for any non-trivial (non-tangherlinian) set of parameters. We have integrated the geodesic equations for the considered solution. It was obtained the multitemporal analogue for the Newton's formula (eq. (3.4.6)), describing the interaction between the massive particle and the multitemporal extended object ("multitemporal hedgehog"), corresponding to the solution. It was shown that in the multitemporal case the inertial and gravitational masses are matrices. (It may be defined also as tensors). We have also obtained the generalization of the Myers-Perry charged black hole solution on the case of a chain of Ricci-flat internal spaces (this solution contains the multitemporal analogue as a special case).

Appendix

Here we present the expression for the tensor Riemann squared (3.2.9) corresponding to the cosmological metric

$$g = -B(t) dt \otimes dt + \sum_{i=1}^{n} A_i(t) g^{(i)},$$

defined on the manifold $M = R \times M_1 \times \ldots \times M_n$, where $g^{(i)}$ is a metric on the manifold M_i, $dim M_i = N_i$, $i = 1, \ldots, n$. By a straightforward calculation the following relation was obtained

$$\begin{aligned} I[g] = \quad & \sum_{i=1}^{n} \{ A_i^{-2} I[g^{(i)}] + A_i^{-3} B^{-1} \dot{A}_i^2 R[g^{(i)}] - \frac{1}{8} N_i B^{-2} A_i^{-4} \dot{A}_i^4 \\ & + \frac{1}{4} N_i B^{-2} (2A_i^{-1} \ddot{A}_i - B^{-1} \dot{B} A_i^{-1} \dot{A}_i - A_i^{-2} \dot{A}_i^2)^2 \} + \frac{1}{8} B^{-2} [\sum_{i=1}^{n} N_i (A_i^{-1} \dot{A}_i)^2]^2. \end{aligned}$$

4. Vacuum Static, Axially Symmetric Fields in D-Dimensional Gravity

4.1. Introduction

Spherically symmetric static solutions of multidimensional gravity have been considered by many authors with a goal to study possible observational windows to extra dimensions [21,77,78]. Among such windows one can name possible variations of fundamental physical constants [50], deviations from Newton's and Coulomb's laws and modified properties of black holes and gravitational radiation as compared with the conventional theory.

Another class of multidimensional models, to be discussed in this section, is the class of axially symmetric models, including spherically symmetric ones as a special case.

Although static, axially symmetric (SAS) configurations are a less popular object of gravitational studies than stationary ones (used for describing fields due to rotating bodies), their properties are of much interest as well. In many papers such solutions are sought and studied, see, for instance, [201-203] and references therein. We will study monopole SAS vacuum configurations in multidimensional gravity and find some features of interest, in particular, membrane and string type sources of fields possessing no curvature singularities.

We consider D-dimensional general relativity and start from the action

$$S = \int d^D x \sqrt{{}^D g}({}^D R + L_m) \tag{4.1.1}$$

where L_m is a matter Lagrangian, in a space with the metric

$$ds_D^2 = g_{\mu\nu} dx^\mu dx^\nu + \mathrm{e}^{2\beta_1} ds_1^2 \tag{4.1.2}$$

where Greek indices range from 0 to 3 and $\beta_1(x^\mu)$ is a scale factor of an internal N-dimensional space with a Ricci-flat ds_1^2 independent of x^μ.

In a 4-dimensional formulation

$$S = \int d^4 x \sqrt{{}^4 g}\,\mathrm{e}^\sigma \Big[R - \Big(\frac{1}{N} - 1\Big)\sigma^\alpha \sigma_\alpha + L_m\Big] \tag{4.1.3}$$

where $\sigma = N\beta_1$ and R is the 4-curvature corresponding to $g_{\mu\nu}$.

There are other 4-dimensional formulations of the theory, connected with (4.1.3) by conformal mappings (*conformal gauges*). The gauge (4.1.3) corresponds to the original theory. The so-called *Einstein gauge*, obtained from (4.1.3) by the conformal mapping

$$\overline{g}_{\mu\nu} = \mathrm{e}^\sigma g_{\mu\nu}, \tag{4.1.4}$$

is more convenient for solving the field equations since the curvature enters into the Lagrangian with a constant factor:

$$S = \int d^4 x \sqrt{{}^4 \overline{g}}\,\mathrm{e}^\sigma [\overline{R} + \alpha_0 \overline{g}^{\alpha\beta} \sigma_\alpha \sigma_\beta + \mathrm{e}^{-\sigma} L_m],$$
$$\alpha_0 = 1/2 + 1/N. \tag{4.1.5}$$

where $\overline{R}$ is the scalar curvature corresponding to $\overline{g}_{\mu\nu}$. Another important gauge, the so-called *atomic* one, in which a test particle moves along geodesics, is defined by

$$g^*_{\mu\nu} = \mathrm{e}^{\sigma/2} g_{\mu\nu} \tag{4.1.6}$$

and is most suitable for interpretaion of measurements, e.g., in the Solar system. However, for studies of singularities and topology the original metric $g_{\mu\nu}$ must be used. For more detailed discussion of the notion of systems of measurement, closely connected with that of conformal gauges, see Ref. [50] and, as applied to multidimensional theory, Refs.[21,204-206].

In what follows we use the Einstein gauge to find the metric (4.1.4) for vacuum SAS configurations. So we start from the equations due to (4.1.5) with $L_m = 0$:

$$\overline{R}_{\mu\nu} = -\alpha_0 \sigma_{,\mu}\sigma_{,\nu}, \tag{4.1.7}$$

$$\Box\sigma = 0 \tag{4.1.8}$$

where $\overline{R}_{\mu\nu}$ and $\Box$ are the Ricci tensor and the D'Alembert operator corresponding to $\overline{g}_{\mu\nu}$.

Vacuum D-dimensional equations are thus reduced to scalar-vacuum ones in 4 dimensions. Although such SAS configurations were repeatedly considered [202,212], it makes sense to return to them to reveal some new features, in particular, those connected with higher dimensions.

4.2. Field equations for axial symmetry

The SAS 4-metric in the Einstein gauge (4.1.4) may be written in the Weyl canonical form [203]

$$d\overline{s}^2 = \mathrm{e}^{2\nu}dt^2 - \mathrm{e}^{-2\nu}[\mathrm{e}^{2\beta}(d\rho^2 + dz^2) + \rho^2 d\phi^2] \tag{4.2.1}$$

The field equations then can be written as

$$\Delta\sigma = 0, \tag{4.2.2}$$

$$\Delta\nu = 0, \tag{4.2.3}$$

$$\beta_z = \rho(2\nu_\rho\nu_z + \alpha_0\sigma_\rho\sigma_z) \tag{4.2.4}$$

$$\beta_\rho = \rho[\nu_\rho^2 - \nu_z^2 + \tfrac{1}{2}\alpha_0\sigma_\rho^2 - \sigma_z^2)] \tag{4.2.5}$$

where the indices z and ρ denote the partial derivatives ∂_ρ and ∂_z, respectively, and Δ is the "flat" Laplace operator in the cylindrical coordinates:

$$\Delta = \rho^{-1}\partial_\rho(\rho\partial_\rho) + \partial_z\partial_z.$$

The integrability condition for (4.2.4) and (4.2.5) is satisfied automatically.

Following the example of [202,206], let us seek solutions in the new coordinates (x, y), connected with ρ and z by

$$\rho^2 = L^2(x^2 + \varepsilon)(1 - y^2), \qquad z = Lxy \tag{4.2.6}$$

where L is a fixed positive constant and $\varepsilon = 0,\ \pm 1$, so that x and y are spherical ($\varepsilon = 0$), prolate spheroidal ($\varepsilon = -1$), or oblate spheroidal ($\varepsilon = +1$) coordinates, respectively. The Laplace operator Δ acquires the form

$$\Delta = \partial_x(x^2 + \varepsilon)\partial_x + \partial_y(1 - y^2)\partial_y. \tag{4.2.7}$$

Separating the variables in Eq.(4.2.3), i.e., putting $\nu(x,y) = \chi(x)\psi(y)$, one obtains

$$[(x^2+\varepsilon)\chi_x]_x + \lambda\chi = 0, \qquad (4.2.8)$$
$$[(1-y^2)\psi_y]_y - \lambda\psi = 0 \qquad (4.2.9)$$

where λ is the separation constant. Solutions to (4.2.9) finite on the symmetry axis $\rho = 0$ are the Legendre polynomials $P_l(y)$, while $\lambda = l(l+1)$ with $l = 0,1,2,\ldots$. The corresponding solutions to (4.2.8) are combinations of Legendre functions of the first and second kinds.

Eq.(4.2.2) is solved in a similar way.

This is the way to obtain solutions of arbitrary multipolarity l or even superpositions of different multipolarities: after writing out the solutions to the linear equations (4.2.3) and (4.2.2), Eqs. (4.2.4) and (4.2.5) are integrable by quadratures. In what follows, however, we restrict ourselves to $l = 0$ (monopole solutions).

4.3. Monopole solutions

The monopole solution to Eq.(4.2.9) may without loss of generality be written in the form

$$e^{\psi} = [(1+y)/(1-y)]^{c_1}, \quad c_1 = \text{const.} \qquad (4.3.1)$$

Regularity at $y = \pm 1$ then requires $c_1 = 0$, so that $\nu = \nu(x)$. Eq.(4.2.8) takes the form $(x^2+\varepsilon)d\chi/dx = \text{const}$. Its integration leads to the following expressions for $\nu(x)$ satisfying the asymptotic flatness condition:

$$\nu = \begin{cases} -\frac{1}{2}b\ln\frac{x+1}{x-1}, & \varepsilon = -1, \\ -b/x, & \varepsilon = 0, \\ -b\cot^{-1}x, & \varepsilon = +1. \end{cases} \qquad (4.3.2)$$

In a similar way $\sigma(x)$ is found:

$$\sigma = \begin{cases} -\frac{1}{2}s\ln\frac{x+1}{x-1}, & \varepsilon = -1, \\ -s/x, & \varepsilon = 0, \\ -s\cot^{-1}x, & \varepsilon = +1. \end{cases} \qquad (4.3.3)$$

Integrating (4.2.4) and (4.2.5), one obtains the expressions for $\beta(x,y)$ satisfying the asymptotic flatness condition $\beta(\infty, y) = 0$

$$e^{2\beta} = \begin{cases} (x^2-1)^K(x^2-y^2)^{-K}, & \varepsilon = -1, \\ \exp[-K(1-y^2)/x^2], & \varepsilon = 0, \\ (x^2+y^2)^K(x^2+1)^{-K}, & \varepsilon = +1 \end{cases} \qquad (4.3.4)$$

with $K = \frac{1}{2}(2b^2 + \alpha_0 s^2) \geq 0$.

4.4. General properties of the solutions

The solutions have been found under the boundary condition providing regularity (local euclidity) at the symmetry axis $\rho = 0$, or $y = \pm 1$.

At spatial infinity the solutions are asymptotically spherically symmetric. Indeed, assuming $y = \cos\theta$ where θ is the conventional polar angle, the SAS line element (4.2.1) transformed by (4.2.6), is spherically symmetric under the condition

$$e^{2\beta} = (x^2 + \varepsilon)/(x^2 + \varepsilon y^2). \tag{4.4.1}$$

The condition (4.4.1) holds for all the solutions in the limit $x \to \infty$ where they have Schwarzschild asymptotics. A particular expression for the Schwarzschild mass in terms of the integration constants is conformal gauge-dependent. Recalling that the mass is most meaningfully defined in the atomic gauge (4.1.6), one can write:

$$\begin{aligned} &g^*_{tt} \approx 1 - 2GM/r, \qquad r \approx Lx; \\ &GM = (b - s/4)L \end{aligned} \tag{4.4.2}$$

As for the whole space, the condition (4.4.1) is fulfilled under the additional requirement

$$K\varepsilon = \tfrac{1}{2}(2b^2 + \alpha_0 s^2)\varepsilon = -1. \tag{4.4.3}$$

As b and s are real, this condition can hold only for $\varepsilon = -1$. Quite naturally, the solution with $\varepsilon = -1$ constrained by (4.4.3) coincides with the well-known generalized Schwarzschild solution [171] with the $(4+N)$-dimensional metric (4.1.2)

$$\begin{aligned} ds^2_D = &\Big(1 - \frac{2k}{R}\Big)^{a_0} \\ &- \Big(1 - \frac{2k}{R}\Big)^{-a_0 - Na_1}\Big[dR^2 + R^2\Big(1 - \frac{2k}{R}\Big)d\Omega^2\Big] \\ &\qquad + \Big(1 - \frac{2k}{R}\Big)^{a_1} ds_1^2, \\ Na_1^2 + a_0^2 &+ (a_0 + Na_1)^2 = 2 \end{aligned} \tag{4.4.4}$$

where the variable R and the integration constants are connected with ours in the following way:

$$x + 1 = R/k; \;\; Na_1 = -s; \;\; a_0 = b + s/2; \;\; K = L. \tag{4.4.5}$$

In [21,78] (see also references therein) solutions with a chain of Ricci-flat internal spaces, generalizing [171], are given; still more general spherical solutions with massless gauge and dilaton fields are discussed, e.g., in [171,185,186,205-207].

The general solution with $\varepsilon = -1$ has a naked singularity at $x = 1$ in all cases, except the spherically symmetric one when, in addition, the scalar field σ is constant (or the extra dimensions are frozen), in agreement with [78]. The singularity at $x = 1$ is anisotropic in all cases except (4.4.4): the metric coefficients behave in different ways when the singularity is approached from different directions. For some sets of integration constants the path to the singularity $y = \text{const}$, $\phi = \text{const}$, $x \to 1$ has an infinite length; however, the explicit conditions of such a behavior are conformal gauge-dependent.

In the case $\varepsilon = 0$ the solution generalizes the well-known Curzon vacuum solution of general relativity [208], extensively studied in a number of papers, see, e.g., Ref.[209] and references therein. The metric can be written in the form (4.2.1) with

$$\nu = -b/x,\ 2\beta = -K\rho^2/(L^2x^4),\ Lx = \sqrt{\rho^2 + z^2}. \tag{4.4.6}$$

In the special case $s = 0$ our solution coincides with the Curzon one up to a re-definition of constants.

The solution is singular at $x = 0$ in all cases except $s = b = 0$ when it reduces to flat space-time. The singularity is anisotropic, such that even the finiteness or infiniteness of some metric coefficients can depend on the direction of approach. As shown in [209], in the Curzon case the true nature of the singularity is revealed in some new coordinates, allowing one to penetrate beyond $x = 0$ (in our notation). It turns out that curvature singularity $x = 0$ has the shape of a ring and some spatial geodesics can pass through it to reach a second spatial infinity on the other side of the ring.

This quasi-wormhole structure is preserved for the present, more general solution, although the exact conditions when this is the case or, on the contrary, $x = 0$ is just a singular center, is conformal gauge-dependent.

A further study of this solution, despite its possible interest, is beyond the scope of this paper. We will instead pay more attention to the solution with $\varepsilon = +1$, which has no curvature singularity and therefore seems more promising; and although a preferred conformal gauge does exist (the one in which the original D-dimensional theory is formulated), it is remarkable that the most important features of the configuration to be discussed do not depend on conformal factors of the form $\exp(\text{const} \times \sigma)$.

4.5. Membranes, strings and wormholes

The non-existence of a curvature singularity for $\varepsilon = +1$ does not necessarily mean that the space-time is globally regular. Let us study the limit $x \to 0$ in some detail.

The functions ν, σ and e^{β} are finite at $x = 0$.

The curve $x = 0$, $y = 0$ as viewed in the coordinates (ρ, z) lies in the plane $z = 0$ and forms a ring $\rho = L$ of finite length (Fig. 20). In the original conformal gauge (4.1.2) the ring radius is $r_0 = L\exp(b\pi/2 + s\pi/4)$.

The surface $x = 0$, $y > 0$ is a disk bounded by the above ring and parametrized by the coordinates y and ϕ. Its 2-dimensional metric is

$$dl^2_{\text{disk}} = L^2 e^{-2\nu-\sigma}[(1 - \xi^2)^K d\xi^2 + \xi^2 d\phi^2] \tag{4.5.1}$$

where $\xi = \sqrt{1 - y^2}$. This metric is flat if and only if $K = 0$, i.e., when the solution is trivial. Otherwise the disk is curved but has a regular center at $y = 1$ (the upper small black circle in Fig. 20). The limit $x \to 0$ corresponds to approaching the disk from the half-space $z > 0$.

Another similar disk, the lower half-space one, corresponds to $y < 0$. The two disks are naturally identified when our oblate spheroidal coordinates are used in flat space (obtained here in the case $K = 0$).

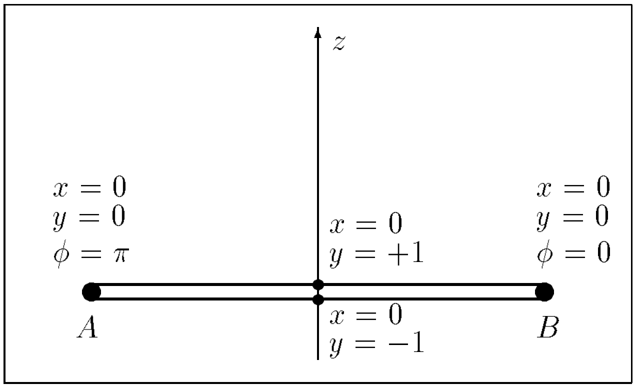

Figure 20: Axial section of the neighborhood of the ring $x = y = 0$. The points A and B, marked by big black circles, belong to the ring, the thick lines connecting them show the upper and lower disks $x = 0,\ y \gtrless 0$.

A possible identification of points $(x = 0,\ y = y_0,\ \phi = \phi_0)$ and $(x = 0,\ y = -y_0,\ \phi = \phi_0)$, where ϕ_0 is arbitrary and $0 < y_0 \le 1$, leads to a finite discontinuity of the extrinsic curvature of the surfaces identified, or, physically, to a finite discontinuity of forces acting on test particles. Such a behavior corresponds to a membrane-like matter distribution. Thus a source of the global vacuum (or scalar-vacuum) gravitational field may be a membrane bounded by the ring $x = y = 0$.

There is another possibility, with no field discontinuity across the surface $x = 0$. Namely, one can continue the (x, y) coordinates to negative x by just replacing in (4.3.2) and (4.3.3) the function $\cot^{-1} x$ (undefined for $x < 0$) by $\pi/2 - \tan^{-1} x$, coinciding with the former at $x > 0$. This results in the appearance of another "copy" of the 3-space, so that a particle crossing the regular disk $x = 0$ along a trajectory with fixed y, threads a path through the ring and can ultimately get to another flat spatial infinity, with a different asymptotic value of ν and σ:

$$\begin{aligned} &\nu(+\infty) = 0, \quad \nu(-\infty) = -\pi b, \\ &\sigma(+\infty) = 0, \quad \sigma(-\infty) = -\pi s, \end{aligned} \tag{4.5.2}$$

The function β is even with respect to x and hence coincides at both asymptotics. We obtain a wormhole configuration, nonsymmetric with respect to its "neck" $x = 0$, having no curvature discontinuities, except maybe the ring $x = y = 0$.

It now remains to study the geometry near the ring. To this end let us consider a section of the ring by an (x, y) surface at fixed ϕ and small x and y. Its 2-dimensional metric near the point $x = y = 0$ is

$$dl^2_{(x,y)} = (x^2 + y^2)^{K+1}(dx^2 + dy^2). \tag{4.5.3}$$

This metric is flat, as is directly verified by the following transformation: introduce the polar coordinates r and ψ $(x = r\cos\psi,\ y = r\sin\psi)$ and further transform them to ξ and η by the formulas

$$r = [(K+2)\xi]^{1/(K+2)}, \ \psi = \eta/(K+2). \tag{4.5.4}$$

The result is

$$dl^2_{(x,y)} = r^{2K+2}(dr^2 + r^2 d\psi^2) = d\xi^2 + \xi^2 d\eta^2 \qquad (4.5.5)$$

Thus we have above all assured that the metric near the ring $x{=}y{=}0$ is locally flat. However, it is locally flat on the ring itself only if the proper radius-circumference relation near the origin (the point A or B in Fig. 20) in (4.5.5) holds, i.e., if η is defined on a segment of length 2π. Let us find out the η range.

Given $x > 0$, the polar angle ψ is defined on the segment $[-\pi/2,\ \pi/2]$, hence $\eta \in [-\pi - K\pi/2,\ \pi + K\pi/2]$. Consequently, the local flatness condition is fulfilled on the ring only in the trivial case $K = 0$. Identifying the points $\eta = \pm\pi$ and returning to the (x, y) coordinates, we then obtain flat space-time provided with oblate spheroidal coordinates with the single parameter L.

For $x > 0,\ K > 0$ there is an excess polar angle, the situation opposite to a top-of-a-cone singularity. Such singularities are conventionally interpreted as cosmic strings, although in those objects a deficient rather than excessive polar angle range is considered. One can conclude that a possible source of the vacuum or scalar-vacuum gravitational field is a disk membrane bounded by a special kind of string.

In the wormhole case x can have either sign, hence

$$\psi \in [-\pi,\ \pi]\ \Rightarrow \eta \in [-(2+K)\pi,\ (2+K)\pi]. \qquad (4.5.6)$$

Thus the axially symmetric wormhole solution contains a string-like ring singularity with a polar angle excess greater than 2π.

The excessive polar angle can have another mathematical meaning. Namely, if the excess is a multiple of 2π, the singularity behaves like a branching point in a Riemannian surface of an analytic function of a properly defined complex variable. In our case the variable is $\zeta = \xi + i\eta$ and the analytic function is $\zeta^{1/(K+2)}$. Conformal mappings with analytic functions represent a natural way of regularizing metrics like (4.5.3); this method was indeed used in similar situations in [205,206,210,211] where the relevant analytic function was logarithmic and the branching multiplicity was potentially (without additional identifications) infinite.

If one postulates that the "string" should behave as a branching point, the integrality condition ($K =$ integer for (4.5.6)) is a quantization-type condition for the parameters of the solution. For instance, in the case $s = 0$, i.e., a purely vacuum configuration (with maybe trivial extra dimensions), the mass is determined by $GM = bL$ and $K = b^2$, so that, given L is a fixed length, the spectrum of masses has the form $GM = L\sqrt{K}$ where K is a positive integer.

4.6. Concluding remarks

The results described appear from solving the field equations for pure vacuum or scalar vacuum in conventional general relativity as well as multidimensional gravity. One can conclude that SAS configurations can have nontrivial structures; those free of curvature singularities are in our view of greatest physical interest. Notably the singularities in SAS solutions are naked, except special spherically symmetric cases (for the vacuum case see (4.4.4) for $a_1 = 0$). For general relativity this is a manifestation of the well-known uniqueness or no-hair theorems; it would be, however, of interest to analyze the situation in dilaton gravity for which spherically symmetric black-hole and non-black-hole solutions are known (see, e.g., [205]) and SAS ones are either known [206], or can be readily obtained, for instance, in D dimensions with Ricci-flat internal spaces.

The above solutions can be of interest for describing late stages of gravitational collapse and/or cosmological dark matter. Their monopole nature probably means that they cannot decay through gravitational-wave emission.

Other generalizations of the present solutions, which are either known or easily obtainable by the known methods and are yet to be investigated in detail, are those with pure imaginary, nonminimally coupled and multiple scalar fields and/or multiple internal spaces.

Chapter 3

Quantum Multidimensional Models. Wormholes

1. Multidimensional Classical and Quantum Wormholes in Models with Cosmological Constant

1.1. Introduction

In quantum cosmology instantons, solutions of the classical Einstein equations in Euclidean space, play an important role giving the main contributions to the path integral [1]. Among them classical wormholes are of special interest, because they are connected with processes changing the topology of the models [51]. We remind that classical wormholes are usually Riemannian metrics consisting of two large regions joined by a narrow throat (handle). They exist for special types of matter [51,214] and do not exist for pure gravity. In quantum cosmology it is generally assumed that on Planck scale processes with topology changes should take place. For this reason Hawking and Page [34] introduced the notion of quantum wormholes as a quantum extension of the classical wormhole paradigma. They proposed to regard quantum wormholes as solutions of the Wheeler-DeWitt (WDW) equation with the following boundary conditions:

(i) the wave function is exponentially damped for large spatial geometry,

(ii) the wave function is regular when the spatial geometry degenerates.

The first condition expresses the fact that space-time should be Euclidean at spatial infinity. The second condition should reflect the fact that space-time is nonsingular when spatial geometry degenerates. For example, the wave function should not oscillate an infinite number of times.

The given approach extends the number of objects which can be treated as wormholes [226-230].

We believe that for the description of quantum gravitational processes at high energies the multidimensional approach is more adequate. Modern theories of unified physical interactions use ideas of hidden (or extra) dimensions. In order to study different phenomena at early stages of the universe one should use these theories or at any rate models keeping their main characteristics. But more reliable conclusions may be done only on the basis of exact solutions which are usually obtained in rather simple cases.

Therefore, at the beginning we consider a cosmological model with n $(n > 1)$ Einstein spaces containing a massless minimally coupled scalar field and a cosmological constant Λ. The gauge covariant form of the WDW equation was proposed in [15]. This model is integrable in the case with only one of the Einstein spaces being not Ricci-flat and vanishing cosmological constant. The general properties of this particular model were investigated in [20] while classical as well as quantum wormhole solutions were found for different models in [231-232]. (In [37] particular integrable cases with milticomponent perfect fluid were considered).

The present section is devoted to the case of nonzero cosmological constant. For the model with one space of positive constant curvature in four dimensional space-time with cosmological constant and axionic matter (which is equivalent to a free minimally coupled scalar field, see, for example, [51] and the paper by Brown et al. in [219]) classical wormhole solutions were obtained in [51,233]. In the case of four dimensional space-time with nontrivial topology $\mathrm{I\!R} \times S^1 \times S^2$ and non-zero cosmological constant this type of solution exists, too [234].

Here, we investigate the case with at least one of the spaces, say M_1 being Ricci-flat. If all the other spaces $M_i, i = 2, \ldots, n$ are also Ricci-flat this model is fully integrable in the classical [19] as well as in the quantum cases [15]. For $\Lambda < 0$ a family of quantum wormhole solutions with continuous and discrete spectra exist. Classical wormholes can be found in this case only in the presense of a scalar field. In the presence of a scalar field classical wormhole solutions exist also for another particular case with fine tuning of the parameters of the model, if $\Lambda < 0$ and all $M_i, i = 2, \ldots, n$ are not Ricci-flat and have the same sign of the curvature. In this case only M_1 has a dynamical behaviour and is considered as our external space. All the other internal spaces $M_i, i = 2, \ldots, n$ are freezed with fixed scale factors $a_{(0)i}$ which are fine tuned to values determined by the cosmological constant. This type of solutions belongs to the class of models with spontaneous compactification. In the case of models without a cosmological constant and with only one non-Ricci-flat factor space solutions with spontaneous compactification were also found in [235].

We would like to note that solutions of the WDW equation in four dimensional models with $\Lambda \neq 0$ and with a conformal scalar field were first obtained in [236] and [237] respectively (see also [50]). They include possibly the first quantum wormhole type solutions in four dimensions as well as DeWitt's solution for the Friedman universe with dust (1967). Vacuum quantum cosmological solutions in four dimensions may be found in [38]. The path integral approach to quantum cosmology [213] for models with cosmological constant in four and five dimensions with nontrivial topology was developed in [239-244].

The section is organized as follows. In section 1.2 the general description of the models considered is given. In section 1.3 classical and quantum wormholes are obtained for all spaces being Ricci-flat. In section 1.4 classical wormholes are considered in the model with spontaneous compactification of extra dimensions. Conclusions complete the paper.

1.2. GENERAL DESCRIPTION OF THE MODEL

The metric of the model

$$g = -exp[2\gamma(\tau)]d\tau \otimes d\tau + \sum_{i=1}^{n} exp[2\beta^i(\tau)]g^{(i)}, \tag{1.2.1}$$

is defined on the manifold

$$M = \mathrm{I\!R} \times M_1 \times \ldots \times M_n, \tag{1.2.2}$$

where the manifold M_i with the metric $g^{(i)}$ is an Einstein space of dimension d_i, i.e.

$$R_{m_i n_i}[g^{(i)}] = \lambda^i g^{(i)}_{m_i n_i}, \tag{1.2.3}$$

$i = 1, \ldots, n$; $n \geq 2$. The total dimension of the space-time M is $D = 1 + \sum_{i=1}^{n} d_i$.

