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Physical and Geometrical Classification in General Relativity*

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1 Introduction

The object of these notes is to discuss briefly some classification procedures in general relativity theory. The geometrical objects to be classified here are: (i) bivectors, (ii) the Weyl tensor, (iii) symmetric second order tensors and (iv) the space-time connection. Topics (i) and (ii) will be mentioned only briefly since they are well-known. Topic (iii) will be dealt with in more detail since it has a number of useful applications. Topic (iv) which will be described in terms of the holonomy group of space-time is, mathematically, little more than classifying the subgroups of the Lorentz group and is perhaps of less interest. However in a number of problems recently it has proved useful and deserves some consideration.

Throughout these notes a standard notation is used, (M, g) denotes our space-time with Lorentz metric g of signature $(-+++)$, round and square brackets denotes the usual symmetrisation and skew-symmetrisation of indices, a semicolon denotes a covariant derivative and a comma a partial derivative. The Riemann, Ricci, Weyl and energy-momentum tensors are denoted by $R^a{}_{bcd}$, $R_{ab}(\equiv R^c{}_{acb})$, C_{abcd} and T_{ab} and the Ricci scalar by
$R = R_{ab}g^{ab}$.

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2 The Classification of General Second Order Tensors

Let $p \in M$ and let S_{ab} be a second order covariant tensor at p. The fact that S is a tensor means that it makes sense to ask for the eigenvectors and eigenvalues of S, that is, to ask for a vector k at p and a real number ρ such that

$$S_{ab}k^b = \rho k_a (= \rho g_{ab} k^b) \tag{1}$$

Note that it makes sense because the answer would be independent of the coordinate system used in (1). One should also note that in general it matters over which index of S the contraction in (1) is performed. Here the second index is chosen. Actually since we will only consider the cases when S is either symmetric or skew-symmetric this will hardly matter in practice, there just being a sign change if S is skew.

The problem in solving (1), that is, finding the eigenvectors and corresponding eigenvalues, is that since g is of Lorentz signature the problem is not posed in the usual form dealt with in the standard algebra texts. One can rectify this by simply rewriting (1) in the form

$$S^a{}_b k^b = \rho k^a (= \rho \delta^a_b k^b) \tag{2}$$

The δ^a_b rather than the g_{ab} on the R.H.S. of (2) casts the problem into the standard form. However, we pay a price for this because the tensor $S^a{}_b$ will have no sensible symmetric or skew-symmetric property even if the original S_{ab} did. Further, the eigenvalues ρ in (1) or (2) may be real or complex. Thus the possibilities for solving (1) are to either (i) work with (2) and suffer the attendant problems but where at least the problem is in standard form or (ii) to work with (1) and try to develop a new technique in order to determine the algebraic structure or (iii) find an alternative representation of S for which classification techniques exist. We will explore these possibilities in these notes.

3 Preliminary Remarks

At $p \in M$ we can introduce two types of tetrad bases for the *tangent space* T_pM to M at p. A *(pseudo-) orthonormal tetrad* $(t,\, x,\, y,\, z)$ satisfies $-t^a t_a = x^a x_a = y^a y_a = z^a z_a = 1$ and a *real null tetrad* $(\ell,\, n,\, x,\, y)$ satisfies $\ell^a \ell_a = n^a n_a = 0 \quad \ell^a n_a = x^a x_a = y^a y_a = 1$ with all other inner products of tetrad members zero in both cases. We can also introduce a *complex null tetrad* $(\ell,\, n,\, m,\, \bar{m})$ at p where, ℓ and n are as in the real null tetrad case and m and (its conjugate) $\bar{m}$ are defined from the real null tetrad by $\sqrt{2}m = x + iy$, $\sqrt{2}\bar{m} = x - iy$. Thus m and $\bar{m}$ are complex null vectors and the only non vanishing inner products amongst $\ell,\ n,\ m,\ \bar{m}$ are $\ell^a n_a = 1 \ m^a \bar{m}_a = 1$. The following completeness relations then hold at p

$$g_{ab} = 2\ell_{(a}n_{b)} + x_a x_b + y_a y_b = 2\ell_{(a}n_{b)} + 2m_{(a}\bar{m}_{b)} = -t_a t_b + x_a x_b + y_a y_b + z_a z_b \tag{3}$$

It will be convenient to know the set of changes of one real null tetrad (ℓ, n, x, y) to another such tetrad (ℓ', n', x', y') for which the *direction* of ℓ is preserved, that is, $\ell' = A\ell$ $(A \in \mathcal{R}, A > 0)$. This condition on ℓ together with the orthogonality relations on (ℓ', n', x', y') for it also to be a real null tetrad means that this set of tetrad transformations (which is just a particular subgroup of the Lorentz group and will be set into proper context later) is given by

$$\begin{aligned} \ell' &= A\ell \\ x' &= \alpha x - \beta y - A\sqrt{2}(\alpha\gamma + \beta\delta)\ell \\ y' &= \beta x + \alpha y + A\sqrt{2}(\alpha\delta - \beta\gamma)\ell \\ n' &= A^{-1}n + \sqrt{2}(\gamma x - \delta y) - A(\gamma^2 + \delta^2)\ell \end{aligned} \qquad (4)$$

where $A, \alpha, \beta, \gamma, \delta \in \mathcal{R}$ with $A > 0$ and $\alpha^2 + \beta^2 = 1$. Thus there are 4 free parameters in (4). These transformations can be more simply expressed in terms of the corresponding complex null tetrad change $(\ell, n, m, \bar{m}) \to (\ell', n', m', \bar{m}')$ again with the direction of ℓ preserved, $\ell' = A\ell$ $(A \in \mathcal{R}, A > 0)$

$$\begin{aligned} \ell' &= A\ell \\ m' &= e^{i\theta}(m - A\bar{B}\ell) \\ n' &= A^{-1}n + Bm + \bar{B}\bar{m} - AB\bar{B}\ell \end{aligned} \qquad (5)$$

where $A, \theta \in \mathcal{R}, A > 0, B \in \mathcal{C}$ – again 4 free parameters. In what follows, these tetrad changes will be used to replace a real (or complex) null tetrad with another in order to simplify the expression for a certain tensor expressed in such a tetrad.

