Universal Behaviour of Interoccurrence Times between Losses in Financial Markets: An Analytical Description

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Abstract – We consider 16 representative financial records (stocks, indices, commodities, and exchange rates) and study the distribution $P_Q(r)$ of the interoccurrence times r between daily losses below negative thresholds $-Q$, for fixed mean interoccurrence time R_Q. We find that in all cases, $P_Q(r)$ follows the form $P_Q(r) \propto 1/[(1+(q-1)\beta r]^{1/(q-1)}$, where β and q are universal constants that depend only on R_Q, but not on a specific asset. While β depends only slightly on R_Q, the q-value increases logarithmically with R_Q, $q = 1 + q_0 \ln(R_Q/2)$, such that for $R_Q \to 2$, $P_Q(r)$ approaches a simple exponential, $P_Q(r) \cong 2^{-r}$. The fact that P_Q does not scale with R_Q is due to the multifractality of the financial markets. The analytic form of P_Q allows also to estimate both the risk function and the value-at-risk, and thus to improve the estimation of the financial risk.

The identification of the laws that govern the dynamical evolution of financial markets is one of the central issues in economy since understanding the laws allows a better estimation of the financial risk. Due to the pioneering work of Granger [1] and Mandelbrot [2] it is well accepted that the financial markets are extraordinary complex and show quite complex multifractal temporal behavior [3–11]. Here we show that despite this complexity universal empirical laws that have a remarkably simple form can be found for the dynamical behavior of the markets. These laws reflect the non-linear memory in the markets and can be used to improve the risk estimation.

In our analysis, we consider the returns X_i of an asset after the i-th unit trading period which might range between seconds and years. The returns are the central quantities in the financial markets and related to the price P_i of an asset by

$$X_i = (P_i - P_{i-1})/P_{i-1}. \tag{1}$$

By definition, positive returns are gains and negative returns are losses. Here we focus on daily closing prices P_i where $i-1$ and i now denote subsequent trading days. It is well known that the returns X_i are linearly uncorrelated such that the autocorrelation function vanishes for time lags s above the order of minutes (see, e.g [12,13]). But the volatility which can be defined as absolute value of the returns, shows long-term persistency, reflecting the nonlinear memory and the multifractality [2–11] of the financial markets.

The time evolution of the markets can be characterized by the set of interoccurrence times r_i between losses below a negative threshold $-Q$, in particular by their mean interoccurrence time R_Q and their distribution function $P_Q(r)$ [14] (see also [15]). For each record, there is a one-by-one correspondence between $-Q$ and R_Q ,

$$\frac{1}{R_Q} = \int_{-Q}^{-\infty} D(r)dr, \tag{2}$$

where $D(r)$ is the distribution of the returns. It has been shown recently by Bogachev and Bunde [14,16] when analyzing 16 assets in 4 asset classes (stocks, stock indices, commodities, currencies) that the interoccurrence times r_i are long-term correlated with an autocorrelation function that decays by a power law, reflecting the nonlinear memory in the market. It has been further shown that for (fixed) interoccurrence times $R_Q \geq 10$ (in units of trading days), the distribution function $P_Q(r)$ only depends on

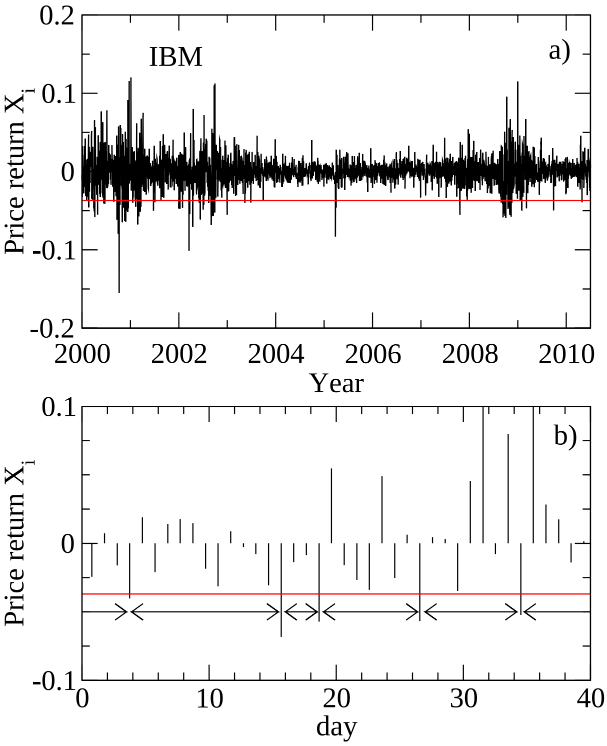

Fig. 1: Illustration of the relative daily price returns X_i of the IBM stock between (a) January 2000 and June 2010 and (b) August 27 and October 23, 2002. The red line shows the threshold $Q \simeq -0.037$, which corresponds to an average interoccurrence time of $R_Q = 70$. In (b) the interoccurrence times are indicated by arrows.

