- The two fields of application of plasma physics, laboratory nuclear fusion research and plasma-astrophysics, may be *described from a single point of view by means of magnetohydrodynamics (MHD).* This yields new insights for the interpretation of phenomena in the Universe due to plasma dynamics.

- For example, the solar system exhibits a secondary layer of magnetic phenomena on top of the gravitational attraction. These plasma physical phenomena exhibit a *global coherence due to the pervading presence of magnetic fields.* At other locations in the Universe (e.g. pulsars), magnetic interactions can not be considered as small perturbations but dominate the physics.

- We can learn much about magnetized astrophysical plasmas from laboratory and solar system studies because there we can *observe both the spatial magnetic structure and the temporal evolution.*

Nuclear Fusion

- Deuterium–tritium reactions: $\mathrm{D}^2 + \mathrm{T}^3 \rightarrow \mathrm{He}^4 \; + \; \mathrm{n}$;
- Lawson criterion: minimum $n\tau_E \sim 3 \times 10^{20}\,\mathrm{m}^{-3}\,\mathrm{s}$ at $\bar{T} \sim 20\,\mathrm{keV}$,
  so that with $n \sim 10^{20}\,\mathrm{m}^{-3} \;\Rightarrow\; \tau_E \sim 3\,\mathrm{s}$;
- Confinement by magnetic fields $\Rightarrow$ tokamaks.

Astrophysical plasmas

- The standard view of nature fails because it skips 18 order of magnitiude;
- Examples: corona / solar wind / magnetospheres $\Rightarrow$ magnetized plasmas.

Definition(s) of plasma

- Amount of ionization (Saha);
- Microscopic definition: collective behavior ($\tau \ll \tau_n$, $\lambda \gg \lambda_{\mathrm{D}}$, $N_{\mathrm{D}} \gg 1$);
- Macroscopic definition: the magnetic field enters.

Theory of motion of single charged particles

- Equation of motion,

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})\,,$$

- in magnetic field yields periodic motion with cyclotron frequency and radius:

$$\Omega \equiv \frac{|q|B}{m}\,, \qquad R \equiv \frac{v_\perp}{\Omega} \approx \frac{\sqrt{2mkT}}{|q|B}\,.$$

- Drifts in curved magnetic fields described by adiabatic invariants.

Kinetic theory

- Boltzmann equation describes distribution functions $f_\alpha(\mathbf{r}, \mathbf{v}, t)$:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = C_\alpha \equiv \left(\frac{\partial f_\alpha}{\partial t}\right)_{\rm coll}\,.$$

- Moment reduction leads to fluid description. Truncation and closure of the infinite chain of equation by transport coefficients.

Theoretical models

- **Single particle equation of motion:**

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$\Rightarrow$ charged particles gyrate around $\mathbf{B}$ with gyro frequency $\Omega \equiv eB/m$ and gyro radius $R \equiv v_\perp/\Omega\,.$

- **Kinetic theory (Vlasov equation):**

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

$\Rightarrow$ charge imbalances by plasma oscillations with frequency $\omega_p \equiv \sqrt{ne^2/(\epsilon_0 m_e)}$ in regions of size of Debye sphere with radius $\lambda_D \equiv \sqrt{\epsilon_0 kT/(ne^2)}\,.$

- **Fluid description:**

$$\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = -\nabla p + \mathbf{j} \times \mathbf{B}$$

$\Rightarrow$ pinching: plasma currents $\mathbf{j}$ create confining $\mathbf{B}$;

$\Rightarrow$ stable Alfvén waves, $v_A = B_0/\sqrt{\mu_0\rho}$, and unstable kinks, $\tau \sim v_A/L$ (μ sec).

The MHD model

- **Postulating the basic equations**

  $\Rightarrow$ perfectly conducting fluid motion interacting with $\mathbf{B}$

- **Concept of Magnetic Flux**

  $\Rightarrow$ flux tubes & flux conservation

- **MHD equations in conservation form**

  $\Rightarrow$ global & local viewpoint

- **Discontinuities**

  $\Rightarrow$ jump conditions at shock fronts

  $\Rightarrow$ boundary conditions at interfaces

- **MHD model problems**

The MHD model (cont'd)

- MHD $\equiv$ gas dynamics + pre-Maxwell equations ($v \ll c$)

  $\Rightarrow$ coupled by Newton's law (with Lorentz force) & Ohm's law ($\mathbf{E}' = 0$).

  $\Rightarrow$ Scale independent: macro description for 90% (known) matter.

  $\Rightarrow$ Complete model: $t = 0$ data + geometry specification $\longrightarrow$ BCs.

- $\nabla \cdot \mathbf{B} = 0 \longrightarrow$ *well-defined magnetic flux*

$$\Psi \equiv \iint_S \mathbf{B} \cdot \mathbf{n} \, d\sigma$$

  $\Rightarrow$ global magnetic flux conservation:

  in tokamak vessel fixed $\Psi_{\rm pol} = {\rm const}$ and $\Psi_{\rm tor} = {\rm const}$.

- Conservation laws: $\partial / \partial t \, (\cdots) + \nabla \cdot (\cdots) = 0$

  $\Rightarrow$ global: mass ρ, momentum $\boldsymbol{\pi}$, total energy $\mathcal{H}$, magnetic flux ψ .

  $\Rightarrow$ local: mass of moving fluid element,

  flux through any surface bounded by contour moving with fluid.

General framework (1)

- One may obtain the one-fluid (resistive) MHD equations by (a) just posing them, (b) by taking moments of the Boltzmann equation.
- Resistive terms negligible if magnetic Reynolds number $R_m \equiv \mu_0 v L/\eta \gg 1$.
- $\Rightarrow$ **Ideal MHD equations:**

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0\,,$$
$$\rho(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}) + \nabla p - \rho\mathbf{g} - \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} = 0\,,$$
$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p + \gamma p\nabla\cdot\mathbf{v} = 0\,,$$
$$\frac{\partial \mathbf{B}}{\partial t} - \nabla\times(\mathbf{v}\times\mathbf{B}) = 0\,, \qquad \nabla\cdot\mathbf{B} = 0\,.$$

- **Initial data** ρ_i, $\mathbf{v}_i$, p_i, $\mathbf{B}_i$ should be given.
- **Boundary conditions** for closed magnetic geometry with wall on plasma (Model I):

$$\mathbf{n}\cdot\mathbf{v} = 0\,, \qquad \mathbf{n}\cdot\mathbf{B} = 0\,.$$

Conservation form

- Defining
  - *momentum density*: $\boldsymbol{\pi} \equiv \rho \mathbf{v}$,
  - *stress tensor*: $\mathbf{T} \equiv \rho \mathbf{v}\mathbf{v} + (p + \frac{1}{2}B^2)\,\mathbf{I} - \mathbf{B}\mathbf{B}$,
  - *total energy density*: $\mathcal{H} \equiv \frac{1}{2}\rho v^2 + \frac{p}{\gamma - 1} + \frac{1}{2}B^2$,
  - *energy flow*: $\mathbf{U} \equiv (\frac{1}{2}\rho v^2 + \frac{\gamma}{\gamma - 1}p)\mathbf{v} + B^2\mathbf{v} - \mathbf{v}\cdot\mathbf{B}\,\mathbf{B}$,
  - *(no name)*: $\mathbf{Y} \equiv \mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}$,

yields **the MHD equations in conservation form**:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{\pi} = 0$$ *(conservation of mass),*

$$\frac{\partial \boldsymbol{\pi}}{\partial t} + \nabla \cdot \mathbf{T} = 0$$ *(conservation of momentum),*

$$\frac{\partial \mathcal{H}}{\partial t} + \nabla \cdot \mathbf{U} = 0$$ *(conservation of energy),*

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \mathbf{Y} = 0$$ *(conservation of magnetic flux).*

