Lecture 1

Discussion of Maxwell's Equations

1.1 Brief history of Maxwell's Equations

Maxwell's equations are a set of four partial differential equations that relate the electric and magnetic fields to their sources, charge density and current density. These four equations, together with the Lorentz force law are the complete set of laws of classical electromagnetism.The four modern day Maxwell's equations appeared in James Clerk Maxwell's 1861 paper "On Physical Lines of Force", (The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science) and on his 1865 paper "A Dynamical Theory of the Electromagnetic Field" (Philosophical Transactions of the Royal Society of London 155, 459-512 (1865).

The transcendental importance of Maxwell's contribution to the understanding of electromagnetic phenomena was highlighted by Albert Einstein in 1940. Einstein wrote:

The precise formulation of the time-space laws was the work of Maxwell. Imagine his feelings when the differential equations he had formulated proved to him that electromagnetic fields spread in the form of polarized waves, and at the speed of light! To few men in the world has such an experience been vouchsafed. It took physicists some decades to grasp the full significance of Maxwell's discovery, so bold was the leap that his genius forced upon the conceptions of his fellow-workers —(Science, May 24, 1940)

1.2 Maxwell's Equations in Vacuum

The relation between electric charge $\rho(\boldsymbol{x},t)$ and current $\boldsymbol{J}(\boldsymbol{x},t)$ densities to the electric $\boldsymbol{E}(\boldsymbol{x},t)$ and magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$ are governed by the four modern Maxwell's equations. The vector $\boldsymbol{x}$ locates a point in space and t is the time coordinate. In differential form and in SI units, Maxwell's equations in vacuum are

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$
$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \tag{EQ (1-1)}$$

where ε_0 and μ_0 are constants representing respectively the electrical permittivity and magnetic permeability of vacuum and c is the speed of light in vacuum. For time dependent sources and fields, these equations couple the electric to the magnetic field. In general these equations are <u>inhomogeneous</u> in regions where sources exist.

1.3 Inhomogeneous Wave Equation

Using the curl equations, the Maxwell first order equations can be combined into second order equations. The result is

$$\nabla \times \nabla \times \boldsymbol{E} + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = -\mu_0 \frac{\partial \boldsymbol{J}}{\partial t}$$
$$\nabla \times \nabla \times \boldsymbol{B} + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \mu_0 \nabla \times \boldsymbol{J} \tag{EQ (1-2)}$$

In addition, the two divergence equations must be satisfied. Solution of EQ (1-2) is in general difficult. In most cases, it is convenient to represent the magnetic field in terms of a vector potential $\boldsymbol{A}(\boldsymbol{x},\boldsymbol{t})$ such that

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad \text{EQ (1-3)}$$

With this definition Faraday's and Ampere's laws becomes

$$\nabla \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) = 0$$
$$\nabla \times \nabla \times \boldsymbol{A} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \qquad \text{EQ (1-4)}$$

From Faraday's law the vector field $\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}$, which is curless, can be derive from the negative gradient of a scalar field $\Phi(\boldsymbol{x},\boldsymbol{t})$, known as the scalar electric potential. Hence, the time dependent electric field can now be given by

$$\boldsymbol{E} = -\nabla\Phi - \frac{\partial \boldsymbol{A}}{\partial t} \qquad \text{EQ (1-5)}$$

After substituting EQ (1-3)and EQ (1-5)the following second order equations for Φ an $\boldsymbol{A}$ are obtained

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla\left[\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right] = -\mu_0 \boldsymbol{J}$$
$$\nabla^2 \Phi + \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}) = -\frac{\rho}{\varepsilon_0} \qquad \text{EQ (1-6)}$$

1.4 Lorentz Gauge

EQ (1-6), taken together, are as powerful and complete as Maxwell's equations. Moreover, in terms of the potentials, the problem has been reduced somewhat, as the electric and magnetic fields each have three components which need to be solved for (six components altogether), while the electric and magnetic potentials have only four components altogether.

