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Lecture Notes:
String Theory and Zeta-function

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Abstract

These lecture notes are based on a revised and LaTexed version of the Master thesis defended at ISAS [1]. The research part being omitted, they include a review of the bosonic closed string *à la* Polyakov and of the one-loop background field method of quantization defined through the zeta-function. In an appendix some basic features of the Riemann's zeta-function are also reviewed. The pedagogical aspects of the material here presented are particularly emphasized. These notes are used, together with the Scherk's article in *Rev. Mod. Phys.* [2] and the first volume of the Polchinski book [3], for the mini-course on String Theory (16-hours of lectures) held at CBPF. In this course the Green-Schwarz-Witten two-volumes book [4] is also used for consultative purposes.

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Introduction.

These lecture notes contain a self-contained and pedagogical exposition of the heat-kernel method and of the generalized Riemann's zeta-functions associated to elliptic operators. Their role in defining one-loop partition functions for Euclidean Field Theories in a given background is explained.

Later these results are employed to investigate the Polyakov functional quantization of the closed bosonic string and to derive its $d = 26$ critical dimensionality by requiring the vanishing of its conformal anomaly.

The material here presented is divided as follows. In the first chapter the heat kernel method and the zeta-function prescription are presented. Their connection is shown in paragraph **1.2**. In paragraph **1.3** the relation between zeta-function and the trace anomaly is made explicit. In paragraph **1.4** the connection with the index theorem for elliptic operators is mentioned.

Chapter two is entirely devoted to explicitly compute the trace anomaly by making use of the heat-kernel techniques.

In chapter three the Polyakov prescription for the quantization of the closed bosonic string is reviewed. In paragraph **3.1** the symmetries of the closed bosonic string are presented. In paragraph **3.2** its functional quantization is introduced. In paragraph **3.3** the results obtained in the first part are applied to derive the critical dimensionality ($d = 26$) of the bosonic string.

Four appendices are included. The real and complex notations for oriented two-dimensional manifolds are presented, as well as the explicit form of the covariant derivatives acting on tensor fields and of the formulas for the cancellation of the conformal anomaly. An extra appendix is devoted in reviewing the basic features of the Riemann's zeta-function, its role in mathematics and physics and the fundamental problems still opened.

The exposition of chapter **1** is mostly based on the works [5]-[8], while chapter **3** has been prepared on the basis of some of the fundamental references for the first period of string theory (the original papers of the $'70s$ and of the $'80s$)[9]-[13]. References concerning the most recent developments can be found in the Polchinski book [3].

Contents

1 The heat kernel method and the zeta-function prescription.

1.1 Introductory remarks and notations.

It is here presented a method which enables to give a precise meaning to the one-loop partition functions of Euclidean Field Theories formulated on a curved space-time background or, equivalently stated, to the evaluation of the determinant of a certain class of differential operators.

The Euclideanized version of the path integral is required in order to deal with well-defined mathematical objects. Moreover, in order to make use of the desirable spectral properties of the compact linear operators [14], the Euclidean manifolds under consideration are assumed to be compact. For the purpose of applying the results here derived to the theory of the closed bosonic string, for simplicity the manifolds we are dealing with will be also assumed boundaryless and oriented.

Let $g_{ab}(x)$ be a riemannian metric (thought to be an external background) for a given manifold $\mathcal{M}$. The real fields Φ over $\mathcal{M}$ ($\Phi : \mathcal{M} \rightarrow \mathbf{R}$) are assumed to belong to a Hilbert space whose scalar product is given by

$$\begin{aligned} <\Phi|\Psi> &=_{def} \int_{\mathcal{M}} dx\sqrt{g(x)}\Phi(x)\Psi(x) \\ g(x) &= det(g_{ab}) \end{aligned} \qquad (1.1)$$

The eigenstates of the position are the generalized vectors $|y>$ such that

$$\forall|\Phi>, \quad <y|\Phi>=\Phi(y),$$

therefore

$$|y>\equiv \Psi(x) = \delta(x,y)\frac{1}{\sqrt{g(y)}}. \qquad (1.2)$$

The two following relations hold, the orthonormality condition

$$<y|z> \ = \ \delta(y,z)\frac{1}{\sqrt{g(y)}} \qquad (1.3)$$

and the completeness relation

$$\int dx\sqrt{g(x)}|x><x| \ = \ \mathbf{1}. \qquad (1.4)$$

Let Ω be an operator ($\Omega : \Phi \mapsto \Phi'$) admitting a complete set of proper eigenstates Φ_n, i.e.

$$\begin{aligned} \Omega|\Phi_n> &= \lambda_n|\Phi_n> \\ \sum_n |\Phi_n><\Phi_n| &= \mathbf{1} \\ <\Phi_n|\Phi_m> &= \delta_{nm}. \end{aligned} \qquad (1.5)$$

The operator Ω is specified by giving all its matrix elements, either $<\Phi|\Omega|\Psi>\ \forall\Phi,\Psi$, or $<x|\Omega|y>\ \forall x,y$.

Let $< x|\Omega|\Psi >= \Omega(x)\Psi(x)$, then

$$< x|\Omega|y >=_{def} \Omega(x,y) = \Omega(x)\delta(x,y)\frac{1}{\sqrt{g(y)}} = \Omega(x) < x|y > . \tag{1.6}$$

Following the finite-dimensional case, the trace $tr(\Omega)$ is formally defined as follows

$$tr(\Omega) = \sum_n \lambda_n = \sum_n < \Phi_n|\Omega|\Psi_n >= \int dx\sqrt{g(x)} < x|\Omega|x > . \tag{1.7}$$

A generic state $|\Phi >$ can be expressed through its mode expansion as

$$|\Phi > \;\; = \;\; \sum_n c_n|\Phi_n > . \tag{1.8}$$

Let us consider now the following partition function

$$\mathcal{Z}(g) = e^{-\frac{1}{\hbar}W(g)} = \int \mathcal{D}\Phi(x) e^{-\frac{1}{\hbar}\int dx\sqrt{g(x)}\Phi(x)\Omega(x)\Phi(x)} \tag{1.9}$$

For the moment the eigenvalues of the operator Ω are supposed to be strictly positive ($\Omega\Phi_n = \lambda_n\Phi_n$, with $\lambda_n > 0 \;\forall n$).

In a d-dimensional space the mass dimension of $\Phi_n(x)$ and of $\Omega(x)$ can be fixed to be

$$\begin{aligned} [\Phi_n(x)] &= \frac{d}{2}, \\ [\Omega(x)] &= 2. \end{aligned} \tag{1.10}$$

As a consequence, the mass dimension of $\Phi(x)$, λ_n and c_n is given by

$$\begin{aligned} [\Phi(x)] &= \frac{d}{2} - 1 \\ [\lambda_n] &= 2 \\ [c_n] &= -1. \end{aligned} \tag{1.11}$$

The measure $\mathcal{D}\Phi(x)$ can be defined through the mode expansion of $\Phi(x)$, following a procedure similar to the one introduced by Fujikawa [15] for computing the chiral anomaly

$$\int \mathcal{D}\Phi(x) \quad =_{def} \prod_n (\mu dc_n), \tag{1.12}$$

where an arbitrary massive factor μ has been inserted in order to deal with a dimensionless measure.

Naively the functional integral will be given by

$$\mathcal{Z}(g) \;\; = \;\; \prod_n (\mu \int dc_n e^{-\frac{\lambda_n c_n{}^2}{\hbar}}). \tag{1.13}$$

¿From the formula

$$\int_{-\infty}^{+\infty} dx e^{-\alpha x^2} \;\; = \;\; \sqrt{\frac{\pi}{\alpha}} \tag{1.14}$$

we can formally compute

$$\mathcal{Z}(g) = \prod_n (\mu\sqrt{\frac{\pi\hbar}{\lambda_n}}) = \prod_n (\frac{\mu^2\pi\hbar}{\lambda_n})^{\frac{1}{2}} = \left(\prod_n \frac{\lambda_n}{\mu^2\pi\hbar}\right)^{-\frac{1}{2}} = \left(\det(\frac{\Omega}{\pi\hbar\mu^2})\right)^{-\frac{1}{2}}. \quad (1.15)$$

In order to give a precise meaning to the partition function we have introduced we need a prescription which allows to deal with the product of an infinite number of eigenvalues; on the other hand the presence of the arbitrary factor μ is unavoidable and simply reflects the arbitrariness in the choice of the normalization constant for the path integral.

In order to make sense of expressions such as (1.15), the zeta-function prescription will be adopted. Before introducing it, let us mention another equivalent approach, worth to know since it is widely used in the literature [10]. This is the so-called proper-time regularization. It is based on the following representation of the logarithm function

$$\log x = -\int_0^{+\infty} \frac{dt}{t}(e^{-tx} - e^{-t}). \quad (1.16)$$

The determinant of a finite-dimensional operator A can be expressed through

$$\log \det A = -\int_0^{+\infty} \frac{dt}{t}(e^{-tA} - e^{-t\mathbf{1}}). \quad (1.17)$$

Similarly, the determinant of the infinite-dimensional operator Ω that we are considering can be regularized by introducing a cutoff ϵ as follows

$$\log \det{}_\epsilon \Omega = -tr \int_\epsilon^{+\infty} \frac{dt}{t}(e^{-t\Omega}) = \sum_n \int_\epsilon^{+\infty} \frac{dt}{t} e^{-t\lambda_n}. \quad (1.18)$$

This method assumes implementing a renormalization prescription, required in order to remove the dependence of the results from the cutoff ϵ.

We finally point out that the presence of the cutoff ϵ spoils the scale invariance that one naively would expect from the (formally infinite) expressions such as

$$F(x) = \int_0^{+\infty} \frac{dt}{t} e^{-tx}.$$

1.2 The heat equation and the zeta-function.

In this paragraph only non-negative, self-adjoint, elliptic operators (for a definition of the elliptic operators see e.g. [16]) will be considered.

To any given operator Ω satisfying the above properties it can be associated the following equation (the heat equation)

$$\frac{\partial}{\partial\tau}G_\Omega(x,y,\tau) = -\Omega(x)G_\Omega(x,y,\tau) \tag{1.19}$$

with the given boundary condition

$$\lim_{\tau\to 0^+} G_\Omega(x,y,\tau) \;=\; \delta(x,y)\frac{1}{\sqrt{g(y)}}. \tag{1.20}$$

The heat kernel $G_\Omega(x,y,\tau)$ represents the diffusion in a parameter time τ of a unit quantity of heat (or ink) placed at the point y at $\tau=0$.

The equation can be formally solved by writing down

$$G_\Omega(x,y,\tau) = \sum_n e^{-\lambda_n\tau}\Phi_n(x)\Phi_n(y) = e^{-\tau\Omega(x)}\delta(x,y)\frac{1}{\sqrt{g(y)}}. \tag{1.21}$$

Another expression for $G_\Omega(x,y,\tau)$ is

$$G_\Omega(x,y,\tau) \;=\; < x|e^{-\tau\Omega}|y > \tag{1.22}$$

since for a generic function $f(\Omega)$ we have

$$< x|f(\Omega)|y > \;=\; f(\Omega(x))\delta(x,y)\frac{1}{\sqrt{g(y)}}. \tag{1.23}$$

A natural choice for defining the trace of $G_\Omega(x,y,\tau)$ is expressed through

$$tr(G_\Omega(x,y,\tau)) =_{def} tr(e^{-\tau\Omega}) = \sum_n e^{-\lambda_n\tau}. \tag{1.24}$$

It follows then

$$tr(G_\Omega(x,y,\tau)) = \int dx\sqrt{g(x)} < x|e^{-\tau\Omega}|x >= \int dx\sqrt{g(x)}G_\Omega(x,x,\tau). \tag{1.25}$$

Let us now introduce the generalized Riemann zeta-function $\zeta_\Omega(s)$ associated to the non-negative operator Ω. It is defined through the position

$$\zeta_\Omega(s) \;=\; \sum_n{}' \frac{1}{{\lambda_n}^s}, \tag{1.26}$$

where the sum is taken over the positive eigenvalues only and $s\in\mathbf{C}$ is a complex variable. The sum is convergent for large $Re\ s>0$; for other values of s $\zeta_\Omega(s)$ must be defined by means of analytic continuation. It can be proven that $\zeta_\Omega(s)$ is a meromorphic function and in particular analytic at $s = 0$. The usual Riemann's zeta-function ζ is defined in terms of the eigenvalues of the harmonic oscillator

$$\zeta = \sum_n \frac{1}{n^s} \tag{1.27}$$

and is absolutely convergent for $Re\ s > 1$. It is one of the most celebrated mathematical functions and is linked with the statistical distribution of the prime numbers (for more information and references see the appendix **4**).

