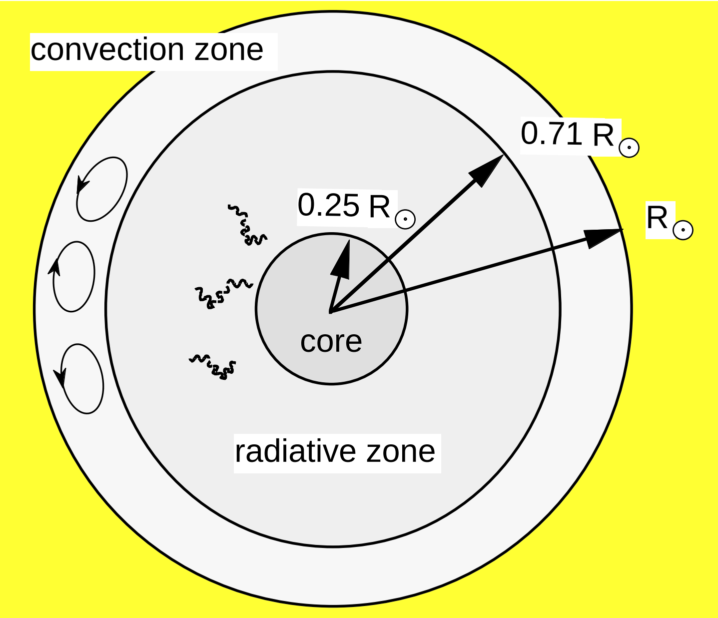
Chapter 7: Waves/instab. in inhomogeneous plasmas

Overview

- **Hydrodynamics of the solar interior:** radiative equilibrium model of the Sun, convection zone; [book: Sec. 7.1]
- **Hydrodynamic waves & instabilities of a gravitating slab:** HD wave equation, convective instabilities, gravito-acoustic waves, helioseismology; [book: Sec. 7.2]
- **MHD wave equation for a gravitating magnetized plasma slab:** derivation MHD wave equation for gravitating slab, gravito-MHD waves; [book: Sec. 7.3]
- **Continuous spectrum and spectral structure:** singular differential equations, Alfvén and slow continua, oscillation theorems; [book: Sec. 7.4]
- **Gravitational instabilities of plasmas with magnetic shear:** energy principle for gravitating slab, interchange instabilities in sheared/shearless magnetic fields. [book: Sec. 7.5]

Structure of the Sun

Standard Solar model:



- $R_\odot = 7.0 \times 10^8\,\mathrm{m},\ M_\odot = 2.0 \times 10^{30}\,\mathrm{kg}.$
- *Solar luminosity* (total power output):

$$L_\odot = 3.86 \times 10^{26}\,\mathrm{W}$$

$\Rightarrow$ heat flux at $1\,\mathrm{AU}$ ($= 1.5 \times 10^{11}\,\mathrm{m}$):

$$\frac{L_\odot}{4\pi \times (1\,\mathrm{AU})^2} = 1.36\,\mathrm{kW\,m^{-2}}$$ *(solar constant).*

- *Produced by p-p fusion reactions* in the core:

$$4\,\mathrm{p} \;\longrightarrow\; \mathrm{He}^4 + 2\,\mathrm{e}^+ + 2\,\nu_e\;(0.5\,\mathrm{MeV}) + 2\,\gamma\;(26.2\,\mathrm{MeV})\,.$$

- Energy γ radiation transported through *radiative zone* ($0.25R_\odot \le r \le 0.713R_\odot$) outward. Takes millions of years per photon, wavelengths shift to visible light.
- Radiative transport exceeded by convection in *convection zone* ($0.71R_\odot \le r \le R_\odot$).

Radiative equilibrium model

HD ($\equiv$ MHD with $\mathbf{B} = 0$) equations:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{v} = 0\,, \tag{1}$$

$$\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} + \nabla p - \rho\mathbf{g} = 0\,, \quad \text{where} \quad \mathbf{g} = -G\,\frac{M(r)}{r^2}\,\mathbf{e}_r\,, \quad \frac{dM}{dr} = 4\pi r^2\rho\,, \tag{2}$$

$$\rho\frac{\mathrm{D}e}{\mathrm{D}t} + p\nabla\cdot\mathbf{v} = \nabla\cdot[\,\lambda\nabla(kT)\,] + \rho\varepsilon\,, \quad \text{where} \quad e \equiv \frac{1}{\gamma-1}\frac{p}{\rho}\,. \tag{3}$$

New: *a) radiative transport governed by thermal conduction coefficient $\lambda(r)$,*

$$\lambda = \lambda(\rho, T) = (16\sigma/(3k)\,T^3/(\kappa\rho)\,, \quad \text{with opacity} \quad \kappa = \kappa(\rho, T)\,. \tag{4}$$

b) thermonuclear energy production per unit mass $\varepsilon(r)$:

$$\varepsilon = \varepsilon(X, \rho, T) = 0.25\,\rho X^2(10^6/T)^{2/3}\exp\{-33.8\,(10^6/T)^{1/3}\}\,. \tag{5}$$

c) equation of state mimicking particle species by mean molecular weight $\mu(r)$:

$$p \approx \frac{1+Z_c}{A}\frac{\rho kT}{m_p} \quad \Rightarrow \quad p = \frac{1}{\mu}\frac{\rho kT}{m_p}\,, \quad \text{where} \quad \mu = \mu(\rho, T)\,. \tag{6}$$

Mixture H^+ ($\mu = \frac{1}{2}$), He^{++} ($\mu = \frac{4}{3}$), heavier ions ($\mu \approx 2$): $\mu \approx (2X + \frac{3}{4}Y + \frac{1}{2}Z)^{-1}$.

Static equilibrium ($\mathbf{v} = 0\,,\ \partial/\partial t = 0$)

- *Hydrostatic equilibrium:*

$$\frac{dp}{dr} = -G\,\frac{\rho M}{r^2}\,, \qquad \frac{dM}{dr} = 4\pi r^2 \rho\,, \quad \left(\rho = \frac{m_p}{k}\,\frac{\mu p}{T} \right)\,. \tag{7}$$

- *Radiative equilibrium:*

$$\frac{1}{r^2}\frac{d}{dr}\Big[\, r^2 \lambda\,\frac{d}{dr}(kT)\,\Big] = -\rho\varepsilon\,, \quad \text{exploiting local luminosity } L$$

$$\Rightarrow\ \frac{dT}{dr} = -\frac{3}{64\pi\sigma}\,\frac{\kappa\rho L}{r^2 T^3}\,, \qquad \frac{dL}{dr} = 4\pi\, r^2 \rho\varepsilon\,. \tag{8}$$

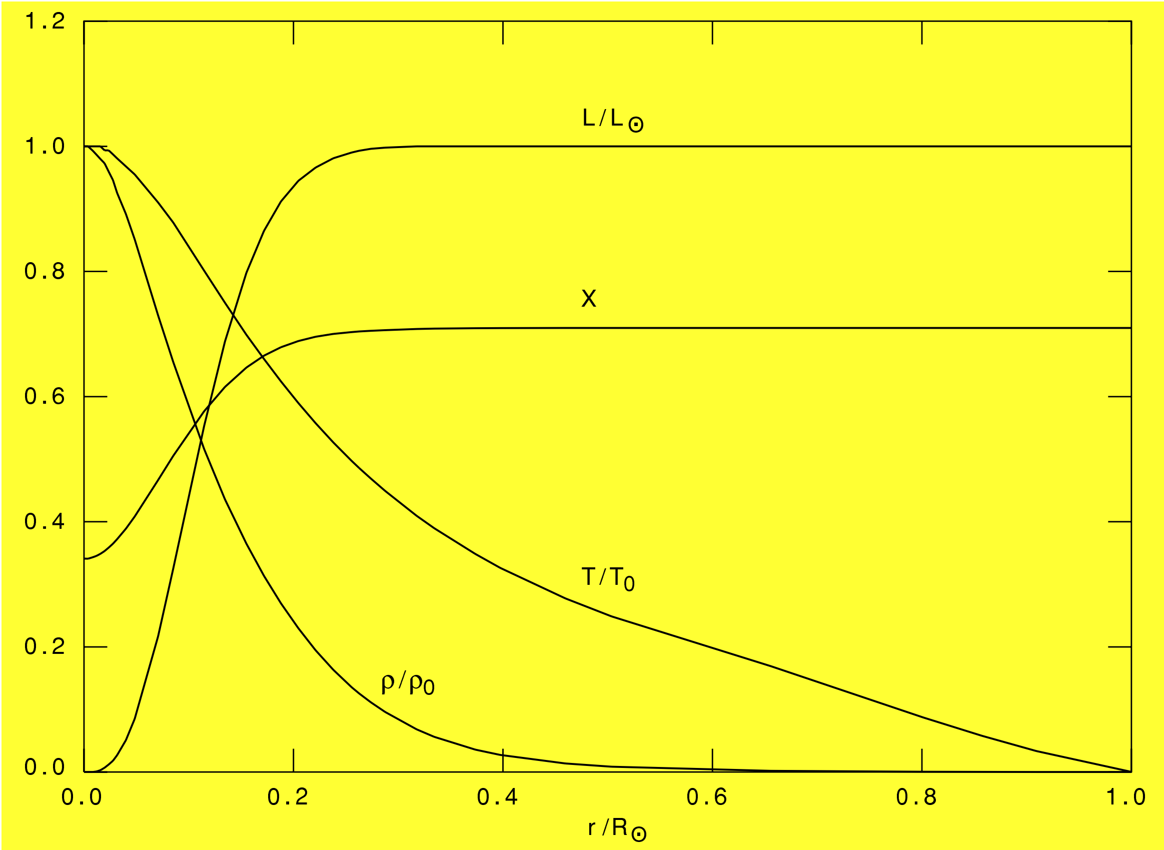
ODEs (7)–(8) are complete when supplemented with $\mu(\rho, T)$, $\kappa(\rho, T)$, and $\varepsilon(\rho, T)$ obtained from microscopic data on abundances, ionization, scattering, etc.

- *Boundary conditions:*

$$p(R_\odot) \approx 0\,, \quad M(0) = 0\,, \quad T(R_\odot) \approx 0\,, \quad L(0) = 0\,. \tag{9}$$

$\Rightarrow$ Provides realistic solutions, where values $p(0)$ and (0) correspond to thermonuclear burn, **but assumption of hydrostatic equilibrium up to the solar surface is wrong!**

Solution (Foukal, Solar Astrophysics, 1990):



Obtained with modified BC *at* $r = 0.713 R_\odot$*:*

$$-\frac{dT}{dr} = -\left(\frac{dT}{dr}\right)_{\rm isentr.}, \qquad (10)$$

since **convection zone is convectively unstable!**

Convection zone

- In outer layers of the Sun, cooling is so strong that absolute value of temperature gradient exceeds threshold given by Schwarzschild criterion for convective stability.
- Strong mixing in convection zone, $\mu \approx \mathrm{const}$, so that equation of state becomes

$$p = \mathcal{R}\rho T\,, \quad \text{with gas constant } \mathcal{R} \equiv (k/m_p)\,\mu^{-1} \tag{11}$$

$$\Rightarrow \quad -T' = \frac{1}{\mathcal{R}}\Big(\frac{p}{\rho^2}\rho' - \frac{1}{\rho}p'\Big) \quad \Big[= \frac{1}{\mathcal{R}}\Big(\frac{p}{\rho^2}\rho' + g\Big)\,, \text{ using equil. } p' = -\rho g \Big]. \tag{12}$$

- Neutrally stable motions only if fluid is *isentropic*:

$$S \equiv p\rho^{-\gamma} = \mathrm{const} \;\Rightarrow\; -(T')_{\mathrm{isentr.}} = -\frac{1}{\mathcal{R}}\frac{\gamma-1}{\gamma}\frac{1}{\rho}p' \quad \Big[= \frac{1}{\mathcal{R}}\frac{\gamma-1}{\gamma}g \Big]. \tag{13}$$

- Convective instability when actual temperature gradient $-T'$ exceeds this value, i.e. when **Schwarzschild criterion for convective stability**,

$$-T' \le -(T')_{\mathrm{isentr.}} \quad \Rightarrow \quad \rho' - \frac{\rho}{\gamma p}p' \le 0 \quad \Big[\Rightarrow \quad \rho' g + \frac{\rho^2 g^2}{\gamma p} \le 0 \Big], \tag{14}$$

violated. $\Rightarrow$ Recover criterion for gravitational stability [in square brackets]:

Convective and gravitational (Rayleigh–Taylor) instabilities are the same!

HD wave equation

- Gravity waves unstable if Schwarzschild criterion for convective instability is violated.
  $\Rightarrow$ Consider more general *solar oscillations*, neglecting $\mathbf{B}$ and spherical geometry:
  Gravito-acoustic waves in planar stratification (depend on vertical coord. x).