At p the set of second order symmetric tensors is a 10-dimensional vector space which can be spanned by the 10 basis symmetric tensors (expressed in terms of a real null tetrad (ℓ, n, x, y) at p) given by

$$\begin{aligned} &\ell_a\ell_b, \quad n_a n_b, \quad 2\ell_{(a}n_{b)}, \quad x_a x_b, \quad y_a y_b, \quad 2x_{(a}y_{b)}, \\ &2\ell_{(a}x_{b)}, \quad 2\ell_{(a}y_{b)}, \quad 2n_{(a}x_{b)}, \quad 2n_{(a}y_{b)}. \end{aligned} \qquad (6)$$

Similarly, at p, the set of bivectors at p is a 6-dimensional vector space which can be spanned by the 6 basis bivectors (expressed in terms of a real null tetrad at p) given by

$$2\ell_{[a}n_{b]}, \quad 2x_{[a}y_{b]}, \quad 2\ell_{[a}x_{b]}, \quad 2\ell_{[a}y_{b]}, \quad 2n_{[a}x_{b]}, \quad 2n_{[a}y_{b]}. \qquad (7)$$

Thus any second order symmetric tensor at p can be written as a linear combination of the basis members (6) and any bivector at p can be written as a linear combination of the members (7).

Now let F_{ab} be a (real) bivector at $p \in M$ and let $\overset{*}{F}_{ab}$ denote its dual. Then we can construct a *complex* bivector $\overset{+}{F}_{ab} = F_{ab} + i\,\overset{*}{F}_{ab}$ called the *complex self dual* of F_{ab}. It has the property that $(\overset{+}{F}_{ab})^* = -i\,\overset{+}{F}_{ab}$. Conversely any *complex* bivector P_{ab} at p satisfying the self dual condition $\overset{*}{P}_{ab}= -iP_{ab}$ can be written as $P_{ab} = F_{ab} + i\overset{*}{F}_{ab}$ for a *real* bivector F_{ab}. Now the set of all complex bivectors at p forms a 6-dimensional complex (or a 12-dimensional real) vector space. However the set of all complex self dual bivectors at p is a 3-dimensional complex (or a 6-dimensional real) vector space. Noting that the 1st and 2nd real bivectors in (7) form a dual pair as do the 3rd and 4th (with a minus sign inserted in front of the 4th) and the 5th and 6th we can define a basis for the 3-dimensional complex vector space of self dual complex bivectors by

$$V_{ab} = 2\ell_{[a}\bar{m}_{b]} \quad M_{ab} = 2\ell_{[a}n_{b]} + 2\bar{m}_{[a}m_{b]} \quad U_{ab} = 2n_{[a}m_{b]}. \tag{8}$$

that is, $\sqrt{2}V_{ab} = \ell_{[a}x_{b]} + i\ell_{[a}x_{b]}^{*} = \ell_{[a}x_{b]} - i\ell_{[a}y_{b]}$, etc. In (8) the basis bivectors V, M and U have been built from a complex null tetrad. When that complex null tetrad is changed under the scheme (5) the corresponding complex bivectors V, M and U change according to

$$\begin{aligned} V'_{ab} &= Ae^{-i\theta}V_{ab} \\ M'_{ab} &= 2A\bar{B}V_{ab} + M_{ab} \\ U'_{ab} &= A\bar{B}^2e^{i\theta}V_{ab} + \bar{B}e^{i\theta}M_{ab} + A^{-1}e^{i\theta}U_{ab} \end{aligned} \tag{9}$$

4 The Classification of 2-spaces in T_pM

The set of 2-dimensional subspaces (2-spaces) of T_pM can be classified according as they contain *no* null directions (in which case they are called *spacelike) exactly one* null direction (in which case they are called *null*) or *exactly two* null directions (in which case they are called *timelike*). It is easy to show that there are no other possibilities. For example, let (ℓ, n, x, y) be a real null tetrad at p and if u and v are any two independent vectors at p let (u, v) denote the 2-space that they span (ie determine) at p. Then (x, y) is spacelike, (ℓ, n) is timelike and (ℓ, x), (ℓ, y), (n, x) and (n, y) are null.

Now given a 2-space at p the set of vectors which are orthogonal to each member of this 2-space is also a 2-space called its *orthogonal complement*. Thus (x, y) and (ℓ, n) are orthogonal complements as are (ℓ, x) and (ℓ, y) and also (n, x) and (n, y). Note that the orthogonal complement of a spacelike 2-space is timelike and vice-versa and the orthogonal complement of a null 2-space is null (and they contain the same null direction).

It is then easy to show that every member of a *spacelike* 2-space is a spacelike vector and that it can be spanned by an orthogonal pair of spacelike vectors. Also a *null* 2-space contains a null vector, say ℓ, and all its multiples and all its other members are spacelike and orthogonal to ℓ. It can thus be spanned by vectors ℓ and x with $\ell^a\ell_a = \ell^a x_a = 0$, $x^a x_a = 1$. A *timelike* 2-space contains timelike, spacelike and null members and can be spanned by independent null vectors, say ℓ and n, normalised so that $\ell^a n_a = 1$.

5 The Classification of Bivectors

Let $p \in M$ and F_{ab} a bivector at p. Regarding F_{ab} as a skew matrix, its rank is therefore an even number 0,2 or 4. If it is 0 then $F_{ab} = 0$. *Suppose then that the rank of F_{ab} is* 2.

Then there exists a 2-space $U \subseteq T_pM$ such that if $k \in U$ then $F_{ab}k^b = 0$. If U is *spacelike* and spanned by orthogonal unit vectors x and y, one completes them to a null tetrad $(\ell,\ n,\ x,\ y)$ and expands F_{ab} at p in terms of the basis (7)

$$F_{ab} = 2\alpha\ell_{[a}n_{b]} + 2\beta x_{[a}y_{b]} + 2\gamma\ell_{[a}x_{b]} + 2\delta\ell_{[a}y_{b]} + 2\mu n_{[a}x_{b]} + 2\nu n_{[a}y_{b]}. \tag{10}$$

for $\alpha, \beta, \gamma, \delta, \mu, \nu \in \mathcal{R}$. The conditions $F_{ab}x^b = F_{ab}y^b = 0$ in (10) then show that $\beta = \gamma = \delta = \mu = \nu = 0$ and so

$$F_{ab} = 2\alpha\ell_{[a}n_{b]} \quad \begin{pmatrix} \alpha \in \mathcal{R} \\ \alpha \neq 0 \end{pmatrix} \tag{11}$$