Here we investigate the general model with cosmological constant Λ and a homogeneous minimally coupled field $\varphi(t)$ with a potential $U(\varphi)$.

The action of the model is adopted in the following form

$$S = \frac{1}{2}\int d^D x \sqrt{|g|}\{R[g] - \partial_M \varphi \partial_N \varphi g^{MN} - 2U(\varphi) - 2\Lambda\} + S_{GH}, \tag{1.2.4}$$

where $R[g]$ is the scalar curvature of the metric $g = g_{MN} dx^M \otimes dx^N$ and S_{GH} is the standard Gibbons-Hawking boundary term [245]. The field equations, corresponding to the action (1.2.4), for the cosmological metric (1.2.1) in the harmonic time gauge $\gamma \equiv \sum_{i=1}^{n} d_i \beta^i$ are equivalent to the Lagrange equations, corresponding to the Lagrangian

$$L = \frac{1}{2}\sum_{i,j=1}^{n}(G_{ij}\dot{\beta}^i\dot{\beta}^j + \dot{\varphi}^2) - V, \tag{1.2.5}$$

with the energy constraint imposed

$$E = \frac{1}{2}\sum_{i,j=1}^{n}(G_{ij}\dot{\beta}^i\dot{\beta}^j + \dot{\varphi}^2) + V = 0. \tag{1.2.6}$$

Here, the overdot denotes differentiation with respect to the harmonic time τ. The components of the minisuperspace metric read

$$G_{ij} = d_i \delta_{ij} - d_i d_j \tag{1.2.7}$$

and the potential is given by

$$V = V(\beta, \varphi) = \exp(2\sum_{i=1}^{n} d_i \beta^i)\left[-\frac{1}{2}\sum_{j=1}^{n}\theta_j e^{-2\beta^j} + U(\varphi) + \Lambda\right], \tag{1.2.8}$$

where $\theta_i = \lambda^i d_i$. If the M_i are spaces of constant curvature, then θ_i may be normalized in such a way that $\theta_i = k_i d_i(d_i - 1)$, $k_i = \pm 1, 0$. We may also consider the generalization of the model with the potential (1.2.8) modified by the substitution

$$U(\varphi) \mapsto \tilde{U}(\varphi, \beta). \tag{1.2.9}$$

This gives us the possibility to investigate models with an arbitrary scalar field potential $\tilde{U}(\varphi,\beta) \equiv U(\varphi)$ as well as (for $\varphi =$const) models with an arbitrary potential $\tilde{U}(\varphi,\beta) \equiv U(\beta)$. Effective potentials of the form $U(\beta)$ may have their origin in an ideal fluid matter source. In special cases the general form $\tilde{U}(\varphi,\beta)$ of the potential leads us to new integrable models. An example of this kind of potential will be presented.

With the general potential $\tilde{U}(\varphi,\beta)$ the equations of motion are

$$-d_i\ddot{\beta}^i + d_i\sum_{k=1}^{n} d_k\ddot{\beta}^k + e^{2\sum_{k=1}^{n} d_k\beta^k}\Bigg[(d_i-1)\,\theta_i e^{-2\beta^i} + d_i\sum_{k\neq i}\theta_k\, e^{-2\beta^k}$$

$$-\frac{\partial\tilde{U}}{\partial\beta^i} - -2d_i\tilde{U} - 2d_i\Lambda\Bigg] = 0\ , \qquad i=1,\ldots,n, \tag{1.2.10}$$

$$\ddot{\varphi} + \frac{\partial\tilde{U}}{\partial\varphi}\exp\left(2\sum_{i=1}^{n} d_i\beta^i\right) = 0. \tag{1.2.11}$$

The constraint reads

$$\frac{1}{2}\left(\sum_{i,j=1}^{n} G_{ij}\dot{\beta}^i\dot{\beta}^j + \dot{\varphi}^2\right) + V = 0. \tag{1.2.12}$$

At the quantum level the constraint (1.2.12) is modified into the WDW equation (see [7])

$$\left[\frac{1}{2}\left(G^{ij}\frac{\partial}{\partial\beta^i}\frac{\partial}{\partial\beta^j} + \frac{\partial^2}{\partial\phi^2}\right) - V(\beta,\varphi)\right]\Psi(\beta,\varphi) = 0, \tag{1.2.13}$$

where $\Psi = \Psi(\beta,\varphi)$ is the wave function of the universe, V is the potential (1.2.8) and

$$G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2-D} \tag{1.2.14}$$

are the components of the matrix inverse to the matrix (G_{ij}) (1.2.7). The minisuperspace metric $G = G_{ij}dx^i \otimes dx^i$ (1.2.7) was diagonalized in [14,15]

$$G = -dz^0 \otimes dz^0 + \sum_{i=1}^{n-1} dz^i \otimes dz^i, \tag{1.2.15}$$

where

$$z^0 = q^{-1}\sum_{j=1}^{n} d_j\beta^j,$$

$$z^i = [d_i/\Sigma_i\Sigma_{i+1}]^{1/2}\sum_{j=i+1}^{n} d_j(\beta^j - \beta^i), \tag{1.2.16}$$

$i = 1,\ldots,n-1$, where

$$q = [(D-1)/(D-2)]^{1/2}, \qquad \Sigma_i = \sum_{j=i}^{n} d_j. \tag{1.2.17}$$

The WDW equation (1.2.13) takes in variables (1.2.16) the following form

$$\left[-\frac{\partial}{\partial z^0}\frac{\partial}{\partial z^0} + \sum_{i=1}^{n-1}\frac{\partial}{\partial z^i}\frac{\partial}{\partial z^i} + \frac{\partial^2}{\partial\varphi^2} - 2V(z,\varphi)\right]\Psi = 0. \tag{1.2.18}$$

1.3. WORMHOLES FOR RICCI-FLAT SPACES

In this chapter we consider the Ricci-flat case ($\theta_i = \lambda_i d^i = 0,\ i = 1, \ldots, n$), with $\Lambda \neq 0$ and $U(\varphi) = 0$. If the M_i are internal spaces they should be compact. This compactness is a necessary condition also for the Hartle-Hawking boundary condition (see below). The compactness of Ricci-flat spaces may be achieved by appropriate boundary conditions. The d-dimensional tore is the simplest example.

Classical solutions

In the considered case the Lagrangian (1.2.5) may be written in the following form

$$L = \frac{1}{2} \sum_{I,J=1}^{n+1} G_{IJ} \dot{\beta}^I \dot{\beta}^J - \Lambda \exp(\sum_{I=1}^{n+1} u_I \beta^I), \tag{1.3.1}$$

where $\beta^{n+1} = \varphi,\ u_i = 2d_i, i = 1, \ldots, n,\ u_{n+1} = 0$ and

$$(G_{IJ}) = \begin{pmatrix} G_{ij} & 0 \\ 0 & 1 \end{pmatrix} \tag{1.3.2}$$

(the matrix (G_{ij}) is defined in eq. (1.2.7)). We consider the coordinates $(z^A) = (z^a, z^n = \beta^{n+1} = \varphi)$, where $z^a,\ a = 0, \ldots, n-1$, are defined in (1.2.16). It is clear that

$$z^A = \sum_{I=1}^{n+1} V_I^A \beta^I, \tag{1.3.3}$$

$A = 0, \ldots, n$, where

$$\left(V_I^A\right) = \begin{pmatrix} V_i^a & 0 \\ 0 & 1 \end{pmatrix} \tag{1.3.4}$$

and the matrix (V_i^a) is defined in (1.2.16). This introduced matrix diagonalizes the minisuperspace metric

$$G_{IJ} = \sum_{A,B=0}^{n} \eta_{AB} V_I^A V_J^B, \tag{1.3.5}$$

$I, J = 1, \ldots, n+1$ and $(\eta_{AB}) = (\eta^{AB}) = \mathrm{diag}(-1, +1, \ldots, +1)$.

In the coordinates (1.3.3) the Lagrangian (1.3.1) reads (q is defined in (1.2.17))

$$L = \frac{1}{2} \sum_{A,B=0}^{n} \eta_{AB} \dot{z}^A \dot{z}^B - \Lambda \exp(2qz^0). \tag{1.3.6}$$

The Lagrange equations for the Lagrangian (1.3.6)

$$-\ddot{z}^0 + 2q\Lambda \exp(2qz^0) = 0, \tag{1.3.7}$$

$$\ddot{z}^A = 0, \qquad A = 1, \ldots, n, \tag{1.3.8}$$

with the energy constraint

$$E = \frac{1}{2} \sum_{A,B=0}^{n} \eta_{AB} \dot{z}^A \dot{z}^B + \Lambda \exp(2qz^0) = 0 \tag{1.3.9}$$

can be readily solved. First integrals of (1.3.8) are

$$\dot{z}^A = p^A, \qquad A = 1, \ldots, n, \tag{1.3.10}$$

where p^A are arbitrary constants of integration. Then the constraint (1.3.9) may be rewritten

$$-\frac{1}{2}(\dot{z}^0)^2 + \mathcal{E} + \Lambda e^{2qz^0} = 0 \tag{1.3.11}$$

with

$$2\mathcal{E} = \sum_{A=1}^{n} (p^A)^2. \tag{1.3.12}$$

We obtain the following solution

$$z^A = p^A \tau + q^A, \qquad A = 1, \ldots, n, \tag{1.3.13}$$

where p^A and q^A are constants and

$$2qz^0 = \ln[\mathcal{E}/\{\Lambda \sinh^2(q\sqrt{2\mathcal{E}}(\tau - \tau_0))\}], \qquad \mathcal{E} \neq 0, \ \Lambda > 0, \tag{1.3.14}$$
$$= \ln[1/\{2q^2\Lambda(\tau - \tau_0)^2\}], \qquad \mathcal{E} = 0, \ \Lambda > 0, \tag{1.3.15}$$
$$= \ln[-\mathcal{E}/\{\Lambda \cosh^2(q\sqrt{2\mathcal{E}}(\tau - \tau_0))\}], \qquad \mathcal{E} > 0, \ \Lambda < 0, \tag{1.3.16}$$

Here τ_0 is an arbitrary constant.

Kasner-like parametrization

First we consider the case $\mathcal{E} > 0$. In this case the relations (1.3.14) and (1.3.16) may be written in the following form

$$2qz^0 = \ln[\mathcal{E}/\{|\Lambda| f_\delta^2(q\sqrt{2\mathcal{E}}(\tau - \tau_0))\}], \tag{1.3.17}$$

where $\delta \equiv \Lambda/|\Lambda| = \pm 1$ and

$$\begin{aligned} f_\delta(x) \equiv \frac{1}{2}(e^x - \delta e^{-x}) &= \sinh x, \qquad \delta = +1, \\ &= \cosh x, \qquad \delta = -1. \end{aligned} \tag{1.3.18}$$

We introduce a new time variable by the relation

$$t = \frac{T}{\sqrt{\delta}} \ln \frac{\exp(q\sqrt{2\mathcal{E}}(\tau - \tau_0)) + \sqrt{\delta}}{\exp(q\sqrt{2\mathcal{E}}(\tau - \tau_0)) - \sqrt{\delta}}, \tag{1.3.19}$$

where

$$T \equiv [(D-2)/2|\Lambda|(D-1)]^{1/2}. \tag{1.3.20}$$

It is not difficult to verify that the following relations take place

$$\begin{aligned}\sinh(t\sqrt{\delta}/T)/\sqrt{\delta} &= 1/f_\delta(q\sqrt{2\mathcal{E}}(\tau-\tau_0)), &(1.3.21)\\ \tanh(t\sqrt{\delta}/2T)/\sqrt{\delta} &= \exp(-q\sqrt{2\mathcal{E}}(\tau-\tau_0)), &(1.3.22)\\ dt &= -Tq\sqrt{2\mathcal{E}}d\tau/f_\delta(q\sqrt{2\mathcal{E}}(\tau-\tau_0)). &(1.3.23)\end{aligned}$$

Now, we introduce the following dimensionless parameters

$$\bar{\alpha}^I \equiv -\sum_{A=1}^{n} \bar{V}^I_A p^A/q\sqrt{2\mathcal{E}}, \tag{1.3.24}$$

where $(\bar{V}^I_A) = (V^A_I)^{-1}$ (see (1.3.3)). It is clear that

$$\left(\bar{V}^I_A\right) = \begin{pmatrix} \bar{V}^i_a & 0 \\ 0 & 1 \end{pmatrix} \tag{1.3.25}$$

where $(\bar{V}^i_a) = (V^a_i)^{-1}$. The relation (1.3.24) is equivalent to the following relations

$$\begin{aligned}\bar{\alpha}^i &= -\sum_{a=1}^{n-1} \bar{V}^i_a p^a/q\sqrt{2\mathcal{E}}, \quad i=1,\ldots,n, &(1.3.26)\\ \bar{\alpha}^{n+1} &= -p^n/q\sqrt{2\mathcal{E}}. &(1.3.27)\end{aligned}$$

It follows from eq. (1.3.5), that

$$\bar{V}^I_A = \sum_{J=1}^{n+1}\sum_{B=0}^{n} G^{IJ} V^B_J \eta_{BA} \tag{1.3.28}$$

where

$$\left(G^{IJ}\right) \equiv (G_{IJ})^{-1} = \begin{pmatrix} G^{ij} & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.3.29}$$

From (1.2.14) and (1.3.28) we have

$$\bar{V}^i_0 = -G^{ij}u_j/2q = (q(D-2))^{-1}; \qquad \bar{V}^{n+1}_0 = 0. \tag{1.3.30}$$

Using the relations (1.3.13), (1.3.17), (1.3.21)-(1.3.23), (1.3.24) and (1.3.30) we get the folowing expressions for the solution of field equations

$$\begin{aligned}g &= -dt\otimes dt + \sum_{i=1}^{n} a_i^2(t) g_{(i)}, &(1.3.31)\\ a_i(t) &= \exp(\beta^i(t)) = A_i[\sinh(rt/T)/r]^{\sigma}[\tanh(rt/2T)/r]^{\bar{\alpha}^i}, &(1.3.32)\\ \exp(\varphi(t)) &= \exp(\beta^{n+1}(t)) = A_{n+1}[\tanh(rt/2T)/r]^{\bar{\alpha}^{n+1}}, &(1.3.33)\end{aligned}$$

where $t>0, r=\sqrt{\delta}=\sqrt{\Lambda/|\Lambda|}=\sqrt{\pm 1}$, $\sigma=(D-1)^{-1}$, $A_i \neq 0$ are constants, $i = 1,\ldots,n$, and the parameters $\bar{\alpha}^I$ satisfy the relations

$$\frac{1}{2}\sum_{I=1}^{n+1} u_I \bar{\alpha}^I = \sum_{i=1}^{n} d_i \bar{\alpha}^i = 0, \tag{1.3.34}$$

$$\sum_{I,J=1}^{n+1} G_{IJ}\bar{\alpha}^I\bar{\alpha}^J = \sum_{i=1}^{n} d_i(\bar{\alpha}^i)^2 + (\bar{\alpha}^{n+1})^2 = (D-2)/(D-1). \tag{1.3.35}$$

The first relation (1.3.34) can be easily proved, using the definition (1.3.24) and the following identity (we remind that due to (1.2.16) and (1.3.3) $u_I = 2qV_I^0$)

$$\sum_{I=1}^{n+1} u_I \bar{V}_A^I = 2q \sum_{I=1}^{n+1} V_I^0 \bar{V}_A^I = 2q\delta_A^0 = 0 \tag{1.3.36}$$

for $A > 0$. The relation (1.3.5) follows immediately from (1.2.7), (1.3.5), (1.3.12), (1.3.24) and (1.3.34).

Now we consider the case $\mathcal{E} = 0$. In this case for $\Lambda > 0$ there exist also an exceptional solution with the following scale factors in (1.3.31)

$$a_i(t) = \bar{A}_i \exp(\pm \sigma t/T). \tag{1.3.37}$$

$i = 1, \ldots, n$, and $\varphi(t) =$const. (This solution can be readily obtained using the formulas (1.3.13) and (1.3.15).)

It is interesting to note that for $\Lambda > 0$ the solution (1.3.37) with the sign $''+''$ is an attractor for the solutions (1.3.32), i.e.

$$a_i(t) \sim \bar{A}_i \exp(\sigma t/T), \qquad i = 1, \ldots, n, \tag{1.3.38}$$

and $\varphi(t) \sim$const for $t \to +\infty$. The relation (1.3.38) is the isotropization condition. We note that the solution (1.3.31)-(1.3.35) with $\bar{\alpha}^{n+1} = 0$ was considered previously in [48]. The special case of this solution with $n = 2$ was considered earlier in [248].

The volume scale factor corresponding to (1.3.32) has the form

$$v = \prod_{i=1}^{n} a_i^{d_i} = (\prod_{i=1}^{n} A_i^{d_i}) \sinh(rt/T)/r \tag{1.3.39}$$

It oscillates for negative value of cosmological constant and exponentially increases as $t \to +\infty$ for positive value. For positive A_i and $\mathcal{E}$ the following identity takes place

$$\prod_{i=1}^{n} A_i^{d_i} = \sqrt{\mathcal{E}/|\Lambda|}. \tag{1.3.40}$$

For small time values we have the following asymptotical relations

$$a_i(t) \sim c_i t^{\alpha_i}, \qquad \exp\varphi(t) \sim c_{n+1} t^{\alpha_{n+1}} \tag{1.3.41}$$

as $t \to 0$, $i = 1, \ldots, n$, where

$$\alpha_i = \bar{\alpha}^i + \sigma, \qquad \alpha_{n+1} = \bar{\alpha}^{n+1}, \tag{1.3.42}$$

are Kasner-like parameters, satisfying (see (1.3.34), (1.3.35)) the relations

$$\sum_{i=1}^{n} d_i \alpha_i = \sum_{i=1}^{n} d_i (\alpha_i)^2 + \alpha_{n+1}^2 = 1. \tag{1.3.43}$$

The behaviour of the scale factors near the singularity coincides with that for the case $\Lambda = 0$ [249,250]. For $\alpha_{n+1} = 0$ see also [19].

We note also that in terms of α_i-parameters the solution (1.3.31) - (1.3.35) reads

$$\begin{aligned} a_i(t) &= \bar{A}_i[\sinh(rt/2T)/r]^{\alpha_i}[\cosh(rt/2T)]^{2\sigma-\alpha_i}, && (1.3.44)\\ \exp(\varphi(t)) &= A_{n+1}[\tanh(rt/2T)/r]^{\alpha_{n+1}}. && (1.3.45)\end{aligned}$$

Let us apply these solutions to the case of two spaces ($n = 2$). From (1.3.43) and (1.3.44) we find for the non-exceptional solutions

$$\begin{aligned} {a_1}^{\pm} &= A_1[\sinh(rt/2T)/r]^{\alpha_1^{\pm}}[\cosh(rt/2T)]^{\alpha_1^{\mp}}, && (1.3.46)\\ {a_2}^{\pm} &= A_2[\sinh(rt/2T)/r]^{\alpha_2^{\mp}}[\cosh(rt/2T)]^{\alpha_2^{\pm}}, && (1.3.47)\end{aligned}$$

where

$$\alpha_1^{\pm} = \frac{d_1 \pm \sqrt{R}}{d_1(d_1+d_2)}, \qquad \alpha_2^{\pm} = \frac{d_2 \pm \sqrt{R}}{d_2(d_1+d_2)}, \tag{1.3.48}$$

and

$$R = d_1 d_2[(d_1+d_2)(1-\alpha_3^2) - 1]. \tag{1.3.49}$$

Graphically these solutions are presented in Figs. 21-22.

In the Euclidean case after the Wick rotation ($t \to -it$) we get the following instanton solutions

$$\begin{aligned} g &= dt \otimes dt + \sum_{i=1}^{n} a_i^2(t) g_{(i)}, && (1.3.50)\\ a_i(t) &= \tilde{A}_i[\sinh(ts/2T)/s]^{\alpha_i}[\cosh(ts/2T)]^{2\sigma-\alpha_i}, && (1.3.51)\\ \exp(\varphi(t)) &= \tilde{A}_{n+1}[\tanh(ts/2T)/s]^{\alpha_{n+1}}. && (1.3.52)\end{aligned}$$

where T is defined by (1.3.20), $s = \sqrt{-\Lambda/|\Lambda|}$ and the parameters α satisfy the relations (1.3.43). For $\Lambda < 0$ we have the special solution ($\mathcal{E} = 0$)

$$a_i(t) = \bar{A}_i \exp(\pm\sigma t/T). \tag{1.3.53}$$

We note that for the Euclidean case the scale factors may be obtained from the corresponding Lorentzian ones by the substitution $\Lambda \mapsto -\Lambda$. For $n = 3$, $d_1 = d_2 = d_3 = 1$, $\Lambda > 0$ the special case of the solution (1.3.50) - (1.3.52) was considered in [251]. We note that for $\Lambda < 0$ there are wormhole-like sections of the total metric (1.3.50). This takes place, for example, if $n = 2$, $\alpha_3^2 \leq 1 - d_2^{-1}$, $1 < d_1 < d_2$, (see Fig. 22). In this case the scalar field is real in Euclidean region.

Now, we consider the solutions of the field equations with complex scalar field and real metric. In this case $\mathcal{E}, p^1, \ldots, p^{n-1}$ are real and hence (see (1.3.12)) p^n is either real or pure imaginary. The case of real p^n was considered above.

For pure imaginary p_n we have three subcases: a) $\mathcal{E} > 0$, b) $\mathcal{E} = 0$, c) $\mathcal{E} < 0$. In the first case a) $\mathcal{E} > 0$ after the reparametrization (1.3.19), (1.3.20) we get the solutions (1.3.32)-(1.3.35) with an imaginary value of $\bar{\alpha}^{n+1}$. The cases b) and c): $\mathcal{E} \leq 0$ take place only for $\Lambda > 0$.

Let us consider the case c) $\mathcal{E} < 0$. Here, we have (see (1.3.26), (1.3.27)) imaginary $\bar{\alpha}^k$:

$$\bar{\alpha}^k = i\ \sigma_k, \qquad k = 1, \ldots, n,\ \bar{\alpha}^{n+1} = \sigma_{n+1}. \tag{1.3.54}$$

The solution may be obtained from (1.3.32)-(1.3.35) substituting (1.3.54) and $t/T \mapsto t/T + i\frac{\pi}{2}$:

$$g = -dt \otimes dt + \sum_{i=1}^{n} a_i^2(t) g_{(i)}, \tag{1.3.55}$$

$$a_i(t) = \hat{A}_i [\cosh(t/T)]^{\sigma} [f(t/2T)]^{\sigma_i}, \tag{1.3.56}$$

$$\varphi(t) = c + 2i\sigma_{n+1} \arctan e^{-t/T}, \tag{1.3.57}$$

where $c, \hat{A}_i \neq 0$ are constants, $i = 1, \ldots, n$, $\sigma = (D-1)^{-1}$, T is defined in (1.3.20), $\Lambda > 0$ and the real parameters σ_I satisfy the relations

$$\sum_{i=1}^{n} d_i \sigma_i = 0, \tag{1.3.58}$$

$$-\sum_{i=1}^{n} d_i \sigma_i^2 + \sigma_{n+1}^2 = (D-2)/(D-1). \tag{1.3.59}$$

Here

$$f(x) \equiv [\tanh(x + i\frac{\pi}{4})]^i = \exp(-2 \arctan e^{-2x}) \tag{1.3.60}$$

is smooth monotonically increasing function bounded by its asymptotics: $e^{-\pi} < f(x) < 1$; $f(x) \to 1$ as $x \to +\infty$ and $f(x) \to e^{-\pi}$ as $x \to -\infty$ (see Fig. 3). The solution (1.3.55)-(1.3.59) may be also obtained from formulas (1.3.13), (1.3.14). The relation between the harmonic and the proper times (1.3.21) is modified for our case $\mathcal{E} < 0$

$$\cosh(t/T) = 1/\sin(q\sqrt{2|\mathcal{E}|}(\tau - \tau_0)). \tag{1.3.61}$$

For the volume scale factor we have

$$v = \prod_{i=1}^{n} a_i^{d_i} = (\prod_{i=1}^{n} \hat{A}_i^{d_i}) \cosh(t/T). \tag{1.3.62}$$

The scalar field varies $\varphi(t)$ varies from $c + i\pi\sigma_{n+1}$ to c as t varies from $-\infty$ to $+\infty$. The solution (1.3.55)-(1.3.59) is non-singular for $t \in (-\infty, +\infty)$. Any scale factor $a_i(t)$ has a minimum for some t_{0i} and

$$a_i(t) \sim A_i^{\pm} \exp(\sigma|t|/T), \tag{1.3.63}$$

for $t \to \pm\infty$.

The Lorentzian solutions considered above have also Euclidean analogues for $\Lambda < 0$

$$g = dt \otimes dt + \sum_{i=1}^{n} a_i^2(t) g_{(i)}, \tag{1.3.64}$$

$$a_i(t) = \hat{A}_i [\cosh(t/T)]^{\sigma} [f(t/2T)]^{\sigma_i}, \tag{1.3.65}$$

$$\varphi(t) = c + 2i\sigma_{n+1} \arctan e^{-t/T}, \tag{1.3.66}$$

with the parameters σ_I satisfying the relations (1.3.58)-(1.3.59). This solution may be iterpreted as classical Euclidean wormhole solution. An interesting special case of solution (1.3.64)-(1.3.66) occurs for $\sigma_i = 0$, $i = 1, \ldots, n$, (this corresponds to $p^i = 0$)

$$a_i(t) = \hat{A}_i[\cosh(t/T)]^{\sigma}, \tag{1.3.67}$$

$$\varphi(t) = c \pm 2iq^{-1} \arctan e^{-t/T}. \tag{1.3.68}$$

All scale factors (1.3.67) have a minimum at $t = 0$ and are symmetric with respect to time inversion: $t \mapsto -t$. We want to stress here that wormhole soluions take place only in the presence of an imaginary scalar field in the Euclidean region. Analytic continuation of the solutions (1.3.67), (1.3.68) into the Lorentzian region leads to real geometry and real scalar field there.

Quantum wormholes

The model introduced above leads to the WDW equation (1.2.18)

$$-2\hat{H}\Psi \equiv \left[-\frac{\partial}{\partial z^0}\frac{\partial}{\partial z^0} + \sum_{i=1}^{n} \frac{\partial}{\partial z^i}\frac{\partial}{\partial z^i} - 2\Lambda \exp(2qz^0) \right] \Psi = 0. \tag{1.3.69}$$

We are seeking the solution of (1.3.69) in the form

$$\Psi(z) = \exp(i\vec{p}\vec{z})\Phi(z^0), \tag{1.3.70}$$

where $\vec{p} = (p^1, \ldots, p^n)$ is a constant vector (generally from C^n), $\vec{z} = (z^1, \ldots, z^{n-1}, z^n = \varphi)$, $\vec{p}\vec{z} \equiv \sum_{i=1}^n p_i z^i$ and $p_i = \sum_{j=1}^n \eta_{ij} p^j = p^i$. The substitution of (1.3.70) into (1.3.69) gives

$$[-\frac{1}{2}(\frac{\partial}{\partial z^0})^2 + V_0(z^0)]\Phi = \mathcal{E}\Phi, \tag{1.3.71}$$

where $\mathcal{E} = \frac{1}{2}\vec{p}\vec{p}$ and $V_0(z^0) = -\Lambda e^{2qz^0}$. The potential $V_0(z^0)$ is plotted on fig. 24 and fig. 25 for $\Lambda > 0$ and $\Lambda < 0$ respectively. The clasiically allowed (Lorentzian) and forbidden (Euclidean) regions are shown there with respect to the energy levels $\mathcal{E}$. Solving (1.3.71), we get

$$\Phi(z^0) = B_{i\sqrt{2\mathcal{E}}/q}(\sqrt{-2\Lambda}q^{-1}e^{qz^0}), \tag{1.3.72}$$

where $i\sqrt{2\mathcal{E}}/q = i|\vec{p}|/q$, and $B = I, K$ are modified Bessel functions. We note, that

$$v = \exp qz^0 = \prod_{i=1}^{n} a_i^{d_i} \tag{1.3.73}$$

is proportional to the spatial volume of the universe.

