Lecture 2

Discontinuous functions

2.1 Power Series Expansion

It is useful to represent a function in terms of a power series. Assuming that such a series exists and that the value of the function $f(u)$ is known at $u = u_0$, then

$$f(u) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{d^n}{du^n} f(u)\right]_{u_0} (u - u_0)^n \qquad \text{EQ (2-1)}$$

This series is known as the Taylor series expansion of $f(u)$ about u_0. Very near u_0 the value of the function is given approximately by the first two terms of the series expansion

$$f(u) \cong f(u_0) + \left[\frac{d}{du} f(u)\right]_{u_0} (u - u_0) \qquad \text{EQ (2-2)}$$

If $u_0 = 0$ the series is called the Maclaurin series

$$f(u) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{d^n}{du^n}f(u)\right]_0 u^n \qquad \text{EQ (2-3)}$$

2.2 Summation of N+1 terms of a power series

$$S_N = \sum_{n=0}^{N} a^n = 1 + a + a^2 + \ldots + a^N$$
$$S_N = \frac{1 - a^{N+1}}{1 - a} \qquad \text{EQ (2-4)}$$

This series is known as the geometric power series. In the limit as N approaches infinity

$$S_\infty = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \qquad |a| < 1 \qquad \text{EQ (2-5)}$$

2.3 Trigonometric formulae

Euler's formula:

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta \qquad i = \sqrt{-1} \qquad \text{EQ (2-6)}$$

The following relations can be used using Euler's formula

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \qquad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$
$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2$$
$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \sin\theta_2\cos\theta_1 \qquad \text{EQ (2-7)}$$
$$(\sin\theta)^2 = \frac{1}{2}(1 - \cos 2\theta) \qquad (\cos\theta)^2 = \frac{1}{2}(1 + \cos 2\theta)$$

2.4 Discontinuous Mathematical Functions

Solutions of simple problems in field theory are facilitated when mathematical discontinuous functions or functions whose derivatives are discontinues, are introduced to describe "point source" distributions, such as point charges, line currents, surface currents, etc.

Among the special functions used here: (1) the Heaviside or step function, (2) the Signum function, (3) the absolute value function, and (4) the Dirac delta function. The definitions presented here emphasize intuitive understanding rather than mathematical rigor.

2.5 Heaviside or step function in rectangular coordinates

Consider the independent variable u as one of the rectangular coordinates whose value spans the interval

$$-\infty \le u \le \infty$$

Definition of the step function $\theta(u)$

$$\theta(u) = \begin{Bmatrix} 1 & if & u > 0 \\ 0 & if & u < 0 \end{Bmatrix} \quad \text{EQ (2-8)}$$

This is a weak definition because the value of the function is left open at $u = 0$. Also

$$\theta(-u) = \begin{Bmatrix} 1 & if & u < 0 \\ 0 & if & u > 0 \end{Bmatrix} \quad \text{EQ (2-9)}$$

and

$$\theta(u - u_0) = \begin{Bmatrix} 1 & if & u - u_0 > 0 & & u > u_0 \\ 0 & if & u - u_0 < 0 & & u < u_0 \end{Bmatrix} \quad \text{EQ (2-10)}$$

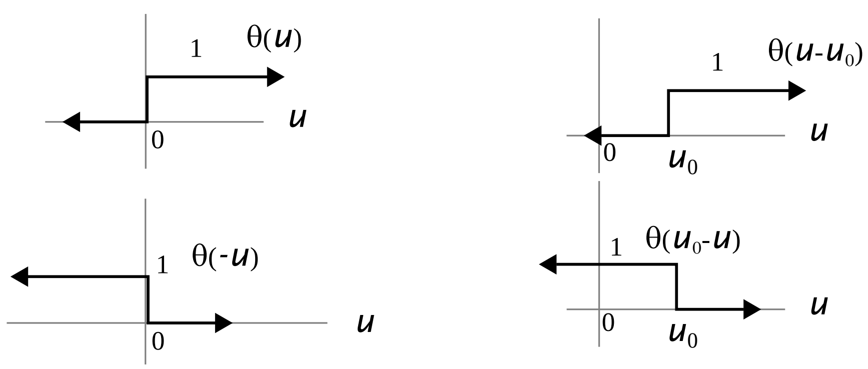


Figure 2-1 **Graphic representation of the Heaviside function**

The Heaviside function can be visualized as the limit of a sequence of continuous functions. For example consider the continuous function

$$\theta_a(u) = \begin{bmatrix} \frac{1}{2} + \frac{u}{a} & |u| < \frac{a}{2} \\ 1 & u > \frac{a}{2} \\ 0 & u < -\frac{a}{2} \end{bmatrix}$$

$$\theta(u) = \lim_{a \to 0} \{\theta_a(u)\}$$

where a is a label associated with the behavior of the function in three regions of space. The Heaviside function can be defined as the limit of the sequence of functions $\theta_a(u)$ as a approaches zero. A plot of a component of the sequence is shown below

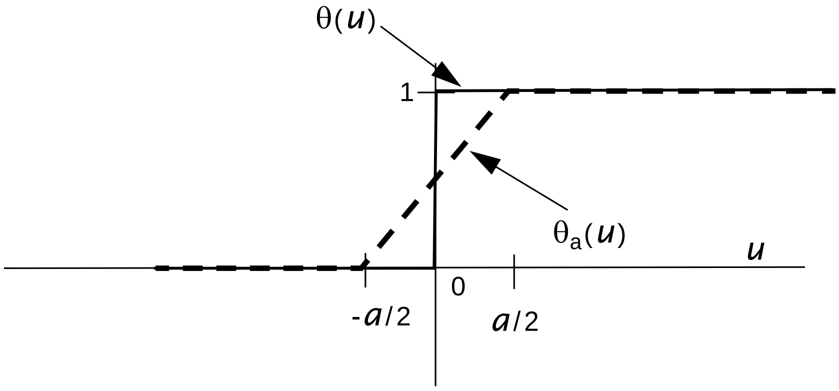


Figure 2-2 **The Heaviside discontinuous function is defined as the limit of a sequence of functions $\theta_a(u)$ as a approaches zero.**

2.6 Some properties of the Heaviside function

$$\theta(u) + \theta(-u) = 1 \qquad \theta(au) = \begin{Bmatrix} \theta(u) & if & a > 0 \\ \theta(-u) & if & a < 0 \end{Bmatrix}$$

$$\theta(au) = \theta(-a)\theta(-u) + \theta(a)\theta(u) \qquad \theta^2(u) = \theta(u) \qquad \text{EQ (2-11)}$$

$$\int_{-\infty}^{\infty} \theta(u' - u) f(u') du' = \int_{u}^{\infty} f(u') du' \qquad \int_{-\infty}^{\infty} \theta(u - u') f(u') du' = \int_{-\infty}^{u} f(u') du$$

2.7 Integration of the step function over a finite integral

Consider the following integral

$$\int_{u_1}^{u_2} f(u)\theta(u)du \qquad \text{EQ (2-12)}$$

Let

$$\int f(u)\theta(u)du = g(u)\theta(u) + C \qquad \text{EQ (2-13)}$$

where C is an integration constant. The above assumption requires that $g(u)$ satisfy the following equation.

$$\frac{\mathrm{d}}{\mathrm{d}u}[\theta(u)g(u) + C] = \theta(u)f(u)$$

$$\delta(u)g(u) + \theta(u)\frac{\mathrm{d}}{\mathrm{d}u}g(u) = \theta(u)f(u)$$

The last equality demands that

$$\delta(u)g(u) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}u}g(u) = f(u)$$

The solution of the first equation is

$$g(0) = 0$$

The solution of the second equation is

$$g(u) - g(0) = \int_0^u f(u')du'$$

$$g(u) = \int_0^u f(u')du' \qquad \text{EQ (2-14)}$$

Hence, combining and Figure 2-12, Figure 2-13, and Figure 2-14 yields

$$\int_{u_1}^{u_2} f(u)\theta(u)du = \left[\theta(u)\int_0^{u_2} f(u')du'\right] - \left[\theta(u)\int_0^{u_1} f(u')du'\right] \qquad \text{EQ (2-15)}$$

More generally

$$\int_{u_1}^{u_2} f(u)\theta(u-u_0)du = \theta(u_2-u_0)\int_{u_0}^{u_2} f(u)du - \theta(u_1-u_0)\int_{u_0}^{u_1} f(u)du$$
$$\int_{u_1}^{u_2} f(u)\theta(u_0-u)du = \theta(u_0-u_2)\int_{u_0}^{u_2} f(u)du - \theta(u_0-u_1)\int_{u_0}^{u_1} f(u)du \qquad \text{EQ (2-16)}$$
$$\int_{u_1}^{u_2} f(u)\theta(u+u_0)du = \theta(u_2+u_0)\int_{-u_0}^{u_2} f(u)du - \theta(u_1+u_0)\int_{-u_0}^{u_1} f(u)du$$