- Equilibrium of plane slab with constant external gravity field $\mathbf{g} = (-g, 0, 0)$:

$$\nabla p_0 = \rho_0 \mathbf{g} \quad \Rightarrow \quad p_0'(x) = -\rho_0(x) g \,. \tag{15}$$

- Linearize HD equations:

$$\frac{\partial \rho_1}{\partial t} + \mathbf{v}_1 \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \mathbf{v}_1 = 0 \,, \tag{16}$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + \nabla p_1 - \rho_1 \mathbf{g} = 0 \,, \tag{17}$$

$$\frac{\partial p_1}{\partial t} + \mathbf{v}_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \mathbf{v}_1 = 0 \,. \tag{18}$$

- With $\mathbf{v}_1 = \partial \boldsymbol{\xi} / \partial t$ $\Rightarrow$ **Wave equation for gravito-acoustic waves in plane slab:**

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} - \nabla(\gamma p \nabla \cdot \boldsymbol{\xi}) - \rho \nabla(\mathbf{g} \cdot \boldsymbol{\xi}) + \rho \mathbf{g} \, \nabla \cdot \boldsymbol{\xi} = 0 \,. \tag{19}$$

- Normal mode solutions $\boldsymbol{\xi}(\mathbf{r},t) = \hat{\boldsymbol{\xi}}(x)\,\mathrm{e}^{i(k_y y + k_z z - \omega t)}$ $\Rightarrow$ matrix EVP:

$$\begin{pmatrix} \rho\omega^2 + \frac{d}{dx}\gamma p \frac{d}{dx} & \mathrm{i}k_y(\frac{d}{dx}\gamma p + \rho g) & \mathrm{i}k_z(\frac{d}{dx}\gamma p + \rho g) \\ \mathrm{i}k_y\big(\gamma p \frac{d}{dx} - \rho g\big) & \rho\omega^2 - k_y^2\gamma p & -k_y k_z \gamma p \\ \mathrm{i}k_z\big(\gamma p \frac{d}{dx} - \rho g\big) & -k_y k_z \gamma p & \rho\omega^2 - k_z^2 \gamma p \end{pmatrix} \begin{pmatrix} \hat{\xi}_x \\ \hat{\xi}_y \\ \hat{\xi}_z \end{pmatrix} = 0\,. \quad (20)$$

- Rotate coordinate system so that $k_z = 0$ $\Rightarrow$ $\rho\omega^2\hat{\xi}_z = 0$, express $\hat{\xi}_y$ in $\hat{\xi}'_x$ and $\hat{\xi}_x$, and insert in first component of Eq. (20) $\Rightarrow$ *Second order ODE for $\hat{\xi}_x$:*

$$\frac{d}{dx}\left(\frac{\gamma p\,\rho\omega^2}{\rho\omega^2 - k_0^2\gamma p}\frac{d\hat{\xi}_x}{dx}\right) + \left[\rho\omega^2 - \frac{k_0^2\rho^2 g^2}{\rho\omega^2 - k_0^2\gamma p} - \big(\frac{k_0^2\gamma p\,\rho g}{\rho\omega^2 - k_0^2\gamma p}\big)'\right]\hat{\xi}_x = 0\,. \quad (21)$$

- Imposing rigid boundary conditions,

$$\hat{\xi}_x(x\!=\!0) = \hat{\xi}_x(x\!=\!a) = 0\,, \quad (22)$$

where $x = 0$ corresponds to center and $x = a$ to surface of the Sun, is OK as long as modes are sufficiently localized *(cavity modes)*.

Convective instabilities

- Since time-scales gravitational instabilities much longer than time-scales acoustic oscillations, assume $\rho\,|\omega^2| \ll k_0^2 \gamma p$. Wave equation (21) simplifies to

$$\frac{d}{dx}\Big(\frac{\rho\omega^2}{k_0^2}\frac{d\hat{\xi}_x}{dx}\Big) - \rho(\,\omega^2 - N^2)\,\hat{\xi}_x = 0\,, \qquad (23)$$

with *Brunt–Väisäläa frequency:*

$$N^2 \equiv -g\,\Big(\frac{1}{\rho}\rho' - \frac{1}{\gamma p}p'\Big) \qquad \Big[= -\frac{1}{\rho}\Big(\rho' g + \frac{\rho^2 g^2}{\gamma p}\Big)\Big]\,. \qquad (24)$$

- Assuming rapid spatially oscillatory modes $\hat{\xi}_x(x) \sim \exp(\mathrm{i}qx)$, with $qa \gg 1$, gives estimate for the eigenfrequencies of local instabilities:

$$\omega^2 \approx {\omega_{\mathrm{c}}}^2 \equiv \frac{k_0^2}{k_0^2 + q^2}\,N^2(x)\,. \qquad (25)$$

$\Rightarrow$ *System locally unstable in range of x where $N^2 < 0$,* which is nothing else but the Schwarzschild criterion. $\Rightarrow$ Convective instabilities grow as $\exp(\sqrt{-{\omega_{\mathrm{c}}}^2}\,t)$.

Gravito-acoustic waves

- *Exponentially stratified medium with constant sound speed:*

$$\rho = \rho_0 \mathrm{e}^{-\alpha x}\,, \quad p = p_0 \mathrm{e}^{-\alpha x} \quad \Rightarrow \; c^2 = \frac{\gamma p}{\rho} = \frac{\gamma p_0}{\rho_0} = \mathrm{const}\,. \tag{26}$$

$$p' = -\alpha p = -\rho g \quad \Rightarrow \; \alpha = \frac{\rho g}{p} = \frac{\rho_0 g}{p_0} = \frac{\gamma g}{c^2} = \mathrm{const}\,. \tag{27}$$

Spectral equation (21) reduces to

$$\frac{c^2\omega^2}{\omega^2 - k_0^2 c^2}\frac{d}{dx}\left(\mathrm{e}^{-\alpha x}\frac{d\hat{\xi}_x}{dx}\right) + \left(\omega^2 - \frac{k_0^2 g^2}{\omega^2 - k_0^2 c^2} + \alpha\,\frac{k_0^2 c^2 g}{\omega^2 - k_0^2 c^2}\right)\mathrm{e}^{-\alpha x}\,\hat{\xi}_x = 0. \tag{28}$$

- Squared Brunt–Väisäläa frequency simplifies to

$$N^2 = \alpha g - \frac{g^2}{c^2} = (\gamma - 1)\frac{g^2}{c^2} > 0 \quad \Rightarrow \; \text{only stable waves.} \tag{29}$$

Eq. (28) transforms to

$$\frac{d^2\hat{\xi}_x}{dx^2} - \alpha\,\frac{d\hat{\xi}_x}{dx} + \frac{\omega^4 - k_0^2 c^2\,\omega^2 + k_0^2 c^2 N^2}{c^2\,\omega^2}\,\hat{\xi}_x = 0\,, \tag{30}$$

which is a differential equation with constant coefficients: solution is trivial.

- Solutions:

$$\hat{\xi}_x = C\mathrm{e}^{(\frac{1}{2}\alpha \pm \mathrm{i}q)x}\,, \qquad q \equiv \sqrt{-\frac{1}{4}\alpha^2 + \frac{\omega^4 - k_0^2c^2\,\omega^2 + k_0^2c^2N^2}{c^2\,\omega^2}}\,. \tag{31}$$

Expression under square root positive for oscillatory solutions satisfying BCs (22) with quantized q:

$$qa = n\pi \quad (n = 1, 2, \ldots)\,. \tag{32}$$

- *Dispersion equation of gravito-acoustic waves* from inversion of Eq. (31) for q :

$$\omega^4 - (k_0^2 + q^2 + \tfrac{1}{4}\alpha^2)c^2\omega^2 + k_0^2c^2N^2 = 0\,, \tag{33}$$

with solutions

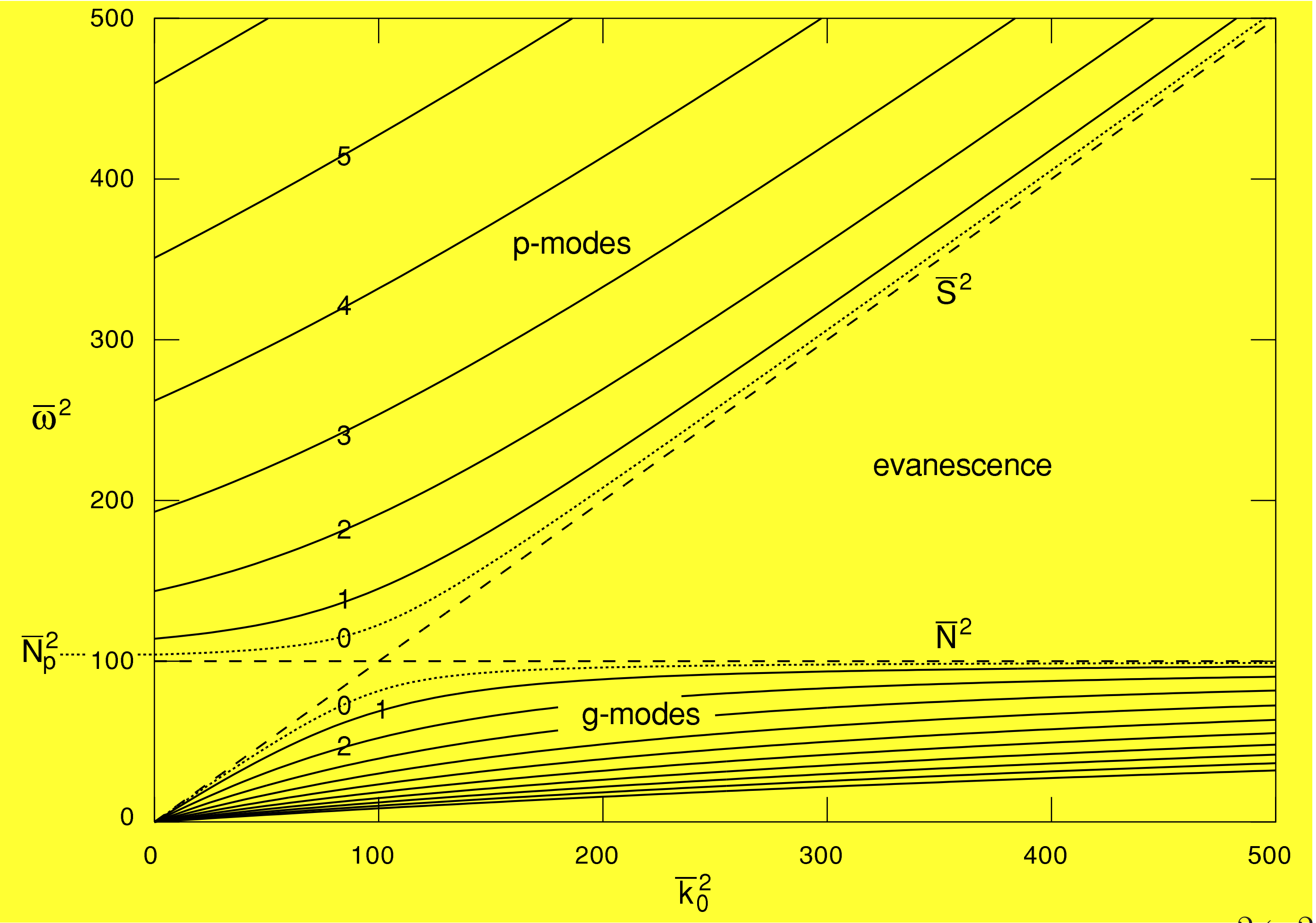
$$\omega_{p,g}^2 = \tfrac{1}{2}k_{\mathrm{eff}}^2c^2\left[1 \pm \sqrt{1 - \frac{4k_0^2N^2}{k_{\mathrm{eff}}^4c^2}}\right]\,, \quad k_{\mathrm{eff}}^2 \equiv k_0^2 + q^2 + \tfrac{1}{4}\alpha^2\,, \tag{34}$$

where k_{eff} is the effective total ‘wave number’ and k_0 is the horizontal wave number.

  – Branch with $+$ sign: *acoustic waves* or *p-modes* (pressure driven);

  – Branch with $-$ sign: *gravity waves* or *g-modes* (gravity driven).