For a given Ω the primed determinant $\det'\Omega$ built up with the positive eigenvalues only is formally given by

$$\begin{aligned} \det{}'\Omega &= \prod_n{}' \lambda_n, \\ \ln\det{}'\Omega &= \sum_n{}' \ln\lambda_n. \end{aligned} \tag{1.28}$$

For $\zeta_\Omega(s)$ we can write

$$\frac{d}{ds}\zeta_\Omega(s) = \frac{d}{ds}\sum_n{}' e^{-s\ln\lambda_n} = -\sum_n{}' \ln\lambda_n e^{-s\ln\lambda_n}. \tag{1.29}$$

At a formal level we obtain

$$\zeta_\Omega{}'(0) = -\ln\det{}'\Omega. \tag{1.30}$$

Therefore, since the definition of ζ_Ω is such that at $s = 0$ ζ_Ω is analytical, it makes sense to regularize $\det'\Omega$ through the prescription

$$\ln\det{}'\Omega \quad =_{def} -\frac{d}{ds}\zeta_\Omega(s)|_{s=0}. \tag{1.31}$$

Let us now make explicit the connection between the zeta-function and the heat kernel by expressing $\zeta_\Omega(s)$ in terms of $G_\Omega(x,y,\tau)$ or, more specifically, its trace.

Let us call $\hat{\Omega}$ the operator obtained by restricting the action of Ω to the subspace $(Ker\Omega)^\perp$ and let $\pi(Ker\Omega)$ be the projector over $Ker\Omega$.

We recall that the operators that we are considering are Fredholm operators, having a finite-dimensional kernel. We obtain

$$\frac{1}{\hat{\Omega}^s} = \int_0^{+\infty} d\tau e^{-\tau\hat{\Omega}^{-s}} = \sum_n{}' \frac{1}{\lambda_n{}^s}|\Phi_n><\Phi_n| \tag{1.32}$$

and therefore

$$tr'\frac{1}{\hat{\Omega}^s} = \sum_n{}' <\Phi_n|\frac{1}{\Omega^s}|\Phi_n> = \sum_n{}' \frac{1}{\lambda_n{}^s} = \zeta_\Omega(s). \tag{1.33}$$

By making use of the Mellin transform

$$\int_0^{+\infty} d\tau\tau^{s-1}e^{-\tau\hat{\Omega}} = \frac{1}{\hat{\Omega}^s}\int_0^{+\infty} d\tau e^{-\tau}\tau^{s-1} \tag{1.34}$$

(which can be easily understood by applying both sides to an eigenvector $|\Phi_n>$), we can write

$$\frac{1}{\hat{\Omega}^s} = \frac{1}{\Gamma(s)}\int_0^{+\infty} d\tau\tau^{s-1}e^{-\tau\hat{\Omega}}. \tag{1.35}$$

Here $\Gamma(s)$ is the Gamma-function given, for $Re\ s > 0$, by

$$\Gamma(s) = \int_0^{+\infty} d\tau\tau^{s-1}e^{-\tau}. \tag{1.36}$$

We can therefore write

$$\begin{aligned}\zeta_\Omega(s) &= tr'\frac{1}{\hat{\Omega}^s} = {\sum_n}' < \Phi_n|\frac{1}{\Gamma(s)}\int_0^{+\infty} d\tau\tau^{s-1}e^{-\tau\hat{\Omega}}|\Phi_n >= \\ &= \sum_n < \Phi_n|\frac{1}{\Gamma(s)}\int_0^{+\infty} d\tau\tau^{s-1}\left(e^{-\tau\Omega} - \pi(Ker\Omega)\right)|\Phi_n > . \end{aligned} \tag{1.37}$$

The final expression

$$\zeta_\Omega(s) = \frac{1}{\Gamma(s)}\int dx\sqrt{g(x)}\int_0^{+\infty} d\tau\tau^{s-1}\left(G_\Omega(x,x,\tau) - dim(Ker\Omega)\right) \tag{1.38}$$

gives the generalized zeta-function in terms of the trace of its corresponding heat kernel.

Let us conclude this paragraph with a remark. The operator $\Omega_{\mathbf{C}}$, acting on complex fields, can be thought to correspond to the "doubling" of a real operator $\Omega_{\mathbf{R}}$ acting on real fields. Naively one would expect $\det\Omega_{\mathbf{C}} = (\det\Omega_{\mathbf{R}})^2$. The zeta-function prescription is consistent with this requirement.

1.3 The connection between zeta-function and the trace anomaly.

In this paragraph the connection between zeta-function and the trace anomaly will be explained .

Let, as before, the operator Ω be associated to $\zeta_\Omega(s)$. The operator $\alpha\Omega$, for a given constant $\alpha > 0$, is in its turn associated to

$$\zeta_{\alpha\Omega}(s) \quad = \quad \frac{1}{\alpha^s}\zeta_\Omega(s). \tag{1.39}$$

It follows that

$$\ln \det{}'\alpha\Omega = -\frac{d}{ds}\left(\zeta_{\alpha\Omega}(s)\right)|_{s=0} = \ln \det{}'\Omega + \ln\alpha\cdot\zeta_\Omega(0). \tag{1.40}$$

We remark in particular that

$$\zeta_{\alpha\Omega}(0) \quad = \quad \zeta_\Omega(0). \tag{1.41}$$

This relation, as it will be shown later, means that the trace anomaly does not depend on the arbitrary parameter μ which normalizes the path integral.

The following classical action

$$\mathcal{S} \quad = \quad \frac{1}{2}\int dx\sqrt{g(x)}g^{ab}\partial_a\Phi(x)\partial_b\Phi(x) \tag{1.42}$$

for a real bosonic field $\Phi(x)$ describing bosonic scalars, is invariant under diffeomorphisms and, in $d = 2$ dimensions, also under the Weyl transformations

$$\begin{aligned} g_{ab}(x) \quad &\mapsto \quad e^{\alpha(x)}g_{ab}(x) \\ \Phi(x) \quad &\mapsto \quad \Phi(x) \quad (unchanged). \end{aligned} \tag{1.43}$$

Taking the parameter of the Weyl transformations being a constant independent of the space-time we get the dilatation invariance

$$0 = \delta\mathcal{S} = \int dx\delta g_{ab}(x)\frac{\delta\mathcal{S}}{\delta g_{ab}(x)} = \delta\alpha\cdot\int dx g_{ab}(x)\frac{\delta\mathcal{S}}{\delta g_{ab}(x)}. \tag{1.44}$$

The stress-energy tensor $T^{ab}(g)$ is defined as

$$T^{ab}(g) \qquad =_{def} \frac{2}{\sqrt{g(x)}}\frac{\delta\mathcal{S}}{\delta g_{ab}(x)}. \tag{1.45}$$

The relation (1.44), due to the property that $\sqrt{g(x)} > 0 \ \forall x$, implies the tracelessness of T^{ab}, i.e.

$$T^a{}_a \quad = \quad 0 \tag{1.46}$$

(the Einstein convention over repeated indices is assumed). Please notice that the above relation is obtained without making use of the equations of motion.

At a quantum level we have to consider the partition function

$$\mathcal{Z}(g) = e^{-\frac{1}{\hbar}W(g)} = \int \mathcal{D}\Phi(x) e^{-\frac{1}{2\hbar}\int dx \sqrt{g(x)} g^{ab}\partial_a\Phi\partial_b\Phi}. \tag{1.47}$$

With the prescripion discussed in the previous paragraph we have

$$\mathcal{Z}(g) \;=\; e^{-\frac{1}{2\hbar}\ln\det'\frac{\Omega}{\mu^2}}, \tag{1.48}$$

where Ω is now the operator

$$\Omega(x) \;=\; -\frac{1}{\sqrt{g(x)}}\partial_a\sqrt{g(x)}g^{ab}\partial_b. \tag{1.49}$$

Therefore

$$W(g) \;=\; = -\frac{\hbar}{2}\frac{d}{ds}\zeta_{\frac{\Omega}{\mu^2}}(s)|_{s=0}. \tag{1.50}$$

Under an infinitesimal dilatation we get

$$\delta W(g) = \frac{\hbar}{2}\delta\alpha\zeta_{\frac{\Omega}{\mu^2}}(0) = \frac{\hbar}{2}\delta\alpha\zeta_{\Omega}(0). \tag{1.51}$$

Since, as we will prove later, $\zeta_\Omega(0)$ is different from zero, this relation means that the dilatation invariance is broken at the quantum level.

Let us now define the quantum energy-momentum tensor T^{ab} through the position

$$T^{ab}(x) \quad =_{def} \frac{2}{\sqrt{g(x)}}\frac{\delta W}{\delta g_{ab}(x)}. \tag{1.52}$$

We obtain

$$\int dx \sqrt{g}(x) T^a{}_a \;=\; \hbar\zeta_\Omega(0), \tag{1.53}$$

which means that the energy-momentum tensor is no longer traceless.

Until now we have just performed formal manipulations, without worrying about how explicitly performing computations. The machinery which enables us to extract information from the operators we are interested in will be developed in the next chapter. Here, for convenience, let us just anticipate some results that will be later proven. If we fix to work in $d = 2$ dimensions the heat kernel $G_\Omega(x, x, \tau)$ can be expanded at small τ as follows

$$G_\Omega(x, x, \tau) \;=\; \frac{a_{-1}}{\tau} + a_0(x) + O(\sqrt{\tau}), \tag{1.54}$$

where the first coefficient a_{-1} is independent of x.

Let us now prove that the value of $\zeta_\Omega(s)$ at $s = 0$ is related to the coefficient $a_0(x)$ appearing in the above expansion, so that (almost) all the information about the trace anomaly is encoded in this coefficient.

Let us consider $\zeta_\Omega(s)$ as given by the equation (1.38). At small s the expansion for $\Gamma(s)^{-1}$ is given by

$$\frac{1}{\Gamma(s)} \;\approx\; s + \gamma s^2 + O(s^3),$$

where γ is the Euler constant (see appendix **4**). In order to have a non-vanishing $\zeta_\Omega(s)$ at $s=0$ we have therefore to look to the contributions of the poles $\frac{1}{s}$ arising from the integration $\int_0^{+\infty} d\tau\tau^{s-1}(...)$. The integration $\int_0^{+\infty} d\tau(...)$ can be decomposed, for an arbitrary τ_0, as $\int_0^{+\infty}(...) = \int_0^{\tau_0}(...) + \int_{\tau_0}^{+\infty}(...)$. The integration $\int_{\tau_0}^{+\infty} d\tau(...)$ gives an analytic function of s, so that the only terms containing poles can arise from the integration $\int_0^{\tau_0} d\tau(...)$.