R_Q but not on the considered asset, and decays asymptotically by a power law, with an R_Q-dependent exponent that reflects the multifractality in the market.

In this Letter we re-analyze the functional form of $P_Q(r)$, for nearly the same assets as in [16] but updated [17], now also including the current financial crises. The financial records were obtained from [18–20]. We show explicitly that in all cases the functional form of $P_Q(r)$, in the whole range of accessible r and R_Q values, can be very well approximated by q-exponentials known from nonextensive statistical mechanics [21, 23], see eq. (3). The q-value increases logarithmically with R_Q, such that for large R_Q, $P_Q(r)$ follows a power law, while for $R_Q = 2$, $P_Q(r)$ becomes a simple exponential. Using the q-exponential form of $P_Q(r)$, we can derive also an analytical form for the risk function $W_Q(t; \Delta t)$, which is defined as probability that at least one event below $-Q$ will occur in the next time interval Δt if the last event occurred t days ago [14]. The risk function plays a major role in the estimation of the value-at-risk which is perhaps the most common tool in evaluating financial risk [26].

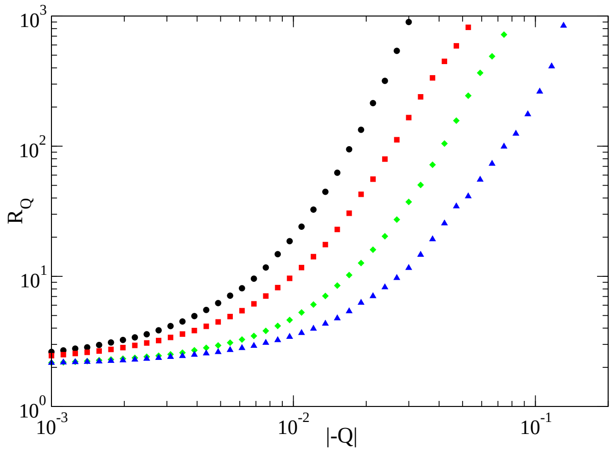

Fig. 2: The mean interoccurrence time R_Q versus the absolute value of the loss threshold $-Q$, for the exchange rate US Dollar against British Pound, the index S&P500, the IBM stock , and crude oil (WTI), from left to right.

Figure 1a shows the returns of the IBM record, between January 2000 and June 2010. The red horizontal line marks the daily losses below $-Q \cong -0.037$ where the mean interoccurrence time R_Q is 70 trading days. Figure 1b shows, for the time window between August 27 and October 23, 2002 the interoccurrence times. The interoccurrence times reflect the dynamics of the asset. In the 'quiet' area between April 2005 and September 2008 daily losses below -0.037 did not occur, i.e. the interoccurrence time is about 2.5 years while in the following volatile period they occurred with high frequency, with interoccurrence times of the order of days and weeks. Figure 2 shows how R_Q depends on Q for one example from each of the assets classes considered here. Since at large losses $D(x)$ decays by a power law [2], the mean interoccurrence times R_Q increase by a power law for large Q. When comparing different records, it is essential to keep R_Q fixed. Figure 3a shows, again for the IBM record, the distribution function $P_Q(r)$ of the interoccurrence times, for five mean interoccurrence times $R_Q = 2$, 5, 10, 30, and 70. The numerical results are the circles. The lines are the best fits by a q-exponential [29],

$$P_Q(r) = \frac{A}{(1 + (q-1)\beta r)^{1/(q-1)}}, \qquad (3)$$

where A is the normalization constant. In the mathematical literature, this distribution is usually referred to as "generalized Pareto" distribution [27]. Occasionally it is also referred to as "Zipf-Mandelbrot" distribution. Given the poor statistics of the data, the fit is excellent. Figure 3b shows the dependence of β (squares) and q (circles) on R_Q. The semilogarithmic plot shows that β first decreases slightly with increasing R_Q and then above $R_Q = 6$ reaches a plateau with $\beta \cong 0.23$. The figure shows also that the q-value decreases logarithmically with R_Q,

