Electrostatics

Charge distributions using discontinuous functions

Unbounded Solutions of Poisson's Equations

1,2,3 D charge distributions

Uniform 1D charge distributions

$$\rho(\mathrm{z}) = \sigma\delta(\mathrm{z} - \mathrm{z}')$$

Infinite plane at z = z'

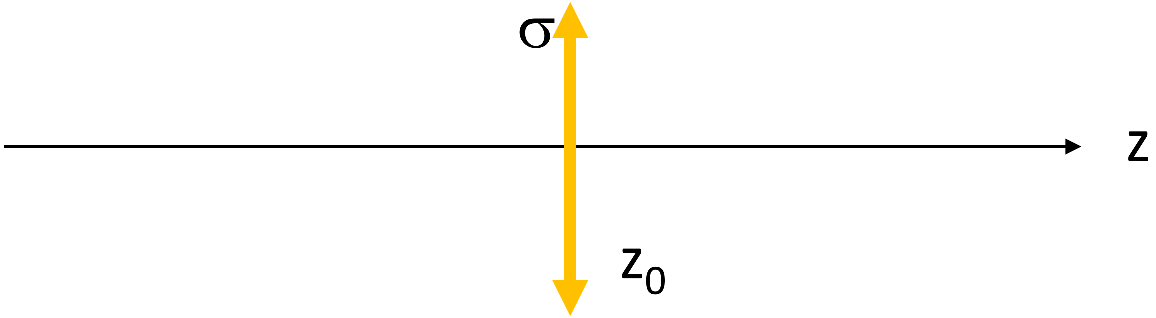
$$\rho(\rho) = \sigma\delta(\rho - \rho')$$

Infinite cylindrical surface of radius ρ'

$$\rho(\mathrm{r}) = \sigma\delta(\mathrm{r} - \mathrm{r}')$$

Spherical surface at radius r'

Electric field generated by $\sigma\delta(z-z_0)$

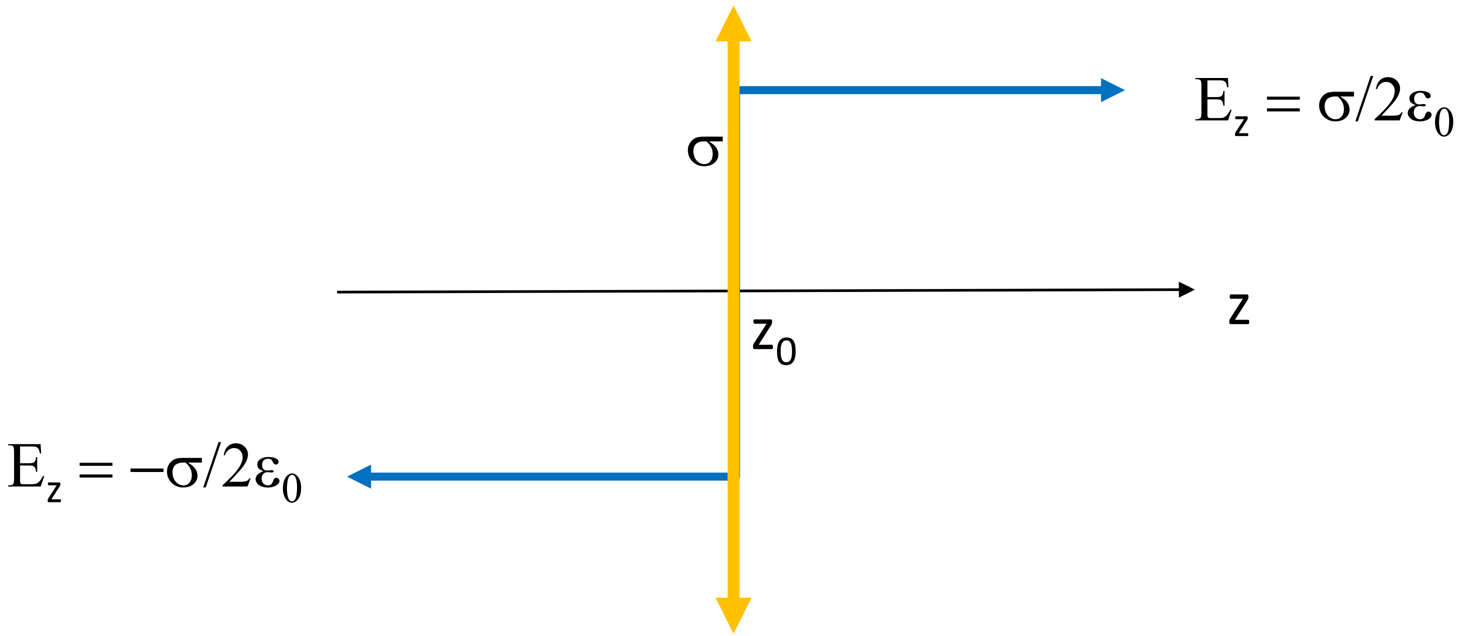


$$\nabla\cdot\mathbf{E}(z)=\frac{\sigma}{\varepsilon_0}\delta(z-z_0) \qquad \nabla\times\mathbf{E}(z)=0$$

$$\frac{\partial E_z}{\partial z}=\frac{\sigma}{\varepsilon_0}\delta(z-z_0) \quad\rightarrow\quad E_z(z)=\frac{\sigma}{2\varepsilon_0}\varepsilon(z-z_0)+C$$

$$E_z(z_0+\Delta z)=-E_z(z_0-\Delta z) \quad\rightarrow\quad C=0$$

$$E_z(z) = \frac{\sigma}{2\varepsilon_0}\,\varepsilon(z - z_0) \qquad E_x = E_y = 0$$



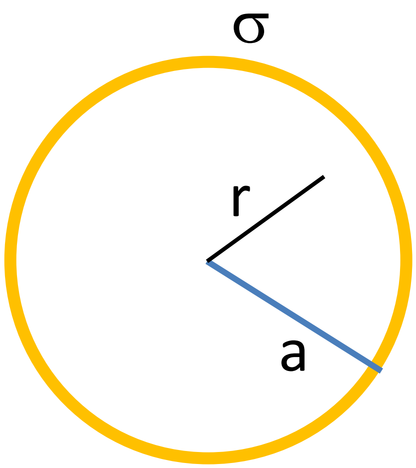
Electric field generated by σδ(r-a)

$$\nabla \cdot \mathbf{E} = \frac{\rho(r)}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0}\delta(r-a)$$

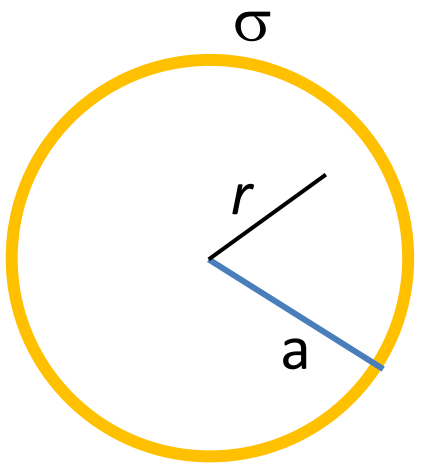
$$\frac{1}{r^2}\frac{d}{dr}[r^2 E_r(r)] = \frac{\sigma}{\varepsilon_0}\delta(r-a)$$

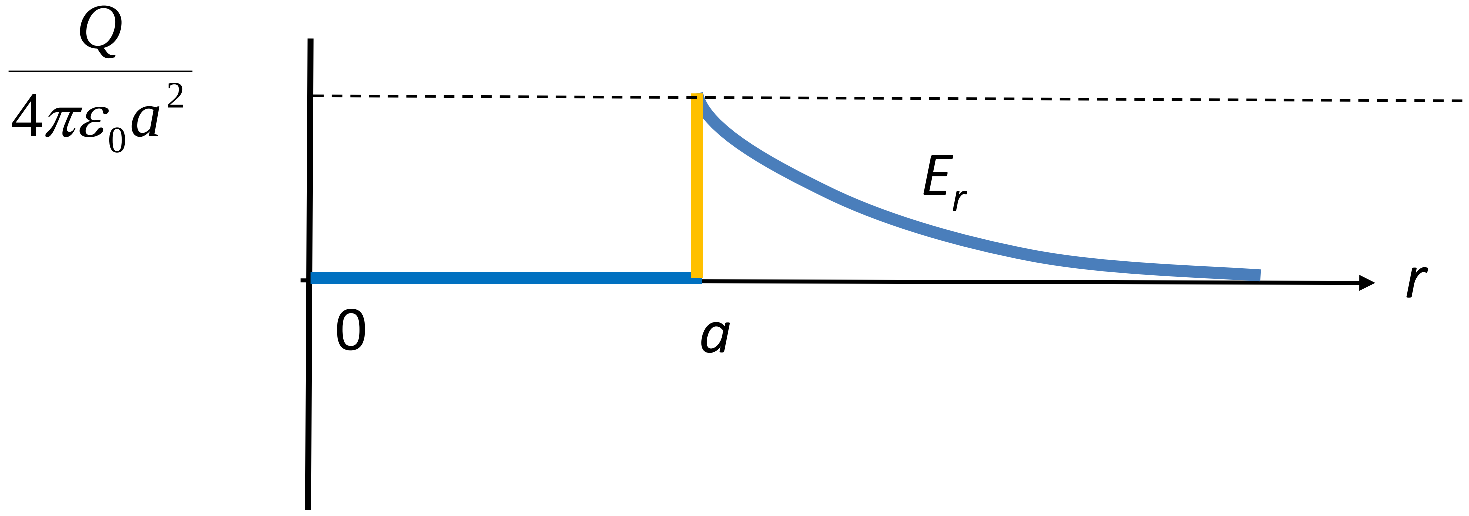
$$\int_0^r dr' \frac{d}{dr'}[r'^2 E_r(r')] = r^2 E_r(r) = \frac{\sigma}{\varepsilon_0}\int_0^r r'^2 \delta(r'-a)dr$$

$$E_r(r) = \frac{\sigma}{\varepsilon_0}\frac{a^2}{r^2}\int_0^r \delta(r'-a)dr = \frac{\sigma}{\varepsilon_0}\frac{a^2}{r^2}\theta(r-a)$$



$$E_r(r) = \frac{Q}{4\pi\varepsilon_0 r^2}\theta(r-a)$$





Electric potential of σδ(r-a)

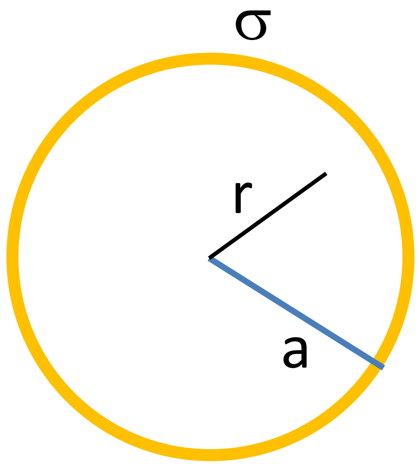
$$\Phi(r) = -\int_{\infty}^{r} E(r')dr' = -\frac{Q}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{\theta(r'-a)}{r'^2}dr' = \frac{Q}{4\pi\varepsilon_0}\theta(r'-a)[\frac{1}{r'}-\frac{1}{a}]_{\infty}^{r}$$

