Chapter 4: The MHD model

Overview

- **The ideal MHD equations:** postulating the basic equations, scale independence, what is a physical model?; [book: Sec. 4.1]
- **Magnetic flux:** flux tubes, global magnetic flux conservation; [book: Sec. 4.2]
- **Conservation laws:** conservation form of the equations, global conservation laws, local conservation laws – conservation of magnetic flux; [book: Sec. 4.3]
- **Discontinuities:** shocks and jump conditions, boundary conditions for interface plasmas; [book: Sec. 4.5]
- **Model problems:** laboratory models I–III, astrophysical models IV–VI. [book: Sec. 4.6]

Postulating the basic equations

Equations of magnetohydrodynamics can be introduced by

- *averaging the kinetic equations* by moment expansion and closure through transport theory (book: Chaps. 2 and 3);
- just *posing them as postulates* for a hypothetical medium called 'plasma' and use physical arguments and mathematical criteria to justify the result (Chaps. 4, . . .).

[There is nothing suspicious about posing the basic equations. That is what is actually done with all basic equations in physics.]

In the second approach, since *the MHD equations describe the motion of a conducting fluid interacting with a magnetic field*, we need to *combine Maxwell's equations with the equations of gas dynamics* and provide *equations describing the interaction*.

- **Maxwell's equations** describe evolution of electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{B}(\mathbf{r},t)$ in response to current density $\mathbf{j}(\mathbf{r},t)$ and space charge $\tau(\mathbf{r},t)$:

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}\,, \qquad \textit{(Faraday)} \qquad (1)$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\,, \qquad c \equiv (\epsilon_0\mu_0)^{-1/2}\,, \qquad \textit{('Ampère')} \qquad (2)$$

$$\nabla\cdot\mathbf{E} = \frac{\tau}{\epsilon_0}\,, \qquad \textit{(Poisson)} \qquad (3)$$

$$\nabla\cdot\mathbf{B} = 0\,. \qquad \textit{(no monopoles)} \qquad (4)$$

- **Gas dynamics equations** describe evolution of density $\rho(\mathbf{r},t)$ and pressure $p(\mathbf{r},t)$:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{v} \equiv \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0\,, \qquad \textit{(mass conservation)} \qquad (5)$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} + \gamma p\nabla\cdot\mathbf{v} \equiv \frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p + \gamma p\nabla\cdot\mathbf{v} = 0\,, \qquad \textit{(entropy conservation)} \qquad (6)$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$$

is the *Lagrangian time-derivative* (moving with the fluid).

- **Coupling between system described by $\{\mathrm{E}, \mathrm{B}\}$ and system described by $\{\rho, p\}$** comes about through equations involving the velocity $\mathbf{v}(\mathbf{r}, t)$ of the fluid:
  'Newton's' *equation of motion for a fluid element* describes the acceleration of a fluid element by pressure gradient, gravity, and electromagnetic contributions,

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{F} \equiv -\nabla p + \rho \mathbf{g} + \mathbf{j} \times \mathbf{B} + \tau \mathbf{E} \,; \qquad \textit{(momentum conservation)} \tag{7}$$

  'Ohm's' law (for a perfectly conducting moving fluid) expresses that *the electric field* $\mathbf{E}'$ *in a co-moving frame vanishes*,

$$\mathbf{E}' \equiv \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \,. \qquad \textit{('Ohm')} \tag{8}$$

- Equations (1)–(8) are complete, but inconsistent for *non-relativistic velocities*:

$$v \ll c \,. \tag{9}$$

  $\Rightarrow$ We need to consider **pre-Maxwell equations**.

Consequences of pre-Maxwell

1. *Maxwell's displacement current negligible* [$\mathcal{O}(v^2/c^2)$] for non-relativistic velocities:

$$\frac{1}{c^2}\left|\frac{\partial \mathbf{E}}{\partial t}\right| \sim \frac{v^2}{c^2}\frac{B}{l_0} \quad \ll \quad \mu_0|\mathbf{j}| \approx |\nabla \times \mathbf{B}| \sim \frac{B}{l_0} \qquad \text{[using Eq. (8)]}\,,$$

indicating length scales by l_0 and time scales by t_0, so that $v \sim l_0/t_0$.

$\Rightarrow$ Recover original Ampère's law:

$$\mathbf{j} = \frac{1}{\mu_0}\nabla \times \mathbf{B}\,. \tag{10}$$

2. *Electrostatic acceleration is also negligible* [$\mathcal{O}(v^2/c^2)$]:

$$\tau|\mathbf{E}| \sim \frac{v^2}{c^2}\frac{B^2}{\mu_0 l_0} \quad \ll \quad |\mathbf{j} \times \mathbf{B}| \sim \frac{B^2}{\mu_0 l_0} \qquad \text{[using Eqs. (3), (8), (10)]}\,.$$

$\Rightarrow$ Space charge effects may be ignored and Poisson's law (3) can be dropped.

3. *Electric field then becomes a secondary quantity,* determined from Eq. (8):

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}\,. \tag{11}$$

$\Rightarrow$ For non-relativistic MHD, $|\mathbf{E}| \sim |\mathbf{v}||\mathbf{B}|$, i.e. $\mathcal{O}(v/c)$ smaller than for EM waves.

Basic equations of ideal MHD

- Exploiting these approximations, and eliminating $\mathbf{E}$ and $\mathbf{j}$ through Eqs. (10) and (11), the basic equations of ideal MHD are recovered *in their most compact form*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \tag{12}$$

$$\rho(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}) + \nabla p - \rho \mathbf{g} - \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = 0\,, \tag{13}$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0\,, \tag{14}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,. \tag{15}$$

$\Rightarrow$ *Set of eight nonlinear partial differential equations (PDEs) for the eight variables $\rho(\mathbf{r}, t)$, $\mathbf{v}(\mathbf{r}, t)$, $p(\mathbf{r}, t)$, and $\mathbf{B}(\mathbf{r}, t)$*.

- The magnetic field equation (15)(b) is to be considered as a initial condition: once satisfied, it remains satisfied for all later times by virtue of Eq. (15)(a).

Thermodynamic variables

- Alternative thermodynamical variables (replacing ρ and p):
  e – internal energy per unit mass ($\sim$ temperature T) and s – entropy per unit mass.
  Defined by the ideal gas relations, with $p = (n_e + n_i)kT$:

$$e \equiv \frac{1}{\gamma - 1}\frac{p}{\rho} \approx C_v\, T\,, \qquad C_v \approx \frac{(1+Z)k}{(\gamma-1)m_i}\,,$$
$$s \equiv C_v \ln S + \text{const}\,, \qquad S \equiv p\rho^{-\gamma}\,. \tag{16}$$

- From Eqs. (12) and (14), we obtain an evolution equation for the internal energy,

$$\frac{\mathrm{D}e}{\mathrm{D}t} + (\gamma - 1)\, e\nabla\cdot\mathbf{v} = 0\,, \tag{17}$$

  and an equation expressing that *the entropy convected by the fluid is constant* (i.e. adiabatic processes: thermal conduction and heat flow are negligible),

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0\,, \quad \text{or} \quad \frac{\mathrm{D}S}{\mathrm{D}t} \equiv \frac{\mathrm{D}}{\mathrm{D}t}(p\rho^{-\gamma}) = 0\,. \tag{18}$$

  This demonstrates that Eq. (14) actually expresses entropy conservation.

Gravity

- In many astrophysical systems, the external gravitational field of a compact object (represented by point mass M_* situated at position $\mathbf{r} = \mathbf{r}_*$ far outside the plasma) is more important than the internal gravitational field. The Poisson equation

$$\nabla^2 \Phi_{\rm gr} = 4\pi G[M_* \delta(\mathbf{r} - \mathbf{r}_*) + \rho(\mathbf{r})] \qquad (19)$$

then has a solution with negligible internal gravitational acceleration (2nd term):

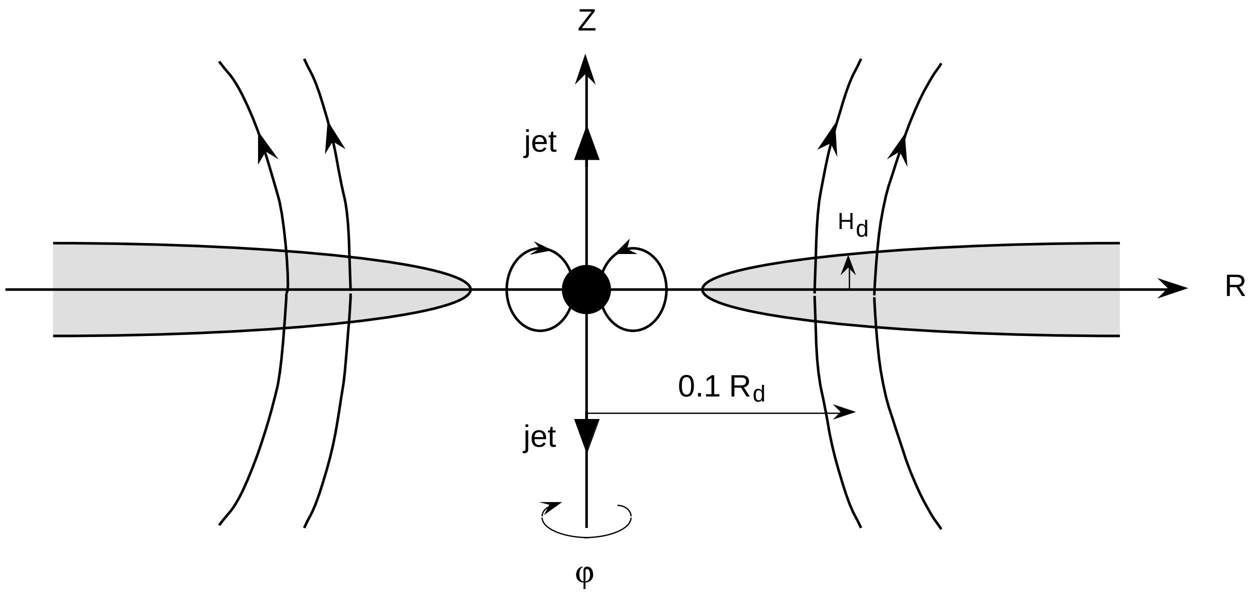
$$\mathbf{g}(\mathbf{r}) = -\nabla\Phi_{\rm gr}(\mathbf{r}) = -GM_* \frac{\mathbf{r} - \mathbf{r}_*}{|\mathbf{r} - \mathbf{r}_*|^3} - G \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}\, d^3 r' \,. \qquad (20)$$

- Estimate gravitational forces $\mathbf{F}_g^{\rm ex/in} \equiv \rho \mathbf{g}^{\rm ex/in}$ compared to Lorentz force $\mathbf{F}_B \equiv \mathbf{j} \times \mathbf{B}$:

