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GEOMETRICAL PROPERTIES OF ELECTROMAGNETIC TIDAL FORCES

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In general, elementary particles as well as extensive bodies have internal degrees of freedom that naturally turn their trajectories into accelerated curves. Hence, we propose to describe the kinematical properties of nongeodesic congruences and study how tidal forces are modified. Once the general scenario is well established, we analyze in details tidal effects due to electromagnetic fields, i.e. the relative acceleration between test charged particles. An algebraic analysis of these fields is developed together with a geometrical interpretation in terms of local field lines. In this framework, we compare general relativity and electrodynamics in terms of operationally equivalent objects.

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1. Introduction

The identification of gravity with the geometrical properties of a Riemannian space–time is one of the most profitable and intriguing ideas in physics, which together with the geodesic hypothesis and Einstein field equations form the core of general relativity. Notwithstanding, geometrize gravity creates some theoretical difficulties such as the definition of gravitational energy, the appearance of space–time singularities and the difficulties encountered on its quantization process (see for instance Refs. 1–4.

To deal with some of the above difficulties, it became an interesting practice to compare general relativity with electrodynamics (see Refs. 5–7). Despite the differences of their mathematical formalism and the unavoidable deficiency of any analogy, there has been many interesting results by comparing general relativity with electromagnetism. In general, one applies well-known techniques used in Maxwell theory to some of the mathematically involved and physically subtle issues of general relativity. Among the profitable analogies, there are the dragging of inertial frames by rotating masses and the so-called gravitomagnetic field,[8–21] motion of spinning particles in a gravitational field,[22,23,a] algebraic classification of Weyl tensor,[b] curvature discontinuities and the Cauchy problem in general relativity,[24–26,c] definition of the super-energy–momentum tensor[27–29] and the quasi-Maxwellian equations for gravitation.[14–30,30–34]

It is possible to group these analogies in two main groups. The first group emphasizes the analogy between the Faraday tensor $F_{\mu\nu}$ and the Christoffel symbol $\Gamma^{\alpha}{}_{\mu\nu}$ in the context of weak field limit of Einstein's equations and assuming slowly moving test particles. In this scenario, by choosing a particular gauge condition, it is possible to show that the gravitational linearized equations can be cast in an equivalent form to Maxwell's equations, which allows to define spatial gravito-electromagnetic fields involving derivatives of the perturbed metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and the geodesic equation reproduces the usual Lorentz force equation (see, for instance Refs. 2 and 11).

The second group is based on the differential structure of Einstein and Maxwell equations and its relation to the Cauchy problem. In this analysis, it is natural to relate the Faraday tensor $F_{\mu\nu}$ with the curvature tensor $R_{\alpha\beta\mu\nu}$ in vacuum. Using Bianchi identities and Lichnerowicz theorem, one can show that gravitational equations are formally equivalent to Maxwell's. In this scheme, it seems reasonable to associate the electric and magnetic parts of the Weyl tensor $E_{\mu\nu}$ and $H_{\mu\nu}$, introduced by A. Matte in the early 1950's,[27–35] with the usual electromagnetic fields E_{μ} and H_{μ}.[1] As a consequence, one can also suggest to interpret the properties of the Weyl tensor in different space–times through a classification of its algebraic properties and its relation with the algebraic properties of $F_{\mu\nu}$.

Recently, it was pointed out by Costa and Herdeiro[36] and independently by Goulart and Falciano[37] that a third approach is needed to clarify some aspects of both theories. They proposed a completely covariant analogy between general relativity and electromagnetism based on tidal tensors, which compares objects with exactly the same operational meaning. Once the Riemannian curvature $R_{\alpha\beta\mu\nu}$ is the tensor that determines the geodesic deviations in gravity, it becomes natural to consider the deviation equation between charged particles that follow trajectories given by the Lorentz force in Maxwell's theory. Thus, one is led to deal with a third order tensor $F_{\alpha\beta;\mu}$ and study accelerated congruences considering particles with constant charge–mass ratio.

In Ref. 36, Costa and Herdeiro give, in analogy with electromagnetic tidal tensors, a clear and meaningful interpretation of the electric and magnetic parts of the Weyl tensor and derived, from very simple considerations, an equation coupling the angular momentum with derivatives of the magnetic field analogously to Papapetrou's equation, which couples the angular momentum with the magnetic part of

[a]See also Sec. A of Ref. 5, equation of motion of spinning particles in electrodynamics and GR.
[b]See, for instance Ref. 1 Sec. 4 — Pirani Criterion.
[c]See Ref. 1 and references therein.

the Weyl tensor. They have also presented interesting realizations of this analogy for particular situations.

Another interesting result was obtained in Ref. 37 by Goulart and Falciano. They studied the irreducible parts of the object $F_{\alpha\beta;\mu}$. It was shown that the electromagnetic analogue of the Weyl tensor is described not only by two traceless symmetric spatial tensors but also by two vectors that were called Q^α and Z^α. The tensorial degrees of freedom are in complete analogy with the electric and magnetic parts of the Weyl tensor, while the vectorial have no gravitational counterpart. The absence of these two vectors can be used to distinguish situations where it is possible or not to map one theory into another. In addition, it was derived dynamical equations for the irreducible parts of $F_{\alpha\beta;\mu}$. The equations obtained are not only analogous to the quasi-Maxwellian equations of gravitation suggested by Bel,[27] Jordan, Lichnerowicz and others in the late 1950's (see Ref. 1 and references therein for a detailed discussion), but also represent the evolution of objects with exactly the same physical content.

In this paper, we shall take a less ambitious program. We are not interested in a direct comparison between dynamical properties of general relativity and electrodynamics. We are concerned mainly in the geometrical structure of the electrodynamic tidal forces. In other words, if in a given configuration of electromagnetic fields a cloud of test charged particles is released what will be the net effect of field gradients on their relative motions? We show that the local behavior of particles is much more complicated than in its corresponding gravitational analogue. We develop a detailed analysis of the deviation properties in electrodynamics and show explicitly in which sense the tidal effects differ from the gravitational ones.