Similarly if U is *timelike* and spanned by null vectors ℓ and n (scaled so that $\ell_a n^a = 1$) a similar argument shows that

$$F_{ab} = 2\beta x_{[a}y_{b]} \quad \begin{pmatrix} \beta \in \mathcal{R} \\ \beta \neq 0 \end{pmatrix} \tag{12}$$

and if U is *null* and spanned by a null vector ℓ and a unit spacelike vector y (so that $\ell^a y_a = 0$) then

$$F_{ab} = 2\gamma\ell_{[a}x_{b]} \quad \begin{pmatrix} \gamma \in \mathcal{R} \\ \gamma \neq 0 \end{pmatrix} \tag{13}$$

There are no other possibilities if rank $F_{ab} = 2$ and in each of the above rank 2 cases F can always be written in the form $F_{ab} = p_a q_b - q_a p_b$ and then F is called *simple.* Note that the vectors p and q are not uniquely determined by F. However, the 2-space spanned by the vectors p and q in any (simple) form for F is well defined and is called the *blade* of F. Simple bivectors are then classified as *spacelike, timelike* or *null* according as their blade is a spacelike, timelike or null 2-space. Further, it is easy to show that the dual $\overset{*}{F}$ of a simple bivector F is also simple and its blade is the orthogonal complement of the blade of F. Thus the dual of a spacelike (simple) bivector is timelike and vice versa and the dual of a null bivector is null. As an example note that the bivectors in (7) are all simple and are, respectively, timelike, spacelike, null, null, null and null.

If F is a *null* bivector then it takes a form like (13) and ℓ, which is the only null direction in its blade, is called the *principal null direction* of F. It follows in this case that $\overset{*}{F} \propto 2\ell_{[a}y_{b]}$ and ℓ is also the principal null direction of $\overset{*}{F}$. If F is *timelike* then it takes a form like (11) and then ℓ and n are called the *principal null directions of* F. If F is *spacelike* then although its blade contains no null directions, it uniquely determines its (timelike) orthogonal complement (ie the blade of its dual). The two principal null directions of its dual are then called the *principal null directions of* F. Thus if F takes a form like (12) its principal null directions are ℓ and n.

Now suppose the rank of F_{ab} is 4. First construct the corresponding self-dual (complex) bivector F_{ab} and expand in terms of the basis (8)

$$\overset{+}{F}_{ab} = \alpha V_{ab} + \beta M_{ab} + \gamma U_{ab} \quad \left(\alpha,\ \beta,\ \gamma\ \in\ \mathcal{C} \right) \tag{14}$$

The idea is to show that by changing the basis in (14) (i.e. by changing the null tetrad $(\ell, n, m, \bar{m})$ according to (5) and hence the basis (8) according to (9)) we can make α and γ equal to zero in (14). After making the change (5) and hence (9) we get

$$\begin{aligned}\overset{+}{F'}_{ab} &= \alpha V'_{ab} + \beta M'_{ab} + \gamma U'_{ab} \\ &= \alpha(Ae^{-i\theta}V_{ab}) + \beta(2A\bar{B}V_{ab} + M_{ab}) \\ &+ \gamma(A\bar{B}^2e^{i\theta}V_{ab} + \bar{B}e^{i\theta}M_{ab} + A^{-1}e^{i\theta}U_{ab})\end{aligned}$$

so

$$\begin{aligned}\overset{+}{F'}_{ab} &= (\alpha Ae^{-i\theta} + 2\beta A\bar{B} + \gamma A\bar{B}^2e^{i\theta})V_{ab} \\ &+ (\beta + \gamma\bar{B}e^{i\theta})M_{ab} + (\gamma A^{-1}e^{i\theta})U_{ab}\end{aligned} \tag{15}$$

Now α ,β and γ are given by (14) and we have the freedom to choose A, θ and B in (5) and (9) (and clearly one of α, β and γ must be non-zero otherwise we would have $F_{ab} = 0$). First, fix A and θ and choose B so that the coefficient of V_{ab} in (15) vanishes. In this new tetrad $\overset{+}{F'}_{ab}$ has no coefficient of V_{ab} (and by switching ℓ and n we can think of this as $\overset{+}{F'}_{ab}$ having no coefficient of U_{ab}). So we can choose the tetrad so that (dropping primes)

$$\overset{+}{F}_{ab} = \alpha V_{ab} + \beta M_{ab} \tag{16}$$

Now in (16) we must have $\beta \neq 0$ otherwise, by taking real parts we would get F_{ab} simple and hence of rank 2 – a contradiction. Using (16), we repeat the tetrad change (5) and (9) and (15) becomes

$$\overset{+}{F'}_{ab} = (\alpha Ae^{-i\theta} + 2\beta A\bar{B})V_{ab} + \beta M_{ab} \tag{17}$$

and we choose B so that $\alpha Ae^{-i\theta} + 2\beta A\bar{B} = 0$. Then (dropping primes again) $\overset{+}{F}_{ab} = \beta M_{ab}$ and taking real parts we recover F_{ab} in the form

$$F_{ab} = 2a\ell_{[a}n_{b]} + 2bx_{[a}y_{b]} \tag{18}$$

where $\beta = a - ib$, $a, b \in \mathcal{R}$. In other words, if F_{ab} has rank 4 it can be written (as in (18)) as the sum of two simple bivector one spacelike and one timelike (in fact they are duals). Then when F_{ab} has rank 4 (i.e. F_{ab} not simple) it is not hard to show that the two 2-spaces $(x,\ y)$ and $(\ell,\ n)$ represented by (18) are uniquely determined by F_{ab}.

This completes the classification of bivectors. It can be summarised as follows:

Case I	$F_{ab} = 0$.
Case II	F_{ab} *rank* 2 ($\Leftrightarrow F_{ab}$ simple) and either F_{ab} spacelike, timelike, null – canonical forms(11), (12), (13)
Case III	F_{ab} *rank* 4 (F_{ab} non-simple) – canonical form (18).

Remarks

(i) Since the rank of F_{ab} is even then the existence of one vector $k^a(\neq 0)$ such that $F_{ab}k^b = 0$ implies that F_{ab} is simple (if $F_{ab} \neq 0$).