  1) *Tokamak* (with radius a of the plasma tube and M_*, R_* referring to the Earth):

$$\begin{aligned}
|\mathbf{F}_B| \equiv |\mathbf{j} \times \mathbf{B}| &\sim \frac{B^2}{\mu_0 a} &&= 7.2 \times 10^{6} \;\; \mathrm{kg\,m^{-2}\,s^{-2}}\,, \\
|\mathbf{F}_g^{\rm ex}| \equiv |\rho \mathbf{g}^{\rm ex}| &\sim \rho G \frac{M_*}{R_*^2} &&= 1.7 \times 10^{-6} \;\; \mathrm{kg\,m^{-2}\,s^{-2}}\,, \\
|\mathbf{F}_g^{\rm in}| \equiv |\rho \mathbf{g}^{\rm in}| &\sim \rho^2 G a &&= 1.9 \times 10^{-24} \, \mathrm{kg\,m^{-2}\,s^{-2}}\,.
\end{aligned} \qquad (21)$$

2) *Accretion disk* (replace $a \to H_d$, and $R_* \to R = 0.1 R_d$):



(a) Young stellar object (YSO)
$(R_d \sim 1\,\mathrm{AU}\,,\, H_d \sim 0.01\,\mathrm{AU}\,,$
$M_* \sim 1 M_\odot\,,\, n = 10^{18}\,\mathrm{m}^{-3}\,)$:

$$|\mathbf{F}_B| = 5.3 \times 10^{-12}\,,$$
$$|\mathbf{F}_g^{\mathrm{ex}}| = 1.0 \times 10^{-9}\,, \qquad (22)$$
$$|\mathbf{F}_g^{\mathrm{in}}| = 2.9 \times 10^{-19}\,.$$

(b) Active galactic nucleus (AGN)
$(R_d \sim 50\,\mathrm{kpc}\,,\, H_d \sim 120\,\mathrm{pc}\,,$
$M_* \sim 10^8 M_\odot\,,\, n = 10^{12}\,\mathrm{m}^{-3}\,)$:

$$|\mathbf{F}_B| = 2.2 \times 10^{-21}\,,$$
$$|\mathbf{F}_g^{\mathrm{ex}}| = 1.0 \times 10^{-27}\,, \qquad (23)$$
$$|\mathbf{F}_g^{\mathrm{in}}| = 6.4 \times 10^{-22}\,.$$

Scale independence

- The MHD equations (12)–(15) can be made dimensionless by means of a choice for *the units of length, mass, and time*, based on typical magnitudes l_0 for length scale, ρ_0 for plasma density, and B_0 for magnetic field at some representative position. The unit of time then follows by exploiting *the Alfvén speed*:

$$v_0 \equiv v_{A,0} \equiv \frac{B_0}{\sqrt{\mu_0 \rho_0}} \quad \Rightarrow \quad t_0 \equiv \frac{l_0}{v_0} \,. \tag{24}$$

- By means of this basic triplet l_0, B_0, t_0 (and derived quantities ρ_0 and v_0), we create dimensionless independent variables and associated differential operators:

$$\bar{l} \equiv l/l_0\,, \quad \bar{t} \equiv t/t_0 \qquad \Rightarrow \qquad \bar{\nabla} \equiv l_0 \nabla\,, \quad \partial/\partial\bar{t} \equiv t_0\, \partial/\partial t\,, \tag{25}$$

  and dimensionless dependent variables:

$$\bar{\rho} \equiv \rho/\rho_0\,, \quad \bar{\mathbf{v}} \equiv \mathbf{v}/v_0\,, \quad \bar{p} \equiv p/(\rho_0 v_0^2)\,, \quad \bar{\mathbf{B}} \equiv \mathbf{B}/B_0\,, \quad \bar{\mathbf{g}} \equiv (l_0/v_0^2)\,\mathbf{g}\,. \tag{26}$$

- Barred equations are now identical to unbarred ones (except that μ_0 is eliminated).

  $\Rightarrow$ *Ideal MHD equations independent of size of the plasma (l_0), magnitude of the magnetic field (B_0), and density (ρ_0), i.e. time scale (t_0).*

Scales of actual plasmas

	l_0 (m)	B_0 (T)	t_0 (s)
tokamak	20	3	3×10^{-6}
magnetosphere Earth	4×10^7	3×10^{-5}	6
solar coronal loop	10^8	3×10^{-2}	15
magnetosphere neutron star	10^6	10^8 *	10^{-2}
accretion disc YSO	1.5×10^9	10^{-4}	7×10^5
accretion disc AGN	4×10^{18}	10^{-4}	2×10^{12}
galactic plasma	10^{21} $(= 10^5$ ly$)$	10^{-8}	10^{15} $(= 3 \times 10^7$ y$)$

* Some recently discovered pulsars, called magnetars, have record magnetic fields of 10^{11} T : the plasma Universe is ever expanding!

Note Tokamak: 1 min $(60\,\mathrm{s})$ $\Rightarrow$ 20×10^6 crossing times ,
Coronal loop: 1 month $(2.6 \times 10^6\,\mathrm{s})$ $\Rightarrow$ 2×10^5 ” .

A crucial question:

Do the MHD equations (12)–(15) provide a complete model for plasma dynamics?

Answer: **NO!**

Two most essential elements of a scientific model are still missing, viz.

1. What is the *physical problem* we want to solve?
2. How does this translate into *conditions on the solutions of the PDEs?*

This brings in the space and time constraints of the *boundary conditions* and *initial data*. Initial data just amount to prescribing arbitrary functions

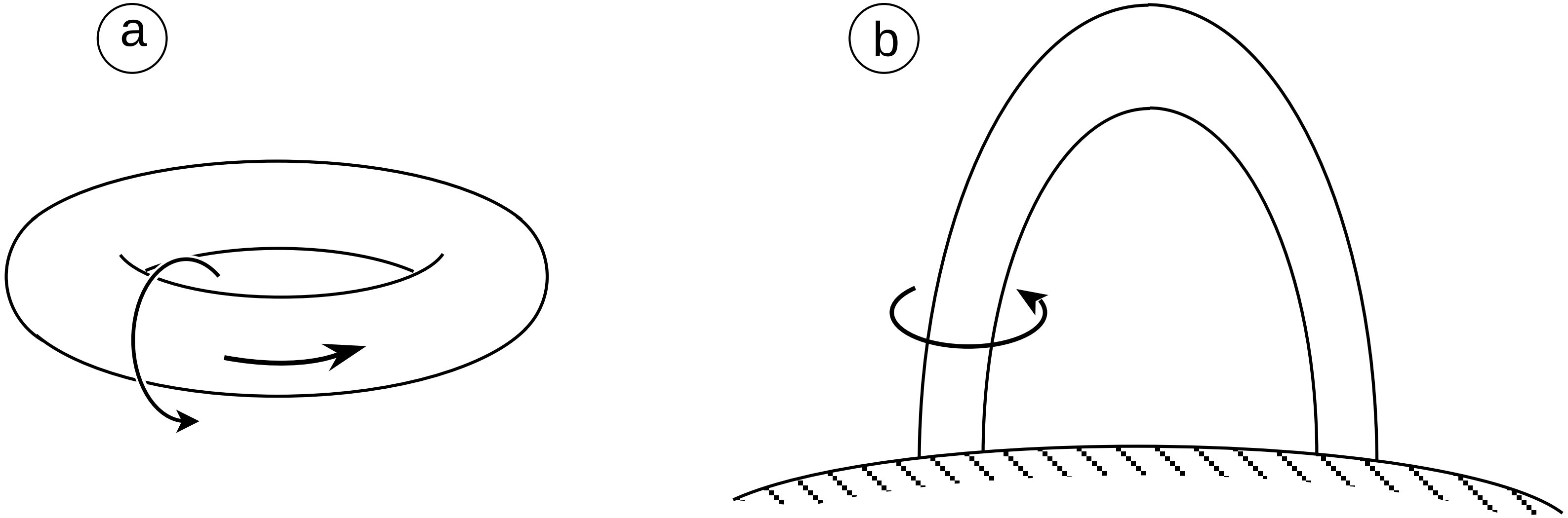
$$\rho_i(\mathbf{r})\,[\,\equiv \rho(\mathbf{r}, t=0)\,]\,, \quad \mathbf{v}_i(\mathbf{r})\,, \quad p_i(\mathbf{r})\,, \quad \mathbf{B}_i(\mathbf{r}) \quad \text{on domain of interest}\,. \tag{27}$$

Boundary conditions is a much more involved issue since it implies specification of a **magnetic confinement geometry**.

$\Rightarrow$ *magnetic flux tubes* (Sec.4.2), *conservation laws* (Sec.4.3), *discontinuities* (Sec.4.4), *formulation of model problems for laboratory and astrophysical plasmas* (Sec.4.5).

Flux tubes

- Magnetic flux tubes are the basic magnetic structures that determine which *boundary conditions* may be posed on the MHD equations.



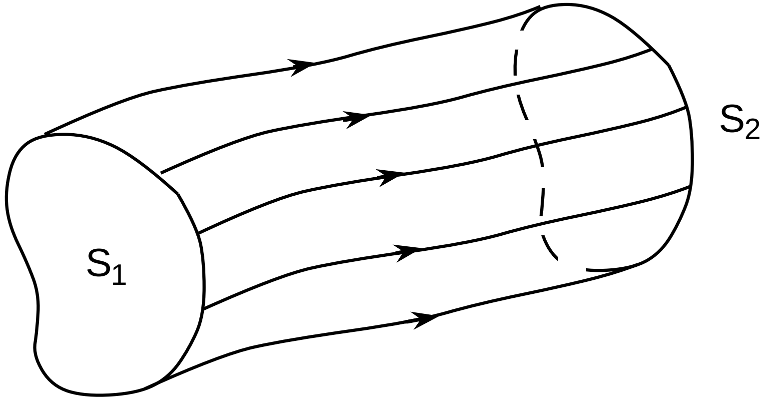
- Two different kinds of flux tubes:
  (a) closed onto itself, like in thermonuclear *tokamak* confinement machines,
  (b) connecting onto a medium of vastly different physical characteristics so that the flux tube may be considered as finite and separated from the other medium by suitable jump conditions, like in *coronal flux tubes*.

Flux tubes (cont'd)

- Magnetic fields confining plasmas are essentially *tubular structures*: The magnetic field equation

$$\nabla \cdot \mathbf{B} = 0 \qquad (28)$$

is not compatible with spherical symmetry. Instead, magnetic flux tubes become the essential constituents.



- Gauss' theorem:

$$\iiint_V \nabla \cdot \mathbf{B}\, d\tau = \oiint \mathbf{B} \cdot \mathbf{n}\, d\sigma = -\iint_{S_1} \mathbf{B}_1 \cdot \mathbf{n}_1\, d\sigma_1 + \iint_{S_2} \mathbf{B}_2 \cdot \mathbf{n}_2\, d\sigma_2 = 0\,,$$

Magnetic flux of all field lines through surface element $d\sigma_1$ is the same as through arbitrary other element $d\sigma_2$ intersecting that field line bundle.

$$\Rightarrow \qquad \Psi \equiv \iint_S \mathbf{B} \cdot \mathbf{n}\, d\sigma \quad \text{is well defined} \qquad (29)$$

(does not depend on how S is taken). Also true for smaller subdividing flux tubes!

