Generation of magnetoacoustic zonal flows by Alfvén waves in a rotating plasma

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Analytical theory of nonlinear generation of magnetoacoustic zonal flows in a rotating plasma is developed. As the primary modes causing such a generation, a totality of the Alfvén waves are considered, along with the kinetic, inertial, and rotational. It is shown that in all these cases of the Alfvén waves the generation is possible if the double plasma rotation frequency exceeds the zonal flow frequency. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2755941]

I. INTRODUCTION

Collective phenomena in a rotating magnetized media, related to the magnetorotational instability (MRI),[1–3] are one of the central topics of recent studies in the physics of liquid metals,[4–10] magnetic geodynamics,[11] as well as in astrophysics and plasma physics.[12–20] The main massive of analytical investigations in this topic is addressed to the linear theory of MRI. In contrast to them, the goal of the present paper is the analysis of nonlinear phenomena in rotating magnetized media taking the magnetized plasma as the example of such media. As a first step in this analysis we consider generation of zonal flows related to the magnetoacoustic oscillation branches by the Alfvén waves.

Generation of zonal flows (in other terminology, convective cells) by Alfvén waves was the topic of a rather wide number of papers, see Refs. 21–24. As it is known for the problem of zonal flow generation, only dispersive Alfvén waves are of interest. Characteristic varieties of such waves are the kinetic Alfvén waves (KAW),[21–23] relevant to plasma with $\beta > M_e/M_i$ and the inertial Alfvén waves (IAW),[24] realized for $\beta < M_e/M_i$, where β is the ratio of plasma pressure to the magnetic field pressure, M_e and M_i are the electron and ion mass, respectively. Both of these varieties of Alfvén waves are considered in our paper. The study of linear waves in a rotating plasma predicts one more variety of dispersive Alfvén waves, the rotational Alfvén waves (RAW), which are also included in our paper as a possible reason of generation of magnetoacoustic zonal flows.

There are two nonlinear mechanisms of zonal flow generation related, respectively, to the so-called vector and scalar nonlinearities.[24–26] However, as it is well-known in the case of nonrotating plasma, the scalar nonlinearity does not enter the plasma dynamic equations describing the Alfvén waves. Due to this fact, studying zonal flow generation by these waves, one should deal only with the vector nonlinearity. Using such a nonlinearity presupposes that the waves should have two projections of the perpendicular wave vector $\mathbf{k}_\perp$, k_x, and k_y, where x is the zonal flow direction, i.e., the "radial" one, z is direction of the equilibrium magnetic field, y is the "poloidal" direction. Meanwhile, in the case of rotating plasma there is a linear mixing of the Alfvén and magnetoacoustic waves. In its turn, it is known that the nonlinear dynamics of the magnetoacoustic waves is governed by the scalar nonlinearity (see, e.g., Ref. 27). Therefore, it is reasonable to suggest that in the rotating plasma a nonlinear mixing of these waves should be revealed which should lead to involving the scalar nonlinearity in the Alfvén wave dynamics. This allows one to consider the simplest case of Alfvén waves with $k_y=0$. This idea is the basis of the present paper. We take these waves to be dependent on x and z and, dealing with only the scalar nonlinearity, show that they are able to generate magnetoacoustic zonal flows.

In Sec. II we represent our basic equations. In Sec. III we transform them in conformity with the problem of zonal flow generation introducing the variables characterizing the primary modes, the zonal flows, and the sidebands. Section IV addresses the analysis of sideband amplitudes. In Sec. V we derive the zonal flow dispersion relation. Its analysis is given in Sec. VI. The results of the paper are discussed in Sec. VII.

II. BASIC EQUATIONS

We consider a cold rotating plasma described by the motion equation

$$\rho\frac{d\mathbf{V}}{dt} = -\frac{1}{8\pi}\nabla \mathbf{B}^2 + \frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B} \tag{1}$$

and the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}) = 0. \tag{2}$$

Here $\mathbf{V}$ is the plasma velocity, $\mathbf{B}$ is the magnetic field, ρ is the plasma mass density, $d/dt=\partial t+(\mathbf{V}\cdot\nabla)$. We are interested in the solution of these equations in a cylindrical coordinate system (R,ϕ,z), where R, ϕ, z are the radial, poloidal, and longitudinal coordinates, respectively. All the physical values are assumed to be independent of the poloidal coordinate, $\partial/\partial\phi=0$. The longitudinal plasma velocity is taken to be vanishing, $V_z=0$. Then Eqs. (1) and (2) lead to

$$\frac{\partial V_R}{\partial t} + V_R\frac{\partial V_R}{\partial R} - \frac{V_\phi^2}{R} = \frac{1}{8\pi\rho}\frac{\partial B_z^2}{\partial R} + \frac{B_z}{4\pi\rho}\frac{\partial B_R}{\partial z}, \tag{3}$$

$$\frac{\partial V_\phi}{\partial t} + V_R\left(\frac{\partial V_\phi}{\partial R} + \frac{V_\phi}{R}\right) = \frac{B_z}{4\pi\rho}\frac{\partial B_\phi}{\partial z}, \tag{4}$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial R}(\rho V_R) = 0. \tag{5}$$

It is assumed that plasma rotates in the poloidal direction with the angular frequency Ω dependent on the radial coordinate, $\Omega=\Omega(R)$. There is the equilibrium magnetic field $\mathbf{B}_0$, directed along the parallel coordinate, $\mathbf{B}_0=B_0\mathbf{e}_z$, where $\mathbf{e}_z$ is the unit vector along z.