The general solution of Eq. (1.3.69) has the following form

$$\Psi(z) = \sum_{B=I,K} \int d^n\vec{p}\, C_B(\vec{p}) e^{i\vec{p}\vec{z}} B_{i|\vec{p}|/q}(\sqrt{-2\Lambda}q^{-1}e^{qz^0}), \tag{1.3.74}$$

where functions C_B ($B = I, K$) belong to an appropriate class. Similar solutions were found for the two-component model ($n = 2$) and $\Lambda > 0$ in [252].

The solutions (1.3.70) are the eigenstates of the quantum-mechanical operators $\hat{\Pi}_{z^i} = -(i/N)\partial/\partial z^i, i = 1, \ldots, n$ with the eigenvalues $(1/N)p_i$ where $N = 1$ for the Lorentzian space-time region and $N = i$ for the Euclidean one.

Due to the well known time problem in quantum cosmology the WDW equation is not really the Schrödinger equation. There is no generally accepted procedure to overcome this problem but for our particular model we can introduce some time coordinate into the quantum equations in analogy to [20].

We split the WDW operator $\hat{H}$ (1.3.69) into two parts

$$\hat{H} = -\hat{H}_0 + \hat{H}_1, \tag{1.3.75}$$

where

$$\hat{H}_0 = -\frac{1}{2}\frac{\partial^2}{\partial {z^0}^2} - \Lambda e^{2qz^0}, \tag{1.3.76}$$

and

$$\hat{H}_1 = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial {z^i}^2}. \tag{1.3.77}$$

Then the WDW equation (1.3.69) becomes

$$\hat{H}_0\Psi = \hat{H}_1\Psi. \tag{1.3.78}$$

Applying $\hat{H}_1$ to the wave function (1.3.70) one gets

$$\hat{H}_1\Psi = \mathcal{E}\Psi. \tag{1.3.79}$$

Now, we take $\mathcal{E}$ to be real. Then, equation (1.3.79) shows that $\mathcal{E}$ can be treated as the energy of the subsystem $\hat{H}_1$ and equation (1.3.79) becomes the Schrödinger equation. From this point of view Ψ gives the stationary states of the subsystem described by the wave equation (in Lorentzian region)

$$i\frac{\partial\tilde{\Psi}}{\partial\tau} = \hat{H}_1\tilde{\Psi}, \tag{1.3.80}$$

where

$$\tilde{\Psi} = e^{-i\mathcal{E}\tau}\Psi. \tag{1.3.81}$$

It can be easily seen that the wave equation

$$i\frac{\partial\tilde{\Psi}}{\partial\tau} = \hat{H}_0\tilde{\Psi} \tag{1.3.82}$$

is reduced to equation (1.3.71).

In the semiclassical limit for the wave function (1.3.81) equations (1.3.80) and (1.3.82) are reduced to the classical equations (1.3.10), (1.3.11). Indeed, the wave function (1.3.81) can be rewritten in the form

$$\tilde{\Psi} = e^{-i\mathcal{E}\tau}e^{iS_1}\Phi(z^0) \tag{1.3.83}$$

with $S_1 = \sum_{i=1}^{n} p_i z^i$. In the semiclassical limit the wave function $\Phi(z^0)$ takes the form

$$\Phi(z^0) = C(z^0)e^{iS_0} \tag{1.3.84}$$

with $C(z^0)$ being a slowly varying function and $S_0(z^0)$ being a rapidly varying phase. Time is defined in the semiclassical limit as an affine parameter along integral curves

$$\frac{\partial}{\partial \tau} = \sum_{i=0}^{n} \frac{\partial(S_0 + S_1)}{\partial z_i} \frac{\partial}{\partial z^i} \tag{1.3.85}$$

where $z_0 = -z^0$, $z_i = z^i$, $i = 1, \ldots, n$, As result we find the equations $\dot{z}^i = p^i$, $i = 1, \ldots, n$, and these coincide with the classical equations (1.3.10). For this reason we used the same notation for the constants of integration in (1.3.10) and for the momenta in the wave function (1.3.70). The velocity along z^0 is found to be $\dot{z}_0 = -\dot{z}^0 = \partial S_0 / \partial z^0$. Using this relation and putting the wave function (1.3.83), (1.3.84) into equation (1.3.82) we reproduce the classical equation (1.3.11).

As shown above the parameter $\mathcal{E}$ can be interpreted as energy. So we may treat the state $\mathcal{E} = 0$ as the ground state of the system. The demand of reality of the geometry leads to real momenta p^i $(i = 1, \ldots, n-1)$ in the Lorentzian region. The scalar field can be real or imaginary there. In the ground state we put all momenta p^i $(i = 1, \ldots, n)$ equal to zero and the ground state wave function reads

$$\Psi_0 = B_0 \left(\sqrt{-2\Lambda} q e^{qz^0} \right). \tag{1.3.86}$$

It is interesting to note that Ψ_0 is invariant with respect to the rotation group O(n) in the space of vectors $\vec{z} = (z^1, \ldots, z^n)$.

In eq. (1.3.86) B_0 denotes the Bessel functions of order zero. A particular solution may be specified by boundary conditions. For example, quantum wormhole boundary conditions were presented in the Introduction. Among the different types of boundary conditions for wave functions describing the universe the most popular is the Hartle-Hawking (HH) boundary condition [213]. According to the HH proposal the ground-state wave function of the universe Ψ_0^{HH} is given by a path integral over all compact Euclidean geometries and the regular matter fields:

$$\Psi_0^{(HH)} = \int d[g]d[\varphi]e^{-I_E}, \tag{1.3.87}$$

where I_E is the Euclidean action. For our model the Euclidean action in harmonic time gauge and in z-coordinates reads

$$I_E = \frac{1}{2} \int_{\tau^*}^{\tau} d\tau \left[-(\dot{z}^0)^2 + \sum_{i=1}^{n} (\dot{z}^i)^2 + 2\Lambda e^{2qz^0} \right] - \frac{1}{2} \frac{\dot{v}}{v} \Big|_{\tau^*} \tag{1.3.88}$$

where v denotes the spacial volume (1.3.73) up to a numerical factor. The upper limit τ corresponds to the boundary of the D-dimensional manifold, where the z^i $(i = 0, \ldots, n)$ have values indicated by the arguments of $\Psi_0^{(HH)}$. The lower limit τ^* corresponds to the point where the D-dimensional manifold closes in a smooth way. The origin of the second term in (1.3.88) was explained in detail, e.g. by Louko [240]. In the semiclassical limit the wave function is given by

$$\Psi_0^{(HH)} \approx e^{-I_E^{cl}}, \tag{1.3.89}$$

where I_E^{cl} should be calculated on the classical Euclidean solutions with boundary conditions defined by the concrete scheme of geometry closing at $\tau = \tau^*$.

Now, we find the relationship between the HH wave function (1.3.87), (1.3.89) and our ground-state wave functions (1.3.86). Let us first consider the case of a negative cosmological constant $\Lambda < 0$. Then, we have the classical Euclidean equations

$$\dot{z}^0 = \pm\sqrt{2|\Lambda|}e^{qz^0}. \tag{1.3.90}$$

The spacial volume v may be presented in the form

$$v = e^{qz^0} = -\left(q\sqrt{2|\Lambda|}\ \tau\right)^{-1}, \qquad -\infty < \tau < 0. \tag{1.3.91}$$

Formula (1.3.91) shows that the geometry closes at the harmonic time $\tau \to -\infty$. It is easy to see from (1.3.91) that the second term in (1.3.88) contributes nothing to I_E. So, on these classical solutions the Euclidean action I_E^{cl} reads

$$I_E^{cl} = \frac{1}{q^2}\frac{1}{\tau} = \frac{-\sqrt{2|\Lambda|}}{q}e^{qz^0} \tag{1.3.92}$$

and we get the semiclassical HH wave function (1.3.89)

$$\Psi_0^{(HH)} \approx \exp\left(\frac{\sqrt{2|\Lambda|}}{q}e^{qz^0}\right) = \exp\left(\frac{\sqrt{2|\Lambda|}}{q}\prod_{i=1}^{n} a_i^{d_i}\right). \tag{1.3.93}$$

Eqn. (**??**) shows that (for the class of real Euclidean geometries)

$$\Psi_0^{(HH)} \to +\infty, \qquad z^0 \to +\infty. \tag{1.3.94}$$

This conditions provides us with the possibility to chose a solution of equation (1.3.69) corresponding to the HH ground state:

$$\Psi_0^{(HH)} = I_0\left(\frac{\sqrt{2|\Lambda|}}{q}e^{qz^0}\right). \tag{1.3.95}$$

The vacuum solution (1.3.95) has the asymptotic form

$$\Psi_0^{(HH)} \to \exp\left(\frac{\sqrt{2|\Lambda|}}{q}e^{qz^0}\right), \qquad z^0 \to +\infty, \tag{1.3.96}$$

which coincides with (1.3.93).

A similar procedure can be performed for a positive cosmological constant $\Lambda > 0$ (see, e.g. [20]). In this case the classical Euclidean equation is

$$\left(\frac{\dot{v}}{v}\right)^2 + 2q^2\Lambda v^2 = 0 \tag{1.3.97}$$

and gives an imaginary geometry. This reflects the fact that the geometry should be purely Lorentzian in the case $\Lambda > 0$ for $\mathcal{E} \geq 0$. The action I_E^{cl} (1.3.88) is indefinite in this case. We can avoid this problem if we perform the analytic continuation $v \to iv$. After this continuation the action (1.3.88) is formally reduced to the action in the case $\Lambda < 0$. Again, it leads to the following asymptotic for the HH wave function: $\Psi_0^{(HH)} \to +\infty$, $v \to +\infty$, which shows that the wave function $I_0[(\sqrt{2\Lambda}/q)v]$ is a solution of eq. corresponding to the HH ground state for the class of Euclidean solutions considered here. Thus, after analytic continuation to real values of v the vacuum state corresponding to the HH boundary condition is

$$\Psi_0^{(HH)} = J_0\left(\frac{\sqrt{2\Lambda}}{q}v\right). \tag{1.3.98}$$

This solution has the asymptotic form

$$\Psi_0^{(HH)} \to \cos\left(\frac{\sqrt{2\Lambda}}{q}v\right) = \cos\left(\frac{\sqrt{2\Lambda}}{q}\prod_{i=1}^{n} a_i^{d_i}\right), \qquad v \to +\infty. \tag{1.3.99}$$

For the Bianchi I universe $(n = 3, d_1 = d_2 = d_3 = 1)$ eq. (1.3.99) is reduced to

$$\Psi_0^{(HH)} \sim \cos\left(2\sqrt{\Lambda/3}\,a_1 a_2 a_3\right), \qquad v \to +\infty. \tag{1.3.100}$$

Similar results for the Bianchi I universe were obtained earlier in papers by Laflamme and Louko [239-240].

Now, let us turn to quantum wormholes. We restrict our consideration to real values of p_i. This corresponds to real geometries in the Lorentzian region. In this case we have $\mathcal{E} \geq 0$.

If $\Lambda > 0$ the wave function Ψ (1.3.70) is not exponentially damped when $v \to \infty$, i.e. the condition (i) for quantum wormholes (see the Introduction) is not satisfied. It oscillates and may be interpreted as corresponding to the classical Lorentzian solution.

For $\Lambda < 0$, the wave function (1.3.70) is exponentially damped for large v only, when $B = K$ in (1.3.72). But in this case the function Φ oscillates an infinite number of times, when $v \to 0$. So, the condition (ii) is not satisfied. The wave function describes the transition between Lorentzian and Euclidean regions.

The functions

$$\Psi_{\vec{p}}(z) = e^{i\vec{p}\vec{z}} K_{i|\vec{p}|/q}(\sqrt{-2\Lambda}q^{-1}e^{qz^0}), \tag{1.3.101}$$

may be used for constructing quantum wormhole solutions. Like in [20,231,246] we consider the superpositions of singular solutions

$$\hat{\Psi}_{\lambda,\vec{n}}(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dk \Psi_{qk\vec{n}}(z)e^{-ik\lambda}, \tag{1.3.102}$$

where $\lambda \in \mathrm{IR}$, $\vec{n}$ is a unit vector $(\vec{n}^2 = 1)$ and the quantum number k is connected with the quantum number $\mathcal{E} = \frac{1}{2} \mid \vec{p} \mid^2$ by the formula $2\mathcal{E} = q^2k^2$. The calculation gives

$$\hat{\Psi}_{\lambda,\vec{n}}(z) = exp[-\frac{\sqrt{-2\Lambda}}{q}e^{qz^0}\cosh(\lambda - q\vec{z}\vec{n})]. \tag{1.3.103}$$

It is not difficult to verify that the formula (1.3.103) leads to solutions of the WDW equation (1.3.69), satisfying the quantum wormholes boundary conditions.

We also note that the functions

$$\Psi_{m,\vec{n}} = H_m(x^0)H_m(x^1)\exp[-\frac{(x^0)^2+(x^1)^2}{2}], \tag{1.3.104}$$

where

$$x^0 = (2/q)^{1/2}(-2\Lambda)^{1/4}\exp(qz^0/2)\cosh(\frac{1}{2}q\vec{z}\vec{n}), \tag{1.3.105}$$

$$x^1 = (2/q)^{1/2}(-2\Lambda)^{1/4}\exp(qz^0/2)\sinh(\frac{1}{2}q\vec{z}\vec{n}), \tag{1.3.106}$$

$m = 0, 1, \ldots,$ are also solutions of the WDW equation with the quantum wormhole boundary conditions. Solutions of such type were previously considered in [20,34,231]. They are called discrete spectrum quantum wormholes.

It is clear from the equation (1.3.71) and fig. 4 that in the case $\Lambda < 0$ a Lorentzian region exists as well as an Euclidean one for $\mathcal{E} > 0$. In the case $\Lambda > 0$ only the Lorentzian region occurs for $\mathcal{E} \geq 0$ and for $\mathcal{E} < 0$ both of these regions exist (see fig. 25). The condition $\mathcal{E} < 0$ leads for pure gravity to a complex geometry in the Lorentzian region. We can avoid this problem by the help of a free scalar field, because in this case $2\mathcal{E} = \sum_{i=1}^{n-1} p_i^2 + p_n^2$ and we can achieve $\mathcal{E} < 0$ for real $p_i (i = 1, \ldots, n-1)$ and imaginary p_n, i.e. for an imaginary scalar field in the Lorentzian region. The wave functions (1.3.70), (1.3.72) with $\Lambda > 0$ and $\mathcal{E} < 0$ describe the transitions between Euclidean and Lorentzian regions, i.e. tunneling universes.

1.4. CLASSICAL WORMHOLES, FINE TUNING OF PARAMETERS

Classical wormhole solutions exist in our model also for another interesting case. This is the case with spontaneous compactification of the internal dimensions. Let the factor space M_1 be our dynamical external space. All the other factor spaces $M_i (i = 2, \ldots, n)$ are considered as internal and static. They should be compact and the internal dimensions have the size of order of Planck's length $L_{PL} \sim 10^{-33}$ cm. The scale factors of the internal factor spaces should be constant: $a_i = e^{\beta^i} \equiv a_{(0)i} (i = 2, \ldots, n)$. It is not difficult to show that in the case of fine tuning of the parameters due to

$$\frac{\theta_i}{d_i a_{(0)i}^2} = \frac{2\Lambda}{D-2} \equiv C_0, \qquad i = 2, \ldots, n, \tag{1.4.1}$$

all dynamical equations (1.2.10) are reduced to one for the scale factor $a_1 = e^{\beta^1}$ and this equation reads

$$\ddot{\beta}^1 = -e^{2\sum\limits_1^n d_k\beta^k}\left[\frac{\theta_1}{d_1}e^{-2\beta^1} - C_0 - \frac{1}{d_1-1}\left(\frac{1}{d_1}\frac{\partial \tilde{U}}{\partial \beta^1} + 2\tilde{U}\right)\right]. \tag{1.4.2}$$

The constraint (1.2.12) has form

$$d_1(d_1-1)\dot{\beta}^{1^2} = \dot{\varphi}^2 - \theta_1 e^{2(d_1-1)\beta^1} e^{2\sum\limits_2^n d_k\beta^k} + e^{2d_1\beta^1} e^{2\sum\limits_2^n d_k\beta^k}\left[2\tilde{U} + \right.$$

$$+C_0(d_1-1)]\,. \tag{1.4.3}$$

From the equations (1.4.1) it follows that all internal spaces should be non-Ricci-flat and $\mathrm{sign}\,\theta_i = \mathrm{sign}\,\Lambda$, $(i = 2,\ldots,n)$. We remind here that overdot denotes differentiation with respect to the harmonic time τ [15]. The minimally coupled scalar field has the specific potential $\tilde{U}(\beta,\varphi) = U(\varphi)\exp\left[-2\sum_{i=1}^{n} d_i\beta^i\right]$. For this potential it is easy to get the first integral of equation (1.2.11)

$$\dot{\varphi}^2 + 2U(\varphi) = \nu^2 \;\; = \;\; const. \tag{1.4.4}$$

This gives

$$\varphi = \nu\tau \;\; + \;\; const \tag{1.4.5}$$

for $U(\varphi) = 0$ and

$$\varphi = \varphi_0 \cos m(\tau - \tau_0) \tag{1.4.6}$$

for $U(\varphi) = \frac{m^2\varphi^2}{2}$, where $\nu = m\varphi_0$ here.

Let us investigate the model where our external space M_1 is Ricci-flat, i.e. $\theta_1 = 0$. Then we can rewrite the equation (1.4.3) as follows

$$(\dot{\beta}^1)^2 = \tilde{\nu}^2 + \tilde{\Lambda}e^{2d_1\beta^1}, \tag{1.4.7}$$

where the constants are

$$\tilde{\nu}^2 = \frac{\nu^2}{d_1(d_1-1)} \tag{1.4.8}$$

and

$$\tilde{\Lambda} = \frac{2\Lambda}{d_1\left(\sum_{k=1}^{n} d_k - 1\right)} \prod_{k=2}^{n} a_{(0)k}^{2d_k}. \tag{1.4.9}$$

It is clear from equation (1.4.7) that the dynamical behavior of the scale factor a_1 depends on the signs of ν^2 and Λ . If $\Lambda > 0$ and $\nu^2 \geq 0$ then a_1 expands from zero to infinity. For $\Lambda > 0$ and $\nu^2 < 0$, a_1 has the turning point at some minimum and this case may be realized for an imaginary scalar field in the Lorentzian region. For a real scalar field in the Lorentzian region (i.e. $\nu^2 > 0$) and $\Lambda < 0$ the scale factor a_1 expands from zero to its maximum and after the turning point shrinks again to zero. For the latter case the solution has a continuation into the Euclidean region with the topology of a wormhole, that means, two asymptotic regions which are connected with each other through a throat.

Let us investigate the case with $\Lambda < 0$ in more details. As $\mathrm{sign}\,\Lambda = \mathrm{sign}\,\theta_i$ $(i = 2,\ldots,n)$ then for $\Lambda < 0$ the curvatures $\theta_i < 0$ also. (As a special case the internal spaces M_i $(i = 2,\ldots,n)$ may be compact spaces of constant negative curvature [247].) The solution of equation (1.4.7) (the Lorentzian region) has the form

$$a_1(\tau) = \left[\tilde{\nu}^2/|\tilde{\Lambda}|\right]^{1/2d_1}\left[\cosh d_1\tilde{\nu}\tau\right]^{-1/d_1}\;, \qquad -\infty < \tau < +\infty. \tag{1.4.10}$$

The synchronous time t and the harmonic time τ are connected by the differential equation

$$e^{\gamma(\tau)}d\tau = dt, \tag{1.4.11}$$

where

$$\gamma(\tau) = \sum_{i=1}^{n} d_i\beta^i. \tag{1.4.12}$$

It is not difficult to get the connection

$$\cosh(d_1\tilde{\nu}\tau) = \left[\cos\left(\sqrt{C_0 d_1}\ t + \ const\right)\right]^{-1}. \tag{1.4.13}$$

With the help of this connection we obtain the expression for the scale factor a_1 with respect to the synchronous time

$$a_1(t) = \left[\tilde{\nu}^2/|\tilde{\Lambda}|\right]^{1/2d_1}\left[\sin\sqrt{C_0 d_1}\ t\right]^{1/d_1}, \quad 0 \le t \le \frac{\pi}{\sqrt{C_0 d_1}}, \tag{1.4.14}$$

where the constant in (1.4.13) was fixed by condition $a(t=0)=0$. For $t \to 0$ we have $a_1 \sim t^{1/d_1}$. Thus the external space M_1 has the behavior of a FRW-universe filled with radiation for $d_1 = 2$ and with ultrastiff matter for $d_1 = 3$. The Lorentzian metric (1.2.1) in synchronous time gauge reads as

$$g = -dt \otimes dt + a_1^2(t)g_{(1)} + \sum_{i=2}^{n} a_{(0)i}^2 g_{(i)} \tag{1.4.15}$$

with a_1 given by (1.4.14).

Our next step is to get the wormhole-type solution for M_1 in the Euclidean region. The transition into the Euclidean space is performed by the Wick rotation $t \to -it$. The exact form of the transformation from the Lorentzian time t_L to the Euclidean "time" t_E can be obtained demanding the existence of wormholes being symmetric with respect to the throat (see Zhuk in [225]). In what follows, for the expression (1.4.14) we should perform the analytic continuation $t_L = \frac{\pi}{2\sqrt{C_0 d_1}} - it_E$. This gives us

$$a_1(t) = \left[\tilde{\nu}^2/|\tilde{\Lambda}|\right]^{1/2d_1}\left[\cosh\sqrt{C_0 d_1}t\right]^{1/d_1}, \quad -\infty < t < +\infty. \tag{1.4.16}$$

Of course, this formula can be obtained also as a solution to the Euclidean analog of the equations (1.4.3), (1.4.7) for an imaginary scalar field in the Euclidean region. The metric of the Euclidean region is given by

$$g = dt \otimes dt + a_1^2(t)g_{(1)} + \sum_{i=2}^{n} a_{(0)i}^2 g_{(i)}, \tag{1.4.17}$$

where $a_1(t)$ is described by (1.4.16). Thus, in Euclidean space we have two asymptotic regions $t \to \pm\infty$ connected through a throat of the size $\left[\tilde{\nu}^2/|\tilde{\Lambda}|\right]^{1/2d_1}$ and this object is a wormhole by definition.

It is clear that the Lorentzian solution (1.4.14) and its Euclidean analog (1.4.16) take place only in the presence of a real scalar field in the Lorentzian region (i.e $\nu^2 > 0$) or equivalently an imaginary scalar field in the Euclidean region.

1.5. CONCLUSIONS

In this section we investigated multidimensional cosmological models with $n(n>1)$ Einstein spaces M_i in the presence of the cosmological constant Λ and a homogeneous minimally coupled scalar field $\varphi(t)$ as a matter source. The problem was to find classical and quantum wormhole solutions. Classical wormholes are solutions of the classical Einstein equations describing Riemannian metrics with two large regions joined by a throat. Quantum wormholes are solutions of the Wheeler-DeWitt (WDW) equation with the proper boundary conditions proposed by Hawking and Page [34].

The model was investigated where one of the factor spaces, say M_1, is Ricci-flat. In the case when all other factor spaces $M_i, i=2,\dots,n$, are Ricci-flat too, the classical Einstein as well as the WDW equations are integrable. For a negative cosmological constant $\Lambda<0$ quantum wormhole solutions were constructed. These solutions exist for pure gravity as well as for the model with a free minimally coupled scalar field. Classical wormhole solutions exist in the Euclidean region for $\Lambda<0$ in the presence of an imaginary as well as real scalar field.

Classical wormhole solutions were also obtained in models with spontaneous compactification. In this case the Ricci-flat factor space M_1 was considered as our external dynamical space. All other factor spaces $M_i, i=2,\dots,n$ are static with constant scale factors $a_{(0)i}=const$ and all of them are fine tuned to each other and to the cosmological constant: $\frac{\theta_i}{d_i a_{(0)i}^2}=\frac{\theta_k}{d_k a_{(0)k}^2}=\frac{2\Lambda}{D-2}, \qquad i,k=2,\dots,n.$

As in the previous model, wormhole solutions exist for a negative cosmological constant $\Lambda<0$. But there are important differences. Firstly, all inner spaces $M_i, i=2,\dots,n$, are non-Ricci-flat and have negative curvature. Secondly, the wormhole solution for the later case exists only in the presence of an imaginary sclar field in the Euclidean region. Thirdly, it seems hardly to be possible in the case $\theta_2,\dots,\theta_n\neq 0$ to integrate the Einstein equations as well as the WDW equation without the demand of spontaneous compactification with fine tuning.

In models with one scale factor having a turning point (at the minimum) the production of the Lorentzian space-time is treated as a quantum tunneling process [253] ("birth from nothing"). The universe appears spontaneously going through the potential barrier with size equal to the size of the Lorentzian universe at the turning point. In our case of multidimensional models this kind of interpretation becomes more complicated. It follows from (1.3.56) that the factor spaces M_i in general reach their minimum expansion positions at different times. The "birth from nothing" for each factor space takes place at a different value of time. If the difference between these events goes to infinity the extra dimensions are in the classically forbidden region forever. This interpretation is in the spirit of the Rubakov-Shaposhnikov idea [254] stating that extra dimensions are unobservable because they are hidden from us by a potential barier.

2. Quantum Multicomponent Model with Perfect Fluid [37]

2.1. Introdution

In this section we shall study the quantum behaviour of the model described in the section 1 of chapter I and analyse the quantum wormhole solutions of the Wheeler-DeWitt (WDW) equation.

The WDW equation for the model in harmonic time gauge reads as follows:

$$(-\frac{1}{2\mu}G^{ij}\partial_i\partial_j + \mu V)\Psi = 0, \tag{2.1.1}$$

where $\Psi = \Psi(x)$ is "the wave function of the Universe", V is the potential (I-1.2.5), $\partial_i = \partial/\partial x^i$ and G^{ij} are defined in (I-1.2.13). The relation (2.1.1) is a result of a trivial quantization of the zero energy constraint (I-1.2.16), written in the form $\mu E = 0$. Here μ is a fundamental quantum parameter of the theory.

In f-gauge (I-1.2.19) the WDW equation should be written in the conformally covariant form [15] (such form of the WDW equation was discussed earlier by Misner [255])

$$(-\frac{1}{2\mu}\Delta[e^{2f}G] + \frac{a_n}{\mu}R[e^{2f}G] + e^{-2f}\mu V)\Psi^f = 0, \tag{2.1.2}$$

where $\Delta[\hat{G}]$ and $R[\hat{G}]$ are the Laplace-Beltrami operator and the scalar curvature of $\hat{G}$ respectively, $a_n = (n-2)/8(n-1)$ and

$$\Psi^f = \exp[(2-n)f/2]\Psi. \tag{2.1.3}$$

Without loss of generality we put $\mu = 1$ below.

2.2. One-component matter

Here we find the quantum analogues of the classical solutions from section I-1.3, i.e. we integrate the WDW equation

$$(-\frac{1}{2}\eta^{ab}\frac{\partial}{\partial z^a}\frac{\partial}{\partial z^b} + V_A)\Psi = 0. \tag{2.2.1}$$

with the potential (I-1.3.8-10). We note, that the WDW equation for 1-component model with n Ricci-flat spaces was considered previously in [17].

a) $u^2 < 0$. In this case the WDW equation (2.2.1) reads

$$[\frac{\partial}{\partial z^0}\frac{\partial}{\partial z^0} - \sum_{i=1}^{n-1}\frac{\partial}{\partial z^i}\frac{\partial}{\partial z^i} + 2\kappa^2 A\exp(2qz^0)]\Psi = 0. \tag{2.2.2}$$

We are seeking solutions of (2.2.2) in the following form

$$\Psi(z) = \exp(i\vec{p}\vec{z})\Phi(z^0), \tag{2.2.3}$$

where $\vec{p} = (p_1, \ldots, p_{n-1})$ is a constant vector (generally from C^{n-1}), $\vec{z} = (z^1, \ldots, z^{n-1})$, $\vec{p}\vec{z} \equiv \sum_{i=1}^{n-1} p_i z^i$. The substitution of (2.2.3) into (2.2.2) gives

$$[-(\frac{\partial}{\partial z^0})^2 - 2\kappa^2 A\exp(2qz^0)]\Phi = 2\mathcal{E}\Phi, \tag{2.2.4}$$

where $2\mathcal{E} = \sum_{i=1}^{n-1} p_i^2$. Solving (2.2.4), we get two linearly independent solutions

$$\Phi(z^0) = B_\nu(\sqrt{-2\kappa^2 A}q^{-1}e^{qz^0}), \tag{2.2.5}$$

where $\nu = i\sqrt{2\mathcal{E}}/q = i|\vec{p}|/q$, and $B_\nu = I_\nu, K_\nu$ is modified Bessel function. We note, that

$$v = \exp qz^0 = \exp(\frac{1}{2}u_i x^i) = \prod_{i=1}^{n} a_i^{u_i/2} \tag{2.2.6}$$

is a natural scale factor for the model ($a_i = e^{x^i}$).

