Physica A 390 (2011) 587–594

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Nonadditive Tsallis entropy applied to the Earth's climate

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ARTICLE INFO

Article history:
Received 14 September 2010
Available online 3 November 2010

Keywords:
Nonadditive Tsallis entropy
Earth's climate
Complex systems
Time series

ABSTRACT

The concepts of nonextensive statistics, which has been applied in the study of complex systems, are used to analyze past records of the Earth's climate. The fluctuations within the record of deuterium content (hence temperature) in the last glacial period appear to follow a q-Gaussian distribution. Analyses of the time-dependent nonadditive entropy indicate transitions between different complexity levels in the data prior to the abrupt change in the system dynamics at the end of the last glaciation. Different fluctuation regimens are evidenced through wavelets analysis. It is also suggested that time-dependent entropy analysis could be useful for indicating the approach to a critical transition of the Earth's climate for which theoretical models are in many cases not available.

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1. Introduction

Nonlinear systems are present in different areas of knowledge, like ecology, medicine, economics and engineering. Within these disciplines, and others, chaotic dynamics and complex behavior usually appear, leading to the emergence of new features and states with high levels of organization. In these systems, the existence of thresholds and that of crossovers are common features. Examples have been detected in epileptic seizures in medicine, financial crashes, catastrophic shifts in ecosystems, to name but a few [1–3]. Analyzing the different levels of complexity is essential to understanding the system's dynamics and to predicting its behavior using evidence such as fingerprints, or early-warning signals, of critical changes in the time series [4–9].

Critical transitions or abrupt changes in the past temperature records of the Earth are among the most striking examples of complexity emergence [9–11]. On the other hand, for the Earth's climate, like for many other natural systems, the models determining the dynamics of the system are unknown, and information is usually obtained from the analysis of highly nonlinear time series (TS) [4,11]. Entropy analysis is one of the complexity measurements or statistical tools that can provide information about the dynamics of a specific system. This analysis has been used to separate healthy from pathological physiological time series [12], in the quantification of spatio-temporal dynamics of the human brain [13], and in similar matters [14].

On the other hand, natural systems, like the Earth's climate, usually have non-equilibrium stationary states whose dynamics can, in some cases, be conveniently addressed through nonextensive statistical mechanics [15]. Indeed, Tsallis statistics has been satisfactorily applied to describe changes in complexity of various systems, such as electroencephalograms [16], detection of different levels of organization (complexities) between periods of intense and normal magnetic activity in the Earth's magnetosphere [17], description of the number of citations by the Institute of Scientific Information [18], and many other recent works [19–24].

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doi:10.1016/j.physa.2010.10.045

The present work uses the nonadditive entropy, on which nonextensive statistical mechanics is based, for analyzing the climate fluctuations in past deuterium records corresponding to the last glacial period, which characterize the climate variability history. The principal goal here is to show that the nonextensive Tsallis statistics allows for a convenient analysis of the climate. Consistently we will identify different complexity levels across the time series of past deuterium content fluctuations, pointing out the suitability of the Tsallis entropy for describing climate transformations.

2. Tsallis entropy

The nonadditive Tsallis entropy is determined by

$$S_q \equiv k \sum_{1}^{W} p_i \ln_q \left(\frac{1}{p_i}\right) \equiv \frac{k}{q-1}\left(1 - \sum_{1}^{W} p_i^q\right). \tag{1}$$

In Eq. (1), $\ln_q(x) = \frac{x^{1-q}-1}{1-q}$ is the q-logarithm function. S_q according to Eq. (1) is a generalization of the Boltzmann–Gibbs (BG) entropy $\left(S_{\mathrm{BG}} = -k_B \sum_{i=1}^{W} p_i \log p_i\right)$ [25], and it is essentially extensive when applied to nonlinear scale-invariant systems with long range correlations [25,26]. In the last expression, p_i is the probability of a microscopic configuration, W being the number of microstates (configurations) and the parameter $q \in \Re$ characterizes the nonextensivity of the system. For $q = 1$ it reproduces the BG result, while it is sub-extensive for $q > 1$ and super-extensive for $q < 1$.