Global magnetic flux conservation

- **Kinematical** concept of flux tube comes from $\nabla \cdot \mathbf{B} = 0$.
- **Dynamical** concept of magnetic flux conservation comes from the *induction equation,* a contraction of Faraday's law, $\partial \mathbf{B}/\partial t = -\nabla \times \mathbf{E}$, and 'Ohm's' law, $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})\,. \tag{30}$$

- Example: *Global magnetic flux conservation inside toroidal tokamak.* 'Ohm's' law at the wall (where the conductivity is perfect because of the plasma in front of it):

$$\mathbf{n}_w \times [\,\mathbf{E} + (\mathbf{v} \times \mathbf{B})\,] \stackrel{(A.2)}{=} \mathbf{n}_w \times \mathbf{E}_t + \mathbf{n}_w \cdot \mathbf{B}\,\mathbf{v} - \mathbf{n}_w \cdot \mathbf{v}\,\mathbf{B} = 0\,. \tag{31}$$

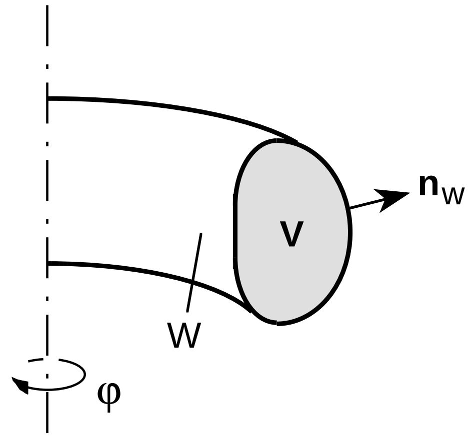
Since $\mathbf{E}_t = 0$, and the boundary condition that there is *no flow across the wall*,

$$\mathbf{n}_w \cdot \mathbf{v} = 0 \qquad \text{(on } W\text{)}\,, \tag{32}$$

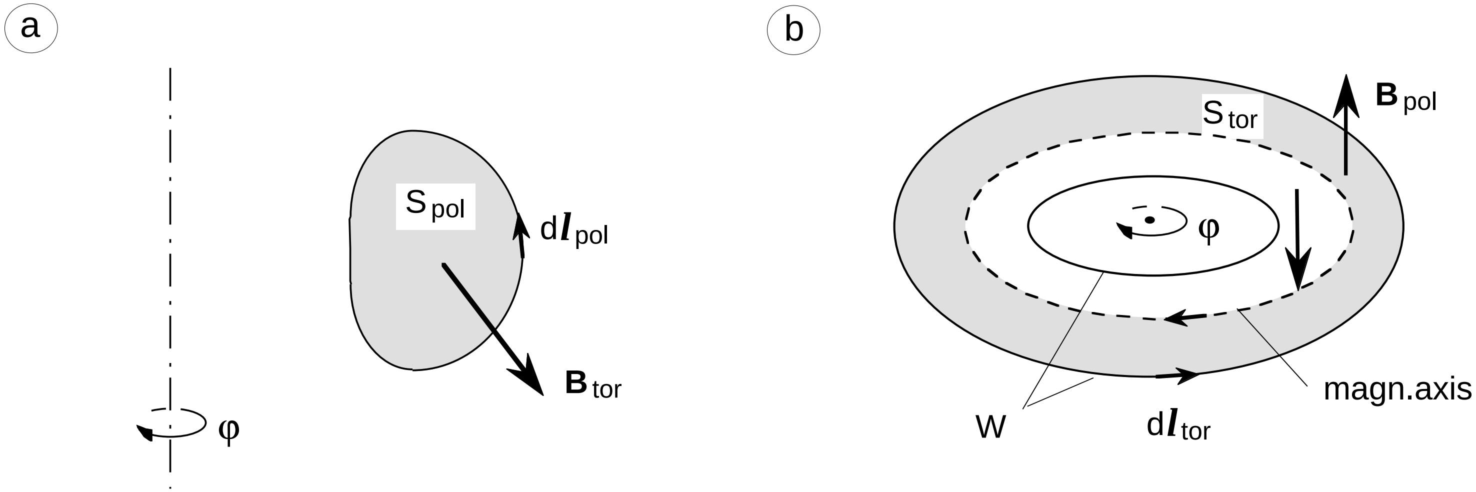
the other contribution has to vanish as well:

$$\mathbf{n}_w \cdot \mathbf{B} = 0 \qquad \text{(on } W\text{)}\,. \tag{33}$$

$\Rightarrow$ *magnetic field lines do not intersect the wall*.



Global magnetic flux conservation (cont'd)



Apply induction equation to toroidal flux:

$$\frac{\partial \Psi_{\rm tor}}{\partial t} \equiv \iint_{S_{\rm pol}} \frac{\partial \mathbf{B}_{\rm tor}}{\partial t} \cdot \mathbf{n}_{\rm tor}\, d\sigma = \iint \nabla \times (\mathbf{v} \times \mathbf{B}_{\rm tor}) \cdot \mathbf{n}_{\rm tor}\, d\sigma = \oint \mathbf{v} \times \mathbf{B}_{\rm tor} \cdot d\boldsymbol{l}_{\rm pol} = 0\,, \tag{34}$$

because $\mathbf{v}$, $\mathbf{B}$, and $\boldsymbol{l}$ tangential to the wall. ⇒ BCs (32) & (33) guarantee $\Psi_{\rm tor} = {\rm const}\,.$

Similarly for poloidal flux:

$$\frac{\partial \Psi_{\rm pol}}{\partial t} \equiv \iint_{S_{\rm tor}} \frac{\partial \mathbf{B}_{\rm pol}}{\partial t} \cdot \mathbf{n}_{\rm pol}\, d\sigma = 0 \quad \Rightarrow \quad \Psi_{\rm pol} = {\rm const}\,. \tag{35}$$

Magnetic flux conservation is the central issue in magnetohydrodynamics.

Conservation form of the MHD equations

- Next step: systematic approach to *local conservation properties.*
- The MHD equations can be brought in conservation form:

$$\frac{\partial}{\partial t}(\cdots) + \nabla \cdot (\cdots) = 0\,. \qquad (36)$$

This yields: *conservation laws*, *jump conditions*, *and powerful numerical algorithms!*

- By intricate vector algebra, one obtains the *conservation form of the ideal MHD equations* (suppressing gravity): ⇓ From now on, putting $\mu_0 \rightarrow 1$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \qquad (37)$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot [\, \rho \mathbf{v}\mathbf{v} + (p + \tfrac{1}{2}B^2)\,\mathbf{I} - \mathbf{B}\mathbf{B}\,] = 0\,, \qquad p = (\gamma - 1)\rho e\,, \qquad (38)$$

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho e + \tfrac{1}{2}B^2) + \nabla \cdot [(\tfrac{1}{2}\rho v^2 + \rho e + p + B^2)\mathbf{v} - \mathbf{v}\cdot\mathbf{B}\mathbf{B}] = 0\,, \qquad (39)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}) = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,. \qquad (40)$$

It remains to analyze the meaning of the different terms.

Transformation

- Defining
  - *momentum density*: $\boldsymbol{\pi} \equiv \rho \mathbf{v}\,,$ (41)
  - *stress tensor*: $\mathbf{T} \equiv \rho \mathbf{v}\mathbf{v} + (p + \tfrac{1}{2}B^2)\,\mathbf{I} - \mathbf{B}\mathbf{B}\,,$ (42)
  - *total energy density*: $\mathcal{H} \equiv \tfrac{1}{2}\rho v^2 + \tfrac{1}{\gamma-1}\,p + \tfrac{1}{2}B^2\,,$ (43)
  - *energy flow*: $\mathbf{U} \equiv (\tfrac{1}{2}\rho v^2 + \tfrac{\gamma}{\gamma-1}\,p)\mathbf{v} + B^2\mathbf{v} - \mathbf{v}\cdot\mathbf{B}\,\mathbf{B}\,,$ (44)
  - *(no name)*: $\mathbf{Y} \equiv \mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}\,,$ (45)

  yields

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\boldsymbol{\pi} = 0 \qquad (46)$$
(conservation of mass),

$$\frac{\partial \boldsymbol{\pi}}{\partial t} + \nabla\cdot\mathbf{T} = 0 \qquad (47)$$
(conservation of momentum),

$$\frac{\partial \mathcal{H}}{\partial t} + \nabla\cdot\mathbf{U} = 0 \qquad (48)$$
(conservation of energy),

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla\cdot\mathbf{Y} = 0 \qquad (49)$$
(conservation of magnetic flux).

Global transformation laws

- Defining
  - *total mass*: $$M \equiv \int \rho \, d\tau \,, \tag{50}$$
  - *total momentum*: $$\mathbf{\Pi} \equiv \int \boldsymbol{\pi} \, d\tau \,, \tag{51}$$
  - *total energy*: $$H \equiv \int \mathcal{H} \, d\tau \,, \tag{52}$$
  - *total magnetic flux*: $$\Psi \equiv \int \mathbf{B} \cdot \tilde{\mathbf{n}} \, d\tilde{\sigma} \,, \tag{53}$$

  gives, by the application of the BCs (32), (33):

$$\dot{M} = \int \dot{\rho} \, d\tau = - \int \nabla \cdot \boldsymbol{\pi} \, d\tau \overset{\text{Gauss}}{=} - \oint \boldsymbol{\pi} \cdot \mathbf{n} \, d\sigma = 0 \,, \tag{54}$$

$$\mathbf{F} = \dot{\mathbf{\Pi}} = \int \dot{\boldsymbol{\pi}} \, d\tau = - \int \nabla \cdot \mathsf{T} \, d\tau \overset{\text{Gauss}}{=} - \oint (p + \tfrac{1}{2}B^2) \, \mathbf{n} \, d\sigma \,, \tag{55}$$

$$\dot{H} = \int \dot{\mathcal{H}} \, d\tau = - \int \nabla \cdot \mathbf{U} \, d\tau \overset{\text{Gauss}}{=} - \oint \mathbf{U} \cdot \mathbf{n} \, d\sigma = 0 \,, \tag{56}$$

$$\dot{\Psi} = \int \dot{\mathbf{B}} \cdot \tilde{\mathbf{n}} \, d\tilde{\sigma} = \int \nabla \times (\mathbf{v} \times \mathbf{B}) \cdot \tilde{\mathbf{n}} \, d\tilde{\sigma} \overset{\text{Stokes!}}{=} \oint \mathbf{v} \times \mathbf{B} \cdot \, d\boldsymbol{l} = 0 \,. \tag{57}$$

$\Rightarrow$ *Total mass, momentum, energy, and flux conserved:* **the system is closed!**