The general solution of eq. (2.2.2) has the following form

$$\Psi(z) = \sum_{B=I,K} \int d^{n-1}\vec{p}\, C_B(\vec{p})\Psi^B_{\vec{p}}(z), \tag{2.2.7}$$

where

$$\Psi^B_{\vec{p}}(z) = e^{i\vec{p}\vec{z}} B_{i|\vec{p}|/q}(\sqrt{-2\kappa^2 A}q^{-1}e^{qz^0}), \tag{2.2.8}$$

and functions C_B ($B = I, K$) belong to an appropriate class of (generalized) functions.

b) $u^2 > 0$. In this case the WDW equation (2.2.1) reads

$$[-\frac{\partial}{\partial z^1}\frac{\partial}{\partial z^1} + \frac{\partial}{\partial z^0}\frac{\partial}{\partial z^0} - \sum_{i=2}^{n-1}\frac{\partial}{\partial z^i}\frac{\partial}{\partial z^i} + 2\kappa^2 A\exp(2qz^1)]\Psi = 0. \tag{2.2.9}$$

An analogous consideration in this case gives the general solution (2.2.7) with

$$\Psi^B_{\vec{p}}(z) = e^{i\vec{p}\vec{z}} B_{i\nu(\vec{p})}(\sqrt{2\kappa^2 A}q^{-1}e^{qz^0}). \tag{2.2.10}$$

Here $\vec{p} = (p_0, p_2, \ldots, p_{n-1})$, $\vec{z} = (z^0, z^2, \ldots, z^{n-1})$, $\nu(\vec{p}) = i\sqrt{2\mathcal{E}}/q$, and $2\mathcal{E} = p_0^2 - \sum_{i=2}^{n-1} p_i^2$.

c) $u^2 = 0$ for $u \neq 0$ the WDW equation reads

$$[-4\partial_+\partial_- + + \sum_{i=1}^{n-1}(\frac{\partial}{\partial z^i})^2 - 2\kappa^2 A\exp(z^+)]\Psi = 0, \tag{2.2.11}$$

where $z^\pm = z^0 \pm z^1$, $\partial_\pm = \partial/\partial z^\pm$. The substitution

$$\Psi(z) = \exp(i\vec{p}\vec{z})\Phi(z^+, z^-), \tag{2.2.12}$$

with $\vec{p} = (p_2, \ldots, p_{n-1})$, $\vec{z} = (z^2, \ldots, z^{n-1})$ entails

$$[4\partial_+\partial_- + 2\mathcal{E} + 2\kappa^2 A\exp(z^+)]\Phi = 0, \tag{2.2.13}$$

where $2\mathcal{E} = \sum_{i=2}^{n-1} p_i^2$. Introducing new variables u^0, u^1, where $u^0 \pm u^1 = u^\pm$ and

$$u^+ = 2\mathcal{E}z^+ + 2\kappa^2 A\exp(z^+), \qquad u^- = z^- \tag{2.2.14}$$

we get the Klein-Gordon equation for Φ with $m^2 = 1$

$$((\frac{\partial}{\partial u^0})^2 - (\frac{\partial}{\partial u^1})^2 + 1)\Phi = 0. \tag{2.2.15}$$

It is quite obvious how to write the general solution of (2.2.15).

Quantum wormholes. In the case a) $u^2 < 0$ for $A < 0$ there exist the so-called quantum wormhole solutions of the WDW equation [34]. We present here a continuous spectrum family of these solutions. The wave functions are following

$$\hat{\Psi}_{\lambda,\vec{n}}(z) = exp[-q^{-1}\sqrt{-2\kappa^2 A}e^{qz^0}\cosh(\lambda - q\vec{z}\vec{n})]. \tag{2.2.16}$$

where $\lambda \in R$ and $\vec{n}$ is unit vector: $(\vec{n})^2 = 1$ ($\vec{n} \in S^{n-1}$). These solutions are related with the solutions (2.2.8) (with $B = K$) by the formula

$$\hat{\Psi}_{\lambda,\vec{n}}(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dk\, \Psi_{qk\vec{n}}(z)e^{-ik\lambda}, \tag{2.2.17}$$

(such trick was suggested in [246], see also Ref. [20]). The solutions (2.2.16) satisfy the quantum wormhole boundary conditions (in terms of parameter v (2.2.6): i) the wave function is exponentially damped for large space geometries ($v \to +\infty$); ii)the wave function is regular when the spatial geometry degenerates ($v \to 0$).

We also note that the the functions

$$\Psi_{m,\vec{n}} = H_m(x^0)H_m(x^1)\exp[-\frac{(x^0)^2 + (x^1)^2}{2}] \tag{2.2.18}$$

where H_m are Hermite polynomials, $m = 0, 1, \ldots,$,

$x^i = (2/q)^{1/2}(-2\kappa^2 A)^{1/4}\exp(qz^0/2)f^i(\frac{1}{2}q\vec{z}\vec{n}), \qquad i = 0, 1,$

$(f^0, f^1) = (sinh, cosh)$ are also solutions of the WDW equation with the quantum wormhole boundary conditions. (They are called discrete spectrum quantum wormholes.) We note that the special cases of the solutions (2.2.16), (2.2.18) for $u_i = 2N_i$ (Λ-term case) and $u_i = 2N_i - 2\delta_i^1$ (1-curvature case, $\lambda^1 \neq 0$) were considered in [48] and [20] respectively (see section 1).

We also note that for b) $u^2 > 0$ and $A > 0$ there also exist quantum wormhole solutions. (In this case z^0 should be replaced by z^1 in (2.2.16), $\vec{z}$ is defined in I-1.3) and $\vec{n}$ belongs to hypersphere.)

2.3. Two spaces with m-component matter

For the model from the subsection 1.3 chapter 1, the WDW equation (2.1.2) in the f-gauge (I-1.3.43) has the following form ($\mu = 1$)

$$(2\frac{\partial}{V_+(z^+)\partial z^+}\frac{\partial}{V_-(z^-)\partial z^-} + 1)\Psi = 0. \tag{2.3.1}$$

Indeed, for $n = 2$ we have $\Delta[e^{2f}G] = e^{-2f}\Delta[G]$, $a_2 = 0$ and $\Psi^f = \Psi$ (see (2.1.3). In w-variable $w = (w^0, w^1)$, where $w^0 \pm w^1 = w^\pm$, where $w^\pm$ are defined in (I-1.3.45), we get the Klein-Gordon equation with $m^2 = 2$

$$[(\frac{\partial}{\partial w^0})^2 - (\frac{\partial}{\partial w^1})^2 + 2]\Psi = 0. \tag{2.3.2}$$

2.4. n-spaces with m component matter

Here we present the solutions of the WDW equation (2.2.1) with the potential (I-1.3.49) , i.e. quantum analogues of the classical solutions from section 1.3 chapter 1. are considered.

Repeating all arguments from section 2.2 (case a)),we get the general solution of (2.2.1)

$$\Psi(z) = \sum_{*=\pm} \int d^{n-1}\vec{p}\, C_*(\vec{p})\Psi^*_{\vec{p}}(z), \tag{2.4.1}$$

where

$$\Psi^*_{\vec{p}}(z) = \exp(i\vec{p}\vec{z})\Phi^*_{\vec{p}}(z^0), \tag{2.4.2}$$

$* = \pm$, and $\Phi^* = \Phi^*_{\vec{p}}(z^0)$ are two linerly independent solutions of the equation

$$[-(\frac{\partial}{\partial z^0})^2 - 2V_A(z^0)]\Phi = 2\mathcal{E}_{\vec{p}}\Phi, \tag{2.4.3}$$

with the notations for $\mathcal{E} = \mathcal{E}_{\vec{p}}$, $\vec{p}$, $\vec{z}$ from section 2.2(a).

We note, that for special values of parameters $A^{(\alpha)}$ and $b^{(\alpha)}$ in the potential (I-1.3.49) the equation (2.4.3) describes the quantum spin systems [256].

3. MULTIDIMENSIONAL CLASSICAL AND QUANTUM COSMOLOGY WITH SCALAR FIELD [257]

3.1. Introduction

Here we present the extension of the classical study of the model described in section 1, chapter I, for the case of one-component perfect fluid and minimally coupled scalar field as $z^n = k\varphi$ coordinate.

The treatment of classical models is only the necessary first step in analyzing the properties of the "Early Universe" and the last stages of the gravitational collapse in a multidimensional approach. In quantum multidimensional cosmology we hope to find answers to such questions as the singular state, the "creation of the Universe", the nature and value of the cosmological constant, some ideas on possible "seeds" of the observable structure of the Universe, the stability of fundamental constants etc. In the third quantization scheme the problems of topological changes may be treated thoroughly. It should be noted that multidimensional schemes may be also used in multicomponent inflationary scenarios [116–117] (see for example [119]).

In Sec.3.2 we integrate the Einstein equations and analyze a class of exceptional (inflationary) solutions. (Solutions with $n = 2$ were considered recently in [119-120]). The isotropization-like and Kasner-like asymptotical behaviours of the solutions are analyzed. Some special cases, such as isotropic (when the pressures in all spaces are equal) and curvature-like ones, are investigated. In the last case there are solutions with so-called spontaneous and dynamical compactifications. The instanton solutions (classical wormholes) with an imaginary scalar field and negative energy density are also obtained.

In Sec. 3.4 we consider our model at the quantum level (for pioneering papers see [273-274]). Here we quantize the scale factors and the scalar field but treat the perfect fluid as a classical object. Such an approach is quite consistent at least in certain special situations such as the Λ-term [48] and curvature [20,231-232] cases.

The Wheeler-DeWitt equation for the model is solved and quantum wormhole solutions are obtained. The multidimensional quantum wormhole solutions of this section may be considered to be a natural extension of the corresponding solutions of [20,231] and [48,52] for the curvature and Λ-term cases, respectively.

In Sec.3.5 a third quantized cosmology is investigated along the line of [20] and [268] for the curvature and cosmological constant cases, respectively. Here we are led to the theory of massless conformally coupled scalar field in a conformally flat generalized Milne universe [268]. In- and out-vacua are defined and a Planckian spectrum for the out-universes created (from an in-vacuum) is obtained using standard relations [279,280]. The temperature is shown to depend upon the equation of state. It should be noted that recently the interest to the third quantized models was stimulated by the papers [281] (see also [62,282-284] and references therein).

3.2. Classical solutions

The Lagrangian of the system is:

$$L = \frac{1}{2}(G_{ij}\dot{x}^i\dot{x}^j + \kappa^2\dot{\varphi}^2) - \kappa^2 A \exp(u_k x^k) \tag{3.2.1}$$

with the energy constraint

$$E = \frac{1}{2}(G_{ij}\dot{x}^i\dot{x}^j + \kappa^2\dot{\varphi}^2) + \kappa^2 A \exp(u_k x^k) = 0. \tag{3.2.2}$$

In the z variables, it may be rewritten as

$$L = \frac{1}{2}\eta_{AB}\dot{z}^A\dot{z}^B - \kappa^2 A \exp(2qz^0), \tag{3.2.3}$$

where the indices $A, B = 0, \ldots, n$. The energy constraint (3.2.2) reads:

$$E = \frac{1}{2}\eta_{AB}\dot{z}^A\dot{z}^B + \kappa^2 A \exp(2qz^0) = 0. \tag{3.2.4}$$

The Lagrange equations for the Lagrangian (3.2.3)

$$-\ddot{z}^0 + 2qA\exp(2qz^0) = 0, \tag{3.2.5}$$

$$\ddot{z}^B = 0, \quad B = 1, \ldots, n, \tag{3.2.6}$$

with the energy constraint (3.2.4) can be easily solved. From (3.19) we have

$$z^B = p^B t + q^B, \tag{3.2.7}$$

where p^B and q^B are constants and $B = 1, \ldots, n$. The first integral of Eq.(3.2.5) reads

$$-\frac{1}{2}(\dot{z}^0)^2 + A\exp(2qz^0) + \mathcal{E} = 0. \tag{3.2.8}$$

Using (3.2.4), (3.2.7) and (3.2.8) we get

$$\mathcal{E} = \frac{1}{2}\sum_{B=1}^{n}(p^B)^2. \tag{3.2.9}$$

We obtain the following solution to Eqs.(3.2.5), (3.2.8)

$$\exp(-2qz^0)$$

$$= (A/\mathcal{E})\sinh^2(q\sqrt{2\mathcal{E}}(t - t_0)), \quad \mathcal{E} > 0,\ A > 0, \tag{3.2.10}$$

$$= (A/|\mathcal{E}|)\sin^2(q\sqrt{2|\mathcal{E}|}(t - t_0)), \quad \mathcal{E} < 0,\ A > 0, \tag{3.2.11}$$

$$= 2q^2 A(t - t_0)^2, \quad \mathcal{E} = 0,\ A > 0, \tag{3.2.12}$$

$$= (|A|/\mathcal{E})\cosh^2(q\sqrt{2\mathcal{E}}(t - t_0)), \quad \mathcal{E} > 0,\ A < 0. \tag{3.2.13}$$

Here t_0 is an arbitrary constant. For real z^B (or, equivalently, for real metric and scalar field) we get from (3.2.9) $\mathcal{E} \geq 0$. The case $\mathcal{E} < 0$ may take place when a pure imaginary scalar field is considered.

Kasner-Like Parametrization for Non-Exceptional Solutions with a Real Scalar Field

We first consider the real case with $\mathcal{E} > 0$. In this case the relations (3.2.10) and (2.2.13) may be written in the following form:

$$\exp(-2qz^0) = \frac{|A|}{\mathcal{E}} f_\delta^2(q\sqrt{2\mathcal{E}}(t - t_0)), \tag{3.2.14}$$

where $\delta \equiv A/|A| = \pm 1$ and

$$\begin{aligned} f_\delta(x) \equiv \frac{1}{2}(e^x - \delta e^{-x}) &= \sinh x, \quad \delta = +1, \\ &= \cosh x, \quad \delta = -1. \end{aligned} \tag{3.2.15}$$

We introduce a new time variable by the relation

$$\tau = \frac{T}{\sqrt{\delta}} \ln \frac{\exp[q\sqrt{2\mathcal{E}}(t - t_0)] + \sqrt{\delta}}{\exp[q\sqrt{2\mathcal{E}}(t - t_0)] - \sqrt{\delta}} \tag{3.2.16}$$

$$= T \ln \coth[\frac{1}{2} q\sqrt{2\mathcal{E}}(t - t_0)], \qquad \delta = +1, \tag{3.2.17}$$

$$= 2T \arctan \exp[-q\sqrt{2\mathcal{E}}(t-t_0)], \quad \delta = -1, \tag{3.2.18}$$

where

$$\begin{aligned} T &= T(u, A) \equiv (2q^2|A|)^{-1/2} \\ &= (\tfrac{1}{2}|A < u, u >_* |)^{-1/2}. \end{aligned} \tag{3.2.19}$$

For $\delta = +1$ the variable $\tau = \tau(t)$ monotonically decreases from $+\infty$ to 0 when $t - t_0$ is varying from 0 to $+\infty$. For $\delta = -1$ it monotonically decreases from πT to 0 when $t - t_0$ is varying from $-\infty$ to $+\infty$.

It is not difficult to verify that

$$\sinh(\tau\sqrt{\delta}/T)/\sqrt{\delta} = 1/f_\delta(q\sqrt{2\mathcal{E}}(t-t_0)), \tag{3.2.20}$$

$$\tanh(\tau\sqrt{\delta}/2T)/\sqrt{\delta} = \exp[-q\sqrt{2\mathcal{E}}(t-t_0)], \tag{3.2.21}$$

$$d\tau = -qT\sqrt{2\mathcal{E}} dt / f_\delta(q\sqrt{2\mathcal{E}}(t-t_0)). \tag{3.2.22}$$

To present the solutions obtained in a more familiar form we now introduce the following dimensionless "Kasner-like" parameters:

$$\beta^i = -e^i_{\hat{a}} p^{\hat{a}}/(q\sqrt{2\mathcal{E}}), \tag{3.2.23}$$

$$\beta_\varphi = -p^n/(q\sqrt{2\mathcal{E}}). \tag{3.2.24}$$

Here and henceforth $\hat{a}, \hat{b} = 1, \ldots, n-1$. From Eqs. (3.2.7) and (3.2.23) we have

$$\begin{aligned} x^i &= -(u^i/2q) z^0 + e^i_{\hat{a}}[p^{\hat{a}}(t - t_0) + \bar{q}^{\hat{a}}] \\ &= -(u^i/4q^2)(2qz^0) - q\sqrt{2\mathcal{E}}(t - t_0)\beta^i + \gamma^i, \end{aligned} \tag{3.2.25}$$

where

$$\gamma^i = e^i_{\hat{a}} \bar{q}^{\hat{a}}, \qquad \bar{q}^{\hat{a}} = q^{\hat{a}} + p^{\hat{a}} t_0. \tag{3.2.26}$$

Using (3.2.14), (3.2.20), (3.2.21) and (3.2.25), we get for the scale factors:

$$a_i = \mathrm{e}^{x^i} = A_i[\sinh(r\tau/T)/r]^{\sigma^i}[\tanh(r\tau/2T)/r]^{\beta^i} \tag{3.2.27}$$

where $r = \sqrt{\delta}$ and

$$\sigma^i = 2u^i/ < u, u >_*, \quad A_i = (\mathcal{E}/|A|)^{\sigma^i/2}\mathrm{e}^{\gamma^i}, \tag{3.2.28}$$

$i = 1, \ldots, n$. In a similar manner we obtain the expression for the scalar field (see (3.2.21), (3.2.24))

$$\mathrm{e}^{\kappa\varphi} = \mathrm{e}^{z^n} = A_\varphi[\tanh(r\tau/2T)/r]^{\beta_\varphi} \tag{3.2.29}$$

where $A_\varphi > 0$ is constant.

We define a bilinear symmetric form $< .,. >$: $R^n \times R^n \to R$ by the relation

$$< \alpha, \beta >= G_{ij}\alpha^i\beta^j, \tag{3.2.30}$$

$\alpha = (\alpha^i)$, $\beta = (\beta^i) \in R^n$. Using the definitions, (3.2.9), (3.2.22), (3.2.24), we obtain the relations between the Kasner-like parameters

$$\begin{aligned} < \beta, \beta > +(\beta_\varphi)^2 &= G_{ij}\beta^i\beta^j + (\beta_\varphi)^2 \\ &= 1/q^2 = -4/ < u, u >_*, \end{aligned} \tag{3.2.31}$$

and

$$u_i\beta^i = e^0_i e^i_{\hat{a}} P^{\hat{a}} = \delta^0_{\hat{a}} P^{\hat{a}} = 0, \tag{3.2.32}$$

where $P^{\hat{a}} = -p^{\hat{a}}\sqrt{2/\mathcal{E}}$.

Similarly to (3.2.32), we get $u_i\gamma^i = 0$ and hence (see (3.2.28))

$$\prod_{i=1}^{n} A_i^{u_i} = \mathcal{E}/|A|. \tag{3.2.33}$$

Thus the additional integral of motion $\mathcal{E}$ is a certain combination of parameters A_i and $|A|$ depending on the equation of state.

We also introduce the "quasi-volume" scale factor

$$v = \prod_{i=1}^{n} a_i^{u_i/2} = \exp(\tfrac{1}{2}u_i x^i), \tag{3.2.34}$$

where

$$< u, u >< 0 \tag{3.2.35}$$

From (3.2.14), (3.2.20), (3.2.33) (see also (3.2.27), (3.2.32)) we have

$$v = \frac{v_0}{r}\sinh\frac{r\tau}{T} = \sqrt{\mathcal{E}/|A|}/[f_\delta(q\sqrt{2\mathcal{E}}(t - t_0))]. \tag{3.2.36}$$

Here

$$v_0 = \prod_{i=1}^{n} A_i^{u_i/2}. \tag{3.2.37}$$

The quasi-volume scale factor oscillates for $A < 0$ (negative energy density) and exponentially increases as $\tau \to +\infty$ for $A > 0$ (positive energy density).

From (3.2.19), (3.2.22), (3.2.34), (3.2.36) we get

$$\begin{aligned} e^{2\gamma_0(t)} dt \otimes dt &= \Bigl(\prod_{i=1}^{n} a_i^{2N_i}\Bigr) f_\delta^2 (q\sqrt{2\mathcal{E}}(t - t_0)) \frac{d\tau \otimes d\tau}{2q^2 \mathcal{E} T^2} \\ &= \Bigl(\prod_{i=1}^{n} a_i^{2N_i - u_i}\Bigr) d\tau \otimes d\tau. \end{aligned} \tag{3.2.38}$$

Thus we get the following solution to the field equations:

$$g = -\Bigl[\prod_{i=1}^{n} (a_i(\tau))^{2N_i - u_i}\Bigr] d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau) g^{(i)}, \tag{3.2.39}$$

$$a_i(\tau) = A_i \Bigl(\frac{1}{r} \sinh \frac{r\tau}{T}\Bigr)^{2u^i/<u,u>_*} \Bigl(\frac{1}{r} \tanh \frac{r\tau}{2T}\Bigr)^{\beta^i}, \tag{3.2.40}$$

$$e^{\kappa\varphi(\tau)} = A_\varphi \Bigl(\frac{1}{r} \tanh \frac{r\tau}{2T}\Bigr)^{\beta_\varphi}, \tag{3.2.41}$$

$$\kappa^2 \rho(\tau) = A \prod_{i=1}^{n} (a_i(\tau))^{u_i - 2N_i}, \tag{3.2.42}$$

$i = 1, \ldots, n$; where $r = \sqrt{A/|A|}$, T is defined in (3.2.19), $A_i, A_\varphi > 0$ are constants, and the parameters β^i, β_φ satisfy the relations

$$\begin{aligned} \sum_{i=1}^{n} u_i \beta^i &= 0, \\ \sum_{i,j=1}^{n} G_{ij} \beta^i \beta^j + (\beta_\varphi)^2 &= -4/<u,u>_* = 1/q^2. \end{aligned} \tag{3.2.43}$$

Here $\tau > 0$ for $A > 0$ and $0 < \tau < \pi T$ for $A < 0$.

We note that the solution (3.2.39-43) without scalar field ($\beta_\varphi = 0$) was obtained previously in [47]. For $u_i = 2N_i$ (Λ-term case) the solution was considered in [52] (for $\beta_\varphi = 0$ see also [48]), where Euclidean wormholes were constructed.

For small values of τ we have the following asymptotic relations

$$a_i(\tau) \sim C_i \tau^{\bar\beta^i}, \qquad e^{\kappa\varphi(\tau)} \sim C_\varphi \tau^{\beta_\varphi} \tag{3.2.44}$$

as $\tau \to 0$, $i = 1, \ldots, n$, where C_i, C_φ are constants and $\bar\beta^i = \beta^i + \sigma^i$ are the new Kasner-like parameters, satisfying the relations

$$u_i \bar\beta^i = 2, \qquad G_{ij} \bar\beta^i \bar\beta^j + \beta_\varphi^2 = 0. \tag{3.2.45}$$

Exceptional solutions

Now we consider the exceptional real solutions corresponding to $\mathcal{E} = 0$ and $A > 0$ (see (3.2.12)). From $\mathcal{E} = 0$ and (3.2.9) we have $p^B = 0$ and hence

$$z^B = q^B \tag{3.2.46}$$

are constant, $B = 1, \ldots, n$. So, $\kappa\varphi = z^n = \mathrm{const}$ in this case, we have for x^i

$$x^i = -(u^i/4q^2)(2qz^0) + \gamma^i, \qquad \gamma^i = e^i_{\hat{a}} q^{\hat{a}}, \tag{3.2.47}$$

($\hat{a} = 1, \ldots, n-1$). Using (3.2.12), (3.2.19) and (3.2.47) for $t > t_0$, we get

$$a_i = \mathrm{e}^{x^i} = [(t - t_0)/T]^{-\sigma^i} \mathrm{e}^{\gamma^i}, \tag{3.2.48}$$

$i = 1, \ldots, n$.

Introducing the new time variable τ by

$$T/(t - t_0) = \exp[\pm(\tau - \tau_0)/T], \qquad t > t_0, \tag{3.2.49}$$

we obtain

$$a_i(\tau) = \bar{A}_i \exp(\pm\sigma^i \tau/T), \tag{3.2.50}$$

where

$$\bar{A}_i = \exp(\mp\sigma^i \tau_0/T) \exp(\gamma^i), \tag{3.2.51}$$

$i = 1, \ldots, n$.

Similarly to (3.2.32) we get $u_i \gamma^i = 0$ and hence (see (3.2.51))

$$\prod_{i=1}^{n} \bar{A}_i^{u_i} = \exp(\mp 2\tau_0/T). \tag{3.2.52}$$

For the quasi-volume from (3.2.50) and (3.2.52) we get

$$v = \prod_{i=1}^{n} a_i^{u_i/2} = \exp[\pm(\tau - \tau_0)/T]. \tag{3.2.53}$$

Thus for $A > 0$ we have a family of exceptional solutions with a constant real scalar field

$$g = -\Big(\prod_{i=1}^{n} (a_i(\tau))^{2N_i - u_i}\Big) d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau) g^{(i)}, \tag{3.2.54}$$

$$a_i(\tau) = \bar{A}_i \exp[\pm 2u^i \tau/(T < u, u >_*)], \tag{3.2.55}$$

$$\varphi(\tau) = \mathrm{const}, \tag{3.2.56}$$

and $\rho(\tau)$ is determined by (3.2.42). Here $\bar{A}_i > 0$ ($i = 1, \ldots, n$) are constants, and T is defined in (3.2.19).

We note that for $A > 0$ the solution (3.2.55) with the $+$ sign is an attractor for the solutions (3.2.40), i.e.,

$$a_i(\tau) \sim \bar{A}_i \exp(\sigma^i \tau/T), \qquad \varphi(\tau) \sim \mathrm{const}, \tag{3.2.57}$$

$i = 1, \ldots, n$, for $\tau \to +\infty$.