2. General Characterization of Congruences

In general relativity, free test particles follow geodesics. However, elementary particles have internal degrees of freedom such as spin or is coupled to external fields and extensive objects have their internal structure that are not taken into account in the geodesic equation. Therefore, one should expect to have geodesics only on special cases and accelerated motion to be the typical behavior which motivates the analysis of specific accelerated congruences.

We start with a normalized timelike congruence in space–time which may be interpreted as the integral lines $x^\mu(s)$ of a normalized vector field v^μ, i.e. ($v_\mu v^\mu = 1$). The proper length s is a natural parametrization since the reading of clocks attached to observers following the timelike congruence will coincides with the proper length s. Also, one can define a parameter σ such that it forms a coordinate basis with s. Each value of σ specifies a given integral curve of the timelike congruence, i.e. $x^\mu = x^\mu(s, \sigma)$.

Locally, the integral lines along σ can be used to connect two infinitesimally close curves of the timelike congruence. Around a given reference curve Γ, the deviation vector η^μ is defined as the vector linking Γ and an arbitrary sufficiently close curve of the congruence with the same value of the length parameter. Thus, its Lie derivative $\pounds_{\vec{v}}\,\vec{\eta} = 0$ vanish, i.e.

$$\eta^{\alpha}{}_{;\beta}v^{\beta} - v^{\alpha}{}_{;\beta}\eta^{\beta} = 0\,.$$

Any normalized timelike congruence where the coordinate basis (s,σ) has been constructed satisfies the properties

$$a^{\alpha}v_{\alpha} = 0\,, \qquad \dot{\eta}^{\alpha}v_{\alpha} = 0\,,$$

where $a^{\alpha} \equiv v^{\alpha}{}_{;\mu}v^{\mu}$ and $\dot{\eta}^{\alpha} \equiv \eta^{\alpha}{}_{;\mu}v^{\mu}$.

However, due to a possible acceleration $a^{\alpha} \neq 0$, in general, the deviation vector shall not be restricted to H_p, the orthocomplement space of v^{μ} defined by projecting all tensor fields with the projector $h_{\mu\nu} = g_{\mu\nu} - v_{\mu}v_{\nu}$. Using the decomposition $\eta^{\alpha} = nv^{\alpha} + {}_{\perp}\eta^{\alpha}$, with $n \equiv \eta^{\alpha}v_{\alpha}$ and ${}_{\perp}\eta^{\alpha} \equiv h^{\alpha}{}_{\beta}\eta^{\beta}$, it is immediate to show that $\dot{n} = a^{\alpha}\eta_{\alpha} \neq 0$. Thus, the deviation vector has its dynamical equation given by

$$\frac{D^2\eta^{\alpha}}{Ds^2} = \left(R^{\alpha}{}_{\mu\beta\nu}v^{\mu}v^{\nu} + a^{\alpha}{}_{;\beta}\right)\eta^{\beta}\,. \tag{1}$$

Note that even though $\dot{\eta}^{\alpha}v_{\alpha} = 0$, the deviation vector acceleration also has a component along v^{μ}, i.e. $\ddot{\eta}^{\alpha}v_{\alpha} = -a^{\alpha}\dot{\eta}_{\alpha} \neq 0$. This is completely analogous to $a^{\alpha}v_{\alpha} = 0$ but $\dot{a}^{\alpha}v_{\alpha} = -a^{\alpha}a_{\alpha} \neq 0$ and is associated with a nonpreservation of angles for accelerated curves.

In general, one is not interested in η^{α} itself but in the evolution of projected objects in the rest space H_p orthogonal to v^{μ} at a given point p.[3] Hence, we shall define objects restricted to the orthocomplement space H_p. As already mentioned, ${}_{\perp}\eta^{\alpha}$ can be viewed as the distance between two neighboring particles and thus the relative velocity ${}_{\perp}\frac{D}{Ds}{}_{\perp}\eta^{\alpha}$ describes the rate of separation between them in H_p. In terms of projected objects, the relative acceleration satisfies

$$a^{\alpha}_{\rm rel} \equiv {}_{\perp}\frac{D}{Ds}{}_{\perp}\frac{D}{Ds}{}_{\perp}\eta^{\alpha} = \left(R^{\alpha}{}_{\mu\beta\nu}v^{\mu}v^{\nu} + a_{\mu;\nu}h^{\alpha\mu}h^{\nu}{}_{\beta} - a^{\alpha}a_{\beta}\right){}_{\perp}\eta^{\beta}\,. \tag{2}$$

The above deviation equation, also known as Jacobi equation, gives the relative acceleration between the reference curve Γ and an infinitesimally arbitrary neighboring curve as measured in H_p. Although the coefficients involving the derivatives of ${}_{\perp}\eta^{\alpha}$ with respect to the parameter can be excessively complicated depending on the choice of coordinates, the second-order differential system is linear with respect to ${}_{\perp}\eta^{\alpha}$.

The definition of η^{α} is such that it connects points in two neighboring curves with the same value of the affine parameter. Therefore, ${}_{\perp}\eta^{\alpha}$ connects points with different values of affine parameter and contrarily to $\dot{\eta}^{\alpha}$ the relative velocity must have a component parallel to v^{μ} as can easily be checked ${}_{\perp}\dot{\eta}^{\alpha}v_{\alpha} = \frac{D}{Ds}({}_{\perp}\eta^{\alpha})v_{\alpha} = -a^{\mu}\eta_{\mu} \neq 0$.

