# Chapter 5: Waves and characteristics

## Overview

- **Physics and accounting:** use example of sound waves to illustrate method of linearization and counting of variables and solutions; [ book: Sec. 5.1 ]

- **MHD waves:** different representations and reductions of the linearized MHD equations, obtaining the three main waves, dispersion diagrams; [ book: Sec. 5.2 ]

- **Phase and group diagrams:** propagation of plane waves and wave packets, asymptotic properties of the three MHD waves; [ book: Sec. 5.3 ]

- **Characteristics:** numerical method, classification of PDEs, application to MHD. [ book: Sec. 5.4 ]

# Sound waves

- Perturb the gas dynamic equations ($\mathbf{B} = 0$),

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \tag{1}$$

$$\rho \Big(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\Big) + \nabla p = 0\,, \tag{2}$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0\,, \tag{3}$$

about infinite, homogeneous gas at rest,

$$\begin{aligned} \rho(\mathbf{r}, t) &= \rho_0 + \rho_1(\mathbf{r}, t) \qquad (\text{where } |\rho_1| \ll \rho_0 = \text{const})\,, \\ p(\mathbf{r}, t) &= p_0 + p_1(\mathbf{r}, t) \qquad (\text{where } |p_1| \ll p_0 = \text{const})\,, \\ \mathbf{v}(\mathbf{r}, t) &= \mathbf{v}_1(\mathbf{r}, t) \qquad (\text{since } \mathbf{v}_0 = 0)\,. \end{aligned} \tag{4}$$

$\Rightarrow$ *Linearised equations of gas dynamics:*

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0\,, \tag{5}$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + \nabla p_1 = 0\,, \tag{6}$$

$$\frac{\partial p_1}{\partial t} + \gamma p_0 \nabla \cdot \mathbf{v}_1 = 0\,. \tag{7}$$

## Wave equation

- Equation for $\rho_1$ does not couple to the other equations: drop. Remaining equations give *wave equation for sound waves*:

$$\frac{\partial^2 \mathbf{v}_1}{\partial t^2} - c^2\, \nabla\nabla\cdot\mathbf{v}_1 = 0\,, \tag{8}$$

where

$$c \equiv \sqrt{\gamma p_0/\rho_0} \tag{9}$$

is *the velocity of sound* of the background medium.

- Plane wave solutions

$$\mathbf{v}_1(\mathbf{r},t) = \sum_{\mathbf{k}} \hat{\mathbf{v}}_{\mathbf{k}}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{10}$$

turn the wave equation (8) into an algebraic equation:

$$(\,\omega^2 \mathsf{I} - c^2\, \mathbf{k}\mathbf{k}\,)\cdot\hat{\mathbf{v}} = 0\,. \tag{11}$$

- For $\mathbf{k} = k\,\mathbf{e}_z$, the solution is:

$$\omega = \pm k\,c\,, \qquad \hat{v}_x = \hat{v}_y = 0\,, \quad \hat{v}_z \text{ arbitrary}\,, \tag{12}$$

$\Rightarrow$ *Sound waves* propagating to the right (+) and to the left (−): *compressible* ($\nabla\cdot\mathbf{v} \neq 0$) and *longitudinal* ($\mathbf{v} \parallel \mathbf{k}$) waves.

## Counting

- There are also other solutions:

$$\omega^2 = 0\,, \qquad \hat{v}_x\,, \hat{v}_y \text{ arbitrary}\,, \qquad \hat{v}_z = 0\,, \tag{13}$$

$\Rightarrow$ *incompressible transverse ($\mathbf{v}_1 \perp \mathbf{k}$) translations.* They do not represent interesting physics, but simply establish completeness of the velocity representation.

- Problem: 1st order system (5)–(7) for $\rho_1$, $\mathbf{v}_1$, $p_1$ has 5 degrees of freedom, whereas 2nd order system (8) for $\mathbf{v}_1$ appears to have 6 degrees of freedom ($\partial^2/\partial t^2 \to -\omega^2$). However, the 2nd order system actually only has 4 degrees of freedom, since $\omega^2$ does not double the number of translations (13). Spurious doubling of the eigenvalue $\omega = 0$ happened when we applied the operator $\partial/\partial t$ to Eq. (6) to eliminate $p_1$.

- Hence, we *lost one degree of freedom* in the reduction to the wave equation in terms of $\mathbf{v}_1$ alone. This happened when we dropped Eq. (5) for $\rho_1$. Inserting $\mathbf{v}_1 = 0$ in the original system gives the signature of this lost mode:

$$\omega\hat{\rho} = 0 \quad \Rightarrow \quad \omega = 0\,, \quad \hat{\rho} \text{ arbitrary}\,, \quad \text{but } \hat{\mathbf{v}} = 0 \text{ and } \hat{p} = 0\,. \tag{14}$$

$\Rightarrow$ *entropy wave*: perturbation of the density and, hence, of the entropy $S \equiv p\rho^{-\gamma}$. Like the translations (13), this mode does not represent important physics but is needed to account for the degrees of freedom of the different representations.

## MHD waves

- Similar analysis for MHD in terms of $\rho$ , $\mathbf{v}$ , $e$ $\left( \equiv \frac{1}{\gamma-1} p/\rho \right)$ , and $\mathbf{B}$ :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \tag{15}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + (\gamma - 1)\nabla(\rho e) + (\nabla \mathbf{B}) \cdot \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{B} = 0\,, \tag{16}$$

$$\frac{\partial e}{\partial t} + \mathbf{v} \cdot \nabla e + (\gamma - 1) e \nabla \cdot \mathbf{v} = 0\,, \tag{17}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{B} + \mathbf{B} \nabla \cdot \mathbf{v} - \mathbf{B} \cdot \nabla \mathbf{v} = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,, \tag{18}$$

- *Linearise about plasma at rest,* $\mathbf{v}_0 = 0\,, \rho_0\,, e_0\,, \mathbf{B}_0 = \text{const}$ :

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0\,, \tag{19}$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + (\gamma - 1)(e_0 \nabla \rho_1 + \rho_0 \nabla e_1) + (\nabla \mathbf{B}_1) \cdot \mathbf{B}_0 - \mathbf{B}_0 \cdot \nabla \mathbf{B}_1 = 0\,, \tag{20}$$

$$\frac{\partial e_1}{\partial t} + (\gamma - 1) e_0 \nabla \cdot \mathbf{v}_1 = 0\,, \tag{21}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} + \mathbf{B}_0 \nabla \cdot \mathbf{v}_1 - \mathbf{B}_0 \cdot \nabla \mathbf{v}_1 = 0\,, \qquad \nabla \cdot \mathbf{B}_1 = 0\,. \tag{22}$$

## Transformation

- *Sound* and vectorial *Alfvén speed*,

$$c \equiv \sqrt{\frac{\gamma p_0}{\rho_0}}\,, \qquad \mathbf{b} \equiv \frac{\mathbf{B}_0}{\sqrt{\rho_0}}\,, \tag{23}$$

and dimensionless variables,

$$\tilde{\rho} \equiv \frac{\rho_1}{\gamma\,\rho_0}\,, \qquad \tilde{\mathbf{v}} \equiv \frac{\mathbf{v}_1}{c}\,, \qquad \tilde{e} \equiv \frac{e_1}{\gamma\,e_0}\,, \qquad \tilde{\mathbf{B}} \equiv \frac{\mathbf{B}_1}{c\sqrt{\rho_0}}\,, \tag{24}$$

