CBPF-MO-002/93

Symmetries in General Relativity*

Graham Hall

Centro Brasileiro de Pesquisas Físicas — CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brasil

1 Introduction

In recent years there has been much interest in symmetries in general relativity. Originally, such an interest arose because of the need to simplify Einstein's equations in the search for exact solutions and the procedure was fairly ad hoc. More recently there has been an attempt to establish a more systematic approach to symmetries and it is to this more general theory that these lecture notes are devoted. It may seem that such a programme aims simply at taking the older results and making them more rigorous, mathematically. This is not the case. Certainly mathematical rigour is important but this should always be the case and is nothing to do with the topic under discussion. What is relevant is that these more general techniques have produced new results as well as simplifying and extending older ones.

The notation well be the usual one. M will denote a space-time with metric g of Lorentz signature $(-,+,+,+)$ which is assumed smooth. The Riemann, Ricci, and Weyl tensors are denoted by $R_{abcd}, R_{ab}(\equiv R^{c}{}_{acb})$ and $C^{a}{}_{bcd}$, respectively, whilst a semi-colon denotes a covariant derivative, a comma a partial derivative and $\mathcal{L}$ a Lie derivative. The space-time M well be assumed *non-flat* in the sense that the Riemann tensor does not vanish over a non-empty open subset of M. Round and square brackets will denote the usual symmetrisation and skew-symmetrisation of indices. R denotes the Ricci scalar, $R = R_{ab}g^{ab}$, and then the Einstein tensor $G_{ab} = R_{ab} - 1/2Rg_{ab}$.

*Permanent Address:
Department of Mathematical Sciences,
University of Aberdeen,
Edward Wright Building,
Dunbar Street,
Aberdeen AB9 2TY,
Scotland, U.K.

2 The Concept of a Symmetry

Let Ω be some geometrical object on M. The concept of a *symmetry of* Ω, roughly speaking, corresponds for each $p \in M$ to the existence of a neighbourhood U of p and a family of smooth *bijective* maps ϕ_t from U into M (where the indexing label t belongs to some open interval of $\mathcal{R}$) such that each of the maps ϕ_t carries Ω from U to $\phi_t(U)$ in a natural way where it agrees with the original object Ω restricted to $\phi_t(U)$. The way this idea is usually implemented in differential geometry (and, in particular, in general relativity) is to start with a smooth vector field X on M. Then for each $p \in M$ there necessarily exists a neighbourhood U of p and $\varepsilon > 0$ such that the integral curve of X starting from any point $q \in UfU$ is defined on $(-\varepsilon, \varepsilon)$ [1]. For each $t \in (-\varepsilon, \varepsilon)$ one then defines a map ϕ_t on U by requiring that for $q \in U$, $\phi_t(q)$ is that point with parameter value t on the integral curve of X starting at q. The vector field X thus gives rise to a family of maps of the type required. To see how this works is practice let Ω be the metric g on M and suppose X is a smooth vector field with corresponding maps ϕ_t as described above. The statement that ϕ_t "carries" g across from the neighbourhood U of p to $\phi_t(U)$ so that it agrees with the metric g originally present in $\phi(U)$ can be expressed precisely in terms of the pullback map ϕ_{t*} associated with ϕ_t. That is, it is assumed that $\phi_{t*}g = g \circ \phi_t$ (usually loosely written as $\phi_{t*}g = g$). This has a nice geometrical interpretation. Suppose U is chosen to be a *coordinate* neighbourhood of p (as it always can by shrinking U if necessary) and denote the coordinates in U by x^a. Then since ϕ_t is a smooth bijection, $y^a \equiv x^a \circ \phi_t^{-1}$ is a coordinate system for $\phi_t(U)$ (the coordinate system on $\phi(U)$ carried across from that on U by ϕ_t). The condition $\phi_{t*}g = g \circ \phi_t$ is then equivalent to the statement that the components of g in U in the coordinate system x^a are equal term by term to the components of g in $\phi_t(U)$ in the coordinate system y^a. It is in this sense that ϕ_t "preserves the metric" and is thus a *local isometry.* The above condition on ϕ_t and g is reflected is the vector field X from which the maps ϕ_t arose and is equivalent to the statement that $\mathcal{L}_X g = 0$ [1]. This equation *(Killing's equation)* is the form most familiar to relativists and then X is called a *Killing vector field.* It is equivalent to the condition $X_{a;b} + X_{b;a} = 0$ in local coordinates.

By similar methods one can set up the concept of a *conformal (Killing) vector field* X on M by insisting that the corresponding maps ϕ_t preserve the conformal metric, that is, $\phi_{t*}g = \alpha(g \circ \phi_t)$ where α is a smooth function on M. This is equivalent to the condition that $\mathcal{L}_X g = 2\phi g_{ab}$ or $X_{a;b} + X_{b;a} = 2\phi g_{ab}$ in local coordinates for some function ϕ. These and other symmetries will be discussed in more detail in the next section. The essential point to be made here is that symmetries in general relativity are described by means of certain vector fields on M but that the actual "symmetries" themselves really come from the maps ϕ_t which arise from the vector field and preserve some geometrical object on M.