Extremization of the continuous version of S_q with appropriate constraints leads to the q-Gaussian distribution which is based on the q-exponential function: $e_q(x) \equiv (1 + (1-q)x)^{\frac{1}{1-q}}$ whenever the argument is positive, and zero otherwise. Note that for $q = 1$ the e_q function reduces to the common $\exp(x)$ function. The q-Gaussian distributions can be written using the q-exponential as follows:

$$G_q(x) = A_q \left(1 + \frac{(q-1)x^2}{(3-q)\sigma_q}\right)^{\frac{1}{1-q}} \tag{2}$$

where σ_q is the q-generalized variance (a convenient characterization of the width of the distribution), and A_q is a normalization constant with values

$$A_q = \begin{cases} \dfrac{\Gamma\left(\frac{5-3q}{(2-2q)}\right)}{\Gamma\left(\frac{2-q}{(1-q)}\right)} \sqrt{\dfrac{(1-q)}{\pi(3-q)\sigma_q}} & q < 1 \\ \sqrt{\dfrac{1}{\pi(3-q)\sigma_q}} & q = 1 \\ \dfrac{\Gamma\left(\frac{1}{(q-1)}\right)}{\Gamma\left(\frac{3-q}{(2q-2)}\right)} \sqrt{\dfrac{(q-1)}{\pi(3-q)\sigma_q}} & q > 1. \end{cases} \tag{3}$$

3. Earth's climate and complexity

The Earth's climate is a natural complex system with many variables and feedback mechanisms leading to emerging properties that are not easy to compare with the available theoretical models. Time series of deuterium content (which characterizes the temperature) variations show clear evidence of abrupt changes which have been taken as critical thresholds or tipping points [11]. Examples are the greenhouse–icehouse transition [27], the abrupt change at the end of the Younger Dryas cold period [28], the end of glaciations [29], among others [4].

In ecosystems, through the study of fluctuations, it is possible to identify regularities in the time series data that may throw some light on the mechanisms involved [30,31]. Fluctuations analysis is also crucial to the *critical slowing down* phenomenon which consists in the fact that the necessary time for returning to equilibrium increases, after a system is perturbed, as the critical point is approached [4]. Following this idea, Kleinen modeled the hemispheric thermohaline circulation system through a simple stochastic model which exhibits changes in the power spectral density as the system moves toward the critical region [32]. Held and Kleinen considered the climate as a system at equilibrium, stochastically perturbed by noise, whose dynamics is determined by the most unstable mode (the critical mode with the smallest decay rate). This mode disappears at the critical threshold causing the fluctuations to follow a 1D autocorrelation (AR-1) process. This idea was satisfactorily tested for the North Atlantic thermohaline circulation system [5]. Also, an increase in the variance of the fluctuations can be taken as a signal of a critical point approaching [33].

Other studies point to non-Gaussian character of the Earth's temperature fluctuations, suggesting that the Earth's climate should be considered as a non-equilibrium system [30]. In this sense, evidences of complexity exist linking the dynamics of the Earth's climate and that of the Sun, where complex events like solar flares, sunspots and others [31] occur. It seems