The development of Appendix EQ (2-16) will be left as an exercise at the end of this chapter. As an application of Figure 2-16, consider the integral

$$\int_{u_1}^{u_2} e^u\theta(u)du = \theta(u_2)\int_0^{u_2} e^u du - \theta(u_1)\int_0^{u_1} e^u du$$
$$\int_{u_1}^{u_2} e^u\theta(u)du = \theta(u_2)(e^{u_2}-1) - \theta(u_1)(e^{u_1}-1)$$

2.8 Signum function

The Signum or sign function is defined as follows

$$\varepsilon(u) = \begin{cases} 1 & if \quad u > 0 \\ -1 & if \quad u < 0 \end{cases} \qquad \text{EQ (2-17)}$$
$$\varepsilon(u) = \theta(u) - \theta(-u) = 2\theta(u) - 1$$

Plots of the signum function are shown below

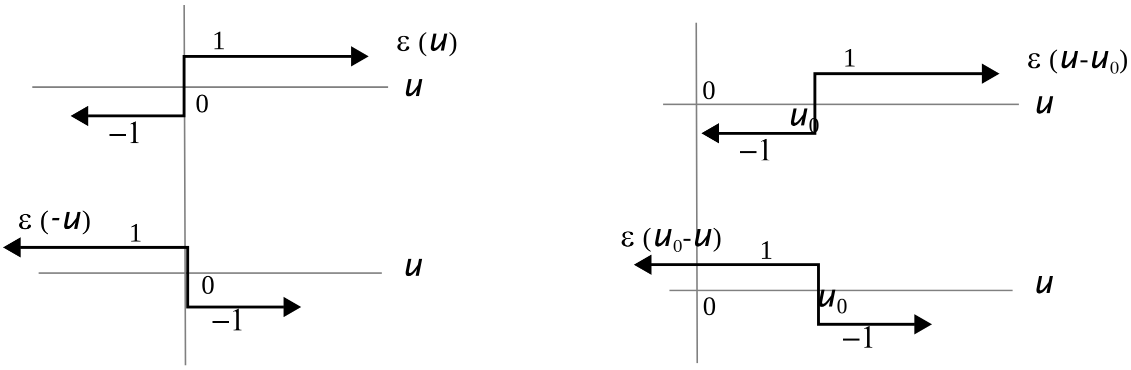


Figure 2-3 **Graphic representation of (a) ε(u) and ε(-u) and (b) ε(u-u_0) and ε(u_0-u)**

2.9 Properties of the sign function

$$\varepsilon(u) + \varepsilon(-u) = 0 \qquad \theta(u) - \theta(-u) = \varepsilon(u)$$
$$\varepsilon(au) = \varepsilon(a)\varepsilon(u) + \varepsilon(-a)\varepsilon(-u) \qquad \varepsilon^2(u) = 1$$

EQ (2-18)

$$\int_{u_1}^{u_2} f(u)\varepsilon(u-u_0)du = \varepsilon(u_2-u_0)\int_{u_0}^{u_2} f(u')du' - \varepsilon(u_1-u_0)\int_{u_0}^{u_1} f(u')du'$$
$$\int_{u_1}^{u_2} f(u)\varepsilon(u_0-u)du = \varepsilon(u_0-u_2)\int_{u_0}^{u_2} f(u')du' - \varepsilon(u_0-u_1)\int_{u_0}^{u_1} f(u')du'$$

EQ (2-19)

2.10 Absolute value function

$$|u| = \begin{Bmatrix} u & if & u > 0 \\ -u & if & u < 0 \\ 0 & if & u = 0 \end{Bmatrix} = u\varepsilon(u)$$
$$|f(u)| = f(u)\varepsilon(f(u))$$

EQ (2-20)

Note that the absolute function is continuous at $f(u) = 0$, however its derivative is discontinuous at $f(u) = 0$.

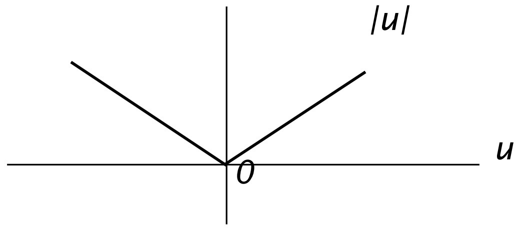


Figure 2-4 **Graphic representation of** $|u|$

2.11 Integration by parts and handling of the absolute value function

$$\int_{-\infty}^{\infty} u e^{-|u|} du = 0 \qquad \textit{Odd function}$$

$$\int_{-\infty}^{\infty} u^2 e^{-|u|} du = 2\int_{0}^{\infty} u^2 e^{-u} du \qquad \textit{Even function}$$

$$u^2 e^{-u} du = 2ue^{-u} du - d[u^2 e^{-u}]$$

$$\int_{-\infty}^{\infty} u^2 e^{-|u|} du = 4\int_{0}^{\infty} ue^{-u} du - 2[u^2 e^{-u}]\Big|_0^{\infty} = 4\int_{0}^{\infty} ue^{-u} du$$

$$\int_{-\infty}^{\infty} u^2 e^{-|u|} du = 4\left[\int_{0}^{\infty} e^{-u} du - \int_{0}^{\infty} d[ue^{-u}]\right] = 4$$

$$\int_{-\infty}^{\infty} u^2 e^{-|u|} du = 4$$

2.12 Symmetric square step

A symmetric square step $SSS(u)$ of with $\Delta u > 0$ and height = +1 can be represented in terms of Heaviside or signum functions as follows

$$SSS(u) = \theta\left(u + \frac{\Delta u}{2}\right) - \theta\left(u - \frac{\Delta u}{2}\right) \qquad \Delta u > 0$$

$$SSS(u) = \frac{\varepsilon\left(u + \frac{\Delta u}{2}\right) - \varepsilon\left(u - \frac{\Delta u}{2}\right)}{2}$$

EQ (2-21)

$$SSS(u) = \theta\left(\frac{\Delta u}{2} - |u|\right)$$

$$SSS(u) = \theta\left(u + \frac{\Delta u}{2}\right)\theta\left(\frac{\Delta u}{2} - u\right)$$

2.13 Symmetric square well

Similarly, a symmetric square well function $SSW(u)$ can be described by

$$SSW(u) = SSS(u) \qquad \Delta u > 0$$

EQ (2-22)

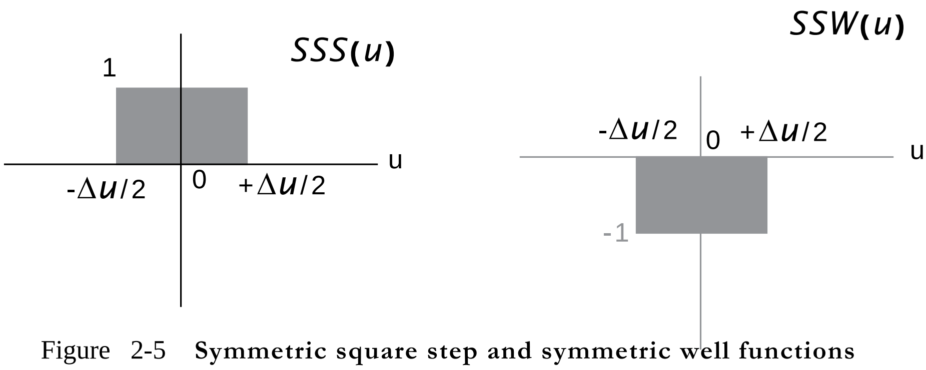


Figure 2-5 **Symmetric square step and symmetric well functions**

2.14 Dirac delta function

The Dirac delta function is defined by

$$\delta(u) = 0 \qquad if \qquad u \neq 0$$

$$\int_{-\eta}^{\eta} \delta(u)du = \varepsilon(\eta)$$

EQ (2-23)

A non-rigorous definition of the Dirac delta function is

$$\delta(u) = \frac{\mathrm{d}}{\mathrm{d}u}\theta(u) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}u}\varepsilon(u) \qquad \text{EQ (2-24)}$$

The delta function can be generated from a sequence of functions. There are two basic properties assigned to the delta function. The first is that the area of the delta function is normalized to one.

$$\int_{-\infty}^{\infty} \delta(u)du = 1 \qquad \text{EQ (2-25)}$$

Using Figure 2-24

$$\int_{-\infty}^{\infty} \delta(u)du = \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}u}\theta(u)du = \int_{-\infty}^{\infty} d\theta(u)$$

$$\int_{-\infty}^{\infty} \delta(u)du = \theta(\infty) - \theta(-\infty) = 1$$

and

$$\int_{-\infty}^{\infty} \delta(u)du = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}u}\varepsilon(u)du = \frac{1}{2}\int_{-\infty}^{\infty} d\varepsilon(u)$$

$$\int_{-\infty}^{\infty} \delta(u)du = \frac{1}{2}[\varepsilon(\infty) - \varepsilon(-\infty)] = 1$$

Hence the non-rigorous definition of the delta function, as defined by Figure 2-24, satisfies the first property. The second property is

$$\int_{-\infty}^{\infty} f(u)\delta(u)du = f(0) \qquad \text{EQ (2-26)}$$

where $f(u)$ is a continuous function of u at and near $u = 0$.