To perform a measurement, an observer must refer the desired quantities with respect to a local Cartesian coordinate axes defined by a tetrad system composed of three orthonormal vectors λ^{μ}_1, λ^{μ}_2, $\lambda^{\mu}_3 \in H_p$ and a unitary vector λ^{μ}_0 parallel to its four-velocity. A necessary condition is that the system of vectors shall be propagated in such a way that it retains its properties so that one can compare a given quantity with respect to this system along the evolution. Note that if the local Cartesian coordinate axes are parallelly propagated then the three vectors λ_1^μ, λ_2^μ, λ_3^μ would in general no longer be orthogonal to v^μ. The proper way to propagate them is then to use the Fermi propagation (see Ref. 7 for a detailed discussion). A vector field $\xi(x)$ is Fermi propagated along the integral curve of v^μ if its Fermi derivative defined by

$$\frac{D_F}{Ds}\xi^\mu = \frac{D}{Ds}\xi^\mu - (\xi_\alpha a^\alpha)v^\mu + (\xi_\alpha v^\alpha)a^\mu$$

is zero. The Fermi derivative has the properties that $\frac{D_F}{Ds} = \frac{D}{Ds}$ only if it is a geodesic, v^μ is Fermi transported ($\frac{D_F}{Ds}v^\mu = 0$) and $\frac{D_F}{Ds}{}_\perp\eta = {}_\perp\frac{D}{Ds}\eta$. Using the Fermi derivative, Eq. (2) can be written as

$$\frac{D_F^2}{Ds^2}{}_\perp\eta^\alpha = \left(R^\alpha{}_{\mu\beta\nu}v^\mu v^\nu + a_{\mu;\nu}h^{\alpha\mu}h^\nu{}_\beta - a^\alpha a_\beta\right){}_\perp\eta^\beta\,. \tag{3}$$

Projecting this equation in the tetrad system and defining the deviation vector projection as $r^i = \lambda^i_\alpha{}_\perp\eta^\alpha$, one finds

$$\frac{d^2r^i}{ds^2} = \left(R^i{}_{0j0} + a^i{}_{;j} - a^i a_j\right)r^j\,. \tag{4}$$

In the above equation $i = 1, 2, 3$. Note that since ${}_\perp\eta^\alpha v_\alpha = 0$, the deviation vector will have only components with respect to the spatial tetrad vectors.

2.1. *Special case of geodesics*

Our main focus is the study of accelerated congruences but it might be helpful to remark some peculiarities of geodesic curves to stress the distinction between the two situations. In a geodesic flow, the acceleration a^μ vanishes and one immediately recovers the well-known geodesic deviation equation which depends only on the curvature

$$\frac{D^2\eta^\alpha}{Ds^2} = R^\alpha{}_{\mu\beta\nu}v^\mu v^\nu\eta^\beta\,. \tag{5}$$

The Jacobi equation then determines the deviation properties of a linearized geodesic flow, where both the relative positions and velocities of the near geodesics are, by hypothesis, infinitesimal.

In general, it is always possible to choose the origin of the affine parameter such that in a given point the deviation vector is perpendicular to the velocity ($\eta^\alpha v_\alpha = 0$). In a geodesic congruence the parallel propagation preserves this relation, i.e. the deviation vector remains always restricted to H_p ($\eta^\alpha = {}_\perp\eta^\alpha$). Furthermore, the covariant derivative commutes with projection

$${}_\perp\frac{D}{Ds}{}_\perp\frac{D}{Ds}{}_\perp\eta^\alpha = \frac{D^2}{Ds^2}{}_\perp\eta^\alpha = \frac{D^2}{Ds^2}\eta^\alpha\,.$$

Thus, Eq. (2) is identical to Eq. (1). Note that the geometric structure of the gravitational tidal forces in empty space is completely determined by the algebraic symmetries of the Weyl tensor (see Appendix for definitions). In particular, using the Fermi transported tetrad field the above equation becomes

$$\frac{d^2\eta^i}{ds^2} = \mathcal{E}^i{}_j\eta^j \quad i,j = 1,2,3\,, \tag{6}$$

where $\mathcal{E}^i{}_j$ represents the electric part of the Weyl tensor[d] along Γ. For a given worldline, $\mathcal{E}^i{}_j$ is a function only of "s". Thus, for a given value of the parameter, the relative acceleration field is determined by the properties of $\mathcal{E}_{ij}$. Synge was the first to show that Eq. (6) may be written in a Hamiltonian form (see, for instance Ref. 38). In particular, he showed by means of integral invariants that the gravitational tidal forces is necessarily irrotational. This result is valid not only in vacuum but also in the presence of matter. One may interpret this result as a natural manifestation of geometry. In a geometric framework of gravity, the Jacobi equation forbids the presence of tangential relative accelerations. We will see that this is not the case in electrodynamics.

3. Electrodynamic Tidal Forces

The linearity of Jacobi equation with respect to ${}_\perp\eta^\alpha$ allows us to interpret the evolution of the deviation vector as a map of H_p into H_p in the following way. To each vector ${}_\perp\eta^\alpha$ of the rest space H_p it is assigned another vector $a^\alpha_{\rm rel}$. The set of all objects $a^\alpha_{\rm rel}({}_\perp\eta)$ forms a spatial vector field in H_p with specific geometrical properties determined by the algebraic structure of the matrix connecting ${}_\perp\eta^\beta$ to $a^\alpha_{\rm rel}$.

We have seen that in vacuum the gravitational contribution to the deviation equation is given by the electric part of the Weyl tensor, which is symmetric and traceless. The other two terms in (4) are related to acceleration, and since gravitational contribution is already well known, we shall concentrate on these two extra terms, in particular, to tidal forces generated by electromagnetic interaction.

Consider an accelerated congruence describing a cloud of test charged particles in a given electromagnetic field. The equation of motion of each individual particle is given by the Lorentz force, i.e.

$$a^\mu = \frac{q}{m}F^\mu{}_\nu v^\nu\,. \tag{7}$$

We will assume that all particles have the same physical properties, i.e. the same charge/mass ratio. Different charge/mass ratio will only rescale the field strength without changing the kinematics of the tidal forces. With respect to v^μ, the Faraday tensor can be decomposed as (see App. A)

$$F^{\mu\nu} = E^{[\mu}v^{\nu]} + \eta^{\mu\nu}{}_{\alpha\beta}v^\alpha H^\beta\,. \tag{8}$$

[d]Note that the electric part of Weyl tensor is completely orthogonal to v^μ.

Thus, the acceleration felt by each particle is nothing but the electric field measured in its rest space, i.e. $a^\mu = q/mE^\mu$. Substituting in (3) we obtain

$$a^\alpha_{\rm rel} = M^\alpha{}_{\beta\perp}\eta^\beta\,, \tag{9}$$

where the connecting matrix $M^\alpha{}_\beta$ is given by

$$M^\alpha{}_\beta = \left[\left(\frac{q}{m}\right)h^{\alpha\mu}h_\beta{}^\nu E_{\mu;\nu} - \left(\frac{q}{m}\right)^2 E^\alpha E_\beta\right]. \tag{10}$$

There are two distinct types of contribution to the relative acceleration. The first includes the projected gradients of the electric field while the latter is purely algebraic and quadratic in this field. Note that the last term involves quadratically the parameter q/m and, consequently, its contribution to the relative acceleration is invariant under the charge inversion map $q \to -q$, which is not the case for the first term.