Many different choices of $\boldsymbol{A}$ and Φ are consistent with a given $\boldsymbol{E}$ and $\boldsymbol{B}$, making these choices physically equivalent – a flexibility known as gauge freedom. Suitable choice of $\boldsymbol{A}$

and Φ can simplify these equations, or can adapt them to suit a particular situation. The so called Lorentz gauge

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0 \qquad \text{EQ (1-7)}$$

transforms EQ (1-6) into the following inhomogeneous wave equations for the potentials

$$\begin{aligned} \nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} &= -\mu_0 \boldsymbol{J} \\ \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} &= -\frac{\rho}{\varepsilon_0} \end{aligned} \qquad \text{EQ (1-8)}$$

1.5 Transverse Gauge

Another gauge that uncouples EQ (1-6) is the transverse gauge. It requires

$$\nabla \cdot \boldsymbol{A} = 0 \qquad \text{EQ (1-9)}$$

With the transverse gauge EQ (1-6) becomes

$$\begin{aligned} \nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \frac{1}{c^2}\frac{\partial}{\partial t}\nabla \Phi &= -\mu_0 \boldsymbol{J} \\ \nabla^2 \Phi &= -\frac{\rho}{\varepsilon_0} \end{aligned} \qquad \text{EQ (1-10)}$$

It can be shown that EQ (1-10) can be written as follows

$$\begin{aligned} \nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} &= -\mu_0 \boldsymbol{J}_\perp \\ \nabla^2 \Phi &= -\frac{\rho}{\varepsilon_0} \end{aligned} \qquad \text{EQ (1-11)}$$

where the source of the vector potential is the transverse component of the volume current density $\boldsymbol{J}(\boldsymbol{x},t)$. The longitudinal component of the current density is defined by

$$\boldsymbol{J}_\parallel = \varepsilon_0 \nabla \frac{\partial \Phi}{\partial t} \qquad \text{EQ (1-12)}$$

Solutions of the inhomogeneous wave equation will be used later to study the radiation characteristics of single electrons moving with relativistic speeds along periodic EM structure. The fields generated by electrons (point charges) are solutions of the inhomogeneous WE. The fields are called the Lienard-Wiechert fields. The radiation produced is denominated spontaneous undulator radiation.

Solutions of the inhomogeneous WE in vacuum will be used to:

(a) Describe the fields generated by a point charge (Lienard-Wiechert fields) travelling along a periodic magnetic structure (undulator or wiggler).

(b) Study the wave amplification properties of a free electron laser.

(c) Change the charge distribution properties (velocity modulation and bunching) of an electron beam.

1.6 Homogeneous wave equation

In regions where the sources of the field are absent, the potentials satisfy the following <u>homogeneous</u> wave equations

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0$$

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = 0 \qquad \text{EQ (1-13)}$$

Each rectangular component of the vector potential and the scalar electric potential satisfies the scalar wave equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] u(\boldsymbol{x}, t) = 0 \qquad \text{EQ (1-14)}$$

A solutions of the scalar wave equation is

$$u(\boldsymbol{x}, t) = u(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) \qquad \text{EQ (1-15)}$$

where ω is constant scalar and $\boldsymbol{k}$ is a constant vector. The solution represents a plane wave travelling along the propagation vector defined by

$$\nabla(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) = \nabla(\boldsymbol{k} \cdot \boldsymbol{x}) = \boldsymbol{k}$$

at the speed of light c. The dispersion relation for the solution is

$$\omega = ck \qquad \text{EQ (1-16)}$$

The proof that the solutions of the scalar wave equation have these properties is left as an exercise at the end of the lecture.

One thing to remember is that not all general solutions of the scalar wave equation are solutions of Maxwell's equations.

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= 0 \qquad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{B} &= 0 \qquad \nabla \times \boldsymbol{B} = c^{-2}\frac{\partial \boldsymbol{E}}{\partial t} \end{aligned} \qquad \text{EQ (1-17)}$$

These equations restrict the properties of time dependent fields in vacuum regions where sources are absent. For example a constant amplitude, monochromatic plane wave described by

$$\boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{E}_0 e^{i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)} \qquad \boldsymbol{B}(\boldsymbol{x}, t) = \boldsymbol{B}_0 e^{i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)}$$

satisfies EQ (1-17) provided that the fields satisfy all of Maxwell's equations. The two divergence equations (Gauss's Laws) require that

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= i\boldsymbol{k} \cdot \boldsymbol{E} = 0 \\ \nabla \cdot \boldsymbol{B} &= i\boldsymbol{k} \cdot \boldsymbol{B} = 0 \end{aligned}$$

That is, the direction of both the electric and magnetic fields must be perpendicular to the direction of the propagation vector $\boldsymbol{k}$. The curl equations (Faraday's Law and Ampere's Law)

$$\begin{aligned} i\boldsymbol{k} \times \boldsymbol{E} &= i\omega \boldsymbol{B} \\ i\boldsymbol{k} \times \boldsymbol{B} &= -i\frac{\omega}{c^2}\boldsymbol{E} \end{aligned} \qquad \text{EQ (1-18)}$$

requires that the electric field an magnetic field be perpendicular to each other. Using the triple cross product, the two equations of EQ (1-18)can be combined into the following equations

$$\left[1-\left(\frac{kc}{\omega}\right)^2\right]\begin{Bmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{Bmatrix} = 0$$

For finite fields, the solution of the above equation imposes the following relation between ω and k

$$\omega = ck \qquad \text{EQ (1-19)}$$

that is, the plane wave travels at the constant speed of light in vacuum.