If τ_0 is chosen to be small enough, the small-τ expansion (1.54) for $G_\Omega(x,x,,\tau)$ can be inserted inside the integration. At this point it can be easily realized that a pole is present with residue $\left(\int dx\sqrt{g}(x)a_0(x) - dim(Ker\Omega)\right)$. The final result is

$$\zeta_\Omega(0) \quad = \quad \left(\int dx\sqrt{g(x)}a_0(x) - dim(Ker\Omega)\right). \tag{1.55}$$

It must be observed that in the case of string theories we are interested in the full Weyl invariance and not just the scale invariance. To extract the dependence of the determinant of the operators under Weyl transformations some extra work and an additional hypothesis is needed. Let us think of an infinitesimal transformation $\Omega \mapsto \Omega+\delta\Omega$, parametrized by the infinitesimal function $\delta\sigma(x)$. Let us suppose that under the transformation the variation of the eigenvalue λ_n would be given by

$$\delta\lambda_n \quad = \quad k\lambda_n < \Phi_n|\delta\hat{\sigma}|\Phi_n >, \tag{1.56}$$

where k is a constant, Φ_n the eigenvectors corresponding to λ_n and $\delta\hat{\sigma}|x>= \delta\sigma(x)|x>$.

We therefore obtain

$$\delta\Omega \quad = \quad k\delta\hat{\sigma}\Omega. \tag{1.57}$$

The formula that one gets under such assumptions can be easily applied to the case of the laplacian operators acting in string theory. We notice that, with the above hypothesis, we get

$$Ker\Omega \quad = \quad Ker(\Omega+\delta\Omega). \tag{1.58}$$

After some manipulations we arrive at the intermediate step

$$\delta\zeta_\Omega(s) \quad = \quad -\frac{s}{\Gamma(s)}\int_0^{+\infty} d\tau\tau^{s-1}\left(dx\sqrt{g(x)}\delta\sigma(x)G_\Omega(x,x,\tau) - tr(\delta\hat{\sigma}\pi(Ker\Omega))\right). \tag{1.59}$$

By inserting now the small-τ expansion for the heat kernel $G_\Omega(x,x,\tau)$ as explained before, we obtain the final result

$$\delta\ln\det{}'\Omega \quad = \quad k\left(\int dx\sqrt{g(x)}\delta\sigma(x)a_0(x) - tr(\delta\hat{\sigma}\pi(Ker\Omega))\right). \tag{1.60}$$

1.4 The connection between the heat kernel and the index theorem.

In order to fully appreciate the mathematical consequences of the heat kernel approach that we have reviewed in the previous paragraphs, it is useful in this final section of the first chapter to sketch an argument showing its connection with a well-celebrated mathematical result known under the name of index theorem [17].

Indeed the relation is such that the heat kernel can be used to furnish an independent proof of the index theorem. We will use this property in the following, where the Riemann-Roch theorem will be re-derived by using the heat kernel formalism.

Let for our purposes consider a compact, oriented, boundaryless manifold $\mathcal{M}$ which is thought to be the base manifold of some vector bundles (say V_+ and V_-). Let us denote as E_+ and respectively as E_- the sets of sections of the vector bundles under consideration, therefore

$$E_\pm \;\; = \;\; \{\Phi | \Phi : \mathcal{M} \to V_\pm\} \tag{1.61}$$

Let us furthermore suppose that $E_\pm$ have the structure of a Hilbert space with a well-defined, positive, scalar product (as a concrete example we can think of $E_\pm$ as the set of tensor fields introduced in appendix **1**). Let us now denote as P an operator connecting E_+ and E_-,

$$P \;\; : \;\; E_+ \to E_-, \tag{1.62}$$

while the adjoint $P^\dagger$ is defined so that $P^\dagger : E_- \to E_+$.

We define as Ω_+ and respectively Ω_- the operators

$$\begin{aligned} \Omega_+ \quad &=_{def} P^\dagger P \quad (\Omega_+ : E_+ \to E_+), \\ \Omega_- \quad &=_{def} P P^\dagger \quad (\Omega_- : E_- \to E_-). \end{aligned} \tag{1.63}$$

It turns out that $\Omega_\pm$ are self-adjoint operators (${\Omega_\pm}^\dagger = \Omega_\pm$). We suppose them to be elliptic in order to guarantee the completeness of their eigenvectors.

Operators fulfilling the above hypothesis satisfy important properties. The first one is expressed by the relations

$$\begin{aligned} Ker\Omega_+ \;\; &= \;\; KerP, \\ Ker\Omega_- \;\; &= \;\; KerP^\dagger, \end{aligned} \tag{1.64}$$

which follows from the fact that any given $\Phi \in E_+$ such that $P\Phi = 0$ implies $\Omega_+\Phi = 0$ and that, conversely, for any given Φ such that $\Omega_+\Phi = 0$, the following chain is implied

$$(\Phi|\Omega_+|\Phi >= 0) \Rightarrow (< P\Phi|P\Phi >= 0) \Rightarrow (P\Phi = 0).$$

A second important property is the isomorphism between the spaces $(Ker\Omega_+)^\perp \subset E_+$ and $(Ker\Omega_-)^\perp \subset E_-$. Such an isomorphism arises due to the fact that for any $\Phi_n \in E_+$ eigenvector of E_+ with positive eigenvalue $\lambda_n > 0$ ($\Omega\Phi_n = \lambda_n\Phi_n$), we obtain that $\Psi_n = P\Phi_n$ ($\Psi_n \in E_-$) is an eigenvector of Ω_- with positive eigenvalue λ_n, due to the relations

$$P\Omega_+ = P(P^\dagger P) = (PP^\dagger)P = \Omega_- P. \tag{1.65}$$

As a further consequence, this not just implies an isomorphism between $(Ker\Omega_+)^\perp$ and $(Ker\Omega_-)^\perp$, as well as a one-to-one correspondence between the positive eigenvalues of Ω_+ and Ω_-, but the full identification of the spectrum of positive eigenvalues of Ω_+ with the spectrum of positive eigenvalues of Ω_-.

Let the eigenvector Φ_n, labeled by n, be chosen in such a way to form an orthonormal basis for $(Ker\Omega_+)^\perp$. It turns out as a consequence that the eigenvectors ${\Psi_n}' = \frac{1}{\sqrt{\lambda_n}}\Psi_n$ form a orthonormal basis for $(Ker\Omega_-)^\perp$.

It is convenient to recall that the kernel of the operators under consideration is finite-dimensional. As a consequence it turns out to be a meaningful expression to introduce the analytic index $\mathcal{I}_A(P)$ associated to the operator P through the following position

$$\mathcal{I}_A(P) \quad =_{def} dim(KerP) - dim(KerP^\dagger). \tag{1.66}$$

The index theorem connects the analytical properties (expressed through the above defined analytic index $\mathcal{I}_A(P)$) of the operators P, $P^\dagger$, with some topological properties, expressed through the so-called topological index, by stating that the two indices are equal.

Let us now illustrate how the scheme developed in the previous paragraphs enables to compute the analytic index $\mathcal{I}_A(P)$.

From the above discussion it is clear that the relation

$$\mathcal{I}_A(P) = \sum_n e^{-\lambda_n\tau} - \sum_m e^{-\lambda_m\tau} \tag{1.67}$$

holds (here the parameters n, m label the eigenvalues of Ω_+ and Ω_- respectively). Such a relation can also be stated as follows

$$\mathcal{I}_A(P) \;\; = \;\; tr\left(G_{\Omega_+}(x,x,\tau)\right) - tr\left(G_{\Omega_-}(x,x,\tau)\right), \tag{1.68}$$

where $G_{\Omega_\pm}(x,y,\tau)$ denote the heat kernel for $\Omega_\pm$. It should be noticed that this equality is valid for arbitrary τ. In particular τ can be chosen to be small enough so that the small-τ expansion for $G_{\Omega_\pm}(x,x,\tau)$ can be inserted in the right hand side of the above expression . Therefore in order to compute $\mathcal{I}_A(P)$ it is enough just to compute (when working in $d = 2$ dimensions) the coefficient $a_0(x)$ appearing in the small-τ expansion for both $G_{\Omega_+}(x,x,\tau)$ and $G_{\Omega_-}(x,x,\tau)$, see (1.54). When applied to the laplacian operators introduced in appendix 1, the index so computed turns out to be a topological quantity (as explained in appendix 2).

Let us close this chapter by pointing out that the discussion concerning the index theorem can be repeated in the framework of the Witten's approach to the index of supersymmetric theories [18]. In this context the supersymmetry is introduced as follows. At first a Hilbert space $\mathcal{H}$ is introduced given by the direct sum of E_+ and E_-, i.e.

$$\mathcal{H} \;\; = \;\; E_+ \oplus E_-. \tag{1.69}$$

It is now possible to introduce a hermitian operator Q acting on $\mathcal{H}$, which plays the role of a "supersymmetry operator", through the position

$$Q \quad =_{def} \left(\begin{array}{cc} 0 & P^\dagger \\ P & 0 \end{array}\right). \tag{1.70}$$

The operator $(-)^F$, given by

$$(-)^F \quad =_{def} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{1.71}$$

plays the role of the "fermion number operator" while H

$$H \;=\; Q^2 \tag{1.72}$$

plays the role of the supersymmetric hamiltonian [18, 19].

The following set of (anti-)commutation relations are satisfied

$$[H, Q] = [H, (-)^F] = \{Q, (-)^F\} = 0. \tag{1.73}$$

It is easily checked that for each positive eigenvalue of H there exists a pairing of a "bosonic" state with a "fermionic" one. This correspondence is broken for the state admitting "zero energy". As a consequence the analytic index $\mathcal{I}_A(P)$ can be expressed as a supersymmetric index given by

$$\mathcal{I}_A(P) \;=\; tr(-)^F. \tag{1.74}$$

This viewpoint is at the basis of an alternative way of looking at the heat kernel expansion, which exploits its connection with the supersymmetric quantum mechanical systems [20].

2 A computation of the trace anomaly.

2.1 The method.

In the previous chapter it has been mentioned that in order to obtain the trace anomaly it is not needed to know the full solution of the heat equation. Actually it is sufficient to know the behaviour of $G_\Omega(x,y,\tau)$ at $x=y$ and at small τ.

In this chapter a method will be presented allowing to compute the coefficient $a_0(x)$ in the expansion (1.54) for a certain class of elliptic operators acting on functions which are sections over a 2-dimensional manifold. The method will be explained in detail because it is interesting in itself and because it can be easily generalized to obtain information for operators acting on objects defined over higher-dimensional manifolds.