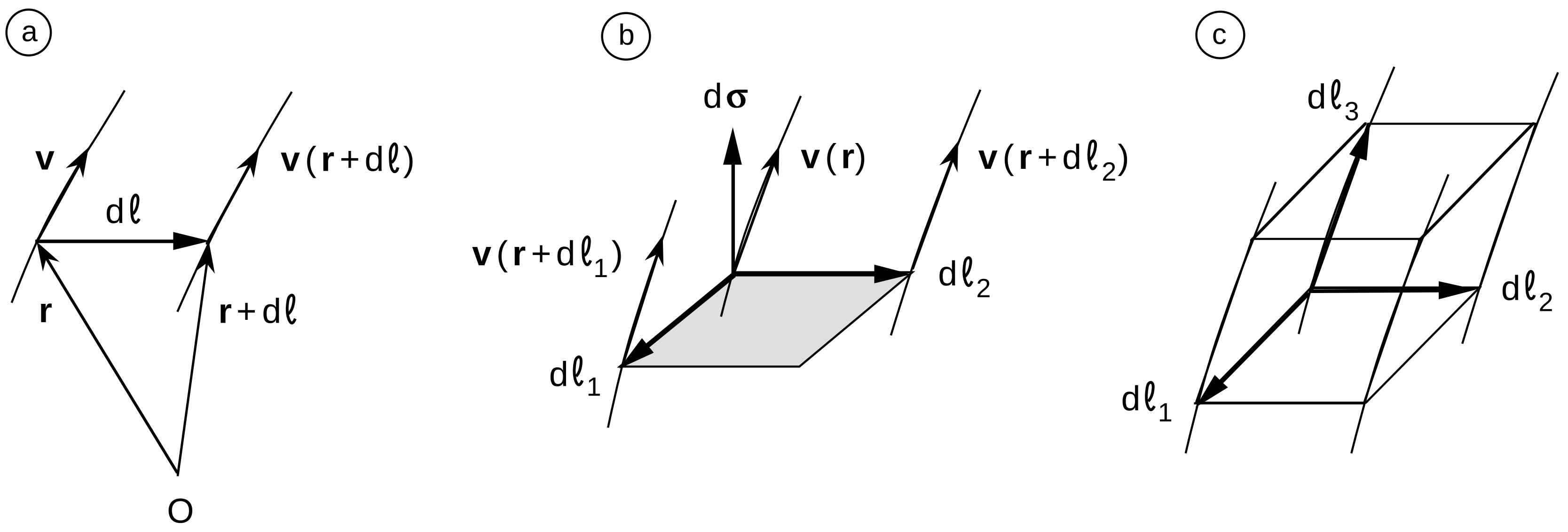
Local transformation laws

- Kinematic expressions for *change in time of line, surface, and volume element*:

$$\frac{\mathrm{D}}{\mathrm{D}t}(d\boldsymbol{l}) = \frac{\mathrm{D}(\mathbf{r}+d\boldsymbol{l})}{\mathrm{D}t} - \frac{\mathrm{D}\mathbf{r}}{\mathrm{D}t} = \mathbf{v}(\mathbf{r}+d\boldsymbol{l}) - \mathbf{v}(\mathbf{r}) = d\boldsymbol{l}\cdot(\nabla\mathbf{v})\,, \tag{58}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}(d\boldsymbol{\sigma}) = -(\nabla\mathbf{v})\cdot d\boldsymbol{\sigma} + \nabla\cdot\mathbf{v}\,d\boldsymbol{\sigma}\,, \quad \text{(crucial for flux conservation!)} \tag{59}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}(d\tau) = \nabla\cdot\mathbf{v}\,d\tau\,. \tag{60}$$



Local transformation laws (cont'd)

- Combine dynamical relations (46)–(48) with kinematics of volume element:

$$\frac{\mathrm{D}}{\mathrm{D}t}(dM) = \frac{\mathrm{D}\rho}{\mathrm{D}t}d\tau + \rho\frac{\mathrm{D}}{\mathrm{D}t}(d\tau) = -\rho\nabla\cdot\mathbf{v}\,d\tau + \rho\nabla\cdot\mathbf{v}\,d\tau = 0\,. \tag{61}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}(d\mathbf{\Pi}) = \frac{\mathrm{D}\boldsymbol{\pi}}{\mathrm{D}t}\,d\tau + \boldsymbol{\pi}\,\frac{\mathrm{D}}{\mathrm{D}t}(d\tau) = (-\nabla p + \mathbf{j}\times\mathbf{B})\,d\tau \neq 0\,, \tag{62}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}(dH) = \frac{\mathrm{D}\mathcal{H}}{\mathrm{D}t}\,d\tau + \mathcal{H}\,\frac{\mathrm{D}}{\mathrm{D}t}(d\tau) = -\nabla\cdot\{[\,(p+\tfrac{1}{2}B^2)\,\mathbf{I} - \mathbf{BB}\,]\cdot\mathbf{v}\}\,d\tau \neq 0\,. \tag{63}$$

⇒ *Mass of co-moving volume element conserved locally,*
momentum and energy change due to force and work on the element.

- However, dynamics (49) for the flux exploits kinematics of surface element:

$$\frac{\mathrm{D}}{\mathrm{D}t}(d\Psi) = \frac{\mathrm{D}}{\mathrm{D}t}(\mathbf{B}\cdot d\boldsymbol{\sigma}) = \frac{\mathrm{D}\mathbf{B}}{\mathrm{D}t}\cdot d\boldsymbol{\sigma} + \mathbf{B}\cdot\frac{\mathrm{D}}{\mathrm{D}t}(d\boldsymbol{\sigma})$$
$$= (\mathbf{B}\cdot\nabla\mathbf{v} - \mathbf{B}\nabla\cdot\mathbf{v})\cdot d\boldsymbol{\sigma} + \mathbf{B}\cdot(\,-(\nabla\mathbf{v})\cdot d\boldsymbol{\sigma} + \nabla\cdot\mathbf{v}\,d\boldsymbol{\sigma}) = 0\,. \tag{64}$$

⇒ **Magnetic flux through arbitrary co-moving surface element constant (!) :**

$$\Psi = \int_C \mathbf{B}\cdot\mathbf{n}\,d\sigma = \mathrm{const} \qquad \text{for any contour } C\,. \tag{65}$$

Jump conditions

Extending the MHD model

- Recall the BCs (32) and (33) for *plasmas surrounded by a solid wall:*

$$\mathbf{n}_w \cdot \mathbf{v} = 0 \qquad (\text{on } W) \quad \Rightarrow \quad \textit{no flow accross the wall,}$$

$$\mathbf{n}_w \cdot \mathbf{B} = 0 \qquad (\text{on } W) \quad \Rightarrow \quad \textit{magnetic field lines do not intersect the wall.}$$

  Under these conditions, conservation laws apply and the system is closed.

- For many applications (both in the laboratory and in astrophysics) this is not enough. One also needs BCs (jump conditions) for *plasmas with an internal boundary* where the magnitudes of the plasma variables 'jump'.

  Example: at the photospheric boundary the density changes $\sim 10^{-9}$.

- Such a boundariy is a special case of a *shock,* i.e. an irreversible (entropy-increasing) transition. In gas dynamics, the *Rankine–Hugoniot relations* relate the variables of the subsonic flow downstream the shock with those of the supersonic flow upstream.

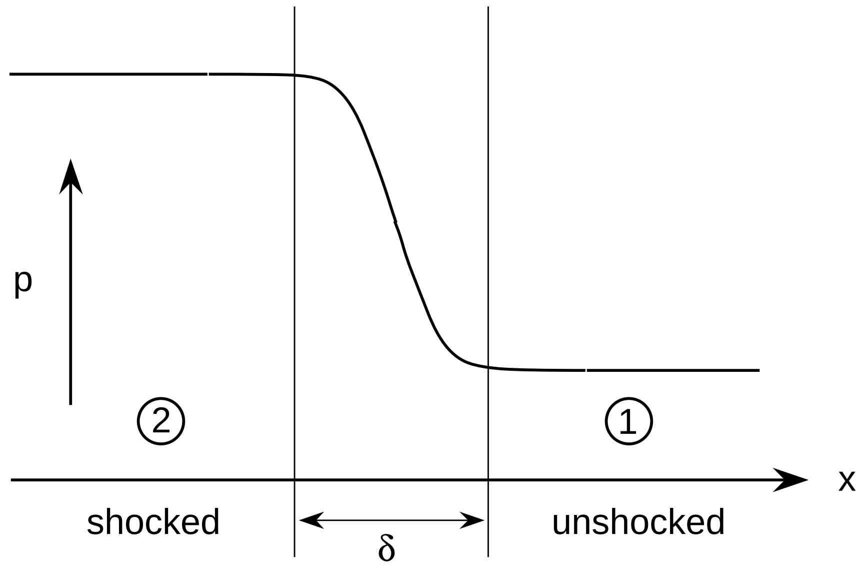
  We will generalize these relations to MHD, but only to get the right form of the jump conditions, not to analyze transonic flows (subject for a much later chapter).

Shock formation

- Excite sound waves in a 1D compressible gas (HD): the local perturbations travel with the sound speed $c \equiv \sqrt{\gamma p / \rho}$.

  $\Rightarrow$ Trajectories in the $x-t$ plane *(characteristics):* $dx/dt = \pm c$.

- Now suddenly increase the pressure, so that p changes in a thin layer of width δ :

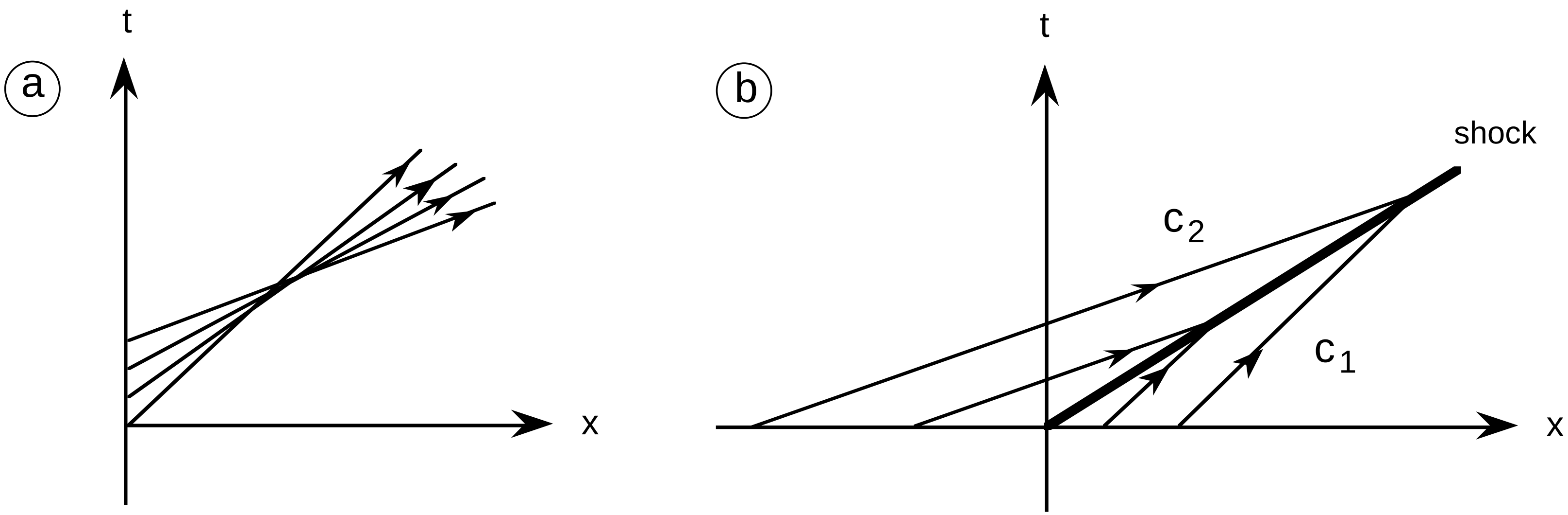


$\Rightarrow$ 'Converging' characteristics in the $x-t$ plane.

