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Geometrizing Relativistic Quantum Mechanics

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Abstract We propose a new approach to describe quantum mechanics as a manifestation of non-Euclidean geometry. In particular, we construct a new geometrical space that we shall call Qwist. A Qwist space has a extra scalar degree of freedom that ultimately will be identified with quantum effects. The geometrical properties of Qwist allow us to formulate a geometrical version of the uncertainty principle. This relativistic uncertainty relation unifies the position-momentum and time-energy uncertainty principles in a unique relation that recover both of them in the non-relativistic limit.

Keywords Foundations of quantum mechanics · Bohm-de Broglie interpretation · Weyl integrable space · Non-Euclidean geometry

1 Introduction

Non-relativistic theories were formulated to describe natural phenomena as a collection of events occurring on space using time as their external parameter. For that it seemed reasonable to formalize the physical arena in an abstract language as a flat, homogeneous and isotropic space. In fact, there was no other but one geometrical theory available. Hence, Euclidean geometry was immediately identified with physical space.

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Apart from its peculiarities, quantum mechanics promptly inherited Euclidean geometry from classical mechanics. Even though some of its formulations do not even have a well defined notion of trajectory, quantum mechanics is defined using the flat Euclidean metric or at most the flat Minkowskian metric when we are dealing with relativistic quantum systems.

One of the novelties of relativistic theories is to describe physical phenomena on a non-Euclidean four dimensional manifold. There is a rupture in the identification of the Euclidean geometry with the physical space. In fact, General Relativity generalize the spacetime structure allowing it to be any possible type of Riemannian geometry.

The geometrical properties of a Riemannian space is completely characterized by a metric tensor. Its connection, the geometrical object that defines the covariant derivative, is identified with the Christoffel symbol, which is completely determined by the metric tensor. These spacetimes have two important properties, namely, all Riemannian spaces are locally Minkowskian and the parallel transport prescription which enables us to compare objects at different locations preserve lengths and angles.

Additionally, there is still a wider class of torsion free geometrical space that was first introduced by Weyl [1] as an attempt to include electromagnetism in the properties of spacetime. In a Weyl space, the covariant derivative is again modified to implement a new gauge symmetry. The main idea was to incorporate electromagnetism in the geometrical degrees of freedom and geometrize both gravitation and electromagnetism, the only known classical long-range interactions.

From a different perspective, London proposed [2] that the geometrical space developed by Weyl could be related to quantum phenomena. London hypothesis was that stationary states of a quantum system such as an hydrogen atom should be associated with special geometrical configurations. In particular, he obtained Bohr's atomic orbits for the hydrogen atom as the only possible integrable orbits on that Weyl space. Therefore, it became admissible that Euclidean geometry could fail not only on large scales but also on small scales. The possibility of this breakdown of Euclidean geometry on small scales was considered by Riemann even before the development of quantum mechanics [3]

> There arises from this problem of searching out the simplest facts by which the metric relations of space can be determined, a problem which in nature of things is not quite definite … These facts are, like all facts, not necessary but of a merely empirical certainty; they are hypothesis; one may therefore inquire into their probability, which is truly very great within the bounds of observation, and thereafter decide concerning the admissibility of protracting them outside the limits of observation, not only toward the immeasurably large, but also toward the immeasurably small.

From London's work up to today, there has been very few but also very interesting analysis relating non-Euclidean spaces and quantum mechanics [4–14]. In general, all these attempts to reproduce quantum phenomena are based on Weyl spaces. After Dirac's work [15], the properties of a Weyl space have been considerably changed with respect to Weyl's original idea. In Appendix we describe in some detail what is nowadays called a Weyl space.

Our aim in this work is to propose a similar but different approach to describe quantum effects. We argue that relativistic quantum mechanics can also be understood as a manifestation of what we shall call a Qwist (quantum weyl integrable spacetime). In particular, we construct a geometrical version of Heisenberg's uncertainty principle, which is related to the variation of a vector's length in Qwist. This length variability also happens in Weyl spaces but with a completely different physical meaning as shall be clear in what follows.

In the next section we shall describe the formal basis of our approach. In advance, it seems convenient to stress that to define a physical theory one has first of all to define its kinematic properties. Given an action principle together with a set of dynamical fields does not completely specify the theory. In particular, each theory has its own internal symmetries, which can be used to construct an equivalent class of observers [16]. The point we would like to stress is that one has to define from the beginning which are the allowed transformations of the dynamical fields.

In particular, one should not be misled by similarities of some of the equations and confuse Qwist with any other geometrical space. Let us now define the structure of the Qwist space that we shall identify with the physical spacetime.

1.1 Qwist Geometry

A Qwist is a geometrical manifold $(g_{\mu\nu}; \Gamma^{\alpha}_{\mu\nu})$ endowed with a metric tensor and a symmetric affine connection $\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{(\mu\nu)}$. Its symmetry group is the Manifold Mapping Group $\mathcal{MMG}$, which allow us to perform an arbitrary coordinate transformation.[1] The connection of this space is defined with an extra degree of freedom given by a scalar field $\Omega(x)$. Thus, Qwist is neither a Riemannian space nor a Weyl geometry (see Appendix for more details).

We shall construct our connection such that the non-metricity condition be given by

$$\nabla_\alpha g_{\mu\nu} \equiv -2\,(\ln \Omega)_{,\alpha}\, g_{\mu\nu}. \tag{1}$$

One can use the above equation to solve for the connection giving

$$\Gamma^{\alpha}_{\mu\nu} = \left\{{}^{\,\alpha}_{\mu\nu}\right\} + \frac{1}{\Omega}\left(\Omega_{,\nu}\,\delta^{\alpha}_{\mu} + \Omega_{,\mu}\,\delta^{\alpha}_{\nu} - \Omega^{,\alpha} g_{\mu\nu}\right). \tag{2}$$

Since the connection is not equal to the Christoffel symbol, it is adequate to distinguish between two kind of covariant derivative. The Qwist covariant derivative is constructed with the connection and we shall denote by

$$\nabla_\alpha \xi^{\mu} \equiv \xi^{\mu}{}_{;\alpha} = \xi^{\mu}{}_{,\alpha} + \Gamma^{\mu}_{\alpha\lambda}\xi^{\lambda},$$

and a Riemannian covariant derivative

$$\xi^{\mu}{}_{//\alpha} = \xi^{\mu}{}_{,\alpha} + \left\{{}^{\,\mu}_{\alpha\lambda}\right\}\xi^{\lambda}.$$

[1] Note that the symmetry group of a Qwist space does not include conformal transformations, which are characteristic to Weyl spaces.