It is made use of a perturbative approach. The operator $\Omega(x)$ that we are considering can be thought to be split into two pieces

$$\Omega(x) = \Omega_0(x) + V(x), \tag{2.75}$$

where $\Omega_0(x)$ is an unperturbed operator for which the exact solution of the heat equation is assumed to be known. The term $e^{-\tau\Omega(x)}$ can be expressed as

$$e^{-\tau\Omega(x)} = \left(e^{-\tau\Omega(x)}e^{\tau\Omega_0(x)}\right)e^{-\tau\Omega_0(x)}. \tag{2.76}$$

In the following the Campbell-Hausdorff formula will be used

$$\begin{aligned} e^A\cdot e^B &= e^{C(A,B)}, \\ C(A,B) &= A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\frac{1}{12}[B,[B,A]]+\ldots \quad . \end{aligned} \tag{2.77}$$

Therefore

$$e^{A+B} = \left(e^{A+B}e^{-B}\right)e^{B} = e^{C(A,B)}e^{B}. \tag{2.78}$$

Identifying $A=-\tau\Omega_0$, $B=-\tau V$ and applying both sides of (2.76) to $\delta(x,y)\frac{1}{\sqrt{g(y)}}$, we can write the following equation

$$G_\Omega(x,y,\tau) = \left(e^{-\tau V-\frac{1}{2}[\tau V,\tau\Omega_0]+\frac{1}{12}[\tau V,[\tau V,\tau\Omega_0]]-\frac{1}{12}[\tau\Omega_0,[\tau\Omega_0,\tau V]]+\ldots}\right)G_{\Omega_0}(x,y,\tau)\frac{1}{\sqrt{g(y)}}, \tag{2.79}$$

with

$$G_{\Omega_0}(x,y,\tau) = e^{-\tau\Omega_0(x)}\delta(x,y). \tag{2.80}$$

When we work in $d=2$ dimensions we can take as $\Omega_0(x)$ the flat laplacian

$$\Omega_0 = -\alpha({\partial_{x_1}}^2+{\partial_{x_2}}^2)+m^2. \tag{2.81}$$

It can be easily verified that the function $G_{\Omega_0}(x,y,\tau)$ which solves the equation

$$\frac{\partial}{\partial\tau}G_{\Omega_0}(x,y,\tau) = -\Omega_0 G_{\Omega_0}(x,y,\tau) \tag{2.82}$$

with the boundary condition

$$\lim_{\tau\to 0^+} G_{\Omega_0}(x,y,\tau) \quad = \quad \delta(x,y) \tag{2.83}$$

is given by

$$G_{\Omega_0}(x,y,\tau) \quad = \quad \frac{1}{4\pi\alpha\tau}e^{-\frac{(x-y)^2}{4\alpha\tau}-m^2\tau}. \tag{2.84}$$

Before going ahead let us just mention that the elliptic operators that we are interested in are the laplacians which play a role in string theory and which are introduced in appendix 2.

It turns out that, when expressed in real coordinates, such laplacians have the following form

$$\Omega(x) \quad = \quad -[A(x)({\partial_{x_1}}^2+{\partial_{x_2}}^2)+B(x)(\partial_{x_1}+i\partial_{x_2})+C(x)]. \tag{2.85}$$

In this chapter it is convenient to work with real coordinates, the final formula being translated into the complex notation used for the strings.

The theories under consideration are invariant under diffeomorphisms. As a consequence we can fix $y=0$ and compute $G_\Omega(x,0,\tau)$ at $x=0$ working in a particular frame of reference. The particular frame is given by the normal coordinates expansion [21] around $x=0$. It implies that for our laplacian operators we can fix

$$A(0)=1 \qquad , \qquad \partial_{x_{1,2}}A(x)|_{x=0}=0. \tag{2.86}$$

Since the heat kernel $G_\Omega(x,y,\tau)$, as evident from the definition given, is scalar both in x and in y, the results obtained, which depend on the particular frame of reference chosen, can be re-expressed at the end in a manifestly covariant form.

We are interested in computing the corrections of $G_{\Omega_0}(0,0,\tau)$ at small τ. We notice that the application of the operator $\partial^2={\partial_{x_1}}^2+{\partial_{x_2}}^2$ to $G_{\Omega_0}(x,0,\tau)$ has the effect of multiplying $G_{\Omega_0}(x,0,\tau)$ by the factor

$$\left(\tfrac{-1}{\alpha\tau}+\tfrac{x^2}{4\alpha^2\tau^2}\right). \tag{2.87}$$

The term $\frac{x^2}{4\alpha^2\tau^2}$ vanishes when we evaluate the trace (i.e. at $x=0$). It is then possible to introduce a sort of τ-dimensionality of the operators. A derivative ∂_x has a τ-dimensionality $[\partial_x]=-\frac{1}{2}$, while a factor x has τ-dimensionality $[x]=\frac{1}{2}$ since, in order to give a non-vanishing contribution when computed $G_{\Omega_0}(x,0,\tau)$ at $x=0$, it must be "eaten" by a derivative.

Let us now expand the functions $A(x)$, $B(x)$, $C(x)$ in Taylor series around $x=0$. We get the following expressions

$$\begin{aligned}
\Omega &= \Omega_1+\Omega_2+\Omega_3,\\
\Omega_1 &= -\left(a+x^af_a+\frac{1}{2}x^ax^b\rho_{ab}+...\right)({\partial_{x_1}}^2+{\partial_{x_2}}^2),\\
\Omega_2 &= -\left(b+x^ad_a+...\right)(\partial_{x_1}+\partial_{x_2})\\
\Omega_3 &= -\left(c+...\right),
\end{aligned} \tag{2.88}$$

where

$$a = A(0), \quad f_a = \frac{\partial}{\partial x^a} A(x)\mid_{x=0}, \quad \rho_{ab} = \frac{\partial^2}{\partial x^a \partial x^b} A(x)\mid_{x=0},$$
$$b = B(0), \quad d_a = \frac{\partial}{\partial x^a} B(x)\mid_{x=0},$$
$$c = C(0), \tag{2.89}$$

while the flat laplacian is now given by

$$\Omega_0 \;=\; -\left(a({\partial_{x_1}}^2 + {\partial_{x_2}}^2) + c\right). \tag{2.90}$$

As explained, we can fix $a = 1$ and $f_a = 0$.

Clearly the only operators in τV giving contribution to the lowest order in τ are those whose τ-dimensionality is ≤ 1. This implies that the only operators which play a role are given by

$$\begin{aligned}\tau V_1 &= -\frac{1}{2} x^a x^b \rho_{ab} \tau ({\partial_{x_1}}^2 + {\partial_{x_2}}^2), \quad (\tau - dim. = 1),\\ \tau V_2 &= -b\tau(\partial_{x_1} + i\partial_{x_2}), \quad (\tau - dim. = \frac{1}{2}),\\ \tau V_3 &= x^a d_a \tau (\partial_{x_1} + i\partial_{x_2}), \quad (\tau - dim. = 1).\end{aligned} \tag{2.91}$$

Let us now evaluate the commutator of these operators with $\tau\Omega_0$. We get

$$\begin{aligned}&[\tau\Omega_0, \tau V_1] = \tau^2[\partial^2, \frac{1}{2} x^a x^b \rho_{ab} \partial^2] = \tau^2({\rho_c}^c \partial^2 + 2x^a \rho_{ac} \partial^2 \partial^c), \quad (\tau - dim. = 1),\\ &[\tau\Omega_0, \tau V_2] = 0,\\ &[\tau\Omega_0, \tau V_3] = \tau^2[\partial^2, x^a d_a(\partial_{x_1} + i\partial_{x_2})] = 2\alpha d_c \partial^c \tau^2 (\partial_{x_1} + i\partial_{x_2}), \quad (\tau - dim. = 1).\end{aligned} \tag{2.92}$$

When we evaluate the commutators of three operators we have

$$\begin{aligned}[\tau\Omega_0, [\tau\Omega_0, \tau V]] &= [\tau\Omega_0, [\tau\Omega_0, \tau V_1]] = [-\tau\partial^2, 2\tau^2 x^a \rho_{ac} \partial^2 \partial^c]] =\\ &= -4\tau^3 \rho_{ac} \partial^a \partial^c \partial^2, \quad (\tau - dim. = 1).\end{aligned} \tag{2.93}$$

In $[\tau V, [\tau V, \tau\Omega_0]]$ the only operator which, by dimensionality argument, can give contribution is V_2; however its commutator with Ω_0 is vanishing.

Clearly all the remaining commutators give no contribution at the lowest order in τ.

At this point we can apply the formula (2.79). The exponential will be expanded in power series and we get

$$\begin{aligned}G_\Omega(0,0,\tau) \;=\; &\{1 - \tau V - \frac{1}{2}[\tau V, \tau\Omega_0] + \frac{1}{12}[\tau V, [\tau V, \tau\Omega_0]] - \frac{1}{12}[\tau\Omega_0, [\tau\Omega_0, \tau V]] + \frac{1}{2!}(\tau V)^2\}\\ &\cdot G_{\Omega_0}(x,0,\tau)\mid_{x=0} \cdot \frac{1}{\sqrt{g(0)}} + O(\sqrt{\tau}).\end{aligned} \tag{2.94}$$

Taking the derivatives ${\partial_{x_1}}^2$, ${\partial_{x_2}}^2$ of $G_{\Omega_0}(x,0,\tau)$ and setting $x = 0$ has the same effect as multiplying $G_{\Omega_0}(0,0,\tau)$ by a factor:

$${\partial_{x_1}}^2 \equiv {\partial_{x_2}}^2 \equiv -\frac{1}{2\tau}, \quad \partial^2 \equiv -\frac{1}{\tau}, \quad \partial^2\partial^2 \equiv \frac{2}{\tau^2}, \quad \frac{1}{2!}\tau^2 {V_2}^2 \equiv \frac{1}{2} b^2 \tau^2 ({\partial_{x_1}}^2 - {\partial_{x_2}}^2) \equiv 0. \tag{2.95}$$

We get the final result

$$G_\Omega(0,0,\tau) \;=\; [1+\frac{1}{6}\tau{\rho_c}^c - \frac{1}{2}\tau(d_{x_1}+id_{x_2})+\tau c]\frac{1}{4\pi\tau}+O(\sqrt{\tau}). \tag{2.96}$$

2.2 The results.

It is convenient to re-express the final result in compact form, both in real and complex notations (for complex notations see the appendices **1** and **2**).

Let

$$\begin{aligned}\Omega_{(x)} \;&=\; -[A(x)({\partial_{x_1}}^2+{\partial_{x_2}}^2)+B(x)(\partial_{x_1}+i\partial_{x_2})+C(x)],\\ &A(0)=1, \quad \partial_{x_{1,2}}A(x)\mid_{x=0}=1,\end{aligned} \tag{2.97}$$

Then

$$G_\Omega(0,0,\tau) \;=\; \frac{1}{4\pi\tau}+\frac{1}{4\pi}\left[\frac{1}{6}({\partial_{x_1}}^2+{\partial_{x_2}}^2)A\mid_{x=0} -\frac{1}{2}(\partial_{x_1}+i\partial_{x_2})B\mid_{x=0}+C(0)\right]+O(\sqrt{\tau}). \tag{2.98}$$

In complex notations we have

$$\Omega_{\mathbf{C}} \;=\; -\left[4A\partial_z\partial_{\overline{z}}+2B\partial_{\overline{z}}+C\right], \tag{2.99}$$

and therefore

$$G_{\Omega_{\mathbf{C}}} \;=\; \frac{1}{2\pi\tau}+\frac{1}{2\pi}\left[\frac{2}{3}\partial_z\partial_{\overline{z}}A\mid_0 -\partial_{\overline{z}}B\mid_0+C(0)\right]+O(\sqrt{\tau}). \tag{2.100}$$

In this formula a factor 2 has been inserted w.r.t. formula (2.98) in order to take into account that $\Omega_{\mathbf{C}}$ is assumed to act on complex fields.

3 The bosonic string theory.

3.1 An introduction to string theory.

In this chapter we will review the Polyakov functional approach to the quantization of the closed, oriented, bosonic string and we will show how in this approach the critical dimensionality $d = 26$ is singled out. Different approaches to the quantization of the non-interacting bosonic string, like the canonical quantization or the BRST-quantization, turn out to be equivalent to the Polyakov formulation. In such approaches the critical dimensionality is recovered by different mathematical structures [22]. For instance in the operatorial formalism the critical dimension is related to the central extension of the Virasoro algebra.

In this introduction the basic ingredients of the classical string theory that are needed to know, will be briefly discussed.

