Magnetohydrodynamic spectral theory of laboratory and astrophysical plasmas

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1. Introduction: Plasmas and the Universe

- Need to account for **abundance of plasma.**
- Magnetohydrodynamics widely applicable due to **scale independence.**

2. MHD spectral theory of static plasmas

- Wave equation with **singularities.**
- Spectral structure **monotonic**.

3. Modifications for stationary plasmas

- Operator **non self-adjoint**
- Yet, **(partial) monotonicity**

4. Instabilities of transonic flows

- Waves and stationary states entangled: **transonic enigma.**
- Transonic flows drive **continuous spectum unstable.**

5. Conclusions

Plasma

- Astronomical observations:

  Plasma is the most abundant ($> 90\%$) state of matter in the Universe!

  (recently: dark matter also has substantial plasma component)

  $\Rightarrow$ **Expect plasmas to play a major role in future cosmology.**

- Crude definition:

  Plasma is a completely ionized gas, consisting of freely moving positively charged nuclei and negatively charged electrons.

- On Earth exceptional, obtained in laboratory thermonuclear fusion experiments at high temperatures ($T \sim 10^8\,\mathrm{K}$). In astrophysics, plasmas ever more prominent (stellar coronae, magnetospheres, pulsars, BH accretion disks, AGN jets, etc.).

- **What is so special about plasma?**

  $\Rightarrow$ **Plasma is essentially a global state because of embedded magnetic field.**

  (meaningless to discuss isolated small piece of plasma)

Standard View of Nature

Nuclear forces

$\Downarrow$

quarks / leptons

nuclei (+) / electrons (−) 10^{-15} m

Electrostatic forces

$\Downarrow$

atoms / molecules 10^{-9} m

(ordinary matter: electrically neutral)

⋮

Gravity

$\Downarrow$

stars / solar system $10^{9}/10^{13}$ m

galaxies / clusters $10^{20}/10^{23}$ m

universe 10^{26} m

However, ...

the Universe does not consist of ordinary matter

- **More than 90 % of visible matter is plasma:**
  Electrically neutral, but nuclei $(+)$ and electrons $(-)$ are not tied in atoms or molecules but **freely move about to form one collective fluid.**

- Unavoidable large scale result is induction of electric currents and **magnetic fields**
  $\Rightarrow$ **Plasma dynamics determined by magnetic flux conservation.**

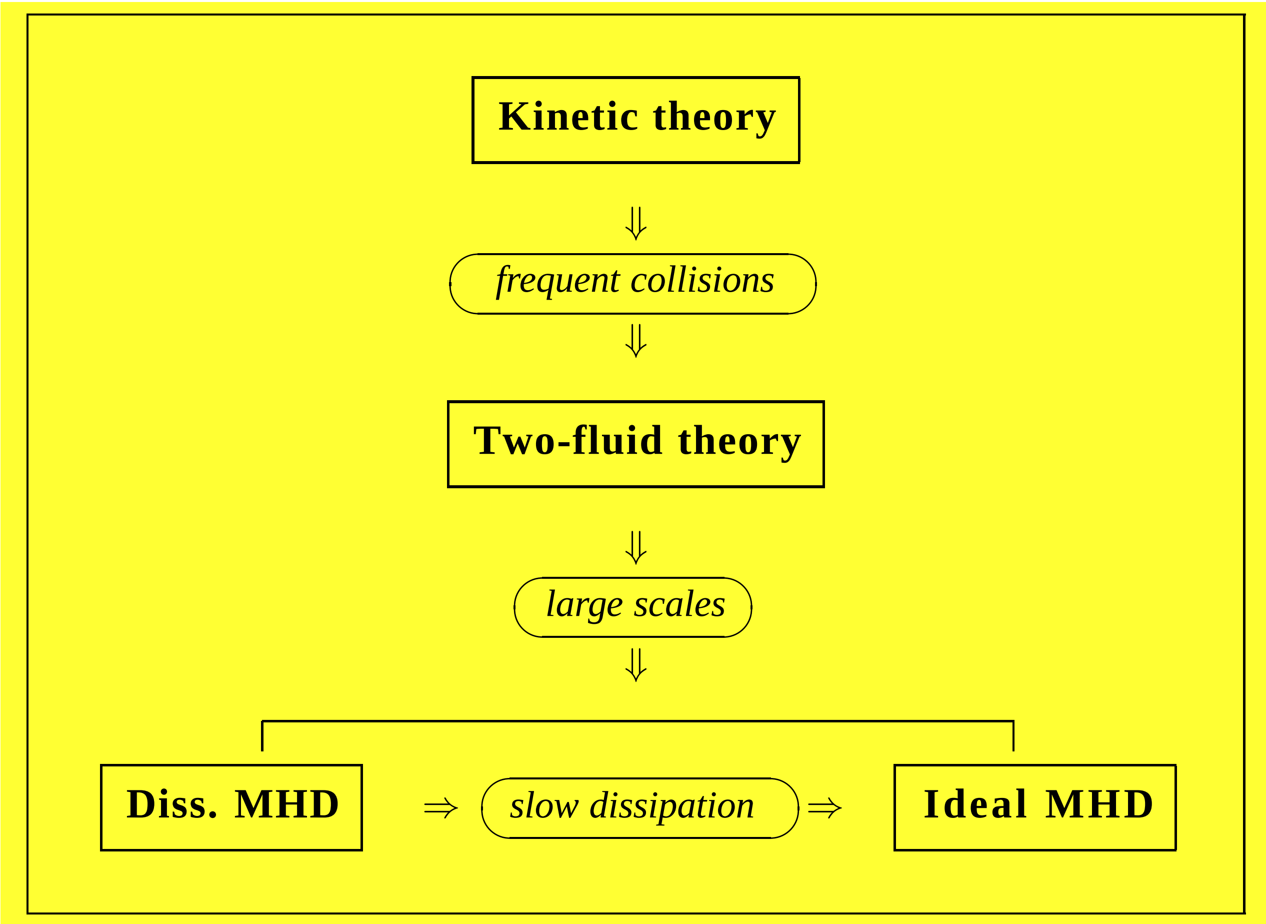
- Spherical symmetry of atomic physics and gravity not present on the plasma scale
  $\Rightarrow \nabla \cdot \mathbf{B} = 0$ **not compatible with spherical symmetry**
  (causing violent disruptions of stellar magnetic configurations).

- Instead, **magnetic confinement geometry of plasma** becomes the basic entity
  $\Rightarrow$ **Either toroidal or infinitely long tubular structures.**

Magnetohydrodynamics (MHD) describes intermediate levels of the Universe!

Theoretical plasma models

Three approaches:



- Local (small scales)

  Boltzmann $f_{e,i}(\mathbf{r}, \mathbf{v}, t)$

  $+$ Maxwell $\mathbf{E}$, $\mathbf{B}$ $(\mathbf{r}, t)$.

- Local and global

  (small and large scales):

  Fluid $n_{e,i}$, $\mathbf{u}_{e,i}$, $p_{e,i}$ $(\mathbf{r}, t)$

  $+$ Maxwell $\mathbf{E}$, $\mathbf{B}$ $(\mathbf{r}, t)$.

- Global (large scales):

  ρ, $\mathbf{v}$, p, $\mathbf{B}$ $(\mathbf{r}, t)$.

Ideal MHD equations

- Average the kinetic equations over space and time, exploit pre-Maxwell equations [displ. current & electr. forces $\mathcal{O}(v^2/c^2) \ll 1$: symmetry between $\mathbf{E}$ and $\mathbf{B}$ broken]
  $\Rightarrow$ Electric field determined from $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ *(perfect conductivity).*

- Eliminating $\mathbf{E}$ and $\mathbf{j}$ yields **nonlinear PDEs of ideal MHD:**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \qquad \textit{(conservation of mass)} \qquad (1)$$

$$\rho\big(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\big) + \nabla p - \rho \mathbf{g} - \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = 0\,, \qquad \textit{(momentum)} \qquad (2)$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0\,, \qquad \textit{(entropy)} \qquad (3)$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,. \qquad \textit{(magnetic flux)} \qquad (4)$$

- **MHD equations are scale-independent:**

Plasma size (l_0), magnetic field strength (B_0), and time scale (t_0) scale out !