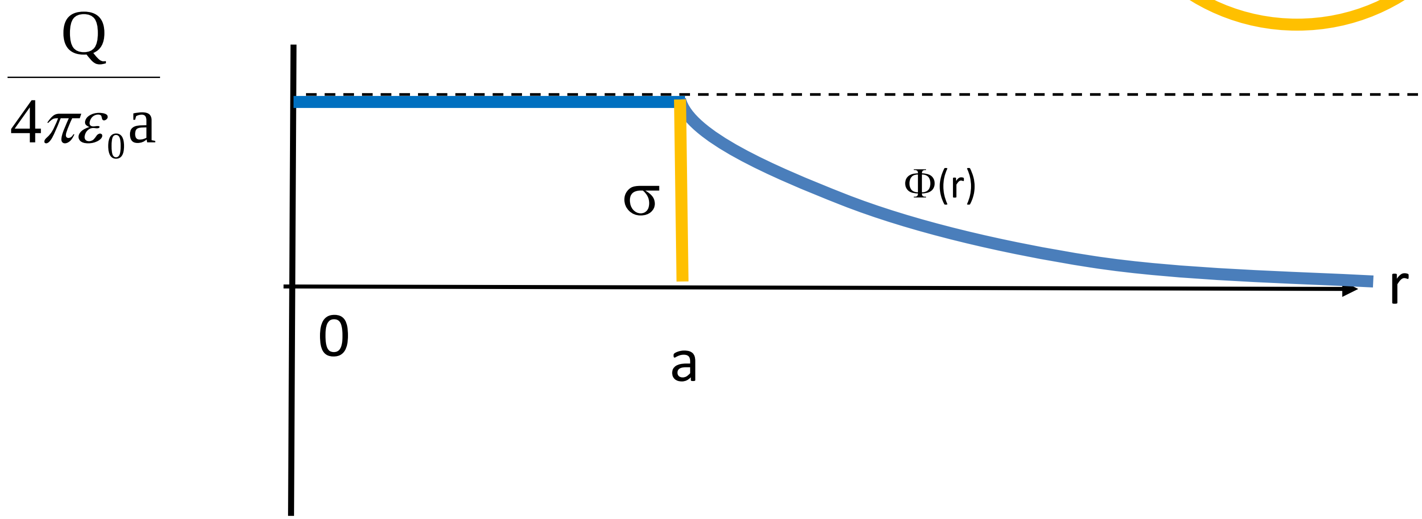
$$\Phi(r) = \frac{Q}{4\pi\varepsilon_0}\left\{[\frac{1}{r}-\frac{1}{a}]\theta(r-a)+\frac{1}{a}\right\}$$

$$\Phi(r) = \frac{Q}{4\pi\varepsilon_0}\left\{[\frac{1}{r}-\frac{1}{a}]\theta(r-a)+\frac{1}{a}[\theta(r-a)+\theta(a-r)]\right\}$$

$$\Phi(r) = \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{r}\theta(r-a)+\frac{1}{a}\theta(a-r)\right\}$$

$$\Phi(r) = \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{r}\theta(r-a) + \frac{1}{a}\theta(a-r)\right\}$$





Electric field for 1D continuous charge distributions

Cartesian coordinate z

$$E_z(z) = \int_{-\infty}^{\infty} G_z(z,z')\rho(z')dz' \quad \rightarrow \quad \frac{dG_z}{dz} = \frac{1}{\varepsilon_0}\delta(z-z')$$

$$G_z(z,z') = \frac{1}{2\varepsilon_0}\varepsilon(z-z') \quad \rightarrow \quad E_z(z) = \frac{1}{2\varepsilon_0}\int_{-\infty}^{\infty}\varepsilon(z-z')\rho(z')dz'$$

Spherical coordinate *r*

$$E_r(r)=\int_0^\infty G_r(r,r')\rho(r')dr' \quad\rightarrow\quad \frac{1}{r^2}\frac{d\left[r^2G_r\right]}{dr}=\frac{1}{\varepsilon_0}\delta(r-r')$$

$$G_r(r,r')=\frac{1}{\varepsilon_0}\frac{r'^2}{r^2}\theta(r-r') \quad\rightarrow\quad E_r(r)=\frac{1}{\varepsilon_0 r^2}\int_0^\infty r'^2\theta(r-r')\rho(r')dr'$$

Cylindrical coordinate ρ

$$E_\rho(\rho)=\int_0^\infty G_\rho(\rho,\rho')\rho(\rho')d\rho' \quad\rightarrow\quad \frac{1}{\rho}\frac{d\left[\rho G_\rho\right]}{d\rho}=\frac{1}{\varepsilon_0}\delta(\rho-\rho')$$

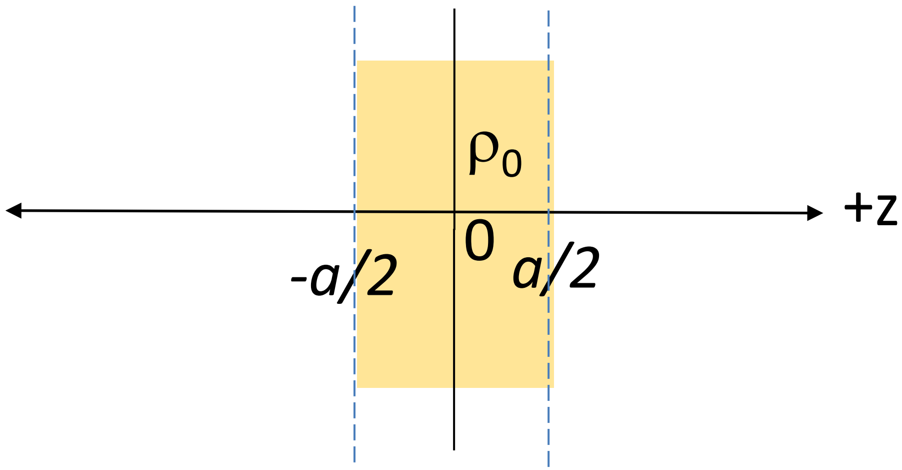
$$G_\rho(\rho,\rho')=\frac{1}{\varepsilon_0}\frac{\rho'}{\rho}\theta(\rho-\rho') \quad\rightarrow\quad E_\rho(\rho)=\frac{1}{\varepsilon_0\rho}\int_0^\infty \rho'\theta(\rho-\rho')\rho(\rho')d\rho'$$

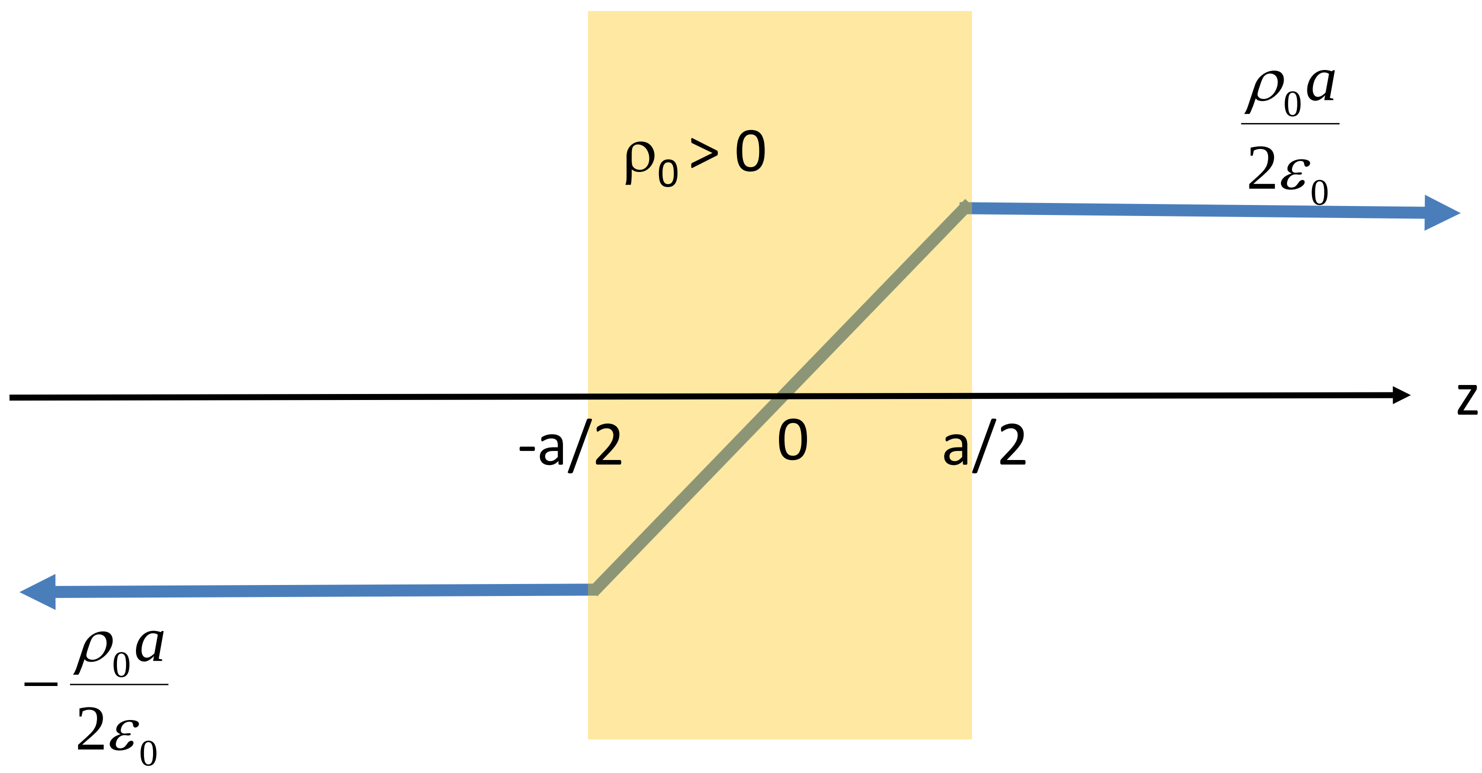
Uniformly charged slab

$$\rho(z') = \rho_0 \theta(\frac{a}{2} - |z'|)$$

$$E_z(z) = \frac{1}{2\varepsilon_0} \int_{-\infty}^{\infty} \varepsilon(z - z')\rho(z')dz' = \frac{\rho_0}{2\varepsilon_0} \int_{-a/2}^{a/2} \varepsilon(z - z')dz'$$

$$E_z(z) = \frac{\rho_0}{2\varepsilon_0} (z' - z)\varepsilon(z - z') \Big|_{-a/2}^{a/2} = \frac{\rho_0}{2\varepsilon_0} \left\{ 2z\theta(\frac{a}{2} - |z|) + a\theta(|z| - \frac{a}{2})\varepsilon(z) \right\}$$





$$E_z(z) = \frac{\rho_0}{2\varepsilon_0}\left\{2z\theta(\frac{a}{2}-|z|)+a\theta(|z|-\frac{a}{2})\varepsilon(z)\right\}$$