The non-metricity condition (1) implies that the length of a vector is not preserved if parallel transported

$$\delta l = -l\Omega^{-1}\Omega_{,\mu}\mathrm{d}x^{\mu}. \tag{3}$$

The above equation has an important physical meaning. It describes how the physical length of a ruler changes from point to point. Note that as long as the extra degree of freedom is a scalar field $\Omega'(x') = \Omega(x)$ there is no gauge freedom in (3).

Furthermore, condition (1) does not suffers from any kind of second clock effect [17, 18] that could be present if instead of the gradient of a scalar function it were a vector field. Considering (3), it is immediate to show that the length of a vector does not change along a closed path

$$\oint \delta l = 0. \tag{4}$$

Thus, this property guarantees that all local measurements such as distances are well defined and can be uniquely determined.

As usual, the curvature tensor can be written in terms of the connection as

$$\mathcal{R}^{\alpha}_{\mu\beta\nu} = \Gamma^{\alpha}_{\mu\beta,\nu} - \Gamma^{\alpha}_{\mu\nu,\beta} + \Gamma^{\varepsilon}_{\mu\beta}\Gamma^{\alpha}_{\nu\varepsilon} - \Gamma^{\varepsilon}_{\mu\nu}\Gamma^{\alpha}_{\varepsilon\beta},$$

which can be used to calculate its traces.

Hence, the Ricci tensor is given by

$$\mathcal{R}_{\mu\nu} = \hat{\mathcal{R}}_{\mu\nu} - \frac{2\nabla_{\nu}(\Omega_{,\mu})}{\Omega} + \frac{4\Omega_{,\mu}\Omega_{,\nu}}{\Omega^2} - g_{\mu\nu}\left[\frac{\Box\Omega}{\Omega} + \frac{\Omega_{,\lambda}\Omega^{,\lambda}}{\Omega^2}\right], \tag{5}$$

where $\Box \equiv \nabla_{\mu}\nabla^{\mu}$ is the d'Alembertian operator and $\hat{\mathcal{R}}_{\mu\nu}$ is the Riemannian part of the Ricci tensor, i.e. the Ricci tensor constructed solely with the Christoffel symbol. For the curvature scalar $\mathcal{R} \equiv g^{\mu\nu}\mathcal{R}_{\mu\nu}$ we find

$$\mathcal{R} = \hat{\mathcal{R}} - 6\Box \ln\Omega - 6(\ln\Omega)_{,\alpha}(\ln\Omega)^{,\alpha} = \hat{\mathcal{R}} - 6\frac{\Box\Omega}{\Omega}, \tag{6}$$

where again $\hat{\mathcal{R}} \equiv g^{\mu\nu}\hat{\mathcal{R}}_{\mu\nu}$ is its corresponding Riemannian part.

It is interesting to show that it is possible to derive the geometrical properties of a Qwist space from a palatini-like variational principle by considering the connection as an independent field. For this purpose, consider the functional

$$I = \int \mathrm{d}^4x\sqrt{-g}\,\Omega^2\mathcal{R}. \tag{7}$$

If we demand this functional to be stationary with respect to variation of the connection we find

$$\delta I = \int \mathrm{d}^4x\sqrt{-g}\,\Omega^2 g^{\mu\nu}\delta\mathcal{R}_{\mu\nu} = \int \mathrm{d}^4x\, Z^{\mu\nu}_{\lambda}\delta\Gamma^{\lambda}_{\mu\nu} = 0, \tag{8}$$

where $Z^{\mu\nu}_{\lambda}$ is given by

$$Z^{\mu\nu}_{\lambda} \equiv (\sqrt{-g}\,g^{\mu\nu}\Omega^2)_{;\lambda} - \frac{1}{2}(\sqrt{-g}\,g^{\mu\alpha}\Omega^2)_{;\alpha}\delta^{\nu}_{\lambda} - \frac{1}{2}(\sqrt{-g}\,g^{\nu\alpha}\Omega^2)_{;\alpha}\delta^{\mu}_{\lambda} = 0. \tag{9}$$

Taking the trace of the above expression yields

$$\left(\sqrt{-g}\, g^{\mu\alpha}\, \Omega^2\right)_{;\,\alpha} = 0. \tag{10}$$

Substituting (10) again in (9) we finally obtain the condition that characterize a Qwist geometry

$$g_{\mu\nu\,;\,\alpha} = -2\,\frac{\Omega_{,\,\alpha}}{\Omega}\, g_{\mu\nu}. \tag{11}$$

In the following sections we will study the dynamics of a spinless charged particle in a Qwist geometry and relate it to a relativistic quantum system. In particular, we will show that this system yields the correct non-relativistic limit, i.e. the Schrödinger equation for a charged particle.

We shall describe quantum mechanics using the Bohm-de Broglie causal interpretation, which are amongst the well-defined formulations that reproduce the same results of the orthodox interpretation but has the advantage of describing matter as point-like particles.

In addition, we propose a geometrical interpretation of Heisenberg's uncertainty principle. This relativistic geometrical version combines both the position-momentum and time-energy relations in an unique principle, which decouples into the usual Heisenberg's uncertainty principles in the non-relativistic limit.

Finally, we discuss how our results are related to Klein-Gordon's equation from a geometrical point of view and conclude in the last section with some final remarks.

2 Relativistic Quantum Mechanics

In this section, we will describe a system composed of a relativistic charged point-like particle interacting with an external electromagnetic field and geometry. Therefore, we are considering a charged particle wandering in a non-Euclidean spacetime that has an independent degree of freedom.

This is perhaps the simplest relativistic system but it will be interesting enough to present the connection between Qwist and quantum mechanics. We decided to include an external electromagnetic field (non-dynamical) insofar as it does not veil the main properties of Qwist. Notwithstanding, if preferable, it is possible to do the same analysis turning the electromagnetic field off, which is equivalent to consider a neutral particle without expense of the quantum physical content.

Our discussion will be based on a variational principle that at the same time provides the dynamical equation for the particle and naturally endows the spacetime with an affine structure that is typical of a Qwist space. As will be clear in what follows, the particle's dynamics is given by integrating a relativistic Hamilton-Jacobi equation where its momentum is described by the derivative of Hamilton's principle function $P_\mu = \partial_\mu S$. In addition, apart from a kind of non-minimal coupling, the geometrical sector shall be given by the Ricci scalar.