Time scale from density (ρ_0) through *Alfvén speed* $v_{\rm A} \equiv B_0/\sqrt{\mu_0 \rho_0} \;\Rightarrow\; t_0 \equiv l_0/v_{\rm A}\,.$

Scales of actual plasmas

	l_0 (m)	B_0 (T)	t_0 (s)
tokamak	20	3	3×10^{-6}
magnetosphere Earth	4×10^{7}	3×10^{-5}	6
solar coronal loop	10^{8}	3×10^{-2}	15
magnetosphere neutron star	10^{6}	10^{8} *	10^{-2}
accretion disc YSO	1.5×10^{9}	10^{-4}	7×10^{5}
accretion disc AGN	4×10^{18}	10^{-4}	2×10^{12}
galactic plasma	10^{21} $(= 10^{5}$ ly$)$	10^{-8}	10^{15} $(= 3 \times 10^{7}$ y$)$

* Some recently discovered pulsars, called magnetars, have record magnetic fields of 10^{11} T : the plasma Universe is ever expanding!

Two approaches in plasma dynamics

- Stay **fully nonlinear** ⇒ **Numerical, large-scale computing**

  (inaccurate with respect to dynamics inside and accross magnetic surfaces).

- Split in **nonlinear equilibrium** and **linear perturbations** ⇒ **Spectral theory**

  (restricted to small amplitudes). **[Followed here: nice mathematics.]**

Angle:

Unify laboratory and astrophysical pictures of MHD waves and instabilities

(exploiting scale-independence MHD equations) ⇒ **MHD Spectroscopy**

- *Originated from MHD spectral theory, and large-scale numerical computations*

  [many authors since 1970s]

  ⇒ *MHD spectroscopy for tokamaks, following example of helioseismolgy.*

  [Goedbloed, Huysmans, Holties, Kerner, and Poedts, Plasma Phys. Contr. Fusion **35**, B277 (1993)]

  ⇒ *Magnetoseismology of accretion disks about compact objects.*

  [Keppens, Casse, and Goedbloed, ApJ **579**, L121 (2002)]

Linearization

- **Static equilibrium** ρ_0, p_0, $\mathbf{B}_0\ (\mathbf{r})$:

$$\mathbf{j}_0 \times \mathbf{B}_0 = \nabla p_0 + \rho_0 \nabla \Phi\,, \qquad \mathbf{j}_0 = \nabla \times \mathbf{B}_0\,, \qquad \nabla \cdot \mathbf{B}_0 = 0\,, \tag{5}$$

with BC $\quad \mathbf{n} \cdot \mathbf{B}_0 = 0 \quad$ (at the wall) .

- **Linear perturbations** $\mathbf{v}_1$, p_1, $\mathbf{B}_1$, $\rho_1\ (\mathbf{r}, t)$:

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \mathbf{j}_1 \times \mathbf{B}_0 + \mathbf{j}_0 \times \mathbf{B}_1 - \rho_1 \nabla \Phi\,, \tag{6}$$

$$\frac{\partial p_1}{\partial t} = -\mathbf{v}_1 \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \mathbf{v}_1\,, \tag{7}$$

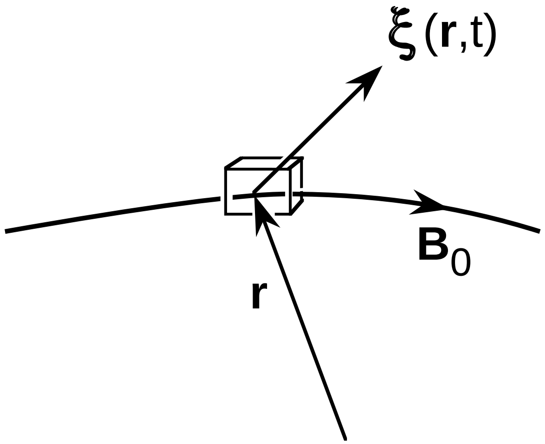
$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0)\,, \qquad \nabla \cdot \mathbf{B}_1 = 0\,, \tag{8}$$

$$\frac{\partial \rho_1}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{v}_1)\,, \tag{9}$$

with BCs $\quad \mathbf{n} \cdot \mathbf{v}_1 = 0\,, \quad \mathbf{n} \cdot \mathbf{B}_1 = 0 \quad$ (at the wall).

Lagrangian reduction

- **Lagrangian displacement vector field $\boldsymbol{\xi}(\mathbf{r}, t)$,**



related to plasma velocity

$$\mathbf{v} = \frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t} \equiv \frac{\partial \boldsymbol{\xi}}{\partial t} + \mathbf{v} \cdot \nabla \boldsymbol{\xi} \quad \Rightarrow \quad \mathbf{v}_1 = \frac{\partial \boldsymbol{\xi}}{\partial t},$$

permits integration of equations for p_1, $\mathbf{B}_1$, ρ_1, and presentation of momentum eq. in terms of $\boldsymbol{\xi}$ alone!

- **Equation of motion with force operator $\mathbf{F}$** *('Schrödinger equation'):*

$$\mathbf{F}(\boldsymbol{\xi}) \equiv -\nabla\pi - \mathbf{B} \times (\nabla \times \mathbf{Q}) + (\nabla \times \mathbf{B}) \times \mathbf{Q} + (\nabla\Phi)\, \nabla \cdot (\rho\boldsymbol{\xi}) = \rho\, \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2}, \quad (10)$$

with shorthand notation

$$\pi \ (\equiv p_1) \ = \ -\gamma p \nabla \cdot \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla p\,,$$

$$\mathbf{Q} \ (\equiv \mathbf{B}_1) \ = \ \nabla \times (\boldsymbol{\xi} \times \mathbf{B})\,.$$

- **For normal modes**, $\hat{\boldsymbol{\xi}}(\mathbf{r})\, \mathrm{e}^{-\mathrm{i}\omega t}$, **eigenvalue problem:**

$$\mathbf{F}(\hat{\boldsymbol{\xi}}) = -\rho\omega^2 \hat{\boldsymbol{\xi}} \quad \Rightarrow \quad \textit{spectrum}\ \{\omega^2\} \ \text{(discrete and continuous)}\,. \quad (11)$$

Hilbert space and quadratic forms

- Inner product for vector fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ (satisfying BCs), and finite norm:

$$\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle \equiv \tfrac{1}{2} \int \rho\, \boldsymbol{\xi}^* \cdot \boldsymbol{\eta}\, dV \,, \qquad \|\boldsymbol{\xi}\| \equiv \langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle^{1/2} < \infty \,. \tag{12}$$

- Force operator $\mathbf{F}$ **self-adjoint linear operator in this Hilbert space:**

$$\tfrac{1}{2} \int \boldsymbol{\eta}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, dV = \tfrac{1}{2} \int \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\eta}^*)\, dV \,.$$

  $\Rightarrow$ **Eigenvalues of $\rho^{-1}\mathbf{F}$ real:** ω^2 either ≥ 0 (stable) or < 0 (unstable).

- **Linearized kinetic energy** related to norm:

$$K \equiv \tfrac{1}{2} \int \rho \mathbf{v}^2\, dV \approx \tfrac{1}{2} \int \rho \dot{\boldsymbol{\xi}}^2\, dV \equiv \|\dot{\boldsymbol{\xi}}\|^2 < \infty \,. \tag{13}$$

  Potential energy from energy conservation, $\dot{W} = -\dot{K}$, and self-adjointness:

$$W = -\tfrac{1}{2} \int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, dV \,. \tag{14}$$

- Intuitive meaning: work done against the force $\mathbf{F}$.