It may seem unexpected that relative accelerations in electrodynamics (10) include both the fields and their derivatives, inasmuch in Newtonian theory appears only first derivatives of the gravitational field $\vec{g}(x)$ in ordinary ocean tides. Nevertheless, General Relativity has exactly the same structure. Recall that the Riemann tensor

$$R^\alpha{}_{\beta\mu\nu} = -\Gamma^\alpha{}_{\beta[\mu,\nu]} + \Gamma^\alpha{}_{\epsilon[\mu}\Gamma^\epsilon{}_{\nu]\beta}\,, \tag{11}$$

involves both derivatives and algebraic terms in the "pseudoforce" $\Gamma^\alpha{}_{\mu\nu}$. In fact, the above equation reveals that both situations (electromagnetic and gravitational) are quite similar: the derivatives appear only linearly, whereas the algebraic terms appear quadratically. Hence, if the inertial and gravitational masses were not precisely equal, an additional parameter would also appears just like in the electromagnetic case. Only in the weak field approximation of both fields the gradient-like terms stand alone.

The fact that electromagnetic "tidal forces" involve similar terms as in its gravitational analogue does not mean that they have the same geometrical properties. In fact, as we will see, in electrodynamics the relative force field presents a more sophisticated structure. This can be seen by studying the properties of (10). The matrix $M^\alpha{}_\beta$ can be separated in three components related to its irreducible parts, each one of them contributing in a different manner to the relative accelerations. Without loss of generality, we fix $q/m = 1$ and set

$$M_{\alpha\beta} \equiv \frac{T}{3}h_{\alpha\beta} + X_{\alpha\beta} + W_{\alpha\beta}\,, \tag{12}$$

with

$$T \equiv h^{\mu\nu}M_{\mu\nu} = E^\mu{}_{;\mu}\,, \tag{13}$$

$$X_{\alpha\beta} \equiv \frac{1}{2}h^\mu{}_{(\alpha}h^\nu{}_{\beta)}E_{\mu;\nu} - E_\alpha E_\beta - \frac{T}{3}h_{\alpha\beta}\,, \tag{14}$$

$$W_{\alpha\beta} \equiv \frac{1}{2}h^\mu{}_{[\alpha}h^\nu{}_{\beta]}E_{\mu;\nu}\,. \tag{15}$$

As long as $M_{\alpha\beta}$ is already in the rest space H_p, there is only its trace T, the traceless symmetric part $X_{\mu\nu} = X_{\nu\mu}$, $X^{\mu}{}_{\mu} = 0$ and its antisymmetric component $W_{\mu\nu} = -W_{\nu\mu}$. Thus, given an external field $F_{\alpha\beta}$ and a trajectory satisfying the Lorentz force, the irreducible parts are determined by a straightforward calculation.

Note that at a given point of the reference curve Γ the irreducible parts are not arbitrary but shall be compatible with Maxwell's equations. Nevertheless, even though these equations constrains the system, the field dynamics alone cannot determine completely the relative acceleration at a given point. In the rest frame of the accelerated particle (see App. A), Maxwell's equations in vacuum projected along and perpendicular to v^{μ} read

$$T = -E^{\mu}E_{\mu} + \omega^{\mu}H_{\mu}\,, \tag{16}$$

$$\begin{aligned} W_{\alpha\beta} = {} & \frac{1}{2}h^{\mu}{}_{[\alpha}h^{\nu}{}_{\beta]}\left(F_{\mu}{}^{\lambda}\hat{k}_{\lambda\nu} + E_{\mu}a_{\nu}\right) \\ & - \frac{1}{2}\eta_{\alpha\beta\mu\nu}\left(\dot{H}^{\mu}v^{\nu} + H^{\mu}E^{\nu}\right) \\ & - \frac{1}{2}v_{[\alpha}\eta_{\beta]\lambda\mu\nu}v^{\lambda}H^{\mu}E^{\nu}\,. \end{aligned} \tag{17}$$

Here $v_{\mu;\nu} \equiv a_{\mu}v_{\nu} + \hat{k}_{\mu\nu}$ and the spatial kinematics term is given by $\hat{k}_{\mu\nu} \equiv \theta/3h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}$. The terms $\theta, \sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ represents the usual expansion, shear and vorticity of the congruence.

As it is well known, Maxwell's equations relate the divergence, the rotational and the time derivative of the electric and magnetic fields. Helmholtz theorem guarantees that specifying the divergence and the rotational together with appropriate boundary conditions determines completely a given field in three dimensions. Notwithstanding, one can also modify the shear $X_{\alpha\beta}$ of a given field without altering its rotational and its divergence.

Consider two distinct fields $\vec{E}_1$ and $\vec{E}_2$ that have the same divergence, rotational and time derivative but different shears. These two different fields will generate the same Maxwell's equations. It is clear that they shall have different boundary conditions. It is worth noting that if the projected gradient is defined with respect to an inertial observer then the trace T and the antisymmetrical part $W_{\alpha\beta}$ of the two field $\vec{E}_1$ and $\vec{E}_2$ are identical but not the symmetric part $X_{\mu\nu}$. Hence, the symmetrical part $X_{\mu\nu}$ has a nonlocal contribution in the sense that it completes the specification of the field by including the dependence on the boundary conditions.

Note that a similar situation also appears in gravitational tidal forces. In vacuum, gravitation is described by the Weyl tensor $W_{\alpha\beta\mu\nu}$ which represents the nonlocal part of the gravitational interaction. The projection of this object in the worldline of an observer gives rise to the electric part $E_{\alpha\beta}$. The fact that the electric part shares the same symmetries with $X_{\mu\nu}$ and also represents a nonlocal component seems an interesting coincidence between these theories.