⇒ *linearised MHD equations* with coefficients $c$ and $\mathbf{b}$:

$$\gamma\,\frac{\partial\tilde{\rho}}{\partial t} + c\,\nabla\cdot\tilde{\mathbf{v}} = 0\,, \tag{25}$$

$$\frac{\partial\tilde{\mathbf{v}}}{\partial t} + c\,\nabla\tilde{\rho} + c\,\nabla\tilde{e} + (\nabla\tilde{\mathbf{B}})\cdot\mathbf{b} - \mathbf{b}\cdot\nabla\tilde{\mathbf{B}} = 0\,, \tag{26}$$

$$\frac{\gamma}{\gamma-1}\,\frac{\partial\tilde{e}}{\partial t} + c\,\nabla\cdot\tilde{\mathbf{v}} = 0\,, \tag{27}$$

$$\frac{\partial\tilde{\mathbf{B}}}{\partial t} + \mathbf{b}\,\nabla\cdot\tilde{\mathbf{v}} - \mathbf{b}\cdot\nabla\tilde{\mathbf{v}} = 0\,, \qquad \nabla\cdot\tilde{\mathbf{B}} = 0\,. \tag{28}$$

## Symmetry

- *Plane wave solutions*, with $\mathbf{b}$ and $\mathbf{k}$ arbitrary now:

$$\tilde{\rho} = \tilde{\rho}(\mathbf{r}, t) = \hat{\rho}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\,, \quad \text{etc.} \tag{29}$$

yields an algebraic system of eigenvalue equations:

$$\begin{aligned}
c\,\mathbf{k}\cdot\hat{\mathbf{v}} &= \gamma\,\omega\,\hat{\rho}\,, \\
\mathbf{k}\,c\,\hat{\rho} + \mathbf{k}\,c\,\hat{e} + (\mathbf{k}\mathbf{b}\cdot - \mathbf{k}\cdot\mathbf{b})\,\hat{\mathbf{B}} &= \omega\,\hat{\mathbf{v}}\,, \\
c\,\mathbf{k}\cdot\hat{\mathbf{v}} &= \tfrac{\gamma}{\gamma-1}\,\omega\,\hat{e}\,, \\
(\mathbf{b}\mathbf{k}\cdot - \mathbf{b}\cdot\mathbf{k})\,\hat{\mathbf{v}} &= \omega\,\hat{\mathbf{B}}\,, \qquad \mathbf{k}\cdot\hat{\mathbf{B}} = 0\,.
\end{aligned} \tag{30}$$

- $\Rightarrow$ **Symmetric eigenvalue problem!** (The equations for $\hat{\rho}$, $\hat{\mathbf{v}}$, $\hat{e}$, and $\hat{\mathbf{B}}$ appear to know about each other.) .

- The symmetry of the linearized system is closely related to an analogous property of the original nonlinear equations: *the nonlinear ideal MHD equations are symmetric hyperbolic partial differential equations*.

## Matrix eigenvalue problem

- Choose $\mathbf{b} = (0, 0, b)\,, \quad \mathbf{k} = (k_\perp, 0, k_\parallel)$ :

$$\begin{pmatrix} 0 & k_\perp c & 0 & k_\parallel c & 0 & 0 & 0 & 0 \\ k_\perp c & 0 & 0 & 0 & k_\perp c & -k_\parallel b & 0 & k_\perp b \\ 0 & 0 & 0 & 0 & 0 & 0 & -k_\parallel b & 0 \\ k_\parallel c & 0 & 0 & 0 & k_\parallel c & 0 & 0 & 0 \\ 0 & k_\perp c & 0 & k_\parallel c & 0 & 0 & 0 & 0 \\ 0 & -k_\parallel b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k_\parallel b & 0 & 0 & 0 & 0 & 0 \\ 0 & k_\perp b & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat\rho \\ \hat v_x \\ \hat v_y \\ \hat v_z \\ \hat e \\ \hat B_x \\ \hat B_y \\ \hat B_z \end{pmatrix} = \omega \begin{pmatrix} \gamma\,\hat\rho \\ \hat v_x \\ \hat v_y \\ \hat v_z \\ \frac{\gamma}{\gamma-1}\,\hat e \\ \hat B_x \\ \hat B_y \\ \hat B_z \end{pmatrix} . \tag{31}$$

$\Rightarrow$ **Another representation of the symmetry of linearized MHD equations.**

- New features of MHD waves compared to sound: occurrence of Alfvén speed $b$ and anisotropy expressed by the two components $k_\parallel$ and $k_\perp$ of the wave vector. We could compute the dispersion equation from the determinant and study the associated waves, but we prefer again to exploit the much simpler velocity representation.

# MHD wave equation

- Ignoring the magnetic field constraint $\mathbf{k}\cdot\hat{\mathbf{B}} = 0$ in the $8\times 8$ eigenvalue problem (31) would yield *one spurious eigenvalue* $\omega = 0$. This may be seen by operating with the projector $\mathbf{k}\cdot$ onto Eq. (30)(d), which gives $\omega\,\mathbf{k}\cdot\hat{\mathbf{B}} = 0$.

- Like in the gas dynamics problem, a *genuine but unimportant marginal entropy mode* is obtained for $\omega = 0$ with $\hat{\mathbf{v}} = 0$, $\hat{p} = 0$, and $\hat{\mathbf{B}} = 0$:

$$\omega = 0\,, \qquad \hat{p} = \hat{e} + \hat{\rho} = 0\,, \qquad \hat{S} = \gamma\hat{e} = -\gamma\hat{\rho} \neq 0\,. \tag{32}$$

- Both of these marginal modes are eliminated by exploiting *the velocity representation.* The perturbations $\rho_1$, $e_1$, $\mathbf{B}_1$ are expressed in terms of $\mathbf{v}_1$ by means of Eqs. (19), (21), and (22), and substituted into the momentum equation (20). This yields the **MHD wave equation for a homogeneous medium**:

$$\frac{\partial^2 \mathbf{v}_1}{\partial t^2} - \Big[\,(\mathbf{b}\cdot\nabla)^2\,\mathbf{I} + (b^2 + c^2)\,\nabla\,\nabla - \mathbf{b}\cdot\nabla\,(\nabla\,\mathbf{b} + \mathbf{b}\,\nabla)\Big]\cdot\mathbf{v}_1 = 0\,. \tag{33}$$

The sound wave equation (8) is obtained for the special case $\mathbf{b} = 0$.

# MHD wave equation (cont'd)

- Inserting plane wave solutions gives the required eigenvalue equation:

$$\left\{ \left[ \omega^2 - (\mathbf{k}\cdot\mathbf{b})^2 \right] \mathbf{I} - (b^2 + c^2)\, \mathbf{k}\,\mathbf{k} + \mathbf{k}\cdot\mathbf{b}\,(\mathbf{k}\,\mathbf{b} + \mathbf{b}\,\mathbf{k}) \right\} \cdot \hat{\mathbf{v}} = 0 \,, \qquad (34)$$

or, in components:

$$\begin{pmatrix} -\,k_\perp^2 (b^2 + c^2) - k_\parallel^2 b^2 & 0 & -k_\perp k_\parallel c^2 \\ 0 & -k_\parallel^2 b^2 & 0 \\ -\,k_\perp k_\parallel c^2 & 0 & -k_\parallel^2 c^2 \end{pmatrix} \begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{pmatrix} = -\omega^2 \begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{pmatrix} . \qquad (35)$$

Hence, a $3 \times 3$ symmetric matrix equation is obtained in terms of the variable $\hat{\mathbf{v}}$, with *quadratic eigenvalue* $\omega^2$, corresponding to the original $6 \times 6$ representation with eigenvalue $\omega$ (resulting from elimination of the two marginal modes).

- Determinant yields the **dispersion equation:**

$$\det = \omega\,(\omega^2 - k_\parallel^2 b^2) \left[ \omega^4 - k^2 (b^2 + c^2)\, \omega^2 + k_\parallel^2 k^2 b^2 c^2 \right] = 0 \qquad (36)$$

(where we have artificially included a factor $\omega$ for the marginal entropy wave).