Dispersion diagram p- and g-modes

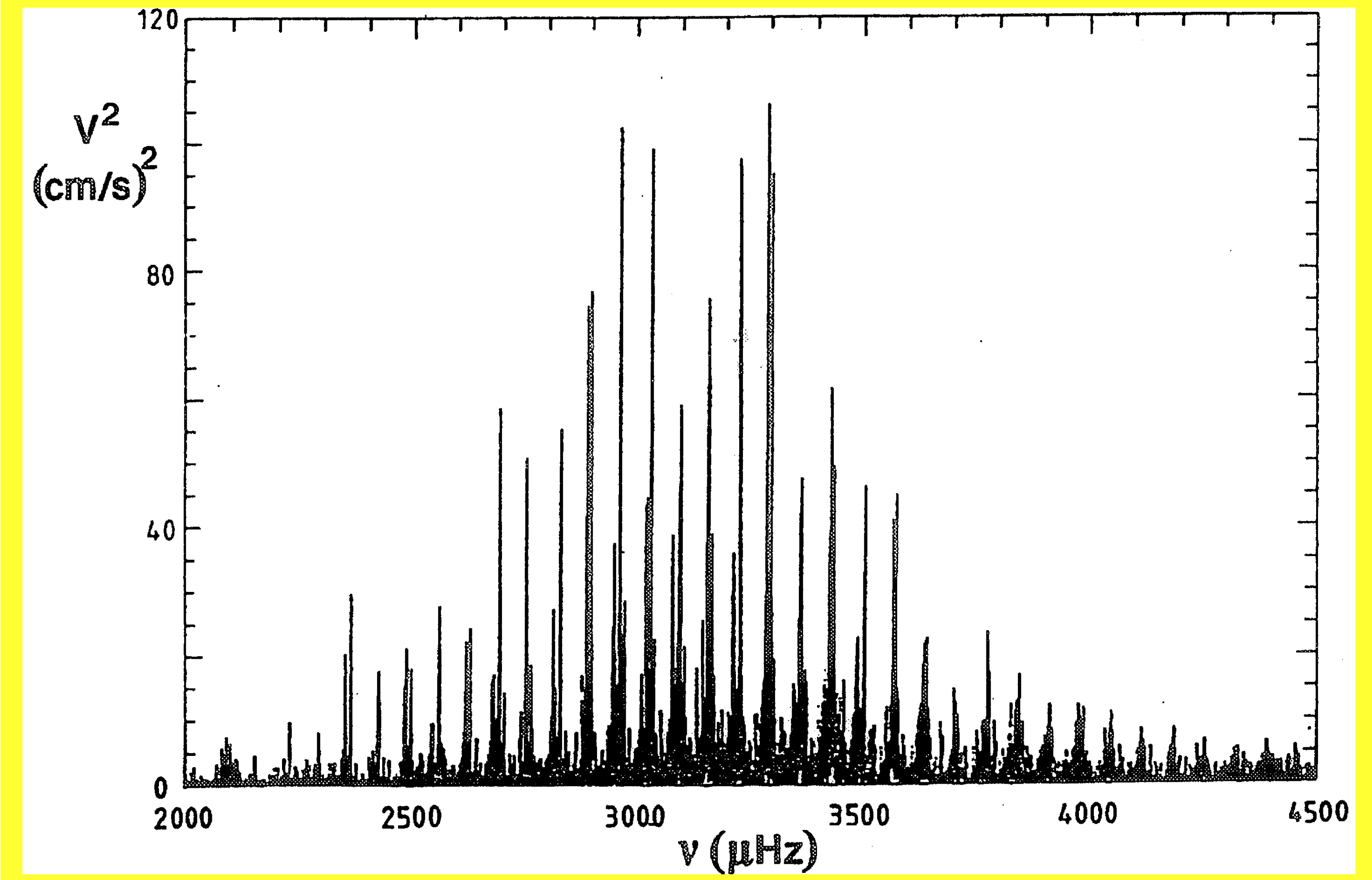


- Frequencies p-modes *increase* monotonically, clustering at ∞: $\omega_p^2(q^2) \uparrow \omega_P^2 = \infty$.

  Frequencies g-modes *decrease* monotonically, clustering at 0: $\omega_g^2(q^2) \downarrow \omega_G^2 = 0$.

Helioseismology

- Power spectrum of solar oscillations, from Doppler velocity measurements in light integrated over solar disk (Christensen-Dalsgaard, *Stellar Oscillations,* 1989):



⇒ Powerful tool for probing the interior of the sun!

- Done by comparison with theoretically calculated spectrum for standard solar model (of course, spherical geometry) (Christensen-Dalsgaard, 1989).

- Orders of magnitude :

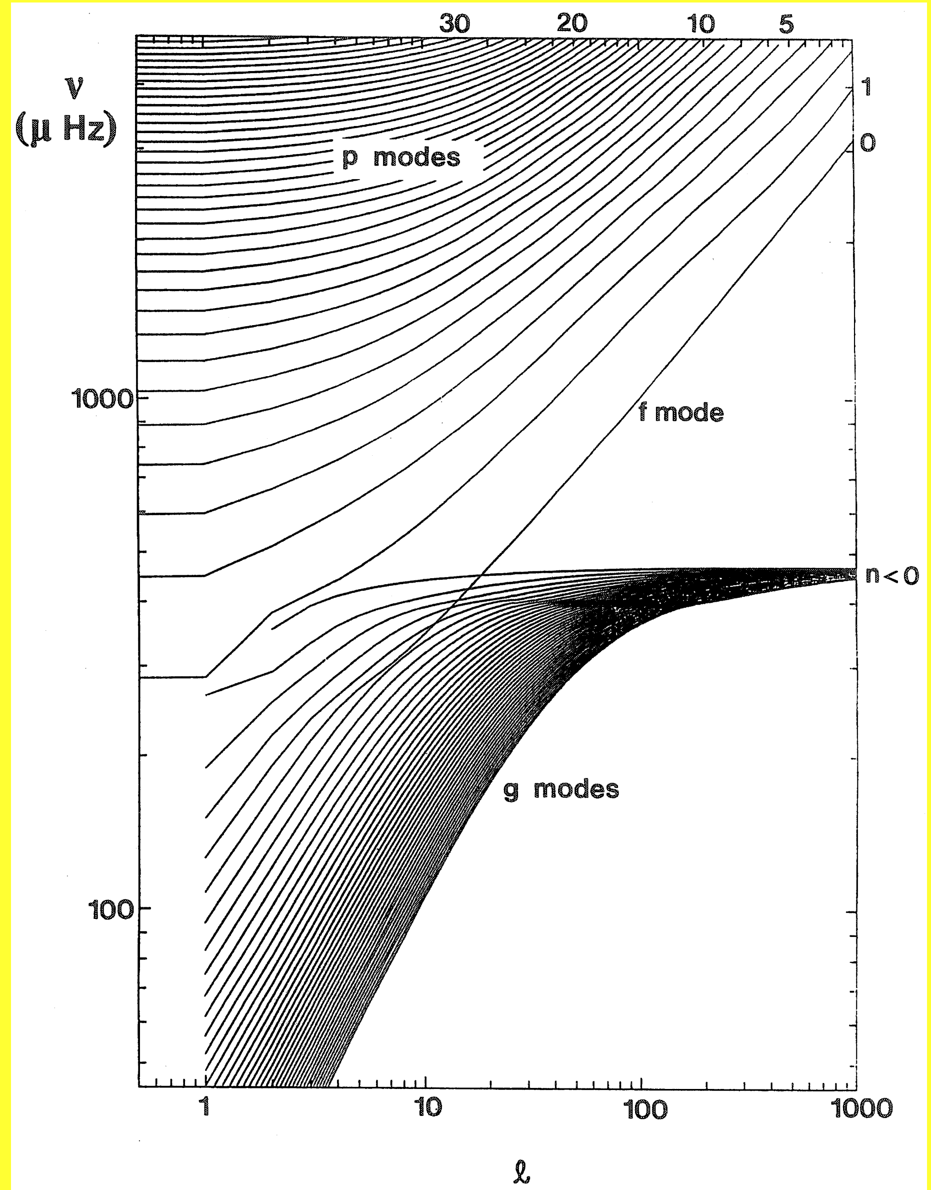
  $\tau \sim 5\,\mathrm{min} \quad \Rightarrow \quad \nu \sim 3\,\mathrm{mHz}$

  $\tilde{v}_r < 1\mathrm{km/s} \;\approx\; 5 \times 10^{-4} R_\odot / 5\,\mathrm{min}$

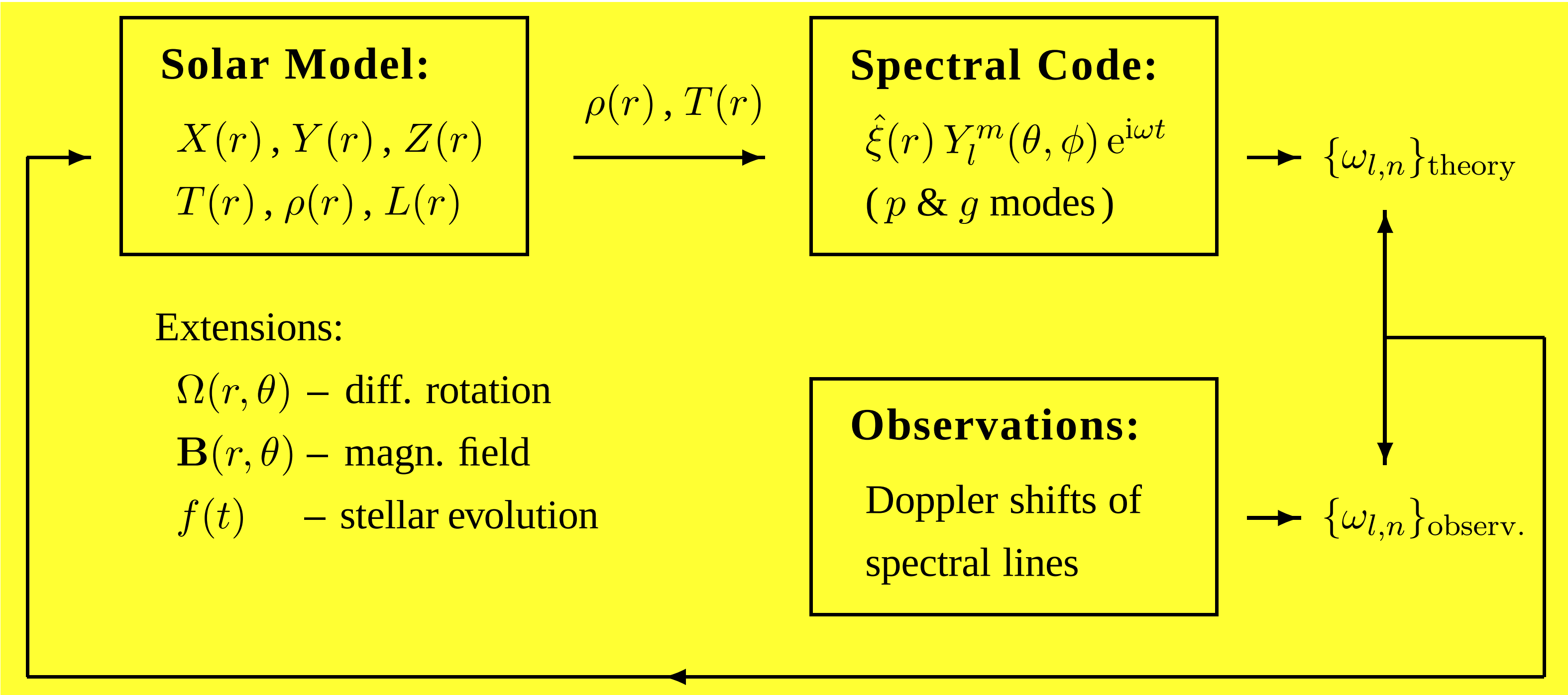
  $\Rightarrow$ linear theory OK!

- p-modes of low order l penetrate deep in the Sun, high l modes are localized on outside. g-modes are *cavity modes* trapped deeper than convection zone and, hence, quite difficult to observe.

- Frequencies deduced from the Doppler shifts of spectral lines agree with calculated ones for p-modes to within 0.1%!



Systematics of helioseismology



- Similar activities:

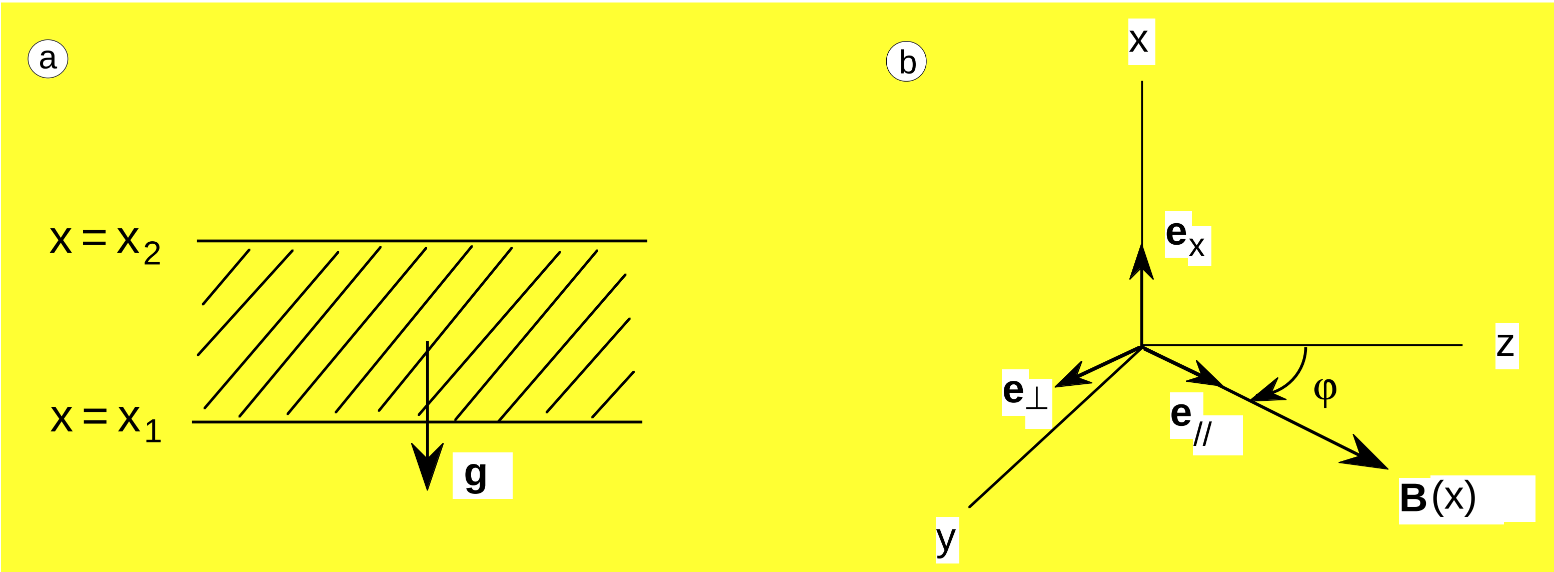
  – *MHD spectroscopy* for laboratory fusion plasmas (Goedbloed et al., 1993),

  – *Sunspot seismology* (Bogdan and Braun, 1995),

  – *Magneto-seismology of accretion disks* (Keppens et al., 2002).