  $\Rightarrow$ **Energy principle with trial functions:** $W[\boldsymbol{\xi}] \geq 0$ (stable) or < 0 (unstable).

Two 'pictures' of MHD spectral theory:

Differential eqs. ('Schrödinger')	**Quadratic forms** ('Heisenberg')		
Equation of motion: $\mathbf{F}(\boldsymbol{\xi}) = \rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2}$	*Hamilton's principle:* $\delta \int_{t_1}^{t_2} \Big(K[\dot{\boldsymbol{\xi}}] - W[\boldsymbol{\xi}] \Big)\, dt = 0$	$\Rightarrow$	Full dynamics: $\boldsymbol{\xi}(\mathbf{r}, t)$
Eigenvalue problem: $\mathbf{F}(\boldsymbol{\xi}) = -\rho\omega^2 \boldsymbol{\xi}$	*Rayleigh's principle:* $\delta \frac{W[\boldsymbol{\xi}]}{I[\boldsymbol{\xi}]} = 0$	$\Rightarrow$	Spectrum $\{\omega^2\}$ & eigenf. $\{\boldsymbol{\xi}(\mathbf{r})\}$
Marginal equation: $\mathbf{F}(\boldsymbol{\xi}) = 0$	*Energy principle:* $W[\boldsymbol{\xi}] \gtrless 0$	$\Rightarrow$	Stability $\binom{\mathrm{y}}{\mathrm{n}}$ & trial $\boldsymbol{\xi}(\mathbf{r})$

Standard theory (static plasmas)

- Solve MHD force operator equation $\mathbf{F}(\boldsymbol{\xi}) = -\rho\omega^2\boldsymbol{\xi}$ ('Schrödinger equation') for **cylindrical flux tube** ('H atom').
- *Project Fourier modes*

$$\boldsymbol{\xi}(r,\theta,z) = \big(\xi_r(r), \xi_\theta(r), \xi_z(r)\big)\, \mathrm{e}^{\mathrm{i}(m\theta + kz)} \tag{15}$$

on magnetic geometry (normal, $\perp$, $\parallel \mathbf{B}$): $\chi\,(\equiv r\xi_r)\,,\ \eta\,,\ \zeta\,.$

- Elimination of $\eta(\chi', \chi)$ and $\zeta(\chi', \chi)$ leads to **radial wave equation:**

$$\frac{d}{dr}\left(\frac{N}{rD}\frac{d\chi}{dr}\right) + \left[U + \frac{V}{D} + \left(\frac{W}{D}\right)'\right]\chi = 0\,, \tag{16}$$

[Hain & Lüst (1958), Goedbloed (1971)]

where N, D, U, V, W are functions of r and ω^2 .

- Two kinds of singularities:

$N \equiv \rho^2(\gamma p + B^2)(\omega^2 - \omega_A^2)\,(\omega^2 - \omega_S^2)$ $\Rightarrow$ **genuine continua** $\{\omega_S^2\}$, $\{\omega_A^2\}$,

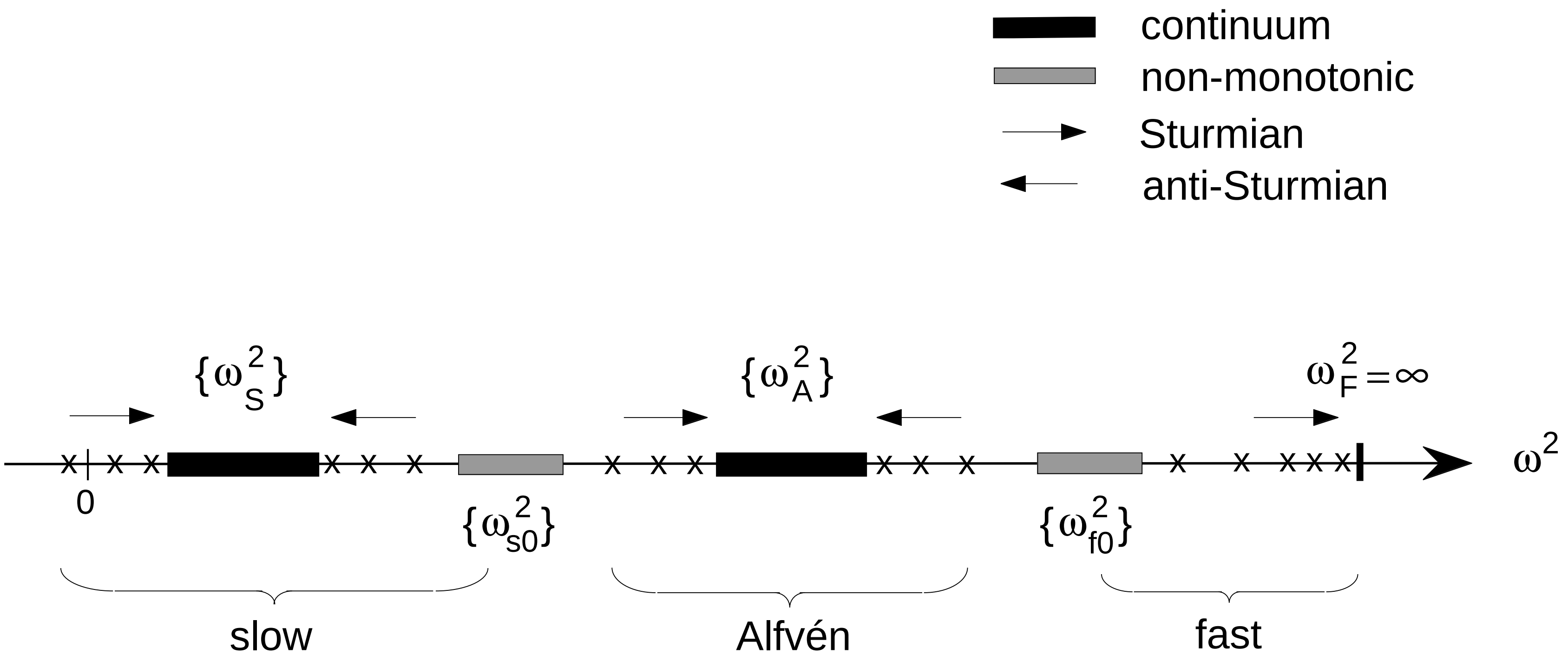
[Uberoi (1972), Grad (1973)]

$D \equiv \rho^2(\omega^2 - \omega_{s0}^2)\,(\omega^2 - \omega_{f0}^2)$ $\Rightarrow$ **apparent continua** $\{\omega_{s0}^2\}$, $\{\omega_{f0}^2\}$.

[Appert at al., Greene (1974), Goedbloed (1998)]

Schematic spectral structure

- **Spectrum 'hangs' on the continuous spectra** $\{\omega_S^2\}$, $\{\omega_A^2\}$, $\{\omega_F^2(\equiv\infty)\}$, where *slow magneto-sonic (ζ), Alfvén (η), or magno-tosonic (ξ) waves become singular*:



- **Monotonicity of spectrum outside singularities proved** by splitting the quadratic form in 3D internal product and 1D Sturm-Liouvile type expression. [Goedbloed (1974)]

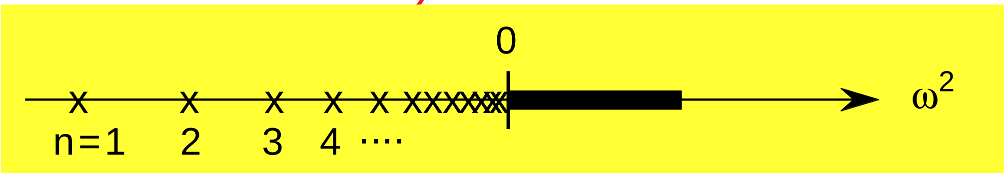
What 'picture' has emerged?

- MHD spectral theory is **powerful organizing principle** for dynamics of macroscopic waves and instabilities in plasmas.