Synchronous-time parametrization

The relations (3.2.55) imply

$$\prod_{i=1}^{n} a_i^{2N_i-u_i} = \bar{P}^2 \exp[\pm 2(\bar{\sigma}-1)\tau/T], \tag{3.2.58}$$

where

$$\bar{P} = \prod_{i=1}^{n} \bar{A}_i^{N_i-u_i/2}, \tag{3.2.59}$$

$$\bar{\sigma} = \frac{2N_i u^i}{< u, u >_*} = \frac{< u^{(\Lambda)}, u >_*}{< u, u >_*}. \tag{3.2.60}$$

Here and henceforth

$$u_i^{(\Lambda)} = 2N_i. \tag{3.2.61}$$

Now we introduce the synchronous time variable t_s satisfying the relation

$$\bar{P}^2 \exp[\pm 2(\bar{\sigma}-1)\tau/T]\, d\tau \otimes d\tau = dt_s \otimes dt_s. \tag{3.2.62}$$

First we consider the case

$$\bar{\sigma} \neq 1 \iff < u^{(\Lambda)} - u, u >_* \neq 0. \tag{3.2.63}$$

Introducing t_s by the formula

$$t_s = \frac{\bar{P}T}{|\bar{\sigma}-1|} \exp[\pm(\bar{\sigma}-1)\tau/T] > 0, \tag{3.2.64}$$

we get for the scale factors

$$a_i = a_i(t_s) = A_i t_s^{\nu^i}, \tag{3.2.65}$$

where

$$\nu^i = \sigma^i/(\bar{\sigma}-1) = 2u^i/ < u^{(\Lambda)} - u, u >_* \tag{3.2.66}$$

and

$$A_i = \bar{A}_i [|\bar{\sigma}-1|/(\bar{P}T)]^{\nu^i}. \tag{3.2.67}$$

The parameters ν^i (3.2.66) satisfy the relation

$$\nu^i (2N_i - u_i) = 2. \tag{3.2.68}$$

Eqs. (3.2.19), (3.2.59), (3.2.67) and (3.2.68) imply

$$\begin{aligned}\prod_{i=1}^{n} A_i^{u_i-2N_i} &= T^2/(\bar{\sigma}-1)^2 \\ &= -2< u, u >_*/(A< u^{(\Lambda)} - u, u >_*^2).\end{aligned} \tag{3.2.69}$$

From (3.2.42), (3.2.65), (3.2.68) and (3.2.69) we get the following formula for the density:

$$\kappa^2\rho = \kappa^2\rho(t_s) = \frac{-2 < u, u >_*}{< u^{(\Lambda)} - u, u >_*^2 t_s^2}. \tag{3.2.70}$$

The metric reads:

$$g = -dt_s \otimes dt_s + \sum_{i=1}^{n} a_i^2(t_s) g^{(i)}, \tag{3.2.71}$$

where the scale factors are determined by (3.2.65), $i = 1, \ldots, n$. Thus Eqs. (3.2.42), (3.2.65), (3.2.70), (3.2.71) and $\varphi = \mathrm{const}$ describe the exceptional solutions for the case (3.2.63). We call them power-law inflationary solutions.

Now let us consider the case

$$\bar{\sigma} = 1 \iff < u^{(\Lambda)} - u, u >_* = 0. \tag{3.2.72}$$

From (3.2.42), (3.2.58) we obtain

$$\kappa^2\rho = A\bar{P}^{-2} = \mathrm{const.} \tag{3.2.73}$$

Introducing the synchronous time $t_s = \bar{P}\tau$, from (3.67) we get

$$a_i(t_s) = \bar{A}_i \exp[\mp \frac{u^i}{\sqrt{- < u, u >_*}} \frac{t_s}{T_0}], \tag{3.2.74}$$

where

$$T_0 = (2\kappa^2\rho)^{-1/2}. \tag{3.2.75}$$

Eqs. (3.2.71), (3.2.73)-(3.2.75) and $\varphi = \mathrm{const}$ describe the exponential-type inflation for the case (3.2.72).

Let us consider the synchronous time parametrization for the solutions (3.2.39)-(3.2.43). The synchronous time t_s and the τ variable are related by

$$t_s = \varepsilon F(\tau), \qquad \frac{dF}{d\tau} = f(\tau), \tag{3.2.76}$$

where $\varepsilon = \pm 1$ and

$$\begin{aligned} f(\tau) &= \prod_{i=1}^{n} (a_i(\tau))^{N_i - u_i/2} \\ &= P[\sinh(r\tau/T)/r]^{\bar{\sigma}-1}[\tanh(r\tau/2T)/r]^{\beta^i N_i}, \end{aligned} \tag{3.2.77}$$

$$\begin{aligned} &\kappa^2\rho(\tau) \\ &= AP^{-2}[\sinh(r\tau/T)/r]^{2-2\bar{\sigma}}[\tanh(r\tau/2T)/r]^{-2\beta^i N_i}, \end{aligned} \tag{3.2.78}$$

with $P = \prod_{i=1}^{n} A_i^{N_i - u_i/2}$ and $\bar{\sigma}$ defined in (3.2.72). From (3.2.78) it follows

$$f(\tau) \sim B\tau^{p-1}, \qquad \tau \to +0, \tag{3.2.79}$$

where $B > 0$ is constant and

$$p = p(\beta) = \bar{\sigma} + \beta^i N_i = (\sigma^i + \beta^i) N_i. \tag{3.2.80}$$

Putting $\varepsilon =$ sign (p), from (3.2.76) and (3.2.79) we get

$$t_s \sim B_1 \tau^p, \qquad \tau \to +0, \tag{3.2.81}$$

with $B_1 = B/|p|$ (here the integration constant in (3.2.76) is properly fixed).

Proposition 1. *Let $1/q^2 - (\beta_\varphi)^2 \geq 0$. Then, for all $\beta = (\beta^i)$ satisfying the relations (3.2.43), we have $p(\beta) \neq 0$ and*

$$i)\ u^i N_i < 0 \ \Rightarrow \ p(\beta) > 0, \tag{3.2.82}$$
$$ii)\ u^i N_i > 0 \ \Rightarrow \ p(\beta) < 0. \tag{3.2.83}$$

Proposition 1 is a special case of a more general Proposition 2 proved in the Appendix:

Proposition 2. *Let two vectors $u = (u_i)$, $v = (v_i) \in R^n$ satisfy the inequalities $< u, u >_* \equiv -4q^2 < 0$ and $< v, v >_* < 0$. Then $u^i v_i =< u, v >_* \neq 0$ and for all $\beta = (\beta^i)$ such as*

$$u_i \beta^i = 0, \qquad G_{ij} \beta^i \beta^j \leq 1/q^2, \tag{3.2.84}$$

the following relation is valid:

$$\text{sign } (u^i v_i) = -\text{sign } ((\sigma^i + \beta^i) v_i), \tag{3.2.85}$$

where $\sigma^i = 2u^i / < u, u >_*$.

For the vector

$$v_i = N_i = \frac{1}{2} u_i^{(\Lambda)} \tag{3.2.86}$$

we have

$$v^i = N^i = G^{ij} N_j = \frac{1}{2 - D} \tag{3.2.87}$$

and hence

$$< v, v >_* = N_i N^i = -\frac{D-1}{D-2} < 0. \tag{3.2.88}$$

Thus Eqs. (3.2.80), (3.2.88) and Proposition 2 imply the Proposition 1.

From (3.2.87) we get

$$u_i N^i = \frac{1}{2 - D} \sum_{i=1}^{n} u_i. \tag{3.2.89}$$

Using (3.2.81), (3.2.89) and Proposition 1, we obtain

$$t_s \to +0 \quad \text{as} \quad \tau \to +0, \tag{3.2.90}$$

for

$$(A) \quad \sum_{i=1}^{n} u_i > 0, \quad p(\beta) > 0 \tag{3.2.91}$$

and

$$t_s \to +\infty \quad \text{as} \quad \tau \to +0, \tag{3.2.92}$$

for

$$(B) \quad \sum_{i=1}^{n} u_i < 0, \quad p(\beta) < 0. \tag{3.2.93}$$

In the limit $\tau \to +0$ we have $\tau \sim (t_s/B_1)^{1/p}$ (see (3.2.81)) and hence (see (3.2.44))

$$a_i(t_s) \sim \bar{B}_i t_s^{\alpha^i}, \quad \exp(\kappa\varphi(t_s)) \sim \bar{B}_\varphi t_s^{\alpha_\varphi} \tag{3.2.94}$$

as $t_s \to +0$ in the case A) (3.2.91) and as $t_s \to +\infty$ in the case B) (3.2.93). Here $\bar{B}_i, \bar{B}_\varphi$ are constants and

$$\alpha^i = (\sigma^i + \beta^i)/p(\beta), \quad \alpha_\varphi = \beta_\varphi/p(\beta), \tag{3.2.95}$$

$i = 1 \dots n$. The parameters α^i, α_φ satisfy the Kasner-like relations

$$\sum_{i=1}^{n} N_i \alpha^i = 1, \tag{3.2.96}$$

$$\sum_{i=1}^{n} N_i (\alpha^i)^2 + \alpha_\varphi^2 = 1. \tag{3.2.97}$$

Eq.(3.2.96) is quite obvious, Eq.(3.2.97) follows from (3.2.96) and the relation

$$G_{ij}\alpha^i\alpha^j + \alpha_\varphi^2 = 0, \tag{3.2.98}$$

which is readily verified using (3.2.28), (3.2.43) and (3.2.95).

The Kasner-like asymptotical behaviour (3.2.94), (3.2.96), (3.2.97) for the case (A) agrees with one of the results of [72]: in the case A) the perfect fluid components with $< u, u >_* < 0$ may be neglected near the singularity $t_s \to +0$ and we are led to the Kasner-like formulas [250] (see also [19]).

Note that for the case $n = 2$ the following relation is valid:

$$\begin{aligned} &\Big[b_2 \frac{u_1}{N_1} - (1+s)\frac{u_2}{N_2}\Big]\Big[b_1 \frac{u_2}{N_2} - (1+s)\frac{u_1}{N_1}\Big] \\ &\qquad = (-s^2)(1+s) < u, u >_* \end{aligned} \tag{3.2.99}$$

where $b_i = 1 - 1/N_i$, $i = 1,2$ and $s = \sqrt{1 - b_1 b_2}$. This implies the relations for the light-cone lines ($<u,u>_* = 0$):

$$l_1 : \qquad b_2(1 - \xi_1) = (1+s)(1 - \xi_2), \tag{3.2.100}$$

$$l_2 : \qquad b_1(1 - \xi_2) = (1+s)(1 - \xi_1), \tag{3.2.101}$$

Isotropization-like behaviour

Here we rewrite the attractor behaviour (3.2.57) for the non-exceptional solutions (3.2.39)-(3.2.43) with $A > 0$ (as $\tau \to +\infty$) in terms of the synchronous time variable t_s. For the function (3.2.77) we have the following asymptotical behaviour

$$f(\tau) \sim P[\tfrac{1}{2}\exp(\tau/T)]^{\bar{\sigma}-1} = \bar{B}\exp[(\bar{\sigma}-1)\tau/T], \tag{3.2.102}$$

when $\tau \to +\infty$ ($\bar{B} = \mathrm{const}$).

First consider the case $\bar{\sigma} = 1$ (see (3.2.72)). Then $f(\tau) \sim \bar{B}$ as $\tau \to +\infty$ and hence (see (3.2.76))

$$t_s = F(\tau) \sim \bar{B}\tau + C, \tag{3.2.103}$$

as $\tau \to +\infty$. (Due to $u^i N_i < 0$ and Proposition 1, $\varepsilon = \mathrm{sign}\ (p) = +1$). The synchronous time t_s monotonically increases from 0 to $+\infty$ as τ varies from 0 to $+\infty$ (see (3.2.81)).

From (3.2.42), (3.2.57) and (3.2.103) for the case $\bar{\sigma} = 1$ we get

$$a_i(t_s) \sim \bar{A}_i \exp[-\frac{u^i}{\sqrt{-<u,u>_*}}\frac{t_s}{T_0}], \tag{3.2.104}$$

$$\varphi(t_s) \sim \mathrm{const}, \tag{3.2.105}$$

$$\rho(t_s) \sim \rho_0 \tag{3.2.106}$$

when $t_s \to +\infty$, where $T_0 = (2\kappa^2\rho_0)^{-1/2}$.

Now consider the case $\bar{\sigma} \neq 1$ (see (3.2.63)). Then

$$F(\tau) \sim \frac{\bar{B}T}{(\bar{\sigma}-1)}\exp[(\bar{\sigma}-1)\tau/T] + C, \tag{3.2.107}$$

where C is a constant.

Consider first the subcase $\bar{\sigma} > 0$ or, equivalently, $u_i N^i < 0$ (or $\sum_{i=1}^n u_i > 0$, see (3.2.89)). We have $t_s = F(\tau)$, since $p > 0$ due to (3.2.91) and $\varepsilon = \mathrm{sign}\ (p) = +1$). In this case t_s monotonically increases from 0 to $T_* > 0$ for $0 < \bar{\sigma} < 1$ and to $+\infty$ for $\bar{\sigma} > 1$ as τ varies from 0 to $+\infty$ (see (3.2.81)). Using (3.2.42), (3.2.57) we get

$$a_i(t_s) \sim A_i(T_* - t_s)^{\nu^i}, \tag{3.2.108}$$

$$\varphi(t_s) \sim \mathrm{const}, \tag{3.2.109}$$

$$\kappa^2\rho(t_s) \sim \frac{-2<u,u>_*}{<u^{(\Lambda)}-u,u>_*^2 (T_* - t_s)^2}. \tag{3.2.110}$$

as $t_s \to T_* - 0$, for $\bar{\sigma} < 1$. For $\bar{\sigma} > 1$ we have an asymptotic behaviour in the limit $t_s \to +\infty$ described by the relations

$$a_i(t_s) \sim A_i t_s^{\nu^i}, \tag{3.2.111}$$

$$\varphi(t_s) \sim \mathrm{const}, \tag{3.2.112}$$

$$\kappa^2\rho(t_s) \sim \frac{-2<u,u>_*}{<u^{(\Lambda)}-u,u>_*^2 t_s^2}. \tag{3.2.113}$$

as $t_s \to +\infty$, where ν^i is defined in (3.2.66).

Now consider the subcase $\bar{\sigma} < 0$ or, equivalently, $u_i N^i > 0$ (or $\sum_{i=1}^n u_i < 0$, see (3.2.89)). Recall that $p < 0$ due to (3.2.93) and $\varepsilon = \text{sign}\,(p) = -1$. Then $t_s = -F(\tau)$ (we put $C = 0$ in (3.2.107)) and t_s monotonically decreases from $+\infty$ to 0 as τ varies from 0 to $+\infty$ (see (3.2.81)). In this subcase we obtain the asymptotic behaviour in the limit $t_s \to +0$ described by (3.2.111)-(3.2.113).

Solutions with a pure imaginary scalar field

Here we consider solutions to the field equations with a complex scalar field and a real metric. In this case $\mathcal{E}, p^1, \ldots, p^{n-1}$ are real and hence (see (3.2.18), (3.2.29)) p^n is either real or pure imaginary. The case of real p^n was considered above.

For pure imaginary p^n we have three subcases: (a) $\mathcal{E} > 0$, (b) $\mathcal{E} = 0$, and (c) $\mathcal{E} < 0$. In the first case (a) after the reparametrization (3.2.16)-(3.2.19) we get the solutions (3.2.39)-(3.2.43) with an imaginary value of β_φ. The cases (b) and (c) take place only for $A > 0$, i.e., positive energy density (see (3.2.11), (3.2.12)).

In the case $\mathcal{E} < 0$ we have (see (3.2.23), (3.2.24)) imaginary β^k:

$$\beta^k = i\hat{\beta}^k, \qquad k = 1, \ldots, n, \tag{3.2.114}$$

and real β_φ. The solution is obtained from (3.2.39)-(3.2.43) by substituting (3.2.114) and $\tau/T \mapsto \tau/T + i\frac{\pi}{2}$:

$$g = -\Big[\prod_{i=1}^n (a_i(\tau))^{2N_i - u_i}\Big]\, d\tau \otimes d\tau + \sum_{i=1}^n a_i^2(\tau) g^{(i)}, \tag{3.2.115}$$

$$a_i(\tau) = \hat{A}_i[\cosh(\tau/T)]^{\sigma^i}[f(\tau/2T)]^{\hat{\beta}^i}, \tag{3.2.116}$$

$$\varphi(\tau) = c + 2i\beta_\varphi \arctan\exp(-\tau/T) \tag{3.2.117}$$

where $c, \hat{A}_i \neq 0$ are constants, $i = 1, \ldots, n$, T is defined in (3.2.19), σ^i are given in (3.2.28), $A > 0$ and the real parameters $\hat{\beta}^i, \beta_\varphi$ satisfy the relations

$$\begin{aligned} \sum_{i=1}^n u_i \hat{\beta}^i &= 0, \\ -\sum_{i,j=1}^n G_{ij}\hat{\beta}^i\hat{\beta}^j + (\beta_\varphi)^2 &= -\frac{4}{< u, u >_*} = \frac{1}{q^2}. \end{aligned} \tag{3.2.118}$$

Here, as in [52],

$$f(x) \equiv [\tanh(x + \frac{i\pi}{4})]^i = \exp(-2\arctan e^{-2x}) \tag{3.2.119}$$

is a smooth monotonically increasing function bounded by its asymptotics: $e^{-\pi} < f(x) < 1$; $f(x) \to 1$ as $x \to +\infty$ and $f(x) \to e^{-\pi}$ as $x \to -\infty$. The solution (3.2.115)-(3.2.118) (with ρ from (3.2.42)) may be also obtained from formulas (3.2.7), (3.2.11). The relation between the harmonic time and the τ variable (3.2.20) for $\mathcal{E} < 0$ is modified:

$$\cosh(\tau/T) = 1/\sin(q\sqrt{2|\mathcal{E}|}(t - t_0)). \tag{3.2.120}$$

For the quasi-volume scale factor we have

$$v = \prod_{i=1}^{n} a_i^{u_i/2} = \Big(\prod_{i=1}^{n} \hat{A}_i^{u_i/2}\Big) \cosh(\tau/T). \tag{3.2.121}$$

The scalar field $\varphi(t)$ varies from $c+i\pi\beta_\varphi$ to c as τ varies from $-\infty$ to $+\infty$. The solution (3.2.115)-(3.2.118) for $\tau \in (-\infty, +\infty)$ is nonsingular. Any scale factor $a_i(\tau)$ for some τ_{0i} has a minimum and

$$a_i(\tau) \sim A_i^{\pm} \exp(\sigma^i |\tau|/T), \tag{3.2.122}$$

for $\tau \to \pm\infty$.

The above "Lorentzian" solutions have "Euclidean" analogues for $A < 0$ as well:

$$g = \Big(\prod_{i=1}^{n} (a_i(\tau))^{2N_i - u_i}\Big) d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau) g^{(i)}, \tag{3.2.123}$$

$$a_i(\tau) = \hat{A}_i [\cosh(\tau/T)]^{\sigma^i} [f(\tau/2T)]^{\hat{\beta}^i}, \tag{3.2.124}$$

$$\varphi(\tau) = c + 2i\beta_\varphi \arctan \exp(-\tau/T), \tag{3.2.125}$$

with the parameters $\hat{\beta}^i, \beta_\varphi$ satisfying the relations (3.2.118). When all spaces $(M_i, g^{(i)})$ are Riemannian, this solution may be interpreted as a classical Euclidean wormhole (instanton) solution. Such solutions play a crucial role in quantum gravity.

An interesting special case of the solution (3.2.123)-(3.2.125) occurs for $\hat{\beta}^i = 0$, $i = 1, \ldots, n$, (this corresponds to $p^{\hat{a}} = 0$):

$$a_i(\tau) = \hat{A}_i [\cosh(\tau/T)]^{\sigma^i}, \tag{3.2.126}$$

$$\varphi(\tau) = c \pm 2iq^{-1} \arctan \exp(-\tau/T). \tag{3.2.127}$$

All scale factors (3.2.126) have a minimum at $\tau = 0$ and are symmetric with respect to time reversion: $\tau \mapsto -\tau$. It is necessary to stress that here, as in [52], wormhole solutions exist only in the presence of an imaginary scalar field.

In this subsection we consider some applications of the above formulas, valid for different equations of state in different factor spaces.

The isotropic case

Consider the isotropic case:

$$u_i = hN_i \Longleftrightarrow u = \frac{h}{2} u^{(\Lambda)}, \qquad p_i = (1-h)\rho, \tag{3.2.128}$$

where $h \neq 0$ is constant. From (3.2.86)-(3.2.88) and (3.2.128) it follows

$$u^i = \frac{h}{2-D}, \qquad < u, u >_* = -h^2 \frac{D-1}{D-2} < 0, \tag{3.2.129}$$

and hence

$$\sigma^i = 2u^i / < u, u >_* = \frac{2}{h(D-1)} = \sigma(h) = \sigma. \tag{3.2.130}$$

The solution (3.2.39)-(3.2.43) reads:

$$g = -(\prod_{i=1}^{n}(a_i(\tau))^{(2-h)N_i})d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau) g^{(i)}, \tag{3.2.131}$$

$$a_i(\tau) = A_i[\sinh(r\tau/T)/r]^{\sigma(h)}[\tanh(r\tau/2T)/r]^{\beta^i}, \tag{3.2.132}$$

$$e^{\kappa\varphi(\tau)} = A_\varphi[\tanh(r\tau/2T)/r]^{\beta_\varphi}, \tag{3.2.133}$$

$$\begin{aligned} \kappa^2\rho(\tau) &= A\prod_{i=1}^{n}(a_i(\tau))^{(h-2)N_i} \\ &= A\Big(\prod_{i=1}^{n} A_i^{(h-2)N_i}\Big)[\sinh(r\tau/T)/r]^{2(h-2)/h}, \end{aligned} \tag{3.2.134}$$

$i = 1, \ldots, n$, where $r = \sqrt{A/|A|}$ and

$$T = |h|^{-1}\left[\frac{|A|(D-1)}{2(D-2)}\right]^{-1/2}, \tag{3.2.135}$$

$A_i, A_\varphi > 0$ are constants and the parameters β^i, β_φ satisfy the relations

$$\begin{aligned} &\sum_{i=1}^{n} N_i\beta^i = 0, \\ &\sum_{i=1}^{n} N_i(\beta^i)^2 + (\beta_\varphi)^2 = \frac{4(D-2)}{h^2(D-1)}. \end{aligned} \tag{3.2.136}$$

A special case of this solution with $h = 2$ (the Λ-term case) was considered in [52].

Consider now the exceptional solutions for $A > 0$. From (3.2.60) and (3.2.129) we have

$$\begin{aligned} \bar{\sigma} &= \sigma^i N_i = 2/h, \\ < u^{(\Lambda)} - u, u >_* &= h(h-2)\frac{D-1}{D-2}. \end{aligned} \tag{3.2.137}$$

From (3.2.137) we get: $< u^{(\Lambda)} - u, u >_* = 0 \iff h = 2$ ($h \neq 0$). The matter in this case corresponds to a cosmological constant: $\Lambda = \kappa^2\rho > 0$. Eqs. (3.2.74), (3.2.129) imply the solution of [52] with metric (3.2.71) and

$$a_i(t_s) = \bar{A}_i \exp\Big[\pm\frac{t_s\sqrt{2\Lambda}}{\sqrt{(D-1)(D-2)}}\Big] \tag{3.2.138}$$

($\varphi = \mathrm{const}$), i.e., we here obtain a case of exponential inflation.

For $h \neq 2$ ($\iff$ $<u^{(\Lambda)} - u, u>_* \neq 0$)

$$\nu^i = \frac{2u^i}{<u^{(\Lambda)} - u, u>_*} = \frac{2}{(2-h)(D-1)} = \nu(h) = \nu. \tag{3.2.139}$$

From (3.2.65), (3.2.70), (3.2.129), (3.2.137) and (3.2.139) we obtain the relations for scale factors and the density:

$$a_i(t_s) = A_i t_s^{\nu(h)}, \tag{3.2.140}$$

$$\kappa^2 \rho(t_s) = \frac{2(D-2)}{(h-2)^2(D-1)t_s^2}, \tag{3.2.141}$$

i.e., power law inflation. For $h < 2$ (or $p > -\rho$) we have an isotropic expansion of all scale factors and for $h > 2$ (or $p < -\rho$) an isotropic contraction (see Fig.1).

Kasner-like behaviour.

In the above case $\sum_{i=1}^{n} u_i = h(D-1)$ and hence (see (3.2.90)-(3.2.93)) a Kasner-like behaviour (3.2.94), (3.2.96-97) takes place as (A) $t_s \to +0$, for $h > 0$ (or $p < \rho$), and (B) $t_s \to +\infty$, for $h < 0$ (or $p > \rho$).

Isotropization-like behaviour

Using the previous results and (3.2.137), we are led to the following attractor behaviour:

$$a_i(t_s) \sim A_i t_s^{\nu(h)}, \tag{3.2.142}$$

$$\kappa^2 \rho(t_s) \sim \frac{2(D-2)}{(h-2)^2(D-1)t_s^2}. \tag{3.2.143}$$

in the limits $t_s \to +\infty$, for $0 < h < 2$ (or $-\rho < p < \rho$) and $t_s \to +0$, for $h < 0$ (or $p > \rho$).

Remark 2. For the dust matter case $h = 1$ ($p = 0$), $\rho > 0$, the solution (3.2.131)-(3.2.136) has the sinchronous-time representation

$$\begin{aligned} g &= -dt_s \otimes dt_s + \sum_{i=1}^{n} a_i^2(t_s) g^{(i)}, \\ a_i(t_s) &= \bar{A}_i t_s^{1/(D-1)+\beta^i/2}(t_s + T_1)^{1/(D-1)-\beta^i/2}, \end{aligned} \tag{3.2.144}$$

$$e^{2\kappa\varphi(t_s)} = A_\varphi [t_s/(t_s + T_1)]^{\beta_\varphi}, \tag{3.2.145}$$

$$\kappa^2 \rho(t_s) = 2(D-2)/[(D-1)t_s(t_s + T_1)], \tag{3.2.146}$$

$i = 1, \ldots, n$; where $0 < t_s < +\infty$, $T_1 > 0$, $\bar{A}_i, A_\varphi > 0$ are constants and the parameters β^i, β_φ satisfy the relations

$$\sum_{i=1}^{n} N_i \beta^i = 0, \qquad \sum_{i=1}^{n} N_i (\beta^i)^2 + (\beta_\varphi)^2 = \frac{4(D-2)}{(D-1)}. \tag{3.2.147}$$

A special case of this solution with $\beta_\varphi = 0$ was considered previously in [14] (for $n = 2$ and $N_1 = \ldots = N_n = 1$ see [6] and [4] respectively.)

A curvature-like fluid component

Consider the perfect fluid matter with

$$u_i = 2h(-\delta_i^1 + N_i) = h u_i^{(1)} \tag{3.2.148}$$

where $h \neq 0$ is constant and $N_1 > 1$. For $h = 1$ this component corresponds to a nonzero curvature term in the first space [15] (see below). A calculation gives

$$u^i = -\frac{2h}{N_1}\delta_1^i, \qquad < u, u >_* = -4h^2 b_1 < 0, \tag{3.2.149}$$

where $b_1 = 1 - \frac{1}{N_1}$ and

$$\begin{aligned} < u, u^{(\Lambda)} >_* &= 2u^i N_i = -4h, \\ < u, u^{(\Lambda)} - u >_* &= 4h(-1 + hb_1), \\ \sigma^i &= \frac{\delta_1^i}{h(N_1 - 1)}. \end{aligned} \tag{3.2.150}$$

Using (3.2.148) and (3.2.150) we get from (3.2.39)-(3.2.43):

$$\begin{aligned} g = &-(a_1(\tau))^{2h}\Bigl(\prod_{i=1}^{n}(a_i(\tau))^{2N_i(1-h)}\Bigr)d\tau \otimes d\tau \\ &+ \sum_{i=1}^{n} a_i^2(\tau) g^{(i)}, \end{aligned} \tag{3.2.151}$$

$$a_1(\tau) = A_1[\sinh(r\tau/T)/r]^{\frac{1}{h(N_1-1)}}[\tanh(r\tau/2T)/r]^{\beta^1}, \tag{3.2.152}$$

$$a_i(\tau) = A_i[\tanh(r\tau/2T)/r]^{\beta^i}, \quad i > 1, \tag{3.2.153}$$

$$\mathrm{e}^{\kappa\varphi(\tau)} = A_\varphi[\tanh(r\tau/2T)/r]^{\beta_\varphi}, \tag{3.2.154}$$

$$\kappa^2\rho(\tau) = A(a_1(\tau))^{-2h}\prod_{i=1}^{n}(a_i(\tau))^{2N_i(h-1)}, \tag{3.2.155}$$

$i = 1, \dots, n$, where $r = \sqrt{A/|A|}$, $T = |h|^{-1}(2|A|b_1)^{-\frac{1}{2}}$, $A_i, A_\varphi > 0$ are constants and the parameters β^i, β_φ satisfy the relations

$$\begin{aligned} &\beta^1 = \frac{1}{1 - N_1}\sum_{i=2}^{n} N_i\beta^i, \\ &\Bigl(\sum_{i=2}^{n} N_i\beta^i\Bigr)^2 + (N_1 - 1)\Bigl[\sum_{i=2}^{n} N_i(\beta^i)^2 + (\beta_\varphi)^2\Bigr] = N_1 h^{-2}. \end{aligned} \tag{3.2.156}$$

For $h \neq h_0 \equiv b_1^{-1} = N_1/(N_1 - 1) > 1$ we have from (3.2.150) $< u, u^{(\Lambda)} - u >_* \neq 0$ and (see (3.2.66))

$$\nu^i = \delta_1^i \nu(h), \qquad \nu(h) = [N_1 + h(1 - N_1)]^{-1}. \tag{3.2.157}$$

The power-law inflationary solution for this case is

$$g = -dt_s \otimes dt_s + A_1^2 t_s^{2\nu(h)} g^{(1)} + \sum_{i=2}^{n} A_i^2 g^{(i)}, \tag{3.2.158}$$

$$\varphi = \text{const}, \tag{3.2.159}$$

$$\kappa^2 \rho(t_s) = \frac{b_1}{2(-1+hb_1)^2 t_s^2}. \tag{3.2.160}$$

The internal space scale factors in this solution are constant (the so-called "spontaneous compactification"). It is easily shown that the constancy of internal scale factors leads to the equation of state (3.2.148).