In order to describe the perturbed magnetic field we use the Maxwell equation

$$\frac{\partial\mathbf{B}}{\partial t} = -c[\nabla\times\mathbf{E}] \tag{6}$$

and the Ohm's law of the form

$$\mathbf{E} + \frac{1}{c}[\mathbf{V}\times\mathbf{B}] + \frac{1}{en_0}\frac{\mathbf{B}}{\mathbf{B}^2}\left\{\mathbf{B}\cdot\left(\nabla p_e + M_e n_0\frac{\partial\mathbf{V}_e}{\partial t}\right)\right\} = 0. \tag{7}$$

Here $\mathbf{E}$ is the electric field, p_e is the electron pressure, $\mathbf{V}_e$ is the electron velocity, n_0 is the equilibrium plasma number density, e is the ion charge, M_e is the electron mass.

Note that Eq. (7) without the last term in its left-hand side is the standard one-fluid MHD (magnetohydrodynamic) formula for Ohm's law. The above terms describe the effects of parallel electron pressure and parallel electron inertia. The origin of these terms will be evident if one represents Eq. (7) in the form of the electron motion equation.

Acting by the operator $\nabla\times$ in Eq. (7), we arrive at the freezing condition of the form

$$\frac{\partial\mathbf{B}}{\partial t} - [\nabla\times[\mathbf{V}\times\mathbf{B}]] - \frac{c}{en_0}\times\left[\nabla\times\frac{\mathbf{B}}{\mathbf{B}^2}\left\{\mathbf{B}\cdot\left(\nabla p_e + M_e n_0\frac{\partial\mathbf{V}_e}{\partial t}\right)\right\}\right] = 0. \tag{8}$$

We also allow the magnetic field to satisfy the Maxwell equation,

$$\nabla\cdot\mathbf{B} = 0. \tag{9}$$

Hence one finds

$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{\partial B_z}{\partial z} = 0. \tag{10}$$

It follows from Eq. (8) that

$$\frac{\partial B_R}{\partial t} - \frac{\partial}{\partial z}(V_R B_z) = 0, \tag{11}$$

$$\frac{\partial B_z}{\partial t} = \frac{1}{R}\frac{\partial}{\partial R}(RV_R B_z), \tag{12}$$

$$\frac{\partial B_\phi}{\partial t} - \frac{\partial}{\partial z}(V_\phi B_z) + \frac{\partial}{\partial R}(V_R B_\phi - V_\phi B_R) + \frac{c}{en_0}\frac{\partial}{\partial R}\left[\frac{B_z}{\mathbf{B}^2}\left\{\mathbf{B}\cdot\left(\nabla p_e + M_e n_0\frac{\partial\mathbf{V}_e}{\partial t}\right)\right\}\right] = 0. \tag{13}$$

We take the electron pressure in the form $p_e=nT_{0e}$, where T_{0e} is the equilibrium electron temperature, n is the number density satisfied the electron continuity equation

$$\frac{\partial n}{\partial t} + n_0\frac{\partial V_{ez}}{\partial z} = 0. \tag{14}$$

The electron longitudinal velocity V_{ez} is expressed in terms of the longitudinal electric current j_z by

$$V_{ez} = -j_z/(en_0). \tag{15}$$

According to Ampere's law,

$$j_z = \frac{c}{4\pi}\frac{1}{R}\frac{\partial}{\partial R}(RB_\phi). \tag{16}$$

III. TRANSFORMATIONS OF BASIC EQUATIONS

A. Separation of variables

We represent each perturbed function X in the form

$$X = \tilde{X} + \hat{X} + \bar{X}. \tag{17}$$

Here $\tilde{X}$ describes the primary modes. It is defined by

$$\tilde{X} = \sum_{\mathbf{k}} [\tilde{X}_+ \exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t) + \tilde{X}_- \exp(-i\mathbf{k}\cdot\mathbf{r} + i\omega t)], \tag{18}$$

where ω and $\mathbf{k}$ are the frequencies and wave vectors of the primary modes, respectively, $\tilde{X}_\pm$ are their amplitudes. It is assumed that ω is a function of $\mathbf{k}$, satisfying the linear dispersion relation, $\omega \equiv \omega_{\mathbf{k}}$.

The function $\bar{X}$ describes the zonal flow. It is given by

$$\bar{X} = \bar{X}_0 \exp(-i\Omega_{\rm ZF} t + iq_x x) + \text{c.c.} \tag{19}$$

Here $\Omega_{\rm ZF}$ and q_x are the oscillation frequency and wave vector of the zonal flow, $\bar{X}_0$ is its amplitude, c.c. is the complex conjugate.

The function $\hat{X}$ describes the secondary small-scale modes (sidebands)

$$\hat{X} = \sum_{\mathbf{k}} [\hat{X}_+ \exp(i\mathbf{k}_+\cdot\mathbf{r} - i\omega_+ t) + \hat{X}_- \exp(i\mathbf{k}_-\cdot\mathbf{r} - i\omega_- t) + \text{c.c.}], \tag{20}$$

where $\omega_\pm = \Omega_{\rm ZF} \pm \omega$, and $\mathbf{k}_\pm = (q_x \pm k_x, \pm k_z)$.