3 Symmetries in General Relativity

Let X be a smooth vector field on M. Then X is called an *affine collineation* of M if the corresponding maps ϕ_t preserve the geodesics of M *and* the affine parameters on these geodesics. This condition is equivalent to X satisfying the condition [1]

$$X^a_{;bc} = R^a{}_{bcd} X^d \tag{1}$$

in every coordinate domain of M. If we decompose $X_{a;b}$ into its symmetric and skew symmetric parts then (1) is equivalent to [2, 3]

$$\begin{aligned} (i)\ X_{a;b} &= h_{ab} + F_{ab} \qquad (h_{ab} = h_{ba},\ \ F_{ab} = -F_{ba}) \\ (ii)\ h_{ab;c} &= 0 \\ (iii)\ F_{ab;c} &= R_{abcd} X^d \end{aligned} \tag{2}$$

Thus h is a global, second order, symmetric covariantly constant tensor and F is called the *affine bivector*. In fact, equation $2(iii)$ follows $2(i)$ and $2(ii)$ and so another equivalent condition for X to be an affine collineation is $\mathcal{L}_X g = 2h$ where $h_{ab;c} = 0$. If the tensor h in (2) is a constant multiple of the metric then X is called *homothetic* (and proper *homothetic* of $h \neq 0$) whilst if $h = 0$ X is Killing.

If, on the other hand, the vector field X preserves the *conformal metric* (see the last section) then X is called a *conformal* vector field, This is equivalent to X satisfying

$$X_{a;b} = \phi g_{ab} + F_{ab} \tag{3}$$

where ϕ is a function and the skew part F_{ab} is called the *conformal bivector*. X is called *proper conformal* if ϕ is not constant on M and *special conformal* if $\phi_{;ab} = 0$ on M (and of ϕ is constant on M then X is homothetic). A conformal vector field satisfies (see e.g. [4])

$$F_{ab;c} = R_{abcd} X^d - 2\phi_{,[a} g_{b]c} \tag{4}$$

and

$$\phi_{;ab} = 1/2\ L_{ab;c} X^c - \phi L_{ab} + R_{c(a} F^c_{b)} \tag{5}$$

where $L_{ab} = R_{ab} - 1/6\ R g_{ab}$. The equation (5) can be rewritten in an interesting form which shows, in a sense, the effect of a local conformal map ϕ_t arising from X on the Ricci tensor. This relation which is equivalent to (5) is

$$\mathcal{L}_x R_{ab} = -2\phi_{;ab} - (\phi_{;cd} g^{cd}) g_{ab} \tag{6}$$

The sets of all affine collineations, all conformal vector fields, all special conformal vector fields, all homothetic vector fields and all Killing vector fields on M each form a finite dimensional Lie algebra under the usual bracket of vector fields and are referred to as the *affine algebra*, the *conformal algebra*, the *special conformal algebra*, the *homothetic algebra* and the *Killing algebra* respectively. The intersection of the conformal and affine algebras is the homothetic algebra.

Another type of symmetry which will be discussed later arises from a vector field X on M whose associated maps ϕ preserve the curvature tensor, $\phi_{t*}\Re = \Re \circ \phi_t$, where $\Re$ denotes the Riemann tensor with components $R^a{}_{bcd}$. This is equivalent to $\mathcal{L}_X R^a{}_{bcd} = 0$ and the geometrical interpretation is similar to that given above for Killing vector fields. Such vector fields are called *curvature collineations*. It follows that every affine collineation is a curvature collineation. Curvature collineations are more difficult to handle than the other types of symmetries discussed so far for several reasons. It is interesting to note here that the special conformal algebra can be characterised within the conformal algebra on M by their members satisfying any one (and hence all) of the following conditions [5]

$$\mathcal{L}_X R^a{}_{bcd} = 0 \qquad \mathcal{L}_X R_{ab} = 0 \qquad \mathcal{L}_X G_{ab} = 0 \tag{7}$$

It follows that a conformal vector field is a curvature collineation if and only if it is special conformal.

It is remarked before closing this section that a *proper affine collineation* (i.e. a vector field X satisfying (2) with h *not* a constant (and hence not any) multiple of g) is a very restrictive object to have on M because such tensors h force, in most cases, severe local reducibility of M. We shall return to this later.

4 Further Properties of Symmetries

This section will be used to tidy up some mathematical points regarding the symmetries discussed above before proceeding with particular details of the individual symmetries.