Uniformly charged cylinder

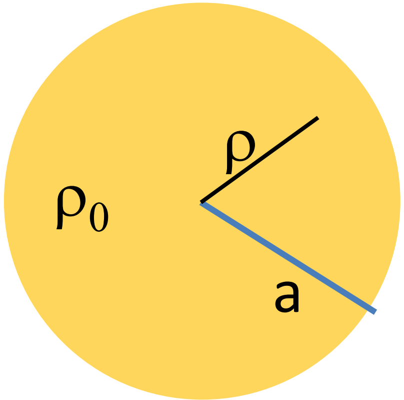
$$\rho(\rho) = \rho_0 \theta(a-\rho)$$

$$E_r(\rho) = \frac{\rho_0}{\varepsilon_0 \rho} \int_0^{\infty} \rho' \theta(\rho - \rho') \theta(a - \rho') d\rho'$$

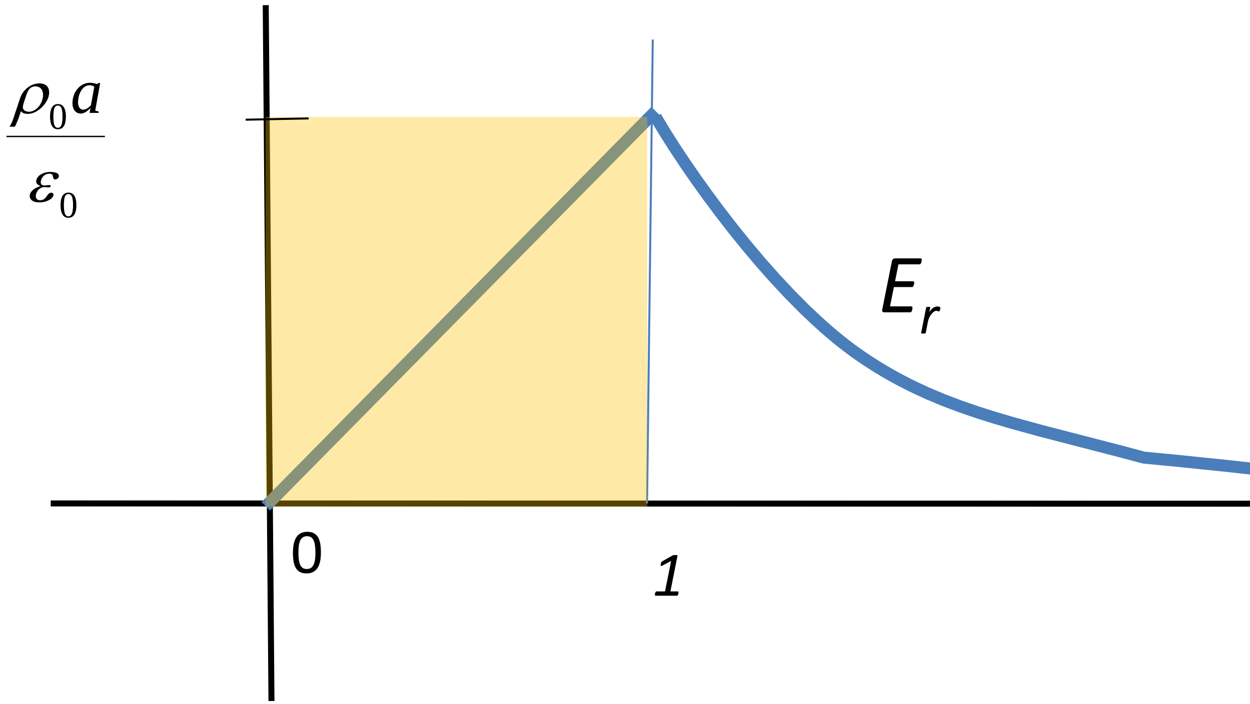
$$E_\rho(\rho) = \frac{\rho_0}{\varepsilon_0 \rho} \int_0^{a} \rho' \theta(\rho - \rho') d\rho'$$

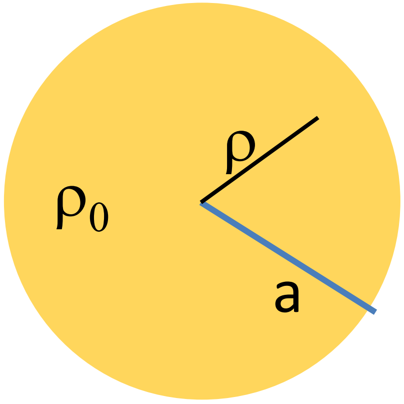
$$E_\rho(\rho) = \frac{\rho_0}{\varepsilon_0 \rho} \left[\frac{(\rho'^2 - \rho^2)}{2} \theta(\rho - \rho') \right]_0^a$$

$$E_\rho(\rho) = \frac{\rho_0 a}{\varepsilon_0} \left[\frac{a}{\rho} \theta(\rho - a) + \frac{\rho}{a} \theta(a - \rho \right]$$



$$E_\rho(\rho) = \frac{\rho_0 a}{\varepsilon_0}\left[\frac{a}{\rho}\theta(\rho - a) + \frac{\rho}{a}\theta(a - \rho\right]$$