B. Relations characterizing the primary modes

According to Eqs. (3)–(5) and (11)–(16) the primary modes are described by the equation system

$$\mp i\omega\tilde{V}_{R\pm} - 2\Omega\tilde{V}_{\phi\pm} = \pm\frac{iB_0}{4\pi\rho_0}(k_z\tilde{B}_{R\pm} - k_x\tilde{B}_{z\pm}), \tag{21}$$

$$\mp i\omega\tilde{V}_{\phi\pm} + \frac{\kappa^2}{2\Omega}\tilde{V}_{R\pm} = \pm\frac{ik_zB_0}{4\pi\rho_0}\tilde{B}_{\phi\pm}, \tag{22}$$

$$\mp i\omega\tilde{B}_{R\pm} \mp ik_zB_0\tilde{V}_{R\pm} = 0, \tag{23}$$

$$\mp i\omega(1 + k_x^2\lambda_e^2)\tilde{B}_{\phi\pm} \mp ik_zB_0\tilde{V}_{\phi\pm} - \frac{d\Omega}{d\ln R}\tilde{B}_{R\pm} - \frac{cT_{0e}}{en_0}k_xk_z\tilde{n}_\pm = 0, \tag{24}$$

$$\tilde{B}_{z\pm} = -k_x\tilde{B}_{R\pm}/k_z, \tag{25}$$

$$\mp i\omega\tilde{\rho}_\pm \pm i\rho_0k_x\tilde{V}_{R\pm} = 0, \tag{26}$$

$$\mp i\omega\tilde{n}_\pm + \frac{ck_xk_z}{4\pi e}\tilde{B}_{\phi\pm} = 0. \tag{27}$$

Here

$$\kappa^2 = \frac{2\Omega}{R}\frac{d(R^2\Omega)}{dR}, \tag{28}$$

ρ_0 is the equilibrium plasma mass density, $\lambda_e^2 = c^2/\omega_{pe}^2$ is the squared collisionless skin length, ω_{pe} is the electron plasma frequency.

By means of Eq. (27) we express $\tilde{n}_\pm$ in terms of $\tilde{B}_{\phi\pm}$,

$$\tilde{n}_\pm = \mp\frac{ick_xk_z}{4\pi e\omega}\tilde{B}_{\phi\pm}. \tag{29}$$

Using Eqs. (29), (15), and (16), Eq. (24) takes the form

$$\mp i\omega\left(1 + k_x^2\lambda_e^2 - \frac{k_z^2v_A^2}{\omega^2}k_x^2\rho_s^2\right)\tilde{B}_{\phi\pm} \mp ik_zB_0\tilde{V}_{\phi\pm} - \frac{d\Omega}{d\ln R}\tilde{B}_{R\pm} = 0, \tag{30}$$

where $\rho_s^2 \equiv T_{0e}/(M_i\omega_{Bi}^2)$ is the squared ion Larmor radius for the electron temperature, $v_A^2 = B_0^2/(4\pi\rho_0)$ is the squared Alfvén velocity, M_i is the ion mass, ω_{Bi} is the ion cyclotron frequency.

We assume $k_x \gg k_z$, $\omega \ll k_xv_A$. It then follows from Eqs. (21)–(23) and (25) that

$$\tilde{V}_{\phi\pm} = -\frac{k_zv_A^2}{\omega}\left(1 - \frac{\kappa^2}{k_x^2v_A^2}\right)\frac{\tilde{B}_{\phi\pm}}{B_0}, \tag{31}$$

$$\tilde{B}_{z\pm} = \frac{B_0}{\rho_0}\tilde{\rho}_\pm = \pm i\frac{2\Omega k_z}{k_x\omega}\tilde{B}_{\phi\pm}, \tag{32}$$

$$\tilde{V}_{R\pm} = \pm i\frac{2\Omega k_z}{k_x^2B_0}\tilde{B}_{\phi\pm}. \tag{33}$$

Substituting Eq. (31) into Eq. (24), we arrive at the dispersion relation for the primary modes

$$D(\omega,\mathbf{k}) \equiv \omega^2 - \frac{k_z^2v_A^2}{1 + k_x^2\lambda_e^2}\left(1 + k_x^2\rho_s^2 - \frac{4\Omega^2}{k_x^2v_A^2}\right) = 0. \tag{34}$$

It is remarkable that the primary mode frequency ω calculated by means of Eq. (34) depends on k_x. Such a dependence leads to that the radial group velocity of these modes, $V_g \equiv \partial\omega/\partial k_x$, proves to be nonvanishing. The effect of finiteness of V_g is crucial for the problem of zonal flow generation considered.

1. Kinetic Alfvén waves

In the limit $\lambda_e \to 0$, $\Omega^2 \to 0$, Eq. (34) yields

$$D(\omega,\mathbf{k}) = \omega^2 - k_z^2v_A^2(1 + k_x^2\rho_s^2) = 0. \tag{35}$$

This limit corresponds to the kinetic Alfvén waves (KAW).

2. Inertial Alfvén waves

For $\rho_s^2 \to 0$, $\Omega^2 \to 0$ one has

$$D(\omega,\mathbf{k}) = \omega^2 - \frac{k_z^2v_A^2}{1 + k_x^2\lambda_e^2} = 0. \tag{36}$$

This dispersion relation describes the inertial Alfvén waves (IAW).

3. Rotational Alfvén waves

At last, taking $\rho_s^2 \to 0$, $\lambda_e^2 \to 0$, we arrive at

$$D(\omega,\mathbf{k}) = \omega^2 - k_z^2 v_A^2[1 - 4\Omega^2/(k_x^2 v_A^2)] = 0. \tag{37}$$

In this case one deals with the rotational Alfvén waves (RAW).