2.15 Integration of the delta function over a finite interval

$$\int_a^b \delta(u)du = \int_a^b \frac{\mathrm{d}}{\mathrm{d}u}\theta(u)du = \int_a^b d\theta(u)$$

$$\int_a^b \delta(u)du = \theta(b)-\theta(b)$$

EQ (2-27)

The above result can be written as follows

$$\int_a^b \delta(u)du = \begin{matrix} 1 \\ -1 \\ 0 \end{matrix} \quad if \quad \begin{matrix} a<0<b \\ b<0<a \\ otherwise \end{matrix}$$

Using a test function that is continuous at u = 0 the following integration can be obtained

$$\int_a^b f(u)\delta(u)du = f(0)\int_a^b d\theta(u)$$

$$\int_a^b f(u)\delta(u)du = f(0)[\theta(b)-\theta(a)]$$

More generally,

$$\int_a^b f(u)\delta(u-u_0)du = f(u_0)\int_a^b d\theta(u)$$

$$\int_a^b f(u)\delta(u-u_0)du = f(u_0)[\theta(b)-\theta(a)]$$

EQ (2-28)

Again, the non-rigorous definition of the delta function obeys the second property. Below is a graphical representation of the delta function.

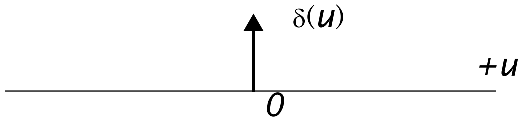


Figure 2-6 **Graphical representation of the Dirac delta function**

2.16 General properties of the delta function

$$
\begin{aligned}
\delta(au) &= \frac{\mathrm{d}}{\mathrm{d}(au)}\theta(au) = \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}u}[\theta(-a)\theta(-u) + \theta(a)\theta(u)] \\
\delta(au) &= \frac{1}{a}\delta(u)[\theta(a) - \theta(-a)] = \frac{1}{a}\delta(u)\varepsilon(a) = \frac{1}{|a|}\delta(u) \\
\delta(-u) &= \frac{1}{|-1|}\delta(u) = \delta(u) \\
u\delta(u) &= 0 \qquad f(u)\delta(u) = f(0)\delta(u) \\
\mathit{if} \qquad & f(u)\delta(u) = 0 \qquad \mathit{then} \qquad f(0) = 0 \\
\mathit{if} \qquad & f(u)\delta(u-u_0) = 0 \qquad \mathit{then} \qquad f(u_0) = 0
\end{aligned}
$$

2.17 Delta Sequences

The delta function can be described as the limit of a set of functions that become concentrated at a single point. A sequence of proper functions ϕ_n can be constructed in such a way that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \phi_n(u) f(u) du = f(0) \qquad \text{EQ (2-29)}$$

That is, ϕ_n approaches the behavior of the delta function as n approaches infinity.

Example (2-1)

A simple sequence that behaves in this manner is the square step sequence

$$\phi_n(u) = \frac{n}{2}\theta\left(\frac{1}{n} - |u|\right) \qquad \text{EQ (2-30)}$$

representing a sequence of rectangular blocks with unit area. The first four functions of this sequence are illustrated below

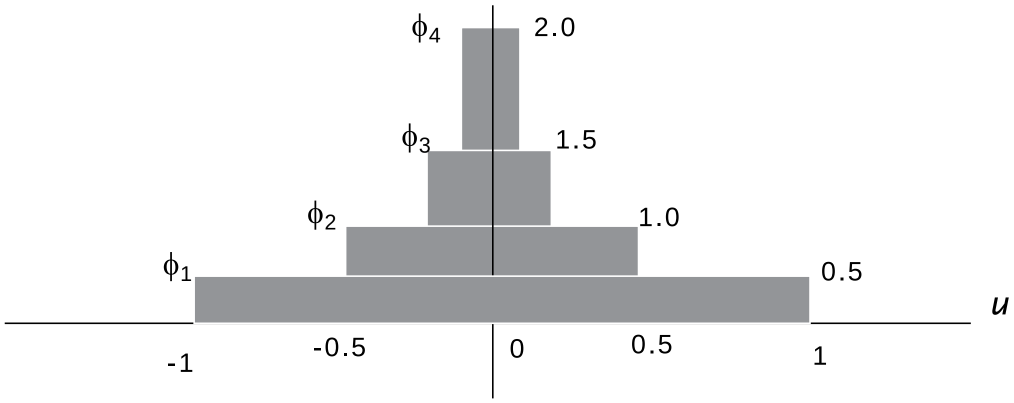


Figure 2-7 **Sequence of square step functions $\phi_n(u)$ representing the delta function $\delta(u)$ as n approaches infinity. The area of the square steps remains fixed at 1.**

To check whether the above sequence represents δ(u), the sequence must satisfy

$$\lim_{n \to \infty} \phi_n(0) \to \infty$$

$$\lim_{n \to \infty} \phi_n(u \neq 0) \to 0$$

$$\int_{-\infty}^{\infty} \phi_n(u) du = 1$$

The first two requirements are met by Figure 2-30 as follows

$$\lim_{n \to \infty} \phi_n(0) = \lim_{n \to \infty} \frac{n}{2}\theta\left(\frac{1}{n}\right) = \{\infty\}$$

$$\lim_{n \to \infty} \phi_n(u \neq 0) = \lim_{n \to \infty} \frac{n}{2}\theta\left(\frac{1}{n} - |u|\right) = \infty\theta(-|u|)$$

$$\lim_{n \to \infty} \phi_n(u \neq 0) = \infty \cdot 0 = 0$$

The integral requirement is also satisfied because

$$\int_{-\infty}^{\infty} \frac{n}{2}\theta\left(\frac{1}{n} - |u|\right) du = \frac{n}{2} \int_{-1/n}^{1/n} du = 1$$

Example (2-2)

Consider the following sequence

$$\phi(u, \eta) = \frac{1}{\eta\sqrt{\pi}} e^{-\left(\frac{u}{\eta}\right)^2}$$

$$\lim_{\eta \to 0} \phi(0, \eta) = \lim_{\eta \to 0}\left[\frac{1}{\eta\sqrt{\pi}}\right] = \infty$$

$$\lim_{\eta \to 0} \phi(u \neq 0, \eta) = \lim_{\eta \to 0}\left[\frac{1}{\eta\sqrt{\pi}} e^{-\left(\frac{u}{\eta}\right)^2}\right] = 0$$

The last equation is true because the exponential function vanishes more rapidly than the factor $1/\eta$ as η becomes small. The integration test yields

$$\int_{-\infty}^{\infty} e^{-\left(\frac{u}{\eta}\right)^2} du = \frac{\eta}{\eta\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\xi^2} d\xi = 1$$

2.18 Dirac delta function of a continuous function as its argument

The delta function has a discontinuity when the value of its argument is zero. Let the argument of the delta function be a continuos function of u such as $f(u)$. Before evaluating the discontinuous function the zeros of $f(u)$ must be found. Let u_n be the n^{th} root of $f(u) = 0$. The value of $f(u)$ near the root u_n is given approximately by the Taylor series expansion

$$f(u) = f(u_n) + \frac{d}{du}f(u)\bigg|_{u_n}(u - u_n) + \frac{1}{2}\frac{d^2}{du^2}f(u)\bigg|_{u_n}(u - u_n)^2 + \ldots$$

$$f(u) \cong 0 + \frac{d}{du}f(u)(u_n)(u - u_n) = a_n(u - u_n) \qquad \text{EQ (2-31)}$$

$$a_n = \frac{d}{du}f(u)\bigg|_{u_n}$$

Consequently, the value of $\delta[f(u)]$ near u_n is

$$\delta[f(u)] \cong \delta[a_n(u-u_n)] = \frac{1}{|a_n|}\delta(u-u_n)$$

$$a_n = \frac{\mathrm{d}}{\mathrm{d}u}f(u)\Big|_{u = u_n} \qquad u \cong u_n$$

It follows that for all values of u the following equation is a representation of the delta function of a function

$$\delta[f(u)] = \sum_n \frac{1}{|a_n|}\delta(u-u_n)$$
$$a_n = \frac{\mathrm{d}}{\mathrm{d}u}f(u)\Big|_{u = u_n}$$

EQ (2-32)

where the summation is carried out for all real roots of $f(u)$.