1.7 Time independent fields

For static distribution of charges and steady state currents the electric field in Maxwell's equations is un-couples from the magnetic field. That is

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= \frac{\rho}{\varepsilon_0} \qquad & \nabla \times \boldsymbol{E} &= 0 \\ \nabla \cdot \boldsymbol{B} &= 0 \qquad & \nabla \times \boldsymbol{B} &= \mu_0 \boldsymbol{J} \end{aligned} \qquad \text{EQ (1-20)}$$

Solutions of these equations will be used to describe the static fields generated by a few devices that are used to change the trajectory of electrons and/or accelerate charged particles. Among these devices are:

(a) Electric and magnetic quadrupoles
(b) Magnetic dipoles and steering coils
(c) Magnetic undulators and wigglers
(d) Electrostatic accelerators
(e) Electrostatic focussing during acceleration

1.8 The Lorentz force

The Lorentz force law itself was actually derived by Maxwell under the name of Equation for Electromotive Force and was one of an earlier set of eight equations by Maxwell. The mathematical statement of the force exerted by an EM field on a point charge q is given by

$$\boldsymbol{F}_q = q[\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}] \qquad \text{EQ (1-21)}$$

where $\boldsymbol{v}(t)$ is the velocity vector of the point charge. The fields must be evaluated at the position of q and at time t.

Reference Books

Jackson, John D. (1998). *Classical Electrodynamics* (3rd ed.). Wiley.

Wolfgang Kurt Hermann, Phillips, Melba. *Classical Electricity* And Magnetism Panofsky.

Griffiths, David J. (1998). *Introduction to Electrodynamics* (3rd ed.). Prentice Hall

Vector identities used in this lecture

$$\begin{aligned}
\nabla\cdot(f\boldsymbol{a}) &= f\nabla\cdot\boldsymbol{a}+\boldsymbol{a}\cdot\nabla f\\
\nabla\times(f\boldsymbol{a}) &= f(\nabla\times\boldsymbol{a})-\boldsymbol{a}\times\nabla f\\
\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) &= \boldsymbol{b}\cdot(\nabla\times\boldsymbol{a})-\boldsymbol{a}\cdot(\nabla\times\boldsymbol{b})\\
\nabla\times(\boldsymbol{a}\times\boldsymbol{b}) &= \boldsymbol{b}\cdot\nabla\boldsymbol{a}-\boldsymbol{a}\cdot\nabla\boldsymbol{b}+\boldsymbol{a}\nabla\cdot\boldsymbol{b}-\boldsymbol{b}\nabla\cdot\boldsymbol{a}\\
\nabla(\boldsymbol{a}\cdot\boldsymbol{b}) &= \boldsymbol{a}\times\nabla\times\boldsymbol{b}+\boldsymbol{b}\times\nabla\times\boldsymbol{a}+(\boldsymbol{a}\cdot\nabla)\boldsymbol{b}+(\boldsymbol{b}\cdot\nabla)\boldsymbol{a}\\
\nabla\times\nabla\times\boldsymbol{a} &= \nabla(\nabla\cdot\boldsymbol{a})-\nabla^2\boldsymbol{a}
\end{aligned}$$

EQ (1-22)

Exercise (1-1) Determine whether the time independent electric field

$$\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

where $\boldsymbol{E}_0$ and $\boldsymbol{k}$ are both vector constants, satisfies Maxwell's equations.

Exercise (1-2) Show that EQ (1-15) is a solution of EQ (1-14)if $k = \omega/c$. Hint: Start with showing that $\nabla u = \boldsymbol{k}u$.

Exercise (1-3) Assume that the vector potential (transverse gauge) is given by

$$\boldsymbol{A} = \boldsymbol{A}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$$

where $\boldsymbol{A}_0$ and $\boldsymbol{k}$ are constant vectors and ω is a scalar constant. Show that: (a) $\boldsymbol{A}$ is a solution each of Maxwell's equations; (b) Identify any constraints regarding the polarization direction of the wave.

Exercise (1-4)