Conserved quantities

- Defining
  - *total mass*: $M \equiv \int \rho \, d\tau \,,$
  - *total momentum*: $\mathbf{\Pi} \equiv \int \boldsymbol{\pi} \, d\tau \,,$
  - *total energy*: $H \equiv \int \mathcal{H} \, d\tau \,,$
  - *total magnetic flux*: $\Psi \equiv \int \mathbf{B} \cdot \tilde{\mathbf{n}} \, d\tilde{\sigma} \,,$

Model I BCs guarantee **global conservation laws** for these quantities:

$$\dot{M} = -\int \nabla \cdot \boldsymbol{\pi} \, d\tau \stackrel{\text{Gauss}}{=} -\oint \boldsymbol{\pi} \cdot \mathbf{n} \, d\sigma = 0 \,,$$

$$\mathbf{F} = \dot{\mathbf{\Pi}} = -\int \nabla \cdot \mathbf{T} \, d\tau \stackrel{\text{Gauss}}{=} -\oint (p + \tfrac{1}{2}B^2) \, \mathbf{n} \, d\sigma \quad (\text{usually} = 0) \,,$$

$$\dot{H} = -\int \nabla \cdot \mathbf{U} \, d\tau \stackrel{\text{Gauss}}{=} -\oint \mathbf{U} \cdot \mathbf{n} \, d\sigma = 0 \,,$$

$$\dot{\Psi} = \int \nabla \times (\mathbf{v} \times \mathbf{B}) \cdot \tilde{\mathbf{n}} \, d\tilde{\sigma} \stackrel{\text{Stokes!}}{=} \oint \mathbf{v} \times \mathbf{B} \cdot \, d\boldsymbol{l} = 0 \,.$$

$\Rightarrow$ **the system is closed!**

Flux conservation

- THE equation for magnetic fields,

$$\nabla \cdot \mathbf{B} = 0$$

implies that field lines are either closed or cover a surface (or volume) ergodically. Most important, it implies that **flux tubes are a well-defined concept.**

- Kinematic expressions for the rate of change of line, surface, and volume element,

$$\frac{\mathrm{D}}{\mathrm{D}t}(d\boldsymbol{l})\,, \quad \frac{\mathrm{D}}{\mathrm{D}t}(d\boldsymbol{\sigma})\,, \quad \frac{\mathrm{D}}{\mathrm{D}t}(d\tau)\,,$$

together with the induction equation,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})\,,$$

yields **infinitesimal flux conservation:**

$$\frac{\mathrm{D}}{\mathrm{D}t}(d\Psi) = 0\,.$$

$\Rightarrow$ **Field lines are frozen in the fluid!**

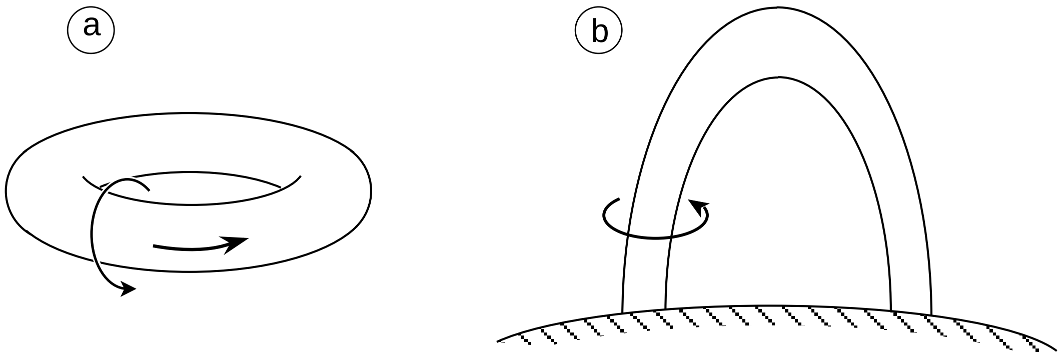
General framework (2)

- **MHD equations** *describe the motion of an electrically conducting fluid interacting with a magnetic field:*

  Gas dynamics $+$ electrodynamics $+$ interaction .

  $(\rho,\,\mathbf{v},\,p)$ $(\mathbf{E},\,\mathbf{B},\,\mathbf{j})$

- **Magnetic geometry** of *generic plasma confinement structures* (like tokamaks or coronal magnetic loops) requires specification of BCs.



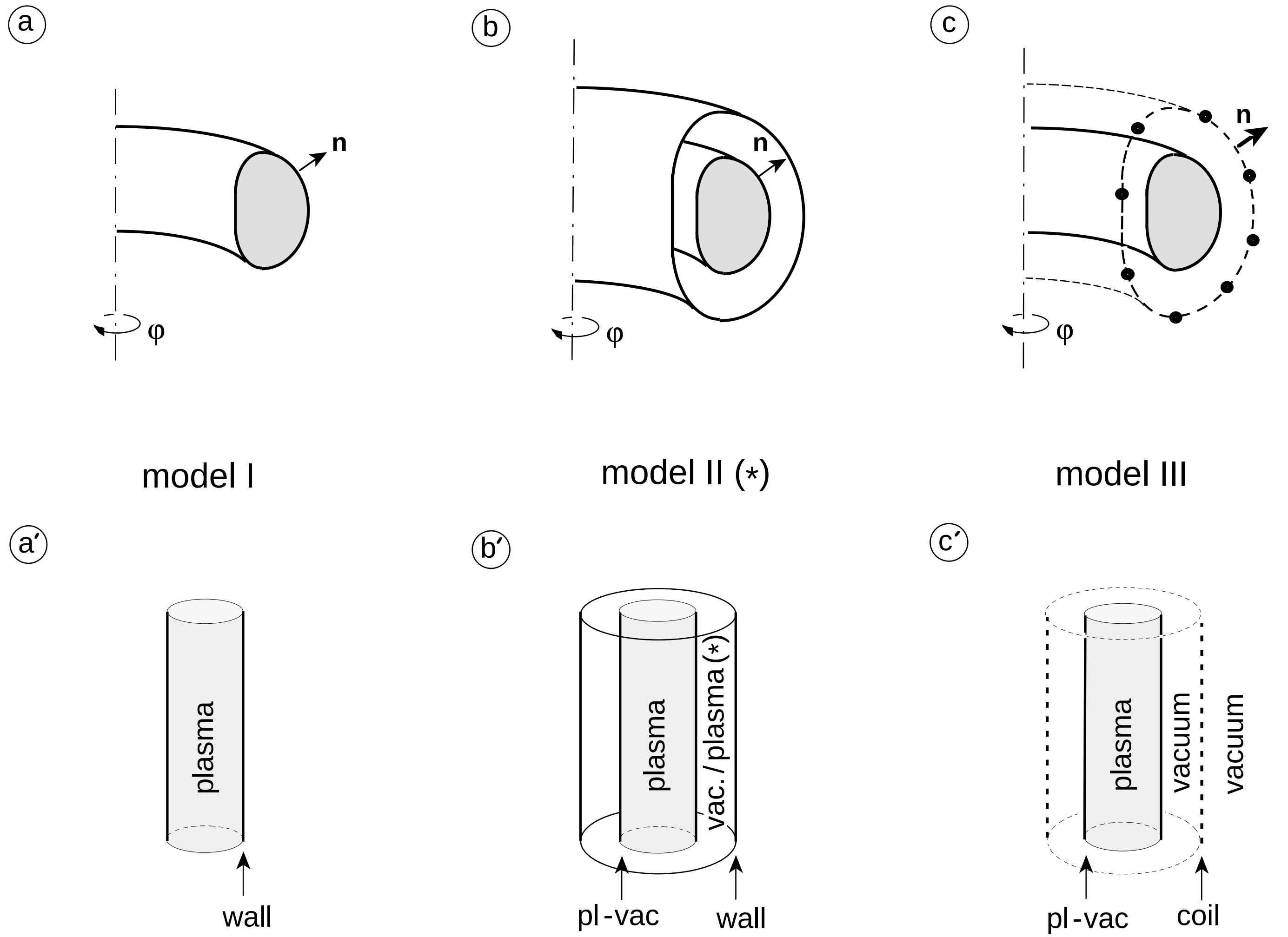
- This usually involves **interfaces** with a **tangential discontinuity** (if B parallel):