## Roots

1) **Entropy waves:**

$$\omega = \omega_E \equiv 0\,, \tag{37}$$

$$\hat{\mathbf{v}} = \hat{\mathbf{B}} = 0\,, \quad \hat{p} = 0\,, \quad \text{but} \quad \hat{s} \neq 0\,. \tag{38}$$

$\Rightarrow$ just perturbation of thermodynamic variables.

2) **Alfvén waves:**

$$\omega^2 = \omega_A^2 \equiv k_\parallel^2 b^2 \quad \rightarrow \quad \omega = \pm\omega_A\,, \tag{39}$$

$$\hat{v}_x = \hat{v}_z = \hat{B}_x = \hat{B}_z = \hat{s} = \hat{p} = 0\,, \quad \hat{B}_y = -\hat{v}_y \neq 0\,. \tag{40}$$

$\Rightarrow$ transverse $\hat{\mathbf{v}}$ and $\hat{\mathbf{B}}$ so that field lines follow the flow.

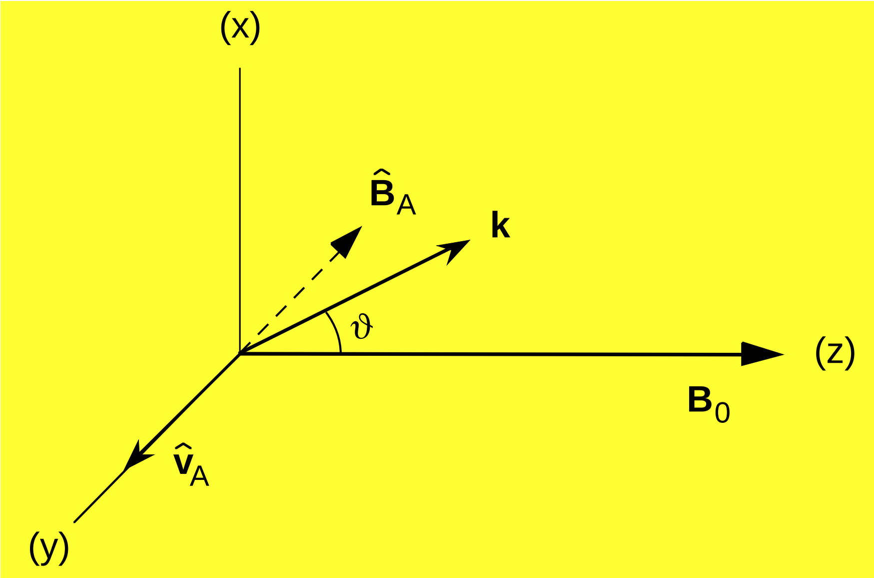
3) **Fast ($+$) and Slow ($-$) magnetoacoustic waves:**

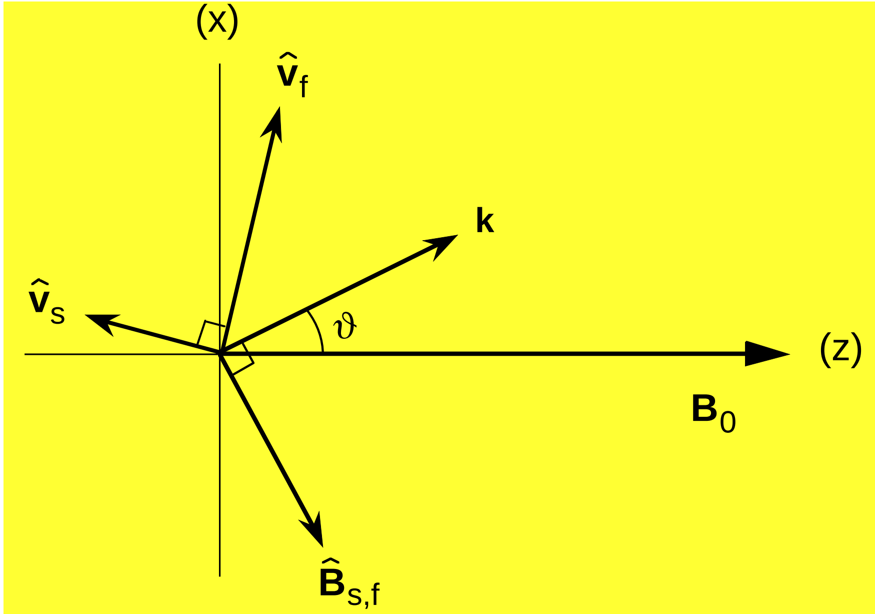
$$\omega^2 = \omega_{s,f}^2 \equiv \tfrac{1}{2}k^2(b^2+c^2)\,\Big[1 \pm \sqrt{1 - \frac{4k_\parallel^2 b^2 c^2}{k^2(b^2+c^2)^2}}\,\Big] \quad \rightarrow \quad \omega = \left\{ \begin{array}{l} \pm\omega_s \\ \pm\omega_f \end{array} \right. \tag{41}$$

$$\hat{v}_y = \hat{B}_y = \hat{s} = 0\,, \quad \text{but} \quad \hat{v}_x\,, \hat{v}_z\,, \hat{p}\,, \hat{B}_x\,, \hat{B}_z \neq 0\,, \tag{42}$$

$\Rightarrow$ perturbations $\hat{\mathbf{v}}$ and $\hat{\mathbf{B}}$ in the plane through $\mathbf{k}$ and $\mathbf{B}_0$.

# Eigenfunctions




**Alfvén waves** **Magnetosonic waves**

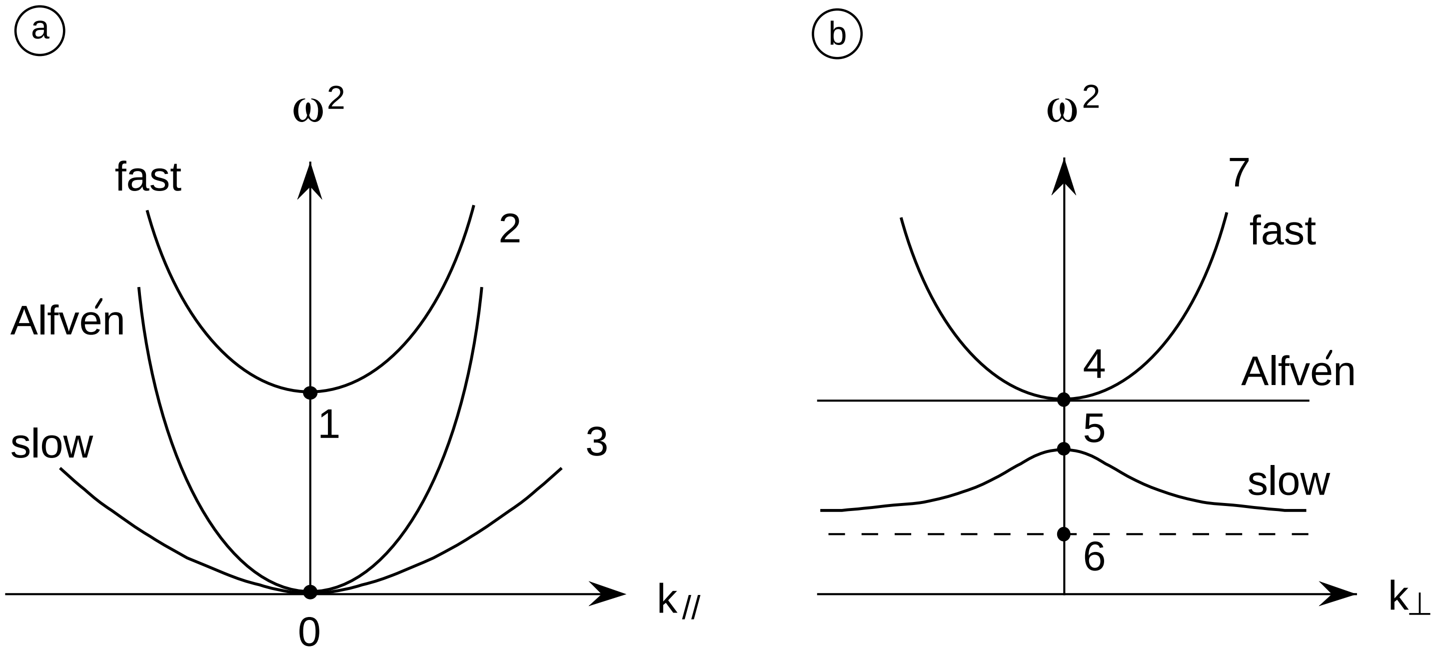
- Note: **the eigenfunctions are mutually orthogonal:**

$$\hat{\mathbf{v}}_s \perp \hat{\mathbf{v}}_A \perp \hat{\mathbf{v}}_f \, . \tag{43}$$

$\Rightarrow$ Arbitrary velocity field may be decomposed at all times (e.g. at $t = 0$) in the three MHD waves: **the initial value problem is a well-posed problem**.