4. Algebraic Properties of Electromagnetic Tides

We shall study the algebraic properties of the electromagnetic tidal forces by analyzing the irreducible parts of $M^{\alpha}{}_{\beta}$. We will define for future use the matrix versions of these three-dimensional tensors as: $\mathbf{X} = X^{\alpha}{}_{\beta}$, $\mathbf{W} = W^{\alpha}{}_{\beta}$, $\mathbf{h} = h^{\alpha}{}_{\beta}$ and the identity matrix $\delta^{\alpha}{}_{\beta}$ by $\mathbf{1}$.

At a point p of space–time, a given $\binom{1}{1}$ tensor $\mathbf{T}$ can be thought as a linear map of the tangent space T_p onto itself. The principal directions of this map and its correspondent eigenvalues are given by[39–41]

$$T^{\alpha}{}_{\beta}\xi^{\beta} = \lambda\xi^{\alpha}\,, \tag{18}$$

where λ is a scalar and ξ^{α} its eigenvector. A fourth order characteristic polynomial for λ is obtained by the condition $\det(\lambda\mathbf{1} - \mathbf{T}) = 0$. In particular, when the four-dimensional tensor $\mathbf{T}$ is completely orthogonal to a given vector v^{μ}, its determinant vanishes identically, i.e. $\det(\mathbf{T}) = 0$. In this case the characteristic polynomial has the form

$$p(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0\,, \tag{19}$$

where the coefficients a_n $(n = 1, 2, 3)$ are functions of the scalar invariants built with traces of powers of $\mathbf{T}$:

$$a_0 = -\frac{1}{6}\big(\mathrm{tr}(\mathbf{T})^3 - 3\,\mathrm{tr}(\mathbf{T}^2)\,\mathrm{tr}(\mathbf{T}) + 2\,\mathrm{tr}(\mathbf{T}^3)\big)\,, \tag{20}$$

$$a_1 = \frac{1}{2}\big(\mathrm{tr}(\mathbf{T})^2 - \mathrm{tr}(\mathbf{T}^2)\big)\,, \tag{21}$$

$$a_2 = -\,\mathrm{tr}(\mathbf{T})\,. \tag{22}$$

A straightforward application of these formula to the irreducible parts (12) leads to the following situations.

(i) *Trace part of* M: $(T/3)\mathbf{h}$

The interpretation of this object is immediate since every vector of the three-space H_p is itself an eigenvector of the operator $(T/3)\mathbf{h}$. As long as $\mathbf{h}$ is a projector, it follows that $\mathbf{h} = \mathbf{h}^n$ for $n \in \mathbb{N}$. Thus, from the characteristic polynomial one immediately obtains a single eigenvalue $\lambda = T/3$. Equations (9) and (12) show that the relative acceleration provided by the trace part of $\mathbf{M}$ is entirely analogous to the acceleration in a central force problem. Through a field line representation, it is immediate to see that this type of electromagnetic tides generate radial lines of force similarly to a charged particle at rest, since $a^{\alpha}_{\mathrm{rel}}$ is always proportional to a given $_{\perp}\eta^{\alpha}$ in the rest space H_p. But note that in this case the relative acceleration increases with the distance.

In Fermi transported basis (4) along Γ, for a given value of the parameter s_0, one obtains

$$\delta^{k}{}_{i}\frac{\partial a^{i}_{\mathrm{rel}}}{\partial r^{k}} = T(s_0)\,, \tag{23}$$

which can be represented also by the integral relation in the rest space H_p

$$\oint \vec{a}_{\mathrm{rel}} \cdot d\vec{\sigma} = T(s_0) \int d^3 r \,. \tag{24}$$

The integral on the left-hand side should be performed around a closed surface with infinitesimal area element given by $d\vec{\sigma}$ in the rest space and the other on the right-hand side gives the Euclidean volume enclosed by this area. The vaccum solution to Einstein's equation could have an analogue behavior due to a nonzero cosmological constant. However, since it must be negligible for noncosmological systems, we always consider a zero cosmological constant in the calculation of ordinary gravitational tidal forces. Thus, the above effect should appears only in electrodynamic tides.

(ii) *Traceless symmetric part of* M: X

It is well known, as discussed in Subsec. 2.1, that the net effect of a gravitational field in vacuum is to produce strain on material bodies, which tend to be deformed while keeping their volume intact. The same effect can be found in the electromagnetic analogue by means of the $\mathbf{X}$ tensor.

As long as $\mathrm{tr}(\mathbf{X}) = 0$, the characteristic polynomial (19) reduces to

$$\lambda^3 - \frac{\lambda}{2}\,\mathrm{tr}(\mathbf{X}^2) - \frac{1}{3}\,\mathrm{tr}(\mathbf{X}^3) = 0 \,. \tag{25}$$

In principle one can solve the above third-order equation and find the possible eigenvalues explicitly in terms of the Lorentz invariant quantities $\mathrm{tr}(\mathbf{X}^2)$ and $\mathrm{tr}(\mathbf{X}^3)$. Note that, given that $\mathbf{X}$ is a symmetrical matrix, all its eigenvalues λ_1, λ_2 and λ_3 are real and it is always possible to diagonalize it. Furthermore, since $\mathbf{X}$ is traceless, one of its eigenvalues can be written as a combination of the others, for instance $\lambda_3 = -(\lambda_1 + \lambda_2)$, which gives the relations

$$\frac{1}{2}\,\mathrm{tr}(\mathbf{X}^2) = (\lambda_1 + \lambda_2)^2 - \lambda_1\lambda_2 \quad \text{and} \quad \frac{1}{3}\,\mathrm{tr}(\mathbf{X}^3) = \lambda_1\lambda_2(\lambda_1 + \lambda_2) \,. \tag{26}$$

Thus, given a background electromagnetic field $F_{\mu\nu}$ and a trajectory Γ it is possible to choose a frame in which $v^\mu = \delta^\mu{}_0$ such that

$$X^\alpha{}_\beta = \mathrm{diag}(0, \lambda_1, \lambda_2, -(\lambda_1 + \lambda_2)) \,, \tag{27}$$

with λ_1 and λ_2 given implicitly by (26). The traceless property of $\mathbf{X}$ implies that field lines will converge along the directions defined by two of the eigenvectors and diverge along the third one, or vice versa, as shown in Fig. 1. This saddle-point-like structure of the field lines is exactly the type of tidal configuration expected to be measured in ordinary gravitational wave detectors, such as LIGO, Virgo or any other.