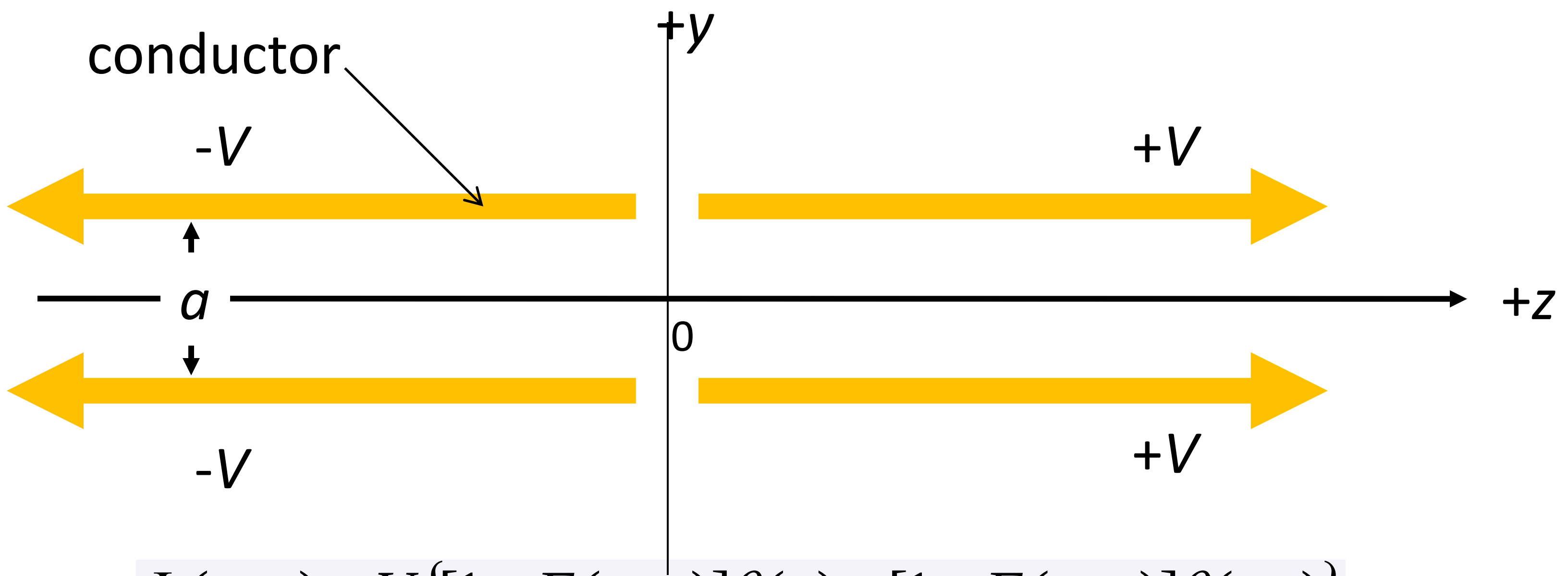
Electron Optic Devices

Electrostatic lens

Quadrupole lenses

Solenoid lenses

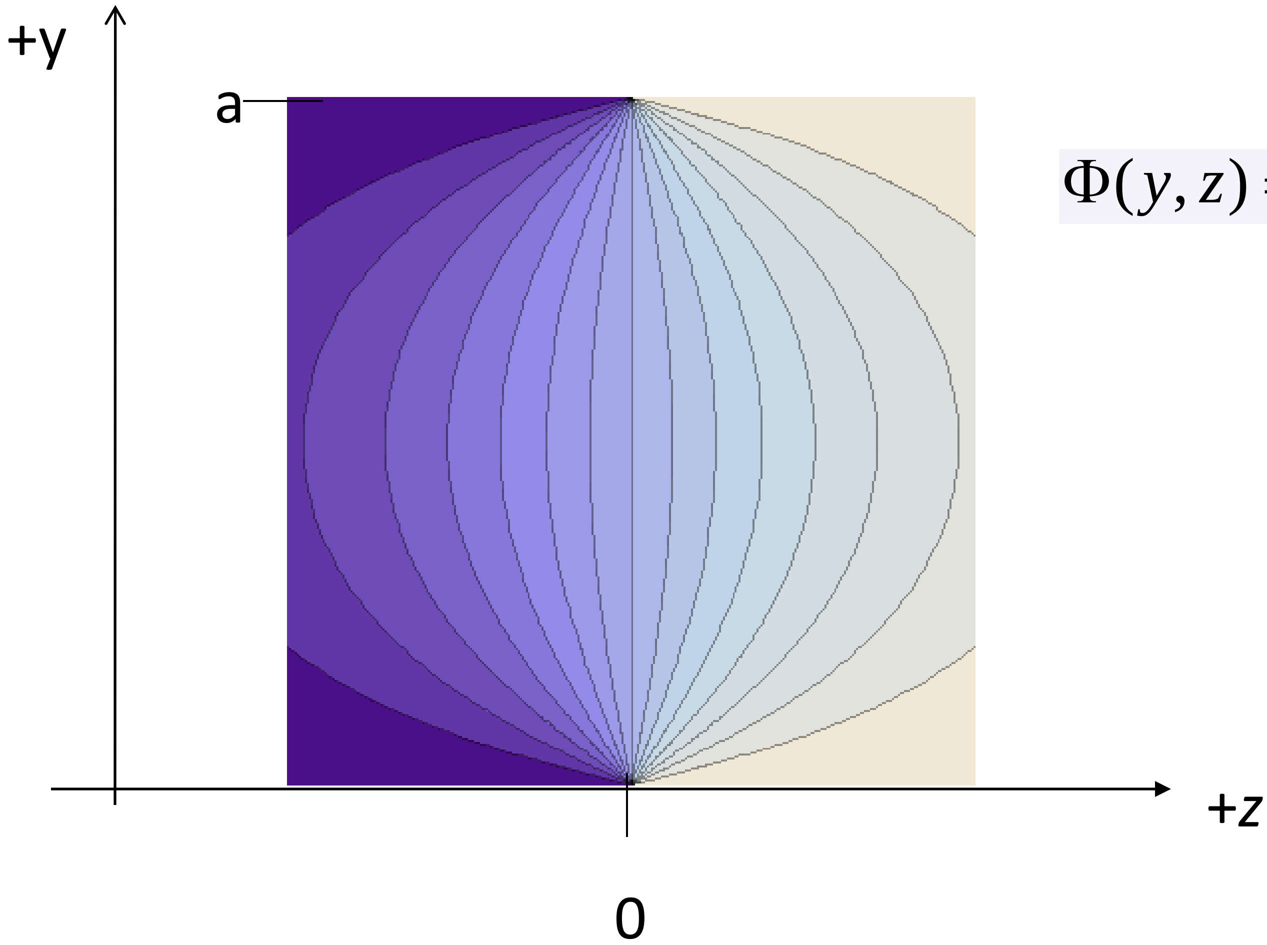
Electrostatic lens



$$\Phi(y,z) = V\{[1+F(y,z)]\theta(z) + [1-F(y,z)]\theta(-z)\}$$

$$F(y,z) = \frac{2}{\pi}\tan^{-1}\left[\frac{\cos(\frac{\pi y}{a})}{\sinh(\frac{\pi z}{a})}\right]$$

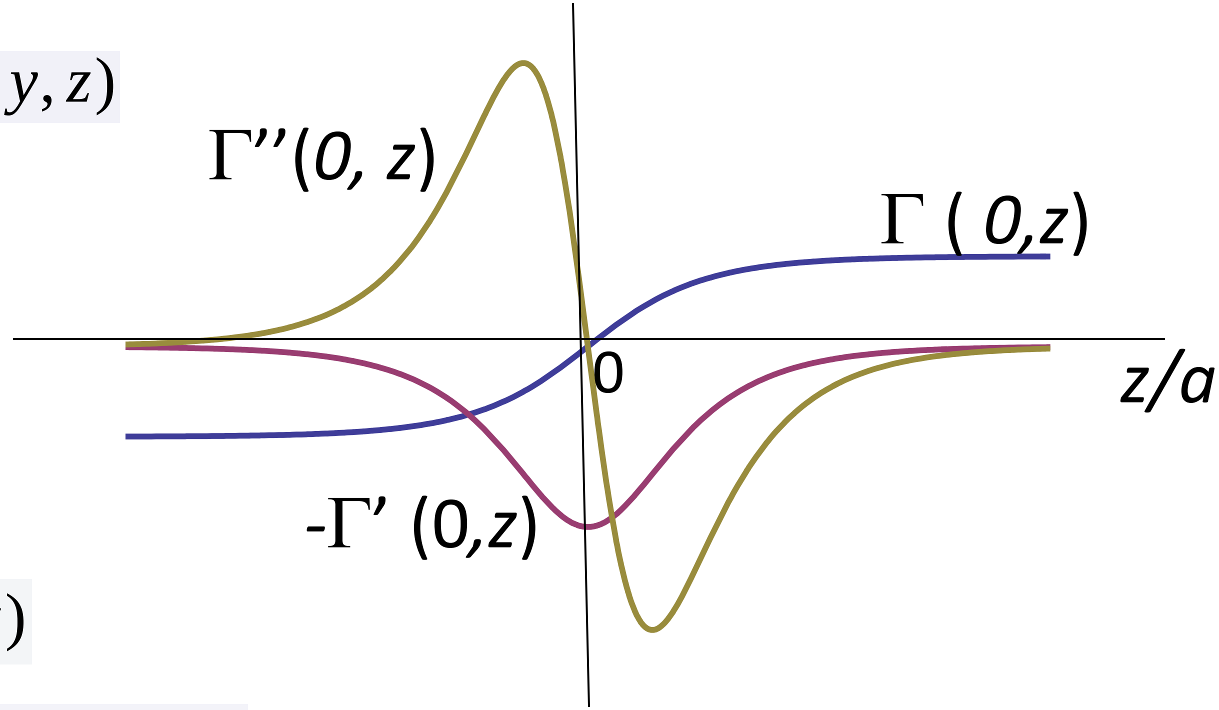
Equipotential lines



$\Phi(y,z) = \text{Constant}$

Electron forces

$$\Phi(y,z) = V\Gamma(y,z)$$



$$\boldsymbol{F} = -|e|\boldsymbol{E}(y,z)$$

$$F_z(y,z) = -|e|\frac{V}{a}\Gamma'(y,z)$$

$$F_y(y,z) = -y|e|\Gamma''(y,z)$$

Electron trajectory equation

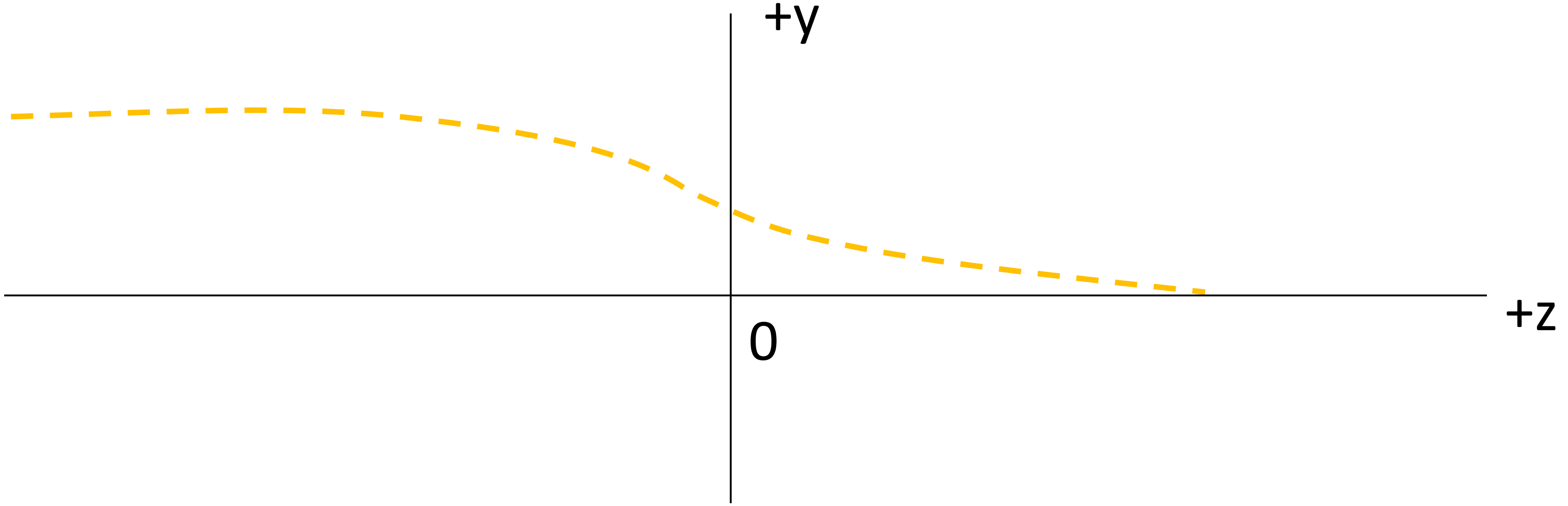
Conservation of energy

$$|e|V = |e|\Phi + \gamma mc^2 \qquad \gamma(y,z) = \frac{1}{\sqrt{1-\beta^2}} = \frac{|e|\left[V - \Phi(y,z)\right]}{mc^2}$$

Trajectory equation

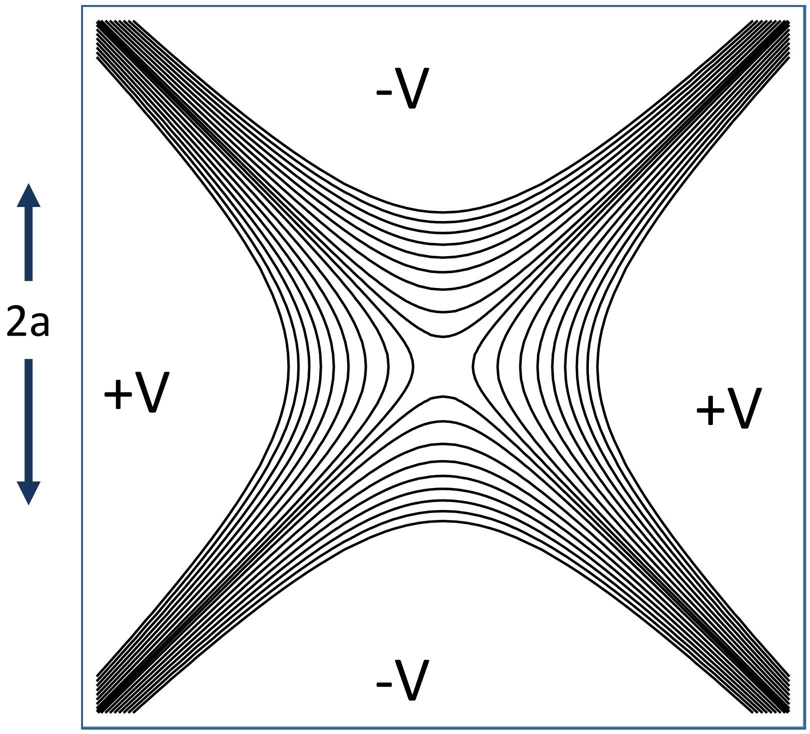
$$y'' = \frac{1+(y')^2}{\gamma\beta^2}\left[\frac{\partial\gamma}{\partial y} - y'\frac{\partial\gamma}{\partial z}\right] \qquad y' = \frac{dy}{dz} \qquad y = y(z)$$