(ii) If F_{ab} satisfies $F_{ab}k^b = \overset{*}{F}_{ab}\, k^b = 0$ then F_{ab} is necessarily null and k^a is the (unique up to scaling) principle null direction of F_{ab} (and $\overset{*}{F}_{ab}$).

(iii) Any bivector (simple or non-simple) which is not null is usually called *non-null.*

The eigenvector structure of these bivector types is easily described in terms of a real null tetrad $(\ell,\ n,\ x,\ y)$.

1. F_{ab} **spacelike**

Canonical form $F_{ab} = x_a y_b - y_a x_b$

The eigenvectors are
ℓ^a (eigenvalue 0)
n^a (eigenvalue 0)
$x^a + iy^a$ (eigenvalue i)
$x^a - iy^a$ (eigenvalue $-i$)

2. F_{ab} **timelike**

Canonical form $F_{ab} = \ell_a n_b - n_a \ell_b$

The eigenvectors are
ℓ^a (eigenvalue 1)
n^a (eigenvalue -1)
x^a (eigenvalue 0)
y^a (eigenvalue 0)

3. F_{ab} **null**

Canonical form $F_{ab} = \ell_a x_b - x_a \ell_b$

The eigenvectors are
ℓ^a (eigenvalue 0)
y^a (eigenvalue 0)

4. F_{ab} **non-simple**

Canonical form $F_{ab} = 2a\ell_{[a}n_{b]} - 2bx_{[a}y_{b]}$

The eigenvector are
ℓ^a (eigenvalue a)
n^a (eigenvalue $-a$)
$x^a + iy^a$ (eigenvalue bi)
$x^a - iy^a$ (eigenvalue $-bi$)

The Segre types here are:

F_{ab} spacelike — $\{(11)z\bar{z}\}$
F_{ab} timelike — $\{11(11)\}$
F_{ab} null — $\{(31)\}$
F_{ab} non-simple — $\{11\ z\bar{z}\}$

6 The Petrov Classification of the Weyl Tensor

The Weyl tensor C_{abcd}, because of its symmetries, can be represented as a 6×6 symmetric matrix C_{AB} according to the block index scheme $1 \leftrightarrow 23$, $2 \leftrightarrow 31$, $3 \leftrightarrow 12$, $4 \leftrightarrow 10$, $5 \leftrightarrow 20$, $6 \leftrightarrow 30$. One can then classify the Weyl tensor by looking for *eigenbivectors* of C_{abcd}, that is, solutions for F_{ab} and λ of the equation

$$C_{abcd}F^{cd} = \lambda F_{ab} \tag{19}$$

which is like finding eigen-6 vectors of C_{AB}, that is, solutions for F^A and μ of the equation $C_{AB}F^B = \mu F_A$ (where block indices are raised and lowered with the "bivector metric" $g_{AB} \leftrightarrow G_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}$). The resulting possible Segre types are just the Petrov canonical types. However a simple approach is made possible by the Weyl symmetry $C^c{}_{abc} = 0$ This means that we can write C_{AB} as

$$C_{AB} = \begin{pmatrix} A & B \\ B^T & -A \end{pmatrix} \begin{pmatrix} A & \text{symmetric} \\ A & \textit{and } B \text{ tracefree} \end{pmatrix}$$

and so we can describe C by the 3×3 real matrices A and B or by the complex 3×3 matrix $D = A + iB$. There are three possible Jordan forms for the matrix D over $\mathcal{C}$, namely

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \quad \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -2\alpha \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Segre $\{111\}$	Segre $\{21\}$	Segre $\{3\}$
Petrov Type I	Petrov Type II	Petrov Type III

where the tracefree condition on A and B (and hence on D) is used. The Segre and Petrov types are indicated. The Petrov type I has a subcase type D (where $\alpha = \beta$) and the type II has a subcase type N (where $\alpha = 0$). They are

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -2\alpha \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Segre $\{11)1\}$	(Segre $\{21\}$)
Petrov Type D	Petrov Type N

A particularly important role is played by null eigenbivectors and by their principal null directions. In fact they lead to an alternative and very useful version of the Petrov classification. This classification is essentially the work of L. Bel and is referred to as the Bel criteria. First define the complex self dual Weyl tensor $\overset{+}{C}_{abcd}$ by

$$\overset{+}{C}_{abcd} = C_{abcd} + i \overset{*}{C}_{abcd} \tag{20}$$

Here $\overset{*}{C}_{abcd}$ is the dual of C_{abcd} (and it does not matter whether the right or left dual is taken since, for the Weyl tensor, they are equal). If $C_{abcd} \neq 0$ then the Bel criteria are as follows

(i) C_{abcd} is (Petrov) type $N \Leftrightarrow \overset{+}{C}_{abcd}\, k^d = 0$ for some vector k. The vector k is necessarily null and unique up to a scaling and is called the repeated principal null direction of C_{abcd}.

(ii) Given that condition (i) does not hold then C_{abcd} is type III $\Leftrightarrow \overset{+}{C}_{abcd}\, k^a k^c = 0$ for some vector k. The vector k is null and unique up to a scaling and is called the repeated principal null direction of C_{abcd}.

(iii) Given that (i) and (ii) do not hold the C_{abcd} is of type II$\Leftrightarrow \overset{+}{C}_{abcd}\, k^a k^c = \alpha k_b k_d$ has a unique (up to a scaling) solution for k with $\alpha \in \mathcal{C}$, $\alpha \neq 0$. The vector k is necessarily null and is called the repeated principal null direction of C_{abcd}.

(iv) The type D characterisation is as above for type II except that now there are required two independent (necessarily null) vectors k satisfying the condition given there and are each referred to as repeated principal null directions.

(v) Otherwise the Petrov type is I and there are exactly 4 distinct null directions k satisfying

$$k_{[e} C_{a]bc[d} k_{f]} k^b k^c = 0 \tag{21}$$

and called principal null directions of the Weyl tensor.