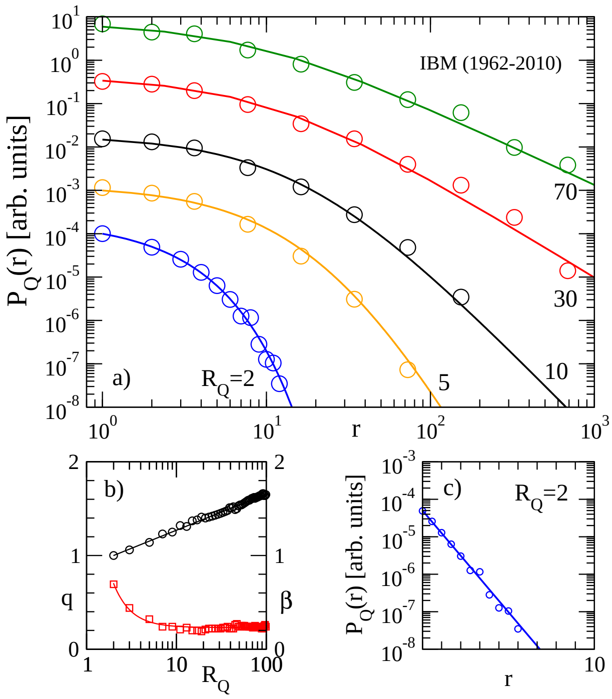

Fig. 3: (a) The distribution function of the interoccurrence times for the relative daily price returns X_i of IBM in the period 1962-2010. The data points belong to $R_Q = 2$, 5, 10, 30, and 70 (in units of days), from bottom to top. The full lines show the fitted q-exponentials. The dependence of the parameters β (squares, lower curve) and q (circles, upper curve) on R_Q in the q-exponential is shown in fig. 3b. Figure 3c confirms that for $R_Q = 2$ the distribution function is a simple exponential. The straight line is 2^{-r}.

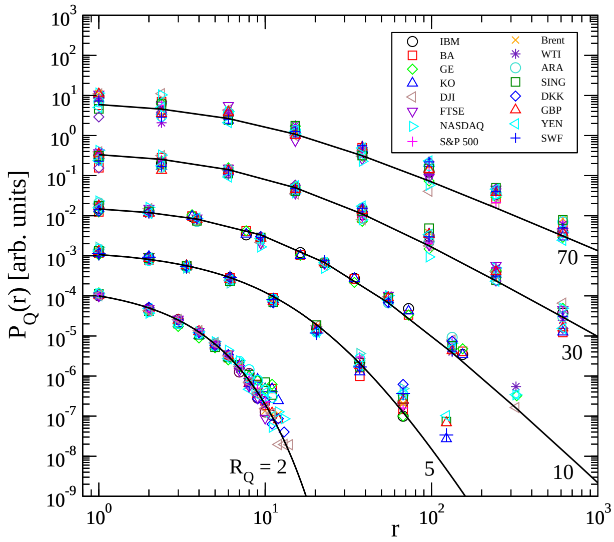

Fig. 4: The distribution function of the interoccurrence times (as in fig. 3a) for the relative daily price returns of 16 examples of financial data, taken from different asset classes (stocks, indices, currencies, commodities). The assets are (i) the stocks of IBM, Boeing (BA), General Electric (GE), Coca-Cola (KO), (ii) the indices Dow Jones (DJI), Financial Times Stock Exchange 100 (FTSE), NASDAQ, S&P 500, (iii) the commodities Brent Crude Oil, West Texas Intermediate (WTI), Amsterdam-Rotterdam-Antwerp gasoline (ARA), Singapore gasoline (SING), and (iv) the exchange rates of the following currencies versus the US Dollar: Danish Crone (DKK), British Pound (GBP), Yen, Swiss Francs (SWF). The full lines show the fitted q-exponentials, which are the same as in fig. 3a.

$$q = 1 + q_0 \ln(R_Q/2)\,, \tag{4}$$

with $q_0 \simeq 0.168$. Accordingly, for $R_Q \to 2$ (where $Q \cong 0$) we have $q \to 1$, and the q-exponential tends to a simple exponential, $P_Q(r) \propto \exp(-\beta r)$, with $\beta \cong \ln 2$, i.e. $P(r) = 2^{-r}$. To show explicitly that for $R_Q = 2$ the distribution of the interoccurrence times has exactly this surprisingly simple form, we have plotted, in fig. 3c, $\ln P_Q(r)$ as a function of r. The figure shows that all symbols are close to a straight line, confirming the simple exponential decay [28].