  $\rho\,,\mathbf{v}'_t\,,\;p\,,\mathbf{B}_t$ jump, but $v'_n = 0\,,\;\; B_n = 0\,,\;\; [\![p + \frac{1}{2}B_t^2]\!] = 0\;\Rightarrow$ Models I–III,

  or a **contact discontinuity** (if B intersects):

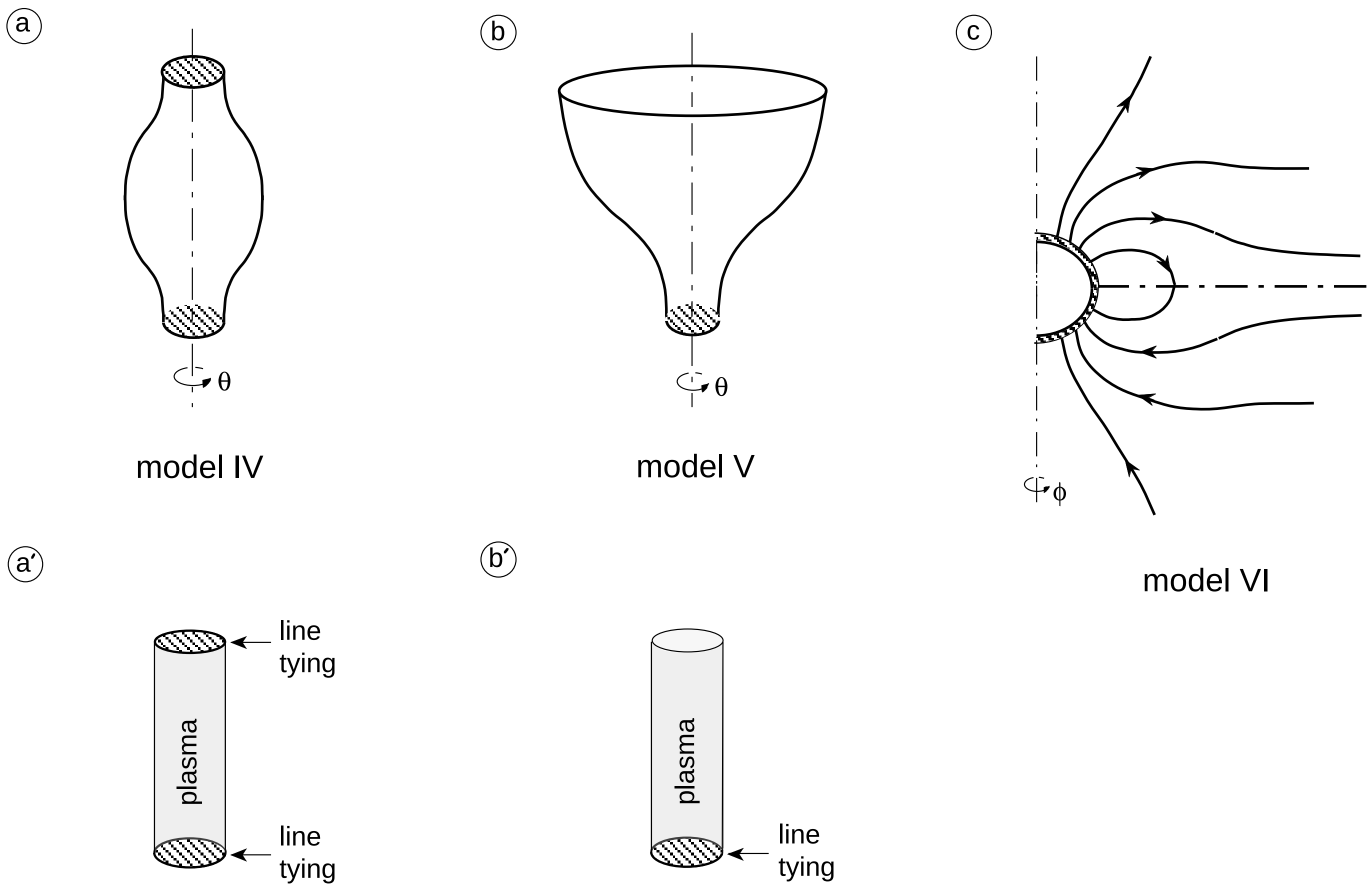
  ρ jumps, other variables continuous, in particular $[\![\mathbf{v}']\!] = 0\;\Rightarrow$ Models IV–VI.

Laboratory plasmas



Model I: Plasma – wall; Model II(*): Plasma – vacuum (* other plasma) – wall;
Model III: Plasma – vacuum – exciting external current coils – vacuum.

Astrophysical plasmas



Model IV: Closed magnetic loop, line-tied at both ends; Model V: Open loop, line-tied at one end and flaring on the other; Model VI: Stellar wind outflow.

Scale independence

- MHD equations are unchanged by changing scales of **length** l_0 , **magnetic field** B_0 , **density** ρ_0 , or **time scale** $t_0 \equiv l_0/v_A$ $\big($with Alfvén speed $v_A \equiv B_0/\sqrt{\mu_0\rho_0}\big)$.

	l_0 (m)	B_0 (T)	t_0 (s)
tokamak	20	3	3×10^{-6}
magnetosphere Earth	4×10^{7}	3×10^{-5}	6
solar coronal loop	10^{8}	3×10^{-2}	15
magnetosphere neutron star	10^{6}	10^{8}	10^{-2}
accretion disc YSO	1.5×10^{9}	10^{-4}	7×10^{5}
accretion disc AGN	4×10^{18}	10^{-4}	2×10^{12}
galactic plasma	10^{21} $(= 10^{5}\ \mathrm{ly})$	10^{-8}	10^{15} $(= 3 \times 10^{7}\ \mathrm{y})$

$\Rightarrow$ **Huge range of applicability of MHD!**

MHD equations

- Roundabout: *'Derived'* by taking moments of the kinetic equations (Chaps. 2 & 3).
- Straightforward: *Posed as reasonable postulates* for medium called 'plasma' (Chap. 4).
- **Conservation laws** for mass, momentum, energy, and *magnetic flux*.
- *BVP: Models I–III* for laboratory and *Models IV–VI* for astrophysical plasmas.

MHD waves

- *IVP: Linearized motion* $\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ about static homogeneous background state
  $\Rightarrow$ **Symmetric** eigenvalue problem for ρ_1 , $\mathbf{v}_1$, e_1 , $\mathbf{B}_1$ (8 single waves).
- Eliminate: *spurious solutions* $\omega\,\mathbf{k}\cdot\mathbf{B} = 0$ (improper account of constraint $\mathbf{k}\cdot\mathbf{B} = 0$),
  unimportant entropy waves $\omega\, S_1 = 0$ (no motion of plasma).
  $\Rightarrow$ **Symmetric** eigenvalue problem for $\mathbf{v}_1$ (3 forward & 3 backward waves):
  $\omega_A = \pm k_\parallel b$ *(Alfvén)*, $\omega_{s,f} = \pm k\sqrt{\tfrac{1}{2}(b^2 + c^2)}\,\left[1 \pm \sqrt{\ldots}\right]$ *(magneto-acoustic)*.
- Eigenvalue only as ω^2 : if < 0 (inhomogeneous plasmas) $\Rightarrow$ *instability* (Chap. 6).