4(a) Differentiability

The space-time M and metric g were assumed smooth. We also assumed that our (affine, conformal, etc) vector field X was smooth. It is now important to ask whether this last assumption is necessary, that is, given that M and g are smooth could it be that such an X exists which is not smooth. If X is an affine collineation (including the homothetic and Killing cases) then clearly we must have X a C^2 vector field in order for (1) and (2) to make sense. It then follows that X *is necessarily smooth.* The result holds for conformal vector fields provided the C^2 condition above is replaced by C^3, in view of equations (3) (4) and (5). However, for curvature collineations, in spite of the smoothness of M and g an X may be constructed which is C^k but not C^{k+1} for any k [6].

4(b) Lie algebra structure and maximum dimension

The sets of affine, conformal, special conformal, homothetic and Killing vector fields constitute Lie algebras under the usual bracket operation. There is a problem with the set of curvature collineations on M since, in view of the remarks in 4(a), the bracket of two such vector fields may not be differentiable and it does not then make sense to ask if it is a curvature collineation [6]. The set of *smooth* curvature collineations on M is, of course, a Lie algebra under the bracket operation.

The equations (2) may be regarded as a set of *first* order differential equations for X, F and h (c.f. [4, 7]). It then follows from an argument using the elementary theory of first order differential equations and topology that two globally defined affine collineations on M are equal everywhere on M if their corresponding quantities X, F and h agree at any point of M. As a consequence the maximum dimension of the affine algebra for a space-time is 4 (from X) + 6 (from F) + 10 (from h) = 20 (and, in general, is

$n + 1/2\ n(n-1) + 1/2\ n(n+1) = n^2 + n$). This maximum dimension (in all cases) can only occur when the connection is *flat*. For the homothetic algebra this maximum dimension is $4 + 6 + 1 = 11$ (and, in general, is $n + 1/2\ n(n-1) + 1 = 1/2\ n(n+1) + 1$) and can only occur when the unique symmetric Levi-Civita connection associated with the metric in flat. For the Killing algebra the maximum dimension is $4 + 6 = 10$ (and, in general, is $n + 1/2\ n(n-1) = 1/2\ n(n+1)$) and can only occur when the metric leads, through its Levi-Civita connection, to a space of constant curvature.

The equations (3) (4) and (5) may be regarded as a set of first order equations for X, F, ϕ and $\phi_{,a}$ and, similarly to the above analysis, two global conformal vector fields on M are equal everywhere on M if their corresponding quantities X, F, ϕ and $\phi_{,a}$ agree at any point of M. As a consequence the maximum dimension of the conformal algebra for a space-time is $4+6+1+4 = 15$ (and, is general, for $n \geq 3$ is $n+1/2\ n(n-1)+1+n = 1/2\ (n+1)(n+2)$) and can only occur if the metric is locally conformally related to a flat metric. The case when $n = 2$ is quite different and need not concern us here.

The converses of the above results regarding when the maximum dimension of each algebra can occur do not hold and will be discussed briefly in section 4(d).

4(c) Local groups and orbits

Let A be either the affine algebra, the conformal (or special conformal) algebra, the homothetic algebra or the Killing algebra. Then A gives rise to a local group G of local diffeomorphisms of M in the following way. Let $k \in N$, let $X_1, \cdots, X_k \in A$ and let $\phi^1_t, \cdots, \phi^k_t$ be the corresponding local maps (the "local one-parameter groups" associated with $X_1, \cdots, X_k$). Then G consists of all local maps (where they are defined) of the form

$$p \rightarrow \phi^1_{t_1}(\phi^2_{t_2}(\cdots \phi^k_{t_k}(p)\cdots)) \qquad p \in M \tag{8}$$

for $t_1, \cdots, t_k \in \mathcal{R}$. The *orbits* in M under G are the equivalence classes in M arising from the equivalence relation $p_1 \sim p_2 \Leftrightarrow \exists\ a \in G$ such that $a(p_1) = p_2$, p_1, $p_2 \in M$. They can be regarded as connected submanifolds of M is a very natural way. So, the orbit containing $p \in M$ consists of all those points in M which can be reached by a finite number of applications of local maps ϕ_t arising from members of A.

4(d) Local and global vector fields

A space-time M may admit a local affine (or conformal or homothetic or Killing) vector field X' say, which is defined in some open subset U of M. It should be remembered that it may not be possible to extend X' to the whole of M. The possibility of such an extension is linked to the global topology of M [7]. Since many of our considerations will be local, this need not concern us except to remark why the converses of the results in section (4b) do not hold. A flat space-time, far example, will always admit 20 local affine collineations in some open neighbourhood of any of its points, but some of these may not be extendible globally to M. Thus a flat space-time may not admit 20 (global) affine collineations. Similar remarks apply to the maximum Killing dimension and spaces of constant curvature and to the maximum conformal dimension and conformally flat-times for the same reason – the *local* (Killing, conformal, etc.) vector fields are guaranteed, but not the global ones.