Example (2-3) Example of integration of a delta function of a function f(u)

Consider the integral

$$\int_{-\infty}^{\infty} \delta(1-u^2)e^{-u}du$$

$$f(u) = 1-u^2$$

The roots of $f(u)$ are (+1, -1) hence

$$|f'(u)||_{u=1} = |-2u||_{u=1} = 2$$
$$|f'(u)||_{u=-1} = |-2u||_{u=-1} = 2$$

finally

$$\int_{-\infty}^{\infty} \delta(1-u^2)e^{-u}du = \int_{-\infty}^{\infty} \frac{1}{2}[\delta(u-1)+\delta(u+1)]e^{-u}du$$

$$\int_{-\infty}^{\infty} \delta(1-u^2)f(u)du = \frac{1}{2}[e^{-1}+e]$$

Consider the function $f\ (u) = 1 - u^2$. The roots of this function are u_1 and u_2. Then

$$f(u) = 1-u^2 \qquad u_1 = 1 \qquad u_2 = -1$$
$$f'(u) = -2u \qquad f'(u_1) = -2 \qquad f'(u_2) = 2$$

Then,

$$\delta[1-u^2] = \sum_{n=1}^{2} \frac{1}{|f'(u_n)|}\delta(u-u_n) = \frac{1}{2}[\delta(u-u_1)+\delta(u-u_2)]$$
$$\delta[1-u^2] = \frac{1}{2}[\delta(u-1)+\delta(u+1)]$$

Example (2-4) Delta function of a periodic function

The function $f(u) = \cos(u)$ has an infinite number of roots located at $u_n = (2n+1)\pi/2$ with n = integer. Hence

$$\delta[\cos(u)] = \sum_{n=-\infty}^{\infty} \frac{1}{\left|\sin\left((2n+1)\frac{\pi}{2}\right)\right|}\delta\left(u-(2n+1)\frac{\pi}{2}\right) = \sum_{n=-\infty}^{\infty} \delta\left(u-(2n+1)\frac{\pi}{2}\right)$$

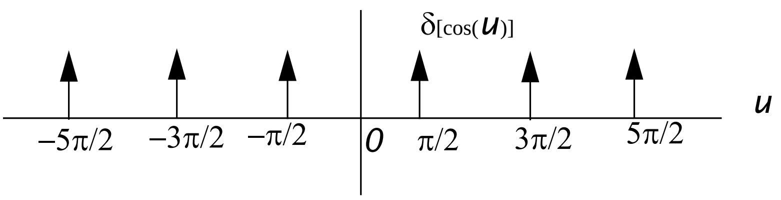


Figure 2-8 **Graphic representation of $\delta[\cos(u)]$**

The representation of the delta function of a function in terms of a summation of delta function containing linear arguments allows for a simplified integration process. That is, for $b > a$

$$\begin{aligned}
\int_a^b g(u)\delta[f(u)]du &= \int_a^b g(u)\sum_i \frac{\delta(u-u_i)}{|a_i|}du \\
\int_a^b g(u)\delta[f(u)]du &= \sum_i \frac{g(u_i)}{|a_i|}\int_a^b \delta(u-u_i)du \\
\int_a^b g(u)\delta[f(u)]du &= \sum_i \frac{g(u_i)}{|a_i|}[\theta(b-u_i)-\theta(a-u_i)]
\end{aligned} \qquad \text{EQ (2-33)}$$

Example (2-5)

Consider the following integration

$$\int_0^\infty e^{-u}\delta[\cos(u)]du$$

with

$$\begin{aligned}
f(u) &= \cos u \qquad & b &= \infty \\
g(u) &= e^{-u} \qquad & a &= 0 \\
u_n &= (2n+1)\frac{\pi}{2} & & \\
|f'(u_n)| &= 1 & &
\end{aligned}$$

then

$$\int_0^\infty e^{-u}\delta[\cos(u)]du = e^{-\frac{\pi}{2}}\sum_{i=-\infty}^{\infty} e^{-n\pi}[\theta(\infty)-\theta(-u_i)]$$

The value of the i^{th} term of the summation will be zero unless $u_i \geq 0$. Hence

$$\int_0^\infty e^{-u}\delta[\cos(u)]du = e^{-\frac{\pi}{2}}\sum_{i=0}^{\infty} e^{-n\pi}$$

$$\int_0^\infty e^{-u}\delta[\cos(u)]du = \frac{e^{-\frac{\pi}{2}}}{1-e^{-\pi}} = \frac{1}{2\sinh(\pi/2)}$$

2.19 Derivative of the delta function

Consider the derivative of the delta function defined by

$$\delta'(u) = \frac{\mathrm{d}}{\mathrm{d}u}\delta(u)$$

A test function $f(u)$ which is continuous and has a continuous derivative at $u = 0$ will be used to evaluate the integral

$$\int_a^b f(u)\delta'(u)du = \int_{-\infty}^{\infty}\theta(u-a)\theta(b-u)f(u)\delta'(u)du$$

$$f(u)\frac{\mathrm{d}}{\mathrm{d}u}\delta(u) = \frac{\mathrm{d}}{\mathrm{d}u}[f(u)\delta(u)] - \delta(u)\frac{\mathrm{d}}{\mathrm{d}u}f(u)$$

Integration by parts yields

$$\int_a^b f(u)\delta'(u)du = \int_a^b d[f(u)\delta(u)] - \int_a^b \delta(u)\frac{\mathrm{d}}{\mathrm{d}u}f(u)du$$

$$\int_a^b f(u)\delta'(u)du = f(u)\delta(u)|_a^b - \frac{\mathrm{d}}{\mathrm{d}u}f(0)\int_a^b \delta(u)du$$

$$\int_a^b f(u)\delta'(u)du = 0 - \frac{\mathrm{d}}{\mathrm{d}u}f(0)[\theta(b)-\theta(a)]$$

The first integral on the right side of the equation is null because the value of the delta function is zero at both limits. Hence

$$\int_a^b f(u)\delta'(u)du = -f'(0) \qquad a < 0 < b$$

$$= 0 \qquad otherwise$$

Hence, under integration the following general relation exists

$$f(u)\delta'(u) = -\delta(u)f'(u) \qquad \text{EQ (2-34)}$$

In particular, with f(u) = u

$$uf'(u) = -\delta(u) \qquad \text{EQ (2-35)}$$

More generally

$$\int_a^b f(u)\delta'(u-u_0)du = -\frac{\mathrm{d}}{\mathrm{d}u}f(u_0)[\theta(b)-\theta(a)] \qquad \text{EQ (2-36)}$$

Thus, the value of the integral is equal to minus the derivative of $f(u)$ evaluated at $u = u_0$.

2.20 Additional Mathematical Relations

Differentiation results

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u}[u\theta(u)] &= \theta(u) + u\frac{\mathrm{d}}{\mathrm{d}u}\theta(u) \\
\frac{\mathrm{d}}{\mathrm{d}u}[u\theta(u)] &= \theta(u) + u\delta(u) = \theta(u) \\
\frac{\mathrm{d}}{\mathrm{d}u}[u\varepsilon(u)] &= \varepsilon(u) + u\frac{\mathrm{d}}{\mathrm{d}u}\varepsilon(u) \\
\frac{\mathrm{d}}{\mathrm{d}u}[u\varepsilon(u)] &= \varepsilon(u) + 2u\delta(u) = \varepsilon(u)
\end{aligned} \qquad \text{EQ (2-37)}$$

$$\frac{d^2}{du^2}[u\varepsilon(u)] = \frac{d}{du}[\varepsilon(u) + 2u\delta(u)]$$

$$\frac{d^2}{du^2}[u\varepsilon(u)] = \frac{d}{du}[\varepsilon(u)] = 2\delta(u)$$

$$\frac{d^2}{du^2}[u\theta(u)] = \frac{d}{du}[\theta(u) + u\delta(u)]$$

$$\frac{d^2}{du^2}[u\theta(u)] = \frac{d}{du}[\theta(u)] = \delta(u)$$

EQ (2-38)