# Dispersion diagrams (schematic)

[ exact diagrams in book: Fig. 5.3, scaling $\bar{\omega} \equiv (l/b)\,\omega\,,\ \bar{k} \equiv k\,l$ ]




- Note: $\omega^2(k_\parallel\!=\!0) = 0$ for Alfvén and slow waves $\Rightarrow$ potential onset of *instability.*
- Asymptotics of $\omega^2(k_\perp \to \infty)$ characterizes *local* behavior of the three waves:

$$\begin{cases} \partial\omega/\partial k_\perp > 0\,, & \omega_f^2 \to \infty & \text{for fast waves,} \\ \partial\omega/\partial k_\perp = 0\,, & \omega_A^2 \to k_\parallel^2 b^2 & \text{for Alfvén waves,} \\ \partial\omega/\partial k_\perp < 0\,, & \omega_s^2 \to k_\parallel^2 \dfrac{b^2 c^2}{b^2 + c^2} & \text{for slow waves.} \end{cases} \qquad (44)$$

## Phase and group velocity

Dispersion equation $\omega = \omega(\mathbf{k}) \;\Rightarrow\;$ two fundamental concepts:

1. A *single plane wave* propagates in the direction of $\mathbf{k}$ with the **phase velocity**

$$\mathbf{v}_{\rm ph} \equiv \frac{\omega}{k}\,\mathbf{n}\,, \qquad \mathbf{n} \equiv \mathbf{k}/k = (\sin\vartheta, 0, \cos\vartheta)\,; \tag{45}$$

$\Rightarrow$ MHD waves are non-dispersive (only depend on angle $\vartheta$, not on $|\mathbf{k}|$):

$$(\mathbf{v}_{\rm ph})_A \;\equiv\; b\cos\vartheta\,\mathbf{n}\,, \tag{46}$$

$$(\mathbf{v}_{\rm ph})_{s,f} \;\equiv\; \sqrt{\tfrac{1}{2}(b^2+c^2)}\,\sqrt{1 \pm \sqrt{1-\sigma\cos^2\vartheta}}\;\mathbf{n}\,, \qquad \sigma \equiv \frac{4b^2c^2}{(b^2+c^2)^2}\,. \tag{47}$$

2. A *wave packet* propagates with the **group velocity**

$$\mathbf{v}_{\rm gr} \equiv \frac{\partial\omega}{\partial\mathbf{k}} \quad \Big[ \equiv \frac{\partial\omega}{\partial k_x}\mathbf{e}_x + \frac{\partial\omega}{\partial k_y}\mathbf{e}_y + \frac{\partial\omega}{\partial k_z}\mathbf{e}_z \Big]\,; \tag{48}$$

$\Rightarrow$ MHD caustics in directions $\mathbf{b}$, and mix of $\mathbf{n}$ and $\mathbf{t}$ ($\perp\,\mathbf{n}$):

$$(\mathbf{v}_{\rm gr})_A \;=\; \mathbf{b}\,, \tag{49}$$

$$(\mathbf{v}_{\rm gr})_{s,f} \;=\; (v_{\rm ph})_{s,f}\left[\mathbf{n} \pm \frac{\sigma\sin\vartheta\cos\vartheta}{2\sqrt{1-\sigma\cos^2\vartheta}\,\left[1\pm\sqrt{1-\sigma\cos^2\vartheta}\right]}\,\mathbf{t}\right]. \tag{50}$$

## Wave packet

Wave packet of plane waves satisfying dispersion equation $\omega = \omega(\mathbf{k})$ :

$$\Psi_i(\mathbf{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} A_i(\mathbf{k})\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega(\mathbf{k})t)}\, d^3k\,. \tag{51}$$

Evolves from initial shape given by *Fourier synthesis,*

$$\Psi_i(\mathbf{r}, 0) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} A_i(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3k\,, \tag{52}$$

where amplitudes $A_i(\mathbf{k})$ are related to initial values $\Psi_i(\mathbf{r}, 0)$ by *Fourier analysis,*

$$A_i(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \Psi_i(\mathbf{r}, 0)\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^3r\,. \tag{53}$$

MHD: $\Psi_i$ – perturbations ($\tilde{\rho}_1$ ,) $\tilde{\mathbf{v}}_1$ (, $\tilde{e}_1$, $\tilde{\mathbf{B}}_1$); $A_i$ – Fourier amplitudes ($\hat{\rho}_1$ ,) $\hat{\mathbf{v}}_1$ (, $\hat{e}_1$, $\hat{\mathbf{B}}_1$).

*Example:* Gaussian wave packet of harmonics centered at some wave vector $\mathbf{k}_0$ ,

$$A_i(\mathbf{k}) = \hat{A}_i\, e^{-\frac{1}{2}|(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{a}|^2}\,, \tag{54}$$

corresponds to initial packet with main harmonic $\mathbf{k}_0$ and modulated amplitude centered at $\mathbf{r} = 0$ :

$$\Psi_i(\mathbf{r}, 0) = e^{i\mathbf{k}_0\cdot\mathbf{r}} \times \frac{\hat{A}_i}{a_x a_y a_z}\, e^{-\frac{1}{2}[(x/a_x)^2 + (y/a_y)^2 + (z/a_z)^2]}\,. \tag{55}$$

## Wave packet (cont'd)

For arbitrary wave packet with localized range of wave vectors, we may expand the dispersion equation about the central value $\mathbf{k}_0$:

$$\omega(\mathbf{k}) \approx \omega_0 + (\mathbf{k} - \mathbf{k}_0) \cdot \left(\frac{\partial \omega}{\partial \mathbf{k}}\right)_{\mathbf{k}_0} , \qquad \omega_0 \equiv \omega(\mathbf{k}_0) . \tag{56}$$