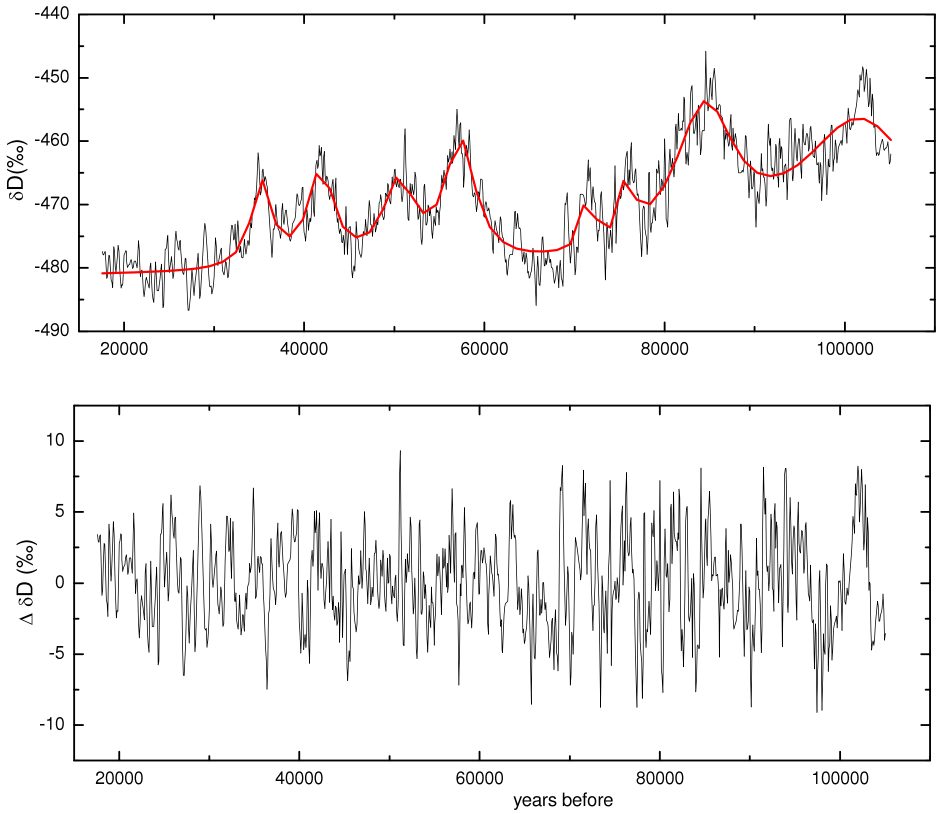

Fig. 1. The upper part is the interpolated time series of deuterium concentrations. The solid line is the fitting function used to obtain the fluctuations shown in the lower part.

that the time intervals of these events (including the fluctuations in the Earth's temperature) follow an inverse power law distribution [31].

4. Results and discussion

The data used in this work were taken from the drillings performed at the *Vostok* station in *East Antarctica* containing time series of different gases during the last glacial–interglacial period (approximately the last 200 kyears) [10,34]. We focused on the deuterium content data for the ice which are taken as proxies of local temperature changes. As the data points are not uniformly spaced, an interpolation algorithm (interp1 function in MATLAB) was used to interpolate the time series with equidistant data. Through this process, the deuterium concentration records were taken uniformly spaced, with about eighty years between points, which is fairly close to the average time step of the original time series.

In order to obtain data fluctuations, the trends in the data were eliminated by fitting them with a multiple-peaks fitting tool. In this method, the principal peaks were selected and then the data were fitted to the sum of multiple Lorentzian functions. Finally, the fit function was subtracted from the interpolated time series producing a fluctuation time series. In Fig. 1 we show the deuterium time series and their respective fluctuations. The solid line represents the fitting function used to remove the trends in the deuterium interpolated TS.

The data obtained from Fig. 1 were binned with a bin width $w = 3.5\,\sigma/n^{1/3}$ (Scott's criterion, with σ being the sample standard deviation and n the time series length). An appropriate look for a possible nonextensive nature of our system can be performed by plotting $\ln_q(P_{\text{norm}}) \equiv [(P_{\text{norm}})^{1-q} - 1]/(1-q)$ (P_{norm} is the normalized probability calculated for each bin where all values of the histogram have been divided by their maximum value) as a function of the square of the fluctuations, i.e. $(\Delta\delta D(‰))^2$. The method is to find the q value which best linearizes $\ln_q(P_{\text{norm}})$ versus $[\Delta\delta D(‰)]^2$. The idea behind it is that if subsystems which generate a complex system with long range correlations are composed, then the extensivity of the system reflects an entropy S_q with a value of $q \neq 1$. Different curves for several values of q were generated and linearly fitted; see Fig. 2. The best linear fit through the R^2 criterion was obtained for $q = 0.93$ as shown in Fig. 2. For comparison, in the same figure we also show two other curves for $q = 0.6$ and $q = 1.2$, both neatly departing from linearity. The inset shows the R^2 parameter obtained from the entire linear fits as a function of q. Note that R^2 describes a bell curve with a maximum at 0.93.