- **Spectral structure centers about three continua** $\{\omega_S^2\}$, $\{\omega_A^2\}$, $\omega_F^2 \equiv \infty$.

  For example, most of tokamak stability theory concerns 'interchange' instabilities, i.e. *cluster spectra emitted by the continua when they extend to* $\omega = 0$ *:*

  $$\omega_A \sim \omega_S \sim k_\parallel B \to 0 \,, \tag{17}$$

  which is the basis of the **ballooning formalism.** [Connor, Hastie, Taylor (1979)]

- The insight has been embedded in **powerful numerical codes** (exploiting advanced eigenvalue solvers like Jacobi-Davidson). [Sleijpen & van der Vorst (1996)]

- Folding in data obtained from large variety of **diagnostics in tokamaks** has led to maturation of MHD spectroscopy for **static laboratory plasmas.**

- MHD spectroscopy for **stationary astrophysical plasmas** still inmature: **spatially resolved observations presently absent, but will emerge in 21th century!**

Linearization for stationary plasmas

- **Stationary equilibrium** now involves background flow $\mathbf{v}$:

$$\nabla \cdot (\rho \mathbf{v}) = 0\,, \tag{18}$$

$$\rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \mathbf{j} \times \mathbf{B}\,, \qquad \mathbf{j} = \nabla \times \mathbf{B}\,, \tag{19}$$

$$\mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0\,, \tag{20}$$

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,. \tag{21}$$

$\Rightarrow$ **Equilibrium becomes a nonlinear problem in its own right.**

- **Linear perturbations** again representable with displacement vector field $\boldsymbol{\xi}$:

$$\mathbf{F}(\boldsymbol{\xi}) + 2\mathrm{i}\rho\omega \mathbf{v} \cdot \nabla \boldsymbol{\xi} + \rho\omega^2 \boldsymbol{\xi} = 0\,, \qquad \text{[Frieman, Rotenberg (1960)]} \tag{22}$$

$$\mathbf{F} \equiv \mathbf{F}_{\rm static}(\boldsymbol{\xi}) + \nabla \cdot [\rho(\mathbf{v} \cdot \nabla \mathbf{v})\boldsymbol{\xi} - \rho \mathbf{v}\mathbf{v} \cdot \nabla \boldsymbol{\xi}]\,, \tag{23}$$

but operator no longer self-adjoint $\Rightarrow$ **complex eigenvalues ω !**

(damped and 'overstable' modes).

Cylindrical model problem

- MHD spectral problem now involves Coriolis effects and Doppler shifts:

$$\mathbf{F}_{\text{static}}(\boldsymbol{\xi}) + \nabla \cdot [\rho(\mathbf{v} \cdot \nabla \mathbf{v})\boldsymbol{\xi}] + \rho(\omega + i\mathbf{v} \cdot \nabla)^2 \boldsymbol{\xi} = 0 \,. \tag{24}$$

- **Radial wave equation** for cylindrical plasma is again ODE:

$$\frac{d}{dr}\left(\frac{\widetilde{N}}{r\widetilde{D}}\frac{d\chi}{dr}\right) + \left[\,\widetilde{U} + \frac{\widetilde{V}}{\widetilde{D}} + \left(\frac{\widetilde{W}}{\widetilde{D}}\right)'\,\right]\chi = 0 \,, \tag{25}$$

[Bondeson et al. (1987), Keppens et al. (2002)]

where $\widetilde{N}$, $\widetilde{D}$, $\widetilde{U}$, $\widetilde{V}$, $\widetilde{W}$ are functions of r and the *Doppler shifted frequency* $\widetilde{\omega}$ *:*

$$\widetilde{\omega} \equiv \omega - \Omega_0(r) \,, \qquad \Omega_0 \equiv m v_\theta / r + k v_z \,. \tag{26}$$

- Forward (+) / backward (−) shifted **genuine** and **apparent** continua (both real!),

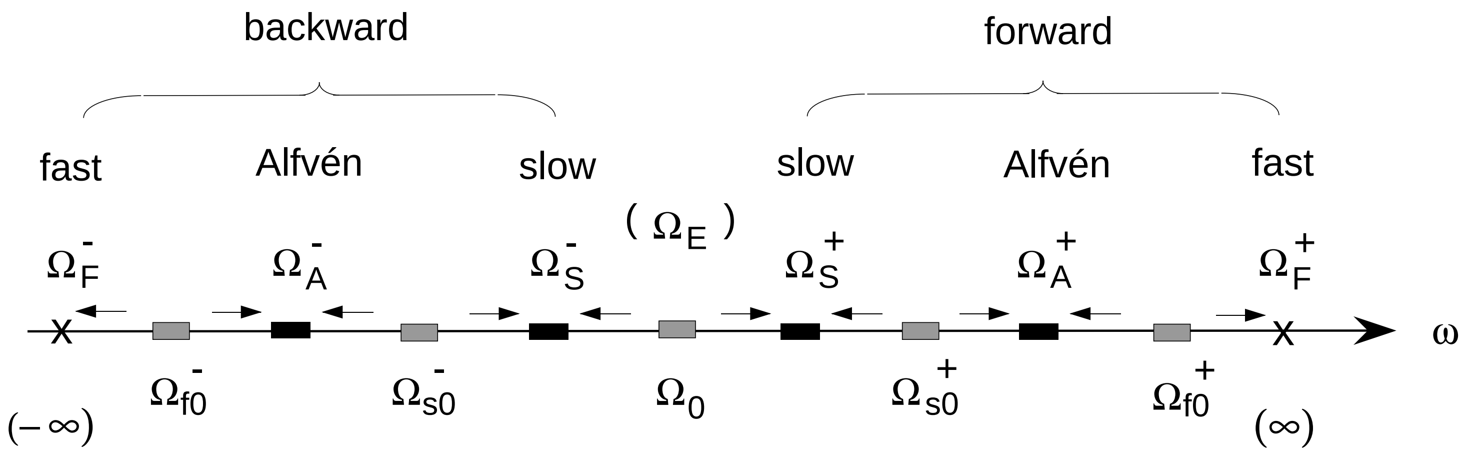
$$\Omega_S^{\pm} \equiv \Omega_0 \pm \omega_S \,, \qquad \Omega_A^{\pm} \equiv \Omega_0 \pm \omega_A \,, \qquad \Omega_F^{\pm} \equiv \pm\infty \,, \tag{27}$$

$$\Omega_{so}^{\pm} \equiv \Omega_0 \pm \omega_{so} \,, \qquad \Omega_{f0}^{\pm} \equiv \Omega_0 \pm \omega_{f0} \,, \tag{28}$$

determine spectral structure: clustering of *complex* eigenvalues towards real axis.

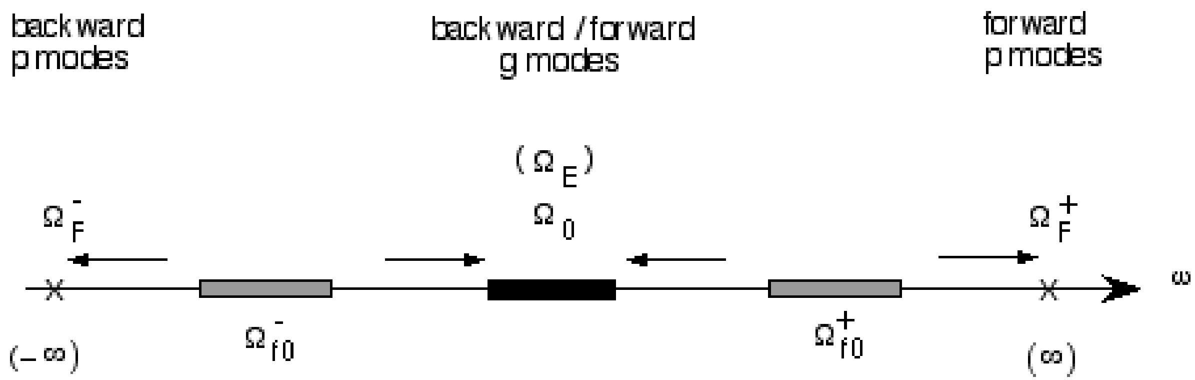
Relationship between MHD and HD spectra

- Structure MHD spectrum, with **real slow, Alfvén, and fast continua** [Goedbloed (2004)]:



$$\Omega_F^{\pm} \equiv \pm\infty$$
$$\Omega_A^{\pm} \equiv \Omega_0 \pm \omega_A$$
$$\Omega_S^{\pm} \equiv \Omega_0 \pm \omega_S$$