A string theory [2, 4] is characterized by the fact that the fundamental dynamical objects are not point-like, but one-dimensional. It is a theory of a curve whose time-evolution sweeps out a 2-dimensional surface (the so-called world-sheet) in a given, external, space-time known as target.

At a classical level the equations of motion are determied by the Nambu-Goto action, expressed by the geometrical area of the swept surface. Such an action is highly non-linear in the coordinates and therefore quite difficult to quantize.

According to the Polyakov prescription [23], an alternative and more convenient starting point consists in assuming a classical action which depends on additional (Lagrange multipliers-like) functions given by the intrinsic metric g_{ab} of the world-sheet surface.

The new action is given by

$$I_0(\overrightarrow{X}, \mathbf{g}) \;=\; \frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X_\mu . \tag{3.101}$$

Here $\mathcal{M}$ is a compact 2-dimensional, oriented surface, $\mathbf{g}$ is a riemannian (positive-definite) metric on $\mathcal{M}$. X^μ is an embedding of $\mathcal{M}$ into a d-dimensional space-time, i. e.

$$\overrightarrow{X} \;:\; \mathcal{M} \rightarrow \mathbf{R}^d . \tag{3.102}$$

The space-time is thought to be flat and Euclidean (it has been assumed that a Wick rotation from Minkowski to the Euclidean has been performed both on the 2-dimensional world-sheet, as well as on the d-dimensional target space).

When using the variational principle we are free to vary independently both X^μ and g_{ab}. The metric g_{ab} is non-dynamical and may be solved for, so that we are led again to the same equations of motion for X^μ as those derived from the Nambu-Goto action.

The string action I_0 admits three groups of invariance at the classical level:
i) invariance under diffeomorphisms of the 2-dimensional manifold $\mathcal{M}$

$$\begin{aligned} x^a &\mapsto (x')^a = x'(x^b)^a, \\ g_{ab}(x) &\mapsto \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b} g_{cd}, \end{aligned} \tag{3.103}$$

ii) conformal (Weyl) invariance under local rescaling of the metric

$$g_{ab}(x) \;\mapsto\; e^{2\sigma(x)} g_{ab}(x), \tag{3.104}$$

iii) invariance under global rotations and translations of $\mathbf{R}^d$ (Euclidean transformations).

As already seen in chapter **1**, the conformal invariance is in general not preserved at the quantum level. As a consequence when, let's say, the partition function is regularized according to the proper-time prescription, the criterion of renormalizability forces us to consider the most general action having coupling of non-negative dimensions, consistent with the symmetries *i)* and *iii)*.[1]

For boundaryless manifolds, the most general action is given by

$$I(\mathbf{X}, \mathbf{g}) \;=\; \int_{\mathcal{M}} d^2x \left(\frac{1}{2} A \sqrt{g(x)} g^{ab} \partial_a X^\mu \partial_b X_\mu + \frac{1}{4\pi} B \sqrt{g(x)} R + C \sqrt{g(x)} \right). \quad (3.105)$$

The second term in the right hand side is a topological invariant quantity giving no contribution to the dynamics while, at the classical level, the compatibility of the equations of motion requires the vanishing of the coefficient C ($C = 0$ for consistency).

[1] the zeta-function prescription, as already remarked, allows to bypass this step since counterterms are not needed being explicitly computed with this method.

3.2 Quantization of the bosonic string.

The Polyakov functional approach to the quantization of the bosonic string is essentially a perturbative approach which postulates a partition function $\mathcal{Z}$ given by

$$\mathcal{Z} \;=\; \mathcal{N}\sum_h \lambda^h \int_{metrics} \mathcal{D}g \int_{embeddings} \mathcal{D}X^\mu \cdot e^{-I_0(\mathbf{X},\mathbf{g})}, \tag{3.106}$$

where h is an integer, the number of handles of the world-sheet surface. The sum over h is a sum over the different topologies of the world-sheet surface and takes into account the effects due to the string interactions; λ is a coupling constant and $\mathcal{N}$ the usual normalization factor.

In order to compute S-matrix amplitudes the insertion of vertex operators, representing states of *in* and *out* strings, is required.

To be definite, it is of course necessary to specify the functional measure. It turns out that a natural measure $\mathcal{D}X^\mu$ for the configuration space $\mathcal{C}$ of strings ($\mathcal{C} = \{\mathbf{X} : \mathcal{M} \to \mathcal{M}\}$) is the one corresponding to the metric

$$\|\delta X^\mu\|^2 \;=\; \int_{\mathcal{M}} d^2x\sqrt{g(x)}\delta X^\mu \delta X^\nu \eta_{\mu\nu} \tag{3.107}$$

between two maps X^μ, $X^{\mu\prime} = X^\mu + \delta X^\mu$ (we recall that in an ordinary Riemannian space of finite dimension N with metric $ds^2 = g_{ij}dx^i dx^j$ the volume element dV is given by $dV = d^N x\sqrt{g(x)}$).

The word "natural" is employed for the following reasons. The (3.107) metric is the simplest expression invariant under global translations of X^μ and under diffeomorphisms, not involving either a derivative of $\mathbf{g}$ or a derivative of X^μ. We further point out that the measure $\mathcal{D}X^\mu$ is not conformally invariant.

A similar procedure is used to single out the measure $\mathcal{D}g$. Let $\mathcal{G}$ be the space of Riemannian metrics $\mathbf{g}$ of the world-sheet manifold. $\mathcal{G}$ turns out to be a convex, non-compact space. A natural metric ("natural" has a similar meaning as before) for $\mathcal{G}$ is expressed by the relation

$$\|\delta g\|^2 \;=\; \int_{\mathcal{M}} d^2x\sqrt{g(x)}(G^{abcd} + ug^{ab}g^{cd})\delta g_{ab}\delta g_{cd} \tag{3.108}$$

where u is an arbitrary positive constant ($u > 0$). G^{abcd} is a projector onto the space of symmetric, traceless tensors

$$G_{ab}{}^{cd} \;=\; \frac{1}{2}(\delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c - g_{ab}g^{cd}). \tag{3.109}$$

$\mathcal{D}g$ is of course the measure associated with such a metric and, just like $\mathcal{D}X^\mu$, is not conformally invariant. We will later prove that for $d = 26$ the conformal anomaly associated with $\mathcal{D}g$ precisely cancels the conformal anomaly associated with $\mathcal{D}X^\mu$.

Both the action and the measure are covariantly defined so that, at least formally, the partition function is invariant under a reparametrization of the world-sheet. Physically equivalent configurations are counted many times, since the configuration space over which one is integrating is the tensor product space $\mathcal{G}\times\mathcal{C}$, while the space of physically equivalent configurations is the quotient space $\mathcal{G}\times\mathcal{C}/diff(\mathcal{M})$.

The functional integral contains an overall infinite factor which has to be removed by restricting the integral to a gauge slice, i.e. to a subspace of metrics meeting each orbit of the local gauge group exactly once. The full diffeomorphisms group $diff(\mathcal{M})$ may be thought to be expressed by a discrete transformation in combination with an element of $diff_0(\mathcal{M})$, the subgroup of diffeomorphisms connected with the identity.

As a matter of fact, it turns out that the "large", discrete diffeomorphisms may be anomalous. Here we will use the Fadeev-Popov technique in order to factor out the integration over $diff_0(\mathcal{M})$. In order to apply the Fadeev-Popov method, let us decompose the most general variation δg_{ab} as follows

$$\delta g_{ab} \;=\; \delta h_{ab} + 2 g_{ab}\delta\tau, \tag{3.110}$$

with

$$\delta h_{ab} \;=\; G_{ab}{}^{cd}\delta g_{cd} \tag{3.111}$$

(δh_{ab} is a symmetric traceless tensor).

Inserting the above decomposition in $\|\delta g\|^2$ we can write

$$\|\delta g\|^2 \;=\; \int_{\mathcal{M}} d^2x\sqrt{g(x)}(G^{abcd}\delta h_{ab}\delta h_{cd} + 16u\delta\tau^2), \tag{3.112}$$

which implies the following relation

$$\mathcal{D}g \;=\; (\mathcal{D}h)\cdot(\mathcal{D}\tau). \tag{3.113}$$

It has to be warned that (3.113) is a decomposition of a tangent vector in an unspecified orthonormal frame which is in general not integrable and does not lead to a coordinate system labeled by h and τ.

The infinitesimal variation δg_{ab} is specified by three arbitrary functions, two of them for the traceless tensor δh_{ab} and one for $\delta\tau$. At this point it is useful to express the generic variation δg_{ab} in terms of an infinitesimal diffeomorphism connected with the identity (such a diffeomorphism involves two arbitrary functions) and of an infinitesimal Weyl transformation involving one arbitrary function (as a matter of fact, not all δg_{ab} can be expressed in such a way, we will however come back on this point later on).

Our infinitesimal diffeomorphism is now specified by the infinitesimal vector field $\delta V^a(x)$. The corresponding variation of g_{ab} is given by

$$\delta_D g_{ab} \;=\; \nabla_a\delta V_b + \nabla_b\delta V_a, \tag{3.114}$$

where ∇_a is the covariant derivative introduced in appendix **2**.

Under an infinitesimal Weyl transformation we have

$$\delta_W g_{ab} \;=\; 2\delta\sigma g_{ab}. \tag{3.115}$$

Therefore

$$\delta g_{ab} \;=\; \delta_D g_{ab} + \delta_W g_{ab}. \tag{3.116}$$

It then follows

$$\begin{aligned} \delta h_{ab} \;&=\; G_{ab}{}^{cd}\delta g_{cd} = 2G_{ab}{}^{cd}\nabla_c\delta V_d = (P\delta V)_{ab}, \\ 2g_{ab}\delta\tau \;&=\; \frac{1}{2}g_{ab}g^{cd}\delta g_{cd} = g_{ab}(2\delta\sigma + g^{cd}\nabla_c\delta V_d), \end{aligned} \tag{3.117}$$

and the latter equality implies

$$\delta\tau \;=\; \delta\sigma + \frac{1}{2}g^{cd}\nabla_c\delta V_d. \tag{3.118}$$

The operator P maps vectors into traceless tensors.

Changing our variables from δh_{ab}, $\delta\tau$ to δV_a, $\delta\sigma$, we can write the measure as

$$(\mathcal{D}h)\cdot(\mathcal{D}\tau) \;=\; (\mathcal{D}\sigma)(\mathcal{D}V)J, \tag{3.119}$$

where J is the jacobian

$$J \;=\; |\frac{\partial(\tau,h)}{\partial(\sigma,V)}| = |\det\begin{pmatrix} \mathbf{1} & * \\ 0 & P\end{pmatrix}| = (detP^\dagger P)^{\frac{1}{2}} \tag{3.120}$$

(the term denoted as "$*$" gives no contribution to the determinant since the matrix is triangular).

$P^\dagger$ is the adjoint of P and maps traceless tensors into vectors.

There are two crucial observations to be made. The first one concerns the zero-modes of the operator P. The vectors δV satisfying $P\delta V = 0$ belong to $Ker(P)$ and are called "conformal Killing vectors". A diffeomorphism generated by such vectors is equivalent to a change in the conformal factor and must be omitted since the deformation of the metric has to be counted only once. The correct jacobian is therefore given by the primed determinant $(\det' P^\dagger P)^{\frac{1}{2}}$, which is built only with the positive eigenvalues.

The second observation concerns the fact that no every deformation δh_{ab} can be expressed as $P\delta V$. This point can be easily understood if we go back to the discussion in paragraph **1.4**. There exists a one-to-one correspondence between the subspace of vector fields given by $(KerP)^\perp$ and the subspace of traceless tensor fields given by $(KerP^\dagger)^\perp$. The procedure used to isolate the volume of the gauge group is incomplete due to this fact, and we are still required to integrate over the deformations which belong to the finite-dimensional subspace $KerP^\dagger$. Such deformations are known as "Teichmüller deformations".