Electron trajectory



Electric quadrupole

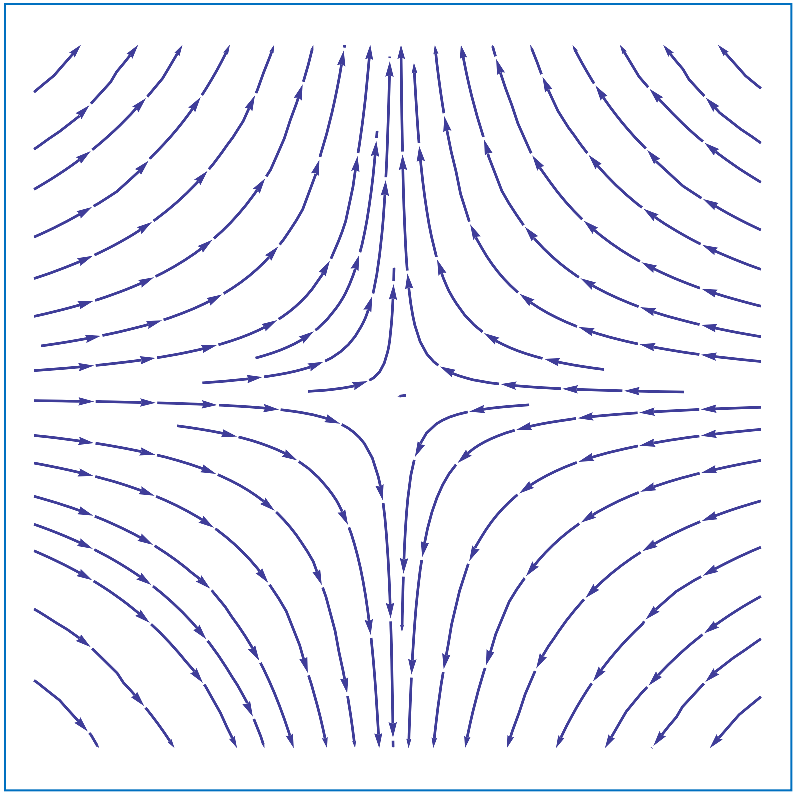
$$\Phi(x,y) = \frac{V}{a^2}(x^2 - y^2)$$



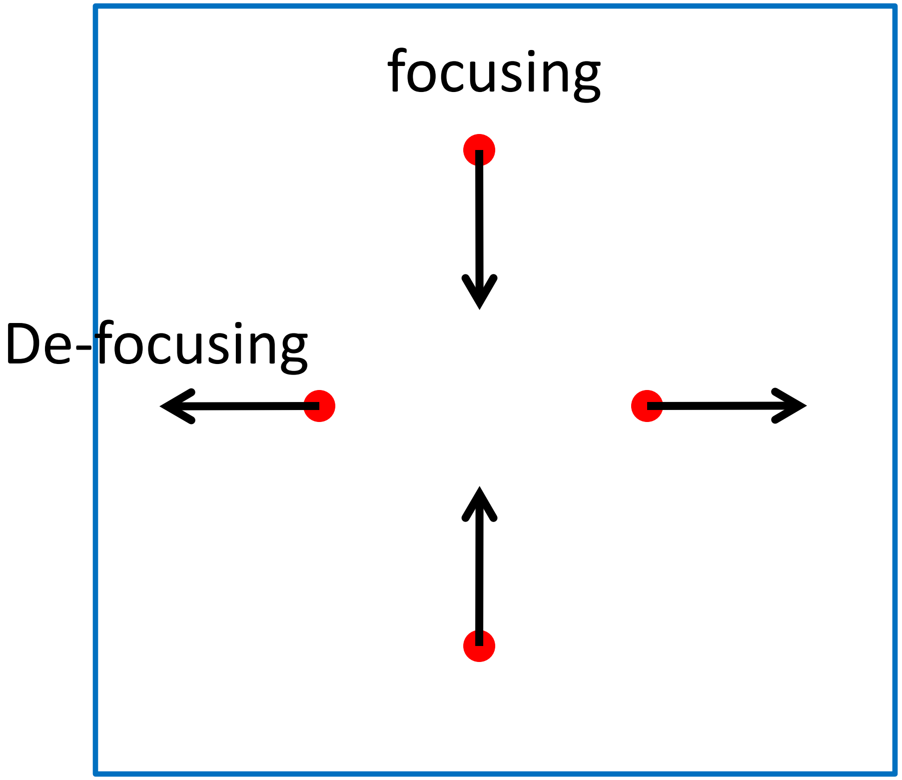
Constant potential lines

Electric Quadrupole

$$\mathbf{E}(x, y) = \frac{2V}{a^2}[-x, y]$$



Electric field lines



Force on electrons

Magnetic quadrupole focusing

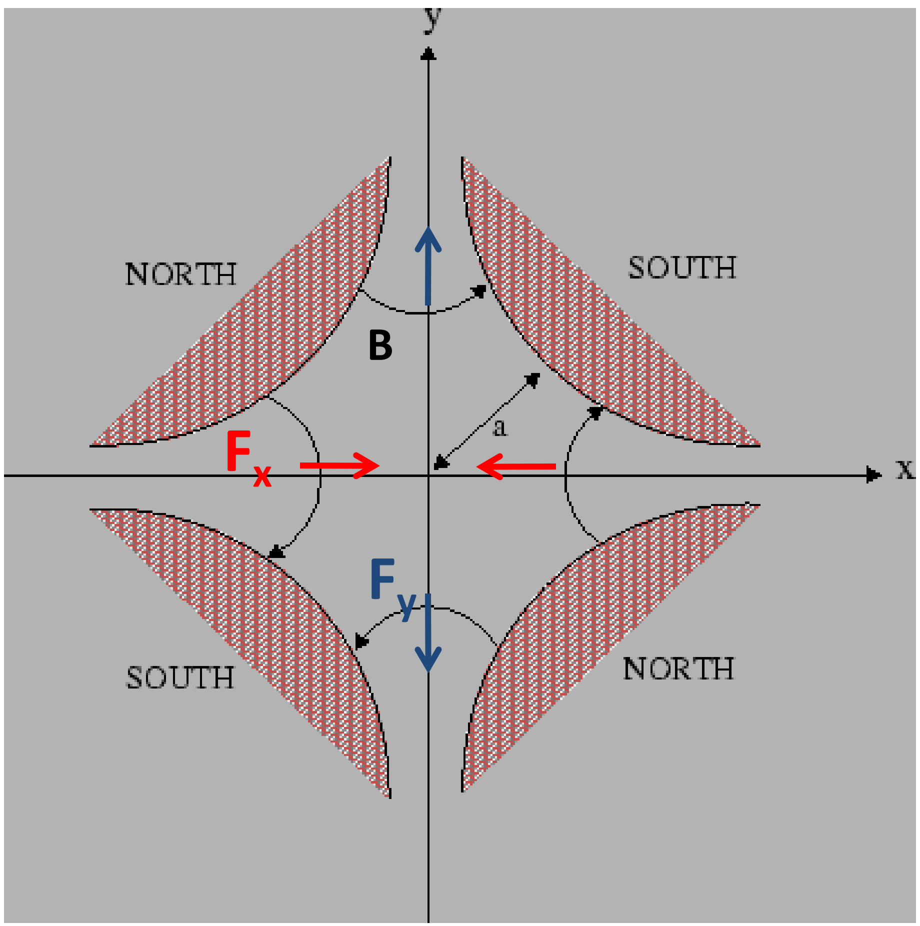
$$\mathbf{A} = -\mathbf{e}_z \frac{B_0}{a} xy \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