Thus, consider the following action that should be justified a posteriori

$$I = \frac{1}{\kappa}\int \mathrm{d}^4x\sqrt{-g}\,\Omega^2\left(\mathcal{L}_g + \mathcal{L}_m\right), \tag{12}$$

with

$$\mathcal{L}_g \equiv \lambda^2 \mathcal{R}, \tag{13}$$

$$\mathcal{L}_m \equiv \frac{g^{\mu\nu}}{\hbar^2}\left(\partial_\mu S - \frac{e}{c}A_\mu\right)\left(\partial_\nu S - \frac{e}{c}A_\nu\right) - \mu^2. \tag{14}$$

In the above expressions g is the determinant of the metric $g_{\mu\nu}$ and $\mathcal{R}$ is the Ricci scalar. As already mentioned, S is the relativistic version of the Hamilton's principle function of the particle which here is coupled to the gauge field $A^\mu = (\varphi, \vec{A})$ describing an external electromagnetic field.

The constants appearing above are the speed of light c, the gravitational constant $\kappa \equiv 16\pi G/c^3$, a dimensionless λ to be determined, the particle's electric charge e and its inverse Compton wavelength $\mu \equiv mc/\hbar$.

We should strongly stress that there is no gravitational interaction in this system. The gravitational constant κ appears as a global factor and it is introduce only to adjust the dimensionality of the action $[I] = \hbar$. Actually, one should not be surprised by the introduction of κ inasmuch this is the only way to change the dimensionality of the curvature scalar to dimension of action.

This system has three dynamical variables to be varied, namely, the dimensionless scalar function Ω, the connection $\Gamma^\lambda_{\mu\nu}$ and S. Variation with respect to the connection give us the geometrical structure of spacetime. Following the derivation of Sect. 1.1, (7)–(11), we have

$$\delta I = \int \mathrm{d}^4x\, Z^{\mu\nu}_\lambda\, \delta\Gamma^\lambda_{\mu\nu} = 0 \quad \Rightarrow \quad g_{\mu\nu;\lambda} = -2\,(\ln\Omega)_{,\lambda}\, g_{\mu\nu}. \tag{15}$$

Varying the Hamilton's function S give us a conservation-like equation,

$$\begin{aligned} \delta I &= -\frac{2}{\kappa\hbar^2}\int \mathrm{d}^4x\sqrt{-g}\,\Omega^2\partial^\mu(\delta S)\left(\partial_\mu S - \frac{e}{c}A_\mu\right) = 0, \\ &= \frac{2}{\kappa\hbar^2}\int \mathrm{d}^4x\sqrt{-g}\,g^{\mu\nu}\left[\Omega^2\left(\partial_\mu S - \frac{e}{c}A_\mu\right)\right]_{//\nu}\delta S = 0, \\ &\Rightarrow \quad g^{\mu\nu}\left[\Omega^2\left(\partial_\mu S - \frac{e}{c}A_\mu\right)\right]_{//\nu} = 0, \end{aligned} \tag{16}$$

where we have dropped a surface term using the four-dimensional Gauss theorem. Note that we have written this expression using a Riemannian covariant derivative, i.e. defined using the Christoffel symbol only.

Finally, variation with respect to Ω gives

$$\left(\partial_\mu S - \frac{e}{c}A_\mu\right)\left(\partial^\mu S - \frac{e}{c}A^\mu\right) - m^2c^2 + \lambda^2\hbar^2\mathcal{R} = 0. \tag{17}$$

Since we are supposing that the Riemannian part of the Qwist geometry is Minkowskian, $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, by choosing $\lambda^2 = 1/6$, the above equation becomes

$$\left(\partial_\mu S - \frac{e}{c}A_\mu\right)\left(\partial^\mu S - \frac{e}{c}A^\mu\right) - m^2c^2 - \hbar^2\frac{\Box\Omega}{\Omega} = 0. \tag{18}$$

Equation (18) generalize the relativistic Hamilton-Jacobi equation with the inclusion of the last term. This extra term which is basically the Weyl curvature scalar respond for all relativistic quantum effects of this system.

The Weyl curvature scalar plays the same role in this relativistic scenario as the Bohmian quantum potential for the non-relativistic quantum mechanics [19–22]. In particular, the classical regime is attained in the limit $\Box\Omega \to 0$.

Given a field configuration for the gauge field A_μ, (16) and (18) define a closed system that is well defined with the specification of appropriate initial conditions. Notwithstanding, since quantum effects are now given by modifications of the space-time structure, it shall be convenient to use a hydrodynamical description and associate the particle's world-line with a time-like congruence.

Following [23–26] we shall define our canonical momentum one-form $\tilde{P} = P_\mu \widetilde{\theta}^\mu$ and de Broglie's mass respectively as

$$P_\mu \equiv \partial_\mu S - \frac{e}{c} A_\mu, \tag{19}$$

$$M \equiv \sqrt{m^2 + \frac{\hbar^2}{6c^2}\mathcal{R}} = m\sqrt{1 + \frac{\hbar^2}{m^2c^2}\frac{\Box\Omega}{\Omega}}. \tag{20}$$

By virtue of (18)–(20) we can define a unitary time-like velocity field by

$$U^\mu = c\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \equiv \frac{1}{M} g^{\mu\nu} P_\nu \quad \Longrightarrow \quad U^\mu U_\mu = c^2 \quad \text{i.e. } \mathrm{d}\lambda^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{21}$$

However, the "Euclidean" particle velocity field is define as

$$V^\mu = c\frac{\mathrm{d}x^\mu}{\mathrm{d}s} \equiv \frac{1}{m} g^{\mu\nu} P_\nu \quad \Longrightarrow \quad V^\mu V_\mu = \frac{M^2}{m^2}c^2. \tag{22}$$

Thus, there is an intrinsic time re-parameterization along the particle's trajectory given by

$$\mathrm{d}s = \frac{\mathrm{d}\lambda}{\sqrt{1 + \frac{\hbar^2}{m^2c^2}\frac{\Box\Omega}{\Omega}}} = \frac{m}{M}\mathrm{d}\lambda. \tag{23}$$

In addition, within this geometrical interpretation, (16) can be viewed as a integrability condition. One can immediately show that for any vector ξ^μ we have

$$\xi^\mu{}_{;\mu} = \frac{1}{\Omega^4}\left(\Omega^4 \xi^\mu\right)_{//\mu} \quad \text{or} \quad \xi_{\mu;\nu}\, g^{\mu\nu} = \frac{1}{\Omega^2}\left(\Omega^2 \xi^\mu\right)_{//\mu},$$

which connects the covariant derivative of Qwist and Riemann geometries. Hence, the continuity-like equation (16) can be written as

$$\frac{1}{\Omega^2}\left(\Omega^2 P^\alpha\right)_{//\alpha} = P_{\alpha\,;\,\beta}\, g^{\alpha\beta} = 0, \tag{24}$$

which can be viewed as an integrability condition on Qwist for the canonical momentum P_μ.