Phase and group velocity

- Plane wave propagates in direction of $\mathbf{k}$ with **the phase velocity,** $\mathbf{v}_{\rm ph} \equiv (\omega/k)\,\mathbf{n}$ **.**
- Wave packet propagates with **group velocity,** $\mathbf{v}_{\rm gr} \equiv \partial\omega/\partial\mathbf{k}$ **.**

  $\Rightarrow$ Phase and group diagrams: response to plane waves and point disturbances.

Characteristics

- Characteristics describe propagation of initial data information through the plasma: *Construction of the characteristics amounts to solution of the IVP.*
- However, *initial data should not be given on a characteristic manifold* since that information would not propagate away from it (by definition). The IVP yields an inhomogeneous algebraic problem for the perturbations normal to the characteristics. $\Rightarrow$ **Duality**: either the homogeneous problem is solvable, giving the characteristics, or the inhomogeneous problem is solvable, giving the solution of the IVP.
- In MHD, the initial data are given on a 3D spatial domain, whereas the characteristics are real 3D manifolds intrinsically involving time (viz. the 2D Friedrichs ray surfaces with time appended as a 3rd coordinate) $\Rightarrow$ **the IVP is well-posed in MHD.**

Intuitive approach to stability

- **Energy**: Stability or instability corresponds to perturbations that *raise or lower the energy* with respect to that of the equilibrium.
- **Force**: Stability or instability corresponds to perturbations creating *a force in the opposite or same direction as the displacement* away from equilibrium.

Force operator formalism

- Starting point: linearized MHD equations for perturbations $\mathbf{v}_1$, π $(\equiv p_1)$, $\mathbf{Q}$ $(\equiv \mathbf{B}_1)$, ρ_1 about an equilibrium $\mathbf{j}_0 \times \mathbf{B}_0 = \nabla p_0 + \rho_o \nabla \Phi$.
- **Lagrangian displacement** $\boldsymbol{\xi}$ permits to express $\mathbf{v}_1 = \partial\boldsymbol{\xi}/\partial t$ and to integrate the equations for π, $\mathbf{Q}$, and ρ_1 with respect to time:

$$\frac{\partial p_1}{\partial t} = -\mathbf{v}_1 \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \mathbf{v}_1 \;\Rightarrow\; \pi = -\gamma p \nabla \cdot \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla p \,,$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0) \;\Rightarrow\; \mathbf{Q} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}) \,,$$

$$\frac{\partial \rho_1}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{v}_1) \;\Rightarrow\; \rho_1 = -\nabla \cdot (\rho_0 \boldsymbol{\xi}) \,.$$

- The equation for $\mathbf{v}_1$ yields the **force operator** *('Schrödinger')* **equation of motion**:

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + (\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 + \mathbf{j}_0 \times \mathbf{B}_1 - \rho_1 \nabla \Phi \,,$$

$$\Downarrow$$

$$\mathbf{F}(\xi) \equiv -\nabla \pi - \mathbf{B} \times (\nabla \times \mathbf{Q}) + \mathbf{j} \times \mathbf{Q} + \nabla \Phi \, \nabla \cdot (\rho \boldsymbol{\xi}) = \rho \frac{\partial^2 \xi}{\partial t^2} \,,$$

with BC at the wall (model I): $\mathbf{n} \cdot \boldsymbol{\xi} = 0 \,.$

Spectral implications:

- Normal modes $\boldsymbol{\xi}(\mathbf{r}, t) = \hat{\boldsymbol{\xi}}(\mathbf{r})\, \mathrm{e}^{-\mathrm{i}\omega t}$ $\Rightarrow$ EVP: $\mathbf{F}(\hat{\boldsymbol{\xi}}) = -\rho \omega^2 \hat{\boldsymbol{\xi}}$,

  collection of eigenvalues $\{\omega^2\} \equiv$ *spectrum of ideal MHD.*

- One proves (by complicated algebra): the operator $\rho^{-1}\mathbf{F}$ is self-adjoint

  $\Rightarrow$ *real eigenvalues* ω^2 $\Rightarrow$ $\omega^2 > 0$ (stable wave) òr $\omega^2 < 0$ (instability).

- Transition through $\omega^2 = 0$ (marginal stability): $\mathbf{F}(\hat{\boldsymbol{\xi}}) = 0 \,.$ In general no solution ($\omega^2 = 0$ is not an eigenvalue) unless special equilibrium parameters are chosen.

Quadratic forms

- Two displacement vector fields with inner product:

$$\begin{aligned} \boldsymbol{\xi} &= \boldsymbol{\xi}(\mathbf{r}, t)\,, \qquad \mathbf{n}\cdot\boldsymbol{\xi} = 0 \;\; \text{(on boundary)} \\ \boldsymbol{\eta} &= \boldsymbol{\eta}(\mathbf{r}, t)\,, \qquad \mathbf{n}\cdot\boldsymbol{\eta} = 0 \;\; \text{(on boundary)} \end{aligned} \quad \Rightarrow \quad \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle \equiv \tfrac{1}{2} \int \rho\, \boldsymbol{\xi}^* \cdot \boldsymbol{\eta}\, dV \,.$$

- Norm: $\|\boldsymbol{\xi}\| \equiv \langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle^{1/2} < \infty$ $\Rightarrow$ **Hilbert space.**

  Relation with **kinetic energy**: $K \equiv \frac{1}{2} \int \rho \mathbf{v}^2\, dV \approx \frac{1}{2} \int \rho |\dot{\boldsymbol{\xi}}|^2\, dV = \langle \dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}} \rangle \,.$

- $\mathbf{F}$ is a *self-adjoint linear operator in Hilbert space* of plasma displacement vectors:

$$\int \boldsymbol{\eta}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, dV = \int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\eta})\, dV \quad \Rightarrow \quad \textbf{mathematics same as in QM!}$$

- Energy conservation and self-adjointness of $\mathbf{F}$:

$$\dot{W} = -\dot{K} = -\int \rho\, \dot{\boldsymbol{\xi}}^* \cdot \ddot{\boldsymbol{\xi}}\, dV = -\int \dot{\boldsymbol{\xi}}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, dV = \frac{d}{dt}\Big[-\tfrac{1}{2} \int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, dV \Big]$$

$\Rightarrow$ **potential energy:** $W = -\frac{1}{2} \int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, dV \,.$ (work of $\boldsymbol{\xi}$ against $\mathbf{F}$).

- Rework:

$$W = \frac{1}{2} \int \left[\gamma p \, (\nabla \cdot \boldsymbol{\xi})^2 + |\mathbf{Q}|^2 + (\boldsymbol{\xi} \cdot \nabla p) \nabla \cdot \boldsymbol{\xi}^* + \mathbf{j} \cdot \boldsymbol{\xi}^* \times \mathbf{Q} - (\boldsymbol{\xi}^* \cdot \nabla \Phi) \, \nabla \cdot (\rho \boldsymbol{\xi}) \, \right] dV .$$

Variational principles

- *Rayleigh-Ritz principle:* Eigenfunctions $\boldsymbol{\xi}$ of $\rho^{-1}\mathbf{F}$ make the Rayleigh quotient Λ stationary:

$$\delta \Lambda = 0 \,, \qquad \Lambda \equiv \frac{W[\boldsymbol{\xi}]}{I[\boldsymbol{\xi}]} \,, \qquad \text{where} \quad I \equiv \|\boldsymbol{\xi}\|^2 = \frac{1}{2} \int \rho \, \boldsymbol{\xi}^* \cdot \boldsymbol{\xi} \, dV \,;$$

  Eigenvalues are the stationary values of Λ: $\omega^2 = \Lambda[\boldsymbol{\xi}]$.