4(e) Zeros of vector fields, fixed points and isotropy

It was pointed out in (4c) that the Lie algebra of affine /conformal /Killing etc. vector fields generate orbits which, being submanifolds, have a well defined dimension (and this dimension may vary as one moves from one orbit to another). If the Lie algebra concerned has dimension m and the orbit through some $p \in M$ has dimension k then necessarily $m \geq k$. If $m > k$ then there exists an s-dimensional subalgebra of this Lie algebra, where $s = m - k$, of vector fields which *vanish* at p. This *isotropy algebra* can be extremely useful in fixing certain properties of the space-time since the corresponding maps ϕ_t of the members of this subalgebra *fix* the point p.

It should be noted that in much of the discussion of this section, *curvature collineations* were not mentioned. They often constitute a special case and have been extensively discussed [6, 8].

5 Further Discussion of Particular Symmetries

Further remarks on the concepts introduced in section 4 can now be given for the individual symmetries.

5(a) Killing symmetries

If M admits a non-trivial Killing vector field X which vanishes at $p \in M$ then the Weyl tensor is necessarily of Petrov type N, D or O at p [9]. In fact it is type N or O (respectively D or O) if the *Killing bivector* $F_{ab} = X_{[a;b]}$ at p (which is necessarily non-zero from 4(b)) is null (respectively non-null). There are also restrictions or the algebraic type of the Ricci (and energy-momentum) tensors at p. If three (or more) independent Killing vectors vanish at p the Weyl tensor is zero at p and the energy-momentum tensor if non-zero at p is of the null fluid or perfect fluid type at p (or M is an Einstein space). It can be shown from this that if M is not of constant curvature it admits at most 7 independent Killing vector fields and that if 7 exist (and with a restrictive clause on the nature of the orbits-see [13] page 458). M is (locally) a homogeneous conformally flat plane wave or an Einstein static universe (or its spacelike equivalent). For a given Killing vector field X on M, the *zero set* of X, that is $\{p \in M : X(p) = 0\}$ is easily described [10, 11].

5(b) Homothetic symmetries

If M admits a Lie algebra of homotheties of dimension m then, by taking linear combinations of them, one can assume it to be spanned by a basis consisting of at most proper homothetic vector with the remainder Killing vector fields. If X is a proper homothetic vector field which vanishes at $p \in M$ then the *zero set of X is either an isolated point or else consists locally of (part of) a null geodesic of zeros of X*. The latter condition characterises the well known (conformally flat or Petrov type N) *plane waves* [10, 12]. At *any* zero of a proper homothetic vector field *all Ricci and Weyl eigenvalues must necessarily vanish* and thus the Ricci (and energy-momentum) tensor has Segre type either $\{(211)\}$, $\{(31)\}$ (both with zero eigenvalue) or O and the Weyl tensor has Petrov type N, III or O [10]. This simplifies greatly the task of classifying metrics with large homothetic algebras [13]. (see appendix I).

5(c) Conformal symmetries

If $p \in M$ and 4 or more independent conformal vector fields vanish at p then the Weyl tensor vanishes at p [14]. It follows if M admits 8 or more independent conformal vector fields then M is conformally flat [14]. The maximum number ($= 7$) of such vector fields that can be admitted when M is not conformally flat is achieved in the homogeneous type N plane waves [11]. A detailed study of the zeros of conformal vector fields is given is [11]. Conformal symmetries present more problems than general affine symmetries because of a lack of "linearity" in the neighbourhood of zeros of such fields. However, under fairly general circumstances one has following interesting result initiated by Bilyolov [15], continued by Defrise-Carter [16] and completed by the present author and one of his students [14, 17]. If (M, g) is a space-time which admits an r-dimensional conformal algebra A and suppose that the Petrov type and the dimension and nature (timelike, spacelike or null) of the orbits associated with A are the same at each $p \in M$ and that M admits no non-globalisable local conformal vector fields. Then for each $p \in M$ there exists an open neigbourhood U of p and a function $\sigma : U \to \mathcal{R}$ such that A, restricted to U, is a Lie algebra of *special* conformal vector fields on U with respect to the metric $g' = e^{2\sigma} g$ on U. If the Petrov type is not O this local scaling function σ can always be chosen so that A restricts to a Lie algebra of homothetic vector fields with respect to g' on U and if (M, g) is not locally conformally related to a plane wave about any $p \in M$ the above local scaling can always be chosen so that A restricts to a Lie algebra of Killing vector fields with respect to g' on U.

Since workers in the area of exact solutions normally make the assumptions made in the above theorem regarding the orbits and Petrov type, this theorem says, roughly, that apart from conformally flat space-times and space-times conformal to a plane wave, conformal vector fields can be "turned into" Killing vector fields by a conformal change of metric. In this sense many properties of conformal vector fields can be obtained from the (easier to study) Killing vector fields. It should, of course, be remembered that if X is a conformal vector field with respect to a metric g then it is also conformal with respect to any metric $g' = e^{2\sigma} g$ conformally related to g. This theorem thus describes those conditions when σ can be chosen to (locally) convert *simultaneously* all the conformal vector fields to Killing (or, in the case of a plane wave, Killing or homothetic) vector fields. Techniques similar to those described here, and others have been used in an attempt to simplify the theoretical study of conformal fields in space-times [11].