C. Zonal flows

Averaging Eqs. (3)–(5) and (10)–(13) over the small-scale oscillations, one has

$$- i\Omega_{\rm ZF}\bar{V}_R - 2\Omega\bar{V}_\phi + iq_x v_A^2 \frac{\bar{B}_z}{B_0} = F_R, \tag{38}$$

$$- i\Omega_{\rm ZF}\bar{V}_\phi + \frac{\kappa^2}{2\Omega}\bar{V}_R = F_\phi, \tag{39}$$

$$\bar{B}_R = 0, \tag{40}$$

$$- i\Omega_{\rm ZF}\bar{B}_z + iq_x B_0 \bar{V}_R = F_z, \tag{41}$$

$$- i\Omega_{\rm ZF}\bar{B}_\phi + iq_x\langle V_R B_\phi - V_\phi B_R\rangle = 0, \tag{42}$$

$$- i\Omega_{\rm ZF}\bar{\rho} + i\rho_0 q_x \bar{V}_R = H_\rho, \tag{43}$$

where the angle brackets mean this averaging. Here

$$F_R = - iq_x\langle \tilde{V}_R\hat{V}_R\rangle - \frac{iq_x}{4\pi\rho_0}\langle \tilde{B}_z\hat{B}_z\rangle + \frac{B_0}{4\pi\rho_0^2}\left\langle \tilde{\rho}\frac{\partial \hat{B}_z}{\partial R} + \hat{\rho}\frac{\partial \tilde{B}_z}{\partial R}\right\rangle + \frac{1}{4\pi\rho_0^2}\left\langle \left(\tilde{B}_z - \frac{B_0}{\rho_0}\tilde{\rho}\right)\frac{\partial \hat{B}_R}{\partial z} + \left(\hat{B}_z - \frac{B_0}{\rho_0}\hat{\rho}\right)\frac{\partial \tilde{B}_R}{\partial z}\right\rangle, \tag{44}$$

$$F_z = - iq_x\langle \tilde{V}_R\hat{B}_z + \hat{V}_R\tilde{B}_z\rangle, \tag{45}$$

$$F_\phi = \frac{1}{4\pi\rho_0}\left\langle \left(\tilde{B}_z - \frac{B_0}{\rho_0}\tilde{\rho}\right)\frac{\partial \hat{B}_\phi}{\partial z} + \left(\hat{B}_z - \frac{B_0}{\rho_0}\hat{\rho}\right)\frac{\partial \tilde{B}_\phi}{\partial z}\right\rangle, \tag{46}$$

$$H_\rho = -\left\langle \tilde{V}_R\frac{\partial \hat{\rho}}{\partial R} + \hat{V}_R\frac{\partial \tilde{\rho}}{\partial R} + \tilde{\rho}\frac{\partial \hat{V}_R}{\partial R} + \hat{\rho}\frac{\partial \tilde{V}_R}{\partial R}\right\rangle. \tag{47}$$

Using Eqs. (38) and (39), we express $\bar{V}_R$ and $\bar{V}_\phi$ in terms of $\bar{B}_z$,

$$\bar{V}_R = \frac{1}{\kappa^2 - \Omega_{\rm ZF}^2}\left(- \Omega_{\rm ZF} q_x v_A^2 \frac{\bar{B}_z}{B_0} - i\Omega_{\rm ZF}F_R + 2\Omega F_\phi\right), \tag{48}$$

$$\bar{V}_\phi = \frac{1}{\kappa^2 - \Omega_{\rm ZF}^2}\left(iq_x v_A^2 \frac{\kappa^2}{2\Omega}\frac{\bar{B}_z}{B_0} - i\Omega_{\rm ZF}F_\phi + \frac{\kappa^2}{2\Omega}F_R\right). \tag{49}$$

Substituting Eq. (48) into Eq. (41), we arrive at

$$(\Omega_{\rm ZF}^2 - q_x^2 v_A^2 - \kappa^2)\bar{B}_z = \frac{q_x B_0}{\Omega_{\rm ZF}}(i\Omega_{\rm ZF}F_R - 2\Omega F_\phi) - \frac{i}{\Omega_{\rm ZF}}(\kappa^2 - \Omega_{\rm ZF}^2)F_z. \tag{50}$$

It is remarkable for our problem that a zonal flow is predicted even in neglecting the primary modes. Such a zonal flow can be called the spontaneous zonal flow. According to Eq. (50), dispersion relation for spontaneous zonal flow is the following:

$$\Omega_{\rm ZF}^2 = q_x^2 v_A^2 + \kappa^2. \tag{51}$$

Evidently, this spontaneous zonal flow oscillation branch is nothing but the fast magnetoacoustic waves modified by the rotation effects. True, since we are interested in the flows with $\Omega_{\rm ZF} \simeq q_x V_g \ll q_x v_A$, the term with $\Omega_{\rm ZF}^2$ in the left-hand side of (50) will be neglected.

Having allowed for the contributions of nonlinear terms into Eqs. (38)–(42), we can neglect them in the following calculations of the sidebands. Therefore, assuming $\kappa^2 \gg \Omega_{\rm ZF}^2$, below we take

$$\bar{V}_\phi = \frac{i\kappa^2}{2\Omega}q_x\frac{\bar{B}_z}{B_0}, \tag{52}$$

$$\bar{V}_R = \frac{\Omega_{\rm ZF}}{q_x}\frac{\bar{B}_z}{B_0}, \tag{53}$$

$$\bar{B}_\phi = 0, \tag{54}$$

$$\bar{\rho} = \rho_0\bar{B}_z/B_0. \tag{55}$$

Similarly, in the following calculations the term H_ρ in Eq. (43) is excessive. The same concerns the nonlinear term in Eq. (42).