We are thus led to write our partition function $\mathcal{Z}$ as follows

$$\mathcal{Z} \;=\; \mathcal{N}\sum_{topolgies}\int \mathcal{D}\sigma\cdot\Omega_{diff_0}{}^\perp(\mathcal{D}X^\mu)(\det' P^\dagger P)^{\frac{1}{2}}\cdot e^{-I_0(\mathbf{X},\hat{g}_t e^{2\sigma})}, \tag{3.121}$$

where $\Omega_{diff_0}{}^\perp$ is the volume of diffeomorphisms which are perpendicular to the conformal Killing vectors, while "t" denotes the Teichmüller deformations ($\mathcal{D}t$ is the integration over the Teichmüller deformations) and $\hat{g}$ is a given reference metric.

Since the action I_0 is quadratic in X^μ, the integration over X^μ can be performed in the usual way. Let $\hat{X}^\mu$ be a classical solution of motion in presence of the background metric $\hat{g}$. We have

$$\int \mathcal{D}X^\mu e^{-I_0(\mathbf{X},\hat{g})} \;=\; e^{-I_0(\hat{\mathbf{X}},\hat{g})}\cdot(\det'\Delta_0)^{-\frac{d}{2}}\cdot\int_{0-modes}\mathcal{D}X_0{}^\mu. \tag{3.122}$$

Here Δ_0 is the laplacian operator (1.49) associated with the action I_0 (see paragraph **1.3**). The symbol $\int_{0-modes}\mathcal{D}X_0{}^\mu$ denotes the eventual integration over the zero-modes of the laplacian operator.

Since our classical action is conformally invariant we have

$$I_0(\hat{\mathbf{X}},\hat{g}) \;=\; I_0(\hat{\mathbf{X}},\hat{g}e^{2\sigma}). \tag{3.123}$$

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3.3 The conformal anomaly of the bosonic string.

In this paragraph the conformal anomaly of the bosonic string (i.e. the dependence of the partition function $\mathcal{Z}$ (3.121) on the conformal factor $\sigma(x)$) will be computed.

If in $\mathcal{Z}$ the integrand is not affected by a change of $\sigma(x)$, $(\sigma(x) \mapsto \sigma(x) + \delta\sigma(x))$, then the measure $\mathcal{D}\sigma(x)$ can be absorbed in the normalization factor $\mathcal{N}$ and the resulting theory is conformally invariant.

In order to leave the derivation as simple as possible, we will not be worried by the complications introduced by the Teichmüller deformations. For this purpose it is sufficient to work in the sector $h = 0$, which means that our base manifolds $\mathcal{M}$ is topologically equivalent to a sphere, known to admit no Teichmüller deformations. Our derivation however can be easily extended to the more general case. In particular it can be proven that the cancellation of the conformal anomaly of the bosonic string in critical dimension is realized independently of the topological sector of the worldsheet. The situation however is no longer the same in the case of superstrings, see [24], where topological obstructions to the conformal invariance can arise.

The partition function (3.121) can, in this simplified case, be re-written as

$$\mathcal{Z} \;=\; \mathcal{N}\int \mathcal{D}\sigma \frac{{\Omega_{diff_0}}^{\perp}}{\Omega_{diff_0}} \cdot \Omega_{diff_0} (\det{}' P^{\dagger}P)^{\frac{1}{2}} (\det{}'\Delta_0)^{-\frac{d}{2}} \int_{0-modes} \mathcal{D}{X_0}^{\mu}, \tag{3.124}$$

where the ratio $\frac{\Omega_{diff_0}}{{\Omega_{diff_0}}^{\perp}}$ is basically Ω_{CKV}, the volume of the conformal Killing vectors. Let us observe at this point that the operator P introduced in (3.117) and mapping vectors (specified by two real components) into traceless tensors (also specified by two components), is expressed in the complex formalism introduced in appendix **2** by the covariant derivative ${\nabla_1}^{z}$ which maps tensors belonging to τ^1 (2 real components) into tensors belonging to τ^2 (also 2 real components). Therefore the operator $P^{\dagger}P$ coincides with the laplacian ${\Delta_1}^{+}$ of appendix **2**.

In the same way Δ_0 is related to ${\Delta_0}^{+}$ of appendix **2** (the only difference is that the latter one acts on complex fields, while we have assumed Δ_0 acting on real fields).

In order to compute the conformal anomaly for the string theory we have to apply the method explained in chapter **1** and **2** to the operators ${\Delta_n}^{+}$, ${\Delta_n}^{-}$ of appendix **2**.

At first we have to compute the small-τ expansion for the heat kernel $G_{{\Delta_n}^{\pm}}(x,x,\tau)$. In order to do so, we have to specialize the formula (2.98) to this class of operators, inserting the correct coefficients.

We recall that the invariance under diffeomorphisms allows us to fix at $x = 0$

$$\sigma(0) = 0, \quad \partial_a\sigma(x)\,|_{x=0} = 0. \tag{3.125}$$

With this choice we can write

$$\begin{aligned} G_{{\Delta_n}^{+}}(0,0,\tau) &= \frac{1}{2\pi\tau} + \frac{3n+1}{12\pi}\cdot(-2\partial^2\sigma)\,|_{x=0} + O(\sqrt{\tau}), \\ G_{{\Delta_n}^{-}}(0,0,\tau) &= \frac{1}{2\pi\tau} - \frac{3n-1}{12\pi}\cdot(-2\partial^2\sigma)\,|_{x=0} + O(\sqrt{\tau}). \end{aligned} \tag{3.126}$$

In order to express $G_{{\Delta_n}^{\pm}}(x,x,\tau)$ in a manifestly covariant form we have to notice that, with the position (3.125), we have

$$R(0) \;=\; -2\partial^2\sigma\,|_{x=0}\,. \tag{3.127}$$

The traces are therefore given by

$$
\begin{aligned}
tr(G_{\Delta_n{}^+}(x,x,\tau)) &= \frac{1}{2\pi\tau}\int d^2x\sqrt{g(x)} + \frac{3n+1}{12\pi}\int d^2x\sqrt{g(x)}R(x) + O(\sqrt{\tau}),\\
tr(G_{\Delta_n{}^-}(x,x,\tau)) &= \frac{1}{2\pi\tau}\int d^2x\sqrt{g(x)} - \frac{3n-1}{12\pi}\int d^2x\sqrt{g(x)}R(x) + O(\sqrt{\tau}).
\end{aligned}
\tag{3.128}
$$

The Riemann-Roch theorem is at this point easily derived. Indeed the analytical index $\mathcal{I}_A(\nabla_n{}^z)$ is given by

$$
\mathcal{I}_A(\nabla_n{}^z) = dim(Ker\nabla_n{}^z) - dim(Ker(\nabla_n{}^z)^\dagger) = tr(G_{\Delta_n{}^+}(x,x,\tau)) - tr(G_{\Delta_{n+1}{}^-}(x,x,\tau)),
\tag{3.129}
$$

therefore

$$
\mathcal{I}_A(\nabla_n{}^z) = (2n+1)\frac{1}{4\pi}\int d^2x\sqrt{g(x)}R(x) = (2n+1)\cdot\chi(\mathcal{M})
\tag{3.130}
$$

(we point out that in such a formula the dimension of the kernel is given by counting the number of *real* components). The latter equality (see appendix **2**) makes use of the Gauß-Bonnet theorem and shows that the analytical index is a topological invariant quantity. By the way, our analysis can prove, without previous knowledge of the Gauß-Bonnet theorem, that $\frac{1}{4\pi}\int\sqrt{g}R$ is indeed an integer and as such a topological quantity, being unmodified by smooth infinitesimal deformations .

For $n=1$ we obtain in particular the relation

$$
\sharp(conf.\quad Kill.\quad vect.) - \sharp(Teichm.\quad deform.) = 6-6h.
\tag{3.131}
$$

The variation of the operator $\Delta_n{}^\pm(\sigma)$ under an infinitesimal Weyl transformation $\sigma\mapsto\sigma+\delta\sigma$ is given by

$$
\delta\Delta_n{}^\pm(\sigma) = \Delta_n{}^\pm(\sigma+\delta\sigma) - \Delta_n{}^\pm(\sigma) = -2(n+1)\delta\sigma\Delta_n{}^\pm(\sigma) + 4(\nabla_n{}^z)^\dagger\delta\sigma\nabla_n{}^z.
\tag{3.132}
$$

The corresponding variation of a positive eigenvalue λ_i of $\Delta_n{}^+(\sigma)$ is

$$
\delta\lambda_i = <\phi_i|\delta\Delta_n{}^+(\sigma)|\phi_i>,
\tag{3.133}
$$

where $|\phi_i>$ is the normalized eigenvector of $\Delta_n{}^+(\sigma)$ corresponding to the eigenvalue λ_i (therefore $|\phi_i>\in(Ker\Delta_n{}^+(\sigma))^\perp$ and is a n-rank tensor).

It follows

$$
\begin{aligned}
\delta\lambda_i &= -2(n+1)<\phi_i|\delta\sigma\Delta_n{}^+(\sigma)|\phi_i> +4n<\nabla_n{}^z\phi_i|\delta\sigma|\nabla_n{}^z\phi_i>=\\
&= -2(n+1)\lambda_i<\phi_i|\delta\sigma|\phi_i> +2n\lambda_i<\psi_i|\delta\sigma|\psi_i>,
\end{aligned}
\tag{3.134}
$$

where $|\psi_i>=\sqrt{\frac{2}{\lambda_i}}\nabla_n{}^z\phi_i$ belongs to τ^{n+1} and is a normalized eigenvector of $\Delta_{n+1}{}^-(\sigma)$ ($\Delta_{n+1}{}^-(\sigma)|\psi_i>=\lambda_i|\psi_i>$, which means $|\psi_i>\in(Ker\Delta_{n+1}{}^-)^\perp$).

We define $\ln\det'\Delta_n{}^+(\sigma)$ by making use of the generalized zeta-function. It is clear by now that in order to compute the variation $\delta\ln\det'\Delta_n{}^+(\sigma)$ we have to repeat, with slight modifications, the procedure explained at the end of the paragraph **1.3**. We get the final result

$$\begin{aligned}\delta \ln \det{}' {\Delta_n}^{+}(\sigma) &= -\frac{6n^2+6n+1}{6\pi}\int d^2x\sqrt{g(x)}R(x)\delta\sigma(x)+ \\ &+2(n+1)tr(\delta\sigma\pi Ker{\nabla_n}^{z}) - 2n\cdot tr(\delta\sigma\pi Ker{\nabla_n}^{z\dagger}). \qquad (3.135)\end{aligned}$$

The above expression is a differential equation which can be integrated. In order to do so we have to express the quantities $g(x)$, $R(x)$ in terms of the reference metric $\hat{g}$. We then have $g = \hat{g}e^{2\sigma}$.

The first term in the r.h.s. can be easily integrated, while the terms containing a trace can be re-expressed as

$$\begin{aligned}(2n+1)tr(\delta\sigma\pi Ker{\Delta_n}^{+}) &= \delta \ln H({\Delta_n}^{+}) \\ -2n\cdot tr(\delta\sigma\pi Ker{\Delta_{n+1}}^{-}) &= \delta \ln H({\Delta_{n+1}}^{-}). \qquad (3.136)\end{aligned}$$

Here $H({\Delta_n}^{+})$ is a finite-dimensional matrix, defined as

$$H_{rs} \quad =_{def} < {\phi_r}^0|{\phi_s}^0 >,$$

where ${\phi_r}^0$, ${\phi_s}^0$ span a basis for $Ker({\Delta_n}^{+})$ and are taken to be independent of the conformal factor. A similar position allows to introduce $H({\Delta_{n+1}}^{-})$, defined in terms of $Ker({\Delta_{n+1}}^{-})$.