- Since $I[\boldsymbol{\xi}] \geq 0$, we may insert trial functions in $W[\boldsymbol{\xi}]$
  $\Rightarrow$ *Energy principle:* Equilibrium is stable if (sufficient) & only if (necessary)

$$W[\boldsymbol{\xi}] > 0$$

  for all displacements $\boldsymbol{\xi}(\mathbf{r})$ that are bound in norm and satisfy the boundary conditions.

Extended energy principle for interface plasmas

- Extension with **surface and vacuum terms:**

$$W[\boldsymbol{\xi}, \hat{\mathbf{Q}}] \;=\; -\tfrac{1}{2}\int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, dV = W^p[\boldsymbol{\xi}] + W^s[\xi_n] + W^v[\hat{\mathbf{Q}}]\,,$$

$$W^p[\xi_n] \;=\; \ldots \qquad \text{(old expression)}\,,$$

$$W^s[\xi_n] \;=\; \tfrac{1}{2}\int |\mathbf{n}\cdot\boldsymbol{\xi}|^2\, \mathbf{n}\cdot[\nabla(p+\tfrac{1}{2}B^2)]\, dS\,,$$

$$W^v[\hat{\mathbf{Q}}] \;=\; \tfrac{1}{2}\int |\hat{\mathbf{Q}}|^2\, d\hat{V}\,,$$

where $\boldsymbol{\xi}$ and $\hat{\mathbf{Q}}$ should satisfy **essential boundary conditions:**

(1) $\boldsymbol{\xi}$ regular *(on plasma volume V)* ,

(2) $\mathbf{n}\cdot\nabla\times(\boldsymbol{\xi}\times\hat{\mathbf{B}}) = \mathbf{n}\cdot\hat{\mathbf{Q}}$ *(1st interface condition on S)* ,

(3) $\mathbf{n}\cdot\hat{\mathbf{Q}} = 0$ *(on outer wall $\hat{W}$)* .

Application to Rayleigh-Taylor instability

- Plasma supported by vacuum field $\hat{\mathbf{B}}$: *Minimize* $W[\boldsymbol{\xi}, \hat{\mathbf{Q}}]$ *with divergence-free trial functions* $\boldsymbol{\xi}$ and $\hat{\mathbf{Q}}$ *satisfying essential BCs.* Fourier modes $\exp\,[\mathrm{i}(k_y y + k_z z)] \;\Rightarrow$

$$W^p = \tfrac{1}{2}\int |\mathbf{Q}|^2\, dV = \tfrac{1}{2}B_0^2 \int_0^a \big[\, k_z^2(|\xi_x|^2 + |\xi_y|^2) + |\xi_x' + \mathrm{i}k_y\xi_y|^2 \,\big]\, dx\,,$$

$$W^s = -\tfrac{1}{2}\rho_0 g \int |\mathbf{n}\cdot\boldsymbol{\xi}|^2\, dS = -\tfrac{1}{2}\rho_0 g |\xi_x(0)|^2\,,$$

$$W^v = \tfrac{1}{2}\int |\hat{\mathbf{Q}}|^2\, d\hat{V} = \tfrac{1}{2}\int_{-b}^0 \big[\, |\hat{Q}_x|^2 + |\hat{Q}_y|^2 + \frac{1}{k_z^2}|\hat{Q}_x' + \mathrm{i}k_y\hat{Q}_y|^2 \,\big]\, dx\,,$$

$$\mathrm{BCs:}\quad \xi_x(a) = 0\,,\quad \hat{Q}_x(0) = i(k_y\hat{B}_y + k_z\hat{B}_z)\,\xi_x(0)\,,\quad \hat{Q}_x(-b) = 0\,,$$

$$\mathrm{Norm:}\quad \mathrm{value}\ \xi_x(0)\,.$$

- $W \equiv \frac{1}{2}\int (F\xi'^2 + G\xi^2)\, dx \;\Rightarrow$ E.L.: $(F\xi')' - G\xi = 0 \;\Rightarrow\; W_{\min} = -\frac{1}{2}F\xi\xi'\big|_{x=0}$:

$$W_{\min} = C^2\big[\, (\mathbf{k}_0\cdot\mathbf{B})^2 - \rho_0 k_0 g \tanh(k_0 a) + (\mathbf{k}_0\cdot\hat{\mathbf{B}})^2 \tanh(k_0 a)\,/\,\tanh(k_0 b)\,\big]\,.$$

$\Rightarrow$ *Gravitational instability balanced by magnetic shear and conducting walls.*

Hydrodynamics of the solar interior

- *HD model of radiative equilibrium* (standard solar model):

$$\frac{dp}{dr} = -G\,\frac{\rho M}{r^2}\,, \qquad \frac{dM}{dr} = 4\pi r^2 \rho\,, \qquad (\,\text{and } \rho \sim \mu\, \mathrm{p/T}\,)\,,$$

$$\frac{dT}{dr} = -\frac{3}{64\pi\sigma}\,\frac{\kappa\rho L}{r^2 T^3}\,, \qquad \frac{dL}{dr} = 4\pi\, r^2 \rho\varepsilon\,,$$

with μ, κ, $\varepsilon\,(\rho, T)$ from microscopic data, and application of the BCs

$$p(R_\odot) \approx 0\,, \quad M(0) = 0\,, \quad T(R_\odot) \approx 0\,, \quad L(0) = 0\,,$$

yields a realistic solution for p, M, T, $L(r)$, *except for the convection zone.*

- In the convection zone, **the Schwarzschild criterion for convective stability,**

$$-T' \le -(T')_{\rm isentr.} \quad \Rightarrow \quad \rho' - \frac{\rho}{\gamma p} p' \le 0 \quad \left[\Rightarrow \quad \rho' g + \frac{\rho^2 g^2}{\gamma p} \le 0 \right],$$

is marginally satisfied, so that improved solution is obtained with the modified BC

$$-T' = -(T')_{\rm isentr.} \quad (\text{at } r = 0.713 R_\odot)\,.$$

Hydrodynamic waves & instabilities of a gravitating slab

- *Wave equation for gravito-acoustic waves in plane slab* (mock-up of solar oscillations):

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} - \nabla(\gamma p \nabla\cdot\boldsymbol{\xi}) - \rho\,\nabla(\mathbf{g}\cdot\boldsymbol{\xi}) + \rho\mathbf{g}\,\nabla\cdot\boldsymbol{\xi} = 0\,.$$

- Normal modes $\hat{\boldsymbol{\xi}}(x)\,\mathrm{e}^{\mathrm{i}(k_y y + k_z z - \omega t)}$, and elimination of $\hat{\xi}_y$ and $\hat{\xi}_z$, yields 2nd order ODE:

$$\frac{d}{dx}\left(\frac{\gamma p\,\rho\omega^2}{\rho\omega^2 - k_0^2\gamma p}\,\frac{d\hat{\xi}_x}{dx}\right) + \left[\rho\omega^2 - \frac{k_0^2\rho^2 g^2}{\rho\omega^2 - k_0^2\gamma p} - \left(\frac{k_0^2\gamma p\,\rho g}{\rho\omega^2 - k_0^2\gamma p}\right)'\right]\hat{\xi}_x = 0\,.$$

$\Rightarrow$ *Convective instabilities* with approximation $\rho\,|\omega^2| \ll k_0^2\gamma p$:

$$\omega^2 \approx \frac{k_0^2}{k_0^2 + q^2}N^2(x)\,,$$ squared BV freq. $N^2 > 0 \;\equiv$ Schwarzschild criterion.

$\Rightarrow$ *Gravito-acoustic waves* for exponential equilibrium:

$$\omega_{p,g}^2 = \tfrac{1}{2}k_{\rm eff}^2 c^2\left[1 \pm \sqrt{1 - \frac{4k_0^2N^2}{k_{\rm eff}^4 c^2}}\right]\,,\quad k_{\rm eff}^2 \equiv k_0^2 + q^2 + \tfrac{1}{4}\alpha^2\,.$$

$+$: p-modes
$-$: g-modes

- *Helioseismology* (spherical geometry): probing solar interior with p-modes.