IV. ANALYSIS OF SIDEBAND AMPLITUDES

Basic equations for sideband amplitudes

Using Eqs. (3)–(5) and (10)–(15), we arrive at the following set of equations for the sideband amplitudes:

$$- i\omega_\pm\hat{V}_{R\pm} - 2\Omega\hat{V}_{\phi\pm} + \frac{B_0}{4\pi\rho_0}ik_{x\pm}\hat{B}_{z\pm} \mp \frac{ik_z B_0}{4\pi\rho_0}\hat{B}_{R\pm} = Y_\pm^{V_R}, \tag{56}$$

$$- i\omega_\pm\hat{V}_{\phi\pm} + \frac{\kappa^2}{2\Omega}\hat{V}_{R\pm} \mp \frac{B_0}{4\pi\rho_0}ik_z\hat{B}_{\phi\pm} = Y_\pm^{V_\phi}, \tag{57}$$

$$- i\omega_\pm\hat{B}_{R\pm} \mp ik_z B_0\hat{V}_{R\pm} = Y_\pm^{B_R}, \tag{58}$$

$$- i\omega_\pm(1 + k_{x\pm}^2\lambda_e^2)\hat{B}_{\phi\pm} \mp ik_z B_0\hat{V}_{\phi\pm} - \hat{B}_{R\pm}\frac{d\Omega}{d\ln R} \mp \frac{cT_{0e}}{en_0}k_z k_{x\pm}\hat{n}_\pm = Y_\pm^{B_\phi}, \tag{59}$$

$$- i\omega_\pm\hat{\rho}_\pm + i\rho_0 k_{x\pm}\hat{V}_{R\pm} = Y_\pm^{\rho}, \tag{60}$$

$$\hat{B}_{z\pm} = \mp \frac{k_{x\pm}}{k_z}\hat{B}_{R\pm}, \tag{61}$$

$$-i\omega_{\pm}\hat{n}_{\pm} \pm \frac{ck_{\pm}k_z}{4\pi e}\hat{B}_{\phi\pm} = 0. \tag{62}$$

Here

$$Y_{\pm}^{V_R} = -ik_{x\pm}\bar{V}_R\tilde{V}_{R\pm}, \tag{63}$$

$$Y_{\pm}^{V_\phi} = -iq_x\bar{V}_\phi\tilde{V}_{R\pm} \mp ik_x\bar{V}_R\tilde{V}_{\phi\pm}, \tag{64}$$

$$Y_{\pm}^{B_R} = \pm ik_z(\bar{V}_R\tilde{B}_{z\pm} + \bar{B}_z\tilde{V}_{R\pm}), \tag{65}$$

$$Y_{\pm}^{B_\phi} = \pm ik_z\bar{B}_z\tilde{V}_{\phi\pm} \pm ik_z\bar{V}_\phi\tilde{B}_{z\pm} - ik_{\pm}(\bar{V}_R\tilde{B}_{\phi\pm} + \bar{B}_\phi\tilde{V}_{R\pm}) + ik_{x\pm}(\bar{V}_\phi\tilde{B}_{R\pm} + \bar{B}_R\tilde{V}_{\phi\pm}), \tag{66}$$

$$Y_{\pm}^{\rho} = -ik_{x\pm}(\bar{\rho}\tilde{V}_{R\pm} + \tilde{\rho}_{\pm}\bar{V}_R). \tag{67}$$

Obtaining Eqs. (63) and (64), we have allowed for $\bar{\rho}=\rho_0\bar{B}_z/B_0$, $\rho_{\pm}=\rho_0\tilde{B}_{z\pm}/B_0$. Using Eq. (62), Eq. (59) reduces to [cf. Eq. (30)]

$$-i\omega_{\pm}\left(1 + k_{x\pm}^2\lambda_e^2 - \frac{k_z^2v_A^2}{\omega_{\pm}^2}k_x^2\rho_s^2\right)\hat{B}_{\phi\pm} \mp ik_zB_0\hat{V}_{\phi\pm} - \hat{B}_{R\pm}\frac{d\Omega}{d\ln R} = Y_{\pm}^{B_\phi}. \tag{68}$$

Since k_x is assumed to be a large parameter, Eq. (56) in allowing for Eq. (61) yields

$$\hat{B}_{R\pm} = \pm\frac{ik_zB_0}{k_{x\pm}^2v_A^2}(2\Omega\hat{V}_{\phi\pm} + Y_{\pm}^{V_R}). \tag{69}$$

It then follows from Eq. (58) that

$$\hat{V}_{R\pm} = -\frac{i\omega_{\pm}}{k_{x\pm}^2v_A^2}(2\Omega\hat{V}_{\phi\pm} + Y_{\pm}^{V_R}) \pm i\frac{Y_{\pm}^{B_R}}{k_zB_0}. \tag{70}$$

Substituting Eq. (70) into Eq. (57) and Eq. (69) into Eq. (59), we arrive at inhomogeneous equation system for $\hat{V}_{\phi\pm}$ and $\hat{B}_{\phi\pm}$,

$$\omega_{\pm}\left(1 + \frac{\kappa^2}{k_{x\pm}^2v_A^2}\right)\hat{V}_{\phi\pm} \pm \frac{k_z}{B_0}v_A^2\hat{B}_{\phi\pm} = iY_{\pm}^{V_\phi} - \frac{\kappa^2}{2\Omega}\left(\frac{\omega_{\pm}}{k_{x\pm}^2v_A^2}Y_{\pm}^{V_R} \mp \frac{Y_{\pm}^{B_R}}{k_zB_0}\right), \tag{71}$$

$$\pm k_zB_0\left(1 + \frac{1}{k_{x\pm}^2v_A^2}\frac{d\Omega^2}{d\ln R}\right)\hat{V}_{\phi\pm} + \omega_{\pm}\left(1 + k_{x\pm}^2\lambda_e^2 - \frac{k_z^2v_A^2}{\omega_{\pm}^2}k_{x\pm}^2\rho_s^2\right)\hat{B}_{\phi\pm} = iY_{\pm}^{B_\phi} \mp \frac{k_zB_0}{k_{x\pm}^2v_A^2}\frac{d\Omega}{d\ln R}Y_{\pm}^{V_R}. \tag{72}$$