Using the relation $\bar{\sigma} = h_0/h$ and the analysis carried out in subsection 3.3 we obtain that the solution (3.2.157)-(3.2.160) is an attractor for non-exceptional solutions with $\rho > 0$ as $t_s \to T_* - 0$, for $h > h_0$; $t_s \to +\infty$, for $0 < h < h_0$ and $t_s \to +0$, for $h < 0$. Thus we have also obtained solutions with the "dynamical compactification".

3.3. The 1-curvature case.

Here we apply the relations obtained to a cosmological model described by the action

$$S = \int d^D x \sqrt{|g|} \{ R[g] - \partial_M \varphi \partial_N \varphi g^{MN} \} \tag{3.3.1}$$

with a scalar field $\varphi = \varphi(t)$ and metric (1.2.1) defined on the manifold (1.2.2), where $(M_i, g^{(i)})$, $i = 2, \ldots, n$, are Ricci-flat spaces and $(M_1, g^{(1)})$ is an Einstein space of nonzero curvature, i.e. $R_{mn}[g^{(1)}] = \lambda^1 g^{(1)}_{mn}, \lambda^1 \neq 0$. Here $n \geq 2$ and $N_i = \dim M_i$. This "1-curvature model" is equivalent to a special case of the above model (3.2.148) with $h = 1$ and $A = -\frac{1}{2}\lambda^1 N_1$ (see [19]). The solution (3.2.151-6) reads for this case:

$$g = (a_1(\tau))^2 [-d\tau \otimes d\tau + g^{(1)}] + \sum_{i=2}^{n} a_i^2(\tau) g^{(i)}, \tag{3.3.2}$$

$$a_1(\tau) = A_1 [\sinh(r\tau/T)/r]^{\frac{1}{(N_1-1)}} [\tanh(r\tau/2T)/r]^{\beta^1}, \tag{3.3.3}$$

$$a_i(\tau) = A_i [\tanh(r\tau/2T)/r]^{\beta^i}, \; i > 1, \tag{3.3.4}$$

$$\mathrm{e}^{\kappa\varphi(\tau)} = A_\varphi [\tanh(r\tau/2T)/r]^{\beta_\varphi}, \tag{3.3.5}$$

$$\kappa^2 \rho(\tau) = A(a_1(\tau))^{-2} \tag{3.3.6}$$

$i = 1, \ldots, n$, where $r = \sqrt{-\lambda^1/|\lambda^1|}$, $T = [|\lambda^1|(N_1-1)]^{-1/2}$, $A_i, A_\varphi > 0$ are constants and the parameters β^i, β_φ satisfy the relations

$$\beta^1 = \frac{1}{1-N_1} \sum_{i=2}^{n} N_i \beta^i,$$
$$\frac{1}{N_1-1} \Big(\sum_{i=2}^{n} N_i \beta^i \Big)^2 + \sum_{i=2}^{n} N_i (\beta^i)^2 + (\beta_\varphi)^2 = \frac{N_1}{N_1-1}. \tag{3.3.7}$$

The power-law inflationary solution for the negative curvature case $\lambda^1 < 0$ reads:

$$g = -dt_s \otimes dt_s + A_1^2 t_s^2 g^{(1)} + \sum_{i=2}^{n} A_i^2 g^{(i)}, \tag{3.3.8}$$

$$\varphi = \mathrm{const}, \tag{3.3.9}$$

where $A_1^2 = |\lambda^1|/(N_1 - 1)$ (see (3.2.69), (3.2.160)). We are led here to the Milne-type solution recently considered in [266].

There is another parametrization of the solution (3.3.2.-7) in terms of an R variable related to τ-variable as

$$F = F(R) = 1 - \left(\frac{R_0}{R}\right)^{N_1 - 1} = \tanh^2 \frac{\tau}{2T}, \quad \lambda^1 < 0, \tag{3.3.10}$$

$$= \left(\frac{R_0}{R}\right)^{N_1 - 1} - 1 = \tan^2 \frac{\tau}{2T}, \quad \lambda^1 > 0. \tag{3.3.11}$$

Here $R > R_0$ for $\lambda^1 < 0$ and $R < R_0$ for $\lambda^1 > 0$; $R_0 = A_1 2^{1/(N_1-1)} \sqrt{(N_1 - 1)/|\lambda^1|}$. In new variables the metric and the scalar field may be written as

$$g = -F^{b-1} dR \otimes dR + F^b R^2 A_1^2 g^{(1)} + \sum_{i=2}^{n} F^{\beta^i} A_i^2 g^{(i)}, \tag{3.3.12}$$

$$\mathrm{e}^{2\kappa\varphi} = A_\varphi^2 F^{\beta_\varphi}, \tag{3.3.13}$$

$A_1^2 = |\lambda^1|/(N_1 - 1)$, $A_i, A_\varphi > 0$ are constants and

$$b = (1 - \sum_{i=2}^{n} N_i \beta^i)/(N_1 - 1) = (N_1 - 1)^{-1} + \beta^1, \tag{3.3.14}$$

and the parameters $\beta^i (i > 1), \beta_\varphi$ satisfy the relations (3.3.7). A special case of the solution (3.3.7), (3.3.12-14) with $\beta_\varphi = 0$ (a constant scalar field) was obtained earlier in [19].

Remark 3. As a special case of the above solution, we get a scalar-vacuum analog of the spherically-symmetric Tangherlini solution [167] with n Ricci-flat internal spaces:

$$g = -f^a dt \otimes dt + f^{b-1} dR \otimes dR + f^b R^2 d\Omega_d^2 + \sum_{i=1}^{n} f^{a_i} B_i g^{(i)}, \tag{3.3.15}$$

$$\mathrm{e}^{2\kappa\varphi} = B_\varphi f^{a_\varphi}, \tag{3.3.16}$$

where $d\Omega_d^2$ is the canonical metric on a d-dimensional sphere S^d ($d \geq 2$), $f = f(R) = 1 - BR^{1-d}$; $B_\varphi, B_i > 0, B$ are constants and the parameters $a, a_1, \ldots, a_n$ satisfy the relations

$$b = (1 - a - \sum_{i=1}^{n} a_i N_i)/(d - 1), \tag{3.3.17}$$

$$(a + \sum_{i=1}^{n} a_i N_i)^2 + (d - 1)(a^2 + a_\varphi^2 + \sum_{i=1}^{n} a_i^2 N_i) = d. \tag{3.3.18}$$

For $a_\varphi = 0$ see also [36,182]. In the parametrization of the harmonic-type variable this solution was presented earlier in [165,182].

Thus, using the above transformations, we can obtain spherically symmetric solutions from cosmological ones.

3.4. Wheeler-DeWitt equation

Now, having studied the classical multidimensional solutions, we start an investigation of their quantum analogs. As usual, quantization of the zero-energy constraint (3.2.4) leads to the Wheeler-DeWitt (WDW) equation in the harmonic time gauge [15,37,52]

$$2\hat{H}\Psi \equiv \left[\frac{\partial}{\partial z^0}\frac{\partial}{\partial z^0} - \sum_{i=1}^{n}\frac{\partial}{\partial z^i}\frac{\partial}{\partial z^i} + 2A\mathrm{e}^{2qz^0}\right]\Psi = 0. \tag{3.4.1}$$

We are seeking a solution to (3.4.1) in the form

$$\Psi(z) = \exp(i\vec{p}\vec{z})\Phi(z^0), \tag{3.4.2}$$

where $\vec{p} = (p^1, \ldots, p^n)$ is a constant vector (generally from C^n), $\vec{z} = (z^1, \ldots, z^{n-1},\ z^n = \kappa\varphi)$, $\vec{p}\vec{z} \equiv \sum_{i=1}^{n} p_i z^i$. Substitution of (3.4.2) into (3.4.1) gives

$$\left[-\frac{1}{2}(\frac{\partial}{\partial z^0})^2 + V_0(z^0)\right]\Phi = \mathcal{E}\Phi, \tag{3.4.3}$$

where $\mathcal{E} = \frac{1}{2}\vec{p}\vec{p}$ and $V_0(z^0) = -A\mathrm{e}^{2qz^0}$. Solving (3.4.3), we get

$$\Phi(z^0) = B_{i\sqrt{2\mathcal{E}}/q}(\sqrt{-2A}q^{-1}\mathrm{e}^{qz^0}) \tag{3.4.4}$$

where $i\sqrt{2\mathcal{E}}/q = i|\vec{p}|/q$, and $B = I, K$ are modified Bessel functions. Note that

$$v = \exp(qz^0) = \prod_{i=1}^{n} a_i^{u_i/2} \tag{3.4.5}$$

is the "quasivolume" (3.2.34).

The general solution of Eq.(3.4.1) has the following form:

$$\Psi(z) = \sum_{B=I,K}\int d^n\vec{p}\ C_B(\vec{p})e^{i\vec{p}\vec{z}}B_{i|\vec{p}|/q}(\frac{\sqrt{-2A}}{q}\mathrm{e}^{qz^0}) \tag{3.4.6}$$

where the functions C_B $(B = I, K)$ belong to an appropriate class. For the Λ-term case this solution was considered in [52,48] and for the two-component model $(n = 2)$ and $\Lambda > 0$ in [95].

In the ground state we put all momenta $p^a (a = 1, \ldots, n)$ equal to zero, and the ground state wave function reads:

$$\Psi_0 = B_0\left(\sqrt{-2A}q^{-1}\mathrm{e}^{qz^0}\right). \tag{3.4.7}$$

It is to be stressed that the function Ψ_0 is invariant with respect to the rotation group O(n).

Remark 4. Applying the arguments of [20,46], one can show that the ground state wave function

$$\Psi_0^{(HH)} = I_0\left(\frac{\sqrt{2|A|}}{q}\exp(qz^0)\right), \qquad A < 0, \tag{3.4.8}$$

$$= J_0\left(\frac{\sqrt{2A}}{q}\exp(qz^0)\right), \qquad A > 0, \tag{3.4.9}$$

satisfies the Hartle-Hawking boundary condition [213]. Special cases of this formula were considered in Refs. [20] (the 1-curvature case) and [52] (the Λ-term case).

From (3.4.3) it follows that in the case $A < 0$ (negative energy density) a Lorentzian domain exists as well as a Euclidean one for $\mathcal{E} > 0$. In the case $A > 0$ only the Lorentzian domain occurs for $\mathcal{E} \geq 0$ but for $\mathcal{E} < 0$ both domains exist. The wave functions (3.4.2), (3.4.4) with $A > 0$ and $\mathcal{E} < 0$ describe transitions between the Euclidean and Lorentzian domains, i.e. tunneling universes.

Quantum wormholes

We consider only real values of p_i. In this case we have $\mathcal{E} \geq 0$.

If $A > 0$, the wave function Ψ (3.4.2) is not exponentially damped when $v \to \infty$, i.e. the condition (i) for quantum wormholes (see the Introduction) is not satisfied. The wave function oscillates and may be interpreted as corresponding to the classical Lorentzian solution. For $A < 0$, the wave function (3.4.2) is exponentially damped for large v only, when $B = K$ in (3.4.4). (Recall that

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}, \qquad K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z},$$

for $z \to \infty$). However, in this case the function Φ oscillates infinitely many times when $v \to 0$. Thus the condition (ii) is not satisfied. The wave function describes a transition between the Lorentzian and Euclidean domains.

The functions

$$\Psi_{\vec{p}}(z) = e^{i\vec{p}\vec{z}}K_{i|\vec{p}|/q}(\sqrt{-2A}q^{-1}e^{qz^0}), \tag{3.4.10}$$

may be used for constructing quantum wormhole solutions. As in [246,276,52], we consider superpositions of the singular solutions

$$\hat{\Psi}_{\lambda,\vec{n}}(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dk \Psi_{qk\vec{n}}(z)e^{-ik\lambda}, \tag{3.4.11}$$

where $\lambda \in R$ and $\vec{n} \in S^{n-1}$ is a unit vector ($\vec{n}^2 = 1$). A calculation gives

$$\hat{\Psi}_{\lambda,\vec{n}}(z) = \exp\Big[-\frac{\sqrt{-2A}}{q}e^{qz^0}\cosh(\lambda - q\vec{z}\vec{n})\Big]. \tag{3.4.12}$$

It is easy to verify that Eq.(3.4.12) leads to solutions of the WDW equation (3.4.1) satisfying the quantum wormholes boundary conditions.

Note that the functions

$$\Psi_{m,\vec{n}} = H_m(x^0)H_m(x^1)\exp[-\frac{(x^0)^2+(x^1)^2}{2}], \tag{3.4.13}$$

where

$$x^0 = (2/q)^{1/2}(-2A)^{1/4}\exp(qz^0/2)\cosh(\tfrac{1}{2}q\vec{z}\vec{n}), \tag{3.4.14}$$

$$x^1 = (2/q)^{1/2}(-2A)^{1/4}\exp(qz^0/2)\sinh(\tfrac{1}{2}q\vec{z}\vec{n}), \tag{3.4.15}$$

$m = 0, 1, \ldots,$ are also solutions to the WDW equation with the quantum wormhole boundary conditions. Solutions of such type were previously considered in [83,41,42,46,48]. They are called discrete spectrum quantum wormholes (see [276]) (and may form a basis in the Hilbert space of the system [277]).

Thus in the case considered quantum wormhole solutions (with respect to quasi-volume (3.4.5)) exist for matter with a negative density ($A < 0$).

3.5. A third-quantized model

Another step in studying topological changes in quantum cosmology can be made in the so-called third quantization approach, where it is meant that the WDW equation is considered within a second-quantized scheme.

Here we put $A > 0$, i.e. the matter density is positive. Consider the case of a real Ψ-field as in [268] for simplicity. The WDW equation (3.4.1) corresponds to the action

$$S = \frac{1}{2}\int d^{n+1}z\Psi\hat{H}\Psi. \tag{3.5.1}$$

Consider two bases of solutions to the WDW equation, $\{\Psi_{\rm in}(\vec{p}), \Psi^*_{\rm in}(\vec{p})\}$ and $\{\Psi_{\rm out}(\vec{p}), \Psi^*_{\rm out}(\vec{p})\}$

$$\Psi_{\rm in}(\vec{p}) = \Psi_{\rm in}(\vec{p}, z) = \left[\frac{\pi}{2q\sinh(\pi|\vec{p}|/q)}\right]^{1/2} J_{-i|\vec{p}|/q}\left(\frac{\sqrt{2A}}{q}e^{qz^0}\right)\frac{{\rm e}^{i\vec{p}\vec{z}}}{(2\pi)^{n/2}}; \tag{3.5.2}$$

$$\Psi_{\rm out}(\vec{p}) = \Psi_{\rm out}(\vec{p}, z) = \tfrac{1}{2}\left(\frac{\pi}{q}\right)^{1/2} H^{(2)}_{i|\vec{p}|/q}\left(\frac{\sqrt{2A}}{q}e^{qz^0}\right)\frac{{\rm e}^{i\vec{p}\vec{z}}}{(2\pi)^{n/2}}. \tag{3.5.3}$$

where J_ν and $H^{(2)}_\nu$ are the Bessel and Hankel functions respectively. These solutions are normalized by the following conditions

$$(\Psi_{\rm in}(\vec{p}), \Psi_{\rm in}(\vec{p}')) = (\Psi_{\rm out}(\vec{p}), \Psi_{\rm out}(\vec{p}')) = \delta\left(\vec{p}-\vec{p}'\right) \tag{3.5.4}$$

where

$$(\Psi_1, \Psi_2) = i\int d^n\vec{z}\left(\Psi_1^*\overleftrightarrow{\partial_0}\Psi_2\right) \tag{3.5.5}$$

is the charge form (indefinite scalar product). Here $\Psi_1 \overleftrightarrow{\partial} \Psi_2 = \Psi_1 \partial \Psi_2 - (\partial \Psi_1) \Psi_2$. Due to the asymptotic behaviour

$$\Psi_{\rm in}(\vec{p}, z) \sim c_{\rm in}(|\vec{p}|) \exp(i\vec{p}\vec{z} - i|\vec{p}|z^0), \quad v \to 0, \tag{3.5.6}$$

$$\Psi_{\rm out}(\vec{p}, z) \sim \frac{c_{\rm out}(|\vec{p}|)}{\sqrt{v}} \exp(i\vec{p}\vec{z} - i\frac{\sqrt{2A}}{q} v), \quad v \to +\infty. \tag{3.5.7}$$

where $\Psi_{\rm in}(\vec{p}, z)$ and $\Psi_{\rm out}(\vec{p}, z)$ are negative-frequency modes of "Kasner"- and "Milne"-types respectively.

The standard quantization procedure [279,280] gives us

$$\begin{aligned} \Psi(z) &= \int d^n\vec{p} \left[a^+_{\rm in}(\vec{p})\Psi^*_{\rm in}(\vec{p}, z) + a_{\rm in}(\vec{p})\Psi_{\rm in}(\vec{p}, z)\right] \\ &= \int d^n\vec{p} \left[a^+_{\rm out}(\vec{p})\Psi^*_{\rm out}(\vec{p}, z) + a_{\rm out}(\vec{p})\Psi_{\rm out}(\vec{p}, z)\right], \end{aligned} \tag{3.5.8}$$

where the non-trivial commutators are

$$[a_{\rm in}(\vec{p}), a^+_{\rm in}(\vec{p'})] = [a_{\rm out}(\vec{p}), a^+_{\rm out}(\vec{p'})] = \delta\left(\vec{p} - \vec{p}'\right). \tag{3.5.9}$$

The "in" and "out" vacuum states satisfy the relations

$$a_{\rm in}(\vec{p})|0, {\rm in} >= a_{\rm out}(\vec{p})|0, {\rm out} >= 0. \tag{3.5.10}$$

The modes (3.5.2) and (3.5.3) are related by the Bogoliubov transformation

$$\Psi_{\rm in}(\vec{p}) = \alpha(\vec{p})\Psi_{\rm out}(\vec{p}) + \beta(\vec{p})\Psi^*_{\rm out}(\vec{p}), \tag{3.5.11}$$

$$\begin{aligned} \alpha(\vec{p}) &= \left[\frac{\exp(\pi|\vec{p}|/q)}{2\sinh(\pi|\vec{p}|/q)}\right]^{1/2}, \\ \beta(\vec{p}) &= \left[\frac{\exp(-\pi|\vec{p}|/q)}{2\sinh(\pi|\vec{p}|/q)}\right]^{1/2}. \end{aligned} \tag{3.5.12}$$

The vacua $|0, {\rm in} >$ and $|0, {\rm out} >$ are unitarily inequivalent. A standard calculation [279,280] gives for the number density of "out-Universes" (of "Milne type") contained in the "in-vacuum" ("Kasner-type" vacuum)

$$n(\vec{p}) = |\beta(\vec{p})|^2 = \left(\exp(2\pi|\vec{p}|/q) - 1\right)^{-1}. \tag{3.5.13}$$

Thus we obtain a Planck distribution of created universes with the temperature

$$T_{\rm Pl} = q/2\pi = \sqrt{- < u, u >_*}/4\pi. \tag{3.5.14}$$

The temperature (3.5.14) depends on the vector $u = (u_i)$ (i.e., on the equation of state of the matter content of the Universe): $T_{\rm Pl} = T_{\rm Pl}(u)$. For example, we get $T_{\rm Pl}(u^{(\Lambda)}) = 2T_{\rm Pl}(u^{(\rm dust)})$. In the Zeldovich matter limit $u \to 0$ we have $T_{\rm Pl} \to +0$.

Remark 5. In [285] a regularization of propagators (in quantum field theory) was introduced using the complex signature matrix

$$(\eta_{ab}(w)) = diag(w, 1, \dots, 1), \tag{3.5.15}$$

where $w \in C \setminus (-\infty, 0]$ is a complex parameter (Wick parameter). Path integrals are originally defined (in covariant manner) for $w > 0$ (i.e. in Euclidean-like region) and then analytically continued to negative w. The Minkowsky space limit corresponds to $w = -1 + i0$ (in notations of [285] $w^{-1} = -a$). The prescription [285] is a natural realization of Wick's rotation. In [286] analogs of the Bogoliubov-Parasiuk theorems [278] for a wide class of propagators regularized by the complex metric (3.5.15) were proved. This formalism may be applied for third-quantized models of the multidimensional cosmology. In this case the corresponding path integrals should be analytically continued from the interval $1 < D < 2$ (D is the dimension), where the minisuperspace metric is Euclidean, to $D = D_0 - i0$, $D_0 = 1 + \sum_{i=1}^{n} N_i$. We note also that recently J.Greensite proposed the idea of treating the space-time signature as a dynamical degree of freedom [287] (see also [288-9]).

3.6. Appendix

Proof of Proposition 2. We introduce the new "diagonalized" variables

$$\beta^a = e^a_i \beta^i, \qquad u_a = e^i_a u_i, \qquad v_a = e^i_a v_i \tag{3.6.1}$$

$$(u_a) = (2q, \vec{0}) \qquad (\sigma^a) = (\sigma^i e^a_i) = (q^{-1}, \vec{0}) \tag{3.6.2}$$

and consequently (see (3.2.84))

$$0 = \beta^i u_i = \beta^a u_a = 2q\beta^0 \;\Rightarrow\; (\beta^a) = (0, \vec{\beta}). \tag{3.6.3}$$

From the second relation in (3.2.84) we get

$$G_{ij}\beta^i\beta^j = \eta_{ab}\beta^a\beta^b = \vec{\beta}^2 \leq 1/q^2. \tag{3.6.4}$$

For the vector $(v_a) = (v_0, \vec{v})$ we have $-v_0^2 + \vec{v}^2 = \; <v, v>_* < 0$ and hence

$$|v_0| > |\vec{v}|, \qquad v_0 \neq 0. \tag{3.6.5}$$

We also obtain from (3.6.2) and (3.6.5)

$$< u, v >_* = -u_0 v_0 = -2qv_0 \neq 0. \tag{3.6.6}$$

Using relations (3.6.2), (3.6.3) and (3.6.5) we get

$$\begin{aligned}(\sigma^i + \beta^i)v_i &= (\sigma^a + \beta^a)v_a \\ &= q^{-1}v_0 + \vec{\beta}\vec{v} = q^{-1}v_0(1 + \frac{q}{v_0}\vec{\beta}\vec{v}).\end{aligned} \tag{3.6.7}$$

Eqs. (3.6.4), (3.6.5) imply the inequality

$$|\frac{q}{v_0}\vec{\beta}\vec{v}| \leq \frac{|\vec{v}|}{v_0} q|\vec{\beta}| \leq \frac{|\vec{v}|}{v_0} < 1. \tag{3.6.8}$$

From (3.6.6)-(3.6.8) (and $q > 0$) we obtain the proposed identity (3.3.85). The proposition is proved.

4. Quantum Dynamics of Inhomogeneous Kaluza-Klein Cosmological Models near the Cosmological Singularity [290]

4.1. Introduction

One of the most difficult problems of modern theoretical physics is the problem of the cosmological singularity. Singularities follow from the classical theory and, as is widely accepted, need quantum gravity to provide its exhaustive description. We do not have any reasonable theory of such a kind yet save, presumably, the superstring theory [23]. And as is known, the last one adds some new features to the existing Einstein gravity. In particular, the superstring theory predicts the dimension of the universe exceeds that of we use to experience at a macroscopic level. In the present Universe additional dimensions are supposed to be compactified to the Planckian size, and display themselves as a set of ordinary matter fields. However, close to the singularity one should expect that all dimensions to play an equal role, and have to be regarded on an equal footing. This enables us to consider more general than Einstein's one multidimensional theories of gravity [21,37,257] in order to study the nature and properties of singularitites.

From the classical point of view properties of general inhomogeneous cosmological Kaluza-Klein models near the singularity were recently considered in Ref. [73] (for more early investigations of the problem see also Refs.[53,54]). It was shown that the properties of metric functions near the singularity may be well-described in the framework of asymptotic models. In this paper we are considering a quantum description of just those models and investigate their behavior near the singularity from the quantum point of view. The main result of this paper is that in the case of $n \leq 9$ (n is the number of spatial dimensions) estimates for mean values of scale functions turns out to be of the same order as in the classical theory. For mean scale factor we get $< a_i >=< g^{Q_i} >\sim cg^{Q_{\min}}$ as $g \to 0$, where g is the metric determinant which near the singularity may serve as a time variable, $Q_{\min} = -\frac{n-3}{n+1}$ is the minimal admissible value of the anisotropy parameters Q_i and c is a slowly varying with g function, including quantum corrections, and differing from the classical one. When considering dimensions exceeding $n = 9$ the situation changes drastically. The potential does not restrict the configuration space and, therefore, we have no states which would be localized on the space of Q_i. If we get ready a localized state (a wave packet) the width of the packet spreads eventually more and more out and simultaneously the center of the wave packet runs to the infinity of the configuration space. In classical theory this signals us that the oscillatory mode becomes unstable and transforms into a Kasner-like behavior. Therefore, different mean values will depend upon the initial state crucially.

The section is organized as follows. In Sec.4.2 we use generalized Kasner variables introduced first in Ref. [61] and adapted to the multidimensional models in Ref. [73] to divide basic variables into two parts. Near the singularity the first part has a behavior like a set of coupled scalar fields while residual variables behave as a set of vector fields and can be neglected in a leading order (in the same manner as it happens for the matter having an equation of state $\epsilon > p$, where p and ϵ are an energy density and pressure respectively) [76,73]. The asymptotic model is derived in Sec.4.3. In Sec.4.4 we consider the quantization of the model. The Wheeler-DeWitt equation turns out to be dependent upon the first group variables only. We solve this equation in a lattice approximation of the coordinate manifold. The probability interpretation is introduced by making use of an explicit selection of a positive frequency sector on the space of solutions to the Wheeler-DeWitt equation [291]. Such procedure implies an ambiguity and, therefore, the same ambiguity will be inherently presented in the obtained quantum gravity. In order to overcome this difficulty we, in Sec.4.5,6, discuss the possibility of the third quantization. We note that the third quantization seems to be the natural scheme providing a description of different possible topologies of the universe [51,292]. We use the scheme proposed in [293] to show that in the course of the evolution the presence of matter, e.g. of an ordinary scalar field, can result in an increasing of quantum topology fluctuations and, therefore, properties of inhomogeneities of the metric may completely be determined by vacuum fluctuations in the third quantized theory. We conclude this paper with some estimates and speculations in Sec.4.7.