We would like to stress that the above configuration is the only type of relative acceleration present in gravity in vacuum. In general relativity, this is an immediate consequence of field geometrization combined with the geodesic hypothesis. The

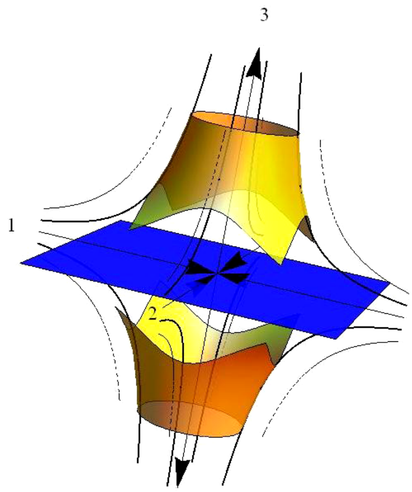

Fig. 1. The saddle-point-like structure of the field lines considering two negatives and one positive eigenvalues. Hence, the fields lines converge along the directions defined by the two eigenvectors associated to negative eigenvalues and diverge along the direction of the third eigenvector.

algebraically similarity of the two situations could persuade someone to wonder if it is possible or not to construct a particular electromagnetic configuration along the trajectory of a charged particle that could mimics gravity. Certainly, this interesting nontrivial question will depend on the properties of $\mathbf{X}$ along the same worldline.

(iii) *Antisymmetric part of* $\mathbf{M}$: $\mathbf{W}$

The eigenvalues of a skew-symmetric matrix can only be pairs of complex conjugate numbers λ, λ^* or zero. Furthermore, for any real skew-symmetric matrix $(\mathbf{A}^n)_{ij} = (-1)^n \mathbf{A}_{ji}$ for $n \in \mathbb{N}$, which implies that $\mathrm{tr}(\mathbf{A}^n) = 0$ for n odd. Thus, the coefficients a_0 and a_2 of the characteristic polynomial equation (19) vanish identically giving

$$\lambda(\lambda^2 - \mathrm{tr}(\mathbf{W}^2)) = 0\,. \tag{28}$$

Given that $\mathrm{tr}(\mathbf{W}^2) < 0$ we obtain three eigenvalues, $\lambda_0 = 0$ and $\lambda\pm = \pm i\sqrt{|\,\mathrm{tr}(\mathbf{W}^2)|}$. The net effect of this type of acceleration can be seen by representing again the deviation equation in Fermi coordinates. We obtain

$$\frac{\partial a^i_{\mathrm{rel}}}{\partial r^k} = W^i{}_k \to \epsilon^{jk}{}_i \frac{\partial a^i_{\mathrm{rel}}}{\partial r^k} = W^j\,, \tag{29}$$

where $W^j \equiv \epsilon^{jk}{}_i W^i{}_k$ represents the rotational of the acceleration field. Thus, in a given point of Γ, the rest space integral relation reads

$$\oint \vec{a}_{\mathrm{rel}} \cdot d\vec{l} = \int \vec{W}(s_0) \cdot d\vec{\sigma}\,, \tag{30}$$

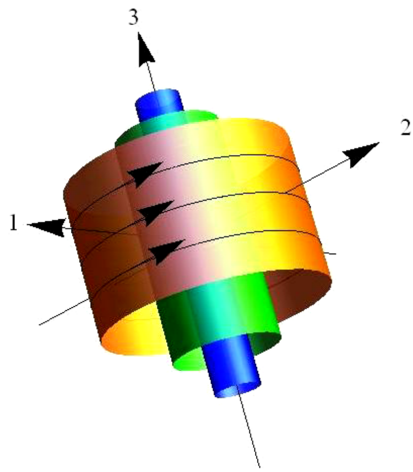

Fig. 2. Local field lines determined by the antisymmetric term $W^{i}_{\ j}$. The concentric cylinders are surfaces of constant relative acceleration intensity. The field strength increases with cylinder radius and the third axis direction coincides with the vector $W^j \equiv \epsilon^{jk}{}_{i} W^{i}{}_{k}$.

where $d\vec{l}$ is an infinitesimal length along a closed line and $d\vec{\sigma}$ is an area element. Thus, as shown in Fig. 2, the antisymmetric part determines a local acceleration field that tends to give rise to tangential forces on material charged bodies. The axis of rotation coincides with the vector W^j. This behavior also only appears due to the nature of the electromagnetic field. As was previously discussed, in the usual geometrical description of gravitation this rotational-type tidal is prohibited by Jacobi equation in vacuum.[38]

5. The Kinematic Restriction and the Gradient-like Limit

So far, we have discussed exact properties of electromagnetic tidal forces. In our previous analysis, we have not assumed any particular configuration of the external field $F_{\mu\nu}$, of the kinematics of charged particles congruence $v_{\mu;\nu}$ or on the magnitude of the charge–mass ratio q/m. In this section we shall make some hypothesis concerning these quantities, which also allow us to recover some results from previous works.

First of all, let us note that the algebraic term $E^\alpha E_\beta$ in (10) has a very peculiar property. As discussed in Sec. 3, the relative acceleration provided by this term is invariant under charge inversion $q \to -q$. This property implies that one can directly measure its effect by considering not one but two equivalent congruences with opposite charges. Since the gradient-like term in (10) inverts its sign under this map, one only needs to perform two experiments with opposite charges and combine their results for a given $_\perp\eta^\alpha$ to obtain the desired term. Thus, it is possible to distinguish experimentally between the algebraic term and the gradient-like term. Note also that the algebraic term appears only due to the noncommutativity between covariant derivative and projection for accelerated curves

$$_\perp\frac{D}{Ds}{}_\perp\frac{D}{Ds}{}_\perp\eta^\alpha \neq \frac{D^2}{Ds^2}{}_\perp\eta^\alpha\,.$$

Another point to consider is that the electric and magnetic fields are defined as projections of the Faraday tensor $F_{\alpha\beta}$ along the observer worldline. In this sense, E_μ and H_μ are observer dependent. So far, we have always considered observers co-moving with the charged test particles. Thus, in an arbitrary configuration of external fields, the kinematics of this congruence can become excessively complicated.