We like to note that q-exponentials play a central role in nonextensive statistical mechanics. In the last years q-exponentials have been found to describe successfully distributions in many complex systems where long range dependencies play some role [23–25]. The q-exponentials extremize, under simple constraints, the nonadditive entropy which generalizes that of Boltzmann-Gibbs [21–23]. It is interesting to note that a related functional form that also appears in the nonextensive statistical mechanics, a q-Gaussian where the argument r in eq. (3) has been substituted by r^2, has been used before [30,31] to approximate the probability density function of the returns X_i. In the mathematical literature, q-Gaussians are usually referred to as "Student distribution". This analytic description allowed Borland to generalize and considerably improve the Black-Scholes option pricing equation by reproducing e.g., the volatility smile [32].

Figure 4 shows, now for all 16 records considered, the distribution function of the interoccurrence times for the same R_Q values as in fig. 3. The figure confirms the earlier finding [16] that for large R_Q values the distribution function behaves in the same universal fashion, for all assets in the four asset classes. The figure shows that the same is true for small R_Q values. The q-exponential fit is the same as in fig. 3. For the same R_Q value, due to finite-size effects the data scatter more for large r values, but still are described remarkably well by the q-exponential fit.

The distribution function of the interoccurrence times is an important tool in risk estimation since it is directly related to the risk function (hazard function) $W_Q(t, \Delta t)$. The risk function is defined as probability that at least one event below Q will occur in the next time interval Δt if the last event occurred t days ago. It can be shown [14]

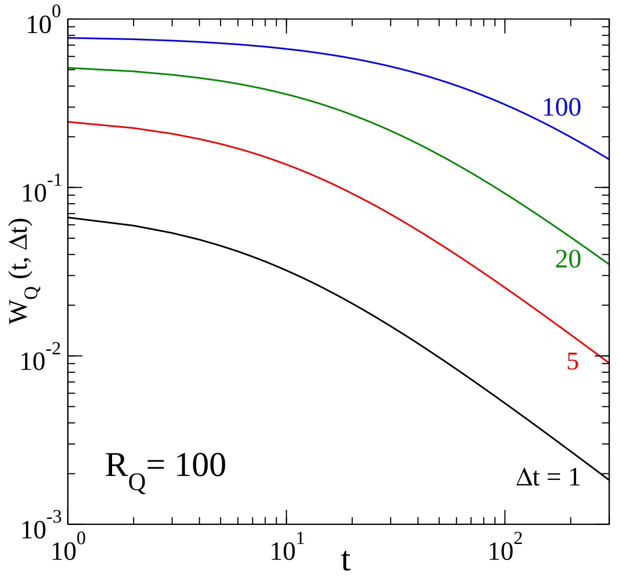

Fig. 5: Universal risk function $W_Q(t, \Delta t)$ from eq. 6a for the interoccurrence time $R_Q = 100$, and for the intervals $\Delta t = 1, 5, 20, 100$ days (from bottom to top).

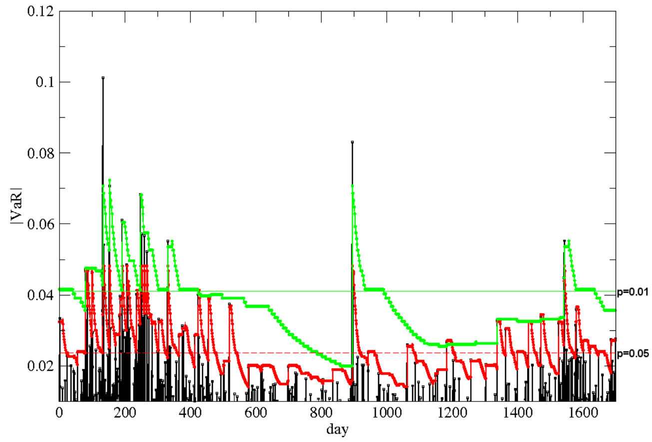

Fig. 6: Value-at-Risk estimate for the IBM stock between 2002-2008 for two confidence probabilities (a) $1 - p = 0.99$ (upper green curves) and (b) $1 - p = 0.95$ (lower red curves). The lowest (black) curve shows the absolute value of the daily losses. Only losses exceeding -0.01 are shown. The two horizontal lines show the zero-th order VaR, when memory effects are being neglected.