$$B_x = B_0 \frac{y}{a} \qquad B_y = B_0 \frac{x}{a}$$

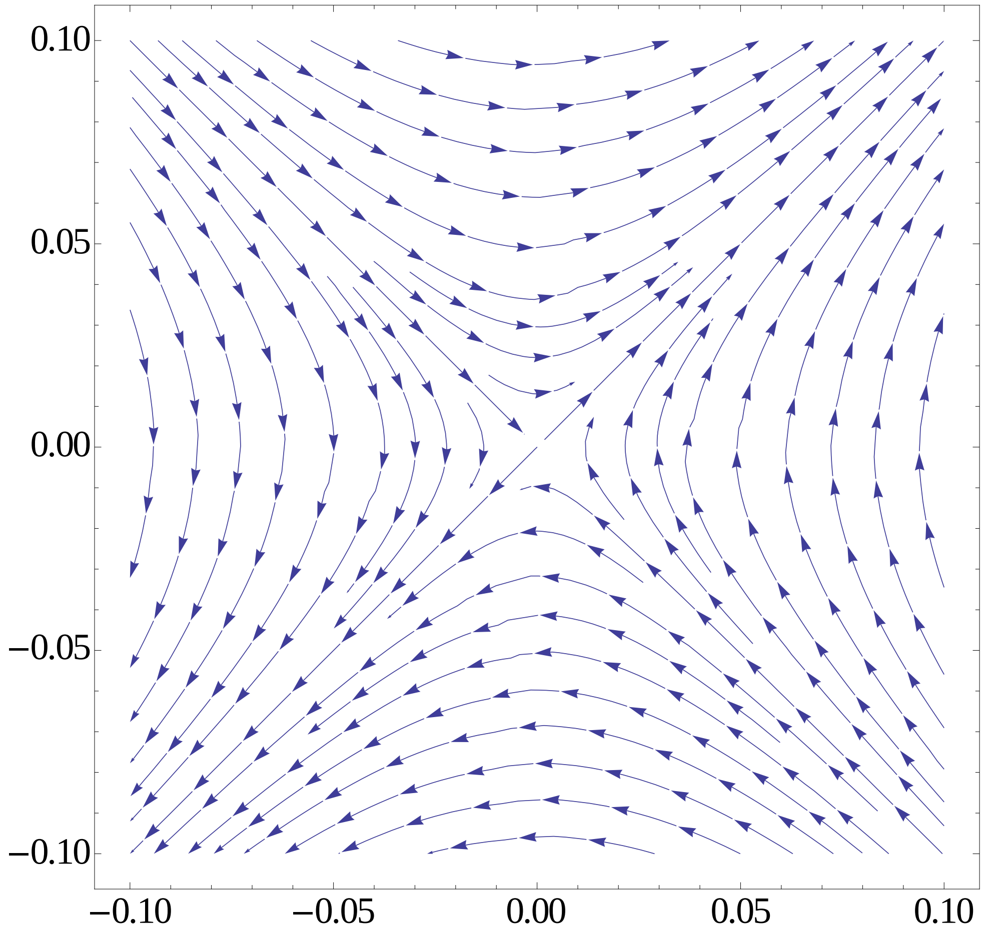
For electrons

$F_x \propto -x$ Focusing force

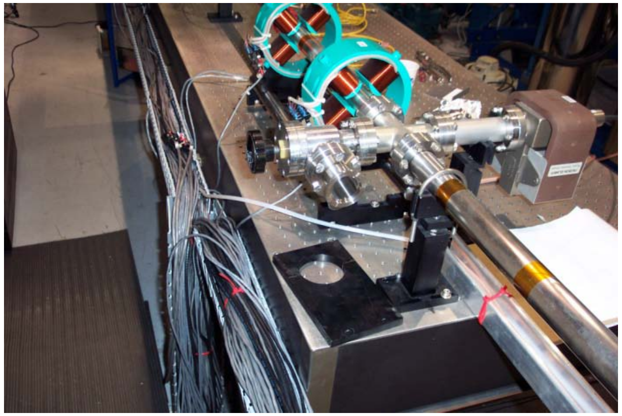
$F_y \propto y$ De - focusing force

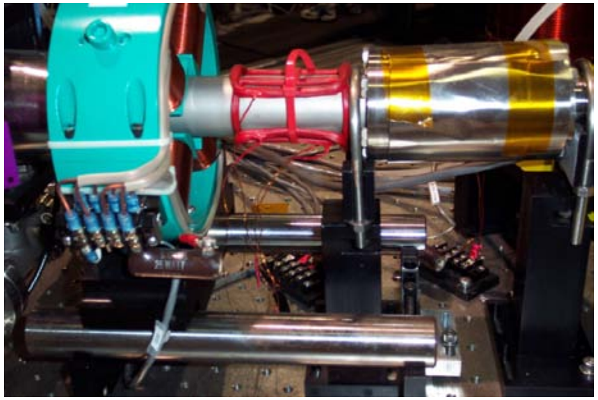


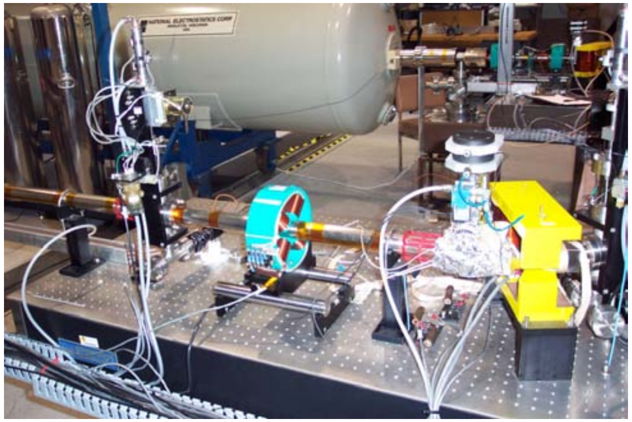
Magnetic field lines



Magnetic quadrupoles used in the CBPF FEL

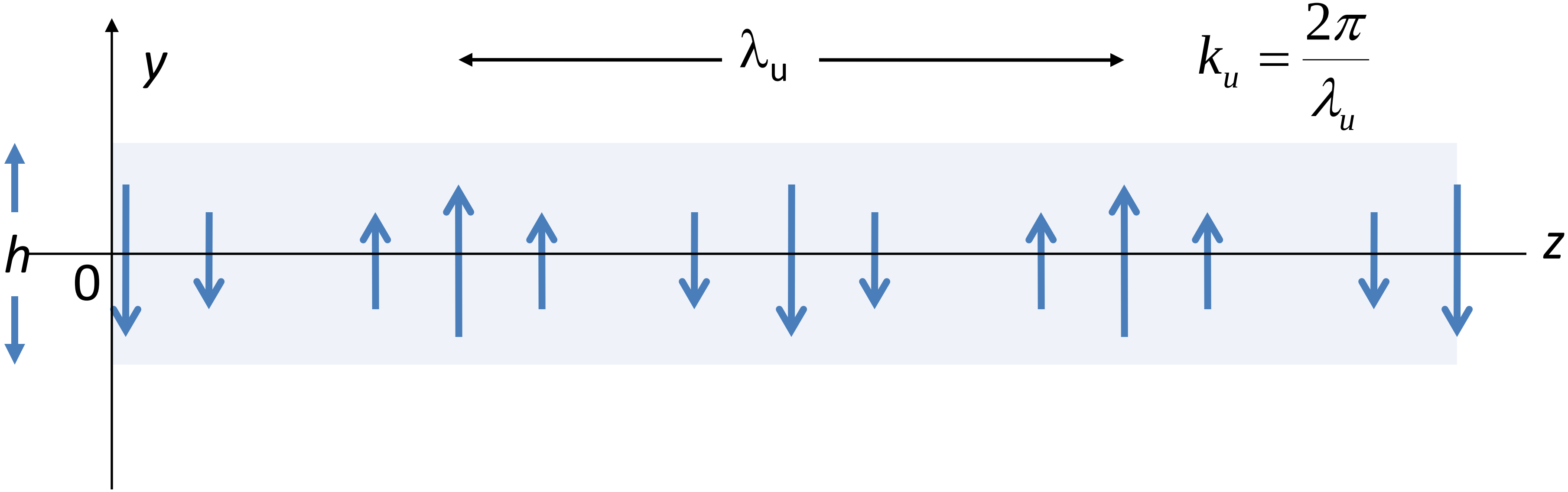






Magnetic Undulator

Periodically Magnetize Slab



$$k_u = \frac{2\pi}{\lambda_u}$$

Magnetization vector $\mathbf{M}(y,z) = \mathbf{e}_y M \cos(k_u z)\theta(h/2 - |y|)$

Magnetic Scalar Potential

$$\nabla^2 \Phi_m(y,z) = \nabla \cdot \mathbf{M} = M[\delta(y-h/2) - \delta(y+h/2]\cos k_u z$$

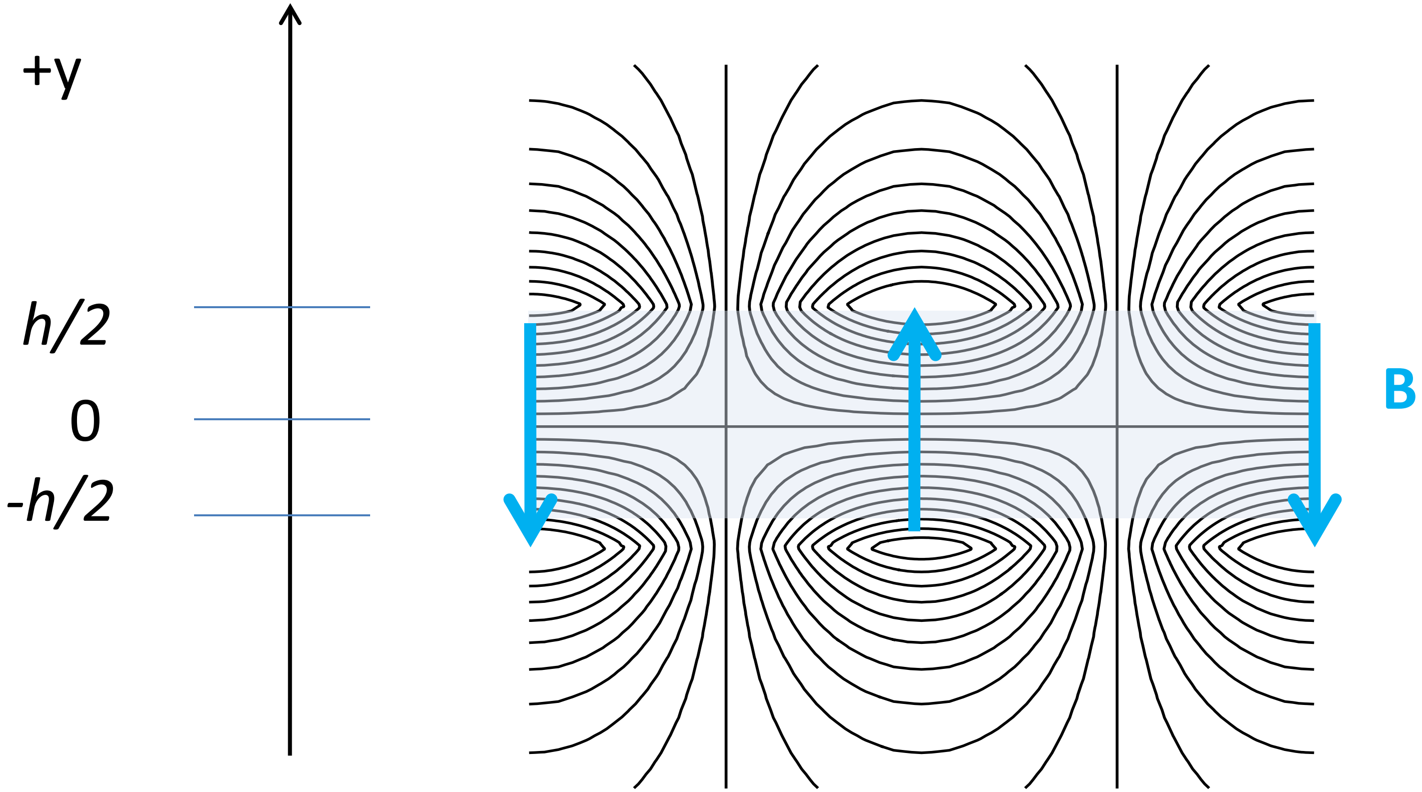
$$\Phi_m = \frac{M}{2k_u}\left[e^{-k_u|y-h/2|} - e^{-k_u|y+h/2|}\right]\cos k_u z$$

$$\mathbf{H} = -\nabla\Phi_m \qquad \mathbf{B} = \mu_0[\mathbf{H} + \mathbf{M}]$$

$$H_y = \frac{M}{2}[\varepsilon(y-h/2)e^{-k_u|y-h/2|} - \varepsilon(y+h/2)e^{-k_u|y+h/2|}]\cos k_u z$$

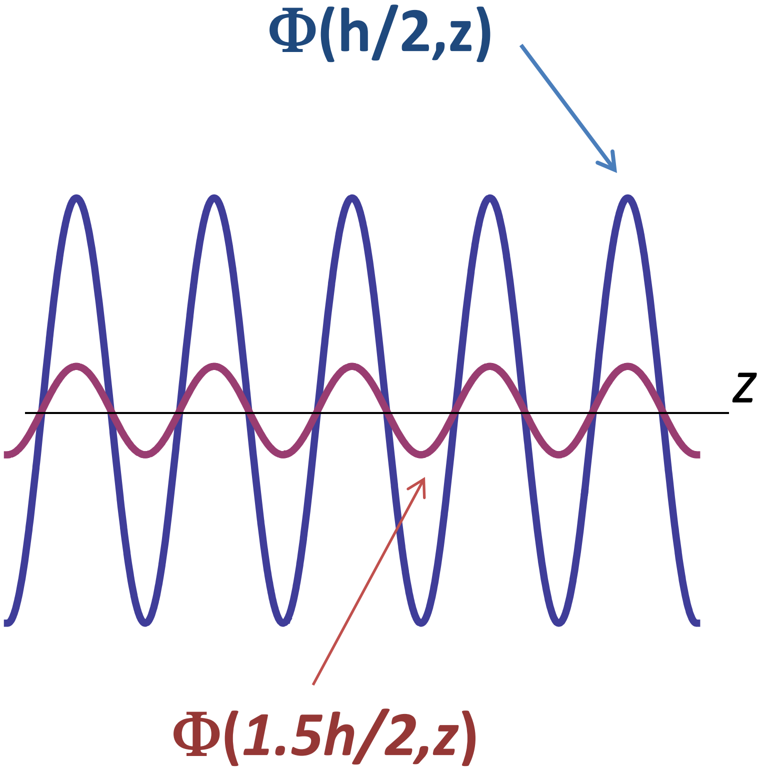
$$H_z = \frac{M}{2}[e^{-k_u|y-h/2|} - e^{-k_u|y+h/2|}]\sin k_u z$$