$\Rightarrow$ Information from different space-time points accumulates, gradients build up until steady state reached where dissipation and nonlinearities balance $\Rightarrow$ **shock.**

Shock formation (cont'd)

- Wihout the non-ideal and nonlinear effects, the characteristics would cross (a). With those effects, in the limit $\delta \to 0$, the characteristics meet at the shock front (b).



$\Rightarrow$ *Moving shock front separates two ideal regions.*

- Neglecting the thickness of the shock (not the shock itself of course), all there remains is to derive jump relations across the infinitesimal layer.

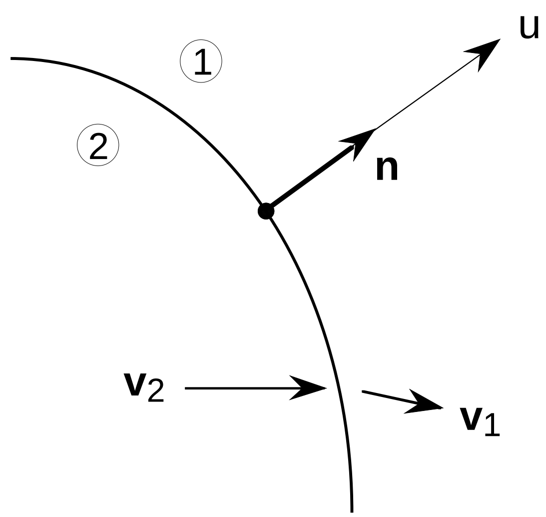
$\Rightarrow$ *Limiting cases of the conservation laws at shock fronts.*

Procedure to derive the jump conditions

Integrate conservation equations across shock from ① (undisturbed) to ② (shocked).

- Only contribution from gradient normal to the front:

$$\lim_{\delta\to 0}\int_1^2 \nabla f\, dl = -\lim_{\delta\to 0}\mathbf{n}\int_1^2 \frac{\partial f}{\partial l}\, dl = \mathbf{n}(f_1 - f_2) \equiv \mathbf{n}\,[\![f]\!]\,. \tag{66}$$



- In frame moving with the shock at normal speed u :

$$\Big(\frac{\mathrm{D}f}{\mathrm{D}t}\Big)_{\rm shock} = \frac{\partial f}{\partial t} - u\frac{\partial f}{\partial l}\ \text{finite} \ll \frac{\partial f}{\partial t} \approx u\frac{\partial f}{\partial l} \sim \infty$$

$$\Rightarrow \quad \lim_{\delta\to 0}\int_1^2 \frac{\partial f}{\partial t}\, dl = u\lim_{\delta\to 0}\int_1^2 \frac{\partial f}{\partial l}\, dl = -u\,[\![f]\!]\,. \tag{67}$$

- Hence, *jump conditions follow from the conservation laws by simply substituting*

$$\nabla f \to \mathbf{n}\,[\![f]\!]\,, \qquad \partial f/\partial t \to -u\,[\![f]\!]\,. \tag{68}$$

⇒ MHD jump conditions

- Conservation of mass,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \Rightarrow \quad -u\,[\![\rho]\!] + \mathbf{n} \cdot [\![\rho \mathbf{v}]\!] = 0\,. \tag{69}$$

- Conservation of momentum,

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot [\,\rho \mathbf{v}\mathbf{v} + (p + \tfrac{1}{2}B^2)\,\mathbf{I} - \mathbf{B}\mathbf{B}\,] = 0$$

$$\Rightarrow \quad -u\,[\![\rho \mathbf{v}]\!] + \mathbf{n} \cdot [\![\rho \mathbf{v}\mathbf{v} + (p + \tfrac{1}{2}B^2)\,\mathbf{I} - \mathbf{B}\mathbf{B}]\!] = 0\,. \tag{70}$$

- Conservation ot total energy,

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho e + \tfrac{1}{2}B^2) + \nabla \cdot [(\tfrac{1}{2}\rho v^2 + \rho e + p + B^2)\mathbf{v} - \mathbf{v} \cdot \mathbf{B}\mathbf{B}] = 0$$

$$\Rightarrow \quad -u\,[\![\tfrac{1}{2}\rho v^2 + \tfrac{1}{\gamma - 1}\,p + \tfrac{1}{2}B^2]\!] + \mathbf{n} \cdot [\![(\tfrac{1}{2}\rho v^2 + \tfrac{\gamma}{\gamma - 1}\,p + B^2)\mathbf{v} - \mathbf{v} \cdot \mathbf{B}\mathbf{B}]\!] = 0\,. \tag{71}$$

- Conservation of magnetic flux,

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}) = 0\,, \qquad \nabla \cdot \mathbf{B} = 0$$

$$\Rightarrow \quad -u\,[\![\mathbf{B}]\!] + \mathbf{n} \cdot [\![\mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}]\!] = 0\,, \qquad \mathbf{n} \cdot [\![\mathbf{B}]\!] = 0\,. \tag{72}$$

MHD jump conditions in the shock frame

- Simplify jump conditions by *transforming to co-moving shock frame,* where relative plasma velocity is $\mathbf{v}' \equiv \mathbf{v} - u\mathbf{n}$, and split vectors in tangential and normal to shock:

$$[\![\rho v_n']\!] = 0\,, \qquad \textit{(mass)} \qquad (73)$$

$$[\![\rho v_n'^{\,2} + p + \tfrac{1}{2}B_t^2]\!] = 0\,, \qquad \textit{(normal momentum)} \qquad (74)$$

$$\rho v_n' \, [\![\mathbf{v}_t']\!] = B_n \, [\![\mathbf{B}_t]\!]\,, \qquad \textit{(tangential momentum)} \qquad (75)$$

$$\rho v_n' [\![\tfrac{1}{2}(v_n'^{\,2} + v_t'^{2}) + (\tfrac{\gamma}{\gamma-1}p + B_t^2)/\rho]\!] = B_n \, [\![\mathbf{v}_t' \cdot \mathbf{B}_t]\!]\,, \qquad \textit{(energy)} \qquad (76)$$

$$[\![B_n]\!] = 0\,, \qquad \textit{(normal flux)} \qquad (77)$$

$$\rho v_n' [\![\mathbf{B}_t/\rho]\!] = B_n [\![\mathbf{v}_t']\!]\,. \qquad \textit{(tangential flux)} \qquad (78)$$

$\Rightarrow$ 6 relations for the 6 jumps $[\![\rho]\!]$, $[\![v_n]\!]$, $[\![\mathbf{v}_t]\!]$, $[\![p]\!]$, $[\![B_n]\!]$, $[\![\mathbf{B}_t]\!]$.

- *Do not use entropy conservation law* since shock is entropy-increasing transition:

not $\dfrac{\partial}{\partial t}(\rho S) + \nabla \cdot (\rho S \mathbf{v}) = 0 \;\;\Rightarrow\;\; \rho v_n' \, [\![S]\!] = 0\,,$ but $[\![S]\!] \equiv [\![\rho^{-\gamma} p]\!] \le 0\,.$ (79)

$\Rightarrow$ This is the only remnant of the dissipative processes in the thin layer.

⇒ Two classes of discontinuities:

(1) *Boundary conditions for moving plasma-plasma interfaces,* where there is no flow accross the discontinuity ($v'_n = 0$) ⇒ will continue with this here.

(2) *Jump conditions for shocks* ($v'_n \neq 0$) ⇒ leave for advanced MHD lectures.

BCs at co-moving interfaces

- When $v'_n = 0\,,$ jump conditions (73)–(78) reduce to:

$$[\![p + \tfrac{1}{2}B_t^2]\!] = 0\,, \qquad \textit{(normal momentum)} \qquad (80)$$

$$B_n\,[\![\mathbf{B}_t]\!] = 0\,, \qquad \textit{(tangential momentum)} \qquad (81)$$

$$B_n\,[\![\mathbf{v}'_t \cdot \mathbf{B}_t]\!] = 0\,, \qquad \textit{(energy)} \qquad (82)$$

$$[\![B_n]\!] = 0\,, \qquad \textit{(normal flux)} \qquad (83)$$

$$B_n\,[\![\mathbf{v}'_t]\!] = 0\,. \qquad \textit{(tangential flux)} \qquad (84)$$

- Two possibilities, depending on whether $\mathbf{B}$ intersects the interface or not:

  (a) *Contact discontinuities* when $B_n \neq 0\,,$

  (b) *Tangential discontinuities* if $B_n = 0\,.$

(a) Contact discontinuities

- For co-moving interfaces with an intersecting magnetic field, $B_n \neq 0\,,$ the jump conditions (80)–(84) only admit a jump of the density (or temperature, or entropy) whereas all other quantities should be continuous:

$$
\begin{aligned}
&\text{– jumping:} && [\![\rho]\!] \neq 0\,, \\
&\text{– continuous:} && v'_n = 0\,, \quad [\![\mathbf{v}'_t]\!] = 0\,, \quad [\![p]\!] = 0\,, \quad [\![B_n]\!] = 0\,, \quad [\![\mathbf{B}_t]\!] = 0\,.
\end{aligned}
\tag{85}
$$

Examples: photospheric footpoints of coronal loops where density jumps,
'divertor' tokamak plasmas with $\mathbf{B}$ intersecting boundary.

- These BCs are most typical for astrophysical plasmas, *modelling plasmas with very different properties of the different spatial regions involved* (e.g. close to a star and far away): difficult! Computing waves in such systems usually requires extreme resolutions to follow the disparate time scales in the problem.

(b) Tangential discontinuities

- For co-moving interfaces with purely tangential magnetic field, $B_n = 0\,,$ the jump conditions (80)–(84) are much less restrictive:

$$\begin{aligned} &\text{– jumping:} \quad [\![\rho]\!] \neq 0\,, \quad [\![\mathbf{v}'_t]\!] \neq 0\,, \quad [\![p]\!] \neq 0\,, \quad [\![\mathbf{B}_t]\!] \neq 0\,, \\ &\text{– continuous:} \quad v'_n = 0\,, \quad B_n = 0\,, \quad [\![p + \tfrac{1}{2}B_t^2]\!] = 0\,. \end{aligned} \tag{86}$$

Examples: tokamak plasma separated from wall by tenuous plasma (or 'vacuum'), dayside magnetosphere where IMF meets Earth's dipole.