Note that the original system equations (16) and (18) are equivalent to (21) and (24). Notwithstanding the mathematical equivalence, one should bear in mind that its physical interpretation is completely different. The Dynamical equations for the two scalar functions, namely S and Ω is now substituted by kinematic relations for the time-like congruence U^μ.

As a matter of fact, specifying our Cauchy-surface Σ_c as a space-like hypersurface where P^μ is time-like and orthogonal everywhere, one can show [26–28] that the integrability condition given by (24) guarantees that P^μ will always remains time-like. This can be proved as follows.

Consider the variation along the congruence U^μ of the quantity $\Omega^2 M\delta V$, where δV is an infinitesimal 3-volume orthogonal to the congruence. It's straightforward to show that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\Omega^{-2}M\delta V\right) &= U^\alpha\partial_\alpha\left(\Omega^{-2}M\right)\delta V + \Omega^{-2}MU^\alpha\partial_\alpha(\delta V)\\ &= U^\alpha\partial_\alpha\left(\Omega^{-2}M\right)\delta V + \Omega^{-2}M\delta V\, U^\alpha{}_{;\alpha}\\ &= \left(\Omega^{-2}MU^\alpha\right)_{;\alpha}\delta V = \left(\Omega^{-2}P^\alpha\right)_{;\alpha}\delta V\\ &= P_{\alpha\,;\,\beta}\, g^{\alpha\beta}\,\Omega^{-2}\delta V = 0\,. \end{aligned}\tag{25}$$

Hence, as long as δV and Ω^{-2} are always positive and (25) shows that $\Omega^{-2}M\delta V = C^{te}$ then M^2 should not change sign. Given that P^μ is time-like on the Cauchy-surface, i.e. $P^\mu P_\mu(\Sigma_c) = M^2(\Sigma_c) > 0$ then it also has to be time-like everywhere $P_\mu P^\mu(x) > 0$.

Our system is suitable to describe relativistic spinless charged particles. Therefore, one might be concerned how to deal with creation-annihilation processes and if it is possible to consistently define a time-like congruence like P^μ in the presence of both particles and anti-particles.

We should emphasize that this analyses deals only with non-interacting particle which in a sense avoid these kind of difficulties. However, this formalism is as good to describe particles as it is to describe anti-particles due to its invariance under time reversal accompanied by a change of sign of the electrical charge e. A time reversal is equivalent to a change of sign of the Hamilton's function $S \to -S$. Hence it's straightforward to show that the system of (16) and (18), remains unchanged if $e \to -e$. We shall come back later to this issues in Sect. 2.3.

Finally, consider the particle's trajectory, which is given by integrating (21). One should already expect that the particle should not follow a geodetic trajectory as long as it has a quantum force acting on it. Given the particle's velocity field we can calculate this quantum force using the kinematic equations. As a matter of convenience, we shall for the moment consider a Cartesian coordinates so that

$$\frac{\mathrm{d}U^\mu}{\mathrm{d}\lambda} = \frac{c^2}{M}\partial^\mu M - \frac{U^\alpha\partial_\alpha M}{M}U^\mu + \frac{e}{M}F^\mu{}_\alpha U^\alpha,$$

where we have define the components of the electromagnetic tensor as $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. We can readily recognize the last term as being the Lorentz's force while the two first ones are intrinsically geometrical terms that we commonly associate with quantum effects.

The geodesic equation written in its covariant form gives

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2}+\Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}=W^\mu+\frac{e}{Mc^2}F^\mu{}_\alpha U^\alpha, \tag{26}$$

with

$$W^\mu\equiv\partial^\mu\ln\left(\frac{M}{\Omega}\right)-\frac{1}{c^2}U^\alpha\partial_\alpha\ln\left(\frac{M}{\Omega^2}\right)U^\mu. \tag{27}$$

Note that the quantum force W^μ also depends on the velocity field $W^\mu = W^\mu(\Omega, U^\lambda)$.

2.1 Non-relativistic Limit

The above system describes a charged relativistic point-like particle interacting with an external electromagnetic field in a Q-wist geometry. This system can be understood as a relativistic generalization of the Schrödinger picture of quantum mechanics. As we shall now show, it is possible to recover the Schrödinger description by redefining the Hamilton's principal function and taking the usual non-relativistic limit $c\to\infty$.

One of the main difference in the description of a relativistic particle is that its energy contains its inertial rest mass as a potential-like energy. Furthermore, inasmuch the energy is related to the Hamilton's function by $E=-\frac{\partial S}{\partial t}$, we define the non-relativistic version of the Hamilton's function by $S_{nr}\equiv S+mc^2t$.

Substituting this ansatz in (18) we find

$$\frac{\partial S_{nr}}{\partial t}\left(1-\frac{eA_0}{mc^2}\right)+\frac{1}{2m}\left(\vec{\nabla}S_{nr}-\frac{e}{c}\vec{A}\right)^2+eA_0-\frac{e^2A_0A^0}{2mc^2}-\frac{\hbar^2}{2m}\frac{\nabla^2\Omega}{\Omega}$$
$$-\frac{1}{2mc^2}\left[\left(\frac{\partial S_{nr}}{\partial t}\right)^2-\frac{\hbar^2}{\Omega}\frac{\partial^2\Omega}{\partial t^2}\right]=0.$$

By considering the limit $c\to\infty$ it is licit to neglect $\frac{eA_0}{mc^2}$ with respect to 1 and $\frac{e^2A_0A^0}{2mc^2}$ to eA_0. Furthermore, the last two terms go away and we identify the spatial part of the Weyl curvature as the non-relativistic Bohmian quantum potential $Q=-\frac{\hbar^2}{2m}\frac{\nabla^2\Omega}{\Omega}$ [19–22].

Thus, in this limit we have

$$\frac{\partial S_{nr}}{\partial t}+\frac{1}{2m}\left(\vec{\nabla}S_{nr}-\frac{e}{c}\vec{A}\right)^2+eA_0+Q=0, \tag{28}$$

which reproduce the first non-relativistic Bohmian equation, namely the Hamilton-Jacobi like equation.