Integration results

$$\int_{u_0}^{u}\theta(u')du' = u'\theta(u')\Big|_a^u = u\theta(u) - u_0\theta(u_0)$$

EQ (2-39)

Also

$$\int_{u_0}^{u}\varepsilon(u')du' = u\varepsilon(u) - u_0\varepsilon(u_0) = |u| - |u_0|$$

$$\int_{u_0}^{u}\delta(u')du' = \theta(u) - \theta(u_0)$$

EQ (2-40)

$$\int_{u_0}^{u}\delta(u')du' = \frac{1}{2}[\varepsilon(u) - \varepsilon(u_0)]$$

and

$$\frac{d^2}{du^2}f(u) = \delta(u)$$

$$f(u) = \begin{Bmatrix} u\theta(u) + cu + d \\ \frac{1}{2}u\varepsilon(u) + au + b \end{Bmatrix}$$

2.21 Solution of Simple differential equations containing the delta function

Example (2-6)

Consider the following first order differential equation

$$\frac{d}{du}\Psi(u) = \delta(u) \qquad \text{EQ (2-41)}$$

It is obvious that $\psi(u)$ must be a discontinuous function of u at u equal to zero, because its derivative generates a discontinuous delta function. Assume that the solution of the above differential equation is

$$\psi(u) = f_>(u)\theta(u) + f_<\theta(-u)$$

The first term represents the solution of the differential equation for $u \geq 0$, while the second term represents the solution for $u \leq 0$. Substitution of the assumed solution into the differential equation yields

$$f_>'(u)\theta(u) + f_<'\theta(-u) + [f_>(u) - f_<]\delta(u) = \delta(u)$$

The order of discontinuity of the delta function is higher than that of the step function, hence the solution of the above equation must require that

$$\begin{aligned} \frac{d}{du}f_>(u) = 0 \rightarrow f_>(u) &= C_> \\ \frac{d}{du}f_<(u) = 0 \rightarrow f_<(u) &= C_< \\ [f_>(u) - f_<]_{u=0} &= 1 \end{aligned} \qquad \text{EQ (2-42)}$$

where $C_>$ and $C_<$ are integration constants. The solution of Figure 2-41 is therefore

$$\psi(u) = C_>\theta(u) + C_<\theta(-u)$$

The two constants of integration are not independent of each other. The third relation in Figure 2-42 $C_> - C_< = 1$ connects the two constants. Hence

$$\psi(u) = C_{>}-\theta(-u)$$

The constant C> is determined by the initial conditions of the problem. For example

$$\begin{aligned} &\textit{if} \qquad \psi(-\infty) = 0 \qquad \textit{then} \\ &\qquad \psi(u) = \theta(u) \end{aligned}$$

Example (2-7) 1D Scattering by a delta function potential

The time independent Schrodinger equation for a delta function potential barrier is

$$\frac{\hbar^2}{2m}\left[\frac{d^2}{dz^2}\psi(z) + k^2\psi(z)\right] = \frac{\hbar^2}{2m}[\lambda\delta(z)\psi(z)] \qquad \text{EQ (2-43)}$$

where $\hbar k$ is the particle momentum and $\psi(z)$ is the wave function of the particle. Since the second derivative of $\psi(z)$ must be discontinuous at $z = 0$, the first derivative of the solution must contain a discontinuous function at $z = 0$. Assume that the solution for all values of z is

$$\psi(z) = f_{>}(z)\theta(z) + f_{<}(z)\theta(-z)$$

where $f_{>}(z)$ and $f_{<}(z)$ are both continuous functions at $z = 0$. The continuity of $f(u)$ at $z = 0$ is represented by the following equation

$$[f_{>}(z)-f_{<}(z)]\big|_{z = 0} = 0 \qquad \text{EQ (2-44)}$$

Substitution of the assumed solution into the differential equation yields the following result

$$\frac{d^2}{dz^2}[f_{>}(z)\theta(z)] = f_{>}(z)\frac{d^2}{dz^2}\theta(z) + \theta(z)\frac{d^2}{dz^2}f_{>}(u)$$
$$+ 2\left[\frac{d}{dz}f_{>}(z)\right]\frac{d}{dz}\theta(z)$$

$$\frac{d^2}{dz^2}[f_{>}(z)\theta(z)] = f_{>}(z)\delta'(z) + \theta(z)f_{>}''(z)$$
$$+ 2[f_{>}'(z)]\delta(z)$$

Using $f_>(z)\delta'(z) = -f_>'(z)\delta(z)$ the result is

$$\frac{d^2}{dz^2}[f_>(z)\theta(z)] = f_>'(z)\delta(z) + \theta(z)f_>''(z)$$
$$\frac{d^2}{dz^2}[f_<(z)\theta(z)] = f_<'(z)\delta(z) + \theta(z)f_<''(z)$$

Therefore substitution of the solution into the differential equation EQ (2-43) yields the following constraints

$$f_>''(z) + k^2 f_>(z)]\theta(z) + [f_<''(z) + k^2 f_<(z)]\theta(-z) + [f_>(z) - f_<(z)]\delta(z) = \lambda\delta(z)[f_>(u)\theta(u) + f_<\theta(-u)]$$

Taking into account the degree of discontinuity of the discontinuos functions, the above relations are satisfied if

$$f_>''(z) + k^2 f_>(z) = 0 \qquad f_<''(z) + k^2 f_<(z) = 0$$
$$f_>'(z) - f_<'(z)]\big|_{z=0} = \lambda[f_>(z)\theta(z) + f_<(z)\theta(-z)]\big|_{z=0}$$

The first two equations demand that $f_>$ and $f_<$ satisfy the homogeneous differential equation of the problem. The last relation describes the discontinuity of the derivative of the solution at the origin. Then, with

$$[\theta(z) + \theta(-z)]_{z=0} = 1$$

the discontinuity of the derivative of the solution at the origin is given by

$$f_>'(z) - f_<'(z)]\big|_{z=0} = \lambda[f_<(z)]\big|_{z=0} \qquad \text{EQ (2-45)}$$

The solution of the homogeneous differential equation is

$$f(u) = e^{\pm iku}$$

Assuming that the incident particle moves along the + z-direction there will be a reflection and a transmitted wave at the origin. Hence for $z > 0$ the transmitted wave function is given by

$$f_>(z) = te^{ikz}$$

where t is the complex transmission amplitude. The wave function of the particle for z < 0 is given by

$$f_<(z) = e^{ikz} + re^{-ikz}$$

where r is the complex reflection amplitude of the wave function. Applying the BC's specified in EQ (2-44) and EQ (2-45) yields

$$t = 1 + r$$
$$t-(1-r) = \frac{\lambda}{ik}t$$

the solution of which is

$$t = \frac{1}{1-\frac{\lambda}{2ik}} \qquad r = \frac{\frac{\lambda}{2ik}}{1-\frac{\lambda}{2ik}}$$

In complex polar notation

$$r = \frac{1}{\sqrt{1+\left(\frac{2k}{\lambda}\right)^2}}e^{i\operatorname{atan}\left(\frac{2k}{\lambda}\right)}$$

$$t = \frac{1}{\sqrt{1+\left(\frac{\lambda}{2k}\right)^2}}e^{i\operatorname{atan}\left(-\frac{\lambda}{2k}\right)}$$

The following result

$$|r|^2 + |t|^2 = 1$$

establishes conservation of energy. The reflection and transmission amplitudes attain the following limiting values when $k >> \lambda$

$$t = 1 \qquad r = 0$$

That is for a high energy particle the transmission through the delta potential is 100%. In the limit k <<λ,

$$t = 0 \qquad r = -1$$

That is, there is a 180^0 phase difference between the phase of the reflected wave function and the phase of the incident wave function. Thus, the particle is fully reflected from the delta potential.