Fig. 3 shows a nonlinear fit according to Eq. (2) (q-Gaussian) performed on the data binned as described above and using a fixed parameter $q = 0.93$ obtained from Fig. 2. In the figure, the solid line represents the best q-Gaussian fit ($R^2 = 0.98$)

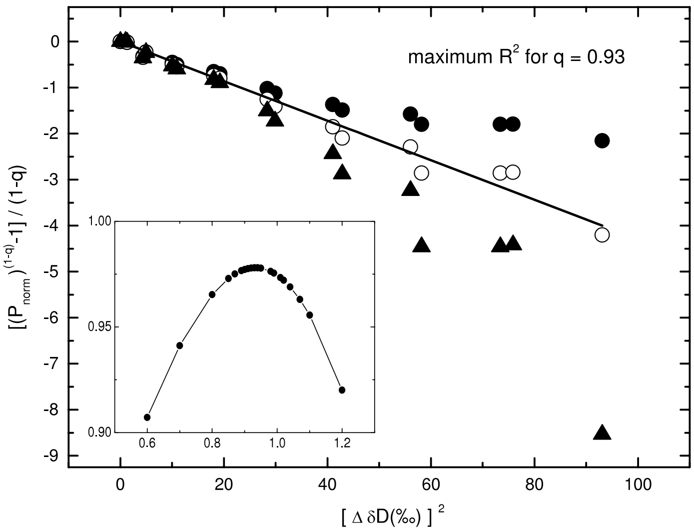

Fig. 2. $[(P_{\mathrm{norm}})^{1-q} - 1](1 - q)$ as a function of q for three different values of q. Open symbols for $q = 0.93$, solid circles for $q = 0.6$ and solid triangles for $q = 1.2$. In the inset we show the R^2 parameter as a function of q.

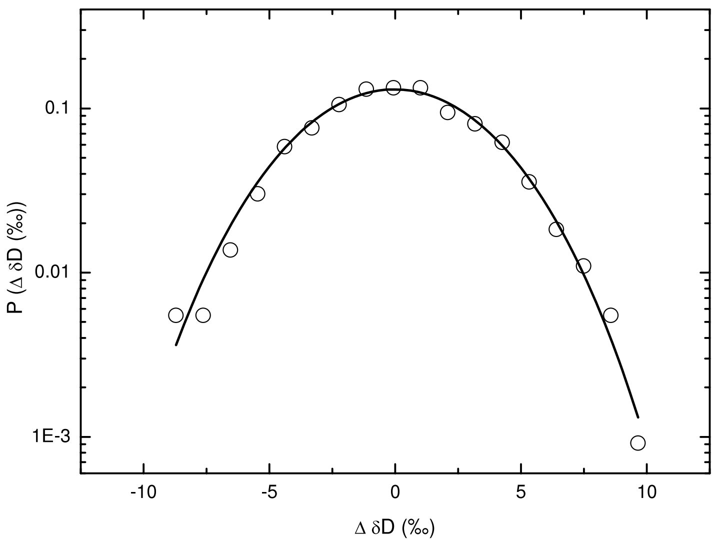

Fig. 3. Binned fluctuations. The solid line represents the fit according to Eq. (2) with $q = 0.93$.

with parameters $\sigma_q = 11.5$ and $q = 0.93$. Figs. 2 and 3 both allow us to conclude that some degree of nonextensivity is present in our system, which justifies the use of nonextensive statistical methods to analyze the time series corresponding to the deuterium fluctuation during the last glaciation.

Time-dependent entropy provides a way to study order/disorder changes associated with different complexity levels in a system. Calculating time-dependent entropy across the time series can be addressed in our case through a time window, inside which S_q is calculating following Eq. (1). This time window is made to slide along the TS, leading to an S_q time series. Let $\Delta\delta(t)$ be the deuterium fluctuation for a specific time t. The set of sequential $\Delta\delta(t_i)$ values ($i = 1, 2, 3, \ldots, N$) for a particular window length (i.e., window length $= N$) is partitioned into intervals, where the interval width is constant for all the calculations along the TS, and equal to $3.5\,\sigma/n^{1/3}$ (σ and n are the standard deviation and length of the fluctuation time series, respectively). In our case n is equal to 1094 which corresponds to the total length of the data set. The calculated Tsallis entropy for a particular window length it is associated with the window medium point (time; odd window length) [35]. Then, the time window is shifted (one point ahead) along the TS, and the same procedure is repeated, generating the whole entropy time series.