The second non-relativistic equation is derived directly from (16),

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left[\Omega^2\left(\frac{\partial S_{nr}}{\partial t}-mc^2-eA_0\right)\right]-\vec{\nabla}.\left[\Omega^2(\vec{\nabla}S_{nr}-\frac{e}{c}\vec{A})\right]=0$$

and again neglecting $\frac{eA_0}{mc^2}$ with respect to 1 and the terms containing $1/c^2$

$$\frac{\partial \Omega^2}{\partial t} + \vec{\nabla}.\left[\Omega^2\left(\frac{\vec{\nabla} S_{nr}}{m} - \frac{e\vec{A}}{mc}\right)\right] = 0. \tag{29}$$

As it's well known, (28) and (29) are equivalent to the Schrödinger equation.[2] Hence it is in fact legitimate to view (16) and (18) as relativistic generalizations of the Schröndinger picture of quantum mechanics for a spinless charged point-like particle.

Note that in this limit the geometrical description degenerates to

$$M \longrightarrow m, \qquad U^\mu \longrightarrow V^\mu, \qquad \mathrm{d}\lambda \longrightarrow \mathrm{d}s.$$

The de Broglie's Mass M goes to the particle's rest mass m while both velocity fields coincide as well as its parameterization. In non-relativistic quantum mechanics one can understand the quantum force as a deviation from an Euclidean geodesic [29]. If we chose a coordinate system which has a vanishing Christoffel symbol, one can show that

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}t^2} = h^{\mu\nu}\frac{\partial}{\partial x^\nu} Q + \frac{e}{m} F^\mu{}_\nu V^\nu,$$

where we have defined the projector tensor $h^{\mu\nu}$ along the velocity field V^μ by $h^{\mu\nu} \equiv g^{\mu\nu} - V^\mu V^\nu$. Taking the non-relativistic limit (see [30–32] for more details), the spatial part becomes

$$m\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} = -\vec{\nabla} Q + e\left(\vec{E} + \vec{v}\times\vec{B}\right),$$

where $\vec{v}$ is the particle's 3-velocity. As expected, the particle feels the usual electromagnetic force plus the non-relativistic quantum force given by $-\vec{\nabla} Q$.

2.2 Relativistic Uncertainty Principle

The uncertainty principle is in the core of the orthodox interpretation of quantum mechanics. In a recent paper [29], we have shown that it is possible to interpret geometrically the uncertainty relation for the position and momentum by virtue of a characteristic length scale defined by a 3-d curvature scalar.

However, as long as we have considered a non-relativistic theory, we have established a purely spatial relation which in the relativistic context is unsatisfactory by the requirement of covariance.

In this section, we shall generalize this "spatial" uncertainty principle to a four dimensional relation. In a relativistic theory only four dimensional quantity acquires physical meaning. Accordingly, the interdependence between space and time compel us to somehow relate the uncertainty principle to the interval $\mathrm{d}s^2 = c^2\mathrm{d}\tau^2 - \mathrm{d}l^2$. In

[2] The above mentioned pair of equations are precisely the Bohm-de Broglie system of equations that one finds when uses the polar form of the wave-function $\psi = \Omega \exp\{\frac{i}{\hbar} S_c\}$ in Schrödinger's equation.

addition, taking the non-relativistic limit from this relation we shall show that we recover both uncertainty relation for position and momentum as well as for time and energy.

As we have shown (6), the Qwist curvature scalar can be decomposed in a Riemannian part plus the contribution of the extra degree of freedom Ω. If we suppose that the Riemannian part is flat, $\hat{\mathcal{R}} = 0$, then

$$\mathcal{R} = -\frac{6}{c^2\Omega}\frac{\partial^2\Omega}{\partial t^2} + 6\frac{\nabla^2\Omega}{\Omega}.$$

Apart from the speed of light c, the above equation shows that Qwist curvature scalar which has dimension of inverse length squared is a sum of two terms one with dimension of inverse time squared and the other with dimension of inverse length squared. Thus, we define the Weyl length and Weyl time by

$$L_w \equiv \left\| 6\frac{\nabla^2\Omega}{\Omega} \right\|^{-1/2}; \qquad T_w \equiv \left\| \frac{6}{\Omega}\frac{\partial^2\Omega}{\partial t^2} \right\|^{-1/2}. \tag{30}$$

In a previous paper [29], we have shown that non-relativistic quantum mechanics can be pictured also as a modification of the 3d Euclidean space which we called Qwis. Through this analysis we were able to construct an uncertainty principle for space and momentum based on the characteristic length above defined L_w.

If in fact the Qwist curvature scalar $\mathcal{R}$ is related to quantum phenomena then the above quantities (30) have to somehow yield a measurement of departure from a classical behavior. Recall (23)

$$\mathrm{d}s = \frac{\mathrm{d}\lambda}{\sqrt{1 + \frac{\hbar^2}{m^2c^2}\frac{\Box\Omega}{\Omega}}}.$$

In the non-relativistic limit, i.e. $c \to \infty$, we have $\mathrm{d}\lambda \to c.\mathrm{d}\tau$. Furthermore, the time derivative of Ω can be neglected compared to its spatial derivative. Thus, we have

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}\lambda} &= \sqrt{1 - \frac{\mathrm{d}l^2}{\mathrm{d}\lambda^2}} = \sqrt{1 - \frac{v^2}{c^2}} \approx 1 - \frac{v^2}{2c^2} = 1 - \frac{p^2}{2m^2c^2} \\ &= \frac{1}{\sqrt{1 + \frac{\hbar^2}{m^2c^2}\frac{\Box\Omega}{\Omega}}} \approx 1 + \frac{\hbar^2}{2m^2c^2}\frac{\nabla^2\Omega}{\Omega}.\end{aligned}$$

Comparing the above equations we find

$$p^2.L_w^2 = \frac{\hbar^2}{6}.$$

To interpret the above equation as an uncertainty-like relation we need an extra assumption. Suppose that any length measurement ΔL can only measure distances bigger than the characteristic Weyl length, i.e.

$$\Delta L \geq L_w = \left\| 6\frac{\nabla^2 \Omega}{\Omega} \right\|^{-1/2}. \tag{31}$$

The reasonability of this hypothesis lies on the notion of a classical measurement. A length measurement is made with a standard ruler which is supposed to be a stiff object. Thus, the notion of classical standard ruler presuppose the validity of Euclidean space (see [29] for a more detailed discussion). With the hypothesis (31), the above relation becomes

$$\Delta p^2 . \Delta L^2 \geq \Delta p^2 . L_w^2 \quad \Rightarrow \quad \Delta p . \Delta L \geq \frac{\hbar}{\sqrt{6}}. \tag{32}$$