The phase difference between the reflected an incident wave functions is given by

$$\tan(\phi) = \frac{Im(r)}{Re(r)} = \frac{2k}{\lambda}$$

2.22 Closure property of complete orthonormal sets

A set of functions $\{\phi_n(u)\}$ defined over some interval $[u_1,u_2]$, and satisfying specified boundary conditions at its ends, is complete if an arbitrary function $f(u)$ can be represented over the interval as a linear combination of the set of functions $\{\phi_n(u)\}$.

$$f(x) = \sum_n q_n \phi_n(u) \qquad \text{EQ (2-46)}$$

provided that the coefficients q_n are chosen appropriately. The set is orthonormal if

$$\int_{interval} \phi_n(u)\phi_m{}^*(u)du = \delta_{n,m} \qquad \text{EQ (2-47)}$$

where $\phi_m{}^*$ is the complex conjugate of ϕ_m, and where the Kronecker symbol $\delta_{n,m}$ is the discrete version of the delta-function

$$\delta_{n,m} = \begin{cases} 1 & if \quad m = n \\ 0 & otherwise \end{cases} \qquad \text{EQ (2-48)}$$

The orthonormal condition can be utilized to obtain the series coefficients as follows

$$\int f(u)\phi_m{}^*(u)dx = \int\left[\sum_n q_n\phi_n(u)\right]\phi_m{}^*(u)du$$
$$\int f(u)\phi_m{}^*(u)dx = \sum_n q_n\delta_{n,m} = q_m \quad \text{EQ (2-49)}$$

Substituting the expression for the coefficients into Appendix EQ (2-48) yields

$$f(u) = \sum_n c_n\phi_n(u) = \int f(u')\left[\sum_n \phi_n(u')\phi_n{}^*(u)\right]du' \quad \text{EQ (2-50)}$$

For the last equation to be correct for an arbitrary function $f(u)$, the relation in square brackets must correspond to the delta function $\delta(u\text{-}u')$. Thus

$$\sum_n \phi_n(u')\phi_n{}^*(u) = \delta(u-u') \quad \text{EQ (2-51)}$$

This last relation represents mathematically the closure property of the set of functions. If the members of a complete orthonormal set are labeled by a continuous variable k instead of a discrete index n the orthonormal condition

$$\int_{-\infty}^{\infty}\phi(k,x)\phi^*(k',x)dx = \delta(k-k') \quad \text{EQ (2-52)}$$

can be used to expand and arbitrary function $f(x)$ in terms of an integral over k. The result is

$$f(x) = \int_{-\infty}^{\infty} c(k)\phi(k,x)dk$$
$$c(k) = \int_{-\infty}^{\infty} f(x)\phi^*(k,x)dx \quad \text{EQ (2-53)}$$

The closure condition for continuous k is

$$\int_{-\infty}^{\infty}\phi(k,x)\phi^*(k,x')dk = \delta(x-x') \quad \text{EQ (2-54)}$$

2.23 Three dimensional Dirac delta function

The previous section have dealt with the definition of the one dimensional delta function in rectangular coordinate. The transition to a three dimensional delta function in rectangular coordinates occurs in a straight forward manner.

$$\delta(\boldsymbol{x} - \boldsymbol{x}') = \delta(x - x')\delta(y - y')\delta(z - z') \qquad \text{EQ (2-55)}$$

The volume integral of the 3D delta function over all space yields the required unity

$$\oint dV \delta(\boldsymbol{x} - \boldsymbol{x}') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - x')\delta(y - y')\delta(z - z')\,dx\,dy\,dz = 1 \qquad \text{EQ (2-56)}$$

2.24 Delta function in spherical coordinates

The conversion from rectangular to spherical coordinates is given by

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\cos\phi \\ z &= r\cos\theta \end{aligned}$$

The conversion will be unique if the following restrictions are imposed on the range of spherical coordinates

$$r \geq 0 \qquad 0 \leq \phi < 2\pi \qquad 0 \leq \theta \leq \pi$$

Integration over all space of the 3-D delta function must yield unity

$$\oint dV \delta(\boldsymbol{x} - \boldsymbol{x}') = \int_0^{\infty} r^2 dr \int_0^{2\pi} d\phi \int_0^{\pi} \delta(\boldsymbol{x} - \boldsymbol{x}')\sin\theta\, d\theta = 1 \qquad \text{EQ (2-57)}$$

Hence the representation of the 3-D delta function in spherical coordinates must be

$$\delta(\boldsymbol{x} - \boldsymbol{x}') = \frac{\delta(r - r')\delta(\phi - \phi')\delta(\theta - \theta')}{r^2\sin\theta} \qquad \text{EQ (2-58)}$$

where here θ is assumed to be one of the independent variables. Using $x = \cos\theta$ as an independent variable the volume integral of the delta functions is expressed by

$$\oint dV\delta(\boldsymbol{x}-\boldsymbol{x}') = \int_0^\infty r^2 dr \int_0^{2\pi} d\phi \int_{-1}^{1} \delta(\boldsymbol{x}-\boldsymbol{x}') dV = 1$$

Hence the representation of the 3D delta function is also given by

$$\delta(\boldsymbol{x}-\boldsymbol{x}') = \frac{\delta(r-r')\delta(\phi-\phi')\delta(\cos\theta-\cos\theta')}{r^2} \quad \text{EQ (2-59)}$$

For the degenerate case ($\boldsymbol{x}' = 0$)

$$\delta(\boldsymbol{x}) = \frac{\delta(r)}{4\pi r^2} \quad \text{EQ (2-60)}$$

The volume integration of $\delta(x)$ leads to the definition of the 1D property of the radial delta function

$$\int_0^\infty \frac{\delta(r)}{4\pi r^2} 4\pi r^2 dr = \int_0^\infty \delta(r) dr = 1 \quad \text{EQ (2-61)}$$

The 1D radial delta function can be obtained from the step function as follows

$$\delta(r) = \frac{\mathrm{d}}{\mathrm{d}r}\theta(r)$$

The strong definition $\theta(0) = 0$ and $\theta(r) = 1$ of the Heaviside function is needed to satisfy Figure 2-61. That is

$$\int_0^r \delta(r')dr' = \int_0^r \frac{\mathrm{d}}{\mathrm{d}r'}\theta(r')dr' = \theta(r)-\theta(0) = 1$$

With the above definition of $\theta(0)=0$, the following plots (a > 0) follow

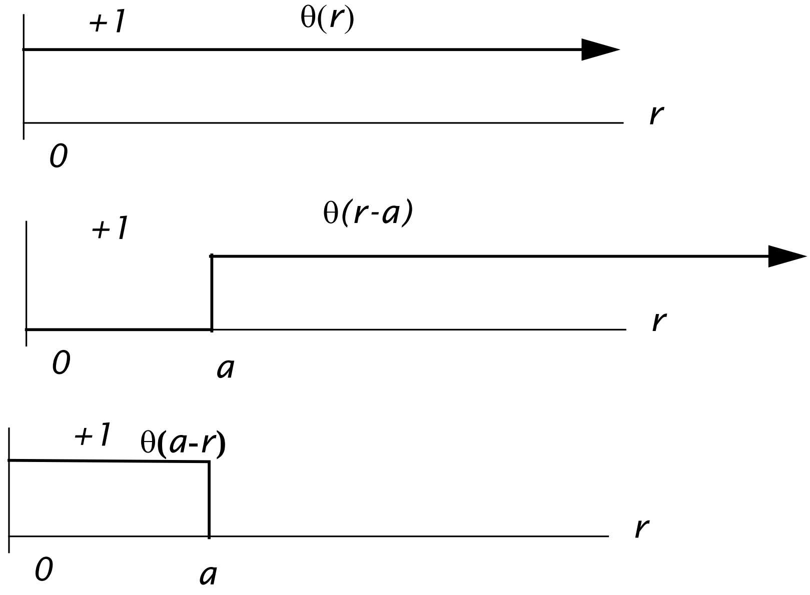


Figure 2-9 **The Heaviside function in spherical radial coordinate**

and the following properties can be derived

$$\begin{aligned} \theta(r-a)+\theta(a-r) &= 1 \\ \theta(r-a)\theta(a-r) &= 0 \end{aligned} \qquad a \ge 0$$

2.25 Delta function in cylindrical coordinates

In terms of cylindrical coordinates (ρ,ϕ,z) the transformation to rectangular coordinates are

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z$$

Again, a unique transformation requires restrictions on the cylindrical coordinates

$$\rho \ge 0 \qquad 0 \le \phi < 2\pi \qquad -\infty < z < \infty$$

Integration the 3-D delta function over all space must yield unity

$$\int dV\delta(\boldsymbol{x}-\boldsymbol{x}') = \int_0^\infty \rho d\rho \int_0^{2\pi} d\phi \int_{-\infty}^\infty \delta(\boldsymbol{x}-\boldsymbol{x}')dz = 1 \qquad \text{EQ (2-62)}$$

Hence the representation of the 3-D delta function in cylindrical coordinates must be

$$\delta(\boldsymbol{x}-\boldsymbol{x}') = \frac{\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')}{\rho} \qquad \text{EQ (2-63)}$$