In the electromagnetic context, the kinematic properties of v^μ, i.e. $v_{\mu;\nu} = a_\mu v_\nu + \hat{k}_{\mu\nu}$, appear explicitly in its tidal forces. This is an immediate consequence of relations (13)–(15) and the definition of E_α. Nevertheless, one can always choose in a given 3-spacelike surface Σ initial conditions $v^\mu(\Sigma)$ such that $\hat{k}_{\alpha\beta}(\Sigma) = 0$. Evidently, due to accelerations, the dynamical evolution will not preserve this condition but it seems reasonable to suppose that it might be approximately valid at least in a small but finite amount of time. In this regime, one obtains the relation $h^{\alpha\mu}h_\beta{}^\nu E_{\mu;\nu} \approx h^{\alpha\mu}h_\beta{}^\nu F_{\mu\lambda;\nu}v^\lambda$. This is the basic assumption made in Ref. 37 to construct an analogy between gravitation and electrodynamics. Finally, if we consider a small charge–mass ratio, i.e. $|q/m| \ll 1$, it is immediate to see that

$$T \to \mathcal{O}\left(\left|\frac{q}{m}\right|^2\right), \tag{31}$$

$$X_{ij} \to \frac{1}{2}(E_{i;j} + E_{j;i}) + \mathcal{O}\left(\left|\frac{q}{m}\right|^2\right), \tag{32}$$

$$W_{ij} \to \frac{1}{2}(E_{i;j} - E_{j;i}) + \mathcal{O}\left(\left|\frac{q}{m}\right|^2\right), \tag{33}$$

which show that in first order the trace part of $\mathbf{M}$ vanishes and the other components are entirely described by derivatives of the electric field as measured in the rest space H_p of the fiducial particle. Thus, the small charge–mass ratio is a gradient-like regime in the sense that the tidal forces are completely described by the gradients of the Faraday tensor.

6. Conclusions

The main purpose of the present work was to study electromagnetic tidal fields acting on charged test particles. Therefore, it seems unavoidable to consider the evolution of accelerated congruences. In the first part, we performed a detailed analysis of the kinematics and the physical properties of accelerated congruences. As a way to stress their peculiarities, we have also analyzed the geodesic case, which is the common gravitational situation.

The algebraic structure of electromagnetic tidal forces are richer than in gravitation. While in gravity appears only one symmetrical traceless tensor field that imprints strain on material bodies, in the electrodynamic case there is all possible irreducible parts **T**, **X** and **W** of the tensor field **M**, which is responsible for the tidal effects. Their effects were considered separately as a map of H_p into H_p that assign to each projected deviation vector ${}_{\perp}\eta^{\beta}$ another vector $a^{\alpha}_{\rm rel}$ in H_p. This map is characterized by the eigenvalues and eigenvectors associated to **T**, **X** and **W**.

In Sec. 5, we have studied the irreducible parts of **M** from a different perspective. Analyzing their physical significance, it was shown that it is possible to separate their contribution by inverting the sign of the charge of the test particles. Furthermore, in the weak charge–mass limit $q/m \ll 1$, tidal effects depends only on derivatives of the electrical field in the rest space of the particle.

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Appendix A. Some Mathematical Machinery

In terms of the Faraday tensor $F^{\mu\nu}$ and its dual $F^{*}_{\mu\nu} \equiv \frac{1}{2}\eta_{\mu\nu\varepsilon\lambda}F^{\varepsilon\lambda}$, Maxwell's equations are given by

$$F^{\mu\nu}{}_{;\nu} = j^{\mu}\,, \tag{A.1}$$

$$\overset{*}{F}{}^{\mu\nu}{}_{;\nu} = 0 \Rightarrow F^{[\alpha\beta;\lambda]} = 0\,. \tag{A.2}$$

In this context, we understand the electric and magnetic fields as projections of the $F^{\mu\nu}$ tensor along the observer worldline v^{ν}, i.e. $E^{\mu} \equiv F^{\mu}{}_{\nu}v^{\nu}$ and $H^{\mu} \equiv \frac{1}{2}\eta^{\mu}{}_{\varepsilon\alpha\beta}v^{\varepsilon}F^{\alpha\beta}$, or inversely $F^{\mu\nu} \equiv E^{[\mu}v^{\nu]} - \eta^{\mu\nu}{}_{\alpha\beta}H^{\alpha}v^{\beta}$. To reobtain the original form of Maxwell's equations it suffices to project Eqs. (A.1), (A.2) along and perpendicularly to the timelike congruence v^{μ}. In particular, in vacuum, the projection along v^{μ} gives

$$\begin{aligned} F^{\mu\nu}{}_{;\nu}v_{\mu} &= (F^{\mu\nu}v_{\mu})_{;\nu} - F^{\mu\nu}v_{\mu;\nu} \\ &= -E^{\mu}{}_{;\mu} - E^{\mu}E_{\mu} - F^{\mu\nu}w_{\mu\nu} = 0 \\ &\Rightarrow E^{\mu}{}_{;\mu} = -E^{\mu}E_{\mu} + w^{\mu}H_{\mu}\,, \end{aligned} \tag{A.3}$$

where we have defined the vector $w^{\mu} \equiv \eta^{\mu}{}_{\nu\alpha\beta}v^{\nu}w^{\alpha\beta}$. The other projection becomes easier if we contract Eq. (A.2) with $\eta_{\alpha\mu\lambda\nu}v^{\lambda}$ before projecting the two remaining indexes. Thus,