For completeness, it is remarked here that the maximum dimension of the *homothetic* algebra in a space-time is 11 and this can only be achieved if it is *flat*. In the non-flat case the maximum dimension is 8 (and can be achieved only by the homogeneous conformally flat plane waves). In the non-conformally flat case the maximum dimension is 7 (and can only be achieved by a type N homogeneous plane wave).

5(d) Affine collineations

In order to study affine collineations we suppose M is *simply connected* or else restrict ourselves to simply connected neighbourhoods for local considerations. Now suppose X is a *proper* affine collineation on M. Then, as remarked earlier, this means that M admits a second order symmetric tensor field h_{ab} which is covariantly constant, $h_{ab;c} = 0$, but *not* a (constant) multiple of the metric g_{ab}. Thus the first step in the study of proper affine collineations is to evaluate the consequences of a space-time admitting such a tensor field. It turns out that [2, 3] (and for a complete solution see [18]) with one exceptional case, M must admit a dual pair of covariantly constant vector fields. (The exceptional case admits a dual pair of covariantly constant bivectors and a pair of recurrent null vector fields). This exceptional case, together with those cases where this covariantly constant vector field is non-null, are locally decomposable space-times and local coordinates can be chosen so that the metric is one of the following

$$ds^2 = \varepsilon dx^{1^2} + \gamma_{\alpha\beta} dx^\alpha dx^\beta \qquad (\alpha, \beta = 2, 3, 4) \tag{9}$$

$$ds^2 = \varepsilon^1 dx^{1^2} + \varepsilon^2 dx^{2^2} + \gamma_{ij} dx^i dx^j \qquad (i, j = 3, 4) \tag{10}$$

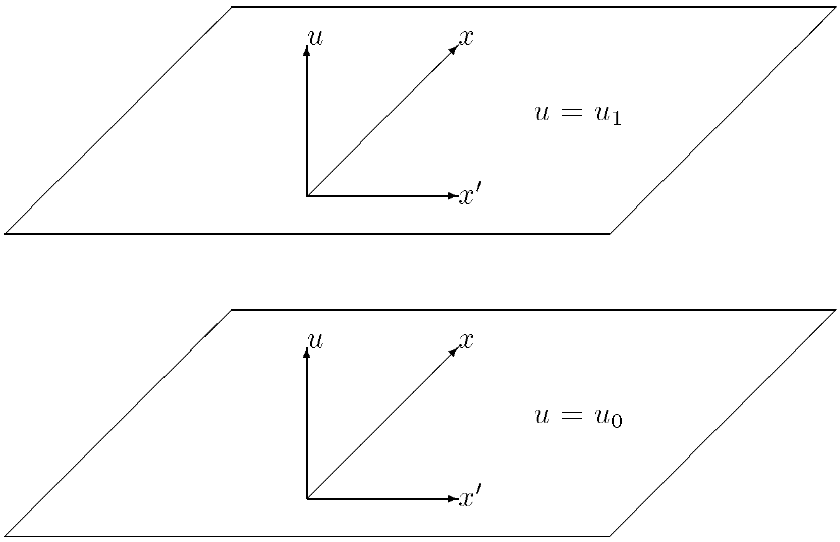
$$ds^2 = Y_{\mu\nu} dx^\mu dx^\nu + Z_{ij} dx^i dx^j \qquad (\mu, \nu = 1, 2, \;\; i, j = 3, 4) \tag{11}$$

Here, $\varepsilon, \varepsilon^1$ and ε^2 are ± 1 and the signatures of the lower dimensional metric are then consistent with the overall Lorentz signature. Also $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}(x^2, x^3, x^4), X_{ij} = X_{ij}(x^3, x^4), Y_{\mu\nu} = Y_{\mu\nu}(x^1, x^2)$ and $Z_{ij} = Z_{ij}(x^3, x^4)$. The lower dimensional metrics are assumed not to decompose further in an obvious sense. These results go a long way to resolving the problem of the existence of proper affine collineations and can either be derived using holonomy theory [2] or by more conventional techniques [18].