For the degenerate case (ρ' =0), the delta function must be independent of angle ϕ, the 3D representation is given by

$$\delta(\boldsymbol{x}) = \frac{\delta(\rho)\delta(z)}{2\pi\rho} \qquad \text{EQ (2-64)}$$

with the understanding that

$$\int_0^\infty \delta(\rho)d\rho = 1 \qquad \text{EQ (2-65)}$$

A strong definition $\theta(0) = 0$ and $\theta(\rho) = 1$ of the Heaviside function is needed to satisfy Figure 2-65. That is

$$\int_0^\rho \delta(\rho')d\rho' = \int_0^\rho \frac{\mathrm{d}}{\mathrm{d}\rho'}\theta(\rho')d\rho' = \theta(\rho)-\theta(0) = 1$$

With the above definition of $\theta(0)$=0, the following plots (a > 0) follow

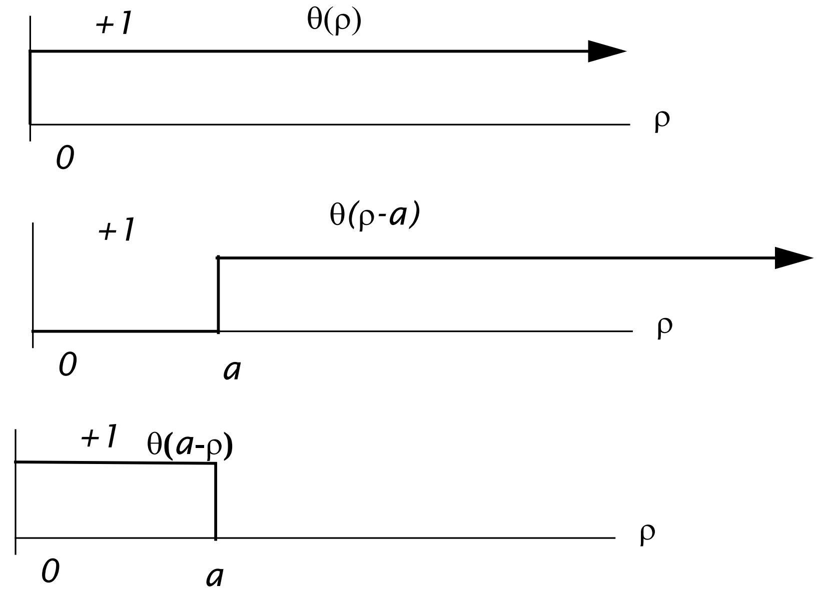


Figure 2-10 **The Heaviside function in cylindrical radial coordinates**

and the following properties can be derived

$$\begin{aligned} \theta(\rho - a) + \theta(a - \rho) &= 1 \\ \theta(\rho - a)\theta(a - \rho) &= 0 \end{aligned} \qquad a \geq 0$$

2.26 Gradient of the Dirac delta function

As it was the case with the 1-D derivative of the delta function, under integration the gradient of the 3D delta function can be represented in terms of an integral of a 3D delta function. Consider the test function $f(x')$. Remembering that $\nabla' = -\nabla$

$$\int f(\mathbf{x}')\nabla'\delta(\mathbf{x} - \mathbf{x}')dV' = -\int f(\mathbf{x}')\nabla\delta(\mathbf{x} - \mathbf{x}')dV$$

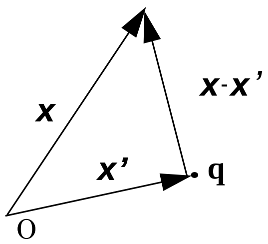
$$\int f(\mathbf{x}')\nabla'\delta(\mathbf{x} - \mathbf{x}')dV' = -\nabla\int f(\mathbf{x}')\delta(\mathbf{x} - \mathbf{x}')dV$$

$$\int f(\mathbf{x}')\nabla'\delta(\mathbf{x} - \mathbf{x}')dV' = -\nabla f(\mathbf{x})$$

Hence the following substitution can be made

$$f(\mathbf{x}')\nabla\delta(\mathbf{x}-\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}')\nabla f(\mathbf{x}) \qquad \text{EQ (2-66)}$$

2.27 Mathematical description of charged sources



Before solving Maxwell's equation it is necessary to develop an appropriate compact mathematical description of the nature of the sources that give rise to the electro-magnetic field. Specifically it will be helpful to describe with precise mathematical functions the spatial and temporal distribution of the volume charge density $\rho(\mathbf{x},t)$ and the electric volume current density $\mathbf{J}(\mathbf{x},t)$ associated with a particular problem in terms of continuous and discontinuous functions. There are some idealized problems that can be solved exactly using special discontinuous functions such as the Heaviside, Signum and Dirac delta function whose definition and mathematical properties were introduced in the Mathematical Review section of this document.

2.28 Static volume charge density distribution for a point charge

The simplest volume charge distribution is that of a point charge. The volume charge density of a point charge q located at position $\mathbf{x}$' can be mathematically described using a three dimensional delta function represented in the most commonly used curved coordinate systems. The following are three representations of the spatial distribution of a 3d point charge using rectangular, cylindrical and spherical coordinates.

$$\rho(\mathbf{x}) = q\delta(\mathbf{x}-\mathbf{x}')$$

$$\rho(\mathbf{x}) = q\delta(x-x')\delta(y-y')\delta(z-z')$$

$$\rho(\mathbf{x}) = \frac{q\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')}{\rho}$$

$$\rho(\mathbf{x}) = \frac{q\delta(r-r')\delta(\phi-\phi')\delta(\theta-\theta')}{r^2\sin\theta}$$

Here the vector coordinate $\mathbf{x'}$ represents the source coordinate and $\mathbf{x}$ represents the field coordinate. Note that the first equation is independent of the coordinate system used. Since the delta function has units corresponding to the reciprocal of its argument, it is clear that all the above relations have units volume charge density, Coulomb/m^3. To check whether the above mathematical description of $\rho(\boldsymbol{x})$ are correct, it is necessary to show that the total charge (i.e. the volume integration of the charge density over all space) is q. Using the appropriate mathematical description of the volume elements for each of the coordinate systems used, integration of the volume charge density over all space yields the desired result. That is,

$$\oint q\delta(\boldsymbol{x}-\boldsymbol{x}')dV' = q$$

In rectangular coordinates

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy[q\delta(x-x')\delta(y-y')\delta(z-z')] = q$$

In cylindrical coordinates

$$\int_{-\infty}^{\infty} \rho d\rho \int_{0}^{2\pi} d\phi \int_{-\infty}^{\infty} dz\left[\frac{q\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')}{\rho}\right] = q$$

In spherical coordinates

$$\int_{0}^{\infty} r^2 dr \int_{0}^{2\pi} d\phi \int_{0}^{\pi} \sin\theta d\theta\left[\frac{q\delta(r-r')\delta(\phi-\phi')\delta(\theta-\theta')}{r^2\sin\theta}\right] = q$$

If the point charge is located at the origin of the coordinate system, the volume charge density must be independent of angles. It can only depend on distance from the origin. Thus

$$\begin{aligned}\rho(\boldsymbol{x}) &= q\delta(x)\delta(y)\delta(z)\\ \rho(\boldsymbol{x}) &= \frac{q\theta(\rho)\delta(\rho)\delta(z)}{2\pi\rho}\\ \rho(\boldsymbol{x}) &= \frac{q\theta(r)\delta(r)}{4\pi r^2}\end{aligned} \qquad \text{EQ (2-67)}$$

Radial Heaviside functions have been inserted into the equations because the coordinates ρ and r must be positive quantities. The reader can check that the total charge described by these functions is q.

2.29 Simple volume charge density distributions

Other "point charge" distributions can be derived using discontinuous mathematical functions.

Example (2-8)

A thin infinite plane having a uniform surface charge density σ_0 can be described in rectangular coordinates as follows

$$\rho(\xi) = \sigma_0\delta(\xi - \xi_0) \qquad \text{EQ (2-68)}$$

where ξ is one of the rectangular coordinates, and ξ_0 is the location of the plane of charge on the ξ-axis. The ξ-dependence of charge per unit area is obtained from

$$\sigma(\xi) = \int_{-\infty}^{\xi} \rho(\xi')d\xi' = \sigma_0\int_{-\infty}^{\xi} \frac{\partial}{\partial\xi'}\theta(\xi' - \xi_0)d\xi'$$

$$\sigma(\xi) = \sigma_0[\theta(\xi - \xi_0) - \theta(-\infty)] = \sigma_0\theta(\xi - \xi_0)$$

Example (2-9) Volume charge density for a uniform line source

Let λ be the constant charge per unit length of an infinitely long line located parallel to the z-axis and passing through the point $(\mathbf{x}_0,\mathbf{y}_0) = (\rho_0,\phi_0)$. The volume charge density can be mathematically described by

$$\begin{aligned} \text{ɔ}(r) &= \lambda\delta(r - r_0) \\ \text{ɔ}(x, y) &= \lambda\delta(x - x_0)\delta(y - y_0) \\ \text{ɔ}(\rho, \phi) &= \frac{\lambda\delta(\rho - \rho_0)\delta(\phi - \phi_0)}{\rho} \end{aligned} \qquad \text{EQ (2-69)}$$

in vector notation, in rectangular coordinates and in cylindrical coordinates.