7 The Classification of Second Order Symmetric Tensors

Now return to section 2 and assume the tensor S_{ab} to be symmetric. So we have equation (2) with $S_{ab} = S_{ba}$. This equation is, at $p \in M$

$$S^a{}_b k^b = \rho k^a \tag{22}$$

and, as it stands, contains no reference to the signature of the metric. However, we have the "consistency condition" that the resulting tensor is symmetric

$$g_{ac} S^c{}_b = g_{bc} S^c{}_a \tag{23}$$

In other words if no metric with Lorentz signature at p can be found to satisfy (23) for a particular algebraic type for $S^a_{\ b}$ then that algebraic type is impossible for S in a space-time. This technique reveals one method of finding the possible algebraic types for S[1]. First suppose that all the eigenvalues of S are *real* (we will deal with the case of complex eigenvalues later). Then the possible Jordan forms for the matrix $S^a_{\ b}$ are

$$\begin{pmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{pmatrix} \quad \begin{pmatrix} \rho_1 & 1 & 0 & 0 \\ 0 & \rho_1 & 0 & 0 \\ 0 & 0 & \rho_2 & 0 \\ 0 & 0 & 0 & \rho_3 \end{pmatrix} \quad \begin{pmatrix} \rho_1 & 1 & 0 & 0 \\ 0 & \rho_1 & 1 & 0 \\ 0 & 0 & \rho_1 & 0 \\ 0 & 0 & 0 & \rho_2 \end{pmatrix}$$

Segre $\{1111\}$ Segre $\{211\}$ Segre $\{31\}$

$$\begin{pmatrix} \rho_1 & 1 & 0 & 0 \\ 0 & \rho_1 & 0 & 0 \\ 0 & 0 & \rho_2 & 1 \\ 0 & 0 & 0 & \rho_2 \end{pmatrix} \quad \begin{pmatrix} \rho_1 & 1 & 0 & 0 \\ 0 & \rho_1 & 1 & 0 \\ 0 & 0 & \rho_1 & 1 \\ 0 & 0 & 0 & \rho_1 \end{pmatrix}$$

Segre $\{22\}$ Segre $\{4\}$

In each case one checks the consistency with (23) and it turns out that for Segre types $\{22\}$ and $\{4\}$ the only solutions of (23) have $\det g_{ab} > 0$ and hence they are inconsistent with Lorentz signature. The other Segre types $\{1111\}\{211\}$ and $\{31\}$ turn out to be consistent. If the eigenvalues of S are not all real then it turns out [1,2] that S must have two real and two complex (i.e. a conjugate pair of) eigenvalues and this is indicated by a "Segre" symbol $\{z\bar{z}11\}$. Thus the only possibilities for S are $\{1111\}$, $\{211\}$, $\{31\}$, $\{z\bar{z}11\}$.

It is straightforward to work out canonical forms for each of these types at $p \in M$ in terms of a real null tetrad at p.

Segre type $\{1,111\}$

Here a real null tetrad (ℓ, n, x, y) at p can be chosen such that

$$S_{ab} = 2\mu_1 \ell_{(a} n_{b)} + \mu_2(\ell_a \ell_b + n_a n_b) + \mu_3 x_a x_b + \mu_4 y_a y_b \tag{24}$$

Here $S^a_{\ b}$ is diagonisable over $\mathcal{R}$ and 4 independent eigenvectors and corresponding eigenvalues are represented by the timelike vector $\ell^a - n^a$ (eigenvalue $\mu_1 - \mu_2$) and the spacelike vectors $\ell^a + n^a(\mu_1 + \mu_2)$, $x^a(\mu_3)$ and $y^a(\mu_4)$. These eigenvectors are mutually orthogonal and $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathcal{R}$.

Segre type $\{211\}$

Here one can choose a real null tetrad $(\ell,\ n,\ x,\ y)$ at p so that

$$S_{ab} = 2\mu_1 \ell_{(a} n_{b)} \pm \ell_a \ell_b + \mu_2 x_a x_b + \mu_3 y_a y_b \tag{25}$$

Here the eigenvectors (and corresponding eigenvalues) can be represented by ℓ^a (with eigenvalue μ_1) $x^a(\mu_2)$ and $y^a(\mu_3)$ and are mutually orthogonal and $\mu_1, \mu_2, \mu_3 \in \mathcal{R}$.

Segre type $\{31\}$

Here one can choose a real null tetrad $(\ell, n,\ x,\ y)$ at p so that

$$S_{ab} = 2\mu_1 \ell_{(a} n_{b)} + 2\ell_{(a} x_{b)} + \mu_1 x_a x_b + \mu_2 y_a y_b \tag{26}$$

Here the eigenvectors can be represented as ℓ^a (with eigenvalue μ_1) and y^a (μ_2) and are orthogonal, and $\mu_1, \mu_2 \in \mathcal{R}$.

Segre type $\{z\bar{z}11\}$

Here one can choose a real null tetrad $(\ell,\ n,\ x,\ y)$ at p so that

$$S_{ab} = 2\mu_1 \ell_{(a} n_{b)} + \mu_2(\ell_a \ell_b - n_a n_b) + \mu_3 x_a x_b + \mu_4 y_a y_b \tag{27}$$

The eigenvectors can be represented by the complex conjugate pair $\ell^a \pm i n^a$ (with respective eigenvalues $\mu_1 \pm i\mu_2$) and the real eigenvectors $x^a(\mu_3)$ and $y^a(\mu_4)$, $\mu_1,\ \mu_2,\ \mu_3,\ \mu_4 \in \mathcal{R}$.

The equations (24), (25), (26), (27) are thus canonical forms for the four possible algebraic forms for S_{ab}. An alternative form for the diagonisable case (24) is

$$S_{ab} = (\mu_2 - \mu_1) t_a t_b + (\mu_1 + \mu_2) z_a z_b + \mu_3 x_a x_b + \mu_4 y_a y_b \tag{28}$$

where a pseudo-orthornormal tetrad (t, z, x, y) is used and which is related to $(\ell,\ n,\ x,\ y)$ by $\sqrt{2} t^a = \ell^a - n^a$ and $\sqrt{2} z^a = \ell^a + n^a$. In this form the eigenvectors (and corresponding eigenvalues) can be represented by $t^a(\mu_1 - \mu_2)$, $z^a(\mu_1 + \mu_2)$, $x^a(\mu_3)$ and $y^a(\mu_4)$.

Notes on the classification [1,2,3]

(i) The type $\{1111\}$ is the *only type that admits a timelike eigenvector.* The corresponding eigenvalue is usually separated from the spacelike eigenvalues by a comma i.e. $\{1,111\}$.