To proceed further, it is useful to note that the set V of covariantly constant tensor fields like h on M form a vector space whose dimension is either 1,2 or 4 and for *proper* affines to exist must be either 2 or 4. It follows that if $\dim V = 2$ (with V spanned by, say, g_{ab} and h_{ab}) then if X and Y are *proper* affine collineations we have

$$\begin{aligned} X_{a;b} &= (\alpha h_{ab} + \beta g_{ab}) + F_{ab} \\ Y_{a;b} &= (\gamma h_{ab} + \delta g_{ab}) + F'_{ab} \end{aligned} \tag{12}$$

where F and F' are the affine bivectors and $\alpha, \beta, \gamma, \delta \in \mathcal{R}, \alpha, \gamma \neq 0$. It then follows that $\gamma X^a - \alpha Y^a$ is *homothetic* or *Killing*, in other words, there is essentially only one proper affine collineation (chosen, say, as X) and all others are linear combinations of this one together with *homothetic* or *Killing* vector fields. This would be the situation for metrics like (9) and (11). A similar situation exists when $\dim V = 4$ (the metrics (10)) only now there are essentially *three* affine collineations is the above sense. [It should be remarked at this point that although the existence of a *proper* affine collineation leads to a covariantly constant tensor field like h above, the converse is false–although it is only in the "exceptional" case referred to earlier that may fail].



As an example consider the case when M admits a covariantly constant timelike vector u^a (and no other such vector fields except, of course, for constant multiples of u) [3]. Then one can choose u^a such that $u^a u_a = -1$ and define (locally) a function u by $u_a = u_{,a}$. In local coordinates the metric then reads

$$ds^2 = -du^2 + \gamma_{\alpha\beta} dx^\alpha dx^\beta \tag{13}$$

where $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}(x^1, x^2, x^3)$ is a positive definite metric in the hypersurfaces u=constant. Now let X be an affine vector field so that (1) - (4) hold. In this case the only covariantly constant second order symmetric tensors are g_{ab} and $u_a u_b$ and so X satisfies

$$X_{a;b} = \alpha g_{ab} + \beta u_a u_b + F_{ab} \tag{14}$$

(and is proper affine if $\beta \neq 0$, homothetic if $\alpha \neq 0$, $\beta = 0$ and Killing if $\alpha = \beta = 0$). Now decompose X parallelly and perpendicularly to u:

$$X^a = -\varepsilon u^a + X'^a \qquad (\varepsilon = X^a u_a) \tag{15}$$

where this equation defines X' and $X'^a u_a = 0$. It is easily shown that $\varepsilon_{,a} = \lambda u_a$ (λ constant) and so $\varepsilon = au + b$ (a, b constant). It then follows that the vector fields εX^a and X'^a are each affine collineations. Further, X' satisfies

$$X'_{a;b} = \alpha(g_{ab} + u_a u_b) + F_{ab} \tag{16}$$

and so the projection of X onto the hypersurfaces u=constant is homothetic (or Killing) with respect to the induced geometry is these hypersurfaces represented by the metric $g_{ab} + u_a u_b$. Thus *any* affine collineation X in such a space-time is the sum of the vector field $-\varepsilon u^a$ (which is affine and is proper affine if $\varepsilon = au + b$ with $a \neq 0$) and a vector field

X' which gives, on projection, a homothetic (or Killing) vector field with respect to the u=constant hypersurface geometries.

Two points should be made regarding this construction. First, the hypersurfaces u=constant are locally isometric under the map that maps points of one of them into another one by following integral curves of the constant vector field u. It then follows that the vector fields X' set up in each of these hypersurfaces are essentially the "same" (i.e. invariant under the isometry just defined) on account of the result $[X', u] = 0$ which follows from (14), (15) and the fact that $u_{a;b} = 0$. Second, it is clear that u^a is an affine collineation (in fact Killing) and uu^a is proper affine. The above results show that there are no other independent ones unless the hypersurfaces of constant u with metric $g_{ab}+u_a u_b$ themselves admit homothetic or Killing vector fields – and in general they will not.

The other cases represented by (9), (10) or (11) are similar. In the cases when M admits a *null* covariantly constant vector field, less can be said but useful information is still available. These techniques readily show that (using Einstein's equations with zero cosmological constant) the existence of a *proper* affine collineation eliminates all vacuum space-times except the *pp*-waves, all perfect fluid space-times for which (in the usual notation) $0 < p \neq \rho > 0$ and all non-null Einstein-Maxwell fields except the "2+2 locally decomposable" case. These strong results justify here, and elsewhere, the advantages of considering symmetries from a general (geometrical) standpoint. These techniques also show that, for a non-flat space-time, the maximum dimension of the affine algebra is 10 and that this can only be achieved in a space-time of constant curvature (where they are all Killing) or in a special type of plane wave, an example of which, in local coordinates, is

$$ds^2 = dx^2 + dy^2 + 2dudv + x^2du^2 \tag{17}$$

which is type N, homogeneous and admits $\partial/\partial v$ and $\partial/\partial y$ as covariantly constant vector fields (and hence $y\partial/\partial y, u\partial/\partial u$ and $u\partial/\partial y + y\partial/\partial u$ are independent proper affine collineations, the other affines being 6 Killing and 1 homothetic vector fields). [Concerning this results there was a slip in the original paper [3] subsequently corrected in [13] where further discussion and examples of high dimensional affine algebras can be found.] (see appendix II).