Gravitating magnetized plasma slab

- Next step is *addition of magnetic field* (not spherical geometry since $\mathbf{B}$ does not fit!):



- Equilibrium (in between two bounding plates):

$$\mathbf{B} = B_y(x)\,\mathbf{e}_y + B_z(x)\,\mathbf{e}_z\,, \qquad \rho = \rho(x)\,, \qquad p = p(x)\,. \tag{35}$$

$$\mathbf{j} = \nabla \times \mathbf{B} = -B_z'(x)\,\mathbf{e}_y + B_y'(x)\,\mathbf{e}_z\,, \tag{36}$$

$$\mathbf{g} = -\nabla\Phi = -\hat{g}\mathbf{e}_x \quad \Rightarrow \quad (p + \tfrac{1}{2}B^2)' = -\rho\hat{g}\,. \tag{37}$$

⇒ *Generic 1D model for inhomogeneous plasmas.*

- Starting point is the general *MHD spectral equation:*

$$\mathbf{F}(\boldsymbol{\xi}) \equiv -\nabla\pi - \mathbf{B}\times(\nabla\times\mathbf{Q}) + (\nabla\times\mathbf{B})\times\mathbf{Q} + \nabla\Phi\,\nabla\cdot(\rho\boldsymbol{\xi}) = \rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\rho\omega^2\boldsymbol{\xi}\,, \quad (38)$$

$$\text{where} \qquad \pi = -\gamma p \nabla\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla p\,, \qquad \mathbf{Q} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B})\,. \quad (39)$$

Aside:

- Recall *homogeneous* plasmas (Chap. 5) with plane wave solutions $\hat{\boldsymbol{\xi}}(\mathbf{k})\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})$:

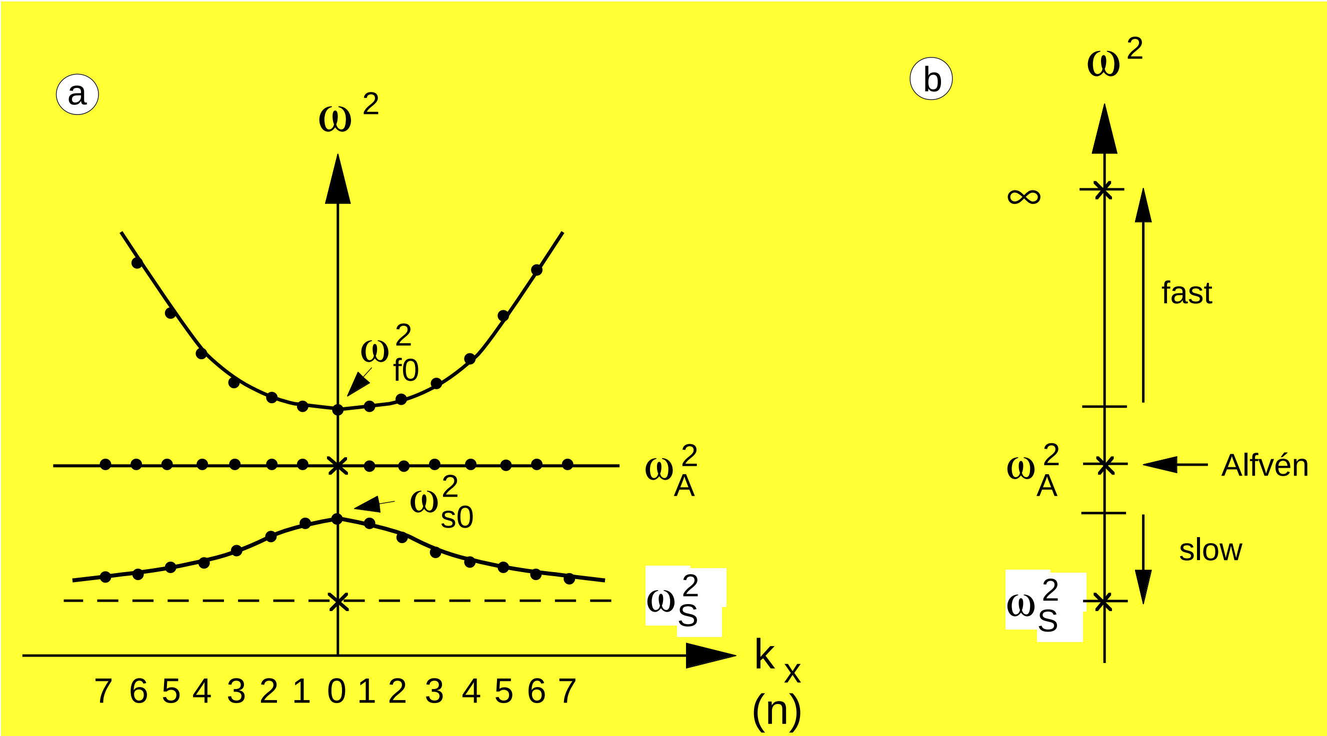
$$\rho^{-1}\mathbf{F}(\hat{\boldsymbol{\xi}}) = \big[-(\mathbf{k}\cdot\mathbf{b})^2\,\mathbf{I} - (b^2+c^2)\,\mathbf{kk} + \mathbf{k}\cdot\mathbf{b}\,(\mathbf{kb}+\mathbf{bk})\big]\cdot\hat{\boldsymbol{\xi}} = -\omega^2\hat{\boldsymbol{\xi}}\,. \quad (40)$$

In components:

$$\begin{pmatrix} -k_x^2(b^2+c^2) - k_z^2 b^2 & -k_x k_y(b^2+c^2) & -k_x k_z c^2 \\ -k_x k_y(b^2+c^2) & -k_y^2(b^2+c^2) - k_z^2 b^2 & -k_y k_z c^2 \\ -k_x k_z c^2 & -k_y k_z c^2 & -k_z^2 c^2 \end{pmatrix} \begin{pmatrix} \xi_x \\ \xi_y \\ \xi_z \end{pmatrix} = -\omega^2 \begin{pmatrix} \xi_x \\ \xi_y \\ \xi_z \end{pmatrix}. \quad (41)$$

Corresponds to Eq. (5.35) [book (5.52)] with $k_y \neq 0$: Coordinate system rotated to distinguish between k_x (becomes differential operator in inhomogeneous systems) and k_y (remains number).

- Dispersion diagram $\omega^2(k_x)$ exhibits relevant asymptotics for $k_x \to \infty$:



Yields the essential spectrum:

$$\omega_F^2 \equiv \lim_{k_x\to\infty} \omega_f^2 \approx \lim_{k_x\to\infty} k_x^2(b^2+c^2) = \infty\,, \quad \textit{(fast cluster point)} \tag{42}$$

$$\omega_A^2 \equiv \lim_{k_x\to\infty} \omega_a^2 = \omega_a^2 = k_\parallel^2 b^2\,, \quad \textit{(Alfvén infinitely degenerate)} \tag{43}$$

$$\omega_S^2 \equiv \lim_{k_x\to\infty} \omega_s^2 = k_\parallel^2 \frac{b^2c^2}{b^2+c^2}\,. \quad \textit{(slow cluster point)} \tag{44}$$

End aside

Back to **inhomogeneous plasmas:**

- *Fourier harmonics* $\hat{\boldsymbol{\xi}}(x; k_y, k_z) \exp\left[\mathrm{i}(k_y y + k_z z)\right]$, keep differential operators d/dx.
- *Field line projection* in normal, perpendicular, and parallel directions:

$$\mathbf{e}_x \equiv \nabla x \,, \qquad \mathbf{e}_\perp(x) \equiv (\mathbf{B}/B) \times \mathbf{e}_x \,, \qquad \mathbf{e}_\parallel(x) \equiv \mathbf{B}/B \,, \tag{45}$$

$$\partial_x \equiv d/dx \,,$$

$$\nabla = \mathbf{e}_x \partial_x + \mathrm{i}\, \mathbf{e}_\perp\, g + \mathrm{i}\, \mathbf{e}_\parallel\, f \,, \quad g(x) \equiv -\mathrm{i}\mathbf{e}_\perp \cdot \nabla = (k_y B_z - k_z B_y)/B \,, \tag{46}$$

$$f(x) \equiv -\mathrm{i}\mathbf{e}_\parallel \cdot \nabla = (k_y B_y + k_z B_z)/B \,,$$

$$\boldsymbol{\xi} = \xi\, \mathbf{e}_x - \mathrm{i}\eta\, \mathbf{e}_\perp - \mathrm{i}\zeta\, \mathbf{e}_\parallel \,. \tag{47}$$

MHD spectral equation (38) $+$ extensive algebra! $\Rightarrow$ *Vector formulation of EVP:*

$$\begin{pmatrix} \frac{d}{dx}(\gamma p + B^2)\frac{d}{dx} - f^2 B^2 & \frac{d}{dx} g(\gamma p + B^2) + g\rho\hat{g} & \frac{d}{dx} f\gamma p + f\rho\hat{g} \\ -\, g(\gamma p + B^2)\frac{d}{dx} + g\rho\hat{g} & -g^2(\gamma p + B^2) - f^2 B^2 & -g f \gamma p \\ -\, f\gamma p \frac{d}{dx} + f\rho\hat{g} & -f\, g\, \gamma p & -f^2 \gamma p \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = -\rho\omega^2 \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} . \tag{48}$$

- Eliminate perpendicular and parallel components η and ζ:

$$\begin{aligned} \eta &= g\,\frac{[(b^2+c^2)\omega^2 - f^2b^2c^2]\,\xi' - \hat{g}\,\omega^2\,\xi}{D}\,, \\ \zeta &= f\,\frac{c^2(\omega^2 - f^2b^2)\,\xi' - \hat{g}\,(\omega^2 - k_0^2b^2)\,\xi}{D}\,. \end{aligned} \tag{49}$$

- Substitute in 1st component $\Rightarrow$ *2nd order ODE formulation EVP* [cf. HD Eq. (21)]:

$$\frac{d}{dx}\frac{N}{D}\frac{d\xi}{dx} + \left[\rho(\omega^2 - f^2b^2) + \rho'\hat{g} - k_0^2\rho\hat{g}^2\,\frac{\omega^2 - f^2b^2}{D} - \left\{\rho\hat{g}\,\frac{\omega^2(\omega^2 - f^2b^2)}{D}\right\}'\,\right]\xi = 0, \tag{50}$$

with singular factors

$$\begin{aligned} N &= N(x;\omega^2) \;\equiv\; \rho(\omega^2 - f^2b^2)\,\left[\,(b^2+c^2)\,\omega^2 - f^2b^2c^2\,\right]\,, \\ D &= D(x;\omega^2) \;\equiv\; \omega^4 - k_0^2(b^2+c^2)\,\omega^2 + k_0^2f^2b^2c^2\,. \end{aligned} \tag{51}$$

Model I boundary conditions:

$$\xi(x_1) = \xi(x_2) = 0\,. \tag{52}$$

- This completes formulation EVP, remains: *study singularities* + *explicit solutions!*

Singular frequencies

- ODE (50) describes all *gravito-magnetohydrodynamic modes* of a gravitating magnetized plasma slab with arbitrary equilibrium profiles.
- Factor in front of highest derivative is crucial for local behavior:

$$\frac{N}{D} = \rho(b^2 + c^2)\,\frac{[\,\omega^2 - \omega_A^2(x)\,]\,[\,\omega^2 - \omega_S^2(x)\,]}{[\,\omega^2 - \omega_{s0}^2(x)\,]\,[\,\omega^2 - \omega_{f0}^2(x)\,]}\,. \tag{53}$$

Alfvén and slow magnetosonic singularities (continuous spectra) for $N \to 0$:

$$\omega_A^2(x) \equiv f^2 b^2 \equiv F^2/\rho\,, \qquad \omega_S^2(x) \equiv f^2 \frac{b^2c^2}{b^2+c^2} \equiv \frac{\gamma p}{\gamma p + B^2}\,\omega_A^2(x)\,. \tag{54}$$

Slow and fast turning point frequencies (apparent singularities) for $D \to 0$:

$$\omega_{s0,f0}^2(x) \equiv \tfrac{1}{2}k_0^2(b^2+c^2)\Big[\,1 \pm \sqrt{1 - \frac{4f^2b^2c^2}{k_0^2(b^2+c^2)^2}}\,\Big]\,. \tag{55}$$

- Function $F(x) \equiv -\mathrm{i}\mathbf{B}\cdot\nabla = \mathbf{k}_0\cdot\mathbf{B}$: gradient operator parallel to magnetic field (important in stability studies!).