Example (2-10) Volume charge density for a charged spherical surface

Consider a charge **Q** uniformly distributed over the surface of a sphere of radius **a**. If the origin of coordinates is chosen to coincide with the center of the sphere the distribution is highly symmetric (i.e.,$\rho(\boldsymbol{x})$ should not depend on angles). In spherical coordinates, the volume charge density must be of the form $\rho(\boldsymbol{x}) = f(r)\delta(r-a)$. The delta function guarantees that charge exists only at r = **a**. The radial function **f**(**r**) can be obtained by requiring that the total charge must be **Q**. Integrating over all space yields

$$\int_0^\infty r^2 dr \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta [f(r)\delta(r-a)] = 4\pi a^2 f(a) = Q$$

$$f(r) = \frac{Q}{4\pi r^2} \rightarrow \rho(x) = \frac{Q}{4\pi r^2}\delta(r-a) \qquad \text{EQ (2-70)}$$

Example (2-11) Volume charge density for a charged plane surface

Let σ be the constant surface charge density (C/m^2) distributed uniformly over an infinite plane. In rectangular coordinates, let the plane be $z = 0$. The volume charge density cannot depend on coordinates transverse to the z-axis (i.e., either x and y in rectangular coordinates or ρ and ϕ in cylindrical coordinates). Therefore, the volume charge density must have the following forms

$$\rho(x, y, x) = A\delta(z)$$
$$\rho(\rho, \phi, z) = A\delta(z)$$

where **A** is a constant. Integration of the volume charge density over z from minus infinity to plus infinity should yield the uniform charge density σ. Thus **A** = σ and

$$\rho(x, y, x) = \sigma\delta(z)$$
$$\rho(\rho, \phi, z) = \sigma\delta(z)$$

Exercises for Chapter 2

Exercise (2-1)

Sketch the functions $f(u) = \theta(u-a) - \theta(u-b))$ and $\theta(u-a)\theta(u-b)$ for $b > a$

Exercise (2-2)

Evaluate:

$$(a) \qquad \int_{-1}^{2} \delta(2u)\cos(u)du$$

$$(b) \qquad \int_{0}^{\infty} \delta'(\sqrt{2}u - 1)\,\mathrm{atan}(u)du$$

$$(c) \qquad \int_{0}^{1} dx \int_{0}^{1} \delta(x-y)dy$$

$$(d) \qquad \int_{-\infty}^{\infty} e^{-|u|}\delta(u^2 + 2u - 3)du$$

Exercise (2-3)

Show that the following sequences of functions are delta functions

$$(a) \qquad \phi_n(u) = \frac{n}{2}e^{-n|u|}$$

$$(b) \qquad \phi_n(u) = \frac{n}{\pi}\frac{1}{1+n^2u^2}$$

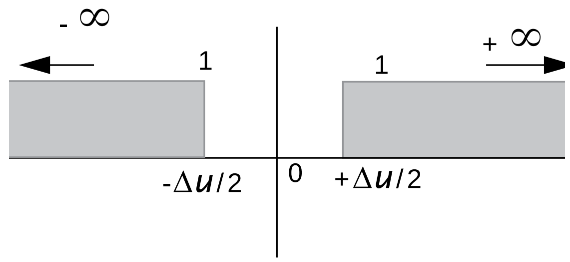
$$(c) \qquad \phi_n(u) = \frac{1-\cos(nu)}{n\pi u^2}$$

Exercise (2-4)

Derive a mathematical description of the square function SSS and square well SSW shown using the signum function.

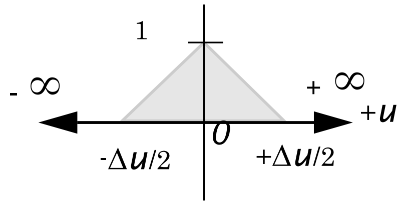
Exercise (2-5)

Derive a mathematical description of the symmetric square well function shown below using the Heaviside and/or the signum function.



Exercise (2-6)

Describe mathematically in one equation the triangular step function shown below, using the discontinuous functions defined early in the chapter.



Exercise (2-7)

Differentiate twice the following functions:$|x|$, $\sin|x|$, $\cos|x|$

Exercise (2-8)

Differentiate the function derived in Appendix EQ (2-22). The result should be represented by only one equation.

Exercise (2-9)

Derive and plot the following function

$$\frac{d}{du}\tan|u|$$

Find and plot $df(u)/du$ given that

$$f(x) = \theta(\pi - |u|)\sin u$$

Exercise (2-10)

Show that $\int_{-\infty}^{u} \delta(u' - u_0)\,du' = \theta(u - u_0)$

Exercise (2-11)

Evaluate

$$(a) \qquad \int_0^5 u\theta(u-2)\,du$$

$$(b) \qquad \int_0^3 u\varepsilon(u-2)\,du$$

$$(c) \qquad \int_{-\infty}^{\infty} e^{-|u|}\theta(u+1)\,du$$

Exercise (2-12)

Evaluate

$$\int_0^{\infty} e^{-u}\delta(\sin(u))\,dx$$

The solution can be simplified using

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$$

Exercise (2-13)

Find and plot

$$f(u) = \int_{u_1}^{u} \theta(u')du'$$

Exercise (2-14)

Show that $\int_{-\infty}^{\infty} e^{u}\theta(u_0 - u)du = 2\theta(u_0)\sinh(u_0)$

Exercise (2-15)

Solve $\frac{\mathrm{d}}{\mathrm{d}u}f(u) = \delta(u)$. Plot $f(u)$-$f(u_0)$, for u_0 = 1 and u_0 = -1.

Exercise (2-16)

Derive the three dimensional representation of the delta function $\delta(x\text{-}x')$ in cylindrical coordinates.

Exercise (2-17)

Integrate the following

$$(a) \qquad \int \theta(u)\cos u$$

$$(b) \qquad \int \theta(u)e^{-u}$$

$$(c) \qquad \int \theta(u)u^{n} \qquad n > 0$$

Exercise (2-18)

Show that

$$\int_{u_1}^{u_2} \delta(u' - u_0)f(u')du' = [\theta(u_2 - u_0) - \theta(u_1 - u_0)]f(u_0)$$

Exercise (2-19)

(a) Show through differentiation that $\psi(u) = \theta(u)f(u)$ is a solution of the differential equation

$$\frac{d}{du}\psi(u) + \alpha\psi(u) = \delta(u)$$

(where $\alpha > 0$), provided that $f(u)$ satisfies

$$f(0) = 1, \qquad \textit{and}$$
$$\frac{d}{du}f(u) + \alpha f(u) = 0$$

(b) Find $f(u)$

Exercise (2-20)

Derive:

$$\int_{u_1}^{u_2} f(u)\varepsilon(u)du = \varepsilon(u_2)\int_0^{u_2} f(u)du - \varepsilon(u_1)\int_0^{u_1} f(u)du$$

Exercise (2-21)

Test the relation

$$\int_{u_1}^{u_2} \theta(u-a)f(u)du = \theta(u_2-a)\int_a^{u_2} f(u)du - \theta(u_1-a)\int_a^{u_1} f(u)du$$

with

$$f(u) = \cos u \qquad a = \frac{\pi}{4} \qquad u_1 = -\pi \qquad u_2 = \pi$$

Exercise (2-22)

(a) Integrate

$$\int_{-\infty}^{u} e^{-q|u'|} du'$$

and show that the result can be written as

$$\int_{-\infty}^{u} e^{-q|u'|} du' = \frac{1}{q}\{1 + \varepsilon(u)[1 - e^{-q|u|}]\}$$

(b) Determine $\psi(u)$ for all u, given that $\psi(-\infty) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}u}\psi(u) = e^{-q|u|} + \delta(u)$$

Answer:

$$\psi(x) = \frac{1}{q}[\theta(u)-\!-\varepsilon(u)e^{-q|u|}]$$

Exercise (2-23)

(a) Find $\psi(u)$ for all values of u given that

$$\frac{\mathrm{d}}{\mathrm{d}u}\psi(u) = e^{-q|u|} + \delta(u)$$

$$\psi(-\infty) = e^{-\frac{3}{q}} \qquad \text{EQ (2-71)}$$

(b) Show that $\psi(\infty) = 2\cosh\frac{1}{q}$