Inserting this approximation in the expression (51) for the wave packet gives

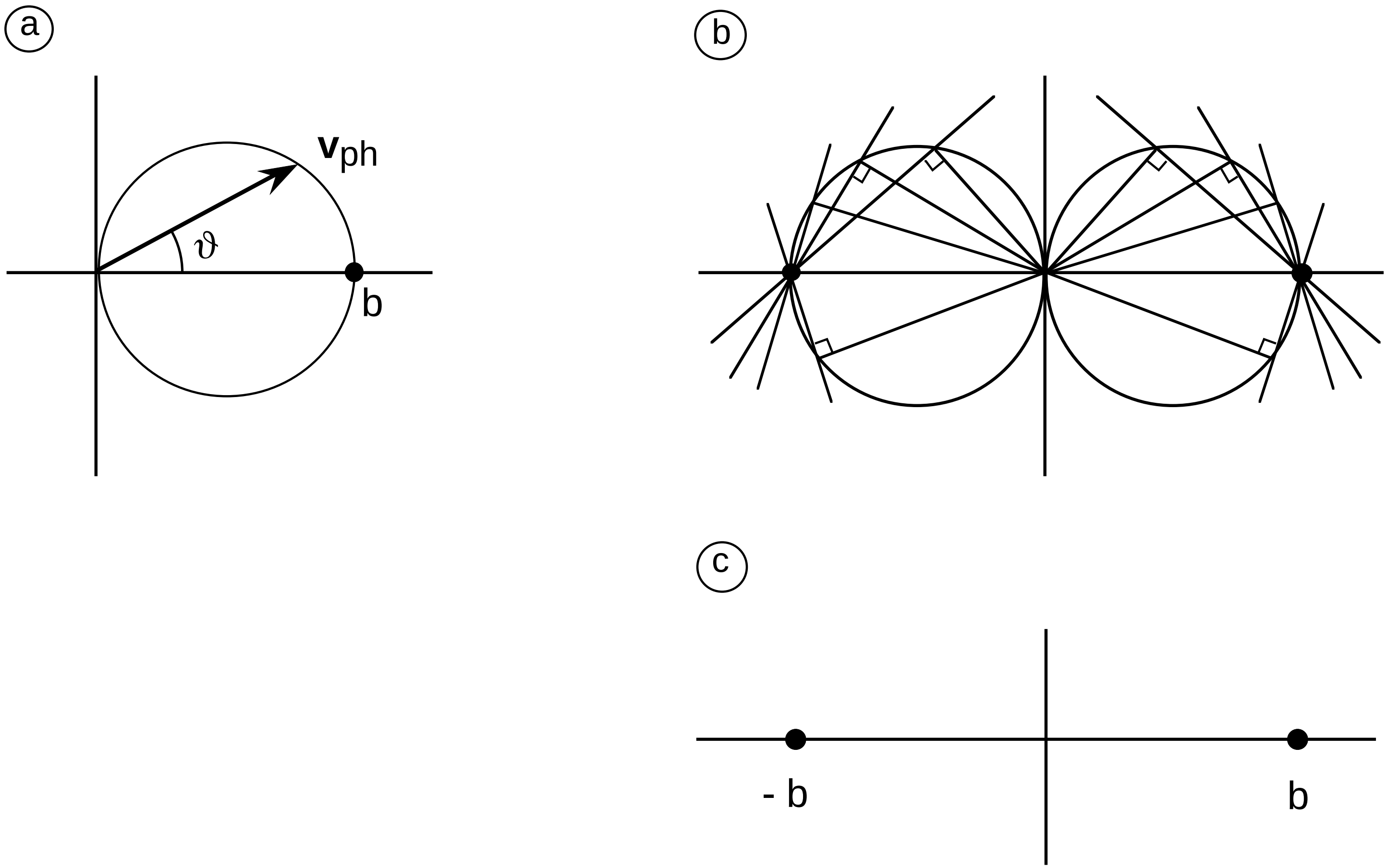
$$\Psi_i(\mathbf{r}, t) \approx e^{i(\mathbf{k}_0 \cdot \mathbf{r} - \omega_0 t)} \times \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} A_i(\mathbf{k})\, e^{i(\mathbf{k} - \mathbf{k}_0) \cdot (\mathbf{r} - (\partial \omega / \partial \mathbf{k})_{\mathbf{k}_0} t)}\, d^3 k , \tag{57}$$

representing a carrier wave $\exp i(\mathbf{k}_0 \cdot \mathbf{r} - \omega_0 t)$ with an amplitude-modulated envelope. Through constructive interference of the plane waves, *the envelope maintains its shape* during an extended interval of time, whereas *the surfaces of constant phase of the envelope move with the group velocity,*

$$\mathbf{v}_{\rm gr} = \left(\frac{d\mathbf{r}}{dt}\right)_{\rm const.\,phase} = \left(\frac{\partial \omega}{\partial \mathbf{k}}\right)_{\mathbf{k}_0} , \tag{58}$$

in agreement with the definition (48).

## Example: Alfvén waves

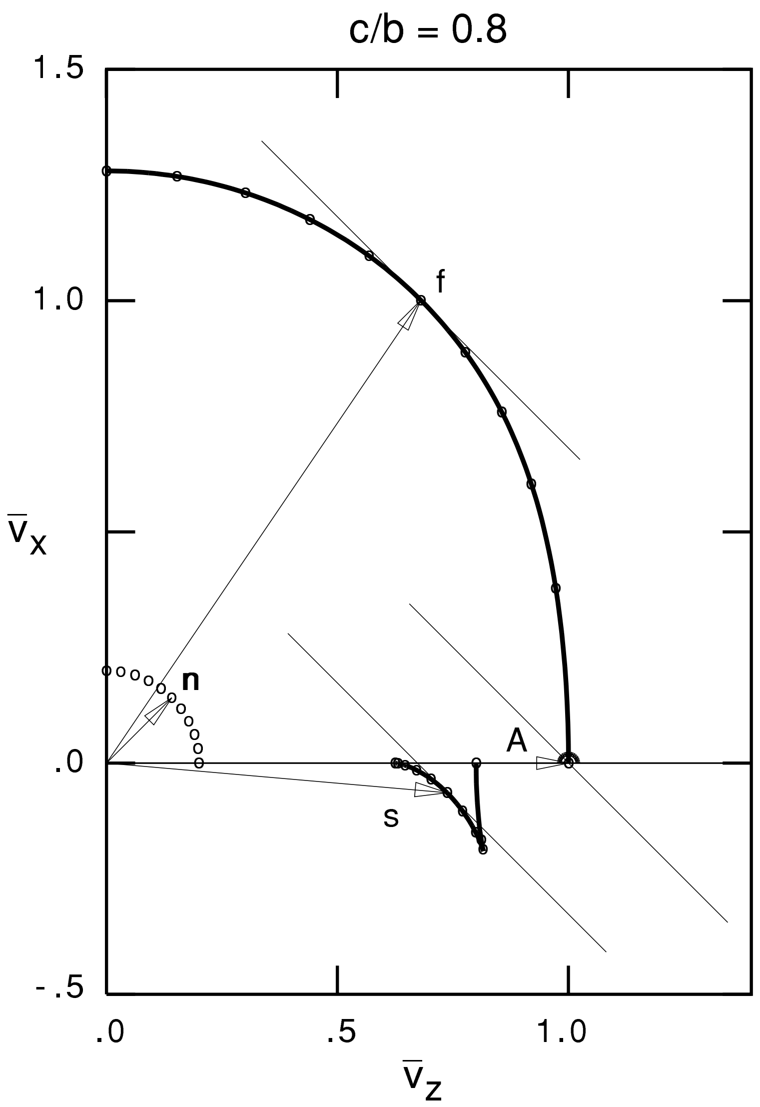



(a) Phase diagram for Alfvén waves is circle $\Rightarrow$ (b) wavefronts pass through points $\pm b$
$\Rightarrow$(c) those points are the group diagram.