- *Plasma–plasma interface BCs* by transforming back to lab frame, $v_n - u \equiv v'_n = 0$:

$$\mathbf{n}\cdot\mathbf{B} = 0 \qquad (\mathbf{B} \parallel \text{interface})\,, \tag{87}$$

$$\mathbf{n}\cdot[\![\mathbf{v}]\!] = 0 \qquad \text{(normal velocity continuous)}\,, \tag{88}$$

$$[\![p + \tfrac{1}{2}B^2]\!] = 0 \qquad \text{(total pressure continuous)}\,. \tag{89}$$

- Jumps tangential components, $[\![\mathbf{B}_t]\!]$ & $[\![\mathbf{v}_t]\!]$, due to *surface current & surface vorticity:*

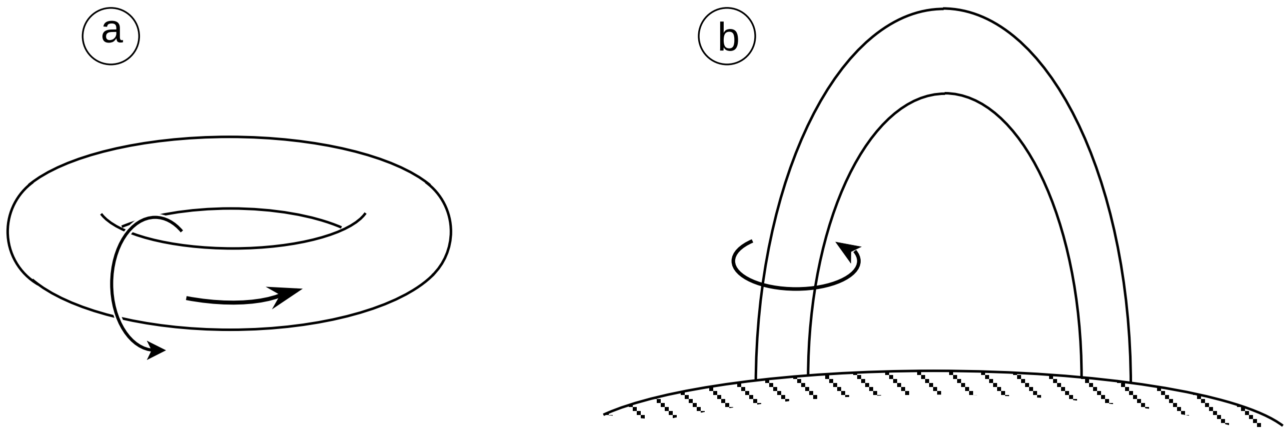
$$\mathbf{j} = \nabla\times\mathbf{B} \quad\Rightarrow\quad \mathbf{j}^\star \equiv \lim_{\delta\to 0,\,|\mathbf{j}|\to\infty}(\delta\,\mathbf{j}) = \mathbf{n}\times[\![\mathbf{B}]\!]\,, \tag{90}$$

$$\boldsymbol{\omega} \equiv \nabla\times\mathbf{v} \quad\Rightarrow\quad \boldsymbol{\omega}^\star \equiv \lim_{\delta\to 0,\,|\boldsymbol{\omega}|\to\infty}(\delta\,\boldsymbol{\omega}) = \mathbf{n}\times[\![\mathbf{v}]\!]\,. \tag{91}$$

Model problems

- We are now prepared to formulate **complete models for plasma dynamics** $\equiv$ **MHD equations** $+$ **specification of magnetic geometries** $\Rightarrow$ **appropriate BCs.**

- For example, recall two generic magnetic structures: (a) tokamak; (b) coronal loop.



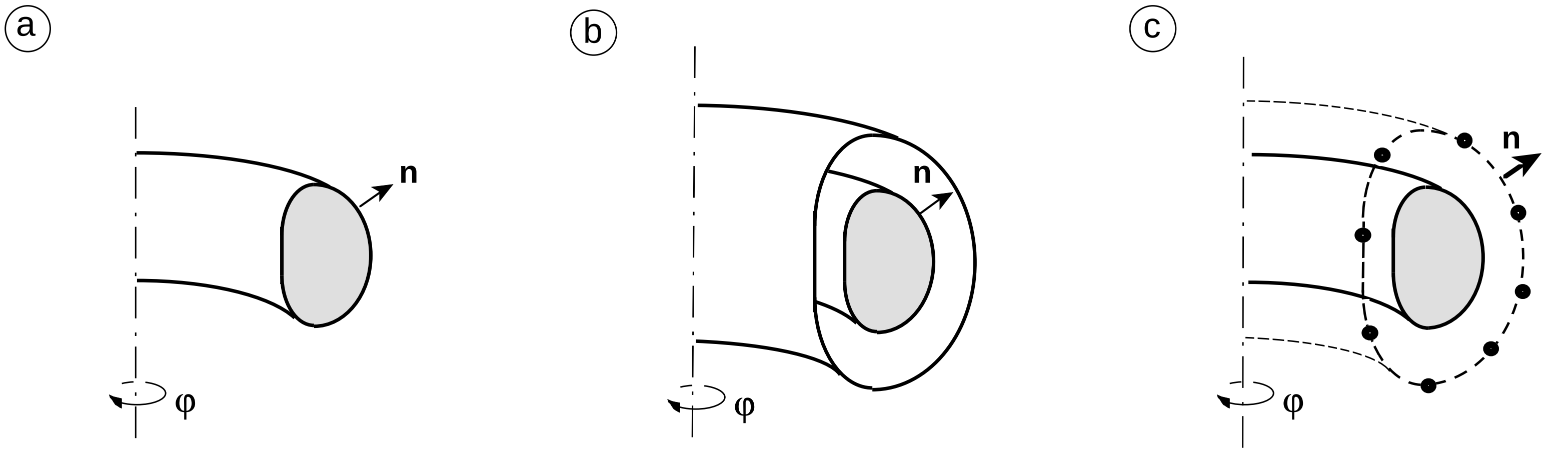
- Generalize this to **six model problems,** separated in two classes:

  $\Rightarrow$ **Models I–III (laboratory plasmas) with tangential discontinuities;**

  $\Rightarrow$ **Models IV–VI (astrophysical plasmas) with contact discontinuities.**

Laboratory plasmas (models I–III)

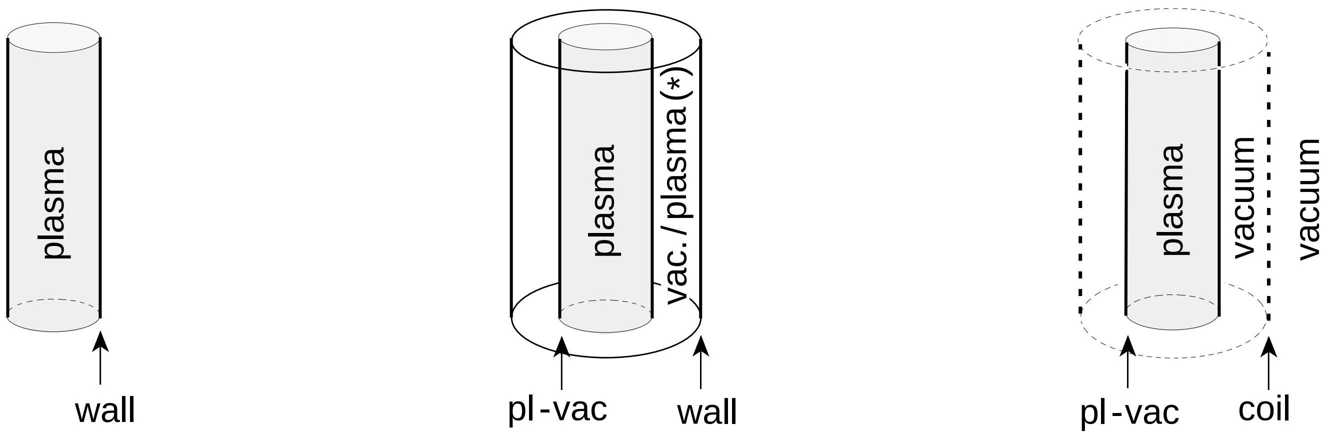
toroidal



model I | model II (∗) | model III

a′ plasma–wall | b′ plasma–vac–wall | c′ plasma–vac–coil–vac

cylindrical



Model I: plasma confined inside rigid wall

- Model I: axisymmetric (2D) plasma contained in a 'donut'-shaped vessel (tokamak) which *confines the magnetic structure to a finite volume.* Vessel $+$ external coils need to be firmly fixed to the laboratory floor since magnetic forces are huge.

  $\Rightarrow$ Plasma–wall, impenetrable wall needs not be conducting (remember why?).

  $\Rightarrow$ Boundary conditions are

$$\mathbf{n} \cdot \mathbf{B} = 0 \qquad \textit{(at the wall)}\,, \tag{92}$$

$$\mathbf{n} \cdot \mathbf{v} = 0 \qquad \textit{(at the wall)}\,. \tag{93}$$

  $\Rightarrow$ just two BCs for 8 variables!

- These BCs guarantee conservation of mass, momentum, energy and magnetic flux: *the system is closed off from the outside world.*

- Most widely used simplification: cylindrical version (1D) with symmetry in θ and z.

  $\Rightarrow$ Non-trivial problem only in the radial direction, therefore: one-dimensional.

Model II: plasma-vacuum system inside rigid wall

- Model II: as I, but plasma separated from wall by vacuum (tokamak with a 'limiter').

  $\Rightarrow$ Plasma–vacuum–wall, wall now perfectly conducting (since vacuum in front).

- Vacuum has no density, velocity, current, only $\hat{\mathbf{B}}$ $\Rightarrow$ pre-Maxwell dynamics:

$$\nabla \times \hat{\mathbf{B}} = 0\,, \qquad \nabla \cdot \hat{\mathbf{B}} = 0\,, \tag{94}$$

$$\nabla \times \hat{\mathbf{E}} = -\frac{\partial \hat{\mathbf{B}}}{\partial t}\,, \qquad \nabla \cdot \hat{\mathbf{E}} = 0\,. \tag{95}$$

  BC at exterior interface (only on $\hat{\mathbf{B}}$, consistent with $\hat{\mathbf{E}}_t = 0$):

$$\mathbf{n} \cdot \hat{\mathbf{B}} = 0 \qquad \textit{(at conducting wall)}\,. \tag{96}$$

- BCs at interior interface ($\mathbf{B}$ not pointing into vacuum and total pressure balance):

$$\mathbf{n} \cdot \mathbf{B} = \mathbf{n} \cdot \hat{\mathbf{B}} = 0 \qquad \textit{(at plasma–vacuum interface)}\,, \tag{97}$$

$$[\![p + \tfrac{1}{2} B^2]\!] = 0 \qquad \textit{(at plasma–vacuum interface)}\,. \tag{98}$$

  $\Rightarrow$ Consequence (not a BC) is jump in $\mathbf{B}_t$, i.e. skin current:

$$\mathbf{j}^\star = \mathbf{n} \times [\![\mathbf{B}]\!] \qquad \textit{(at plasma–vacuum interface)}\,. \tag{99}$$

Model II*: plasma-plasma system inside rigid wall

- Variant of Model II with vacuum replaced by tenuous plasma (negligible density, with or without current), where again the impenetrable wall needs not be conducting.

  $\Rightarrow$ Applicable to tokamaks to incorporate effects of outer plasma.

  $\Rightarrow$ Also for astrophysical plasmas (coronal loops) where 'wall' is assumed far away.