(ii) Eigenvalue degeneracy is usually indicated by enclosing the "equal" eigenvalues inside round brackets. For example, in (24), if $\mu_1 - \mu_2 = \mu_1 + \mu_2 (\Leftrightarrow \mu_2 = 0)$ we would write $\{(1,1)11\}$ and if $\mu_3 = \mu_4$ we would write $\{1,1(11)\}$. If S_{ab} has type $\{(1,111)\}$ then $S_{ab} \propto g_{ab}$.

(iii) The types $\{211\}, \{31\}$ and their degeneracies have unique eigendirections (spanned by ℓ in (25) and (26)). There are *no* (real) null eigendirections in type $\{z\bar{z}11\}$ or $\{z\bar{z}(11)\}$, none in types $\{1,111\}$, $\{1,1(11)\}$ and $\{1,(111)\}$, exactly 2 in types $\{(1,1)11\}$ and $\{(1,1)(11)\}$ and infinitely many in types $\{(1,11)1\}$ and $\{(1,111)\}$.

(iv) It should be stressed that for this (and the other classifications discussed so far) that the classification applies to the tensor S at the point p and will, in general, vary as p changes in M.

There is an alternative approach to finding the canonical forms for S_{ab} which does not involve the theory of Jordan forms (in fact there are several). One writes S_{ab} as a linear combination of the basis symmetric tensors (6) and considers whether a *null* eigenvector is or is not admitted. One then simplifies the tetrad $(\ell,\ n,\ x,\ y)$ gradually by means of the "null rotations" (4) to achieve the canonical forms (24)-(27). However one really needs to know that these canonical forms are distinct in the sense that one cannot transform one into another by tetrad rotations. This is, perhaps, best done by showing that they correspond to different Segre types. The details are in [1,2].

8 Alternative Approaches

There are several alternative approaches to the problem of classifying second order symmetric tensors at a point in space-time. Churchill suggested many years ago [4] that one might consider the invariant 2-spaces associated with the matrix $S^a{}_b$ (rather than eigenvectors which are the invariant 1-spaces of this matrix). A convenient way to study this problem is, starting from S_{ab}, to define the corresponding tracefree tensor $\tilde{S}_{ab} = S_{ab} - 1/4(S_{cd}g^{cd})g_{ab}$ and then the tensor S_{abcd} where [5,6]

$$S_{abcd} = \tilde{S}_{a[c}g_{d]b} + \tilde{S}_{b[d}g_{c]a} \tag{29}$$

and finally the tensor $\overset{+}{S}_{abcd} = S_{abcd} + iS^*_{abcd}$ (right dual!). The tensor $\overset{+}{S}_{abcd}$ is clearly uniquely determined by $\tilde{S}_{ab}$ and conversely $\overset{+}{S}_{abcd}$ uniquely determines $\tilde{S}_{ab}$ since $\tilde{S}_{ab} = \overset{+}{S}{}^c{}_{acb}$. The invariant 2-spaces of S are closely related to the eigenbivectors of $\overset{+}{S}_{abcd}$ and the classification of $\overset{+}{S}_{abcd}$ by eigenbivectors then recovers the original classification of S_{ab} (more precisely of $\tilde{S}_{ab}$).

Other approaches to the problem involve a spinor approach based on equation (29) [7]. Also one can consider a "projective" type of classification based either on the tensor S_{ab} or the tensor S_{abcd} [1,5] or by direct techniques from algebraic geometry and the theory of quadric surfaces [6,7].

It is remarked that the method mentioned in the first paragraph of this section and based on equation (29) leads to the idea of Bel-type criteria for $\overset{+}{S}_{abcd}$ [8]. It is also noted that one of the first complete solutions to this classification problem was given by Plebanski [15].

9 Physical Applications

In this section the symmetric tensor S_{ab} will be taken to be the energy-momentum tensor T_{ab}, and it should be noted that the Ricci tensor and the energy-momentum tensor will have the same Segre type, including degeneracies, because of Einstein's equations $R_{ab} - 1/2Rg_{ab} = kT_{ab}$.

(a) Energy conditions

If one imposes the dominant energy conditions $T_{ab}u^au^b \geq 0$ and $T^a_bu^b$ non-spacelike for each timelike vector u^a at $p \in M$, then T_{ab} is forbidden from having either the Segre type $\{31\}$ or $\{z\bar{z}11\}$ or their degeneracies at p and restricts the other two types by the eigenvalue inequalities [1,3]

$$\mu_1 \leq 0, \quad \mu_2 \geq 0, \quad \mu_1 - \mu_2 \leq \mu_3 \leq \mu_2 - \mu_1, \quad \mu_1 - \mu_2 \leq \mu_4 \leq \mu_2 - \mu_1$$

for type $\{1,111\}$ or its degeneracies, and

$$\mu_1 \leq 0, \quad \mu_1 \leq \mu_2 \leq -\mu_1, \quad \mu_1 \leq \mu_3 \leq -\mu_1$$

together with positive in (25) for type $\{211\}$ or its degeneracies.

(b) Perfect fluids
Here we have pressure p, density ρ, fluid flow vector u^a ($u^a u_a = -1$) and

$$T_{ab} = (p+\rho)u_a u_b + p g_{ab} \tag{30}$$

The eigenvectors and eigenvalues are $u^a(-\rho)$ and any vector orthogonal to u^a (eigenvalue p). Hence we have Segre type $\{1,(111)\}$ (one can recover the form (28) from (30) by using the third completeness relation in (3)). The energy conditions give $\rho \geq p \geq -\rho$

(c) Null Einstein-Maxwell fields
Here the Maxwell tensor F_{ab} is a null bivector of the form (13) where ℓ^a is the "radiation" (null) direction. The energy-momentum tensor at $p \in M$ is of the form

$$T_{ab} = \frac{2}{k}(F_{ac}F^c_{\ b} - \frac{1}{4}F_{cd}F^{dc}g_{ab}) = v\ell_a\ell_b \tag{31}$$

for $v \in \mathcal{R}, v > 0$. From (25) the Segre type is $\{(211)\}$ with all eigenvalues zero and the energy conditions are satisfied since $v > 0$.