# Group diagram: queer behavior

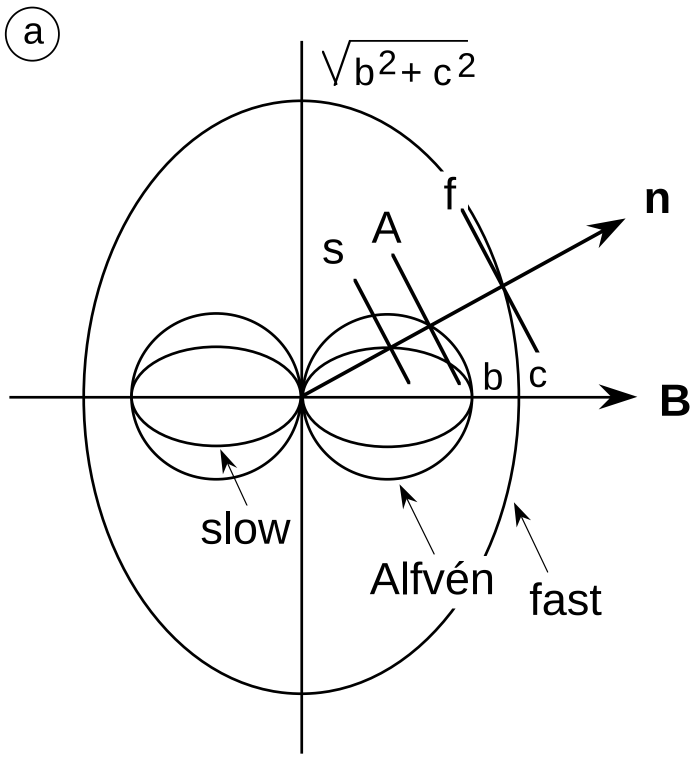
- *Group diagrams with $\mathbf{v}_{\rm gr}$ relative to $\mathbf{n}$ for the three MHD waves in the first quadrant.*

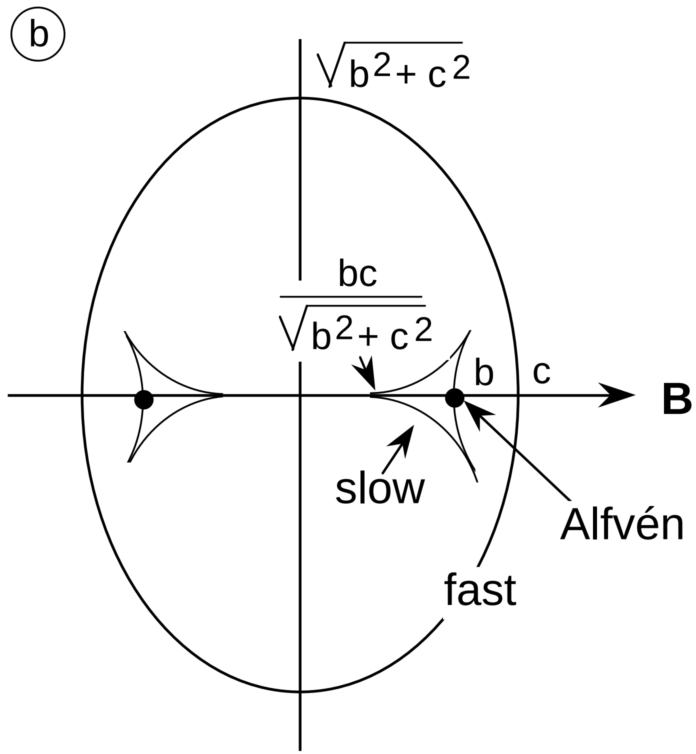
  Group velocities exhibit *mutually exclusive directions of propagation:* When $\mathbf{n}$ goes from $\vartheta = 0$ ($\parallel \mathbf{B}$) to $\vartheta = \pi/2$ ($\perp \mathbf{B}$), the fast group velocity changes from parallel to perpendicular (though it does not remain parallel to $\mathbf{n}$), the Alfvén group velocity remains purely parallel, but the slow group velocity initially changes *clockwise* from parallel to some negative angle and then back again to purely parallel. In the perpendicular direction, *slow wave packages propagate opposite to direction of $\mathbf{n}$!*



# Friedrichs diagrams (schematic)

[ exact diagrams in book: Fig. 5.5, parameter $c/b = \frac{1}{2}\gamma\beta\,,\ \beta \equiv 2p/B^2$ ]




**Phase diagram**
*(plane waves)*

**Group diagram**
*(point disturbances)*

## Summary

- [ *Entropy waves:* non-propagating density / entropy perturbations; ]
- *Alfvén waves:* incompressible velocity perturbations $\perp$ plane of $\mathbf{k}$ & $\mathbf{B}$, preferably propagating $\|$ $\mathbf{B}$ ;
- *Fast magnetoacoustic waves:* compressible velocity perturbations in the plane of $\mathbf{k}$ & $\mathbf{B}$, generalization of sound waves with contributions of the magnetic pressure, propagating in all directions but fastest $\perp$ $\mathbf{B}$ ;
- *Slow magnetoacoustic waves:* compressible velocity perturbations in plane of $\mathbf{k}$ & $\mathbf{B}$ , kind of sound waves with impeded propagation $\perp$ $\mathbf{B}$ (orthogonal to fast modes).

## Connection with next subject

Group diagram has a much wider applicability than just wave propagation in infinite homogeneous plasmas: Construction of wave packet involves contributions of large $\mathbf{k}$ (small wavelengths) so that the **concept of group velocity is essentially a local one**. It returns in *non-linear MHD of inhomogeneous plasmas*, where the associated concept of **characteristics** describes the propagation of initial data information through the plasma.

**Example: point perturbation triggers MHD waves in uniform plasma** (friedrichs.qt)

## Method

- *Linear advection equation* in one spatial dimension with unknown $\Psi(x,t)$,

$$\frac{\partial \Psi}{\partial t} + u\,\frac{\partial \Psi}{\partial x} = 0\,, \tag{59}$$

and given advection velocity $u$. For $u = \mathrm{const}$, the solution is trivial:

$$\Psi = f(x - ut)\,, \quad \text{where} \quad f = \Psi_0 \equiv \Psi(x, t = 0)\,. \tag{60}$$

$\Rightarrow$ Initial data $\Psi_0$ propagate along *characteristics:* parallel straight lines $dx/dt = u$.

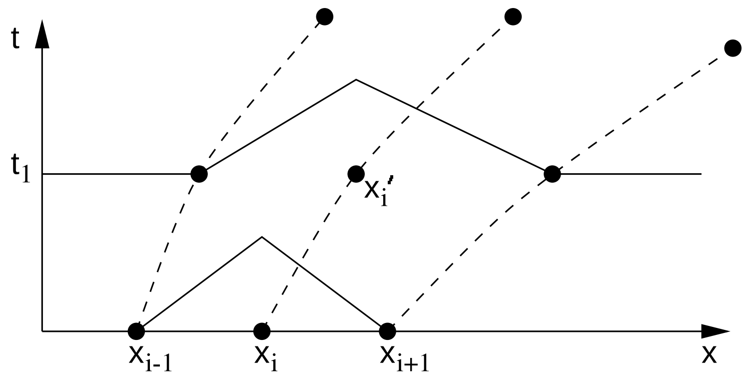
- For $u$ not constant, characteristics become solutions of the ODEs

$$\frac{dx}{dt} = u(x,t)\,. \tag{61}$$

Along these curves, solution $\Psi(x,t)$ of (59) is const:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} \equiv \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial x}\frac{dx}{dt} = 0\,. \tag{62}$$

$\Rightarrow$ For given initial data, the solution can be determined at any time $t_1 > 0$ by constructing *characteristics through suitable set of points.* E.g., $\Psi(x_i', t_1) = \Psi_0(x_i)$ for ‘tent’ function.



## Method (cont'd)

- The method of characteristics generalizes to *nonlinear* partial differential equations: basis of modern developments in *computational (magneto-)fluid dynamics* [C(M)FD].
- Example: *Quasi-linear advection equation* when $u$ is also a function of the unknown $\Psi$ itself. With $u = \Psi$, we obtain *Burgers' equation:*

$$\frac{\partial \Psi}{\partial t} + \Psi \frac{\partial \Psi}{\partial x} = \nu \frac{\partial^2 \Psi}{\partial x^2} \,, \tag{63}$$

where viscous RHS models balance between nonlinear and dissipative processes. At first neglecting this small term, the characteristics are the solutions of the ODE

$$\frac{dx}{dt} = \Psi(x(t), t) \,, \tag{64}$$

which are just a set of straight lines with slopes determined by the initial data. For large times, the characteristics will cross, but the build-up of large gradients is counteracted by smoothing through the dissipative RHS term. This occurs in a very narrow region, so that a valid solution with a *shock* is obtained in the limit $\nu \to 0$.