Since the value $\kappa^2/(k_{x\pm}^2v_A^2)$ is the small parameter, Eq. (71) reduces to

$$\hat{V}_{\phi\pm} = \mp\frac{k_zv_A^2}{B_0\omega_{\pm}}\left(1 - \frac{\kappa^2}{k_{x\pm}^2v_A^2}\right)\tilde{B}_{\phi\pm} + \frac{1}{\omega_{\pm}}\left[iY_{\pm}^{V_\phi} - \frac{\kappa^2}{2\Omega}\left(\frac{\omega_{\pm}Y_{\pm}^{V_R}}{k_{x\pm}^2v_A^2} \mp \frac{Y_{\pm}^{B_R}}{K_zB_0}\right)\right]. \tag{73}$$

Substituting Eq. (73) into Eq. (72) and allowing for $(d\Omega^2/d\ln R)/(k_{x\pm}^2v_A^2)$ and k_z^2/k_x^2 are also small parameters yields

$$\hat{B}_{\phi\pm} = Z_{\pm}/D_{\pm}. \tag{74}$$

Here [cf. Eq. (73)]

$$D_{\pm} = \omega_{\pm}^2 - \frac{k_z^2v_A^2}{1 + k_{x\pm}^2\lambda_e^2}\left(1 + k_{x\pm}^2\rho_s^2 - \frac{4\Omega^2}{k_{x\pm}^2v_A^2}\right), \tag{75}$$

$$Z_{\pm} = \frac{1}{1 + k_{x\pm}^2\lambda_e^2}\left[i(\omega_{\pm}Y_{\pm}^{B_\phi} \mp k_zB_0Y_{\pm}^{V_\phi}) - \frac{\kappa^2}{2\Omega}\left(Y_{\pm}^{B_R} \mp \frac{k_zB_0\omega_{\pm}}{k_{x\pm}^2v_A^2}Y_{\pm}^{V_R}\right)\right]. \tag{76}$$

It will be seen from our following calculations that the values $D_{\pm}$ are a small parameter. Therefore, substituting Eq. (76) into Eq. (73) leads to

$$\hat{V}_{\phi\pm} = \mp\frac{k_zv_A^2}{B_0\omega_{\pm}}\left(1 - \frac{\kappa^2}{k_{x\pm}^2v_A^2}\right)\frac{Z_{\pm}}{D_{\pm}}. \tag{77}$$

Similarly, Eqs. (58), (60), (61), (69), and (70) yield

$$\hat{B}_{z\pm} = \frac{B_0}{\rho_0}\hat{\rho}_{\pm} = \frac{B_0k_{x\pm}\hat{V}_{R\pm}}{\omega_{\pm}} = \pm i\frac{2\Omega k_z}{k_{x\pm}\omega_{\pm}}\frac{Z_{\pm}}{D_{\pm}}. \tag{78}$$

Now we note that in terms of $\bar{B}_z$, $\tilde{B}_{\phi\pm}$ Eqs. (63)–(66) take the form

$$Y_{\pm}^{V_R} = \pm\frac{2\Omega\Omega_{\rm ZF}k_zk_{x\pm}}{q_xk_x^2}\frac{\bar{B}_z\tilde{B}_{\phi\pm}}{B_0^2}, \tag{79}$$

$$Y_{\pm}^{V_\phi} = \pm ik_zv_A^2\left[\frac{k_x}{q_x}\frac{\Omega_{\rm ZF}}{\omega} - \frac{\kappa^2}{v_A^2k_x^2}\left(1 + \frac{k_x}{q_x}\frac{\Omega_{\rm ZF}}{\omega}\right)\right]\frac{\bar{B}_z\tilde{B}_{\phi\pm}}{B_0^2}, \tag{80}$$

$$Y_{\pm}^{B_R} = -\frac{2\Omega k_z^2}{k_x^2}\left(1 + \frac{k_x}{q_x}\frac{\Omega_{\rm ZF}}{\omega}\right)\frac{\bar{B}_z}{B_0^2}\tilde{B}_{\phi\pm}, \tag{81}$$

$$Y_{\pm}^{B_\phi} = -i\left(k_{x\pm}\frac{\Omega_{\rm ZF}}{q_x} \pm \frac{k_z^2v_A^2}{\omega}\right)\frac{\bar{B}_z}{B_0}\tilde{B}_{\phi\pm}. \tag{82}$$

Using Eqs. (79)–(82), Eq. (76) reduces to

$$Z_{\pm} = c_{\pm}\frac{\bar{B}_z\tilde{B}_{\phi\pm}}{B_0}, \tag{83}$$

where

$$c_\pm = \frac{1}{1+k_{x\pm}^2\lambda_e^2}\left[\omega_\pm\left(\frac{k_{x\pm}\Omega_{ZF}}{q_x} \pm \frac{k_z^2 v_A^2}{\omega}\right) + k_z^2 v_A^2 \frac{k_x}{q_x}\frac{\Omega_{ZF}}{\omega}\right]. \quad (84)$$

Here we have neglected the terms small as k_z^2/k_x^2.