$$\begin{aligned} &(F_{\mu\lambda;\nu} + F_{\nu\mu;\lambda} + F_{\lambda\nu;\mu})v^{\lambda}h^{\mu}{}_{\alpha}h^{\nu}{}_{\beta} \\ &\quad = \Big(\big(F_{\mu\lambda}v^{\lambda}\big)_{;\nu} - F_{\mu\lambda}v^{\lambda}{}_{;\nu} - \dot{F}_{\mu\nu} - \big(F_{\nu\lambda}v^{\lambda}\big)_{;\nu} + F_{\nu\lambda}v^{\lambda}{}_{;\nu}\Big)h^{\mu}{}_{\alpha}h^{\nu}{}_{\beta} = 0 \\ &\quad = h^{\mu}{}_{[\alpha}h^{\nu}{}_{\beta]}\big(E_{\mu;\nu} - F_{\mu}{}^{\lambda}\hat{k}_{\lambda\nu} - E_{\mu}a_{\nu}\big) + h^{\mu}{}_{\alpha}h^{\nu}{}_{\beta}\eta_{\mu\nu\rho\tau}\big(\dot{H}^{\rho}v^{\tau} + H^{\rho}a^{\tau}\big) \end{aligned}$$

$$\Rightarrow \frac{1}{2}h^{\mu}{}_{[\alpha}h^{\nu}{}_{\beta]}E_{\mu;\nu}$$
$$= \frac{1}{2}h^{\mu}{}_{[\alpha}h^{\nu}{}_{\beta]}\left(F_{\mu}{}^{\lambda}\hat{k}_{\lambda\nu} + E_{\mu}a_{\nu} - \frac{1}{2}\eta_{\mu\nu\rho\tau}\left(\dot{H}^{\rho}v^{\tau} + H^{\rho}a^{\tau}\right)\right), \qquad \text{(A.4)}$$

where $v_{\mu;\nu} \equiv a_{\mu}v_{\nu} + \hat{k}_{\mu\nu}$ with $\hat{k}_{\mu\nu} = \frac{\theta}{3}h_{\mu\nu} + \sigma_{\mu\nu} + w_{\mu\nu}$.

References

1. V. D. Zakharov, *Gravitational Waves in Einstein's Theory* (Israel Program for Scientific Translations, 1973).
2. I. Ciufolini and J. A. Wheeler, *Gravitation and Inertia* (Princeton University Press, 1995).
3. S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, 1973).
4. R. Feynman, *Lectures on Gravitation* (Addison-Wesley, 1995).
5. R. Ruffini and C. Sigismondi (eds.), *Nonlinear Gravitodynamics*: *The Lense-Thirring Effect* (A documentary introduction to current research) (World Scientific, 2003).
6. B. Mashhoon, Gravitoelectromagnetism: A brief review, in *Measuring Gravitomagnetism*: *A Challenging Enterprise*, ed. L. Iorio (Nova, 2007), pp. 29–39.
7. R. T. Jantzen, P. Carini and D. Bini, *Ann. Phys.* **215**, 1 (1992).
8. V. B. Braginsky, C. M. Caves and K. S. Thorne, *Phys. Rev. D* **15**, 2051 (1977).
9. E. G. Harris, *Am. J. Phys.* **59**, 5 (1991).
10. R. L. Forward, *Proc. IRE* **49**, 892 (1961).
11. H. C. Ohanian and R. Ruffini, *Gravitation and Spacetime*, 2nd edn. (W. W. Norton & Company, 1994).
12. I. Ciufolini, *Nature* **449**, 41 (2007).
13. I. Ciufolini and E. C. Pavlis, *Nature* **31**, 958 (2004).
14. I. Ciufolini and E. C. Pavlis, *New Astron.* **10**, 636 (2005).
15. B. Mashhoon, *Phys. Lett. A* **139**, 103 (1989).
16. B. Mashhoon, *Phys. Lett. A* **173**, 347 (1993).
17. L. Iorio, *New Astron.* **10**, 603 (2005).
18. C. M. Will, *Phys. Rev. D* **67**, 062003 (2003).
19. S. M. Kopeikin, *Int. J. Mod. Phys. D* **15**, 305 (2006).
20. K. Nodtvedt, *Int. J. Theor. Phys.* **27**, 1395 (1988).
21. R. Maartens and B. A. Bassett, *Class. Quantum Grav.* **15**, 705 (1998).
22. A. Papapetrou, *Proc. R. Soc. London, Ser. A* **209**, 248 (1951).
23. E. Corinaldesi and A. Papapetrou, *Proc. R. Soc. London, Ser. A* **209**, 259 (1951).
24. A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics* (W. A. Benjamin, 1967).
25. A. Lichnerowicz, *Ann. Mat. Pura Appl.* **50**, 1 (1960).
26. Y. Choquet-Bruhat, C. Dewitt-Morette and M. Dillard-Bleick, *Analysis, Manifolds and Physics*, Revised edition (Elsevier, 1996).
27. L. Bel, *Gen. Relativ. Gravit.* **32**, 2047 (2000).
28. B. Mashhoon, J. C. McClune and H. Quevedo, *Class. Quantum Grav.* **16**, 1137 (1999).
29. B. Mashhoon, J. C. McClune and H. Quevedo, *Phys. Lett. A* **231**, 47 (1997).
30. P. Jordan, J. Ehlers and W. Kundt, *Abh. Math. – Naturw. Kl. Akad. Wiss. Mainz* **2** (1960).
31. W. Kundt and M. Trumper, *Abh. Math. Nat. Kl.* **12** (1960).

32. G. F. R. Ellis, General relativity and cosmology, in *Proc. Int. School of Physics, Enrico Fermi XLVII* (Academic, London, 1971), p. 104.
33. M. Novello and J. M. Salim, *Fundam. Cosmic Phys.* **8**, 201 (1983).
34. S. W. Hawking, *Astrophys. J.* **145**, 544 (1966).
35. A. Matte, *Can. J. Math.* **5**, 1 (1953).
36. L. F. Costa and C. Herdeiro, *Phys. Rev. D* **78**, 024021 (2008).
37. E. Goulart and F. T. Falciano, *Int. J. Mod. Phys. A* **24**, 4589 (2009).
38. F. A. Pirani, *Gen. Relativ. Gravit.* **41**, 1215 (2009) [Republication of: On the physical significance of the Riemann tensor].
39. G. S. Hall and D. A. Negm, *Int. J. Theor. Phys.* **25**, 405 (1986).
40. C. Bona, B. Coll and J. A. Morales, *J. Math. Phys.* **33**, 670 (1992).
41. G. S. Hall, *J. Phys. A*: *Math. Gen.* **9**, 541 (1976).