4.2. Generalized Kasner Solution, Generalized Kasner Variables

Aiming to obtain a quantum description of inhomogeneous Kaluza-Klein models we start with the canonical formulation of multidimensional gravity. In this formulation basic variables are the spatial Riemann metric components $g_{\alpha\beta}$ and the matter source which will be taken in the form of a scalar field ϕ and its conjugate momenta $\Pi^{\alpha\beta}=\sqrt{g}(K^{\alpha\beta}-g^{\alpha\beta}K)$ and Π_ϕ. These variables are functions specified on the $n-$manifold S $(\alpha=1,...,n)$ and $K^{\alpha\beta}$ is the extrinsic curvature of S. For the sake of simplicity we shall consider S to be compact i.e. $\partial S=0$ (one may consider S to be the n-dimensional sphere though this will not have any significance for our investigation). The action has the following form in Planck units (see for example [294])

$$I=\int_S(\Pi^{ij}\frac{\partial g_{ij}}{\partial t}+\Pi_\phi\frac{\partial\phi}{\partial t}-NH^0-N_\alpha H^\alpha)d^nxdt, \tag{4.2.1}$$

where

$$H^0=\frac{1}{\sqrt{g}}\left\{\Pi^\alpha_\beta\Pi^\beta_\alpha-\frac{1}{n-1}(\Pi^\alpha_\alpha)^2+\frac{1}{2}\Pi^2_\phi+g(W(\phi)-R)\right\}, \tag{4.2.2}$$

$$H^\alpha=-2\Pi^{\alpha\beta}_{|\beta}+g^{\alpha\beta}\partial_\beta\phi\Pi_\phi, \tag{4.2.3}$$

here

$$W(\phi)=\frac{1}{2}\left\{g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi+V(\phi)\right\}. \tag{4.2.4}$$

It turns out to be convenient to use the so-called generalized Kasner-like parametrization of the dynamical variables [61,73]. The metric components and their conjugate momenta are represented as follows

$$g_{\alpha\beta}=\sum_a\exp\{q^a\}\,l^a_\alpha l^a_\beta\,,\qquad \Pi^\alpha_\beta=\sum_a p_a L^\alpha_a l^a_\beta\,, \tag{4.2.5}$$

where $L^\alpha_a l^b_\alpha=\delta^b_a$ $(a,b=0,...,(n-1))$, and the vectors l^a_α contain only $n(n-1)$ arbitrary functions of spatial coordinates. Further parametrization may be taken in the form [73]

$$l^a_\alpha=U^a_bS^b_\alpha,\qquad U^a_b\in SO(n),\qquad S^a_\alpha=\delta^a_\alpha+R^a_\alpha \tag{4.2.6}$$

where R^a_α denotes a triangle matrix ($R^a_\alpha = 0$ as $a < \alpha$.Substituting Eq.(4.2.5), (4.2.6) into (4.2.1) we find the following expression for the action functional

$$I = \int_S (p_a \frac{\partial q^a}{\partial t} + T^\alpha_a \frac{\partial R^a_\alpha}{\partial t} + \Pi_\phi \frac{\partial \phi}{\partial t} - NH^0 - N_\alpha H^\alpha) d^n x dt, \tag{4.2.7}$$

where $T^\alpha_a = 2\sum_b p_b L^\alpha_b U^b_a$ and the Hamiltonian constraint takes the form

$$H^0 = \frac{1}{\sqrt{g}} \left\{ \sum p_a^2 - \frac{1}{n-1} (\sum p_a)^2 + \frac{1}{2}\Pi^2_\phi + V \right\}. \tag{4.2.8}$$

In the case of $n = 3$ the functions R^a_α are connected purely with transformations of a coordinate system and may be removed by resolving momentum constraints $H^\alpha = 0$ [61]. However, in the multidimensional case the functions R^a_α contain $\frac{n(n-3)}{2}$ dynamical functions as well.

4.3. Asymptotic model in the case of arbitrary small times

As it was shown, [61,55,73] (see also [294]), in the vicinity of a singularity the potential term in (4.2.8) can be modeled by potential walls. To this end we represent the potential in the following form

$$V = \sum_{A=1}^{k} \lambda_A g^{\sigma_A}, \tag{4.3.1}$$

where λ_A is a set of functions of all dynamical variables and of their derivatives and σ_A is given by the expression

$$\sigma_{abc} = 1 + Q_a - Q_b - Q_c, \qquad b \neq c, \tag{4.3.2}$$

where Q_a are the anisotropy parameters $Q_a = \frac{q^a}{\sum q}$. Then assuming the finiteness of the functions λ_A and considering the limit $g \to 0$ we find that the potential V may be modeled by potential walls

$$g^{\sigma_A} \to \theta_\infty[\sigma_A(Q)] = \begin{cases} +\infty\ , & \sigma_A < 0, \\ 0\ , & \sigma_A > 0 \end{cases} \tag{4.3.3}$$

Thus, if we put the expressions (4.3.3) into (4.3.8) we find that the Hamiltonian constraint does not depend on the variables R^a_α and its conjugate momenta T^α_a. Of course this is an approximation and the real potential reserves a dependence of this group of variables and, therefore, one should consider the model (4.3.3), (4.2.8) as a first step in an approximation procedure. Then the rest of dynamical variables as well as an ordinary matter sources with the state equation satisfying the inequality $\epsilon > p$ may be accounted in subsequent steps of the approximation procedure [76].

Now we can remove the rest of dynamical functions T^α_a, R^a_α from the action (4.2.7) by putting $N^\alpha = 0$. Then we get the reduced dynamical system

$$I = \int_S \left\{ p_a \frac{\partial q^a}{\partial t} + \Pi_\phi \frac{\partial \phi}{\partial t} - \lambda \left\{ \sum p^2 - \frac{1}{n-1} (\sum p)^2 + \frac{1}{2}\Pi^2_\phi + U(Q) \right\} \right\} d^n x dt, \tag{4.3.4}$$

where λ is expressed via the lapse function as $\lambda = \frac{N}{\sqrt{g}}$.

The configuration space M of the system (4.3.4) (called also superspace) can be represented in the form of the direct product $M = \prod_{x \in S} M_x$. Moreover, every local space M_x is the ordinary $n+1$-dimensional pseudo-Euclidean space. Indeed, one can choose on M a new harmonic set of variables related to the old ones as follows

$$q^a = A^a_j z^j + z^0, \qquad z^n = \sqrt{\frac{2}{n(n-1)}}\phi, \tag{4.3.5}$$

where $j = 1, ..., n-1$, $a = 0, ...n-1$ and the matrix A^a_j is a constant [73] which obeys the conditions

$$\sum_a A^a_j = 0, \qquad \sum_a A^a_j A^a_k = n(n-1)\delta_{jk} \ , \tag{4.3.6}$$

and can be expressed in the explicit form as

$$A^a_j = \sqrt{\frac{n(n-1)}{j(j+1)}}(\theta^a_j - j\delta^a_j), \qquad \theta^a_j = \left\{ \begin{array}{ll} 1 \ , & j > a \ , \\ 0 \ , & j \leq a \ . \end{array} \right. . \tag{4.3.7}$$

Then the action (4.3.4) takes the form formally coincided with the action for a continueous set of relativistic particles

$$I = \int_S \left\{ P_r \frac{\partial z^r}{\partial t} - \lambda'(P_i^2 + U - P_0^2) \right\} d^n x dt, \tag{4.3.8}$$

where $r = 0, ..., n$, $i = 1, ..., n$, $\lambda' = \frac{\lambda}{n(n-1)}$ and the kinetic term, that determines a metric on M_x, turns out to be coincided with that of the ordinary flat $n+1$-dimensional pseudo-Euclidean spacetime manifold.

4.4. Quantization and the probability interpretation

As it was mentioned above the action (4.3.8) resembles the action for a continueous set of relativistic particles. Therefore, quantization of such a system may be carried out in the complete analogy with that of relativistic particles [295]. The zero-energy Hamiltonian constraint leads to the set of the Wheeler-DeWitt equations [273]

$$(-\Delta_x + U_x + \xi P_x)\Psi = 0, \qquad x \in S, \tag{4.4.1}$$

where Ψ is the wave function of the universe, Δ_x denotes a Laplace operator on M_x : $\Delta_x = \frac{1}{\sqrt{-G}}\partial_A \sqrt{-G} G^{AB}\partial_B$, G_{AB} is the metric on M_x determined by the interval

$$\delta\Gamma(x)^2 = \frac{1}{4\lambda'}((\delta z^i(x))^2 - (\delta z^0(x))^2), \tag{4.4.2}$$

P_x is the curvature scalar of M_x. The value of ξ should be chosen as $\xi = \frac{n-1}{4n}$ to provide a conformal invariance of Eq.(4.4.1) which reflects the arbitrariness in the choice of the lapse function λ . Indeed, the transformation

$$G_{AB} \rightarrow \widetilde{G}_{AB} = e^{-2\Omega} G_{AB}, \qquad \Psi \rightarrow \widetilde{\Psi} = e^{\frac{n-1}{2}\Omega}\Psi$$

transforms the Eq.(4.1) into

$$(-\widetilde{\Delta}_x + e^{2\Omega}U_x + \frac{n-1}{4n}\widetilde{P}_x)\widetilde{\Psi} = 0.$$

and the theory becomes independent on a particular choice of λ.

To solve the equation (4.4.1) we shall consider a lattice approximation. To this end we shall suppose the existence of a sufficiently small minimal scale of inhomogeneity for all fields l_{min} , so that the coordinates x will take discrete values only. The continueous limit one obtains tending l_{min} to zero, though, from the other side, one may think of the lattice model as of a background model and treat the scales less than l_{min} as small perturbations.

The system of equations (4.4.1) turns out to be uncoupled, for each from these equations contains a set of functions which are specified at a distinct point x of S . We shall call such sets as $x-$sets. Therefore, the space H of solutions to this system takes the form of the tensor product of spaces H_x ($H = \prod_{x\in S} H_x$) as that of M, where H_x is the space of solutions to a distinct $x-$ equation (4.4.1). Accordingly, all $x-$sets of degrees of freedom may independently be considered. Therefore, at first it will be convenient to work out the probability interpretation and all the technique on the example of one local $x-$set of degrees of freedom and after that to generalize it to the case of all degrees of freedom.

The space of solutions to the WDW equation for a distinct x-set of degrees of freedom.

Every local x-equation (4.4.1) admits the conserved current $J_A(\Psi,\Psi) = i[\Psi^*\nabla_A\Psi - \Psi\nabla_A\Psi^*]$ which may be used to determine the inner product in the space H_x

$$<\varphi \mid \chi> = i\int_{\Sigma_x} J_A(\varphi,\chi)d\Sigma_x^A, \tag{4.4.3}$$

where Σ_x is an arbitrary space-like surface on M_x and ∇_A denotes a covariant derivative on x-metric (4.4.2).

To construct a complete set of solutions to the local Eq.(4.4.1) it turns out to be convenient by making use of the so-called Misner-Chitre like variables [61,73] ($\vec{y} = y^j$, $j = 1,...,n-1$)

$$z^0 = -e^{-\tau}\frac{1+y^2}{1-y^2}, \qquad \vec{z} = -2e^{-\tau}\frac{\vec{y}}{1-y^2}, \qquad y = \mid \vec{y} \mid \leq 1. \tag{4.4.4}$$

In these variables the anisotropy parameters become independent of the timelike variable τ

$$Q_a(y) = \frac{1}{n}\left\{1 + \frac{2A_j^a y^j}{1+y^2}\right\} \tag{4.4.5}$$

and that of the potential $U(Q)$ in Eq.(4.4.1). The metric (4.4.2) in the new variables takes the form

$$\delta\Gamma(x)^2 = \frac{e^{-2\tau}}{4\lambda'}(\frac{4(\delta y^j)^2}{(1-y^2)^2} + e^{2\tau}(\delta z^n)^2 - (\delta\tau)^2). \tag{4.4.6}$$

For the sake of simplicity we shall use the gauge $4\lambda' e^{2\tau} = 1$ in what follows.

The part of the configuration space M_x related to the variables $\vec{y}$ is a realization of the $(n-1)$-dimensional Lobachevsky space and the potential U cuts a part K of it [54,53,73]

$$\sigma_{abc} = 1 + Q_a - Q_b - Q_c \geq 0, \qquad a \neq b \neq c \tag{4.4.7}$$

which in the case $n \leq 9$ has a finite volume. We shall suppose that there is a set of solutions to the eigenvalue problem for the Laplace - Beltrami operator

$$(\Delta_y + k_J^2 + \frac{(n-2)^2}{4})\varphi_J(z) = 0, \qquad \varphi_J \mid_{\partial K} = 0, \tag{4.4.8}$$

where the Laplace operator Δ_y is constructed via the metric $dl^2 = h_{ij}dy^i dy^j = \frac{4(dy)^2}{(1-y^2)^2}$ and J collects all indices numbering the eigenfunctions φ_J. In the case of $n < 10$ the region K has a finite volume and J takes discrete values ($J = 0, 1, 2, \ldots$), while for $n \geq 10$ the volume of K is infinite and the spectrum of the Laplace - Beltrami operator becomes a continueous one. The functions φ_j obey the orthogonality and normalization relations

$$(\varphi_I, \varphi_J) = \int_K \varphi_I^*(y)\, \varphi_J(y) d\mu(y) = \delta_{IJ}, \tag{4.4.9}$$

where $d\mu(y) = \frac{1}{c}\sqrt{h} d^{n-1}y = \frac{2^{n-1}}{c}\frac{d^{n-1}y}{(1-y^2)^{n-1}}$, and c is the volume of K. The completeness conditions are

$$\sum_I \varphi_I^*(y)\, \varphi_I(y') = \frac{\delta\,(y-y')}{\sqrt{h}}.$$

Then a complete orthonormal set $\left\{u_p, u_p^*\right\}$ of solutions to $x-$equation (4.4.1) is constituted by functions of the form

$$u_p = \exp(-\frac{1}{2}\tau)\chi_p(\tau)\Phi_p\,(y,z) \quad , \quad \Phi_p\,(y,z) = (2\pi)^{-1/2}\varphi_J(y)\exp(i\epsilon z^n) \tag{4.4.10}$$

where $p = (J, \epsilon)$. Functions $\chi_p(\tau)$ satisfy the equation following from (4.4.1):

$$\frac{d^2\chi_p}{d\tau^2} + \omega_p^2(\tau)\chi_p = 0, \qquad \omega_p^2(\tau) = k_J^2 + \epsilon^2 e^{-2\tau} \tag{4.4.11}$$

with the normalization condition $\chi_p^* \frac{d\chi_p}{d\tau} - \chi_p \frac{d\chi_p^*}{d\tau} = -i$, and are expressed via the Bessel functions. The initial conditions to Eq.(4.4.11) at a moment τ_0 are to be taken in the form $\chi_p(\tau_0) = \frac{1}{\sqrt{\omega_p(\tau_0)}}$, $\chi_p'(\tau_0) = -i\omega_p\,(\tau_0)\,\chi_p(\tau_0)$.

The set of solutions (4.4.10) is orthonormal in the sense of the scalar product (4.4.3), i.e. they satisfy the relations

$$< u_p \mid u_q >= - < u_p^* \mid u_q^* >= \delta_{pq}, \qquad < u_p \mid u_q^* >= 0. \tag{4.4.12}$$

Thus, an arbitrary solution f to the local Wheeler-DeWitt equation (4.4.1) can be represented in the form

$$f = \sum_p A_p^+ u_p + A_p^- u_p^*, \tag{4.4.13}$$

where $A_p^{\pm}$ are arbitrary constants which are to be specified by initial conditions.

Probability interpretation and the case of all degrees of freedom

Since the norm determined by the scalar product (4.4.3) turns out to be sign-indefinite we face up with the difficulty of probability interpretation. The simplest way to define a positive-definite inner product is to separate a submanifold H_x^+ on the space H_x which is of "positive frequency". If we suppose $A_p^- = 0$ in (4.4.13), then the normalization condition for f takes the form

$$< f \mid f >= \sum_p \mid A_p^+ \mid^2 = 1, \tag{4.4.14}$$

and meets no difficulties. Thus, the subspace of physical states H_x^+ becomes the ordinary Hilbert space and we can adopt the standard probability interpretation [295].

Now the generalization to the case of all degrees of freedom may be carried out straightforwardly. The positive frequency sector H^+ in the total space of solutions H we determine as the direct product of positive frequency local submanifolds $H^+ = \prod_{x\in S} H_x^+$. Thus, the wave function takes the form

$$\Psi = \sum_{[p(x)]} F_{p(x)} U_{p(x)}, \qquad U_{p(x)} = \prod_{x\in S} u_{p(x)} \tag{4.4.15}$$

with the scalar product induced by (4.4.12)

$$\langle \chi | \psi \rangle = \sum_{[p(x)]} B^*_{p(x)} A_{p(x)} \ , \tag{4.4.16}$$

where $\chi = \sum B_{p(x)} U_{p(x)}$ and $\psi = \sum A_{p(x)} U_{p(x)}$ are arbitrary vectors from H^+ .

Dispite that Eq. (4.4.15) and (4.4.16) give already well defined probability interpretation it is necessary to mention that the procedure of the choice of H_x^+ in the H_x is not uniquely defined. We can use a Bogoliubov transformation to construct a new set of modes

$$v_{p,x} = \sum_q \left\{ \alpha\left(x\right)_{pq} u_q + \beta\left(x\right)_{pq} u_q^* \right\} \tag{4.4.17}$$

where we add the label x to point out the possible dependence on $x \in S$ and while $\beta\left(x\right)_{pq} \neq 0$ different sets of modes (4.4.17) define different submanifolds H_x^+. The situation will be worse still when considering the total space H. Therefore, the probability interpretation turns out to be crucially dependent upon the particular choice of the physical sector H^+ in H . Here we face with the main inherent difficulty of quantum cosmology which, apparently, cannot be solved in the framework of the ordinary "one-particle" quantum gravity. To overcome this difficulty it is necessary to use the procedure of second (or "third") quantization of the wave function of the Universe [51,296,292,297].

4.5. Third quantization

In addition to provide a probability interpretation third quantization has another goal. This theory allows describe processes connected with topology changes. The simplest processes of such a kind was widely discussed earlier in connection with wormholes and baby universes [51] and in the context of a description of a quantum creation of the

Universe from nothing [21,296,297]. In the present section we use a new approach pointed out in Refs. [292,293] which generalizes the third quantization and allows to describe arbitrary topologies of the universe. That generalization follows from the fact that the system of WDW equations (4.4.1) is uncoupled in the leading order. Therefore, one may secondly quantize every x-set of degrees of freedom independently from each other. In quantum gravity this corresponds to the situation when the number of points of the physically observable space, specified at a particular point of the basic coordinate manifold S, turns out to be a variable and topology of the physical space may be different from that of S [292-3] (below we shall follow Ref. [293]).

Let us consider a distinct $x-$set of the degrees of freedom. While we do not account for interactions between these sets we can describe quantum states of each set by a local wave functions Ψ_x . When the third quantization is imposed the wave functions Ψ_x become field operators and can be expanded in the form (4.4.13) (for simplicity we consider Ψ_x to be a real scalar function):

$$\Psi_x = \sum C(p,x)u(p,x) + C^+(p,x)u^*(p,x), \tag{4.5.1}$$

where $u(p,x)$ is the set of modes (4.4.12) and the label x we add to point out the possible dependence on spatial coordinates. Now we consider the operators $C(p,x)$ and $C^+(p,x)$ to satisfy the standard commutation relations

$$[C(p,x), C^+(q,x')] = \delta_{p,q}\delta(x,x'). \tag{4.5.2}$$

The field operators Ψ_x act on a Hilbert space of states which has the well known structure in the Fock representation. The vacuum state is defined by the relations $C(x,p) \mid 0 >= 0$ (for all $x \in S$), $< 0|0 >= 1$. Acting by the creation operators $C^+(p,x)$ on the vacuum state we can construct states describing the Universe with arbitrary spatial topologies. In particular, the states of the type (4.4.15) describing the Universe whose spatial topology coinsides with the topology of S take the structure

$$|f >= \sum_{[p(x)]} F_{p(x)}|1_{p(x)} >, \qquad |1_{p(x)} >= \frac{1}{Z_1}\prod_{x\in S} C^+(x,p(x))|0 >, \tag{4.5.3}$$

where Z is a normalization constant and the wave function (4.4.15) describing the simple-topology Universe can be found as

$$< 0|\Psi|f >=< 0|\prod_{x\in S}\Psi_x|f >= \sum_{[p(x)]} F_{p(x)}U_{p(x)}. \tag{4.5.4}$$

The states describing the Universe with n disconnected spatial components have the structure

$$|n >= |1_{p_1(x)}, ..., 1_{p_n(x)} >= \frac{1}{Z_n}\prod_{i=1}^{n}\prod_{x\in S} C^+(x,p_i(x))|0 > \tag{4.5.5}$$

(we recall that in the model under consideration in virtue of the existence of l_{min} the coordinates x take discrete values). Besides these states describing simplest topologies the approach considered allows to construct nontrivial topologies as well. This is due to the fact that the tensor product in (4.5.3), (4.5.5) may be defined either over the whole coordinate manifold S or over part of it $D \subset S$. In this manner, taking sufficiently small pieces D_i of the coordinate manifold S we can glue arbitrarily complex physical spaces. In order to construct the states of such a kind it is convenient to introduce a set of operators as follows

$$a(D,p(D)) = \prod_{x\in D} C(x,p(x)), \qquad a^+(D,p(D)) = \prod_{x\in D} C^+(x,p(x)). \tag{4.5.6}$$

These operators have a clear interpretation, e.g. the operator $a^+(D,p(D))$ creates the whole region $D \in S$ having the quantum numbers $p(D)$. Thus, in the general case states of the Universe will be described by vectors of the type

$$|\Phi >= c_0|0> + \sum_I c_I a_I^+|0> + \sum_{I,J} c_{IJ} a_I^+ a_J^+|0> + ... \tag{4.5.7}$$

Now consider an interpretation of the scheme suggested in [293]. Ordinary measurements are usually performed only on a part K of the coordinate manifold S. There are two possibilities. The first one is that an observer measures all of the quantum state of the region K and, the second, more probable one is when the observer measures only a part of the state. In the second case the observer considers K as if it were a part of the ordinary flat space. Therefore, the part of the quantum state which will be measured, appears to be in a mixed state. This means the loss of quantum coherence widely discussed in Refs. [51]. In order to describe measurements of the second type we define the following density matrix for the region K

$$\rho^{nm}(K) = \frac{1}{N(K)} < \Phi|a^+(K,n(K))a(K,m(K))|\Phi >, \tag{4.5.8}$$

where $|\Phi>$ is an arbitrary state vector of the (4.5.7) type and $N(K)$ is a normalization function which measures the difference between the real spatial topology and the coordinate manifold S. If we consider the smallest region K which contains only one point x of the space S the normalization function $N(x)$ in (4.5.8) will play the role of a "density" of the physical space. For the states (4.5.3), (4.5.5) we have $N(x) = 1$ and $N(x) = n$ respectively. Thus, if $A(K)$ is any observable we find $< A >= \frac{1}{N}Tr(A\rho)$.

4.6. Topology fluctuations and quantum creation of the Universe from nothing

Since the WDW Eq.(4.4.1) has an explicit "time"-dependent form one could expect the existence of quantum polarization effects (topology fluctuation or the so-called spacetime foam [298-9]). These effects can be calculated either by singling out the asymptotic *in* and *out* regions on the configuration space M for which we can determine positive-frequency solutions to Eq.(4.4.1) (see for example [297]), or by using the diagonalization of the Hamiltonian technique [280] by means of calculating depending on time Bogoliubov's coefficients. Let us consider solutions (4.4.10) of the arbitrary local x-equation (4.4.1). The function χ_p can be decomposed in positive and negative frequency parts

$$\chi_p = \frac{1}{\sqrt{2\omega_p}}(\alpha_p e^{-i\theta_p} + \beta_p e^{i\theta_p}), \qquad \frac{d\chi_p}{d\tau} = -i\sqrt{\frac{\omega_p}{2}}(\alpha_p e^{-i\theta_p} - \beta_p e^{i\theta_p}), \tag{4.6.1}$$

where $\theta_p = \int_{\tau_0}^{\tau} \omega_p d\tau$. The functions α_p and β_p satisfy identity $|\alpha_p|^2 - |\beta_p|^2 = 1$ and define the depending on time Bogoliubov coefficients [280]. The depending on time creation and annihilation operators take the form

$$\begin{aligned} b_\tau(x,p) &= \alpha_p(\tau) C(x,p) + \beta_p^*(\tau) C^+(x,p) \quad , \\ b_\tau^+(x,p) &= \alpha_p^*(\tau) C^+(x,p) + \beta_p(\tau) C(x,p) \quad . \end{aligned} \tag{4.6.2}$$

In terms of these operators the super-Hamiltonian of the field Ψ_x (the Hamiltonian density) becomes diagonal

$$E_x = \int_{\Sigma^\tau} \Theta_{\tau A} d\Sigma_x^A = \frac{1}{2}\sum_p \omega_p(\tau)\left(b_\tau^+(x,p)\, b_\tau(x,p) + b_\tau(x,p)\, b_\tau^+(x,p)\right) \ , \tag{4.6.3}$$

where $\Theta_{AB} = \nabla_A \Psi_x \nabla_B \Psi_x - \frac{1}{2} G_{AB}\left(\nabla_C \Psi_x \nabla^C \Psi_x - (U + \xi P)\Psi_x^2\right)$ and $d\Sigma_x^\tau = \sqrt{G^n} d^{n-1} y dz$, G^n is the metric on Σ_x^τ induced by (4.4.6) . The ground state of the Hamiltonian is deternined by the conditions $b_\tau(x,p)|0_\tau\rangle = 0$ for all x and p and is also depending on time. The excitations of (4.6.3) are interpreted as points of physical space having the coordinate $x \in S$.

Now we determine two asymptotic regions as *in* ($\tau \to -\infty$) and *out* ($\tau_0 \to +\infty$). In these regions the functions α_p and β_p take constant values. Substituting the initial conditions $\alpha_p = 1$, $\beta_p = 0$ as $\tau_0 \to -\infty$ in (4.4.11), (4.6.1) we find that in the *out* region the Bogoliubov coefficients are

$$\alpha_p = (\exp(\pi k_J)/2sh(\pi k_J))^{\frac{1}{2}}, \qquad \beta_p = (\exp(-\pi k_J)/2sh(\pi k_J))^{\frac{1}{2}} . \tag{4.6.4}$$

Then, for example, if the initial state of the "superspace"-Hamiltonian (4.6.3) is the ground state $|0_{in}\rangle$, in the *out* region the density matrix (4.5.8) takes form

$$\rho^{pq}(K) = \prod_{x \in K} \rho^{p(x)q(x)}(x), \tag{4.6.5}$$

where $\rho(x)$ is the one-point density matrix

$$\rho^{pq}(x) = \frac{1}{N(x)}|\beta_p|^2 \delta(p,q) = \frac{1}{N(x)}\frac{1}{e^{2\pi k_J} - 1}\delta(p,q) \ , \tag{4.6.6}$$

with $N(x)$ being the normalization function $N(x) = \sum_p \frac{1}{e^{2\pi k_J - 1}}$. This one-point density matrix does not depend on spatial coordinates and has the Plankian form with the temperature $T = \frac{1}{2\pi}$ and therefore, the density matrix (4.6.5) describes a Universe which in average turns out to be homogeneous.

4.7. Estimates and concluding remarks

In this manner the Universe appears to be homogeneous just after topology fluctuations are accounted for. If, on the contrary, one does not consider topology fluctuations, properties of inhomogeneities of the metric depend crucially upon the choice of initial data. Despite this, when $n \leq 9$, near the singularity the behavior of lengths in time shows universal features. This occurs, in the first place, due to the fact that the main contrubution to the mean scales $\left\langle g^{Q_a}\right\rangle$ is given just by those regions of the configuration space in which the anisotropy parameters Q_a take the minimal values. They are the points $Q_a^* = -\frac{n-3}{n+1}$ lying on the boundary ∂K (see, for more detail Ref. [73]). Since at the boundary the eigenfunctions $\varphi_J = 0$, in the neighborhood of ∂K we have $\varphi_J \approx k_J(Q - Q^*)$ and the probability density can be estimated as $P\left(Q\right) \sim \left(Q - Q^*\right)^n$(we recall that in classical theory we had $P_{cl}\left(Q\right) \sim \left(Q - Q^*\right)^{n-2}$ and the need to average out the scale function appeared as a result of a stochastic behavor of the metric functions in space and time). Thus, in the same way as in Ref. [73] for $n > 3$ in the limit $g \to 0$ we find for moments of the scale functions ($M > 0$)

$$\left\langle g^{MQ_a}\right\rangle = C_a\left(M, \tau\right) \frac{g_*^{MQ^*}}{\left(M \ln 1/g_*\right)^{n+1}} ,$$

in the case $n > 3$ and for $n = 3$ the esimate

$$\left\langle g^{MQ_a}\right\rangle = C_a\left(M, \tau\right)\left(M \ln 1/g_*\right)^{-5/2}$$

where $g_* = g\left(\tau, Q^*\right)$ and C_a is a slowlly varying in time function which includes information of initial quantum state. Thus, one can see that in quantum theory the average lengths are also increasing.