At the end we get

$$\begin{aligned}\ln \det{}' {\Delta_n}^{+}(\sigma) &= -\frac{6n^2+6n+1}{12\pi}\int_{\mathcal{M}} d^2x\sqrt{\hat{g}(x)}\left(\hat{g}^{ab}\partial_a\sigma\partial_b\sigma + \hat{R}\sigma\right)+ \\ &+\ln\det H({\Delta_n}^{+} + \ln\det H({\Delta_{n+1}}^{-}) + F(\hat{g}), \qquad (3.137)\end{aligned}$$

where $F(\hat{g})$ is a term which is independent of the conformal factor.

We are now ready to compute the conformal anomaly. A careful analysis conducted in [25, 11] and taking into account the contribution of the conformal Killing vectors and of the Teichmüller deformations shows that the correct Fadeev-Popov determinant is given by

$$J = \left(\frac{\det{}'{\Delta_1}^{+}}{\det H({\Delta_n}^{+})\cdot \det H({\Delta_{n+1}}^{-})}\right)^{\frac{1}{2}}. \qquad (3.138)$$

The contribution of the jacobian to the conformal anomaly is therefore contained in the term

$$-\frac{13}{12\pi}\int_{\mathcal{M}} d^2x\sqrt{\hat{g}}\left(\hat{g}^{ab}\partial_a\sigma\partial_b\sigma + \hat{R}\sigma\right). \qquad (3.139)$$

The last thing which has still to be computed is the contribution $(\det{}'\Delta_0)^{-\frac{d}{2}}\int_{0-modes}\mathcal{D}{X_0}^{\mu}$. For what concerns $\det{}'\Delta_0$, it is obtained from (3.135) by setting $n=0$ and an overall factor $\frac{1}{2}$ which is due to the fact that Δ_0 acts on real fields. We therefore obtain

$$\begin{aligned}\delta \ln \det{}'{\Delta_0}^{+} &= -\frac{2}{12\pi}\int\sqrt{g}R\delta\sigma + 2tr(\delta\sigma\pi Ker{\Delta_0}^{+}), \\ \delta \ln \det{}'\Delta_0 &= -\frac{1}{12\pi}\int\sqrt{g}R\delta\sigma + 2tr(\delta\sigma\pi Ker\Delta_0). \qquad (3.140)\end{aligned}$$

The second term in the r.h.s. which depends on $Ker\Delta_0$ is compensated by the variation $\delta\int_{0-modes}\mathcal{D}X_0{}^\mu$ of the integration over the zero-modes of the laplacian as the following analysis prove. Let us expand a function $\varphi\in\tau^0$ in an orthonormal basis as done in paragraph **1.1**. We get at $\sigma(x)$ and respectively $\sigma(x)'=\sigma(x)+\delta\sigma(x)$,

$$\begin{aligned}\varphi=\sum_n c_n|\phi_n>_\sigma, &\qquad <\phi_n|\phi_m>_\sigma=\delta_{nm},\\ \varphi=\sum_n c_n{}'|\phi_n{}'>_{\sigma'}, &\qquad <\phi_n{}'|\phi_m{}'>_{\sigma'}=\delta_{nm}.\end{aligned}\tag{3.141}$$

We have to take into account only the zero-modes. The eigenvectors which correspond to a zero eigenvalue for the laplacian operator $\Delta_0{}^+$ are independent of $\sigma(x)$. Therefore, if $|\phi_0>$, $|\phi_0{}'>$ are zero-modes at σ and σ' respectively, it then follows

$$|\phi_0{}'> \;=\; (1+\delta\lambda)|\phi_0>,\tag{3.142}$$

where $\delta\lambda$ is a constant.

Since

$$\int d^2x\sqrt{\hat{g}}(1+2\delta\sigma(x))(1+\delta\lambda)\phi_0{}^*(1+\delta\lambda)\phi_0=\int d^2x\sqrt{\hat{g}}\phi_0{}^*\phi_0=1,\tag{3.143}$$

we have

$$\delta\lambda \;=\; -\frac{\int d^2x\sqrt{\hat{g}}\delta\sigma\phi_0{}^*\phi_0}{\int d^2x\sqrt{\hat{g}}\phi_0{}^*\phi_0}.\tag{3.144}$$

We obtain $c_0{}'=c_0(1-\delta\lambda)$ and therefore

$$\int_{0-modes\quad at\quad\sigma'}-\int_{0-modes\quad at\quad\sigma} \;=\; \int dc_0{}'-\int dc_0=-\delta\lambda\int dc_0.\tag{3.145}$$

$\int dc_0$ is a constant factor appearing in the normalization of the partition function. The variation inside the partition function is then precisely given by $tr(\delta\sigma Ker\Delta_0{}^+)$. Since in our case the laplacian operator Δ_0 acts on d fields, we get that at the end

$$\int_{0-modes\quad at\quad\sigma'}-\int_{0-modes\quad at\quad\sigma} \;=\; d\cdot tr(\delta\sigma\pi Ker\Delta_0),\tag{3.146}$$

which precisely cancels the corresponding term appearing in $(\det{}'\Delta_0)^{-\frac{d}{2}}$.

The final result is that in $d=26$ dimensions the contribution of the laplacian operator precisely cancels the contribution of the Fadeev-Popov determinant and the partition function in this critical dimension is conformally invariant.

In dimensions $d\neq 26$, different from the critical one, we get that the partition function $\mathcal{Z}$ contains a term which is the Liouville model, given by

$$\int\mathcal{D}\sigma e^{\frac{d-26}{24\pi}\int d^2x\sqrt{\hat{g}}\left(\hat{g}^{ab}\partial_a\sigma\partial_b\sigma+\hat{R}\sigma+Ae^\sigma\right)}.\tag{3.147}$$

The term proportional to the constant A is present in the regularized theory, in order to have the most general action invariant under diffeomorphisms (it should be recalled the discussion at the end of paragraph **3.1**).

Appendix 1.

Real and complex notation for $2D$, oriented, manifolds.

Every oriented 2-dimensional real manifold is a complex manifold of complex dimension 1. If such a manifols is also connected, it is then called a *Riemann surface*.

Let x_1, x_2 be the real coordinates which, in a given chart, specify a point of our surface. The connection between real and complex notation is given by the following relations

$$\begin{aligned}
&z = x_1 + ix_2, \qquad \overline{z} = x_1 - ix_2;\\
&\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}), \qquad \partial_{\overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2});\\
&4\partial_z\partial_{\overline{z}} = {\partial_{x_1}}^2 + {\partial_{x_2}}^2 = \Delta_0.
\end{aligned}$$

The condition $\partial_{\overline{z}} f = 0$ (or respectively $\partial_z f = 0$) is the Cauchy-Riemann equation which implies the analiticity (antianaliticity) of f.

A basic feature of the 2 dimensions can be stated as follows; for any given riemannian metric $g_{ab}(x_1, x_2)$ it is always possible to find a reparametrization $x_a \mapsto (x')_a = x'(x_b)$ (a change of chart) which at least locally, i.e. in a given chart, makes the metric conformally euclidean. We can therefore make use of the reparametrization invariance to put ourselves locally in the so-called “conformal coordinate system”, with the metric given by

$$ds^2 \;\; = \;\; e^{2\sigma(x_1,x_2)}({dx_1}^2 + {dx_2}^2) = e^{2\sigma} dz d\overline{z}.$$

In the new system we can write the metric ds^2 in terms of complex coordinates as

$$ds^2 \;\; = \;\; g_{z\overline{z}} dz d\overline{z} + g_{\overline{z}z} d\overline{z} dz,$$

with metric tensor components given by

$$\begin{aligned}
g_{zz} &= g_{\overline{z}\overline{z}} = 0,\\
g_{z\overline{z}} &= g_{\overline{z}z} = \frac{1}{2}e^{2\sigma}.
\end{aligned}$$

We remark that an analytic change of coordinates $z \mapsto z' = f(z)$, with $\partial_{\overline{z}} f = 0$, corresponds to the most general transformation of coordinates preserving the conformal nature of our local coordinate system.Under such an analytic change of coordinates the metric component $g_{z\overline{z}}$ transforms as a tensor, so that $g_{z\overline{z}} dz d\overline{z}$ is a scalar.

In general an n-rank tensor field T (for n integer) is a function of z, $\overline{z}$ which under the analytic reparametrization $z \mapsto z' = z'(z)$ transforms as follows

$$T \;\; \mapsto \;\; T' = \left(\frac{\partial z'}{\partial z}\right)^n T.$$

If $n \geq 0$ such a tensor will be denoted as $T^{z\ldots z}$, with n upper indices, while if $n < 0$ it will be denoted with $-n$ lower indices as $T_{z\ldots z}$.

It is just sufficient to consider tensors having only z-type of indices since the metric $g_{z\overline{z}}$ can be used to trade a $\overline{z}$ index for a z index.

The space of n-rank tensor fields $T^{z\cdots z}$ will be denoted as τ^n.

τ^n has the structure of a Hilbert space with an inner product specified, for $S, T \in \tau^n$, by

$$< S|T > \;\; =_{def} \;\; \int d^2z\sqrt{g}(g_{z\overline{z}})^n S^* T$$

(if S is an n-rank tensor in z, S^* is an n-rank tensor in $\overline{z}$).

Such a definition makes our inner product an invariant quantity under reparametrization.

The conventions used in the previous formula are specified by the following relations

$$\begin{aligned}
&\sqrt{\det g_{ab}(x_1, x_2)} = \sqrt{g(x_1, x_2)} = e^{2\sigma}, \\
&\sqrt{\det g(z, \overline{z})} = \sqrt{g(z, \overline{z})} = g_{z\overline{z}} = \frac{1}{2}e^{2\sigma}, \\
&g^{z\overline{z}} = (g_{z\overline{z}})^{-1} = 2e^{-2\sigma}, \\
&d^2z = 2dx_1 dx_2.
\end{aligned}$$

Appendix 2.

Covariant derivatives.

In appendix **1** we have introduced the space τ^n of n-rank tensor fields. Here we will introduce the covariant derivatives which transform a tensor field into a tensor field.

It can be easily checked that the operator ${\nabla_n}^z$, expressed in local coordinates by

$$ {\nabla_n}^z \;=\; g^{z\overline{z}}\partial_{\overline{z}}, $$

has the property of mapping n-rank tensors into $(n+1)$-rank tensors

$$ {\nabla_n}^z \quad : \tau^n \to \tau^{n+1}, $$

which makes it a "raising operator".

Conversely, the operator ${\nabla_z}^n$, expressed by

$$ {\nabla_z}^n = (g^{z\overline{z}})^n\partial_z(g_{z\overline{z}})^n = \partial_z + 2n\partial_z\sigma $$

in the local coordinate system, satisfies

$$ {\nabla_z}^n \quad : \quad \tau^n \to \tau^{n-1}; $$

therefore ${\nabla_z}^n$ is a "lowering operator".