Gravito-MHD waves

- Again, *exponentially stratified medium with constant sound and Alfvén speed:*

$$\rho = \rho_0 \mathrm{e}^{-\alpha x}, \quad p = p_0 \mathrm{e}^{-\alpha x}, \quad \mathbf{B} = B_0 \mathrm{e}^{-\frac{1}{2}\alpha x}\mathbf{e}_z \quad \Rightarrow \quad c^2 = \frac{\gamma p_0}{\rho_0}, \quad b^2 = \frac{B_0^2}{\rho_0}. \tag{56}$$

All singularities squeezed into constants, spectral equation (50) reduces to:

$$\frac{N_0}{\rho_0 D_0}\frac{d}{dx}\left(\mathrm{e}^{-\alpha x}\frac{d\xi}{dx}\right) + \Big[\omega^2 - f^2 b^2 - k_0^2 \hat{g}^2 \frac{\omega^2 - f^2 b^2}{D_0} - \alpha\hat{g} + \alpha\hat{g}\frac{\omega^2(\omega^2 - f^2 b^2)}{D_0}\Big]\mathrm{e}^{-\alpha x}\xi = 0\,. \tag{57}$$

- As in HD, solved by $\xi = C\exp[(\frac{1}{2}\alpha \pm \mathrm{i}q)x]$, with quantized vertical 'wave number'

$$q \equiv \sqrt{-\frac{1}{4}\alpha^2 + \frac{\rho_0}{N_0}\big[\,(\omega^2 - f^2 b^2)(D_0 + k_0^2 c^2 N_B^2) + \alpha\hat{g}\, g^2 b^2 \omega^2\,\big]} = n\,\frac{\pi}{a}, \tag{58}$$

where Brunt–Väisäläa frequency now contains magnetic contribution:

$$N_B^2 = \alpha\hat{g} - \frac{\hat{g}^2}{c^2} = \frac{(\gamma-1)\beta - 1}{1+\beta}\,\frac{\hat{g}^2}{c^2}, \qquad \alpha = \frac{\rho_0 \hat{g}}{p_0 + \frac{1}{2}B_0^2}, \quad \beta = \frac{2p_0^2}{B_0^2}, \tag{59}$$

so that instability ($N_B^2 < 0$) occurs for $0 \le \beta < (\gamma - 1)^{-1}$.

- *Dispersion equation of gravito-MHD waves* follows from inversion of Eq. (58) for q:

$$(\omega^2 - f^2 b^2)\big[\,\omega^4 - k_{\rm eff}^2 (b^2 + c^2)\omega^2 + k_{\rm eff}^2 f^2 b^2 c^2 + k_0^2 c^2 N_B^2\,\big] + \alpha \hat{g}\, g^2 b^2 \omega^2 = 0\,. \qquad (60)$$

- Cubic equation easily solved for *Parallel propagation* ($\mathbf{k}_0 \parallel \mathbf{B} \;\Rightarrow\; f = k_0\,,\; g = 0$):

$$\omega_1^2 = k_0^2 b^2\,, \qquad \omega_{2,3}^2 = \tfrac{1}{2} k_{\rm eff}^2 (b^2 + c^2)\left[1 \pm \sqrt{1 - \frac{4k_0^2 c^2 (k_{\rm eff}^2 b^2 + N_B^2)}{k_{\rm eff}^4 (b^2 + c^2)^2}}\,\right]. \qquad (61)$$

  'Unaffected' Alfvén waves and two gravitationally modified magnetosonic waves. Solution with $-$ sign corresponds to **Parker instability.**

- *Perpendicular propagation* ($\mathbf{k}_0 \perp \mathbf{B} \;\Rightarrow\; f = 0\,,\; g = k_0$):

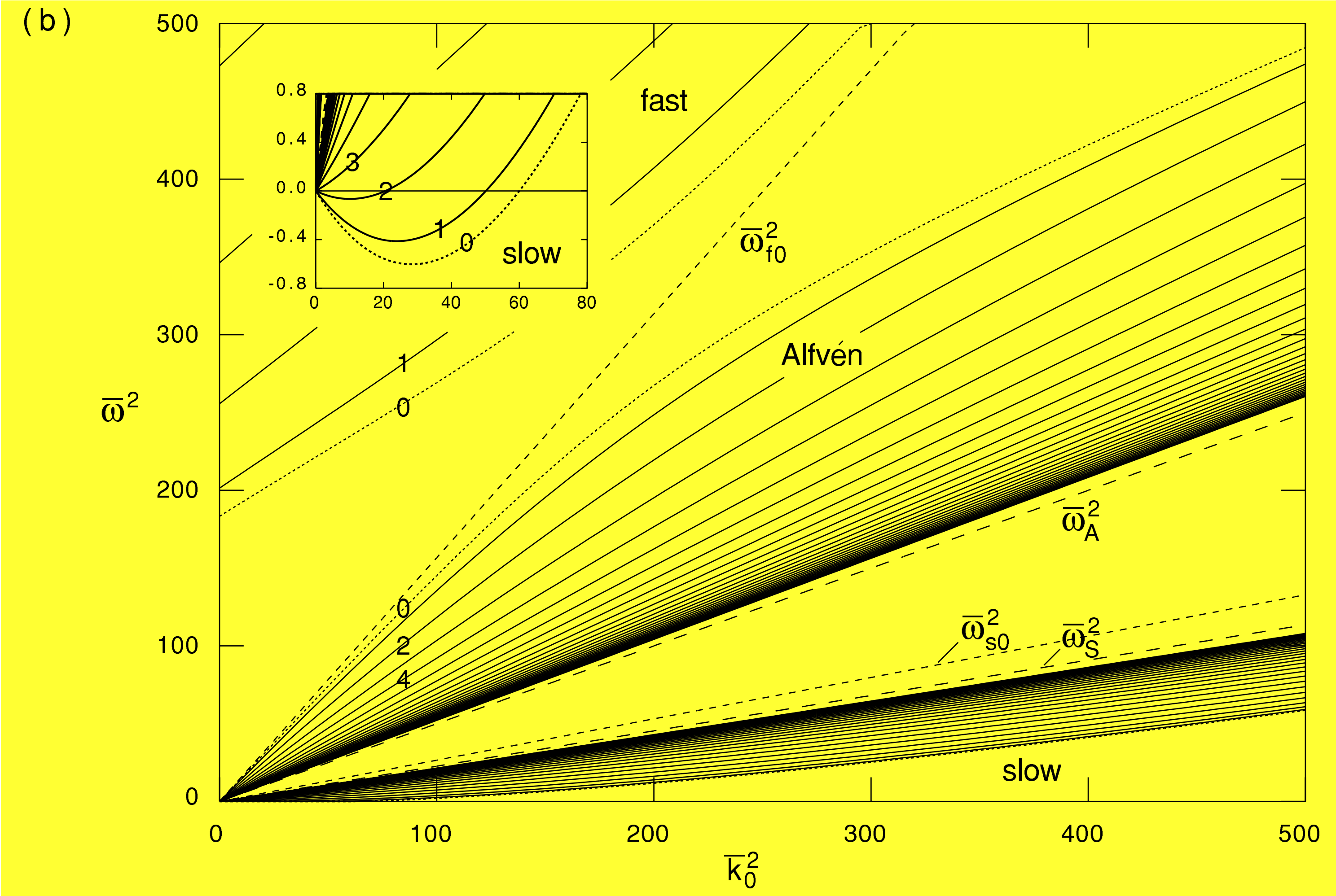
$$\omega_1^2 = 0\,, \qquad \omega_{2,3}^2 = \tfrac{1}{2} k_{\rm eff}^2 (b^2 + c^2)\left[1 \pm \sqrt{1 - \frac{4k_0^2 N_m^2}{k_{\rm eff}^4 (b^2 + c^2)}}\,\right], \qquad (62)$$

  with magnetically modified Brunt–Väisäläa frequency:

$$N_m^2 = \alpha \hat{g} - \frac{\hat{g}^2}{b^2 + c^2} = \frac{(\gamma - 1)\beta + 1}{1 + \beta}\,\frac{\hat{g}^2}{b^2 + c^2} \quad (\,> 0\,!\,)\,. \qquad (63)$$

  Unusual: more stable than parallel propagation! (Note that ω_1^2 is marginal.)

Dispersion diagram for oblique gravito-MHD modes



- Note differences with homogeneous plasma:

  $\omega^2 \uparrow \omega_F^2 \equiv \infty$ *(fast)*, $\omega^2 \downarrow \omega_A^2$ *(Alfvén)*, $\omega^2 \uparrow \omega_S^2$ *(slow, with unstable* $n = 0, 1, 2$*).*

Rayleigh–Taylor instability in diffuse plasma

- Start from MHD wave equation (50). Keep inhomogeneity (no tricks with exponential profiles!), but simplify by assuming *incompressibility:*

$$c^2 \equiv \frac{\gamma p}{\rho} \rightarrow \infty\,, \text{ so that } \begin{cases} N \rightarrow \rho c^2(\omega^2 - \omega_A^2)^2 \\ D \rightarrow -k_0^2 c^2 (\omega^2 - \omega_A^2) \end{cases} \Rightarrow \frac{N}{D} \rightarrow -\frac{\rho}{k_0^2}(\omega^2 - \omega_A^2)\,.$$

- Similarly, terms in Eq. (50) with finite numerators but infinite D vanish, resulting in the much simpler **MHD wave equation for incompressible gravitating plasma slab:**

$$\frac{d}{dx}\left[\rho(\omega^2 - \omega_A^2)\frac{d\xi}{dx}\right] - k_0^2\left[\rho(\omega^2 - \omega_A^2) + \rho'\hat{g}\right]\xi = 0\,, \qquad \omega_A^2 \equiv k_\parallel^2 B^2/\rho\,,$$

  of course with the usual BCs $\xi(0) = \xi(a) = 0$.

- Tangential components in the limit $c^2 \rightarrow \infty$ simplify to

$$\eta \rightarrow -\frac{g}{k_0^2}\xi'\,, \quad \zeta \rightarrow -\frac{f}{k_0^2}\xi' \qquad \big(\text{recall: } \; g \equiv k_\perp\,,\; f \equiv k_\parallel\big)\,,$$

  where only one component is needed since the other follows from incompressibility:

$$\nabla\cdot\boldsymbol{\xi} = \xi' + g\eta + f\zeta = 0 \qquad \big(\text{but } \; c^2\nabla\cdot\boldsymbol{\xi} \; \text{ is finite!}\big)\,.$$

- Simplify even further, but keep essential *inhomogeneity of the density:*

$$B^2 = B_0^2 \,, \qquad \rho = \rho_0 + \rho_0' x \,, \qquad \text{with } B_0^2 \,,\ \rho_0 \,,\ \rho_0' = \text{const} \,.$$

- Of course, the equilibrium equation (37), $(p + \frac{1}{2}B^2)' = -\rho\hat{g}$, must be satisfied. Integration yields the pressure profile, but this result is not needed since $p(x)$ does not appear in the wave equation (peculiarity of incompressibility).

- Now exploit **scale-independence:** Proper scaling of the equations should result in an eigenvalue problem that is essentially independent of the thickness a of the slab, the magnitude B_0 of the magnetic field, and the density ρ_0 of the plasma:

$$\bar{x} \equiv x/a \,, \quad \bar{k}_0 \equiv k_0 a \,, \quad \bar{k}_\parallel \equiv k_0 a \cos\vartheta \,, \quad \bar{\omega}^2 \equiv \omega^2 \rho_0 a^2 / B_0^2 \,.$$

Also introduce dimensionless parameters for density gradient and gravity:

$$\sigma \equiv \rho_0' a/\rho_0 \,, \quad \tau \equiv \rho_0 \hat{g} a / B_0^2 \qquad \big(\text{B–V freq. } \bar{N}_B \equiv -\sigma\tau\big) \,.$$

- The wave equation then becomes:

$$\frac{d}{d\bar{x}}\left[\big((1+\sigma\bar{x})\,\bar{\omega}^2 - \bar{k}_0^2\cos^2\vartheta\big)\frac{d\xi}{d\bar{x}} \right] - \bar{k}_0^2 \left[\, (1+\sigma\bar{x})\,\bar{\omega}^2 - \bar{k}_0^2\cos^2\vartheta + \sigma\tau \,\right] \xi = 0 \,,$$

to be solved on a *unit interval* $0 \le \bar{x} \le 1$, with BCs $\xi(0) = \xi(1) = 0$, so that the scaled eigenvalues depend on **four parameters:** $\bar{\omega}^2 = \bar{\omega}^2(\bar{k}_0^2, \cos^2\vartheta; \sigma, \tau)$.

- Formalise the problem so that it will be easier to generalise and also to implement numerically. Of course, we also drop the bars now:

$$\frac{d}{dx}\left[P(x;\omega^2)\,\frac{d\xi}{dx}\right] - Q(x;\omega^2)\,\xi = 0\,, \qquad \xi(0) = \xi(1) = 0\,,$$

where

$$P \equiv -(1+\sigma x)\,\omega^2 + k_0^2\cos^2\vartheta\,, \qquad Q \equiv k_0^2\,(P - \sigma\tau)\,.$$

- Very efficient numerical routines exist to *integrate a system of n nonlinear first order differential equations* $y_i' = y_i'(y_1, y_2, \ldots y_n, t)$, $i = 1, 2, \ldots n$. Hence, transform the above 2nd order ODE into a system of two ($n = 2$) 1st order (linear) ODEs by defining an auxiliary variable $\psi \equiv P\xi'$:

$$\begin{aligned} \xi' &= \psi/P\,, \\ \psi' &= Q\xi\,, \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} y_1' &= y_2/P\,, \\ y_2' &= Q y_1\,. \end{aligned}$$

- Not accidentally, the independent variable in the numerical routines is called t : such problems come from *initial value problems*, $y_i(0) = c_i$, with known constants c_i. However, we have a *boundary value problem,* $y_1(0) = y_1(1) = 0$, with unknown eigenvalue ω^2 !

- Hence turn the BVP into an IVP: *guess* a value for ω^2, insert that in the functions $P(x;\omega^2)$ and $Q(x;\omega^2)$, integrate the 1st order system from $x = 0$ to $x = 1$ starting with the initial values

$$\begin{aligned} \xi(0) &\equiv y_1(0) &= 0\,, \\ \xi'(0) &\equiv y_2(0)/P(0) &= 1\,, \end{aligned}$$

  and find the solution $\xi(x) \equiv y_1(x)$. In general, that solution will not satisfy the BC $\xi(1) \equiv y_1(1) = 0$. $\Rightarrow$ Guess a new value for ω^2 that is closer to satisfying the BC. This method is called the **shooting method.**

- The shooting method requires an algorithm for iterating on the eigenvalue parameter such that the solution $\xi(x)$ approaches the correct BV at $x = 1$. Such algorithm is provided by the **oscillation theorem** (Goedbloed and Sakanaka, 1974): the number of zeros of the solution of the MHD wave equation is monotonic in the parameter ω^2.