5(e) Special conformal vector fields

If X is a special conformal vector field on M then X satisfies (3) with $\phi_{;ab} = 0$. The vector field X is called *proper special conformal* if ϕ is *not* constant on M and so $\phi_{,a}$ is then *not* identically zero and *covariantly constant*. Thus some of the techniques of the previous section can be used here to give a rather complete account of the situation [5]. It turns out that a space-time M can admit essentially only one proper special conformal vector field X in the sense that any other such vector is a linear combination of X and homothetic (including Killing) vector fields. If M is not conformally flat the maximum dimension of the special conformal algebra is 7 whilst if M is conformally flat but non-flat its maximum dimension is 8. Both maxima can be achieved by certain types of plane waves.

5(f) Curvature collineations

The study of these symmetries is much more involved and the reader is referred to references [6, 8].

6 Further Remarks

The discussion in 5(d) shows thay many space-times cannot possibly admit proper affine collineations and that in 5(e) can be extended to show in a similarly way that proper special conformal vector field cannot exist in many space-times. It is further remarked here that it is a consequence of Brinkmann's theorem that the only vacuum space-times that can admit proper conformal vector field are the *pp*-waves. Also, for most space-times, a curvature collineation is necessarily a homothetic vector field. For a full bibliography see [17,19].

7 Acknowledgements

The author wishes to point out that these lecture notes were informally written and intended as a summary of the way he likes to do things. Hence the lack of references to other researchers in this area. His indebtedness to others has already been recorded in the more extended bibliographies in references [6, 8, 11, 17, 18, 19].

He would also like to take this opportunity to thank Prof. Marcelo ebouças, Filipe Paiva and Joel B. Fonseca-Neto for their kindness and hospitality and also for many valuable discussions during his most enjoyable stay at CBPF in Rio de Janeiro.

In appendices I and II it is assumed that the dimension and nature (timelike, spacelike or null) of the orbits of the Killing and homothetic algebras are the same at all points of M (this assumption is usually made implicitly is the literature).

Appendix I

Homothetic Algebras of High Dimension [13]

In a non flat space-time M the maximum dimension of the homothetic algebra is 8. In this case a proper homothetic vector field must exist together with a Killing algebra of dimension 7. By the above assumption there is thus a single Killing orbit ($= M$) and hence a $3(= 7 - 4)$ parameter Killing isotropy at each $p \in M$. M is then a conformally flat homogeneous plane wave. If the homothetic algebra has dimension 7 one has either a type N homogeneous plane wave or a conformally flat (homogeneous or non-homogeneous) plane wave or a Robertson-Walker "type" (see [13]) space-time. If the homothetic algebra has dimension 6,5 or 4 a significant amount of information is still available [13].

Appendix II

Affine Algebras of High Dimension [13]

In a non-flat space-time M the maximum dimension of the affine algebra is 10 and can be achieved only in space-times of constant curvature and certain plane waves (see [13] and section 5(d) of these notes). A 9-dimensional affine algebra requires M to be a conformally flat, homogeneous plane wave or a certain type of non-homogeneous, type N, non-vacuum plane wave. Examples of space-times with an 8-dimensional affine algebra include the Einstein static-type universe and certain types of plane waves.

Appendix III

In this section the consequences for M when M has an s-dimensional isotropy algebra at p (for some symmetry) are explored. Some of these already have been mentioned briefly (and the general concept of isotropy was set up in section (4e)).

(a) Killing Isotropy

In this case we have *s independent Killing vectors vanishing at p.* Let $\overset{1}{F}_{ab}, \cdots, \overset{s}{F}_{ab}$ be their corresponding bivectors at p. If $s \geq 1$ (that is, if the isotropy is not trivial) the Petrov type at p is N, D or O and in order that the Petrov type be D one must have $s = 1$ or 2 with the bivectors $\overset{1}{F}_{ab}$ (and $\overset{2}{F}_{ab}$) at p each *non-null* with the same pair of principal null directions (equal to those of the Weyl tensor) and in order that the Petrov type be N one must have $s = 1$ or 2 with the bivectors $\overset{1}{F}_{ab}$ (and $\overset{2}{F}_{ab}$) at p each *null* with the same principal null direction (equal to that of the Weyl tensor). In all other cases where $s = 1$ or 2 (and always if $s \geq 3$) the Weyl tensor is zero at p. Consequently, if the Weyl tensor has Petrov type I, II or III at p there can be *no* non-trivial Killing isotropy at p. The situation for the Ricci (or energy-momentum) tensor at p when $s \geq 1$ is more complicated and is described in the table. It turns out that if $s \geq 4$ then the Ricci and energy-momentum tensors are necessarily zero and that if $s = 1$, 2 or 3 eigenvalue degeneracies necessarily occur. In the table the first column gives the "dimension" of the isotropy (equal to s above) and the second column the independent Killing bivectors of an independent set of Killing vectors which vanish at p. The third column gives the "name" of the symmetry and the final column gives the allowed Segre types of the Ricci and energy-momentum tensors at p where it is understood that the allowed types are the ones given, their degeneracies and zero.