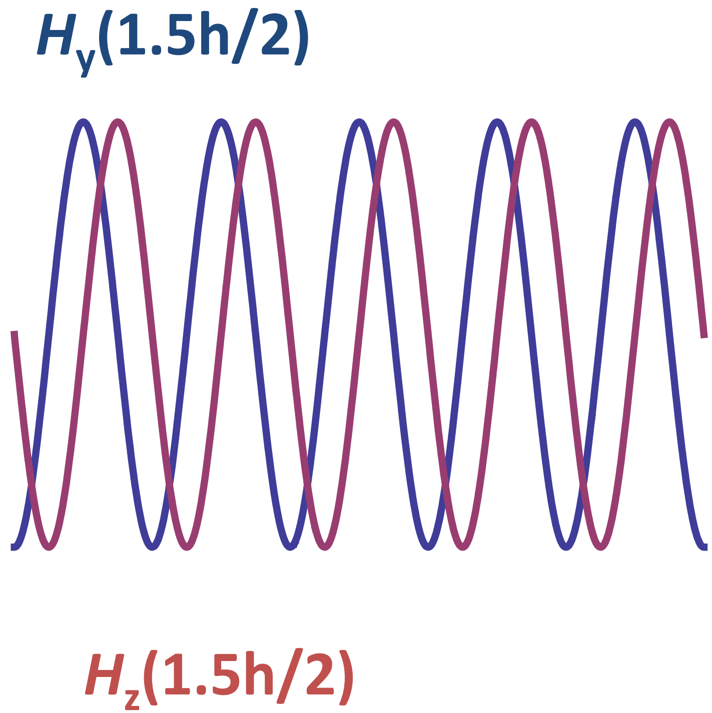
Magnetic Scalar Potential



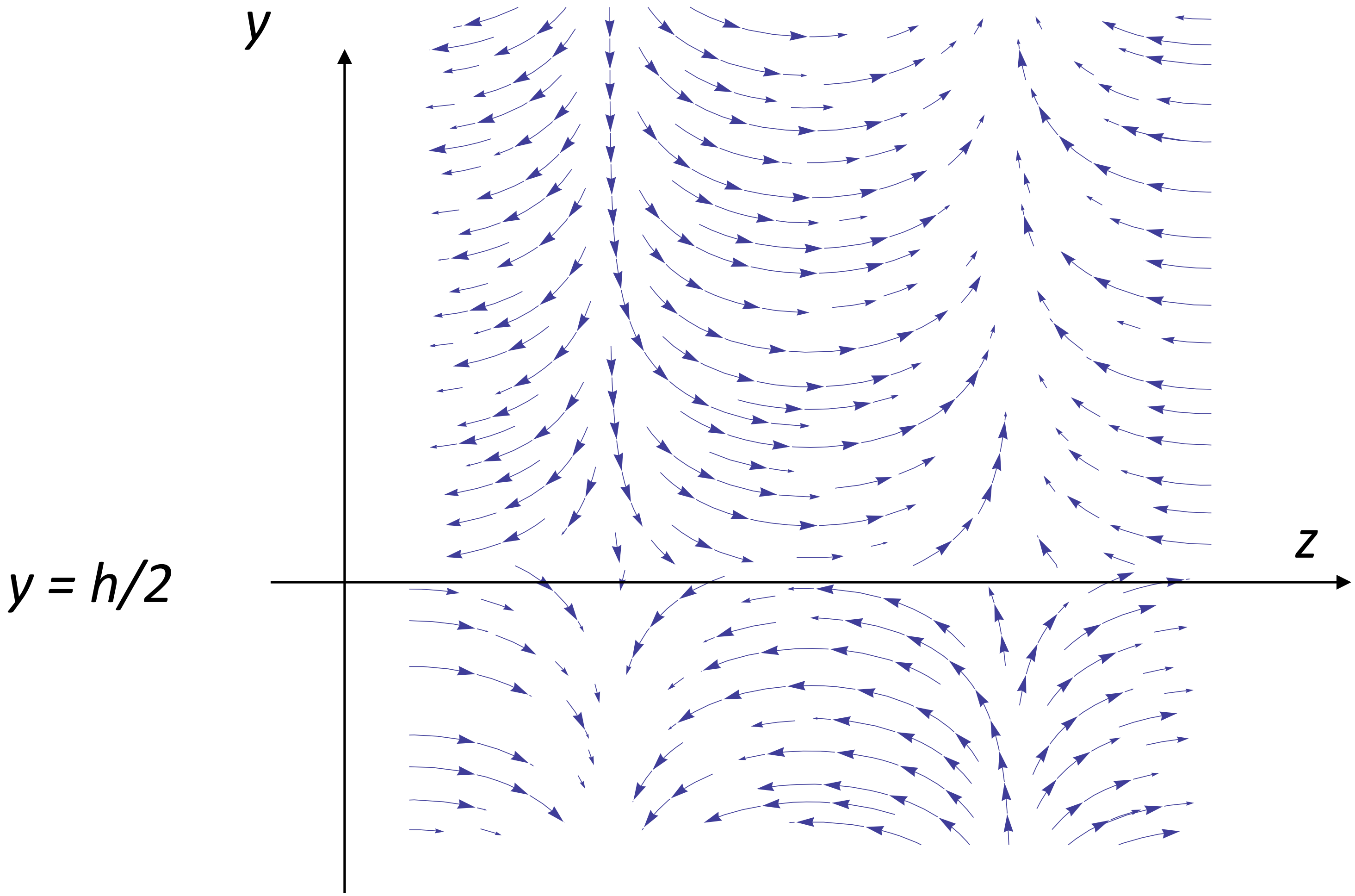
Equipotential lines

Magnetic Fields



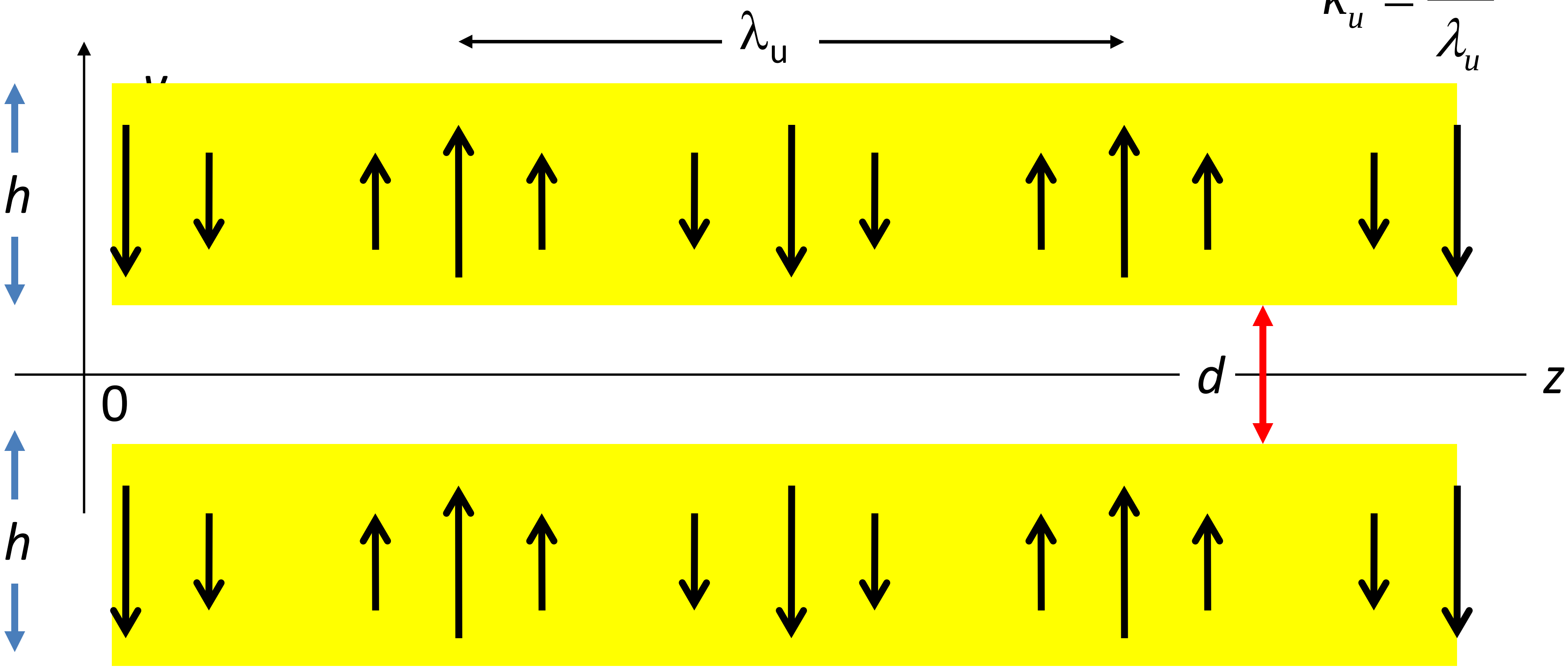


H-Field lines



Parallel Magnetized Slabs

$$k_u = \frac{2\pi}{\lambda_u}$$



Magnetic pole density

$$\mathbf{M}(y,z) = \mathbf{e}_y M\left[ST - SB\right]\cos(k_u z)$$

$$ST = \theta\left[y - d/2\right] - \theta\left[y - (h + d/2)\right]$$

$$SB = \theta\left[y + (h + d/2)\right] - \theta(y + d/2)$$

$$\rho_m = M\left[DT - DB]\right]\cos(k_u z)$$

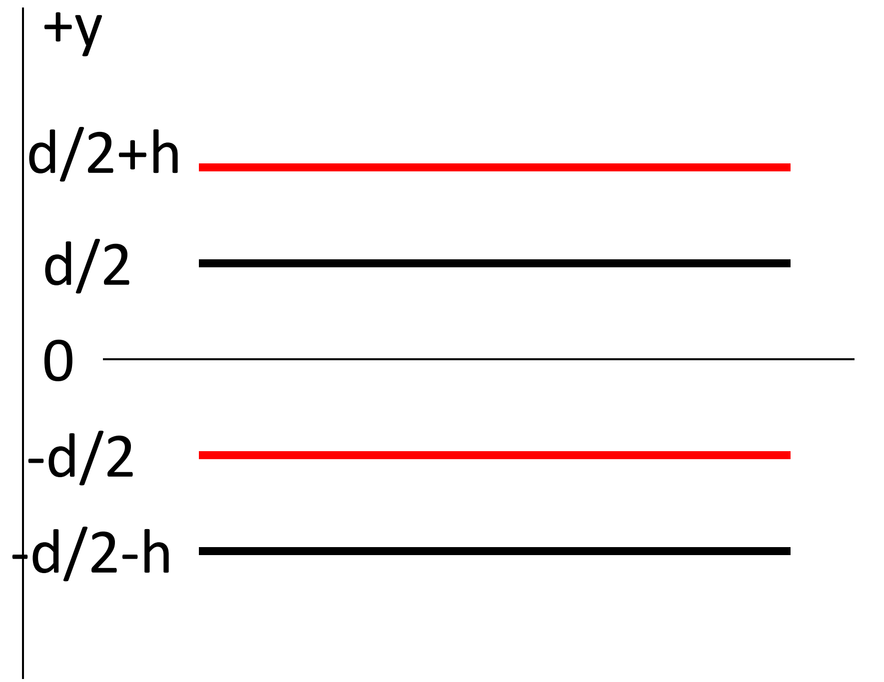
$$DT = \delta\left[y - d/2\right] - \delta\left[y - (h + d/2)\right]$$