It is important to discuss how the parameters involved in the calculations of S_q modify the results. As cited in Ref. [16] there are no specific rules for the choices of these parameters; nevertheless some insights can be taken into account regarding their values. The bin width is important in calculating S_q and its associated rule is that a correct bin width should

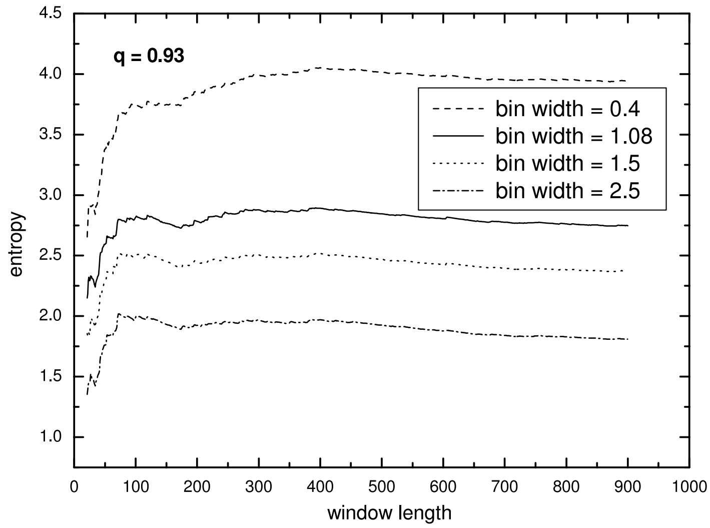

Fig. 4. Dependence of the entropy on the window length size (N) for $q = 0.93$ and four different bin widths.

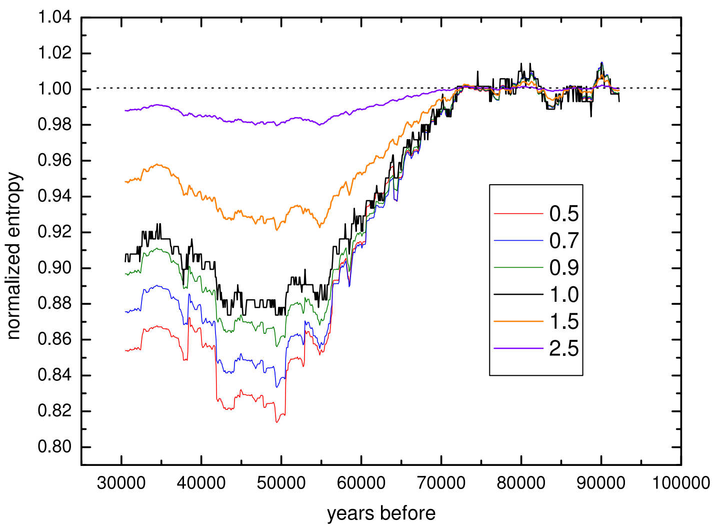

Fig. 5. Tsallis entropy for different values of the q parameter as indicated in the figure.

provide a smooth histogram [16]. In view of this, several bin widths (w) were tested and the best result (lower noise along the entropy time series) was obtained by taking the width of the bin according to Scott's criterion.

Also the window length can introduce errors in the results [36]. In Fig. 4, we show the entropy S_q ($q = 0.93$) as function of window length and for four different bin sizes, including the one ($w = 1.08$) used in Figs. 1 and 2. In the figure, S_q was calculated starting at a fixed time (around 90 kyears before the present) and increasing the length of the windows from 21 up to $N = 900$ (moving toward the critical transition). Note that S_q sharply increases at low sizes of the window while remaining almost constant for lengths above a certain value. It should be stressed that the constancy of S_q can in principle depend on the bin width. Thus, the S_q calculation was performed with a bin size and window length that do not depend on the window size, dispensing with bias correction [28].