- HD spectrum for ordinary shear flow fluid, with **real flow continuum** [Case (1960)]:



$$\Omega_0 \equiv m v_\theta / r + k v_z$$

New insights

- In HD limit ($\mathbf{B} \rightarrow 0$), Alfvén and slow continua collapse into flow continuum:

$$\Omega_A^{\pm} \rightarrow \Omega_0 \,, \qquad \Omega_S^{\pm} \rightarrow \Omega_0 \qquad (\Omega_F^{\pm} \text{ remains at } \pm\infty)\,. \tag{29}$$

  Vice versa, flow continuum is absorbed by the four MHD continua when $B \neq 0$.

  $\Rightarrow$ In contrast to common beliefs, **there is no separate flow continuum in MHD!**

  [Goedbloed et al (2004b)]

- **Monotonicity of MHD spectrum along real axis** could be proved for plane slab, cylindrical plasma (with Coriolis force) required construction of a new quadratic form Q taking the place of ω in the complex plane. [Goedbloed et al (2004a)]

- Significant progress in understanding of **full complex spectrum** (MRI, RT, KH, . . .) **of thin cylindrical slice as model for accretion disk around compact object.**

  [Keppens et al. (2002), Blokland et al. (2005)]

Making up the balance

- In astrophysical plasmas **stationary equilibrium flows** must appear center stage.

  ⇒ Non-selfadjoint operators, Doppler shifts, *eigenvalues on unknown paths in the complex ω-plane,* centrifugal & Coriolis effects in curved velocity fields.

- We could trace the central structure (continua and monotonicity) along the real axis, and are able to accurately compute full complex spectra.

- However, the big challenge is yet to come: **transonic flows!**

  ⇒ Requires at least 2D since hyperbolicity and shocks break symmetry.

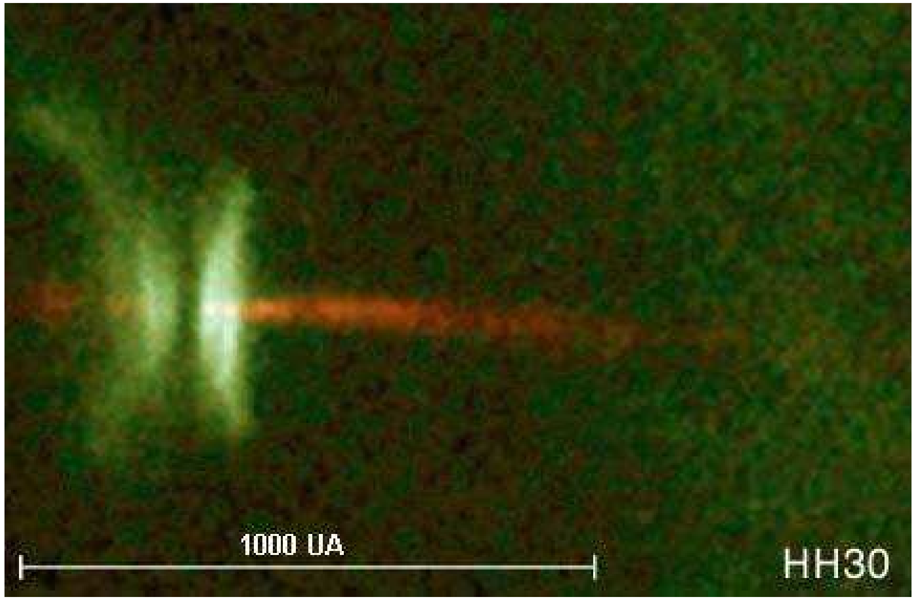
⇒ MHD spectral theory of 2D transonic flows

- Physical problem: In accretion disks around neutron stars or black holes, accretion–ejection requires **anomalous dissipation, i.e. small-scale instabilities.** Standard magneto-rotational instability (MRI) only operates for low magnetic field strength.

  ⇒ Are there **instabilities that operate at arbitrary field strength?**

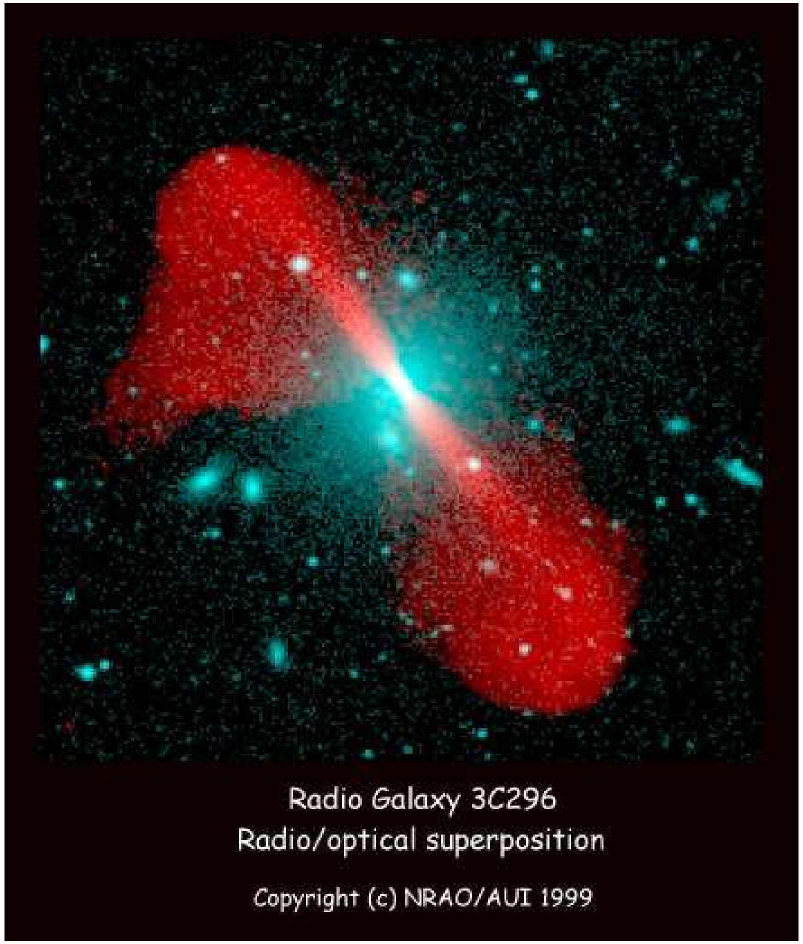
Accretion disk and jets (YSO & AGN)

Young stellar object ($M_* \sim 1 M_\odot$):



disk: dark strip, jets: red.

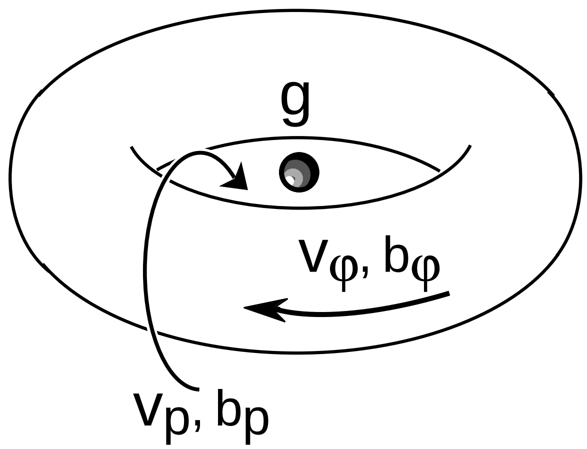
Active galactic nucleus ($M_* \sim 10^8 M_\odot$):



disk: blue (optical), jets: red (radio).