# Classification of PDEs

- Quasi-linear second order PDE in two dimensions:

$$A(\Phi_x, \Phi_y, x, y)\, \Phi_{xx} + 2B(\ldots)\, \Phi_{xy} + C(\ldots)\, \Phi_{yy} = D(\ldots)\,, \qquad (65)$$

With $\Psi_1 \equiv \Phi_x\,, \Psi_2 \equiv \Phi_y \;\Rightarrow\;$ equivalent system of first order equations:

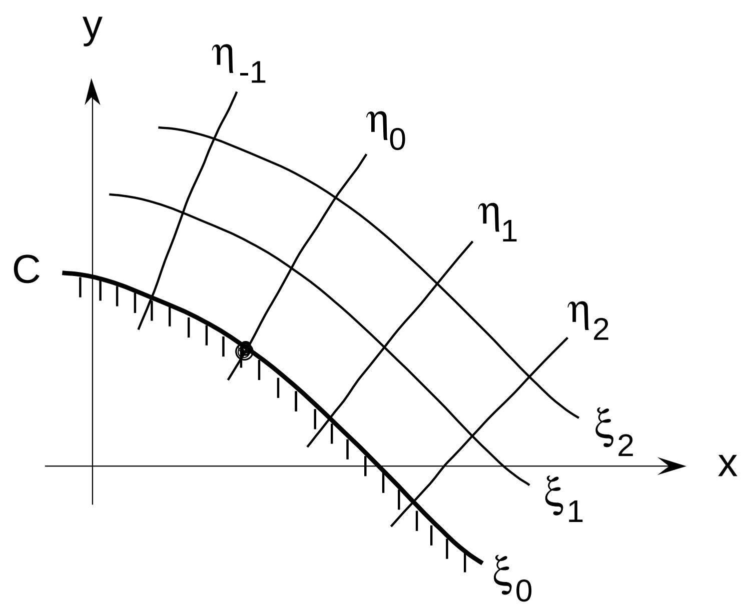
$$\begin{aligned} A(\Psi_1, \Psi_2, x, y)\, \Psi_{1x} + B(\ldots)\, \Psi_{1y} + B(\ldots)\, \Psi_{2x} + C(\ldots)\, \Psi_{2y} &= D(\ldots)\,, \\ \Psi_{1y} - \Psi_{2x} &= 0\,. \end{aligned} \qquad (66)$$

*Cauchy problem:* find $\Psi_1$ and $\Psi_2$ away from boundary $C$ when they are given on it.

- Replace coordinates $x, y$ by boundary fitted coordinates $\xi, \eta$, where boundary $C$ is given by $\xi(x, y) = \xi_0$. Boundary data become

$$\Psi_1(\xi_0, \eta) = f_1(\eta)\,, \quad \Psi_2(\xi_0, \eta) = f_2(\eta)\,. \qquad (67)$$

We wish to investigate under which conditions $\Psi_1(\xi, \eta)$ and $\Psi_2(\xi, \eta)$ may be obtained by means of a power series solution about a particular point $(\xi_0, \eta_0)$ on the boundary.



# Classification (cont'd)

- Power series:

$$\begin{aligned}\Psi_1(\xi,\eta) &= \Psi_1(\xi_0,\eta_0) + (\xi-\xi_0)\left(\frac{\partial\Psi_1}{\partial\xi}\right)_0 + (\eta-\eta_0)\left(\frac{\partial\Psi_1}{\partial\eta}\right)_0 + \cdots\,,\\ \Psi_2(\xi,\eta) &= \Psi_2(\xi_0,\eta_0) + (\xi-\xi_0)\left(\frac{\partial\Psi_2}{\partial\xi}\right)_0 + (\eta-\eta_0)\left(\frac{\partial\Psi_2}{\partial\eta}\right)_0 + \cdots\,.\end{aligned} \tag{68}$$

  Green expressions known from boundary conditions (67) $\Rightarrow$ we need to investigate under which circumstances remaining expressions $(\partial\Psi_i/\partial\xi)_0$ can be calculated.

- Transform PDEs (66) to $\xi$-$\eta$ coordinates:

$$\begin{aligned}(A\xi_x + B\xi_y)\frac{\partial\Psi_1}{\partial\xi} + (B\xi_x + C\xi_y)\frac{\partial\Psi_2}{\partial\xi} \\ = D - (A\eta_x + B\eta_y)\frac{\partial\Psi_1}{\partial\eta} - (B\eta_x + C\eta_y)\frac{\partial\Psi_2}{\partial\eta}\,, \end{aligned} \tag{69}$$

$$\xi_y\frac{\partial\Psi_1}{\partial\xi} - \xi_x\frac{\partial\Psi_2}{\partial\xi} = -\eta_y\frac{\partial\Psi_1}{\partial\eta} + \eta_x\frac{\partial\Psi_2}{\partial\eta}\,.$$

  The unknown derivatives $\partial\Psi_1/\partial\xi$ and $\partial\Psi_2/\partial\xi$ may be determined from Eqs. (69) **if the determinant of the coefficients on the left hand side does not vanish**.

## Classification (cont'd)

- Vice versa, condition that the determinant vanishes,

$$\begin{vmatrix} A\xi_x + B\xi_y & B\xi_x + C\xi_y \\ \xi_y & -\xi_x \end{vmatrix} = -A\xi_x^2 - 2B\xi_x\xi_y - C\xi_y^2 = 0\,, \qquad (70)$$

  defines two directions in every point of the plane, *the characteristic directions*, **along which posing Cauchy boundary conditions does not determine the solution:**

$$\left.\frac{dy}{dx}\right|_{\mathrm{char}} = -\frac{\xi_x}{\xi_y} = \frac{B \pm \sqrt{B^2 - AC}}{A}\,. \qquad (71)$$

- Three cases:
  (a) $B^2 > AC \;\Rightarrow$ characteristics are real: *hyperbolic* equation
  (example: wave equation $\Phi_{xx} - (1/c^2)\Phi_{tt} = 0$);
  (b) $B^2 = AC \;\Rightarrow$ characteristics are real but coincide: *parabolic* equation
  (example: heat equation $\Phi_{xx} - (1/\lambda)\Phi_t = 0$);
  (c) $B^2 < AC \;\Rightarrow$ characteristics are complex: *elliptic* equation
  (example: Laplace's equation $\Phi_{xx} + \Phi_{yy} = 0$).

# Apply to MHD equations

- Instead of 2-vector $(\Psi_1, \Psi_2)$ : 8-vector $\Psi_i$ $(i = 1, \cdots 8)$ for variables $\rho$, $\mathbf{v}$, $e$, $\mathbf{B}$ $(\mathbf{r}, t)$.
- We will prove: MHD equations are *symmetric hyperbolic* PDEs; they posses complete set of *real characteristics related to the eigenvalues of the linearized system.*
- Apply same method as before: Assume boundary data for $\rho$, $\mathbf{v}$, $e$, $\mathbf{B}$ to be given on a 3-dimensional manifold in 4-dimensional space-time $\mathbf{r}$, $t$:

$$\xi(\mathbf{r}, t) = \xi_0 \,. \tag{72}$$

  (Visualize as being swept out by motion of 2-D surfaces in ordinary 3-D space $(\mathbf{r})$ when time $t$ progresses.)
- **Duality**: – If this manifold is characteristic $\Rightarrow$ Cauchy problem ill-posed on it;
  – If this manifold is not characteristic $\Rightarrow$ Cauchy problem well-posed on it.
- Hence, for IVP in MHD (where $\rho(\mathbf{r}, 0)$, $\mathbf{v}(\mathbf{r}, 0)$, $e(\mathbf{r}, 0)$, $\mathbf{B}(\mathbf{r}, 0)$ are given on domain in ordinary 3-space) to be well-posed, ordinary 3-space should not be a characteristic.
- We will prove that the characteristics in MHD are real 3-dimensional manifolds involving time, so that *the IVP in MHD is well-posed.*