Fig. 5 shows S_q (normalized) for different values of q, where we have used a window length $N = 321$ (odd window length). This length has been determined through Fig. 4: notice that the entropy essentially stops depending on the window size for sizes above this value. The entropies for different q parameters exhibit a minimum close to 50 kyears. The average values of S_q far away (above 70 kyears before the present) were used to normalize the data in order to compare the temporal sensitivity of different S_q entropies. The decrease of the entropy well before the onset of the interglacial transition suggests a change in the organization level of the climate system and the development of a new emerging state, with a lower complexity level, as evidenced by the loss of entropy. The entropy tends to increase after the minimum, indicating a recovery of the complexity of the system while it approaches the instability from the last glacial period to the interglacial one.

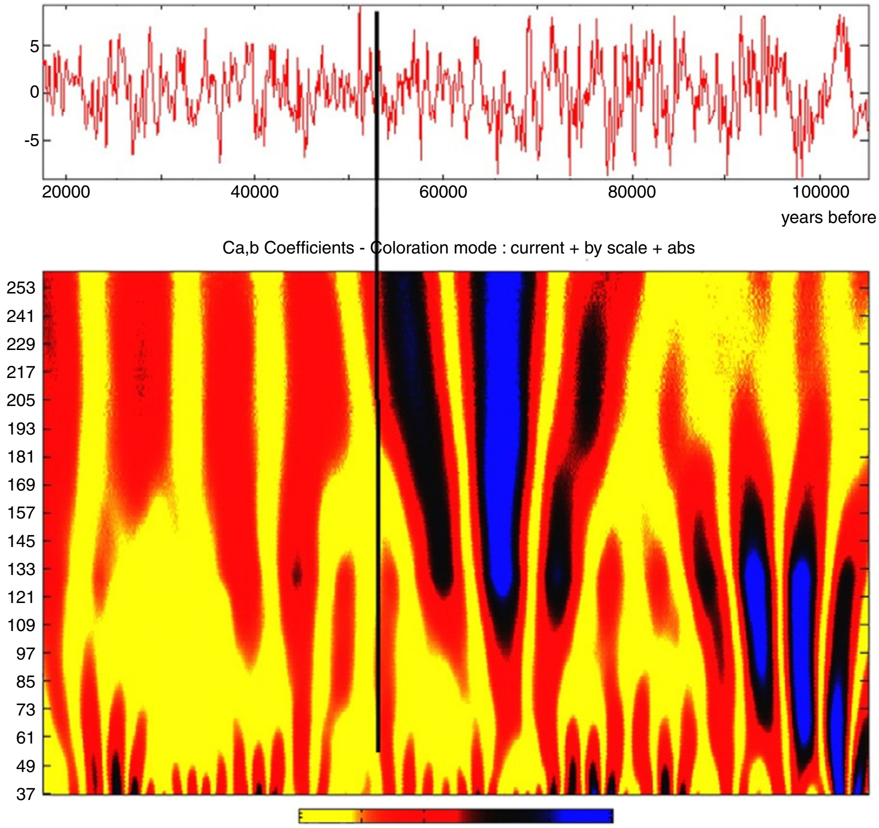

Fig. 6. Continuous wavelet transform of the data for the last glaciation. The upper part shows the fluctuations while the lower one represents the coefficients.

The different trends in S_q (for different q values) suggest, at around $t \approx 50$ kyears before today, a change of pattern in the time series of fluctuations. To investigate this point with an alternative technique, we have used a continuous wavelet transform, which enables us to obtain the wavelet coefficients which represent the regression of the original signal on the wavelets (the symlets family). These coefficients are shown in Fig. 6, where the x-axis is the time, the y axis is the scale of the regression, and the colors represent the coefficient intensities. This plot represents a time-scale view of the signal. The pattern obtained can be separated into two regions (shown by solid line in the figure) just around the point of minimum entropy, i.e., about 50 kyears before the present.

In Fig. 7 we show the original time series, S_q for $q = 0.93$ (for three different window lengths), and the lag-1 autocorrelation coefficient (AR-1 with a sliding window of 251 points) for the data. The AR-1 coefficient was calculated by fitting the data inside a moving time window according to $y_{t+1} = cy_t$. As previously noted in Refs. [5,11], the AR-1 coefficient increases as the system approaches the critical point between the glacial and the interglacial periods. The time dependence of our AR-1 coefficient can be compared with the one found for the same time series used in Ref. [11] (see figure1I and figure 1J therein [11]). The quantitative difference between our AR-1 data and those found in Ref. [11] is due to the fact that during our AR-1 calculations, each AR-1 value was associated with the middle point of the sliding window [35], while in Ref. [11] it was associated with the end of the window.