- The oscillation theorem holds with an important physical proviso: $P(x;\omega^2)$ should not vanish or go to ∞ on $0 \leq x \leq 1$ $\Rightarrow$ *singularities are to be excluded.*

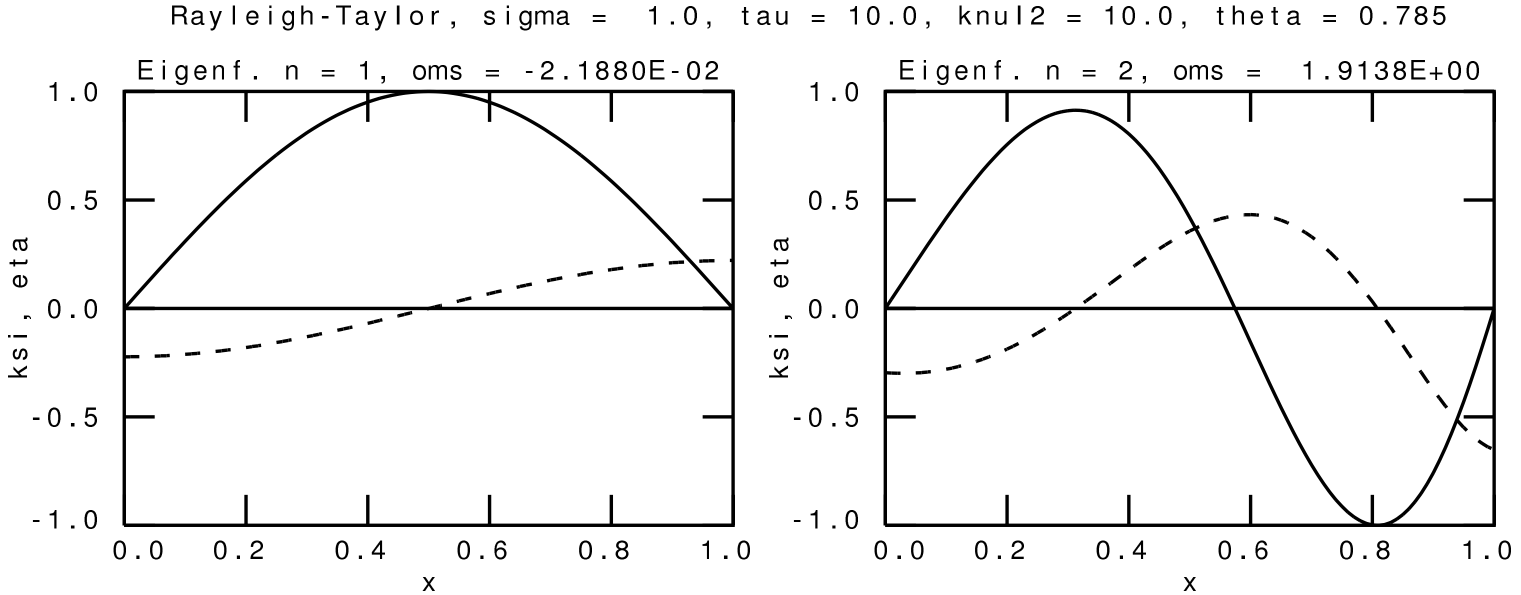
- When does that happen? When $\omega^2 = \omega_A^2(x) \equiv k_\parallel^2 B^2/\rho(x) = 0$ somewhere: the **continuous spectrum** $\omega_A^2(0) \geq \omega^2 \geq \omega_A^2(1)$ is to be avoided.

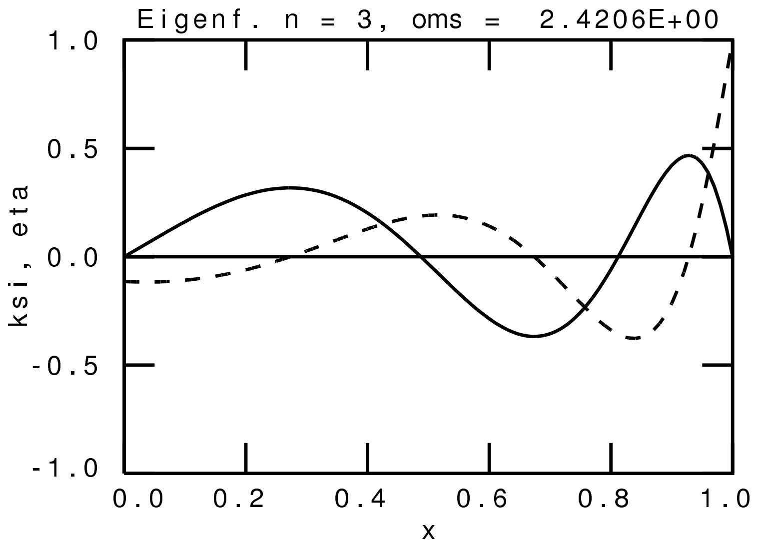
- We are now in an excellent shape to start solving the problem. One piece is still missing: suppose we wish to investigate *instability,* how negative should the initial guess for ω^2 be: $-\infty$? That would be awkward. It would be nice if we had an *estimate of the largest growth rate* that can be expected.

- Recall the second approach to stability, the one with **quadratic forms.** Rather than going back to the general expressions, construct one from the 2nd order ODE by multiplying with ξ and integrating over the interval:
$$\int_0^1 \left[\xi(P\xi')' - Q\xi^2\right] dx = [P\xi\xi']_0^1 - \int_0^1 (P\xi'^2 + Q\xi^2)\, dx = 0\,.$$
Since the boundary term vanishes for eigenfunctions, and P should be positive everywhere, Q must be negative in at least some region. Inserting the linear density profile, this gives a perfect estimate of the range of eigenvalues to be expected.

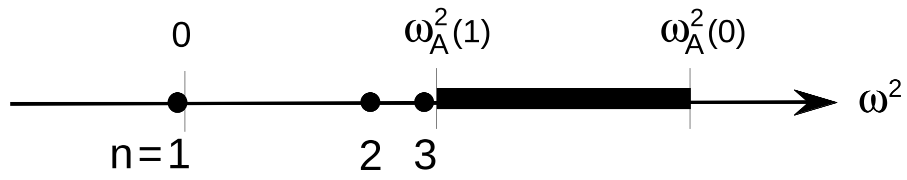
- Here is the **exercise:** Find a numerical library routine for solving a system of ODEs and a convenient plotting library. Using these tools, *compute a number of the lowest eigenvalues of the discrete spectrum of modes of the incompressible plane plasma layer with linear density profile and constant magnetic field for relevant values of the parameters,* i.e. in a range where Rayleigh–Taylor instabilities occur. Discuss the different effects that occur. Note that finding the relevant parameters belongs to the exercise (that is precisely what a physicist should do when formulating a problem).

- **A solution** ($k_0^2 = 10$, $\vartheta = \frac{1}{4}\pi$, $\sigma = 1$, $\tau = 10$):





Spectrum:



Singularities

- We obtained insight in MHD spectrum for exponential equilibrium, with constant 'singular' frequencies ($N = 0$, $D = 0$), but evaded two important difficult problems:
  – Singular frequencies depend on x $\Rightarrow$ *continuous spectrum,*
  – Magnetic field is not uni-directional $\Rightarrow$ *magnetic shear.*

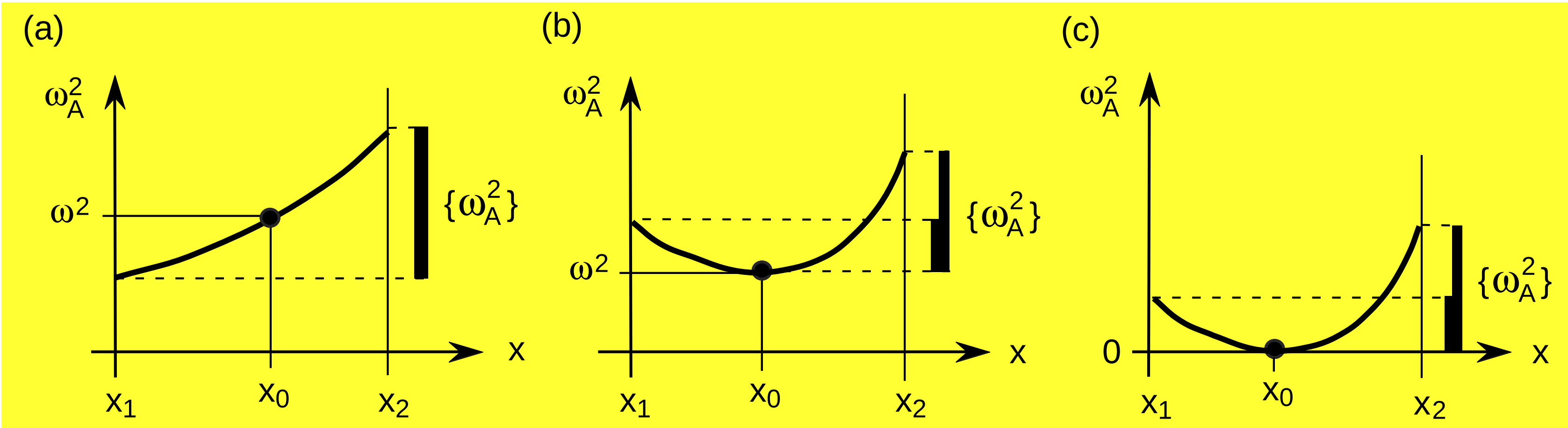
- A problem reduced to a *non-singular 2nd order ODE may be considered solved,* because one can obtain the answers numerically to any relevant degree of accuracy. For example, consider numerical solution by *shooting* of the ODE

$$\frac{d}{dx}\left[P(x;\omega^2)\,\frac{d\xi}{dx}\right] - Q(x;\omega^2)\,\xi = 0\,, \quad \text{with BCs} \quad \xi(x_1) = \xi(x_2) = 0\,. \tag{64}$$

  (1) Specify equilibrium ρ, p, B_y, $B_z(x)$, satisfying $(p + \frac{1}{2}B^2)' = -\rho\hat{g}$, and choose a particular value $\omega^2 = \omega^{2(0)}$ $\Rightarrow$ $P(x;\omega^2)$ and $Q(x;\omega^2)$ known;
  (2) Solve Eq. (64) by means of standard library routine, starting from left BC and stepping towards right end of interval $[x_1, x_2]$;
  (3) Since right BC will not be satisfied, choose new value $\omega^2 = \omega^{2(1)}$, using *oscillation theorem* (see below), that brings solution closer to satisfying BC in next iteration.

- Shooting works if ODE is non-singular: Problem left is **treatment of the singularities.**

- **Three kinds of singularity:**



Case (a): If $\omega_A^2(x)$ is monotonic, any ω^2 in the range $\omega_A^2(x_1) \leq \omega^2 \leq \omega_A^2(x_2)$ leads to a singular point $x_1 \leq x_0 \leq x_2$, where the Alfvén factor may be expanded:

$$\omega^2 - \omega_A^2 \approx -(\omega_A^{2\,\prime})_0\,(x - x_0) \quad \Rightarrow \quad P \sim x - x_0\,. \tag{65}$$

This range yields *continuous spectrum* $\{\omega_A^2\}$ of Alfvén modes $\big(\{\omega_S^2\}$ of slow modes$\big)$.

- *Proof*: Since $P \sim s \equiv x - x_0$ singularity gives logarithmic contributions, try solution

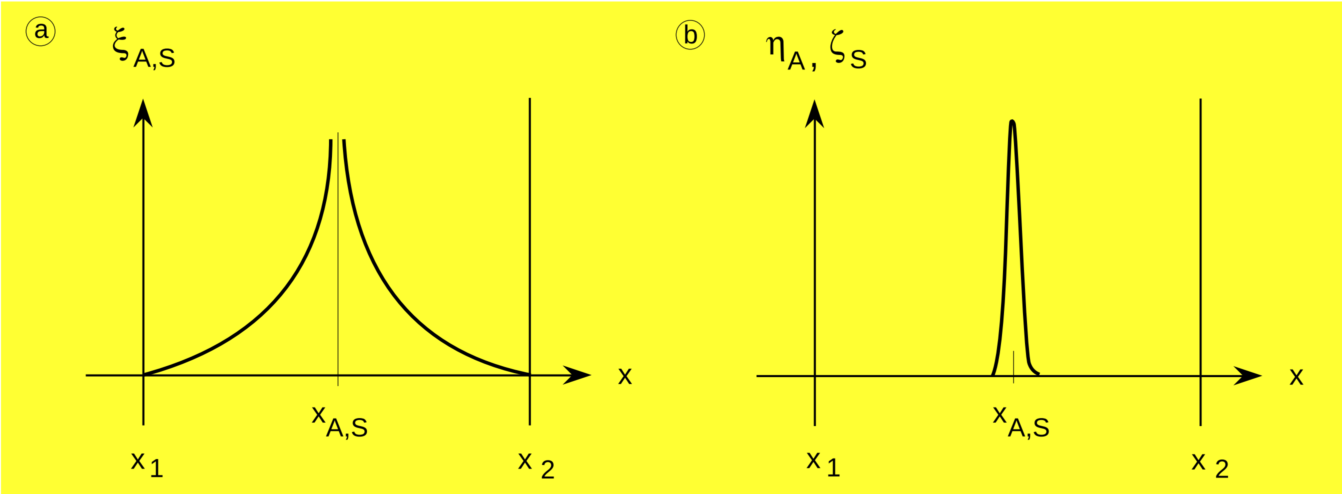
$$\xi = [A_1 u + C_1(u \ln|s| + v)]\,H(-s) + [A_2 u + C_2(u \ln|s| + v)]\,H(s)\,, \tag{66}$$

with $u(x)$ and $v(x)$ regular, and constants $A_{1,2}$ and $C_{1,2}$ to be determined by BCs. By substitution into ODE (64), using $H'(s) = \delta(s)$ and $s\delta(s) = 0$, one finds:

$$A_1 \neq A_2 \text{ (small solution may jump)}\,, \quad C_1 = C_2 \text{ (large solution ‘continuous’)}\,.$$

$\Rightarrow$ With 3 constants one can always satisfy 2 BCs for any $\omega^2 \in \{\omega_A^2(x)\}$; *QED*.