V. DERIVATION OF ZONAL FLOW DISPERSION RELATION

A. Starting form of zonal flow dispersion relation

Using the results of Secs. III and IV, one can show that

$$\frac{B_0}{4\pi\rho_0}\left\langle \tilde{\rho}\frac{\partial \tilde{B}_z}{\partial R} + \hat{\rho}\frac{\partial \hat{B}_z}{\partial R}\right\rangle = \frac{iq_x}{4\pi\rho_0}\langle \tilde{B}_z\hat{B}_z\rangle. \quad (85)$$

Equation (85) means that the contributions of the second and third terms into the right-hand side of Eq. (44) are mutually cancelled. In addition, allowing for Eqs. (32) and (78), the contribution of the last term into the right-hand side of Eq. (54) vanishes. As a result, we arrive at

$$F_\phi = 0, \quad (86)$$

$$F_R = -iq_x\langle \tilde{V}_R\hat{V}_R\rangle, \quad (87)$$

$$F_z = -iq_x\langle \tilde{V}_R\hat{B}_z + \hat{V}_R\tilde{B}_z\rangle. \quad (88)$$

Turning to Eq. (50), let us estimate the relative role of the terms with F_R and F_z for zonal flow generation. Then we assume $\Omega_{ZF} \simeq q_x V_g$, where $V_g \equiv \partial\omega/\partial k_x$ is the radial group velocity of the primary modes, and take the estimate $\partial\omega/\partial k_x \simeq \omega/k_x$. Then using Eqs. (87) and (88), we obtain

$$\frac{\Omega_{ZF}F_R B_0}{q_x v_A^2 F_z} \simeq \frac{k_z^2}{k_x^2}. \quad (89)$$

Allowing for Eq. (86) and neglecting the terms of order k_z^2/k_x^2, Eq. (50) reduces to

$$(q_x^2 v_A^2 + \kappa^2)\bar{B}_z = \frac{i(\kappa^2 - \Omega_{ZF}^2)}{\Omega_{ZF}}F_z. \quad (90)$$

Here we have also taken into account that $\Omega_{ZF} \ll q_x v_A$.

Using Eqs. (32), (33), and (51), we find

$$\langle \tilde{V}_R\hat{B}_z + \hat{V}_R\tilde{B}_z\rangle = \frac{4\Omega^2 k_z^2}{B_0 k_x\omega}\mu, \quad (91)$$

where

$$\mu = \frac{Z_+\tilde{B}_{\phi-}}{k_{x+}^2 D_+}\left(1+\frac{\omega k_{x+}}{k_x\omega_+}\right) + \frac{Z_-\tilde{B}_{\phi+}}{k_{x-}^2 D_-}\left(1+\frac{\omega k_{x-}}{k_x\omega_-}\right). \quad (92)$$

Using Eq. (83), we represent Eq. (92) in the form

$$\mu = \frac{|\tilde{B}_{\phi+}|^2\bar{B}_z}{B_0}\Lambda, \quad (93)$$

where

$$\Lambda = \frac{c_+}{k_{x+}^2 D_+}\left(1+\frac{k_{x+}\omega}{k_x\omega_+}\right) + \frac{c_-}{k_{x-}^2 D_-}\left(1+\frac{k_{x-}\omega}{k_x\omega_-}\right). \quad (94)$$

Thus, according to Eqs. (88), (91), and (93),

$$F_z = -i\frac{4\Omega^2 k_z^2 q_x}{k_x\omega B_0^2}\Lambda|\tilde{B}_{\phi+}|^2\bar{B}_z. \quad (95)$$

Substituting Eq. (95) into Eq. (90), we arrive at the zonal flow dispersion relation

$$q_x^2 v_A^2 + \kappa^2 = \frac{4\Omega^2}{\Omega_{ZF}}\frac{q_x k_z^2(\kappa^2 - \Omega_{ZF}^2)}{k_x\omega B_0^2}\Lambda|\tilde{B}_{\phi+}|^2. \quad (96)$$

B. Transformations of zonal flow dispersion relation

Using Eq. (75) and allowing for smallness of the parameters q_x/k_x, Ω_{ZF}/ω, we obtain

$$\frac{1}{D_\pm} = \pm\frac{1}{2\omega(\Omega_{ZF} - q_x V_g)} \times\left[1 \mp \frac{1}{\Omega_{ZF} - q_x V_g}\left(\frac{\Omega_{ZF}^2}{2\omega} + \frac{3}{2}V_g\frac{q_x^2}{k_x}\right)\right]. \quad (97)$$

Similarly, one has

$$1 + \frac{k_{x\pm}\omega}{k_x\omega_\pm} = 2 \pm \left(\frac{q_x}{k_x} - \frac{\Omega_{ZF}}{\omega}\right), \quad (98)$$

$$\frac{1}{k_{x\pm}^2} = \frac{1}{k_x^2}\left(1 \mp \frac{2q_x}{k_x}\right), \quad (99)$$

$$c_\pm = \frac{1}{1+k_x^2\lambda_e^2}\left\{\left[k_z^2 v_A^2\left(1+\frac{k_x}{q_x}\frac{\Omega_{ZF}}{\omega}\right) + \frac{k_x}{q_x}\Omega_{ZF}\right] \times\left(1 \mp \frac{2\lambda_e^2 k_x q_x}{1+k_x^2\lambda_e^2}\right) \pm \Omega_{ZF}\left(\omega + \frac{k_z^2 v_A^2}{\omega} + \Omega_{ZF}\frac{k_x}{q_x}\right)\right\}. \quad (100)$$