It is easy to verify that, with the scalar product introduced in the previous appendix, the raising and lowering operators are mutually adjoint

$$ ({\nabla_n}^z)^\dagger \;=\; -{\nabla_z}^{n+1}. $$

With our raising and lowering operators we can build two different kinds of self-adjoint elliptic operators, denoted as ${\Delta_n}^\pm$, such that

$$ {\Delta_n}^\pm \quad : \quad \tau^n \to \tau^n. $$

They are defined, respectively, by

$$ \begin{aligned} {\Delta_n}^+ &=_{def} \; -2{\nabla_z}^{n+1}\cdot{\nabla_n}^z, \\ {\Delta_n}^- &=_{def} \; -2{\nabla_{n-1}}^z\cdot{\nabla_z}^n. \end{aligned} $$

In the conformal coordinate system they are given by

$$ \begin{aligned} {\Delta_n}^+ &=_{def} \; -4e^{-2\sigma}[\partial_z\partial_{\overline{z}} + 2n(\partial_z\sigma)\partial_{\overline{z}}], \\ {\Delta_n}^- &=_{def} \; -4e^{-2\sigma}[\partial_z\partial_{\overline{z}} + 2n(\partial_z\sigma)\partial_{\overline{z}} + 2n(\partial_z\partial_{\overline{z}}\sigma)]. \end{aligned} $$

In real coordinates we have

$$ \begin{aligned} {\Delta_n}^+ &=_{def} \; -e^{-2\sigma}[\Delta_0 + 2n((\partial_{x_1} - i\partial_{x_2})\sigma)(\partial_{x_1} + i\partial_{x_2})], \\ {\Delta_n}^- &=_{def} \; -e^{-2\sigma}[\Delta_0 + 2n((\partial_{x_1} - i\partial_{x_2})\sigma)(\partial_{x_1} + i\partial_{x_2}) + 2n\Delta\sigma]. \end{aligned} $$

${\Delta_n}^{\pm}$ are called generalized laplacians because in the limit of flat metric they coincide with the usual flat laplacian (this explains the choice of 2 as normalizing factor in their definition).

We finally introduce the curvature R, which is expressed by the following relation

$$\nabla_z{}^{n+1}\nabla_n{}^z - \nabla_{n-1}{}^z\nabla_z{}^n \;=\; \frac{n}{2}R.$$

Despite the form of the left hand side, R is a function and not an operator.

R turns out to be a scalar object which is expressed, in the local coordinate system, by

$$R \;=\; -2e^{-2\sigma}\Delta_0\sigma.$$

The Gauß-Bonnet theorem ensures us that the quantity $\frac{1}{4\pi}\int_{\mathcal{M}} d^2x\sqrt{g}R$ is a topological invariant quantity which precisely equals the Euler characteristic $\chi(\mathcal{M})$ of the manifold $\mathcal{M}$

$$\chi(\mathcal{M}) \;=\; \frac{1}{4\pi}\int_{\mathcal{M}} d^2x\sqrt{g}R.$$

We recall here that a compact, boundaryless, orientable, real 2-dimensional manifold is topologically equivalent to a sphere with some handles. For such a manifold $\chi(\mathcal{M})$ is given by

$$\chi(\mathcal{M}) \;=\; 2-2h,$$

where h is the number of handles.

A standard way of introducing (and computing) the Euler characteristic $\chi(\mathcal{M})$ is through the formula

$$\chi(\mathcal{M}) \;=\; V-E+F,$$

where V, E, and F denote respectively the number of vertices, edges and faces in any given triangularization of the manifold $\mathcal{M}$ (this number being independent of the chosen triangularization).

Appendix 3.

Cancellation of the conformal anomaly for the bosonic string.

In this appendix we present the final computation which shows the cancellation of the conformal anomaly in the bosonic string.

Notations and explanations have been furnished in chapter **3**.

The operators that we are interested to compute are given by Δ_0 and ${\Delta_1}^+$. Indeed for a d-dimensional bosonic string the conformal invariance is guaranteed whenever the expression

$$\left(\ln \det {\Delta_1}^+\right)^{\frac{1}{2}} + (\ln \det \Delta_0)^{-\frac{d}{2}}$$

is conformally invariant.

The computation for Δ_0 can be recovered from the computation of ${\Delta_0}^+$ since we have

$$\ln \det \Delta_0 \;\; = \;\; \frac{1}{2} \ln \det {\Delta_0}^+.$$

The variations (see (3.137)) of ${\Delta_0}^+$, ${\Delta_1}^+$ are given by

$$\begin{aligned} \delta \ln \det {\Delta_0}^+ &= -\frac{2}{12\pi} \int \sqrt{g} R \delta\sigma, \\ \delta \ln \det {\Delta_1}^+ &= -\frac{26}{12\pi} \int \sqrt{g} R \delta\sigma. \end{aligned}$$

Therefore the cancellation of the conformal anomaly requires the vanishing of the following formula

$$\frac{1}{2}\left(-\frac{26}{12\pi}\right) - \frac{d}{2}\left(\frac{1}{2} \times \frac{-2}{12\pi}\right),$$

which implies the critical dimensionality

$$d \;\; = \;\; 26.$$

Appendix 4.

Basic properties of the Riemann's zeta-function.

In this appendix the basic features and the mathematical importance of the Riemann's zeta-function to the number theory will be briefly reviewed. The still open problem known as Riemann's hypothesis will be stated.

A nice discussion concerning the relevance of zeta-function to physics can be found e.g. in the Gutzwiller's book [26]. This appendix is limited to state the main properties of zeta-function, following as main references [27] and [28], without bothering with actual proofs. Some of the statements made below (for instance its connection with heat kernel and the Gamma function, the computation of its value at specific points) however, have already been proven in Chapter **1** and **2** of these Lecture Notes.

The Riemann's zeta-function can be introduced in two equivalent ways, either as an infinite sum over positive integers

$$\zeta(s) \quad =_{def} \quad \sum_{n=1}^{\infty} n^{-s},$$

or as an infinite product over the prime numbers

$$\zeta(s) \quad =_{def} \quad \prod_p (1-p^{-s})^{-1} \quad (p \quad primes).$$

The equivalence of the two above definitions can be easily checked to be a consequence of the unique factorization of any integer number in its prime factors.

The latter definition is also known as Euler's product for Riemann's zeta-function, quite a peculiar name since Euler introduced it in 1748, one century before Riemann.

The Riemann's zeta function is absolutely convergent for $Re \quad s > 1$, (such a convergence can be easily understood by comparing the infinite sum with the continuous integral $\int_1^\infty dx \cdot x^{-s}$).

The function $\zeta(s)$ extends by analytic continuation to a meromorphic function in the complex plane $\mathbf{C}$, where the only singularity is a pole of order 1 at $s=1$, with residue equal to 1 (that is $\zeta(s) - \frac{1}{s-1}$ is an entire function). Based on its product decomposition Euler proved the existence of the pole at $s=1$ and as a corollary he obtained an independent proof of the existence of infinite prime numbers.

The constant γ, introduced through the limit

$$\gamma \quad = \quad \lim_{N\to\infty}\left(\frac{1}{1}+\frac{1}{2}+...+\frac{1}{N}-\int_1^N dx\cdot\frac{1}{x}\right)$$

is known as Euler's constant.

Already at this stage the Riemann's zeta-function appears related to fundamental properties in prime numbers theory. The connection however is even more explicit. The prime number distribution function, denoted as $\pi(x)$, is according to a conjecture first stated by Gauß and Legendre and later proved by Hadamard and de La Vallée Poussin at the end of the nineteenth century, well approximated by the function $Li(x) = \int_0^x dt\cdot\frac{1}{\log t}$. In the mid-nineteenth century Tchebychev introduced the function

$$\Pi(x) \quad = \quad \pi(x) + \frac{1}{2}\pi(\sqrt{x}) + \frac{1}{3}\pi(\sqrt[3]{x}) + ...$$

and from the Euler's product deduced its integral representation

$$\frac{1}{s}\log\zeta(s) = \int_1^\infty \Pi(x)x^{-s-1}dx.$$

In a memoir published in 1859 Riemann obtained an analytic formula for the number of primes up to a pre-assigned limit. Such a formula is expressed in terms of the zeros of the zeta-function, namely the solutions $\rho \in \mathbf{C}$ of the equation $\zeta(\rho) = 0$.

The zeta-function has zeros (referred to as "trivial zeros") at the negative even integers $-2, -4, \ldots$. Another class of zeros (the "non-trivial ones") are encountered for complex numbers whose real part is equal to $\frac{1}{2}$. The Riemann's hypothesis is the conjecture that *all* non-trivial zeros of the Riemann's zeta-function have real part equal to $\frac{1}{2}$. According to many mathematicians [28], the Riemann's Hypothesis is considered the most important open problem in pure mathematics. Its validity is equivalent to saying that the deviation of the number of primes from $Li(x)$ is

$$\pi(x) = Li(x) + O(\sqrt{x}\log x),$$

so that its failure would create havoc in the distribution of prime numbers. For a more complete account on that see e.g. [28].

Let us now collect some useful formulas concerning the connection of the zeta-function with the Gamma-function, as well as functional equations satisfied by the zeta-function. Such formulas can be used to compute the values of $\zeta(s)$ at some specific points s as later reported.

The Gamma function $\Gamma(s)$ can be introduced through the Mellin integral

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx.$$

It is a meromorphic function on $\mathbf{C}$ with simple poles at $s = 0, -1, -2, -3, \ldots$ and residue $\frac{(-1)^k}{k}$ at $s = -k$.

Since it satisfies the relations

$$\begin{aligned}\Gamma(1) &= 1,\\ \Gamma(s+1) &= s\Gamma(s),\end{aligned}$$

it can be considered the analytic continuation of the factorial.

It satisfies the following functional equations, known respectively as "complement formula"

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s},$$

and "duplication formula"

$$\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}) = \pi^{\frac{1}{2}}2^{1-s}\Gamma(s).$$

The connection between the Riemann's zeta-function and the Gamma function is guaranteed by the following equation

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty}\frac{x^{s-1}}{e^x-1}\cdot dx.$$

It turns out that $\zeta(s)$ satisfies the functional equation

$$\zeta(1-s) \quad = \quad 2^{1-s}\pi^{-s}\Gamma(s)\cos\frac{\pi s}{2}\zeta(s).$$

The above relations allows us to compute the values of the zeta-function for specific integer values of s. We have indeed, apart the pole at $s=1$,

$$\lim_{s\to 1}\zeta(s) \quad = \quad \frac{1}{s-1}+\gamma+O(s),$$

that

$$\zeta(-2)=\zeta(-4)=\zeta(-6)=...=0,$$

while

$$\zeta(2)=\frac{\pi^2}{6}, \quad \zeta(4)=\frac{\pi^4}{90}, \quad \zeta(6)=\frac{\pi^6}{945}, \quad \zeta(8)=\frac{\pi^8}{9450}, \quad \cdots \quad ,$$

$$\zeta(-1)=-\frac{1}{12}, \quad \zeta(-3)=\frac{1}{120}, \quad \zeta(-5)=-\frac{1}{252}, \quad ...$$

and

$$\zeta(0)=-\frac{1}{2}, \quad \frac{d\zeta(s)}{ds}\mid_{s=0}=-\frac{1}{2}\log 2\pi.$$

As explained in Chapter **1**, specific values of the zeta-function can be used to regularize otherwise infinite expressions. This procedure has been vastly used in the context of String Theory (see e.g. [2]).

To give some examples the value of $\zeta(s)$ at $s=-1$ can be used to make sense of an expression like the infinite sum of positive integers and we get

$$1+2+3+...+n+... \quad = \quad -\frac{1}{12}.$$

On the other hand the infinite sum

$$\frac{d\zeta(s)}{ds} \quad = \quad -\sum_n(\log n)n^{-s},$$

formally evaluated at $s=0$, can be used to regularize the following infinite sum

$$\sum_n \log n \quad = \quad \frac{1}{2}\log(2\pi).$$

By formally exponentiating both sides of the above equation we can even regularize the factorial of infinite to be

$$\infty! \quad = \quad \sqrt{2\pi}.$$

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