- Associated *'singular 'eigenfuctions'* have their most important component in the tangential direction (schematic):



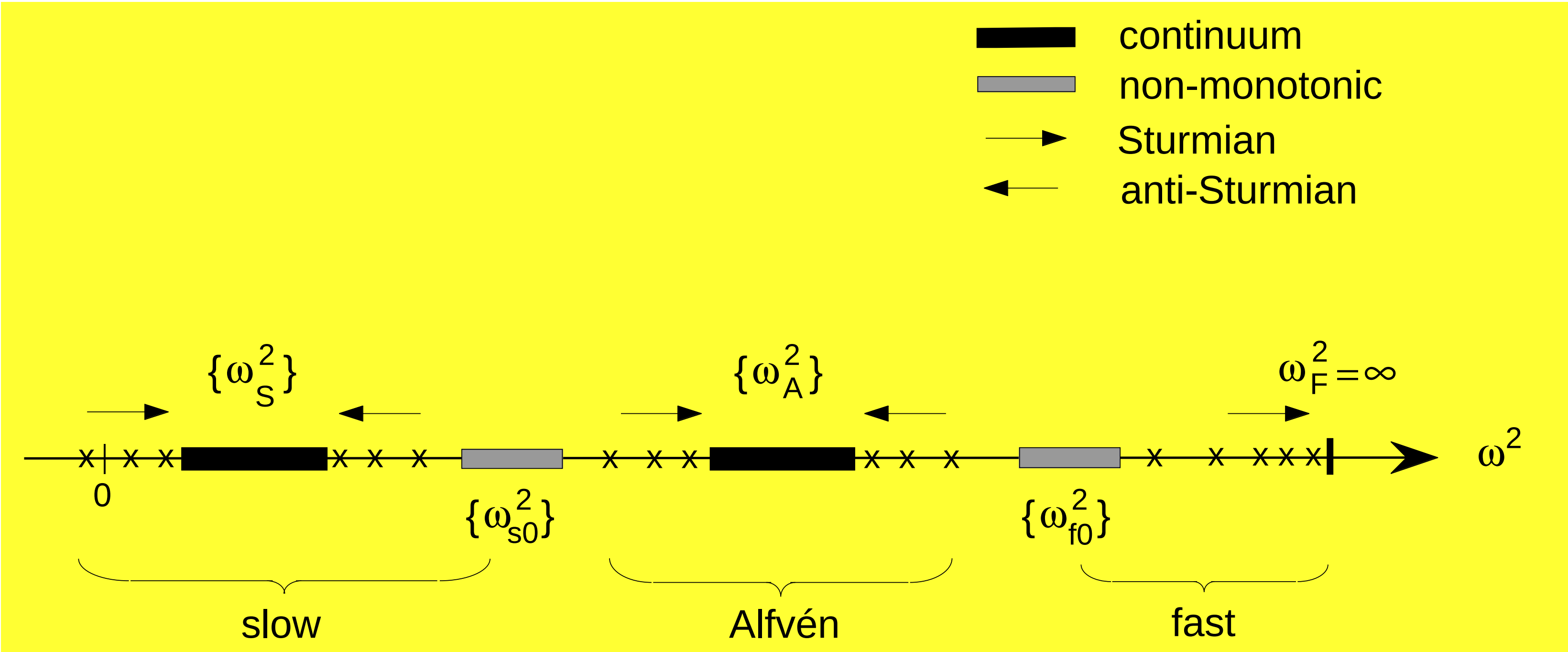
(a) normal and (b) tangential components of improper Alfvén and slow continuum modes: *Alfvén modes dominantly perpendicular, slow modes dominantly parallel.*

- Physical significance of these singularities: Solution of IVP by means of Laplace transform, where ω is assumed complex and continuation is obtained by deforming Laplace contour around singularity, yields *damping of MHD waves*, analogous to Landau damping of plasma oscillations for the Vlasov equation (Sec. 2.3). [Velocity variable v of Vlasov problem corresponds to spatial coordinate x of MHD problem.]

- This damping can be described as *phase mixing* of highly localized shear flows.
$\Rightarrow$ *Alfvén wave heating* is one of the mechanisms proposed for coronal heating, and also investigated for heating in tokamaks (e.g. in TCA-BR, São Paulo).

Schematic structure of the MHD spectrum

- *Quadratic $N = 0$ singularities* (7-26, cases (b) & (c)) may produce *clusterpoints* of the discrete spectrum at the tips of the continua.
- *$D = 0$ singularities are only apparent ones:* coefficients of expansion around such 'singularity' produce cancellation of terms such that solution is completely regular. Frequencies $\{\omega^2_{s0,f0}(x)\}$ act as *separators* of the different parts of the spectrum:



- *Oscillation theorem:* Outside $\{\omega^2_{A,S}\}$, $\{\omega^2_{s0,f0}\}$, *the discrete spectrum is monotonic* in number of nodes of eigenfunctions (either Sturmian or anti-Sturmian).

Energy principle for gravitating plasma slab

Start from general expression for potential energy:

$$W = \tfrac{1}{2} \int \big[\, \gamma p \,|\nabla \cdot \boldsymbol{\xi}|^2 + |\mathbf{Q}|^2 + (\boldsymbol{\xi} \cdot \nabla p) \nabla \cdot \boldsymbol{\xi}^* + \mathbf{j} \cdot \boldsymbol{\xi}^* \times \mathbf{Q} - (\boldsymbol{\xi}^* \cdot \nabla \Phi)\, \nabla \cdot (\rho \boldsymbol{\xi}) \,\big] dV \,.$$

[corrected from the book (p.366)!]

- Evaluate compressibility and magnetic field perturbation for gravitating slab:

$$\nabla \cdot \boldsymbol{\xi} = \xi' + g\eta + f\zeta \,, \tag{67}$$

$$Q_x = \mathrm{i} f B \,\xi \,, \quad Q_y = -(B_y \xi)' + k_z B\, \eta \,, \quad Q_z = -(B_z \xi)' - k_y B\, \eta \,. \tag{68}$$

- Insert and work out:

$$W = \tfrac{1}{2} \int \Big[\frac{f^2 B^2}{k_0^2} \xi'^2 + \big(f^2 B^2 - \rho' \hat{g} - \frac{\rho^2 \hat{g}^2}{\gamma p} \big) \xi^2 + B^2 \big(k_0 \eta + \frac{g}{k_0} \xi' \big)^2 + \gamma p \big(\nabla \cdot \xi - \frac{\rho \hat{g}}{\gamma p} \xi \big)^2 \Big] \, dV \,. \tag{69}$$

- Minimization with respect to the tangential variables is trivial:

$$\eta = -\frac{g}{k_0^2} \xi' \,, \quad \text{and} \quad \nabla \cdot \boldsymbol{\xi} = \frac{\rho \hat{g}}{\gamma p} \xi \;\; \Rightarrow \;\; \zeta = -\frac{f}{k_0^2} \xi' + \frac{\rho \hat{g}}{\gamma p f} \xi \,. \tag{70}$$

so that only first two terms of W remain $\Rightarrow$ *standard minimization problem!*

- Recall: Standard quadratic form $W = \frac{1}{2}\int_{x_1}^{x_2} (P_0\xi'^2 + Q_0\xi^2)\, dx$ is minimised

  by solution of the Euler-Lagrange equation $\frac{d}{dx}\big(P_0 \frac{d}{dx}\xi\big) - Q_0\,\xi = 0$,

  subject to the boundary conditions $\xi(x_1) = \xi(x_2) = 0$.

- Here, the Euler-Lagrange equation,

$$\frac{d}{dx}\left(\frac{F^2}{k_0^2}\frac{d\xi}{dx}\right) - \left(F^2 - \rho'\hat{g} - \frac{\rho^2\hat{g}^2}{\gamma p}\right)\xi = 0\,, \qquad F \equiv fB \equiv -i\mathbf{B}\cdot\nabla\,, \tag{71}$$

  is just *the marginal wave equation* (Eq. (50) with $\omega^2 = 0$).

- Note: In contrast to the spectral equation (50), the marginal equation (71) contains *no eigenvalue* so that BVP ($\xi(x_1) = 0$ ànd $\xi(x_2) = 0$) cannot be solved in general. All one can do is *'shoot' once:* Start from the left, satisfying left BC, integrate to the right, and check whether or not more zeros are encountered on the interval (x_1, x_2). From this fact alone, one can draw proper conclusion with respect to stability.

- Next, proper treatment of singularities: *When $F = 0$ somewhere on (x_1, x_2)*, Alfvén and slow continua fold over (7-26, case (c)) and reach the origin, $\lim_{\omega^2\to 0} N/D = F^2/k_0^2 \to 0$, and *the Euler-Lagrange equation for stability becomes singular.* Hence, two cases:

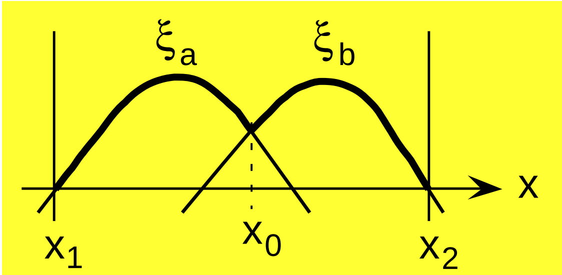
A. Absence of singularities ($F \neq 0$ for $x_1 \leq x \leq x_2$).

Relatively straightforward:

- Insert solution of EL equation back into W, and integrate by parts:

$$W = \tfrac{1}{2} \int_{x_1}^{x_2} \left[P_0 \, \xi'^2 + \xi \, (P_0 \xi')' \right] dx = \tfrac{1}{2} [P_0 \, \xi \, \xi']_{x_1}^{x_2} \, .$$

If (x_1, x_2) larger than distance between two consecutive zeros of the EL solution, split the interval and construct composite trial function of left and right EL solutions:



$$W = \tfrac{1}{2} \big(P_0 \, \xi (\xi_a' - \xi_b') \big) \Big|_{x=x_0} < 0 \;\Rightarrow\; \text{unstable!}$$

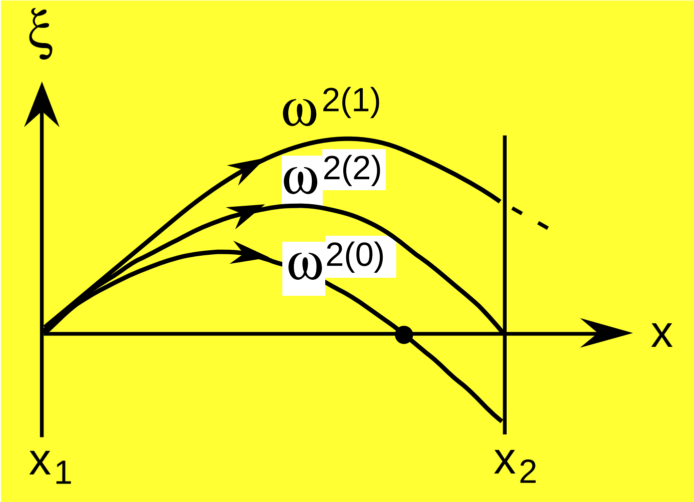
If (x_1, x_2) smaller than distance between consecutive zeros $\Rightarrow$ stable.

- Hence, Newcomb's first theorem (1960, derived for cylinder):

(1) If the 'solution' ξ_0 of the marginal Euler-Lagrange equation (71) that satisfies the left boundary condition $\xi_0(x_1) = 0$ has *another zero* on the interval (x_1, x_2), then a trial function ξ_1 can be constructed (the composite function shown) that satisfies both boundary conditions and the energy $W(\xi_1) < 0$ $\Rightarrow$ *System is unstable;*

(2) If ξ_0 has *no other zeros* on the interval, that construction fails and $W(\xi_1) \geq 0$ for all trial functions $\Rightarrow$ *System is stable.*

- Connection with the spectral problem:
  'Shooting' method to get instability growth rates.
  Special case of solving for discrete eigenvalues outside ranges of $\omega_A^2(x)$, $\omega_S^2(x)$, $\omega_{s0}^2(x)$, $\omega_{f0}^2(x)$:
  Guess initial value $\omega^2 = \omega^{2(i)}$, solve from the left, keep adjusting ω^2 until right BC is also satisfied.
  Works because *discrete spectrum is monotonic.*



- Oscillation theorem for MHD spectrum (Goedbloed & Sakanaka, 1974):
  If x_1 and x_2 are consecutive zeros of ξ_1 satisfying MHD wave equation (50) for ω_1^2, then solutions ξ_2 of MHD wave equation for ω_2^2 oscillate faster than ξ_1 if $\omega_2^2 > \omega_1^2$ and $N/D > 0$ (Sturmian), and slower if $N/D < 0$ (anti-Sturmian).
  [May be proved by means of self-adjointness property of force operator.]