that W_Q is related to P_Q by

$$W_Q(t; \Delta t) = \frac{\int_t^{t+\Delta t} P_Q(r)dr}{\int_t^{\infty} P_Q(r)dr} . \tag{5}$$

Since the PDF follows a q-exponential, the cumulated distribution and thus the risk function is easy to obtain. The simple integration yields

$$W_Q(t; \Delta t) = 1 - \left(1 + \frac{\beta(q-1)\Delta t}{1 + \beta(q-1)t}\right)^{\frac{q-2}{q-1}} . \tag{6a}$$

Figure 5 shows the graph of the risk function for $R_Q = 100$ and $\Delta t = 1,\ 5,\ 20$, and 100. It is easy to show that also $W_Q(t; \Delta t)$ can be expressed in terms of q-exponentials (see [29]),

$$W_Q(t; \Delta t) = 1 - \frac{e_{\tilde{q}}^{-(\beta/\tilde{q})(t+\Delta t)}}{e_{\tilde{q}}^{-(\beta/\tilde{q})\,t}} , \tag{6b}$$

with $\tilde{q} = 1/(2 - q)$. It is easy to see that for exponentially decaying distribution functions, W_Q is a constant $W_Q(t; \Delta t) = 1 - \exp(-\beta \Delta t)$, while for distributions decaying by a power law, $W_Q(t, \Delta t) \propto \Delta t/t$ for $\Delta t \ll t$.

We like to emphasize again that $W_Q(t; \Delta t)$ is universal and can be used for all assets in risk estimation. A prominent example is the Value-at-Risk (VaR). The VaR can be defined as the relative loss $-Q$ that, in a given time unit, can only be exceeded with a certain small probability p. That means, the expected returns are, with probability $1 - p$, greater than $-Q$. Typical values of $1 - p$ are 0.95 and 0.99, and typical time units are 1 and 10 days. In the following we consider as time unit 1 day. In a zero-th order approximation, when memory effects are being neglected, the VaR can be simply determined from the distribution $D(r)$ of the daily returns,

$$\int_{-Q}^{-\infty} D(r)dr = p. \tag{7}$$

To obtain the VaR from eq. (6) we follow the algorithm suggested by Bogachev and Bunde [16] and proceed iteratively:

(i) In the first step, we choose p and determine from eq. (7) the corresponding loss $-Q$. Next, we determine the time t that has elapsed after the last return below $-Q$ and use eq. (6) to determine a new probability W_{-Q}. If W_{-Q} is within a certain error interval $p \pm \Delta p$ the algorithm is stopped.

(ii) If W_{-Q} is above $p + \Delta p$, we multiply $-Q$ by $(1 + \Delta Q)$ and determine the new elapsed time t after the last event below $-Q(1 + \Delta Q)$. If W is still above $p + \Delta p$ we repeat this step until, in the n-th step , $W_{-Q(1+\Delta Q)^n}$ is below $p + \Delta p$. Then we stop the algorithm and choose as the estimate of the VaR, $-Q = -Q(1 + \Delta Q)^n$.

(iii) If W_{-Q} is below $p - \Delta p$, we proceed as in (ii) but with a negative increment $\Delta Q < 0$ until we are above $p - \Delta p$.

Figure 6 shows the VaR of the IBM stock for two confidence probabilities, $p = 0.05$ and $p = 0.01$. To determine the VaR, we have chosen $\Delta p = 0.0001$ and $|\Delta Q| = 0.02$. The figure shows that the estimation of the VaR has been improved by taking into account the non linear memory effects. The VaR is strongly enhanced after large losses and decreases with increasing time after the last big loss.

In summary we have studied the distribution of the interoccurrence times between daily losses below some threshold $-Q$ that are characterized by a certain mean interoccurrence time R_Q. We have shown that for fixed R_Q, the distribution is universal (independent of the considered asset) and described by q-exponentials. The q-value increases logarithmically with $R_Q/2$, such that for $R_Q = 2$ we have $q = 1$ when the q-exponential reduces to a simple exponential. q-exponentials occur naturally in nonextensive statistical mechanics and are known to describe well a large number of distribution in complex systems with long range dependencies. We consider it as a challenge to understand the common underlying mechanisms for this behavior, which then may also lead to a deeper understanding of the financial markets.

In a practical application, we have shown that, the analytic form of the distribution function allows to calculate the related risk function analytically and how it can be used to improve estimations of the value-at-risk. Moreover, the analytic form of the distribution function represents a non-trivial test for market models.

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