- BCs at exterior interface for outer plasma:

$$\mathbf{n} \cdot \hat{\mathbf{B}} = 0 \qquad \textit{(at the wall)}\,,$$

$$\mathbf{n} \cdot \hat{\mathbf{v}} = 0 \qquad \textit{(at the wall)}\,.$$

- BCs at interior interface for tangential plasma-plasma discontinuity:

$$\mathbf{n} \cdot \mathbf{B} = \mathbf{n} \cdot \hat{\mathbf{B}} = 0 \qquad \textit{(at plasma–plasma interface)}\,,$$

$$\mathbf{n} \cdot [\![\mathbf{v}]\!] = 0 \qquad \textit{(at plasma–plasma interface)}\,,$$

$$[\![p + \tfrac{1}{2}B^2]\!] = 0 \qquad \textit{(at plasma–plasma interface)}\,.$$

Note: Model II obtained by just dropping conditions on $\mathbf{v}$ and $\hat{\mathbf{v}}$.

Model III: plasma-vacuum system with external currents

- Model III is an *open* plasma–vacuum configuration excited by magnetic fields $\hat{\mathbf{B}}(t)$ that are externally created by a coil (antenna) with skin current.

  $\Rightarrow$ Open system: forced oscillations *pump energy into the plasma.*

  $\Rightarrow$ Applications in laboratory and astrophysical plasmas: original creation of the confining magnetic fields and excitation of MHD waves.

- BCs at coil surface:

$$\mathbf{n} \cdot [\![\hat{\mathbf{B}}]\!] = 0 \qquad \textit{(at coil surface)}\,, \tag{100}$$

$$\mathbf{n} \times [\![\hat{\mathbf{B}}]\!] = \mathbf{j}_c^\star(\mathbf{r}, t) \qquad \textit{(at coil surface)}\,. \tag{101}$$

  where $\mathbf{j}_c^\star(\mathbf{r}, t)$ is the prescribed skin current in the coil.

- Magnetic field outside coil subject to exterior BC (96) at wall (possibly moved to ∞), combined with plasma-vacuum interface conditions (97) and (98):

$$\mathbf{n} \cdot \mathbf{B} = \mathbf{n} \cdot \hat{\mathbf{B}} = 0 \qquad \textit{(at plasma–vacuum interface)}\,,$$

$$[\![p + \tfrac{1}{2}B^2]\!] = 0 \qquad \textit{(at plasma–vacuum interface)}\,.$$

Energy conservation for interface plasmas

- Total energy for model II *(closed system):*

$$H = \int \mathcal{H}^{\mathrm{p}}\, d\tau^{\mathrm{p}} + \int \mathcal{H}^{\mathrm{v}}\, d\tau^{\mathrm{v}}\,, \qquad \mathcal{H}^{\mathrm{p}} \equiv \tfrac{1}{2}\rho v^2 + \tfrac{1}{\gamma-1}\,p + \tfrac{1}{2}B^2\,, \quad \mathcal{H}^{\mathrm{v}} \equiv \tfrac{1}{2}\hat{B}^2\,. \tag{102}$$

Some algebra gives:

$$\frac{\mathrm{D}H^{\mathrm{p}}}{\mathrm{D}t} = -\int (p + \tfrac{1}{2}B^2)\,\mathbf{v}\cdot\mathbf{n}\, d\sigma\,, \qquad \frac{\mathrm{D}H^{\mathrm{v}}}{\mathrm{D}t} = \int \tfrac{1}{2}\hat{B}^2\,\mathbf{v}\cdot\mathbf{n}\, d\sigma\,. \tag{103}$$

Application of pressure balance BC yields energy conservation:

$$\frac{\mathrm{D}H}{\mathrm{D}t} = \int [\![p + \tfrac{1}{2}B^2]\!]\,\mathbf{v}\cdot\mathbf{n}\, d\sigma \overset{(98)}{=} 0\,, \quad QED. \tag{104}$$

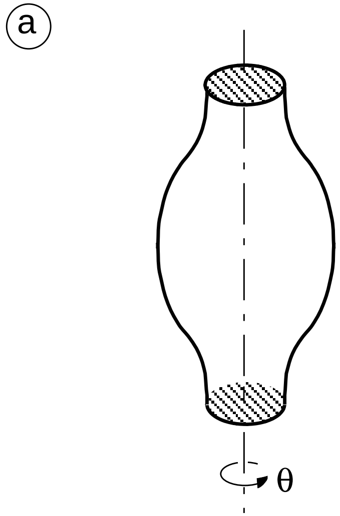
- For model III *(open system),* rate of change of total energy interior to coil surface (assuming $\hat{\mathbf{B}}^{\mathrm{ext}} = 0$):

$$\frac{\mathrm{D}H^{\mathrm{int}}}{\mathrm{D}t} = -\int \hat{\mathbf{S}}\cdot\mathbf{n}\, d\sigma_c \equiv -\int \hat{\mathbf{E}}\times\hat{\mathbf{B}}\cdot\mathbf{n}\, d\sigma_c \overset{(101)}{=} \int \hat{\mathbf{E}}\cdot\mathbf{j}_c^\star\, d\sigma_c\,. \tag{105}$$

$\Rightarrow$ Poynting flux $\hat{\mathbf{S}} \equiv \hat{\mathbf{E}}\times\hat{\mathbf{B}}$: power transferred to system by surface current $\mathbf{j}_c^\star$.

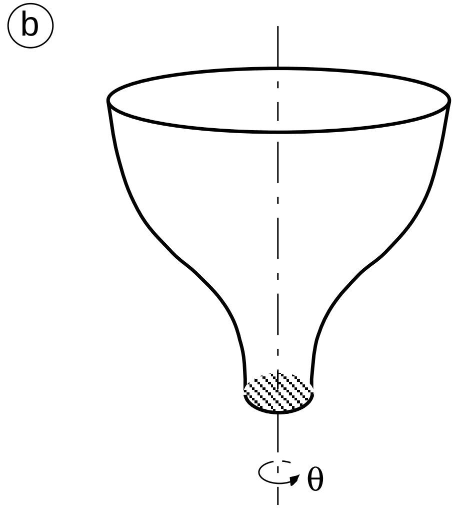
Astrophysical plasmas (models IV–VI)

2D



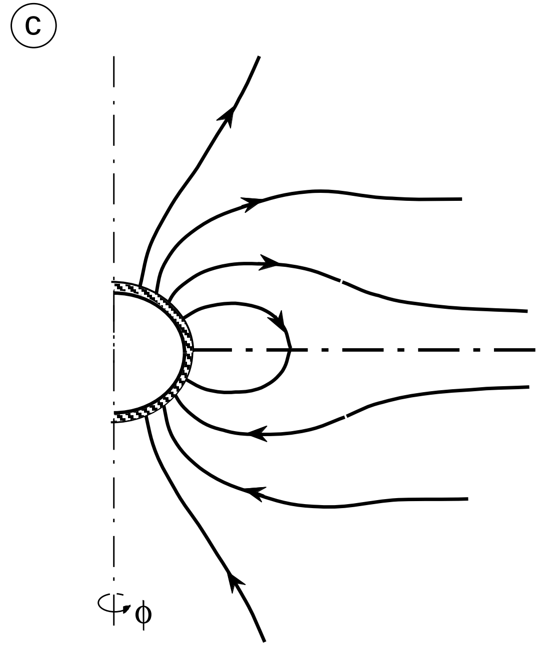
model IV

closed loop



model V

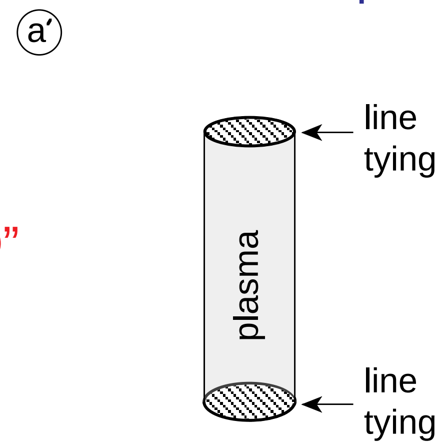
open loop

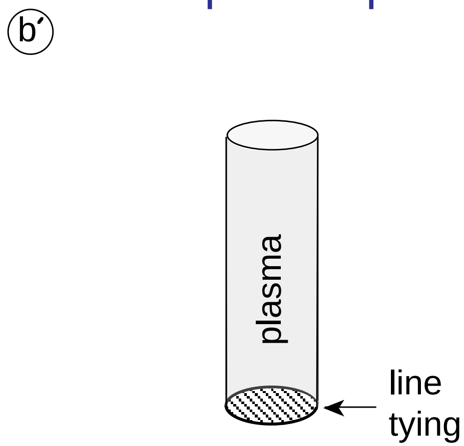


model VI

stellar wind

“1D”





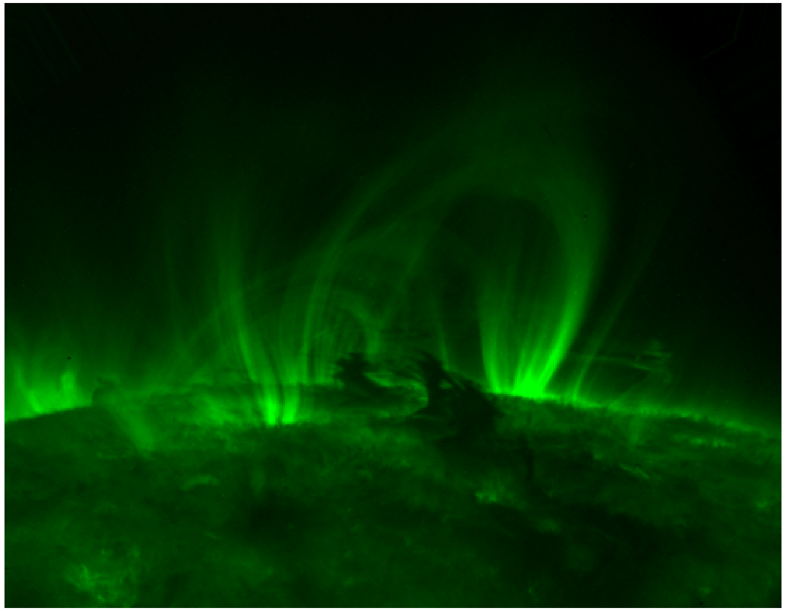
Model IV: 'closed' coronal magnetic loop

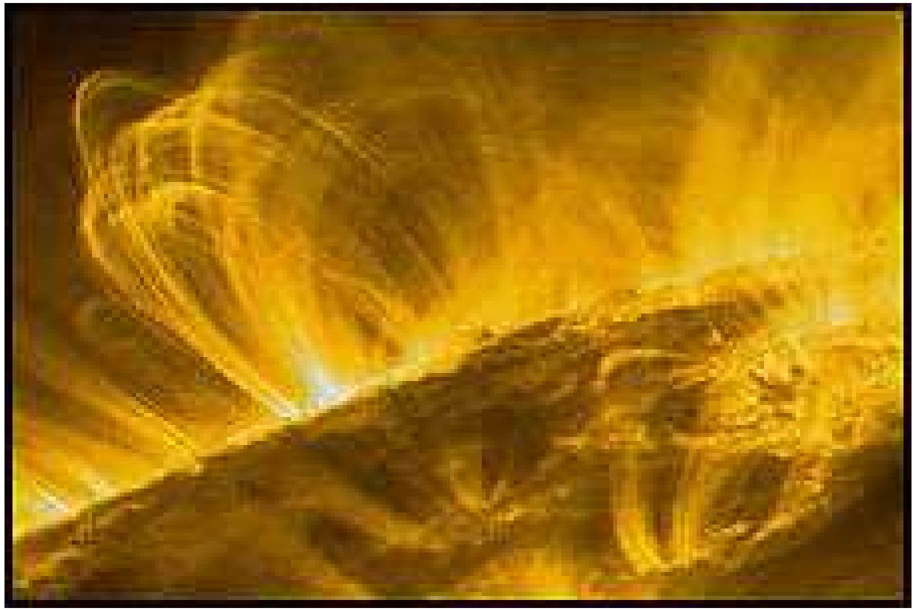
- In model IV, the field lines of finite plasma column (coronal loop) are line-tied on both sides to plasma of such high density (photosphere) that it is effectively immobile.