(d) Non-null Einstein-Maxwell fields
Here the Maxwell tensor is non-null and so takes the form (11), (12), or (18) and so

$$T_{ab} = \alpha\,(2\ell_{(a}n_{b)} - x_a x_b - y_a y_b) \tag{32}$$

for $\alpha \in \mathcal{R}$, $\alpha < 0$. The Segre type is $\{(1,1)(11)\}$ with eigenvalues α and $-\alpha$ and the energy conditions are satisfied since $\alpha < 0$.

(e) General fluid space-times
Here the energy-momentum tensor is

$$T_{ab} = (p - \xi\theta + \rho)u_a u_b + (p-\xi\theta)g_{ab} - 2\eta\sigma_{ab} + 2u_{(a}q_{b)} \tag{33}$$

where u^a is the fluid flow vector, $u^a u_a = -1$, energy density ρ with respect to u^a, dynamic viscosity η, bulk viscosity ξ, isotropic pressure p, shear tensor $\sigma_{ab}(=\sigma_{ba})$, expansion θ and heat flow vector q^a. Also $u^a q_a = 0$, $\sigma_{ab}u^b = 0$ and $\sigma^a_a = 0$. Here the situation requires a more detailed analysis which can be found in [3,9].

(f) Perfect fluid and null Einstein-Maxwell field combined
Here one has [3,9]

$$T_{ab} = (p+\rho)u_a u_b + p g_{ab} + v\ell_a\ell_b \tag{34}$$

and the Segre type is $\{1,1(11)\}$

(g) Perfect fluid and non-null Einstein-Maxwell field combined
Here one has [3,9]

$$T_{ab} = (p+\rho)u_a u_b + p g_{ab} + \alpha(2\ell_{(a}n_{b)} - x_a x_b - y_a y_b) \tag{35}$$

If ℓ^a, n^a and u^a are coplanar the Segre type is $\{1,1(11)\}$ and otherwise it is $\{1,111\}$.

(h) Static space-times
Here M must admit a timelike hypersurface orthogonal Killing vector field X which then turns out to be a (timelike) eigenvector of T_{ab} [10]. It follows that T_{ab} has Segre type $\{1,111\}$ or some degeneracy of this type [2].

There are many other physical applications of this classification including a description of a complete set of algebraic Rainich conditions, applications to the "inheritance of symmetries" problem, geodesic deviation and scattering problems, "peeling" problems (see appendix) and the algebraic degeneracies produced in T_{ab} at points in M where there is a Killing, homothetic or affine isotropy. More details and references can be found in [3]. Again it should be stressed that the classification applies to T_{ab} at some *point* $p \in M$ and the Segre type will, is general, change as p changes.

10 Holonomy and the Classification of Connections in Space-Time

Let $p \in M$ and for a fixed $k, 1 \leq k \leq \infty$ let $C_k(p)$ denote the set of all piecewise C^k closed curves starting and ending at p. Each $c \in C_k(p)$ gives rise by means of parallel transport along c, using the unique symmetric Levi-Civita connnection Γ associated with the space-time metric g, to an isomorphism of the tangent space T_pM to M at p onto itself. If, for $c \in C_k(p)$, $f(c)$ denotes the associated isomorphism of T_pM then, using a standard notation, $f(c^{-1}) = f(c)^{-1}$ and $f(c_1 \circ c_2) = f(c_1) \circ f(c_2)(c_1, c_2 \in C_k(p))$ and it follows that the set of isomorphisms of T_pM arising from all members of $C_k(p)$ *is a subgroup of the Lorentz group* $\mathcal{L}$. This subgroup is called the *k-holonomy group of M at p*. Since M is connected it is necessarily path connected and it follows that the k-holonomy groups at distinct points of M are isomorphic (in fact, conjugate) to each other. Thus it makes sense to speak of the *k-holonomy group of M*. Finally it can be shown that the k-holonomy group of M is independent of k ($1 \leq k \leq \infty$) and so one speaks of the *holonomy group Φ of M*.

It can be shown that if M is simply connected then the holonomy group Φ is connected and hence a connected Lie subgroup of the proper Lorentz group $\mathcal{L}^0(\mathcal{L}^0$ is the component of the identity of $\mathcal{L}$). Since there is a one-to-one correspondence between connected subgroups of a Lie group G and the subalgebras of the Lie algebra of G it follows that if one wishes to classify simply connected space-times by their holonomy group, it is sufficient to classify the subalgebras of the Lie algebra A of $\mathcal{L}^0$ and then to see which of them give (connected) subgroups of $\mathcal{L}$ which can actually be a holonomy group of some (simply connected) space-time. [Alternatively one could drop the simply connected restriction and concentrate on the *restricted* holonomy group (rather than the holonomy group) where the curves c are homotopic to zero [11]]. If we represent $\mathcal{L}$ by

Table 1:

Type	Subalgebra	Con	Rec	Constant Tensors
R_2	$\ell \wedge n$	$x,\ y$	ℓ, n	$g_{ab},\ x_a x_b,\ y_a y_b,\ x_{(a} y_{b)}$
R_3	$\ell \wedge x$	ℓ, y	–	$g_{ab},\ \ell_a \ell_b,\ y_a y_b\ \ell_{(a} y_{b)}$
R_4	$x \wedge y$	ℓ, n	–	$g_{ab},\ \ell_a \ell_b,\ n_a n_b,\ \ell_{(a} n_{b)}$
R_5	$\ell \wedge n + \rho(x \wedge y)$	——	——	——————
R_6	$\ell \wedge n, \ell \wedge x$	y	ℓ	$g_{ab},\ y_a y_b$
R_7	$\ell \wedge n, x \wedge y$	–	ℓ, n	$g_{ab},\ \ell_{(a} n_{b)}$
R_8	$\ell \wedge x, \ell \wedge y$	ℓ	–	$g_{ab},\ \ell_a \ell_b$
R_9	$\ell \wedge n, \ell \wedge x, \ell \wedge y$	–	ℓ	g_{ab}
R_{10}	$\ell \wedge n, \ell \wedge x,\ n \wedge x$	y	–	$g_{ab}, y_a y_b$
R_{11}	$\ell \wedge x, \ell \wedge y,\ x \wedge y$	ℓ	–	$g_{ab}, \ell_a \ell_b$
R_{12}	$\ell \wedge x,\ \ell \wedge y,\ \ell \wedge n + \rho(x \wedge y)$	–	ℓ	g_{ab}
R_{13}	$x \wedge y,\ x \wedge z,\ y \wedge z$	u	–	$g_{ab}, u_a u_b$
R_{14}	$\ell \wedge n,\ x \wedge y,\ \ell \wedge x,\ \ell \wedge y$	–	ℓ	g_{ab}
R_{15}	A	–	–	g_{ab}