Model

- **Transonically rotating magnetized (thick) disk about compact object:**
  'superposition' of tokamak and black hole.



- Assume:
  Accretion speed $\ll$ rotation speeds of the disk.
- Investigate:
  Stationary 2D equilibrium $+$ local instabilities.
- Will find new instabilities driven by transonic transitions of the flow that involve singular **trans-slow Alfvén modes with a continuous spectrum** that **'live' on the curved two-dimensional surfaces spanned by plasma velocity and magnetic field.**
- *Gravitational parameter,* measuring deviation from Keplerian flow (where $\Gamma = 1$):

$$\Gamma(\psi) \equiv \frac{\rho G M_*}{R_0 M^2 B^2} \quad \left[\approx \frac{G M_*}{R v_\varphi^2} \;\; \textit{for parallel flow} \right] .$$

- Analysis presented here, amplified with two numerical codes:
  (a) Transonic equilibria: FINESSE [Beliën et al. (2002)]; (b) Transonic spectra: PHOENIX.

Variational principle for stationary MHD equilibria

- **Two fields:** Poloidal flux ψ, squared poloidal Alfvén Mach number $M^2 \equiv \rho v_p^2/B_p^2$.

- Stationary axisymmetric states obtained by minimizing Lagrangian:

$$\delta \int \mathcal{L}\, dV = 0\,, \qquad \boxed{\mathcal{L} \equiv \frac{1}{2R^2}(1-M^2)|\nabla\psi|^2 - \frac{\Pi_1}{M^2} - \frac{\Pi_2}{\gamma M^{2\gamma}} + \frac{\Pi_3}{1-M^2}}\,,$$

  Nonlinear composite functions $\Pi_j\big(\Lambda_i(\psi); R, Z\big)$ of *five arbitrary flux functions* $\Lambda_i(\psi)$ [streamfunction χ, Bernoulli H, entropy S, vorticity-current density K, electr. pot. Φ] relating the primitive variables ρ, p, v_p, v_φ, B_p, B_φ.

- Euler-Lagrange equations: PDE for $\psi(R,Z)$, algebraic for $M^2(R,Z)$.

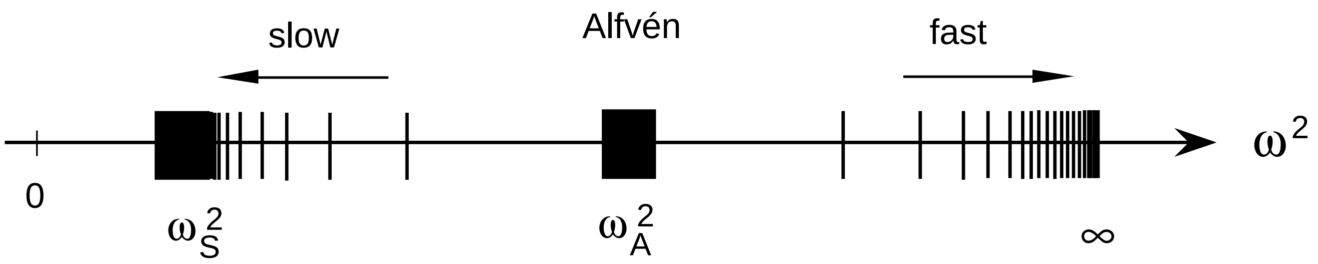
- Substitute solution M^2 back into PDE for ψ $\Rightarrow$ *Characteristics* $dx'/dy' = \pm\sqrt{\Delta}$, determined by

$$\Delta = \frac{\gamma p + B^2}{B_p^2}\,\frac{M^2 - M_c^2}{(M^2 - M_s^2)(M^2 - M_f^2)} \qquad \begin{cases} < 0: \ \text{elliptic} \\ \geq 0: \ \text{hyperbolic} \end{cases}.$$

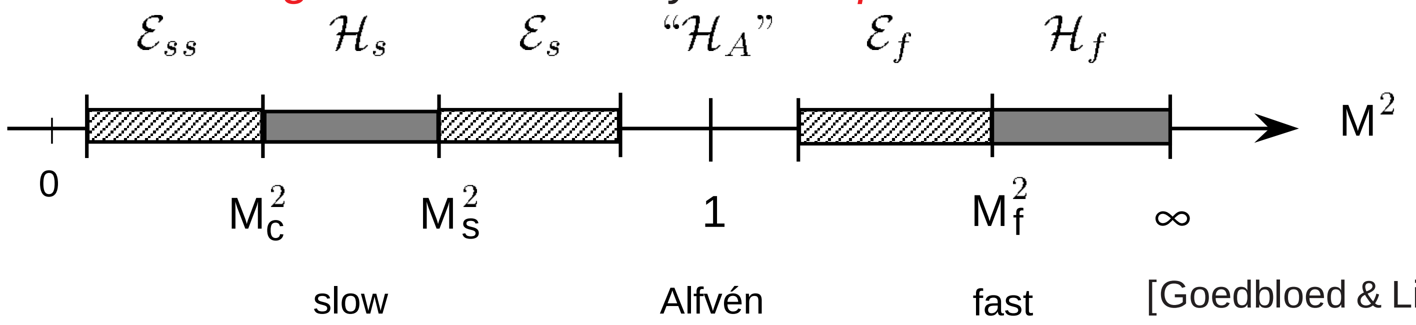
  $\Rightarrow$ Hyperbolicity for $M_c^2 < M^2 < M_s^2$ (slow regime), and $M^2 > M_f^2$ (fast regime).

Transonic enigma

- **At transonic transitions, flows change character from elliptic to hyperbolic.**
  - – Standard tokamak equilibrium solvers diverge in the *hyperbolic* regimes!
  - – 'Remedy': calculate in *elliptic* regimes beyond the hyperbolic ones.
- **Temporal waves & spatial nonlinear stationary states become interconnected!**
  - – *Wave spectra* cluster at *continuous spectra* $\{\pm\omega_S\}$, $\{\pm\omega_A\}$, $\pm\infty(\omega_F)$:



  - – *Hyperbolic flow regimes* delimited by *critical poloidal Alfvén Mach numbers:*



[Goedbloed & Lifschitz (1997)]

Transonic continuum modes

- **Singular modes** localized about *single magnetic / flow surface:*

$$\eta(\psi,\vartheta,\varphi) \approx \delta(\psi-\psi_0)\,\hat\eta(\vartheta)\,e^{in\varphi} \quad \Rightarrow \text{EVP}: \boxed{\hat{\mathbf{A}}\cdot\hat{\mathbf{V}} = \hat{\mathbf{B}}\cdot\hat{\mathbf{V}}}\,, \quad \hat{\mathbf{V}} \equiv (\hat\eta\,,\hat\zeta)^{\mathrm{T}}\,,$$

$$\hat{\mathbf{A}} \equiv \begin{pmatrix} \mathcal{F}\dfrac{R^2B_p^2}{B^2}\mathcal{F} - (M^2-M_c^2)\dfrac{B^2}{\rho^2}\Big[\partial\Big(\dfrac{\rho R B_\varphi}{B^2}\Big)\Big]^2 & -i(M^2-M_c^2)\dfrac{B^2}{\rho^2}\Big[\partial\Big(\dfrac{\rho R B_\varphi}{B^2}\Big)\Big]\mathcal{F}\rho \\ i\rho\mathcal{F}(M^2-M_c^2)\dfrac{B^2}{\rho^2}\Big[\partial\Big(\dfrac{\rho R B_\varphi}{B^2}\Big)\Big] & \mathcal{F}M_c^2B^2\mathcal{F} + \rho\Big[\partial\Big((M^2-M_c^2)\dfrac{B^2}{\rho^2}\,\partial\rho\Big)\Big] \end{pmatrix},$$