MHD wave equation for gravitating plasma slab

- Similar to HD wave equation, with $\rho(x)$, $p(x)$, $\mathbf{g} = -\hat{g}\mathbf{e}_x$, but adding a *sheared* $\mathbf{B} = B_z(x)\,\mathbf{e}_y + B_z(x)\mathbf{e}_z$. Should satisfy equilibrium, $(p + \frac{1}{2}B^2)' = -\rho\hat{g}$.

- Start from $\mathbf{F}(\boldsymbol{\xi}) = -\rho\omega^2\boldsymbol{\xi}$, assume Fourier harmonics $\sim \exp\left[\mathrm{i}(k_y y + k_z z)\right]$, and exploit *field line projection* $\boldsymbol{\xi} = \xi(x)\mathbf{e}_x - \mathrm{i}\eta(x)\mathbf{e}_\perp(x) - \mathrm{i}\zeta(x)\mathbf{e}_\parallel(x)$ $\Rightarrow$ *Vector EVP:*

$$\begin{pmatrix} \partial_x(\gamma p + B^2)\partial_x - f^2B^2 & \partial_x g(\gamma p + B^2) + g\rho\hat{g} & \partial_x f\gamma p + f\rho\hat{g} \\ -\,g(\gamma p + B^2)\partial_x + g\rho\hat{g} & -g^2(\gamma p + B^2) - f^2B^2 & -gf\gamma p \\ -\,f\gamma p\partial_x + f\rho\hat{g} & -f\,g\,\gamma p & -f^2\gamma p \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = -\rho\omega^2 \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}.$$

- Eliminate η and ζ $\Rightarrow$ *Scalar EVP (2nd order ODE for ξ):*

$$\frac{d}{dx}\frac{N}{D}\frac{d\xi}{dx} + \left[\rho(\omega^2 - f^2b^2) + \rho'\hat{g} - k_0^2\rho\hat{g}^2\,\frac{\omega^2 - f^2b^2}{D} - \left\{\rho\hat{g}\,\frac{\omega^2(\omega^2 - f^2b^2)}{D}\right\}'\,\right]\xi = 0\,,$$

where $N \equiv \rho(b^2 + c^2)\,[\,\omega^2 - \omega_A^2(x)\,]\,[\,\omega^2 - \omega_S^2(x)\,]$ *(Alfvén/slow singularities),*

$D \equiv [\,\omega^2 - \omega_{s0}^2(x)\,]\,[\,\omega^2 - \omega_{f0}^2(x)\,]$ *(apparent singularities).*

- For exponential equilibrium $\Rightarrow$ cubic dispersion equation for *Gravito-MHD waves:*

$$(\omega^2 - f^2 b^2)[\,\omega^4 - k_{\rm eff}^2 (b^2 + c^2)\omega^2 + k_{\rm eff}^2 f^2 b^2 c^2 + k_0^2 c^2 N_B^2\,] + \alpha \hat{g}\, g^2 b^2 \omega^2 = 0\,.$$

- Simple explicit limits:

$$\mathbf{k}_0 \parallel \mathbf{B}:\ \ \omega_{2,3}^2 = \tfrac{1}{2} k_{\rm eff}^2 (b^2 + c^2) \left[1 \pm \sqrt{1 - \frac{4k_0^2 c^2 (k_{\rm eff}^2 b^2 + N_B^2)}{k_{\rm eff}^4 (b^2 + c^2)^2}} \right]$$

If $\omega_3^2 < 0$: *Parker inst.*

$$\mathbf{k}_0 \perp \mathbf{B}:\ \ \omega_{2,3}^2 = \tfrac{1}{2} k_{\rm eff}^2 (b^2 + c^2) \left[1 \pm \sqrt{1 - \frac{4k_0^2 N_m^2}{k_{\rm eff}^4 (b^2 + c^2)}} \right]$$

N_m^2: mod. BV freq.
$\omega_{2,3}^2 > 0$: *Stable.*

- For arbitrary directions of $\mathbf{k}_0$, the three waves couple, but the cubic dispersion equation can be solved explicitly by means of Cardano's solutions and plotted (see Book, Fig. 7.10). Important (as in any numerical investigation): introduction of *dimensionless parameters exploiting scale independence.*

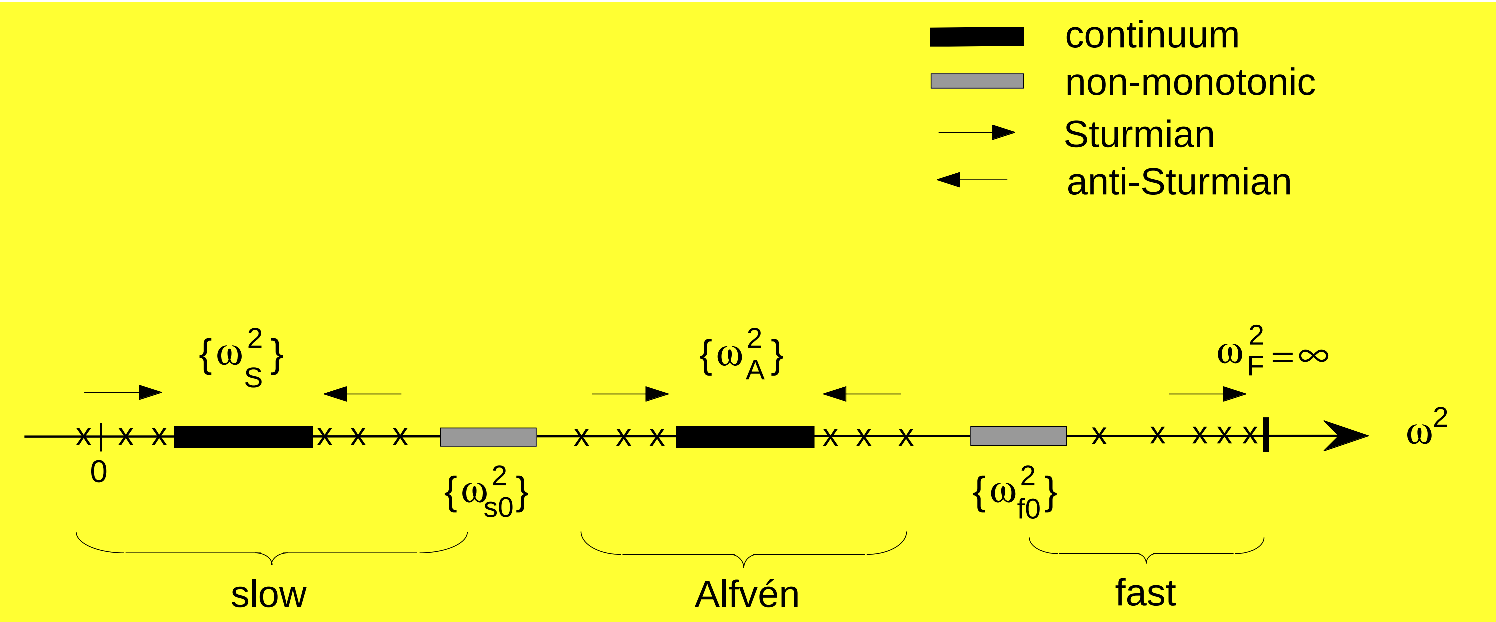
Continuous spectrum and spectral structure

- Main problem is the proper treatment of the singularities $N = 0$:

  Collections of *Alfvén & slow frequencies* $\{\omega_A^2(x)\}$ & $\{\omega_S^2(x)\}$ on $x_1 < x < x_2$ constitute *continuous spectra!*

- *At those frequencies,* the component ξ of the 'eigenfunction' has a 'small' jump and $\ln$ contribution, and *the components η or ζ exhibit δ-function* and x^{-1} singularity.