$$\mathcal{L} = \{T \in GL(4, \mathcal{R}) : T'\eta T = \eta\} \tag{36}$$

where "stroke" means "transpose" and $\eta = diag(-1, 1, 1, 1)$ then the Lie algebra A of $\mathcal{L}^0$ (which equals the Lie algebra of $\mathcal{L}$) can be represented by the vector space of matrices which are skew self adjoint with respect to η, together with the "multiplication" given by matrix commutation. Roughly speaking we can thus represent A by the 6-dimensional vector space of bivectors from which $\mathcal{L}^0$ can be recovered by exponentiation. [That $\mathcal{L}^0$ is an "exponential" Lie group, i.e. it can be "obtained" by exponentiating its Lie algebra is special for $\mathcal{L}^0$ and not a general result for (connected) Lie groups].

The subalgebras of A are well known and have been conveniently labelled $R_1 - R_{15}$ in [12]. In terms of a null tetrad $(\ell,\ n, x, y)$ (or in the case of R_{13} an orthonormal tetrad $(u,\ x,\ y,\ z)$) they are listed in the table. In this table each type is given with its subalgebra, the independent global covariantly constant vector fields admitted (under "Con"), the independent (properly) recurrent global vector fields admitted (under "Rec") and the independent global covariantly constant second order symmetric tensors admitted (under "Constant Tensors"). In the R_5 and R_{12} cases, $\rho \in R$, $\rho \neq 0$. *However R_5 cannot be the holonomy group of any space-time* and hence the line! [11] [Incidentally, a global vector field k on M is called *recurrent* if $k_{a;b} = k_a q_b$ for some 1-form q_a. It is called *properly recurrent* if it cannot be globally scaled so as to be covariantly constant.] All the types in the table except R_5 *can be realised* as the holonomy group of some space-time [11]. The trivial (flat) case R_1 is omitted.

In the table, the types $R_2, R_3, R_4, R_6, R_7, R_{10}, R_{13}$ are *locally decomposable.* The type R_{15} is *generic* in a topological sense and the holonomy groups of the other types can be described in terms of standard groups [11]. Only the types R_8, R_{14} and R_{15} can apply to non-flat *vacuum* space-times [11,12]. A particularly useful application of this classification is to the study of affine collineations on space-times.

Appendix

A "Peeling" Theorem for Segre Types

A rather nice interpretation of the Petrov types is given by the well known "peeling" theorem of Sachs and Penrose. They show how in an asymptotic expansion of the (vacuum) Riemann tensor for a bounded source the Petrov types "peel off" in powers of $1/r$ along some null ray with affine parameter r. A simular calculation can be done for Segre types at least for the case of an Einstein-Maxwell field due to a bounded charge-current distribution [13]. In this case a similar expansion for the Maxwell electromagnetic tensor F_{ab} can be written down [14]

$$F_{ab} = \frac{N_{ab}}{r} + \frac{L_{ab}}{r^2} + \frac{G_{ab}}{r^3} + \frac{H_{ab}}{r^4} + O(r^{-5}) \tag{37}$$

where the expansion is along some null ray with affine parameter r, where N is a null bivector with principal null direction ℓ, L is a bivector satisfying $L_{ab}\ell^b \propto \ell_a$ and G and H are bivectors about which no further information is required. Now compute the energy-momentum tensor using the first (general) equation is (31) to get

$$T_{ab} = \frac{\overset{2}{T}_{ab}}{r^2} + \frac{\overset{3}{T}_{ab}}{r^3} + \frac{\overset{4}{T}_{ab}}{r^4} + \frac{\overset{5}{T}_{ab}}{r^5} + O(r^{-6}) \tag{38}$$

where one finds that $\overset{2}{T}_{ab}$ has Segre type $\{(2111)\}$ with zero eigenvalue, $\overset{3}{T}_{ab}$ has Segre type $\{(31)\}$ with zero eigenvalue, $\overset{4}{T}_{ab}$ satisfies $\overset{4}{T}_{ab}\,\ell^b \propto \ell_a$ and $\overset{5}{T}_{ab}$ satisfies $\overset{5}{T}_{ab}\,\ell^a\ell^b = 0$. Thus one has a similar "peeling" of Segre types. The connection between this result and the above one for the Riemann tensor can be clarified by introducing the fourth order tensor T_{abcd} associated with T_{ab} through its trace-free part $\tilde{T}_{ab} = T_{ab} - 1/4T^c_c g_{ab}$

$$T_{abcd} = \tilde{T}_{a[c}g_{d]b} + \tilde{T}_{b[d}g_{c]a} \tag{39}$$

One then finds from (38) and (39)

$$T_{abcd} = \frac{\overset{2}{T}_{abcd}}{r^2} + \frac{\overset{3}{T}_{abcd}}{r^3} + \frac{\overset{4}{T}_{abcd}}{r^4} + \frac{\overset{5}{T}_{abcd}}{r^5} + O(r^{-6}) \tag{40}$$

where the $\overset{i}{T}_{abcd}$ $(2 \le i \le 5)$ satisfy

$$\begin{array}{ll} \overset{2}{T}_{abcd}\,\ell^d = 0 & \overset{3}{T}_{abcd}\,\ell^d = P_{ab}\ell_c \\ \overset{4}{T}_{abcd}\,\ell^a\ell^c \propto \ell_b\ell_d & \ell^b\ell^c\ell_{[e}\,\overset{5}{T}_{a]bc[d}\,\ell_{f]} = 0 \end{array} \tag{41}$$

where P_{ab} is a null bivector with principal null direction ℓ. If one recalls the Sachs-Penrose peeling theorem and how the individual "peeled off" types are described using the Bel criteria, one sees a direct analogy with equations (40) and (41) where the "Bel criteria" for T_{abcd} are used [1,8].

Acknowledgements

The author wishes to point out that these lecture notes were informally written and intended as a summary of the way he likes to do things. Hence the lack of references to other researchers in this area. His indebtedness to others has already been recorded in the more extended bibliographies in references [1,3].

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