# Application (cont'd)

- Cover 4-space $(\mathbf{r}, t)$ by boundary-fitted coordinates $\xi$, $\eta$, $\zeta$, $\tau$, and try power series:

$$\rho(\xi,\eta,\zeta,\tau) = \rho_0(\eta_0,\zeta_0,\tau_0) + (\xi-\xi_0)\left(\frac{\partial\rho}{\partial\xi}\right)_0 + (\eta-\eta_0)\left(\frac{\partial\rho}{\partial\eta}\right)_0 + (\zeta-\zeta_0)\left(\frac{\partial\rho}{\partial\zeta}\right)_0 + (\tau-\tau_0)\left(\frac{\partial\rho}{\partial\tau}\right)_0 + \cdots \quad \big(\text{etc. for } \mathbf{v},\, e,\, \mathbf{B}\big)\,. \tag{73}$$

- Problem solvable if unknowns $(\partial\rho/\partial\xi)_0\,,\ (\partial\mathbf{v}/\partial\xi)_0\,,\ (\partial e/\partial\xi)_0\,,\ (\partial\mathbf{B}/\partial\xi)_0$ can be constructed from MHD equations. Indicate those by a prime:

$$\begin{aligned}\nabla f &= \nabla\xi\, f' + \nabla\eta\,\frac{\partial f}{\partial\eta} + \nabla\zeta\,\frac{\partial f}{\partial\zeta} + \nabla\tau\,\frac{\partial f}{\partial\tau}\,,\\ \frac{\mathrm{D}f}{\mathrm{D}t} &= (\xi_t + \mathbf{v}\cdot\nabla\xi)\, f' + (\eta_t + \mathbf{v}\cdot\nabla\eta)\,\frac{\partial f}{\partial\eta} + (\zeta_t + \mathbf{v}\cdot\nabla\zeta)\,\frac{\partial f}{\partial\zeta} + (\tau_t + \mathbf{v}\cdot\nabla\tau)\,\frac{\partial f}{\partial\tau}.\end{aligned} \tag{74}$$

- Translation recipe (similar to shock recipe of Sec.4.5):

$$\begin{aligned}\nabla f &\rightarrow \mathbf{n} f' + \cdots\,, \qquad \mathbf{n} \equiv \nabla\xi : \ \textit{normal to the characteristic},\\ \frac{\mathrm{D}f}{\mathrm{D}t} &\rightarrow -u f' + \cdots\,, \qquad -u \equiv \xi_t + \mathbf{v}\cdot\nabla\xi : \ \textit{characteristic speed}.\end{aligned} \tag{75}$$

## Application (cont'd)

- This give:

$$
\begin{aligned}
-u\rho' + \rho\,\mathbf{n}\cdot\mathbf{v}' &= \cdots\,, \\
-\rho u\mathbf{v}' + (\gamma-1)\,\mathbf{n}\,(e\rho' + \rho e') + (\mathbf{n}\,\mathbf{B}\cdot - \mathbf{n}\cdot\mathbf{B})\,\mathbf{B}' &= \cdots\,, \\
-ue' + (\gamma-1)e\,\mathbf{n}\cdot\mathbf{v}' &= \cdots\,, \\
-u\mathbf{B}' + (\mathbf{B}\,\mathbf{n}\cdot - \mathbf{n}\cdot\mathbf{B})\,\mathbf{v}' &= \cdots\,, \qquad \mathbf{n}\cdot\mathbf{B}' = \cdots\,.
\end{aligned}
\tag{76}
$$

*LHS analogous to EVP (30) for linear MHD waves,* where $\mathbf{k}\rightarrow\mathbf{n}$ and $\omega\rightarrow u$ !

- **Duality:** – Values $\rho'$ , $\mathbf{v}'$ , $e'$ , $\mathbf{B}'$ may not be found if

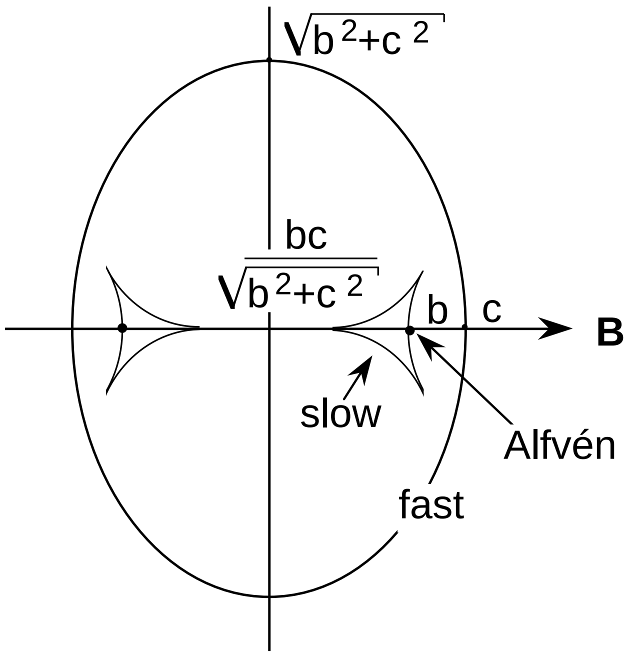
$$\Delta \equiv u(u^2 - b_n^2)\left[\,u^4 - (b^2 + c^2)u^2 + b_n^2 c^2\,\right] = 0 \quad\Rightarrow\quad \xi_0 \text{ characteristic}\,; \tag{77}$$

– Values $\rho'$ , $\mathbf{v}'$ , $e'$ , $\mathbf{B}'$ may be found if

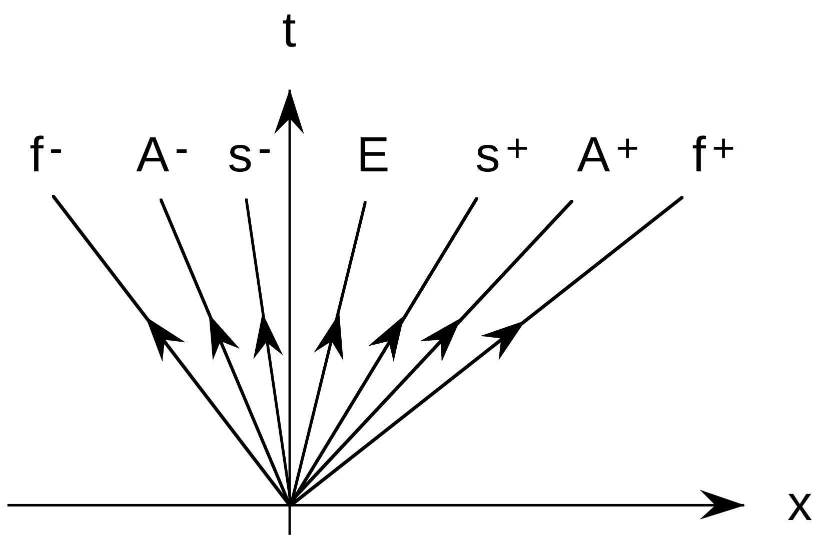
$\Delta \neq 0 \quad \Rightarrow \quad \xi_0$ not characteristic (solutions may be propagated away from it) .

$\Rightarrow$ *7 real characteristics*, corresponding to 7 linear waves (entropy, Alfvén, slow, fast). **The equations of ideal MHD are symmetric hyperbolic equations**, and **the initial value problem is well-posed** (Friedrichs).

## Application (cont'd)




Group diagram is the *ray surface*, i.e. the spatial part of characteristic manifold at certain time $t_0$.




*$x$-$t$ cross-sections of 7 characteristics*
($x$-axis oblique with respect to $\mathbf{B}$; inclination of entropy mode E indicates plasma background flow).

- *Locality* of group diagrams and characteristics neglects *global plasma inhomogeneity.*
  $\Rightarrow$ Next topic is *waves and instabilities in inhomogeneous plasmas*.