In addition, it is also possible to derive the uncertainty relation for time and energy. For the moment we shall turn off the electromagnetic interaction ($A_\mu = 0$). Without the electromagnetic potential, (18) can be recast as

$$E^2 = m^2c^4 + p^2c^2 + \frac{\hbar^2}{\Omega}\frac{\partial^2 \Omega}{\partial t^2} - \hbar^2 c^2 \frac{\nabla^2 \Omega}{\Omega}$$

$$\Rightarrow \quad E = mc^2 \sqrt{1 + \frac{1}{c^2}\left(\frac{p^2}{m^2} - \frac{\hbar^2 \nabla^2 \Omega}{m^2 \Omega}\right) + \frac{1}{c^4}\frac{\hbar^2}{m^2 \Omega}\frac{\partial^2 \Omega}{\partial t^2}}.$$

In Sect. 2.1, we have argued that the non-relativistic Hamilton's principle function should be related to the relativistic one by $S_{nr} = S + mc^2 t$ which can be interpreted as $E_{nr} = E - mc^2$. Therefore, expanding in power of c^{-2}, the above equation becomes

$$E_{nr} = \frac{p^2}{2m} - \frac{\hbar^2 \nabla^2 \Omega}{2m\Omega} + \frac{\hbar^2}{2mc^2 \Omega}\frac{\partial^2 \Omega}{\partial t^2} - \frac{1}{8m^3c^2}\left(p^2 - \hbar^2 \Omega^{-1} \nabla^2\right) + \mathcal{O}(1/c^4). \tag{33}$$

In zeroth order,

$$E_{nr} = \frac{p^2}{2m} - \frac{\hbar^2 \nabla^2 \Omega}{2m\Omega}.$$

Note that the non-relativistic energy includes a classical term $p^2/2m$ plus a geometrical term $-\frac{\hbar^2 \nabla^2 \Omega}{2m\Omega}$. Using this result in the second order term we find

$$0 = \frac{\hbar^2}{2mc^2 \Omega}\frac{\partial^2 \Omega}{\partial t^2} - \frac{E_{cl}^2}{2mc^2}$$

which gives

$$E_{cl}^2 . T_w^2 = \frac{\hbar^2}{6}. \tag{34}$$

A time measurement is by definition the length of a 4-d trajectory in a fix spatial point. However, in Qwist appears an intrinsic time re-parameterization (23) which is certainly not include in the Euclidean definition of a standard clock. Generalizing the hypothesis that a length measurement has to be greater than the weyl length L_w, we shall suppose that a time measurement realized by a standard clock has to be greater than the weyl time T_w, i.e. $\Delta t \geq T_w$. Through this hypothesis, the above relation becomes

$$\Delta E_{cl}.\Delta t \geq \Delta E_{cl}.T_w \quad \Rightarrow \quad \Delta E_{cl}.\Delta t \geq \frac{\hbar}{\sqrt{6}}. \tag{35}$$

Note that the uncertainty relation for position and momentum (32) is naturally incorporated into the uncertainty relation for time and energy. Actually, it is the entanglement between space and time that allows us to derive the above relation (35).

In orthodox quantum mechanics, the impossibility of deriving the uncertainty relation for time and energy is commonly associated with the role played by time as an external parameter, i.e. a lack of a time operator. Our geometrical approach does not deal with operators and in fact treats space and time variables on equal footing in the relativistic sense.

2.3 Geometrical Interpretation of Klein-Gordon's Equation

The geometrical approach developed above describes a relativistic particle interacting with a non-Euclidean geometry. The deviation from a pure classical behavior comes from the non-vanishing of the Ricci scalar $\mathcal{R}$. Once $\Box\Omega \to 0$, the classical regime is recovered. Hence, the extra degree of freedom of the Qwist spacetime, namely the scalar function Ω is responsible for the departure from classical behavior.

Interesting enough, our geometrical variational method is formally equivalent to a Klein-Gordon system. Suppose that instead of treating separately the particle's and the geometrical degrees of freedom, we have defined a wave-function by combining both $\psi = \phi.\exp\{\frac{i}{\hbar}S\}$. The ϕ field has dimension of inverse length as commonly is done in field theory and is related to our dimensionless geometrical degree of freedom by $\Omega = \sqrt{\kappa\hbar}\phi$.

With respect to this hybrid object ψ, as should have been expected, (16) and (18) can be combined into one equation, a massive Klein-Gordon equation

$$D_\mu D^\mu \psi - m^2 c^2 \psi = 0,$$

where we have defined the gauge covariant derivative operator $D_\mu \equiv i\hbar\partial_\mu + eA_\mu$. As it is well known, the above equation can be derived using a variational principle with Lagrangian density

$$\mathcal{L} = -\hbar^{-1}\left(\bar{D}_\mu \psi^* D^\mu \psi - m^2 c^2 \psi^* \psi\right), \tag{36}$$

where ψ and ψ^* should be treated as independent dynamical variables. Modulus a total derivative, (36) defines an action

$$I_\psi = \frac{1}{\kappa}\int d^4x\sqrt{-g}\,\Omega^2\left[\frac{\Box\Omega}{\Omega} - \frac{g^{\mu\nu}}{\hbar^2}\left(\partial_\mu S - \frac{e}{c}A_\mu\right)\left(\partial_\nu S - \frac{e}{c}A_\nu\right) + \frac{m^2c^2}{\hbar^2}\right]. \tag{37}$$

Note that the above action is precisely (12), i.e. $I = I_\psi$, if we impose beforehand a Weyl affine structure for the spacetime and identify the scalar curvature with the first term involving derivatives of Ω. Thus, in a sense, our approach is more general as long as the affine structure is derived as a palatini-like variational principle.

This Lagrangian density naturally defines a conserved current

$$J^\mu \equiv -\frac{1}{2\kappa\hbar}\psi^* \overleftrightarrow{D}^\mu \psi = \Omega^2\left(\partial^\mu S - eA^\mu\right) = \Omega^2 g^{\mu\nu} P_\nu, \tag{38}$$

and a energy-momentum tensor

$$T_{\mu\nu} \equiv \frac{2c}{\hbar}\bar{D}_\mu\psi^*\bar{D}_\nu\psi - \frac{c}{\hbar}\left(\bar{D}_\lambda\psi^*\bar{D}^\lambda\psi + m^2\psi^*\psi\right)g_{\mu\nu}. \tag{39}$$

The Klein-Gordon equation can be casted in a Schrödinger-like form by defining a two-component wave-function (see [26, 33] for more detail). In this approach, it is possible to identify positive and negative energy solutions. The energy is defined as the eigenvalue of the Hamiltonian of this Schrödinger picture and it is numerically equal to the spatial integral of the 00-component of the energy-momentum tensor, i.e. $E = \int d^3x\, T_{00}$.