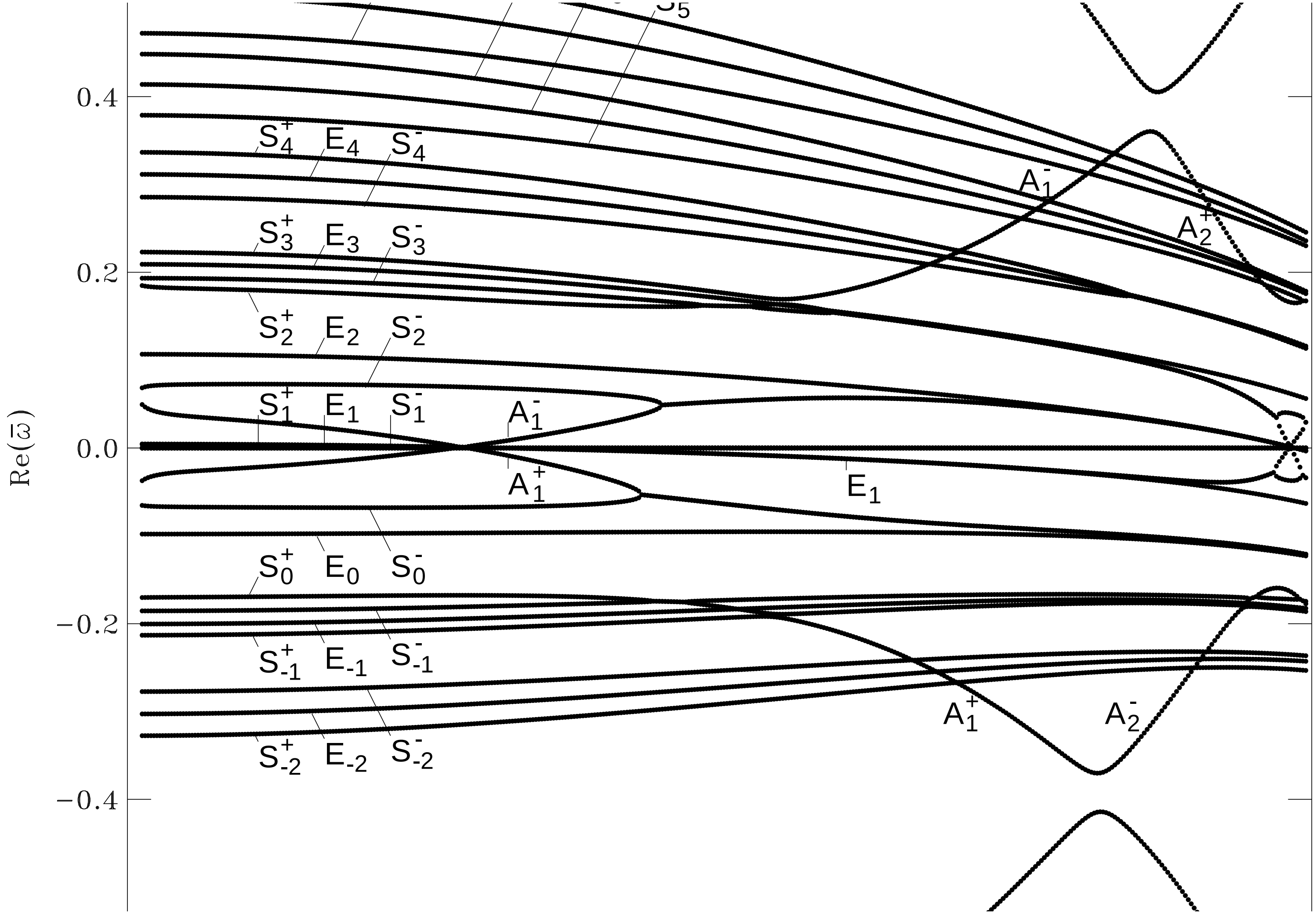
$$\hat{\mathbf{B}} \equiv \begin{pmatrix} (\sqrt{\rho}\,\widetilde\omega - \mathcal{F}M)\dfrac{R^2B_p^2}{B^2}(\sqrt{\rho}\,\widetilde\omega - M\mathcal{F}) & -i\alpha\sqrt{\rho}\,\widetilde\omega \\ i\alpha\sqrt{\rho}\,\widetilde\omega & (\sqrt{\rho}\,\widetilde\omega - \mathcal{F}M)B^2(\sqrt{\rho}\,\widetilde\omega - M\mathcal{F}) \end{pmatrix},$$

with $\widetilde\omega \equiv \omega - n\Omega$ the Doppler shifted frequency in frame rotating with ang. vel. Ω.

- **Always unstable in trans-slow ($M^2 > M_c^2$) flow regime!**

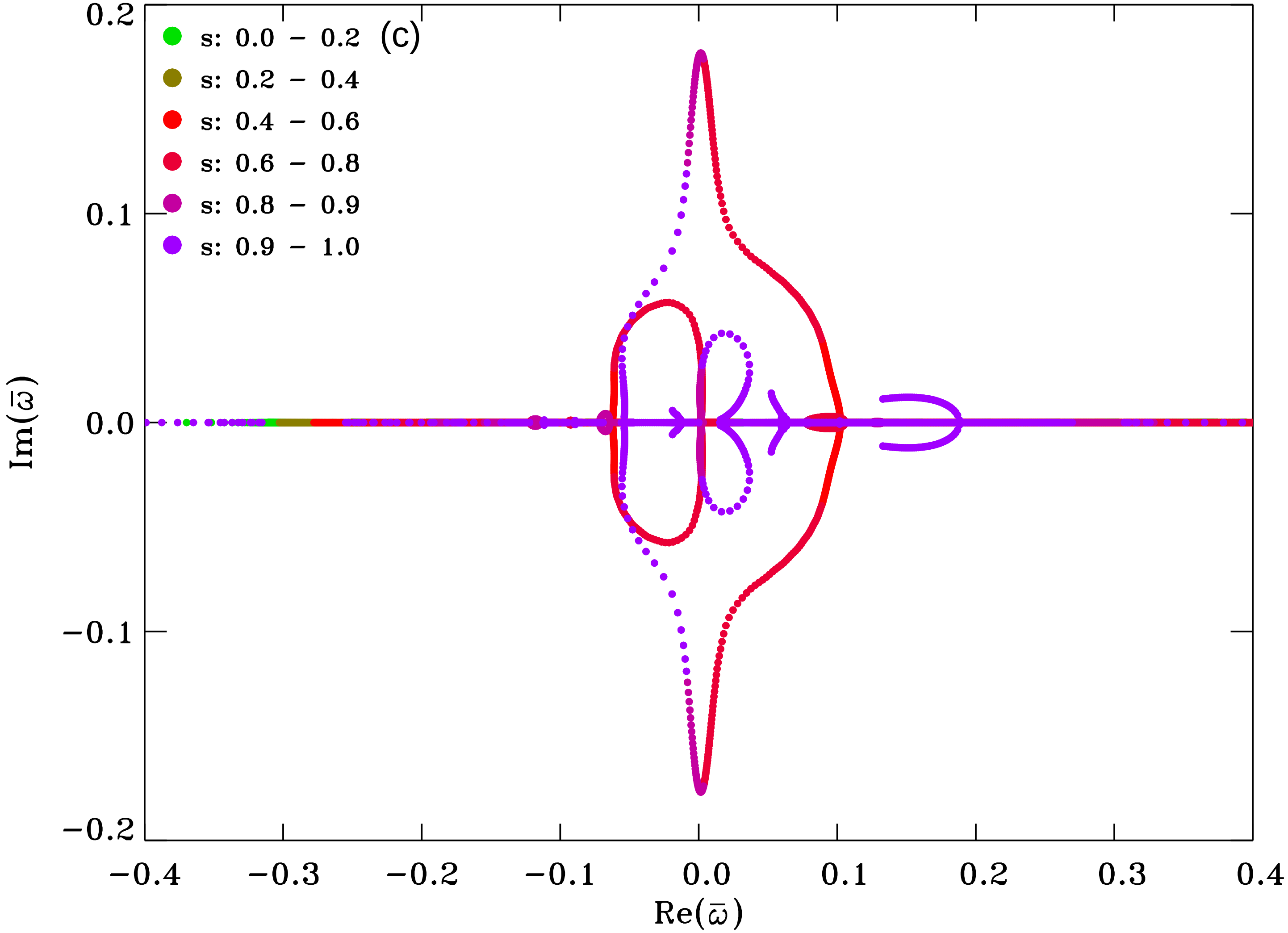
Mode labelling

$A_m^{\pm}$ *branches strongly interact with* $S_{m-1}^{\pm}$ *and* $S_{m+1}^{\pm}$*:*