  $\Rightarrow$ Line-tying boundary conditions:

$$\mathbf{v} = 0 \qquad \textit{(at photospheric end planes)}\,. \tag{106}$$

  $\Rightarrow$ Applies to waves in solar coronal flux tubes, no back-reaction on photosphere:





- In this model, loops are straightened out to 2D configuration (depending on r and z). Also neglecting fanning out of field lines $\Rightarrow$ quasi-1D (finite length cylinder).

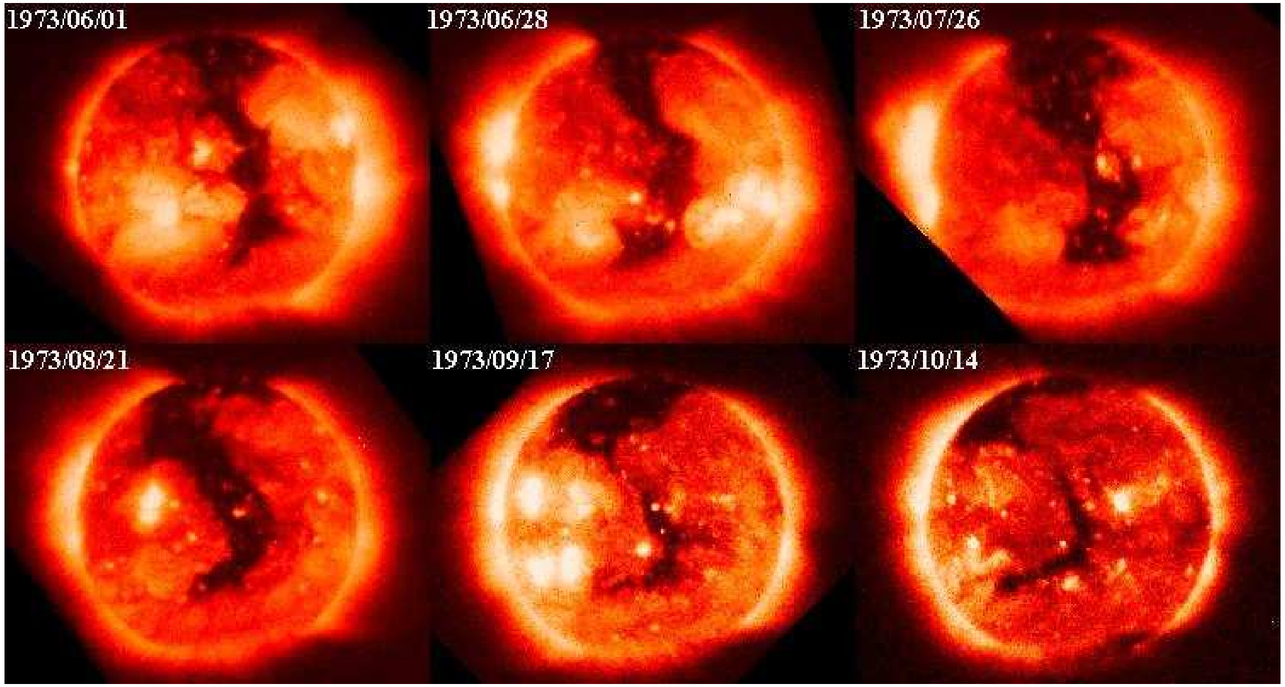
Model V: open coronal magnetic loop

- In model V, the magnetic field lines of a semi-infinite plasma column are line-tied on one side to a massive plasma.

  ⇒ Line-tying boundary condition:

$$\mathbf{v} = 0 \qquad \textit{(at photospheric end plane)}\,.$$

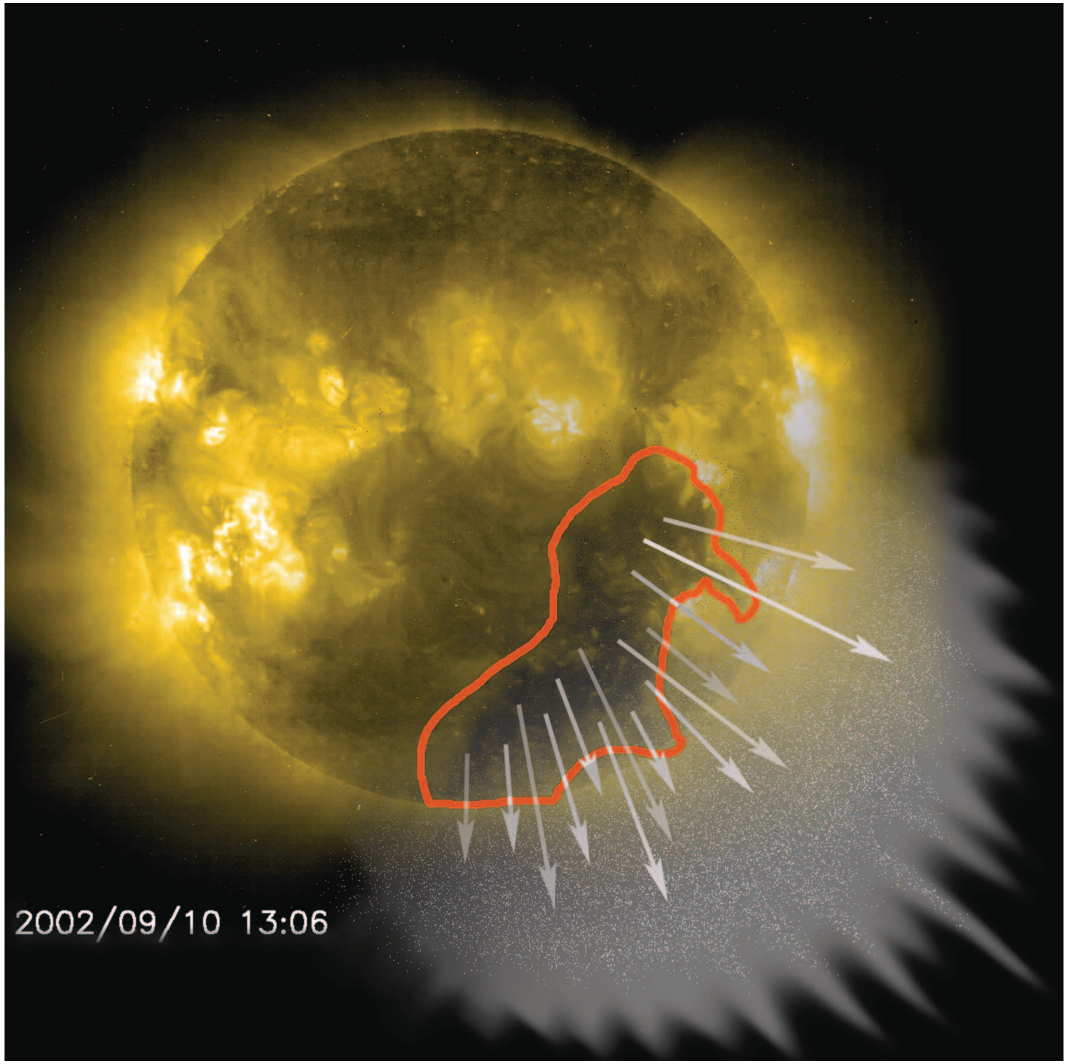
  ⇒ Applies to dynamics in coronal holes, where (fast) solar wind escapes freely:



- *Truly open* variants of models IV & V: photospheric excitation ($\mathbf{v}(t) \neq 0$ prescribed).

Model V (cont'd)

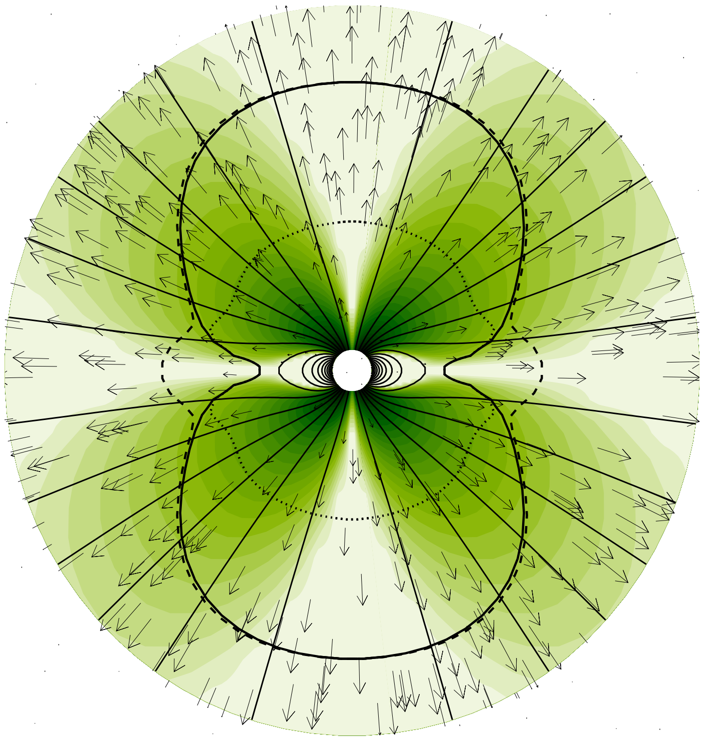
- Actual case:



Model VI: Stellar wind

- In model VI, a plasma is ejected from photosphere of a star and accelerated along the open magnetic field lines into outer space.

  ⇒ Combines closed & open loops (models IV & V), line-tied at dense photosphere, but stress on outflow rather than waves (requires more advanced discussion).



- Output from an actual simulation with the Versatile Advection code: 2D (axisymm.) magnetized wind with 'wind' and 'dead' zone. Sun at the center, field lines drawn, velocity vectors, density coloring. Dotted, drawn, dashed: slow, Alfvén, fast critical surfaces.

  [Keppens & Goedbloed, Ap. J. **530**, 1036 (2000)]