From Fig. 7 one can appreciate the increase of the AR-1 coefficient starting around the point at which the entropy S_q passes through a minimum and increases like AR-1. The same behavior is found using three different window sizes, with the smallest one showing that this behavior persists near to the transition point. The increase of the AR-1 coefficient indicates that the system "remembers" its previous state, signalling that the temporal correlations start playing an important role in the dynamics of the system [4]. Therefore, a picture emerges in which as the system approaches the transition, the effect of natural perturbations on the physical state is determinant. The increase of the variance of the fluctuations broadens the distribution, thus increasing S_q. Finally, it should be noted that, in this region, the behavior that we have described becomes magnified by q values different from 1, as should happen for a system with long temporal correlations.

5. Conclusions

In summary, our work shows that nonextensive statistics can be useful in the analysis of Earth's climate time series. This procedure might enable some more accurate analysis of phenomena such as global warming. The fluctuation data

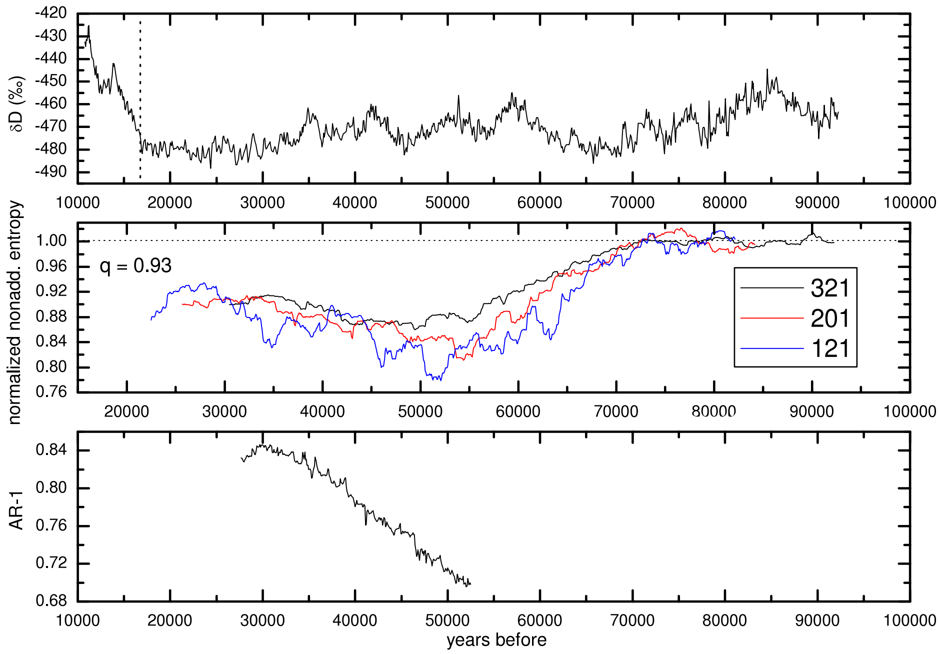

Fig. 7. Comparison between the time entropy ($q = 0.93$) and the autocorrelation coefficient AR-1 (lag $= 1$). The dotted line (upper graph) identifies the onset of the transition.

corresponding to the deuterium concentration during the last glaciation are distributed like a q-Gaussian with $q = 0.93$. The analysis of the time series of Tsallis entropies corresponding to the last glacial–interglacial period suggests the existence of different complexity levels during the transition from the last glacial period to the interglacial one.

The time dependence of the Tsallis entropy shows a behavior consistent with that of the lag-1 autocorrelation coefficient. This suggest that it can also be used as an early warning of the proximity of a critical transition in the Earth's climate system, although more studies and tests on different systems remain necessary. The time dependence of the nonadditive entropy exhibits a loss of complexity during the last glacial period. The S_q entropies decrease, passing through a minimum, after which they increase again as the system approaches the critical transition.

Acknowledgements

The authors are grateful for the support from the National Council for Scientific and Technological Development (CNPq) of the Brazilian Ministry of Science and Technology.

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