- Consequence: *Unstable discrete modes are always Sturmian* because $N/D < 0$ for $\omega^2 < 0$. Hence, numerically solving instability eigenvalue problem (growth rates and eigenfunctions) is not much more complicated than solving for marginal stability.

B. Presence of singularities ($F \equiv k_0 \cdot \mathbf{B}(x_0) = 0$ for some $x_1 \leq x_0 \leq x_2$).

Physical meaning:

- Horizontal wavevector $\mathbf{k}_0 \perp \mathbf{B}$, so that *perturbations do not disturb magnetic field:* Magnetic energy of Alfvén wave perturbations vanishes there because field lines are not bent. At these positions, driving forces of instability are minimally counterbalanced by magnetic tensions so that instabilities tend to localize there.
- By *magnetic shear* ($F' \neq 0$), this region of minimal field line bending can be limited. (Recall shear stabilization of Rayleigh–Taylor instability of interface plasmas, where magnetic shear was entirely localized to surface layer.)
- Introduce angles $\varphi(x)$ between $\mathbf{B}$ and z-axis, and θ between $\mathbf{k}_0$ and z-axis:

$$F \equiv \mathbf{k}_0 \cdot \mathbf{B} = k_0 B(x) \cos[\,\varphi(x) - \theta\,] , \quad \text{where} \;\; \varphi(x_0) - \theta = \pm\pi/2 . \tag{72}$$

Expand Euler-Lagrange equation around singularity $x = x_0$:

$$\rho\omega_A^2 \equiv F^2(x) \approx (F'^2)_0\, s^2 = k_0^2 (B^2 \varphi'^2)_0\, s^2 , \qquad s \equiv x - x_0 , \tag{73}$$

$$\Rightarrow \; \frac{d}{ds}\left[s^2\,(1 + \cdots)\,\frac{d\xi}{ds}\right] - q_0\,(1 + \cdots)\,\xi = 0 , \qquad q_0 \equiv -\left(\frac{\rho'\hat{g} + \dfrac{\rho^2 \hat{g}^2}{\gamma p}}{B^2 \varphi'^2} \right)_0 . \tag{74}$$

Intermezzo on singular differential equations

- Standard theory of *regular singularities* in the complex plane ($s \to z$):

$$N \sim z^l \quad \Rightarrow \quad \xi'' + \frac{1}{z}\, p(z)\, \xi' - \frac{1}{z^2}\, q(z)\, \xi = 0\,, \quad p(z)\ \&\ q(x) \text{ analytic:} \tag{75}$$

$$\begin{cases} p(z) \equiv z\,\dfrac{P'}{P} = p_0 + p_1 z + \cdots \\ q(z) \equiv z^2\,\dfrac{Q}{P} = q_0 + q_1 z + \cdots \end{cases}, \quad \text{where} \quad \begin{cases} \text{cont. spectrum:} \\ p_0 = 1\,,\ q_0 = 0 \quad (l = 1) \\ \text{marg. stability:} \\ p_0 = 2\,,\ q_0 \neq 0 \quad (l = 2) \end{cases}. \tag{76}$$

- Insert *Frobenius expansion,* $\xi = z^\nu \sum_{n=0}^\infty a_n z^n$ (index ν may be complex), in Eq. (75) and balance different powers:

$$\begin{aligned} z^{\nu-2} &: \quad [\nu^2 + (p_0 - 1)\nu - q_0]\, a_0 = 0 \quad \Rightarrow \begin{cases} \nu_1 = \nu_2 = 0 \quad (l = 1) \\ \nu_{1,2} = -\frac{1}{2} \pm \sqrt{1 + 4q_0} \quad (l = 2) \end{cases}, \\ z^{\nu-1} &: \quad [(\nu+1)\nu + p_0(\nu+1) - q_0]\, a_1 = (-\nu p_1 + q_1)\, a_0\,, \quad \text{etc.} \end{aligned} \tag{77}$$

- Index equation for marginal stability $(l = 2)$ discriminates between:

$$\begin{aligned} &\text{(a) } 1 + 4q_0 < 0 : \ \text{indices complex} \ \Rightarrow \text{local stability criteria,} \\ &\text{(b) } 1 + 4q_0 > 0 : \ \text{indices real} \ \Rightarrow \text{global stability theory.} \end{aligned} \tag{78}$$

End intermezzo

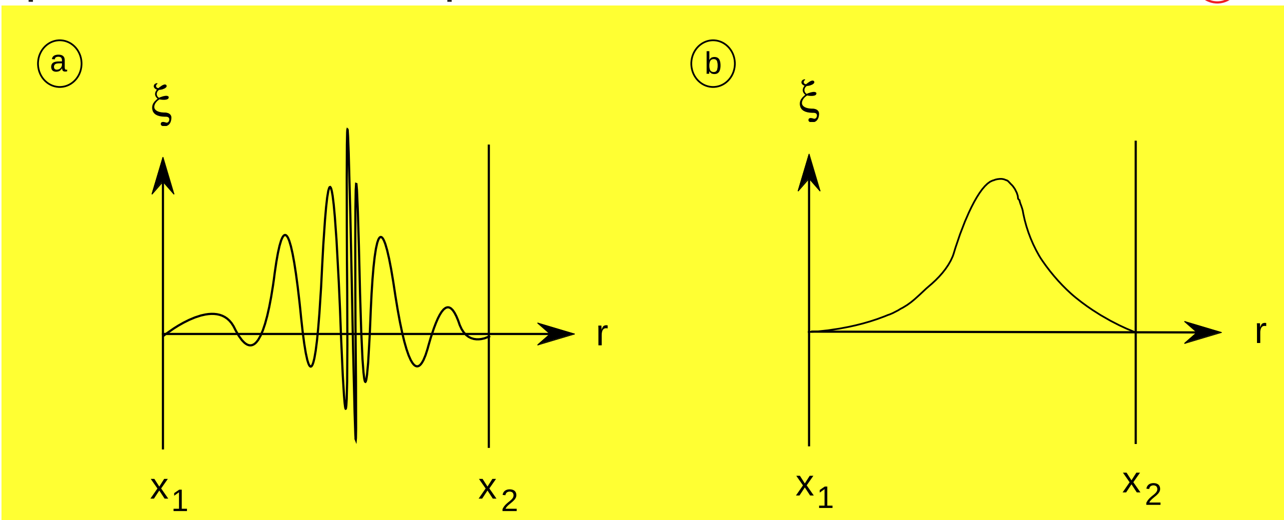
(a) Complex indices ($1 + 4q_0 < 0$)

- Quite extreme oscillatory behavior at singularity $s = 0$:

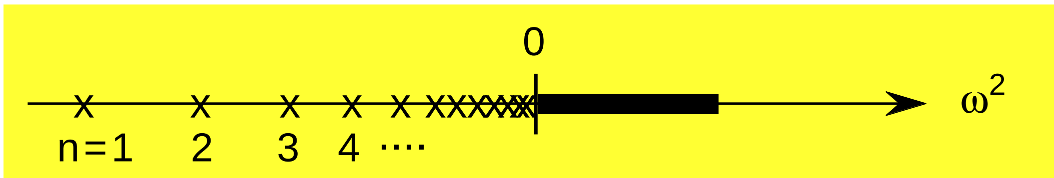
$$\begin{aligned} \xi_1 &= s^{-1/2+iw} + s^{-1/2-iw} &&= 2s^{-1/2}\cos(w \ln s)\,, \\ \xi_2 &= i(s^{-1/2+iw} - s^{-1/2-iw}) &&= -2s^{-1/2}\sin(w \ln s)\,, \end{aligned} \tag{79}$$

where $w \equiv \frac{1}{2}\sqrt{-(1+4q_0)}$.

- Oscillation theorem: *Marginal solutions ⓐ oscillate infinitely rapidly* ($n \to \infty$) and their amplitude also blows up when $s \to 0$; *Actual instabilities ⓑ are global* ($n = 1$):



$\Rightarrow$ $\omega^2 = 0$ is clusterpoint of unstable discrete eigenvalues:



- To avoid these instabilities, one should demand that $1 + 4q_0 > 0$ (real indices). This leads to *necessary stability criterion for interchange modes:*

$$\rho' \hat{g} + \frac{\rho^2 \hat{g}^2}{\gamma p} \;\left(\equiv -\rho N_{\rm B}^2 \right) \;\leq \tfrac{1}{4} B^2 \varphi'^2 \,. \tag{80}$$

  This is the *Schwarzschild criterion, modified by stabilizing shear* (RHS).

- Three terms represent driving force of *gravitational or Rayleigh–Taylor instability* (heavy fluid on top of a lighter one), *modified by adiabatic effects* (term with γ), and *stabilized by magnetic shear* (RHS): Glass of gravitationally unstable plasma may be turned upside down without contents dropping out, if magnetic shear is large enough!

- In cylinder geometry, with Fourier modes $e^{{\rm i}(m\theta + kz)}$, similar condition is known as *Suydam's criterion:*

$$p' + \tfrac{1}{8} r B_z^2 \left(\frac{\mu'}{\mu} \right)^2 > 0 \qquad \left(\mu \equiv \frac{B_\theta}{r B_z} \right) \,. \tag{81}$$

  Violation implies highly localized instabilities close to singular surface ($k + \mu m = 0$), where field lines can be interchanged without appreciable bending. The criterion provides simple explicit condition that may be tested easily and that, for laboratory fusion research, suggests measures (increasing shear or lowering pressure gradient) to improve stability. Toroidal version of this condition: *Mercier criterion.*

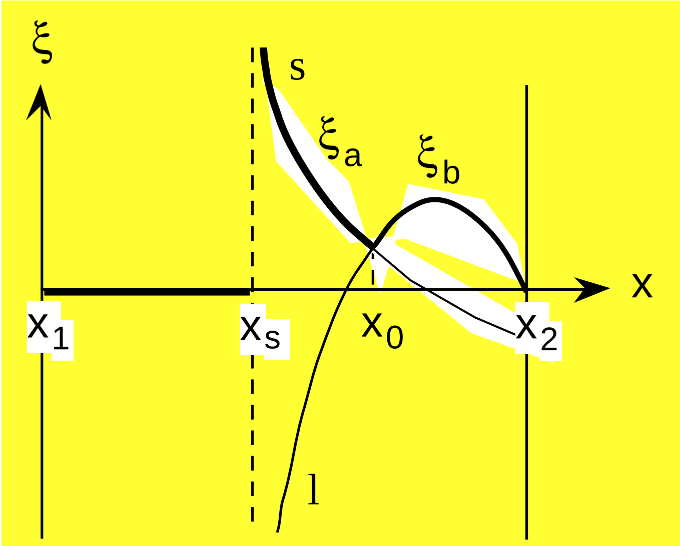
(a) Real indices ($1 + 4q_0 > 0$)

- At the singularity $s = 0$, the two solutions now behave as

$$\begin{aligned} \xi_s &\sim s^{\nu_s}, \qquad \nu_s = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4q_0} > -\tfrac{1}{2} \qquad \text{(`small' solution)}, \\ \xi_l &\sim s^{\nu_\ell}, \qquad \nu_\ell = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1+4q_0} < -\tfrac{1}{2} \qquad \text{(large solution)}. \end{aligned} \tag{82}$$

  Hence, large solution ξ_ℓ always blows up at $s = 0$, whereas 'small' solution may or may not blow up depending on whether square root is smaller or larger than 1.

- Just special case of continuous spectrum ($\omega^2 > 0$) singularities discussed before: 'Small' solution may jump, large solution should be continuous, so that the singularity $x = x_s$ effectively splits the interval (x_1, x_2) in two *independent subintervals* (x_1, x_s) and (x_s, x_2), *with respect to stability!* Hence, we may construct trial functions like shown.



- Energy contribution of 'small' solution at x_s is negligible: *'smallness' counts as zero.*

- Hence, Newcomb's final theorem:

  For specified values of k_y *and* k_z *such that* $F \equiv k_y B_y + k_z B_z = 0$ *at some point* $x = x_s$ *of the interval* (x_1, x_2)*, the gravitating plasma slab is stable if, and only if, (1) the interchange criterion (80) is satisfied at* $x = x_s$*; (2) the non-trivial solution* ξ_L *of the Euler-Lagrange equation (71) that is "small" to the left of* $x = x_s$ *does not vanish in the open interval* (x_1, x_s)*; (3) the non-trivial solution* ξ_R *that is "small" to the right of* $x = x_s$ *does not vanish in the open interval* (x_s, x_2)*.*

- In principle, this solves all stability problems of the gravitating plasma slab. Of course, application provides quite a bit more of physical insight!

- However, *calculating growth rates with the complete wave equation (50)* provides more information on the instabilities, and it is even simpler since *that equation is non-singular as long as* $\omega^2 < 0$*.*

- Finally, *spectral theory of MHD waves and instabilities* has significantly advanced our understanding of the overall connection of these problems. *Computational MHD* has contributed separately by providing superior new discretization methods that may be generalized to *arbitrary geometries* and applied to *extensions of the MHD model.*