It can be shown that the negative energy solution can be mapped into the positive energy solution by a charge conjugation operation. Therefore, as it is well known, we can associate the negative energy solutions to anti-particle states. This charge conjugation is intrinsically related to the invariance of the system by a change of $\psi \to \psi^*$. Hence, charge conjugation can be imitated by a change $S \to -S$ and $e \to -e$.

Notwithstanding, as previously emphasized, our analysis deals only with non-interacting particle. Thus, even thought this formalism is as adequate to particles as to anti-particles, we have not yet established how interacting process like creation-annihilation should be described.

3 Conclusions

It had been shown in the literature that exist a very interesting connection between non-Euclidean geometries and quantum phenomena. The predominant mechanism to describe quantum effects by geometrical degrees of freedom has been based on Weyl space. In the present work, we proposed a new geometrical approach based on a new geometrical space that we called Qwist.

In Qwist, its extra scalar degree of freedom produce a length variability, which is responsible for change in size of extended object. Furthermore, this is a physical and in principle measurable effect. The physical interpretation of this length variability allowed us to formulate a relativistic and geometrical version of the uncertainty principle.

The non-Euclidean properties of Qwist provide two characteristics dimensions, namely a Weyl length and a Weyl time defined by

$$L_w \equiv \left\| \frac{\nabla^2\Omega}{\Omega} \right\|^{-1/2}; \qquad T_w \equiv \left\| \frac{1}{\Omega}\frac{\partial^2\Omega}{\partial t^2} \right\|^{-1/2}.$$

These quantities quantify the departure from an Euclidean geometry, which can be used to restrict the validity of a classical measurement. Therefore, there should have some restriction on determining the properties of the system if there is a significant departure from Euclidean geometry, i.e. there is a significant manifestation of non-Euclideanity in Qwist.

To support the idea that action (12) correctly describes the dynamics of a relativistic charged "quantum" particle, we have studied its non-relativistic limit and shown that it recovers the usual Schrödinger quantum dynamics. In addition, we reformulated our dynamical variables to connect this relativistic system with the Klein-Gordon equation. In particular, it was necessary to define a new complex field that from our point of view mix geometrical degrees of freedom with the particle's Hamilton principle function.

The present formalism is adequate to describe particles as well as anti-particles. However, it not yet clear how interactions between particles and anti-particles shall be included in this scenario. In addition, it is still an open issue the meaning of a many-particles system and its physical interpretation in view of the Qwist geometrical interpretation.

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Appendix: Weyl Geometry

The Weyl space is construct so as to incorporate the gauge transformation of a vector field ($w_\mu \to w_\mu + \Lambda,_\mu$) analogous to the electromagnetic gauge transformation [34–38]. This new vector field is associated with the non-zero covariant derivative of the metric tensor. The gauge transformation of the Weyl space is a combine transformation of the metric and of the weyl vector field. In a weyl gauge transformation we have

$$g_{\alpha\beta} \longrightarrow g'_{\alpha\beta} = e^{2\Lambda(x)} g_{\alpha\beta}, \tag{40}$$

$$w_\mu \longrightarrow w'_\mu = w_\mu + \Lambda,_\mu, \tag{41}$$

while the connection is constructed so that the covariant derivative of the metric is zero $Dg = 0$ (see [34, 35]), or in components

$$D_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - g_{\rho\beta}\Gamma^\rho_{\mu\alpha} - g_{\alpha\rho}\Gamma^\rho_{\mu\beta} - 2w_\mu g_{\alpha\beta} = 0. \tag{42}$$

This equation can be solved to give

$$\Gamma^\alpha_{\mu\nu} = \left\{ {}^{\alpha}_{\mu\nu} \right\} - \left(\delta^\alpha_\mu w_\nu + \delta^\alpha_\nu w_\mu - g_{\mu\nu} w^\alpha\right), \tag{43}$$

where $\left\{ {}^{\alpha}_{\mu\nu} \right\}$ is the Christoffel symbol. Note that the covariant derivative defined in (42) is the usual covariant derivative constructed with the connection plus an extra term related to the weyl vector field (w_μ). To distinguish it from the conventional covariant derivative, one call it co-covariant derivative and defined it as follows. If a tensor is transformed under a weyl gauge transformation into $A_\mu \to A'_\mu = e^{n\Lambda} A_\mu$ then it is called a tensor of power n and its co-covariant derivative is given by

$$D_\mu \xi_\alpha \equiv \nabla_\mu \xi_\alpha - n\xi_\alpha \Lambda_\mu, \tag{44}$$

where the $\vec{\nabla}$ defines the covariant derivative constructed with the connection $\nabla_\nu \xi_\mu \equiv \xi_{\mu,\nu} - \Gamma^\alpha_{\nu\mu}\, \xi_\alpha$. Hence, we have defined the metric $g_{\mu\nu}$ as a tensor of power 2 in the weyl sense.

From (42) one can easily show that

$$\nabla_\mu g_{\alpha\beta} = 2w_\mu g_{\alpha\beta}. \tag{45}$$

Thus, it is also straightforward to show that the length of a vector $l \equiv \sqrt{g_{\mu\nu} l^\mu l^\nu}$ parallel transported will change from point to point such that in an infinitesimal displacement is given by

$$\delta l = l w_\mu \mathrm{d}x^\mu. \tag{46}$$

Note that the length of a vector is not gauge invariant. As a matter of fact, under a weyl gauge transformation we have $l \to l' = e^{\Lambda(x)} l$. Thus, by an adequate gauge choice one can always make $\delta l = 0$ in a point. It suffice to choose $\Lambda = -\int w_\mu \mathrm{d}x^\mu$, then

$$\delta l' = \delta\Lambda\, l' + e^\Lambda \delta l = l' \left(\Lambda_{,\mu} + w_\mu\right) \mathrm{d}x^\mu = 0.$$

However, its variation along a closed curve will not be zero since in general the curvature $K_{\mu\nu} \equiv w_{\mu,\nu} - w_{\nu,\mu} \neq 0$. There is a special case of Weyl space when the Weyl vector w_μ is the gradient of a function, i.e. $w_\mu = \partial_\mu f$. In this case, the length of a vector will not change along a closed curve. Furthermore, one can choose $\Lambda = -f$ so that $w'_\mu = f_{,\mu} + \Lambda_{,\mu} = 0$, which could motivate us to associate this subclass of Weyl spaces with conformal transformations of the metric tensor.