In the case of $n > 9$ the volume of K is infinite and the eigenfunction (4.4.8) proves to be non-normalizable and, therefore, we have no states which would be localized on K. If we get ready a localized state (a wave packet) the width of the packet spreads eventually more and more out and simultaneously the center of the wave packet runs to the infinity of the configuration space. In classical theory this signals us that the oscillatory mode becomes unstable and transforms into a Kasner-like behavior. Therefore, different mean values depend upon the initial state crucially.

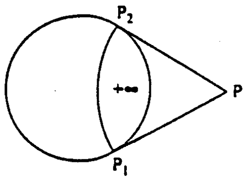

Figure 1. An example of a billiard for $n = 3, m_+ = 1$.

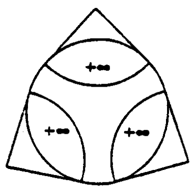

Figure 2. Billiard with infinite volume for $n = 3, m_+ = 3$.

Billiard representation for multidimensional cosmology

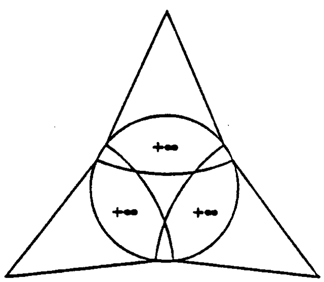

Figure 3. Compact billiard for $n = 3, m_+ = 3$.

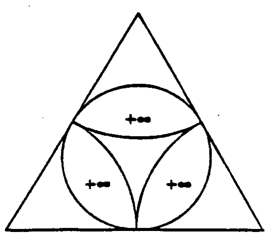

Figure 4. Billiard corresponding to Bianchi-IX model (non-compact with finite volume).

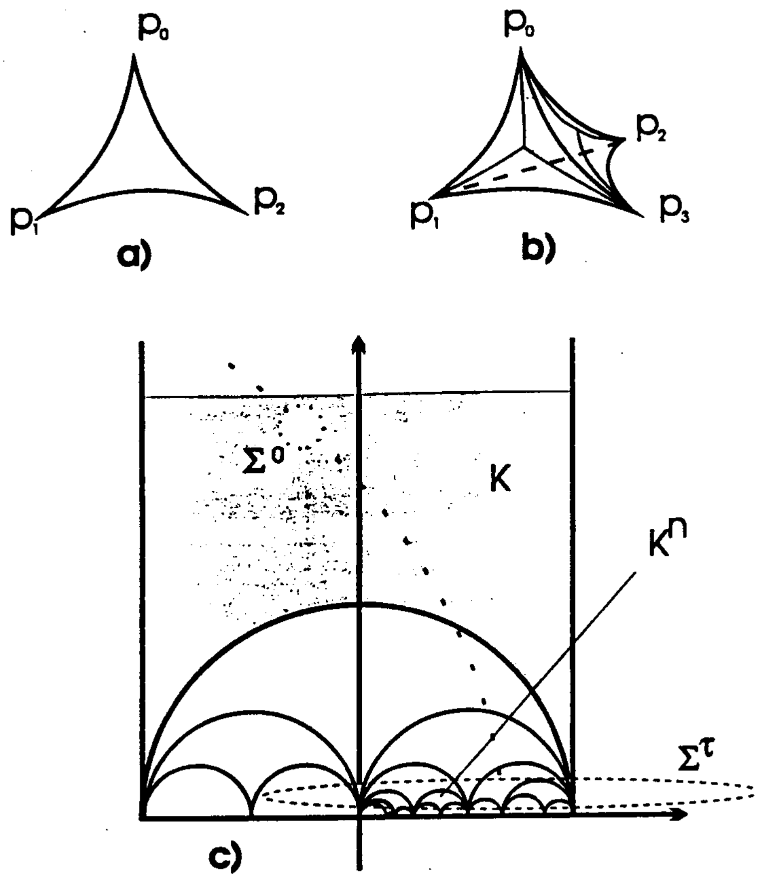

Fig. 5

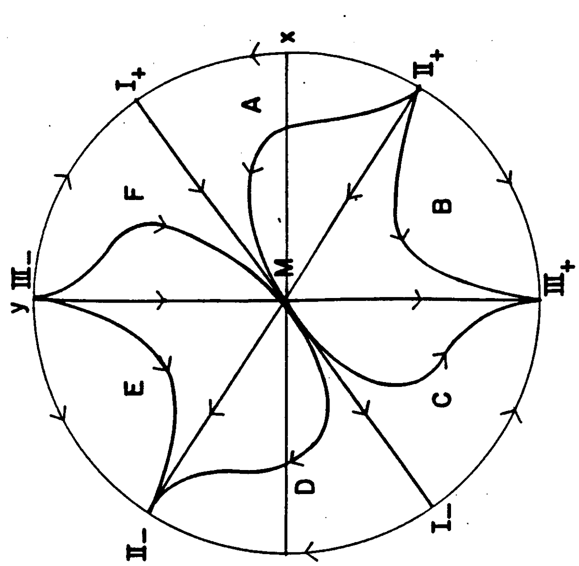

Figure 7

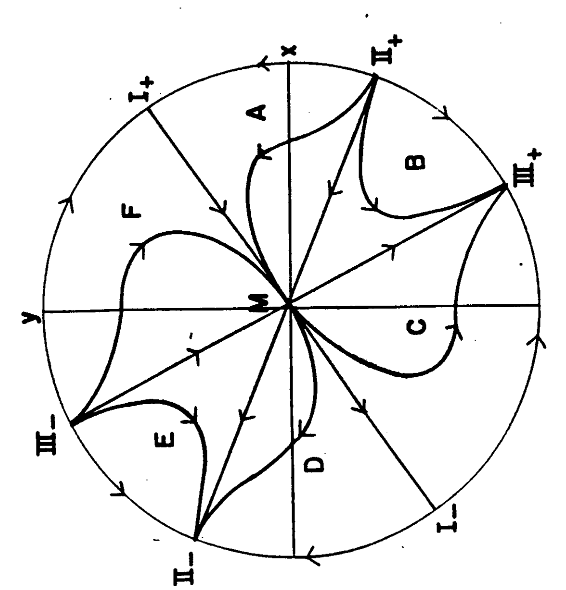

Figure 6

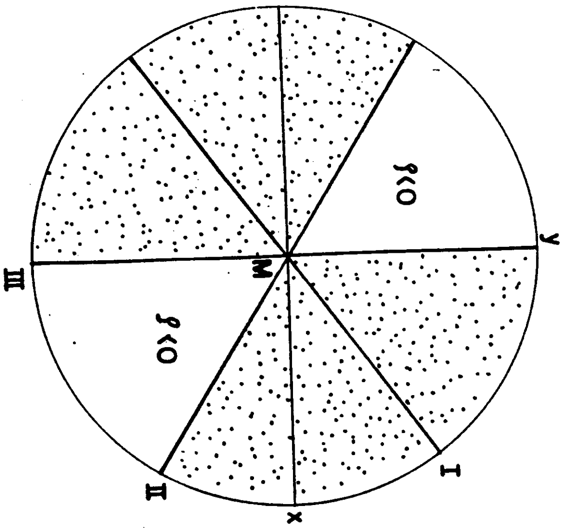

Figure 9

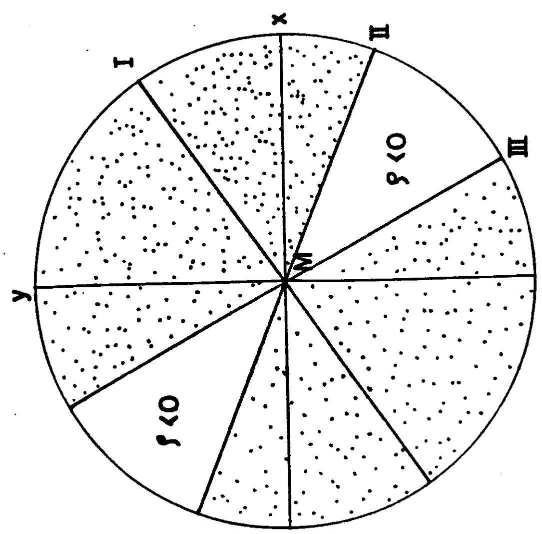

Figure 8

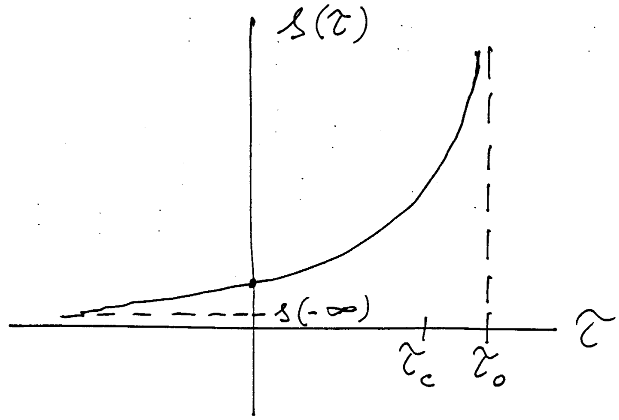

Fig. 10.

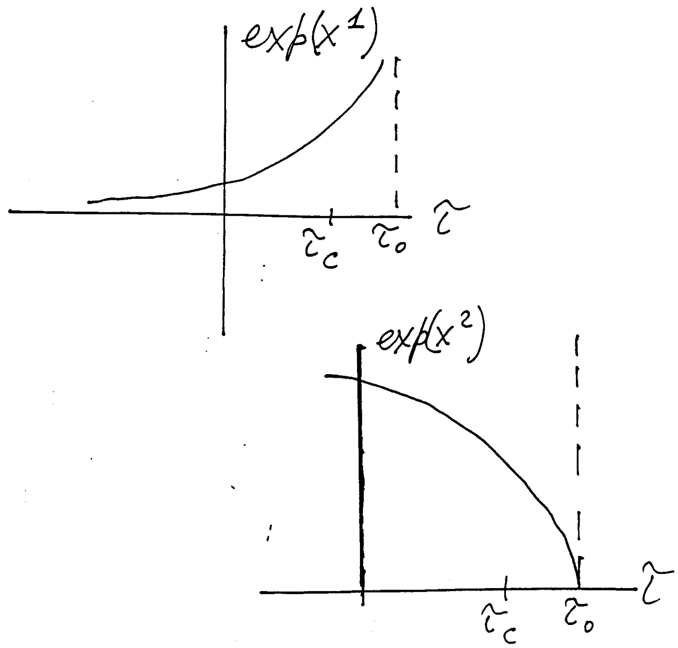

Fig. 11.

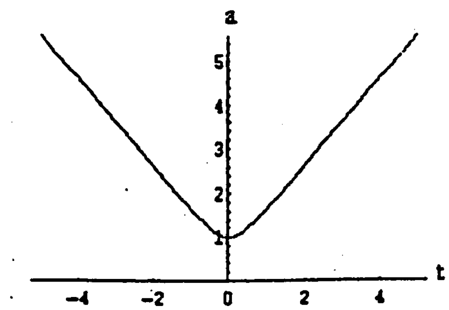

Figure 12

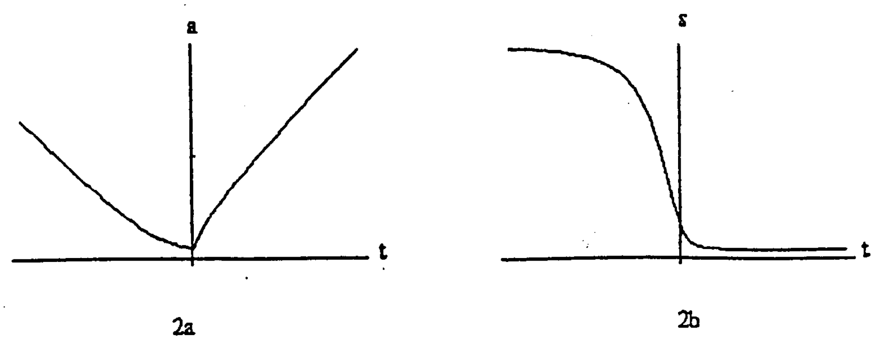

Figure 13

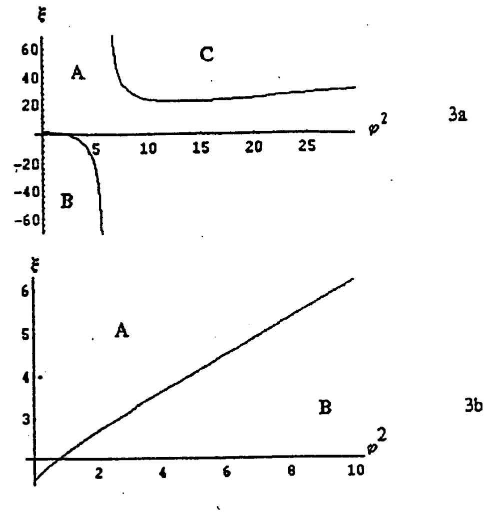

Figure 14

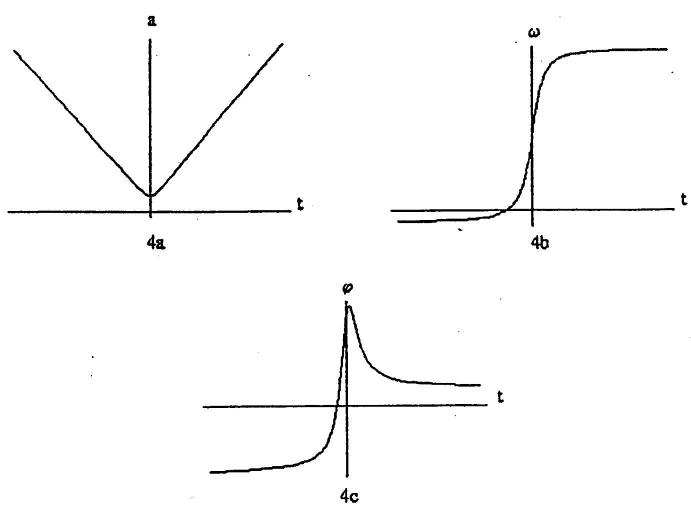

Figure 15

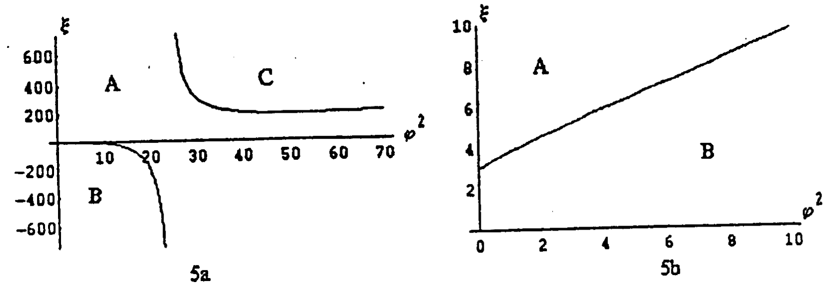

Figure 16

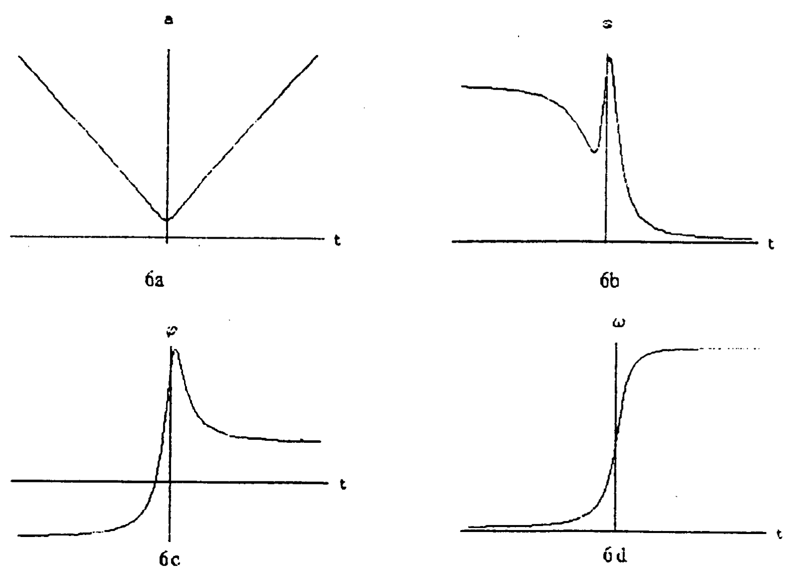

Figure 17

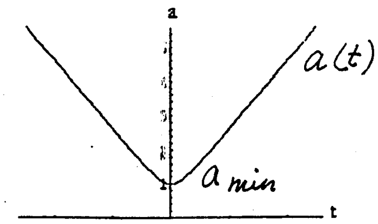

Figure 18

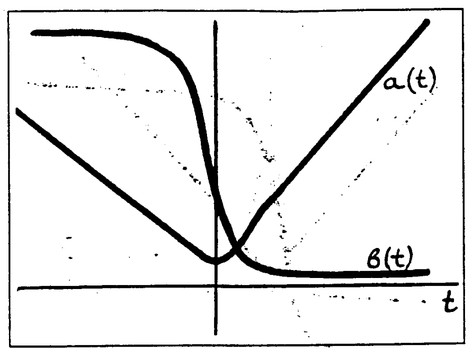

Figure 19

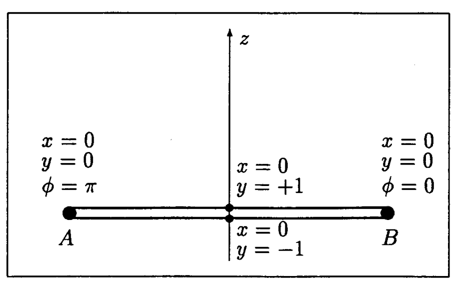

Figure 20

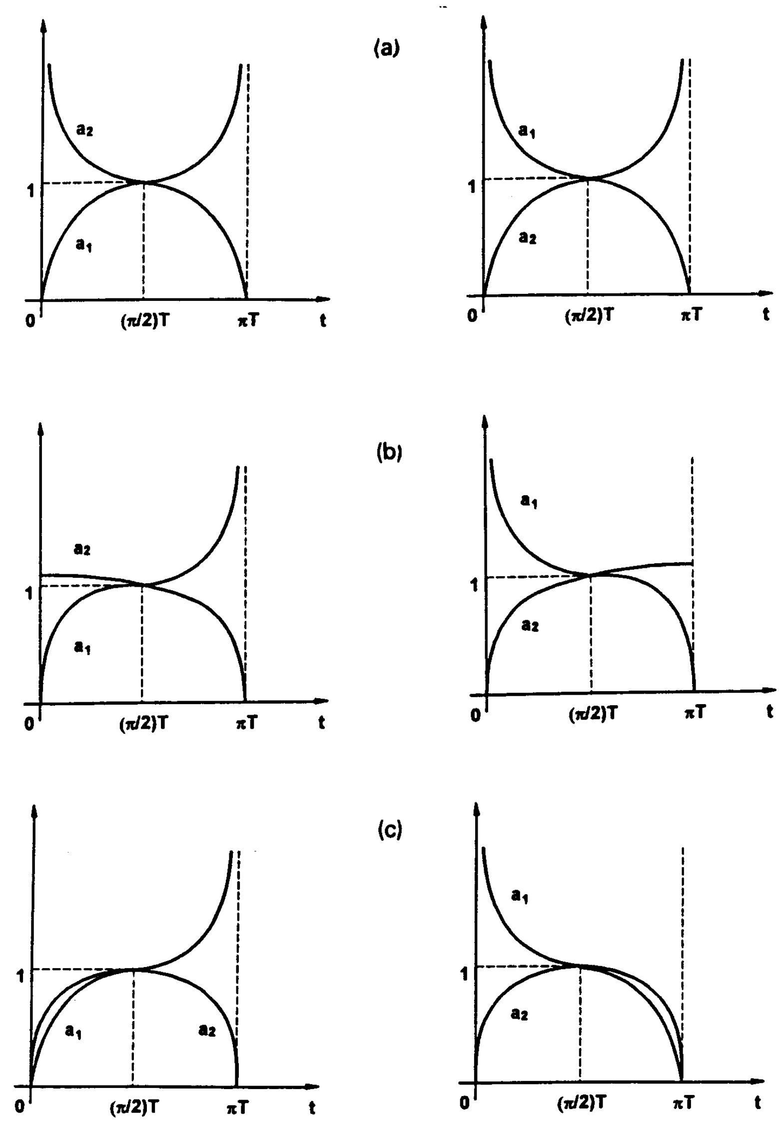

Figure 21

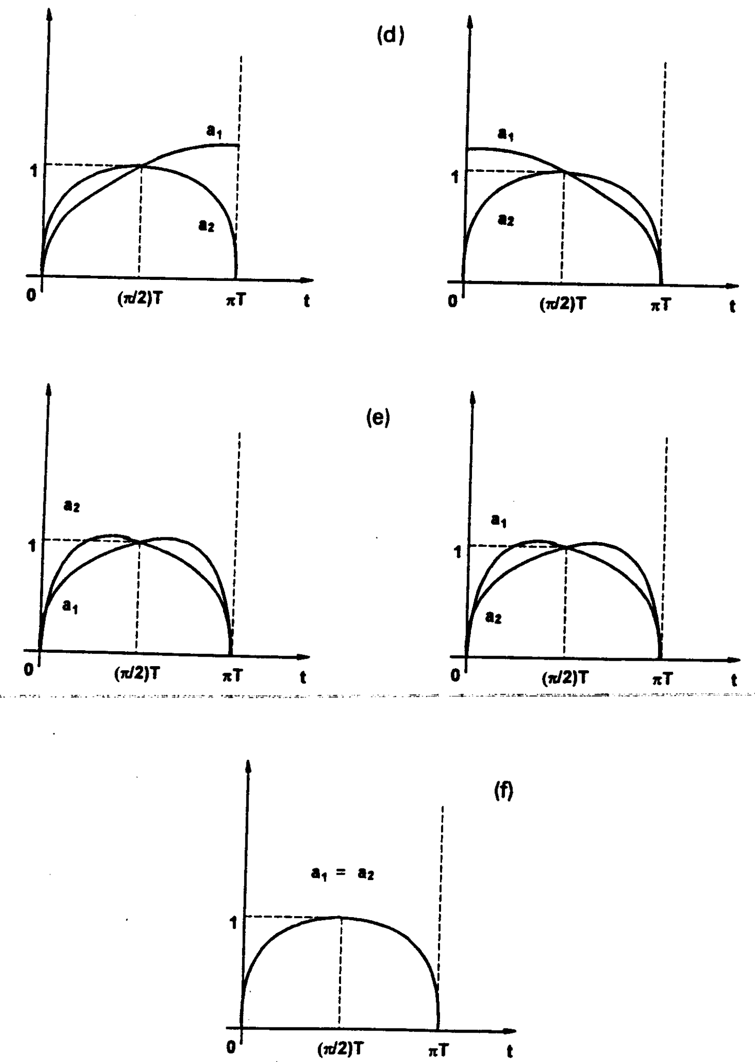

Figure 21 – continued

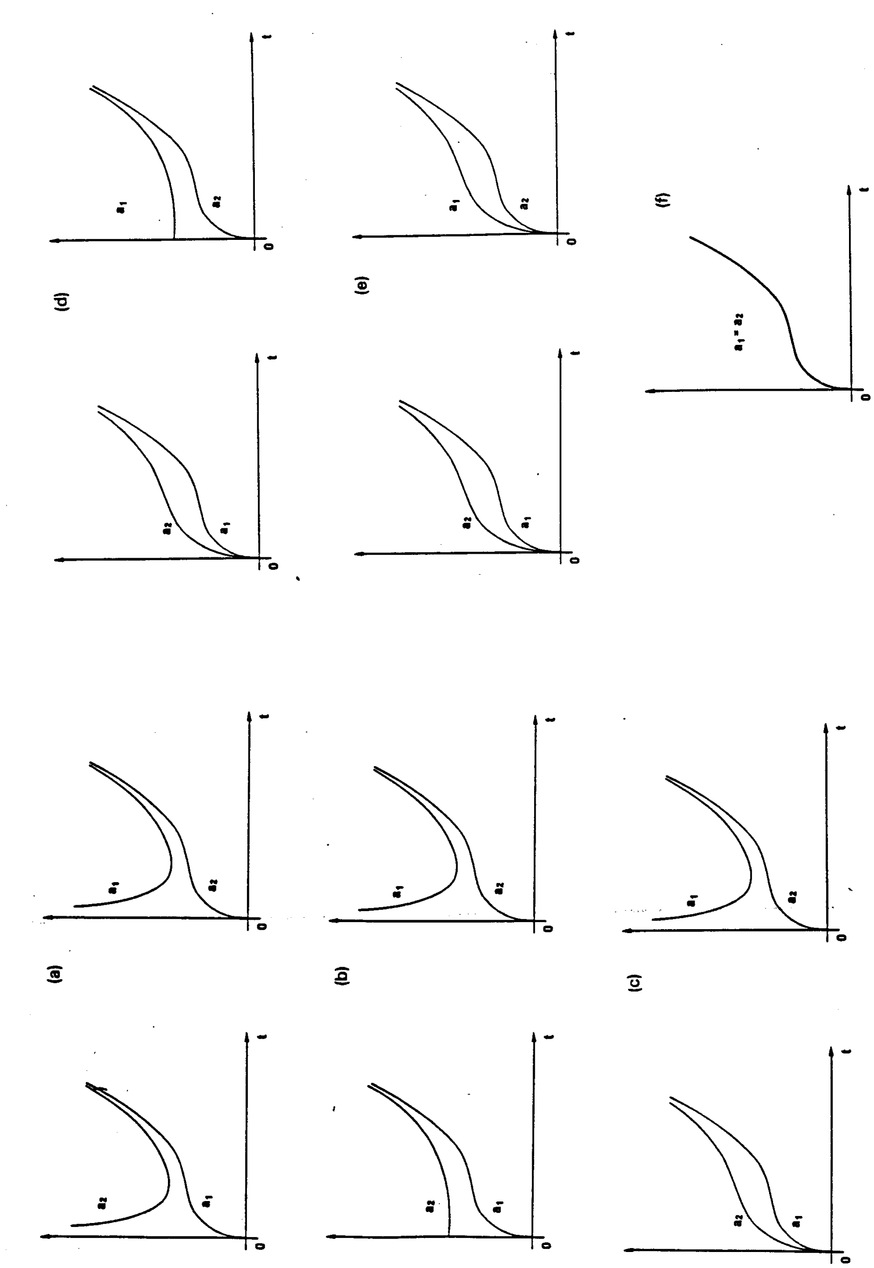

Figure 22

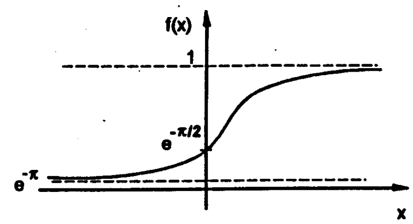

Figure 23

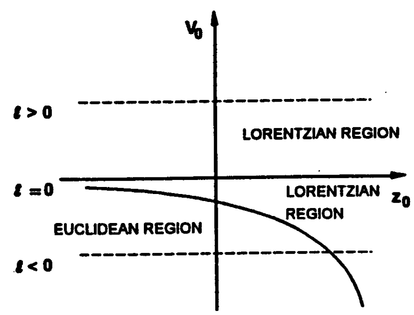

Figure 24

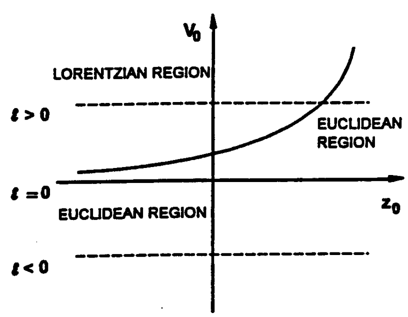

Figure 25

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