- The continuous spectra determine the *overall structure of the MHD spectrum:*



Gravitational instabilities

- Energy W minimized by solutions of the Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{F^2}{k_0^2}\frac{d\xi}{dx}\right) - \left(F^2 - \rho'\hat{g} - \frac{\rho^2\hat{g}^2}{\gamma p}\right)\xi = 0\,, \qquad F \equiv fB \equiv -\mathrm{i}\mathbf{B}\cdot\nabla\,,$$

which is a special case ($\omega^2 = 0$) of the spectral wave equation.

- *Absence of singularities: Oscillatory ξ $\Rightarrow$ unstable; ξ without zeros $\Rightarrow$ stable.* [Related to oscillation theorem for wave equation: spectrum monotonic for $\omega^2 < 0$.]

- *Presence of singularities $F = 0$: Solutions $\xi \sim s^\nu$ with (a) complex ν, or (b) real ν.* (a) Complex index implies *infinitely many unstable modes* ($\omega^2 = 0$ is a cluster point for $n = 1, 2, \ldots \infty$), in violation of necessary stability criterion for interchanges:

$$\rho'\hat{g} + \frac{\rho^2\hat{g}^2}{\gamma p} \;\; \left(\equiv -\rho N_{\mathrm{B}}^2\right) \;\leq \tfrac{1}{4}B^2\varphi'^2\,.$$

(b) If satisfied: *investigation of finite n solutions* needed. Since this involves extensive numerics, one may as well compute the growth rates from the full spectral equation.

Phenomenology of magnetic structures and dynamics

- **Standard solar model** (nuclear fusion in the core)

  $\Rightarrow$ helioseismology and detection of neutrinos (neutrino problem solved!).

- **Origin solar magnetism**: dynamo in convection zone (convectively unstable)

  $\Rightarrow$ solar cycle, sunspots (sunspot seismology);

  $\Rightarrow$ progress in computational MHD of flux tube dynamo.

- **Abundance of observations of solar magnetic structures and dynamics**

  $\Rightarrow$ photosphere / chromosphere / corona (granulation, solar flares, CMEs, etc.);

  $\Rightarrow$ see SOHO and TRACE websites.

- **Planetary magnetic fields, solar wind and space weather**

  $\Rightarrow$ geodynamo, journey through the solar system;

  $\Rightarrow$ space weather affects all planets: observed and simulated!

- **Astro-plasma physics**

  $\Rightarrow$ simulation of launching collimated astrophysical jets.

Consequences

- Abundant observations demonstrate importance of **magnetic flux conservation** and **dynamics of magnetic flux tubes**

  $\Rightarrow$ needs further elaboration of *1D cylindrical flux tube model* (Chapter 9).

- Solar wind and all astrophysical plasmas exhibit **flow-dominated plasmas**

  $\Rightarrow$ systematic theory of *waves and instabilities in plasmas with flow* (Chapter F).

- Magnetic flux tubes usually evolve to **very complex magnetic geometries**

  $\Rightarrow$ requires at least *2D toroidal geometry inspired by tokamak theory* (Chapter T).

- The embarrassing riches of observations, both in the laboratory and in astrophysics, will require the full toolbox of **advanced MHD** for future interpretation

  $\Rightarrow$ *dissipative effects* (reconnection, extended MHD models);

  $\Rightarrow$ *computational methods* (for both spectral theory and nonlinear evolution);

  $\Rightarrow$ *nonlinear dynamics* (transonic flow, turbulence).

Chapter 1. Introduction

- What is the importance of plasma physics?
- How are plasmas confined in the laboratory and in nature?
- Why are plasmas important in astrophysics?
- What is a plasma? [Give definitions with as many qualifications as are relevant.]
- Can you describe the ingredients for a valid macroscopic model of plasmas?

Chapter 2. Elements of Plasma Physics

- What are the different theoretical approaches of plasmas?
- What is the most important periodicities of the dynamics of magnetized plasmas? Memorize the expressions for these frequencies (if you hate memorizing, as most physicists do, just use dimensional arguments and your physical insight to figure out what the main variable should be, and how the frequency should scale with fundamental constants like the electron charge, etc.) and derive estimates of their orders of magnitude for different plasmas. [Use Tables B.3 and B.4 of MHDapp.]

- What is a magnetic mirror? Give astrophysical examples.
- What are the fundamental kinetic equations describing the dynamics of charged particles? What are the unknowns and what do they describe?
- How are those equations completed to describe the full picture of interaction of plasmas with electromagnetic fields?
- How are the kinetic equations reduced to variables in space and time only? What do you gain and what do you loose in this reduction?
- How is the infinite set of obtained equations truncated and then closed?
- What are plasma oscillations? What is the driving force of these oscillations? Again, try to figure out the analytical expression for the frequency and derive estimates (using Tables B.3 and B.4) for its order of magnitude for different plasmas.
- What are the main assumptions in the reduction of kinetic equations to fluid equations (two-fluid and MHD)?
- What are the assumptions in the reduction of the two-fluid to the one-fluid (MHD) picture? Which basic variables remain in the MHD picture?

- Write down the MHD equations. Again, try to do that by using physical arguments. [No panic if you do not succeed: just look them up.]
- Derive the dispersion equation for the Alfvén wave from these equations, making as many simplifications as you want. (Of course, with the constraint that the wave itself should survive).
- Obtain the basic equations for MHD equilibrium (again, from the general MHD equations), and apply them to the z-pinch. [Recall: the variable z refers to the direction of the current.] Inserting numbers from Table B.5, obtain the magnitude of the central pressure. How does it compare with the magnitude of the magnetic pressure? Where is the latter pressure exerted?
- How is the external kink mode stabilized in a tokamak? Can you indicate astrophysical situations where a similar stabilization mechanism might be effective?

Chapter 4. The ideal MHD model

- Describe how the ideal MHD equations can be posed as postulates, starting from the equations for conductive fluids and Maxwell's equations. Which effects are neglected in this reduction?

- What is the minimal number of basic unknowns of the ideal MHD model?
- Compare orders of magnitude of the different terms in the MHD momentum equation (again, exploiting the input of Tables B.3 and B.4). What do you conclude on the magnitude of the $\mathbf{j} \times \mathbf{B}$ force?
- How can one make the MHD equations dimensionless and what are the implications?
- Give the order of magnitude of the Alfvén speed for the different plasmas listed in Table 4.1.
- How are the MHD equations completed to a full model for magnetized plasmas?
- What is the basic constituent (= building block) of a magnetized plasma? Which simple fundamental equation is responsible for it?
- What is the most important conservation law of magnetized plasmas and which of the four MHD equations is responsible for it?
- What is the most powerful expression of the mentioned conservation law in the dynamics of magnetized plasmas? [Hint: Apply your "most important conservation law" to the "basic constituent" of the preceding question.]
- Which conservation laws are expressed by the remaining three MHD equations?

- How does one obtain jump conditions for discontinuities and shocks from the MHD differential equations?
- In contact discontinuities, the density jumps at the interface between two different plasmas. Can you give an astrophysically relevant example?
- In tangential discontinuities, most variables jump at the interface between the two plasmas, except for the normal velocity, the normal magnetic field, and the total pressure. Can you explain why? Can you give an example?
- In open magnetic flux tubes on the Sun, the plasma is line-tied at the photospheric end, but free to move on the other end. Can you indicate where those open flux tubes originate on the Sun and what is the physical significance of the open ends?

Chapter 5. Waves and characteristics

- Starting from the gas dynamic equations (MHD equations with $\mathbf{B} = 0$), can you indicate the steps involved in deriving the dispersion equation for sound waves?
- When the magnetic field is added, a similar reduction can be made to obtain the dispersion equation for the three MHD waves. Indicate what are the basic wave speeds encountered. How many unknowns would you encounter in the eigenvalue problem based on the primitive variables?