Then in the case of the monochromatic primary mode Eq. (94) reduces to

$$\Lambda = -\frac{q_x^2 V_g^2}{\omega^2 k_x^2(\Omega_{ZF} - q_x V_g)^2(1+k_x^2\lambda_e^2)}\left(1+\frac{3\omega}{V_g k_x}\right) \times\left[k_z^2 v_A^2\left(1+\frac{k_x V_g}{\omega}\right) + k_x V_g\omega\right]. \quad (101)$$

For such Λ Eq. (96) yields

$$(\Omega_{ZF} - q_x V_g)^2 = -\frac{4\Omega^2\kappa^2 k_z^2 q_x^2 f}{(q_x^2 v_A^2 + \kappa^2)k_x^4}\left(1 - \frac{\Omega_{ZF}^2}{\kappa^2}\right)\frac{|\tilde{B}_\phi|^2}{B_0^2}, \quad (102)$$

where

$$f = \left(3 + \frac{k_x V_g}{\omega}\right)\left[\frac{1 + k_x V_g/\omega}{1 + k_x^2\rho_s^2 - 4\Omega^2/(k_x^2 v_A^2)} + \frac{k_x V_g}{\omega(1+k_x^2\lambda_e^2)}\right]. \quad (103)$$

VI. ANALYSIS OF ZONAL FLOW DISPERSION RELATION

It is necessary for zonal flow generation that the right-hand side of Eq. (102) should be negative. One can see that its sign is determined by signs of the factor f and the difference $1-\Omega_{ZF}^2/\kappa^2$.

One finds from Eq. (35) that in the case of KAW

$$V_g = k_z^2 v_A^2 k_x \rho_s^2/\omega. \tag{104}$$

Then f takes the form

$$f=(3+4k_x^2\rho_s^2)(1+3k_x^2\rho_s^2)/(1+k_x^2\rho_s^2)^3. \tag{105}$$

Instead of this, for IAW one has from Eq. (36)

$$V_g = -\frac{k_z^2 v_A^2 k_x \lambda_e^2}{\omega(1+k_x^2\lambda_e^2)^2}. \tag{106}$$

In this case

$$f=(3+2k_x^2\lambda_e^2)/(1+k_x^2\lambda_e^2)^3. \tag{107}$$

As last, according to Eq. (37), in the case of RAW

$$V_g = -4\Omega^2 k_z^2/(k_x^3\omega). \tag{108}$$

Since we are interested in weakly dispersive RAW, $\Omega^2/(k_x^2 v_A^2) \ll 1$, the factor f does not depend approximately on dispersion and is given by

$$f=3. \tag{109}$$

Thus, for all varieties of the Alfvén waves of interest one has

$$f>0. \tag{110}$$

Then we conclude that the zonal flow generation by these waves takes place if $\kappa^2 > \Omega_{ZF}^2$, i.e.,

$$\kappa^2 > q_x^2 V_g^2. \tag{111}$$

Taking $\kappa^2 \simeq 4\Omega^2$, this condition can be interpreted as the fact that the generation considered is possible if the double plasma rotation frequency exceeds the zonal flow frequency.

VII. DISCUSSIONS

We have made a step in the development of analytical theory of nonlinear phenomena related to the MRI analyzing generation of magnetoacoustic zonal flows by the Alfvén waves. Then, in addition to the RAW inherent to the standard one-fluid MHD approach, we have included in our analysis the KAW and IAW. This has been attained by adding the effects of electron pressure and electron inertia into the parallel Ohm's law, see Eq. (7). Thereby, we have modified the traditional MHD freezing condition by the above effects, see Eq. (8).

Specificity of our problem is that it is multivariable. This requires the choice of the basis variables for the primary modes and the flows. We have suggested that the primary modes, i.e., the Alfvén waves, can be characterized by the poloidal projection of the perturbed magnetic field $\tilde{B}_\phi$. This value plays the role of basis variable of the primary modes. In contrast to this, the generated flows are relevant to the magnetoacoustic oscillation branches. Therefore, they can be characterized by the compressible part of their magnetic field $\bar{B}_z$. This value is chosen as the basis variable of the flows. Then our main massive of calculations has consisted of obtaining the sideband amplitudes in terms of $\tilde{B}_\phi$ and $\bar{B}_z$. The results of these calculations are given by Eqs. (74), (77), and (78) complemented by Eqs. (75), (83), and (84).

Using the expressions for the sideband amplitudes, we obtain the zonal flow dispersion relation given by Eq. (96) with the parameter Λ defined by Eq. (94). In the case of monochromatic primary modes Eq. (94) reduces to Eq. (101). By means of Eq. (101) we derive the zonal flow dispersion relation of the form Eq. (102) dependent on the factor f and the difference $1-\Omega_{ZF}^2/\kappa^2$. Then we have shown that for all varieties of the primary modes considered the factor f is positive, so that the possibility of zonal flow generation is determined only by the above difference. Requiring it to be positive, we arrive at the criterion (111) qualitatively evidencing that the zonal flow generation can take place if the double plasma rotation frequency exceeds the zonal flow frequency.

We have restricted ourselves to the simplest case of the primary modes with $k_y=0$. It seems to be of interest to study the problem of generation of magnetoacoustic zonal flows by Alfvén waves with $k_y \neq 0$. Then, in addition to the scalar nonlinearity, the vector nonlinearity should be involved.

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