Accretion disk ($\Gamma = 2$)

Complex eigenvalues ($n = 1$), parametrized with flux surface label $s \equiv \psi^{1/2}$: *Locked modes* ($\mathrm{Re}\,\bar{\omega} = 0$) *with huge exponential growth rate!*



- MHD spectroscopy of extended magnetic flux tubes describes dynamics of magnetic strings with traveling Alfvén wave excitations **that can be observed.**

- **The continuous spectra of transonic equilibria are explosively unstable for large central mass**, facilitating both accretion & ejection of jets from accretion disks about compact objects.

- With the advent of ever improving spatial resolution of astrophysical observations, **MHD spectroscopy is expected to become a fruitful theoretical guide.**

Listen to the music of the future (courtesy Igor Semenov)

and

appreciate the mathematical power of Hilbert space (1912) describing it!

Euler-Lagrange equations

- *Nonlinear PDE* for poloidal flux $\psi(R,Z)$:

$$\nabla\cdot\left(\frac{1-M^2}{R^2}\nabla\psi\right)+\frac{1}{M^2}\frac{\partial\Pi_1}{\partial\psi}-\frac{1}{\gamma M^{2\gamma}}\frac{\partial\Pi_2}{\partial\psi}-\frac{1}{M^2-1}\frac{\partial\Pi_3}{\partial\psi}=0\,. \qquad (30)$$

- *Algebraic Bernoulli equation* for squared poloidal Alfvén Mach number $M^2(R,Z)$:

$$\frac{1}{2R^2}|\nabla\psi|^2-\frac{\Pi_1}{M^4}+\frac{\Pi_2}{M^{2(\gamma+1)}}+\frac{\Pi_3}{(M^2-1)^2}=0\,. \qquad (31)$$

- Solution of Bernoulli equation depends on both ψ *and* $\nabla\psi$ $\Rightarrow$ non-trivial conditions for hyperbolicity/ellipticity when inserted back into PDE for ψ.

Variational principle for stationary TF equilibria

- Axisymmetric equilibria from two-fluid Lagrangian: [Goedbloed, PoP **11**, L81 (2004)]

$$\delta \int \mathcal{L}_{\rm TF}(\chi_\alpha, \nabla\chi_\alpha, \rho_\alpha, \psi, \nabla\psi, \widetilde{\phi}, \nabla\widetilde{\phi}, \widetilde{\mathcal{V}}, \nabla\widetilde{\mathcal{V}}; R, Z)\, dV = 0 \qquad (\alpha = e, i)\,,$$

$$\mathcal{L}_{\rm TF} \equiv \frac{1}{2\rho_e R^2}\,|\nabla\chi_e|^2 + \frac{1}{2\rho_i R^2}\,|\nabla\chi_i|^2 - \frac{1}{2\mu_0 R^2}\,|\nabla\psi|^2 + \tfrac{1}{2}\epsilon_0|\nabla\widetilde{\phi}|^2 - \frac{1}{8\pi G}\,|\nabla\widetilde{\mathcal{V}}|^2$$
$$+ \frac{1}{2\mu_0 R^2}\,I^2 + \rho_e\,F_e - \frac{1}{\gamma-1}{\rho_e}^{\gamma} S_e + \rho_i\,F_i - \frac{1}{\gamma-1}{\rho_i}^{\gamma} S_i\,. \qquad (32)$$

- *Seven fields* χ_e, χ_i, ρ_e, ρ_i, ψ, ϕ, $\mathcal{V}$ and nonlinear composites,

$$I(\chi_e,\chi_i) \;\equiv\; RB_\varphi = I_0 + \mu_0[(e/m_e)\chi_e - (Ze/m_i)\chi_i]\,,$$

$$F_e(\chi_e,\psi,\widetilde{\phi},\widetilde{\mathcal{V}};R,Z) \;\equiv\; H_e - \frac{1}{2R^2}(L_e - \frac{e}{m_e}\psi)^2 + \frac{e}{m_e}(\phi_* + \widetilde{\phi}) - \mathcal{V}_* - \widetilde{\mathcal{V}}\,,$$

$$F_i(\chi_i,\psi,\widetilde{\phi},\widetilde{\mathcal{V}};R,Z) \;\equiv\; H_i - \frac{1}{2R^2}(L_i + \frac{Ze}{m_i}\psi)^2 - \frac{Ze}{m_i}(\phi_* + \widetilde{\phi}) - \mathcal{V}_* - \widetilde{\mathcal{V}}\,,$$

of *six arbitrary stream functions* $H_{e,i}$, $L_{e,i}$, $S_{e,i}(\chi_{e,i})$ and three potentials ψ, ϕ, $\mathcal{V}$.

Euler-Lagrange equations (TF)

- PDEs for χ_α :

$$\nabla \cdot \Big(\frac{1}{\rho_e R^2}\nabla\chi_e\Big) = \frac{e}{m_e R^2} I + \rho_e\big[H_e{}' - \frac{1}{R^2}(L_e - \frac{e}{m_e}\psi)L_e{}'\big] - \frac{\rho_e{}^\gamma}{\gamma-1} S_e{}' \,, \quad (33)$$

$$\nabla \cdot \Big(\frac{1}{\rho_i R^2}\nabla\chi_i\Big) = -\frac{Ze}{m_i R^2} I + \rho_i\big[H_i{}' - \frac{1}{R^2}(L_i + \frac{Ze}{m_i}\psi)L_i{}'\big] - \frac{\rho_i{}^\gamma}{\gamma-1} S_i{}' \,, \quad (34)$$

- Bernoulli equations for ρ_α :

$$\frac{1}{2R^2}|\nabla\chi_e|^2 - \rho_e^2 F_e + \frac{\gamma}{\gamma-1}\rho_e{}^{\gamma+1} S_e = 0 \quad \Rightarrow \rho_e \Uparrow \,, \quad (35)$$

$$\frac{1}{2R^2}|\nabla\chi_i|^2 - \rho_i^2 F_i + \frac{\gamma}{\gamma-1}\rho_i{}^{\gamma+1} S_i = 0 \quad \Rightarrow \rho_i \Uparrow \,, \quad (36)$$

- PDEs for ψ , $\widetilde{\phi}$, $\widetilde{\mathcal{V}}$:

$$\frac{R^2}{\mu_0}\nabla \cdot \Big(\frac{1}{R^2}\nabla\psi\Big) = R j_\varphi \equiv -\frac{e}{m_e}\rho_e\big(L_e - \frac{e}{m_e}\psi\big)^2 + \frac{Ze}{m_i}\rho_i\big(L_i + \frac{Ze}{m_i}\psi\big)^2 \,, \quad (37)$$

$$\epsilon_0 \nabla^2\widetilde{\phi} = -\tau \equiv \frac{e}{m_e}\rho_e - \frac{Ze}{m_i}\rho_i \,, \quad (38)$$

$$\frac{1}{4\pi G}\nabla^2\widetilde{\mathcal{V}} = \rho \equiv \rho_e + \rho_i \,. \quad (39)$$