- Now reduce to the velocity representation. Why is that possible at all? How many modes do you expect? Can you name them? Which mode is the most significant one and why? Can you derive an expression for the frequency of that wave?
- Look up the general dispersion equation for the MHD waves. First, sketch the frequency as a function of the parallel wave number, keeping the perpendicular component fixed, and, next, make a sketch for the opposite case. What physical consequences can you draw from these two plots? In particular, pay attention to the asymptotic behavior for large wave numbers. Why would that be important?
- What is the difference between the phase velocity and the group velocity of waves? Can you describe the construction of the group diagram of Alfvén waves? What is the physical significance of that result?

Chapter 6. Spectral Theory

- Present two alternative approaches to the stability study of plasmas.
- Instead of the velocity representation of Chap. 5, a more powerful representation for inhomogeneous plasmas is obtained by means of the displacement variable ξ. Define that variable and show how the two representations are connected.

- How does $\boldsymbol{\xi}$ permit the integration of some of the linearized MHD equations, and how does it transform the equation of motion into the force operator form?
- Indicate how the introduction of normal modes transforms the equation of motion into a spectral problem. What is the relation with the stability problem of plasmas?
- Starting from the explicit form of the force operator (look up!), indicate what the different terms represent. How do you obtain the Alfvén wave from this expression?
- What is the most important property of the force operator for the spectrum of waves and instabilities of linearized MHD?
- Can you derive a formal expression for the potential energy W of perturbations in terms of the force operator? [Hint: exploit energy conservation and introduce the kinetic energy for the perturbations.] Also, give arguments for the sign and the constant in front of that expression.
- In the proof of the above-mentioned property of the force operator, quadratic forms are exploited. How can one obtain a more useful expression (than the formal one) for the potential energy from it?
- What is the meaning of the potential energy expression $W[\boldsymbol{\xi}]$ for stability? Indicate how you obtain necessary and sufficient stability criteria from it.

- Starting from the general expression for W (look up!), derive a necessary and sufficient stability condition for internal gravitational instabilities of a fluid ($\mathbf{B} = 0$).

- Consider a plane plasma slab bounded by a wall at $x = a$, supported against a constant vertical gravity field $\mathbf{g} = -\hat{g}\mathbf{e}_x$ by a vacuum magnetic field bounded by a wall at $x = -b$. Assume the plasma to be incompressible with constant density ρ_0 and magnetic field $\mathbf{B} = B_0\mathbf{e}_z$. Compute the equilibrium pressure distribution. Assume the vacuum magnetic field $\hat{\mathbf{B}}$ to be at an angle with the z-axis. How is the magnitude of $\hat{\mathbf{B}}$ related to the magnitude of $\mathbf{B}$?

- Now determine the stability of this configuration: Look up the expressions for the potential energy $W^p_{\rm inc}[\boldsymbol{\xi}]$ of an incompressible plasma, the surface energy $W^s[\mathbf{n} \cdot \boldsymbol{\xi}]$, and the vacuum potential energy $W^v[\hat{\mathbf{Q}}]$, and the essential boundary conditions connecting $\boldsymbol{\xi}$ and $\hat{\mathbf{Q}}$ at $x = a$, $x = 0$, and $x = -b$. Assume Fourier modes for the ignorable coordinates y and z. Reduce the total energy W to one-dimensional form, and minimize it with respect to the transverse components of $\boldsymbol{\xi}$ and $\hat{\mathbf{Q}}$. Carry out the final minimization by solving the Euler-Lagrange equations for ξ_x and $\hat{Q}_x$, and substitute those solutions back into W.

- Discuss the stability of the configuration from this final expression for W for short wavelength perturbations (so that the walls are effectively at infinity).

Chapter 7. Waves/instabilities inhomogeneous plasmas

- Describe the standard solar model: How and where is the solar luminosity produced? How is the energy produced transported through the Sun? What are the basic ingredients of radiative equilibrium? Where is the spherically symmetric transport exceeded by another kind of dynamics? Can you exploit that knowledge to formulate physical outer boundary conditions for the radiative equilibrium model?
- In the convection zone of the Sun, the Schwarzschild criterion for convective stability limits the absolute value of the temperature gradient that can be sustained there. What is the physical quantity that controls this limit? Can you transform this limit to an expression for the density and pressure gradients? How is this expression transformed again when the equilibrium condition is exploited?
- Derive the wave equation for gravito-acoustic waves in a plane slab geometry. How do you transform the vector equation into a second order ODE for the vertical displacement? Indicate the solution method for convective instabilities with growth rates that are much smaller than the time-scale of sound waves. Derive the dispersion equation of all gravito-acoustic oscillations for an exponentially stratified atmosphere with constant sound speed. What is the qualitative (physical and spectral) difference between the two kinds of solutions obtained?

- Derive the equilibrium condition for a plane slab with arbitrary direction of the magnetic field in a constant vertical gravitational field. How many arbitrary functions are required to specify the equilibrium?
- Indicate the basic steps in the derivation of the wave equation for the gravito-MHD waves of a plane gravitating slab in a magnetic field. Indicate what kind of singularities are associated with the coefficient in front of the highest derivative of the second order ODE obtained.
- Starting from this second order ODE (look up the explicit form), derive the full dispersion equation for the gravito-MHD waves of an exponentially stratified plasma slab with constant sound and Alfvén speeds. Obtain the solutions for parallel and perpendicular propagation. Which of the two cases corresponds to the Parker instability? What is the mechanism of this instability? Can you give an astrophysical application?
- Returning to the singularities of the spectral wave equation for the gravitating plasma slab, could you indicate how one solves such an equation? In the presence of singularities, the solutions of the eigenvalue problem are called 'improper'. Do you know why? What part of the spectrum do they correspond to? There are two kind of singularities that are physically quite different. What are the dominant components of these two types of solutions?

- Sketch the structure of an MHD spectrum for a weakly inhomogeneous plasma slab where the different parts are well separated. There are three sub-structures that correspond to the three types of MHD waves that are also present in a homogeneous plasma. What do they represent physically? What role do the continuous spectra play in this structure?
- Transform the general expression for the energy of MHD perturbations (look up the explicit form) to an expression for the plane gravitating plasma slab, using the field line projection scheme. Minimize with respect to the tangential components of the displacement vector. Obtain the Euler-Lagrange equation that provides the minimizing solutions of the resulting 1D energy integral.
- For a general equilibrium distribution, suppose you solve the mentioned Euler-Lagrange equation starting from one end of the vertical interval to the other end. Suppose you do not encounter singularities, but the solution ξ oscillates. What does this imply for stability of the gravitating slab? Can you indicate the connection with the eigenfrequency of the wave or instability?
- Suppose you encounter a singularity. How do you exploit this fact to reduce the problem to a local stability criterion? Which are the two competing contributions with respect to stability of the plasma slab?

Chapter 8. Magnetic structures

- Describe the main physical processes associated with each of the three internal layers of the Sun. In which of the three layers is the dynamo mechanism supposed to be located and why? Describe how such a dynamo would generate a dipolar magnetic field.
- Describe the main physical processes associated with each of the three external layers (the 'atmosphere') of the Sun. How is energy supposed to be transported upward in the corona and what could be the reason for the spectacular temperature rise there?
- Sketch the structure of the planetary magnetospheres due to the interaction of the solar wind with the dipolar magnetic fields of the planets. Disruptions of the magnetic structures on the solar surface (can you mention at least two kinds?) lead to magnetic disturbances of the magnetosphere of, e.g., the Earth. These disturbance are part of intense research on 'space weather'. Can you indicate why such a research program would be needed?