However, even in this special case when the weyl vector is the gradient of a function, the Weyl space is not equivalent to a conformal transformation of the associated Riemannian space.

A conformal transformation of the metric tensor takes for example $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\Lambda} g_{\mu\nu}$. This transformation maps a Riemannian space $\mathcal{M}$ characterized by $(g_{\mu\nu}, \{^{\alpha}_{\mu\nu}\})$ into another Riemannian space $\widetilde{\mathcal{M}}$ with $(\tilde{g}_{\mu\nu}, \widetilde{\{^{\alpha}_{\mu\nu}\}})$. Thus, we have

$$\widetilde{\left\{^{\alpha}_{\mu\nu}\right\}} \equiv \frac{1}{2}\tilde{g}^{\alpha\xi}\left(\tilde{g}_{\xi\nu,\mu} + \tilde{g}_{\mu\xi,\nu} - \tilde{g}_{\mu\nu,\xi}\right) = \left\{^{\alpha}_{\mu\nu}\right\} + \left(\delta^\alpha_\nu\, \Lambda_{,\mu} + \delta^\alpha_\mu\, \Lambda_{,\nu} - g_{\mu\nu}\Lambda^{,\alpha}\right)$$

which can be inverted

$$\left\{^{\alpha}_{\mu\nu}\right\} = \widetilde{\left\{^{\alpha}_{\mu\nu}\right\}} - \left(\delta^\alpha_\nu\, \Lambda_{,\mu} + \delta^\alpha_\mu \Lambda_{,\nu} - g_{\mu\nu}\, \Lambda^{,\alpha}\right) \tag{47}$$

becoming very similar to (43).

However, the new Riemannian manifold $\widetilde{\mathcal{M}}$ still satisfies the metricity condition, namely, $\widetilde{\nabla}_\alpha \tilde{g}_{\mu\nu} = 0$. The covariant derivative of the transformed metric tensor is still zero. The conformal transformation does not change the metricity condition. Note that if we blindly calculate $\nabla_\alpha \tilde{g}_{\mu\nu}$ we find

$$\nabla_\alpha \left(\widetilde{g}_{\mu\nu}\right) = \nabla_\alpha \left(e^{2\Lambda}\tilde{g}_{\mu\nu}\right) = 2\Lambda_{,\alpha} g_{\mu\nu} \tag{48}$$

which is also very similar to (45). However, contrary to (45), (48) has no physical meaning. It is the covariant derivative in $\mathcal{M}$ applied to the metric tensor of $\widetilde{\mathcal{M}}$.

References

1. Weyl, H.: Sitz. Preuss. Akad. Wiss. **26**, 465 (1918)
2. London, F.: Z. Phys. **42**, 375 (1927)
3. Riemann: In: Smith, D.E. (ed.) A Source Book in Mathematics, vol. 2. Dover, New York (1959)
4. Israelit, M.: Found. Phys. **28**, 205 (1998)
5. Israelit, M.: Found. Phys. **29**, 1303 (1999)
6. Israelit, M.: Found. Phys. **32**, 295 (2002)
7. Israelit, M.: Found. Phys. **32**, 945 (2002)
8. Canuto, V., Adams, P.J., Hsieh, S.H., Tsiang, E.: Phys. Rev. D **16**, 1643 (1977)
9. Rosen, N.: Found. Phys. **13**, 363 (1983)
10. Koch, B.: arxiv:0810.2786 [hep-th] (2008)
11. Koch, B.: arxiv:0901.4106 [gr-qc] (2009)
12. Santamato, E.: Phys. Rev. D **29**, 216–222 (1984)
13. Holland, P., Kyprianidis, A., Vigier, J.P.: Phys. Lett. A **107**, 376 (1985)
14. Gueret, Ph., Holland, P., Kyprianidis, A., Vigier, J.P.: Phys. Lett. A **107**, 379 (1985)
15. Dirac, P.M.: Proc. R. Soc. Lond. A **333**, 403 (1973)
16. Anderson, J.L.: Principles of Relativity Physics. Academic Press, London (1967)
17. Novello, M., Oliveira, L.A.R., Salim, J.M., Elbaz, E.: Int. J. Mod. Phys. D **1**(4), 641–677 (1992)
18. Perlick, V.: Class. Quantum Gravity **8**, 1369 (1991)
19. Bohm, D.: Phys. Rev. **85**, 166 (1952)
20. Bohm, D.: Phys. Rev. **85**, 180 (1952)
21. Bohm, D., Hiley, B.J.: The Undivided Universe. Routledge, London (1993)
22. Holland, P.R.: The Quantum Theory of Motion. Cambridge University Press, Cambridge (1993)
23. de Broglie, L.: Non-Linear Wave Mechanics: A Causal Interpretation. Elsevier, Amsterdam (1960)
24. Halbwachs, F.: Théorie Relativiste des Fluides a Spin. Gauthier-Villars, Paris (1960)
25. Holland, P.R.: Found. Phys. **17**, 345–363 (1987)
26. Cufaro-Petroni, N., Dewdney, C., Holland, P., Kyprianidis, T., Vigier, J.P.: Phys. Lett. A **106**, 368–370 (1984)
27. Kyprianidis, A.: Phys. Lett. A **111**, 111–116 (1985)
28. Santamato, E.: J. Math. Phys. **25**, 2477–2480 (1984)
29. Novello, M., Salim, J.M., Falciano, F.T.: arxiv:0901.3741 [gr-qc] (2009)
30. Landau, L.D., Lifshitz, E.M.: The Classical Theory of Fields. Elsevier, Oxford (1975)
31. Synge, J.L.: Relativity: The Special Theory. North-Holland, Amsterdam (1958)
32. Tolman, R.C.: Relativity Thermodynamics and Cosmology. Oxford University Press, Oxford (1962)
33. Feshbach, H., Villars, F.: Rev. Mod. Phys. **30**, 24 (1958)
34. Rosen, N.: Found. Phys. **12**, 213 (1982)
35. Wheeler, J.T.: Phys. Rev. D **41**, 431 (1990)
36. Shojai, F., Shojai, A.: Gravit. Cosmol. **9**, 163 (2003)
37. Shojai, F., Shojai, A.: arxiv:gr-qc/0404102 (2004)
38. Carroll, R.: arxiv:0705.3921 [gr-qc] (2008)