0		No isotropy	$\{1,111\}$, $\{211\}$, $\{31\}$, $\{z\bar{z}11\}$
1	$\ell\wedge n$	Boost	$\{(1,1)11\}$
1	$\ell\wedge x$	Null Rotation	$\{(21)1\}$, $\{(31)\}$
1	$x\wedge y$	Spacelike Rotation	$\{1,1(11)\}$, $\{z\bar{z}(11)\}$, $\{2(11)\}$
2	$\ell\wedge n$ $x\wedge y$	Boost and spacelike rotation	$\{(1,1)(11)\}$
2	$\ell\wedge x$ $\ell\wedge y$	2-parameter set of null rotations	$\{(211)\}$
3	$\ell\wedge n$ $\ell\wedge x$ $n\wedge x$	3-dimensional Lorentz algebra	$\{(1,11)1\}$
3	$\ell\wedge x$ $\ell\wedge y$ $x\wedge y$	2-parameter set of null rotations and a spacelike rotation	$\{(211)\}$
3	$x\wedge y$ $x\wedge z$ $y\wedge z$	3-dimensional algebra for rotation group	$\{1,(111)\}$
6	$\ell\wedge n$ $\ell\wedge x$ $\ell\wedge y$ $x\wedge y$ $n\wedge x$ $n\wedge y$	6-dimensional algebra for full Lorentz Group	$\{(1,111)\}$

These are the only possibilities which occur.

(b) Homothetic Isotropy

Since a basis can be chosen for any homothetic algebra which contains at most one *proper* homothetic vector field it is sufficient to study the situation at a zero p of a single proper homothetic vector field X. Just as for Killing vector fields one still has $\mathcal{L}_X T_{ab} = \mathcal{L}_X C^a{}_{bcd} = 0$ and if $X(p)=0$ one finds that either M is flat in some neighbourhood of p or that either (i) T_{ab} is of type $\{(211)\}$ with zero eigenvalue, or zero and C_{abcd} is of Petrov type N or O or, (ii) T_{ab} is of type $\{(31)\}$ with zero eigenvalue, or zero and C_{abcd} is of type III or O. The homothetic bivector is either timelike or non-simple at p. In the case (i) M is locally isometric to a plane wave. For more details see [11].

(c) Conformal Isotropy

If X is a conformal vector field on M then one has $\mathcal{L}_X C^a{}_{bcd} = 0$. So if X vanishes at p then either $\phi(p) \neq 0$ (in which case p is called a *homothetic zero* of X) or $\phi(p) = 0$ (in which case p is called an *isometric* zero of X). Here one has information on the Weyl tensor type at p only and the results for a conformal zero are as for the homothetic case if p is a homothetic zero and are as for the Killing case if p is an isometric zero. For more details see [11].

(d) Affine Isotropy (Details in preparation)

Appendix IV

Local vs Global Actions

The "symmetry" vector fields described in these notes give rise to local groups of local transformations. They do not, in general, give rise to a global Lie group action on M. The theorem of Palais states that if M admits a (non-trivial) finite-dimensional Lie algebra of "symmetry" vector fields then they lead to a global Lie group action on M if and only if each such vector field on M is *complete*. For details see R.S. Palais. Mem. Am. Math. Soc. N^0 22, (1957).

Appendix V

Global Extension of Local Symmetries

Symmetry vector fields may be defined only locally (that is in some neighbourhood of each $p \in M$). For Killing, homothetic affine and conformal vector fields if these "local algebras" when precisely defined are of the same dimension and if M is simply connected, these local symmetry vector fields can be extended to global ones (for details see [7]).

Appendix VI

Miscellaneous Results on Symmetries

It can be shown that, *in general* (in a way that can be made topologically precise) the symmetric metric connection of a space-time M uniquely determines its metric up to a constant conformal factor [2]. Similarly it can be shown that the curvature tensor $R^a{}_{bcd}$ on M arising from the space-time metric g uniquely determines g up to a constant conformal factor [20]. As a consequence it is true that *in general*, for a space-time, either it admits no affine collineations or it does in which case they are necessarily homothetic vector fields, and similarly for curvature collineations (c.f [21]). Another consequence of these results is that, roughly speaking, the only *vacuum* space-times which admit *proper* affine collineations and curvature collineations which are *not* homothetic are the *pp*-waves.

Another result of a similar nature follows from Brinkmann's theorem which says that if g and $e^{\sigma}g$ are vacuum metrics for a space-time M (where $\sigma : M \rightarrow \mathcal{R}$) then either σ is constant or g (and $e^{\sigma}g$) are *pp*-wave metrics for M. A consequences of this (again roughly speaking) is that the only vaccum space-times which admit a conformal vector field which is *not* homothetic are the *pp*-waves.

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