$$DB = \delta\left[y + (h + d/2)\right] - \delta(y + d/2)$$

Magnetic potential

Magnetic potential due to a single magnetic surface located at *y = u*

$$\Phi_m(u) = (\pm)\frac{M}{2k_u}\left[e^{-k_u|y-u|}\right]\cos k_u z$$



$$\Phi_m(y,z) = \Phi_m(d/2+h) - \Phi_m(d/2) + \Phi_m(-d/2) - \Phi_m(-d/2-h)$$

Magnetic Intensity

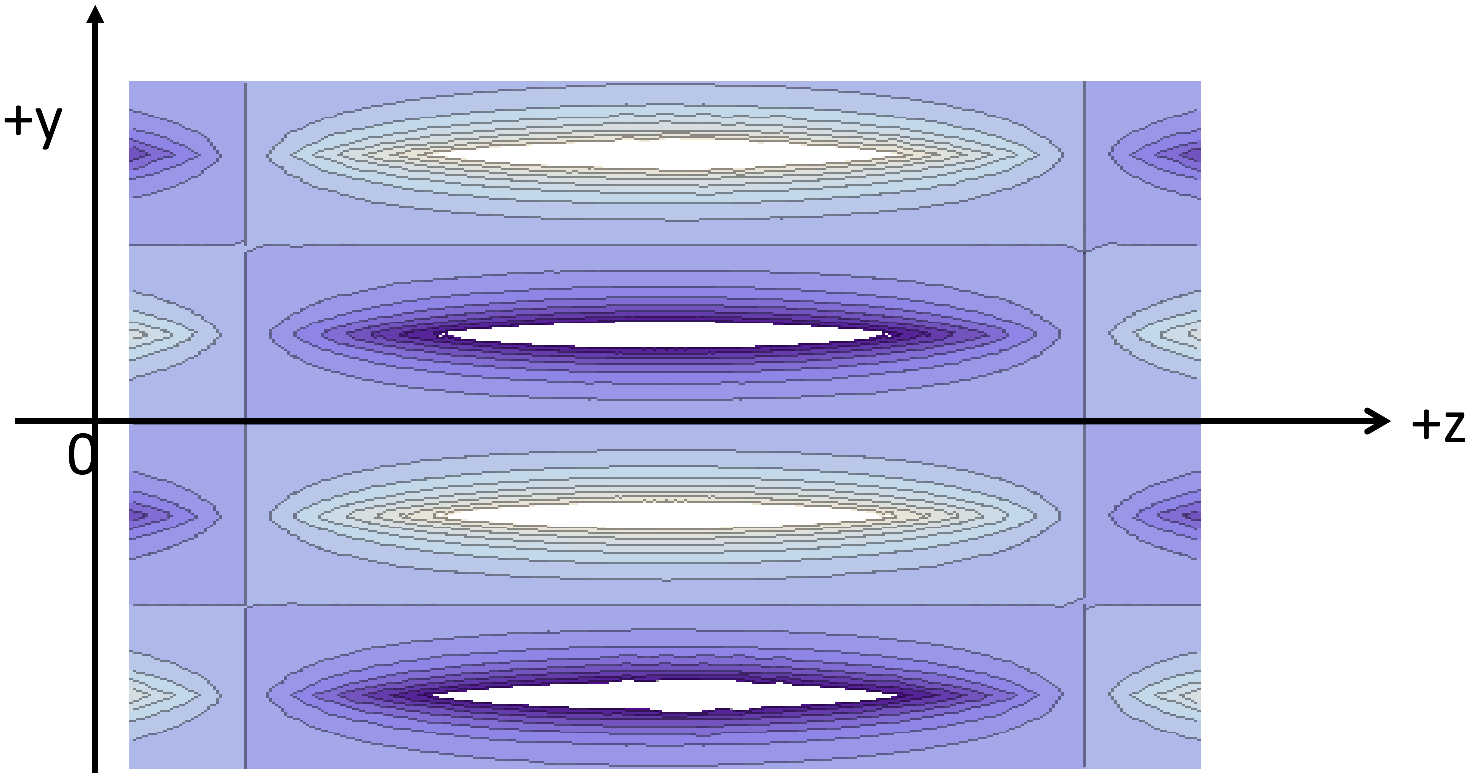
$$H_y(u) = (\pm)\frac{M}{2}\varepsilon(y-u)\left[e^{-k_u|y-u|}\right]\cos k_u z$$

$$H_z(u) = (\pm)\frac{M}{2}\left[e^{-k_u|y-u|}\right]\sin k_u z$$

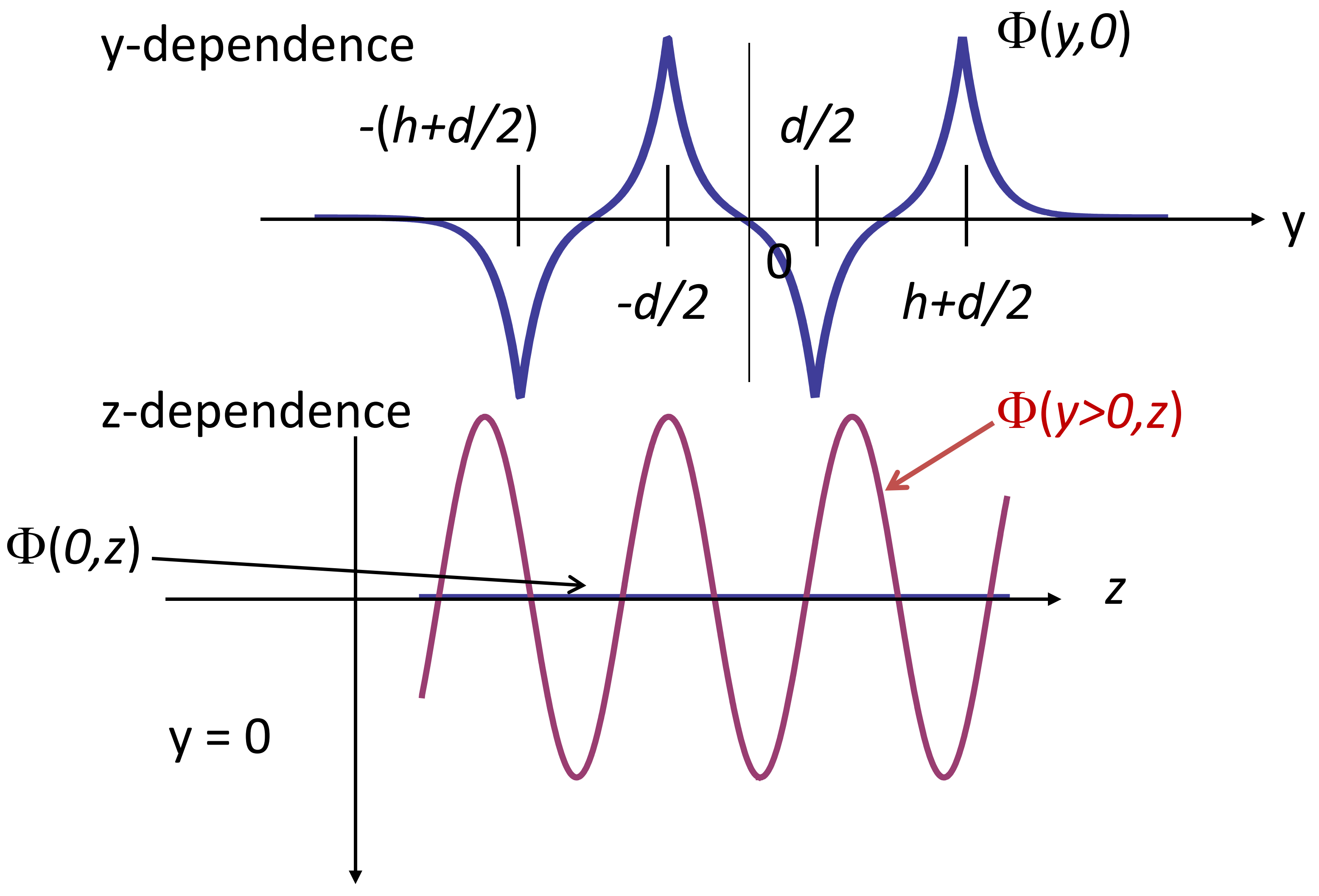
$$H_y(y,z) = H_y(d/2+h) - H_y(d/2) + H_y(-d/2) - H_y(-d/2-h)$$

$$H_z(y,z) = H_z(d/2+h) - H_z(d/2) + H_z(-d/2) - H_z(-d/2-h)$$

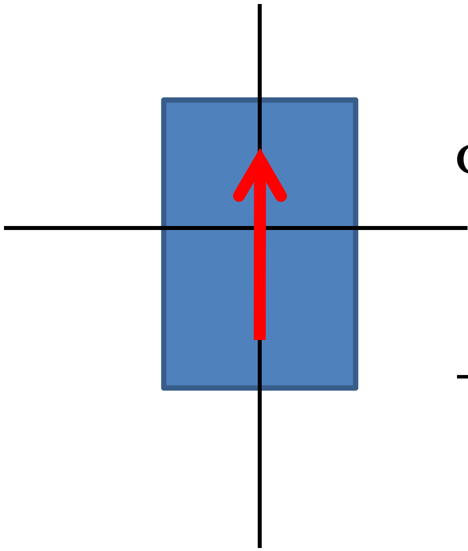
Magnetic Equipotential Lines



Magnetic potential



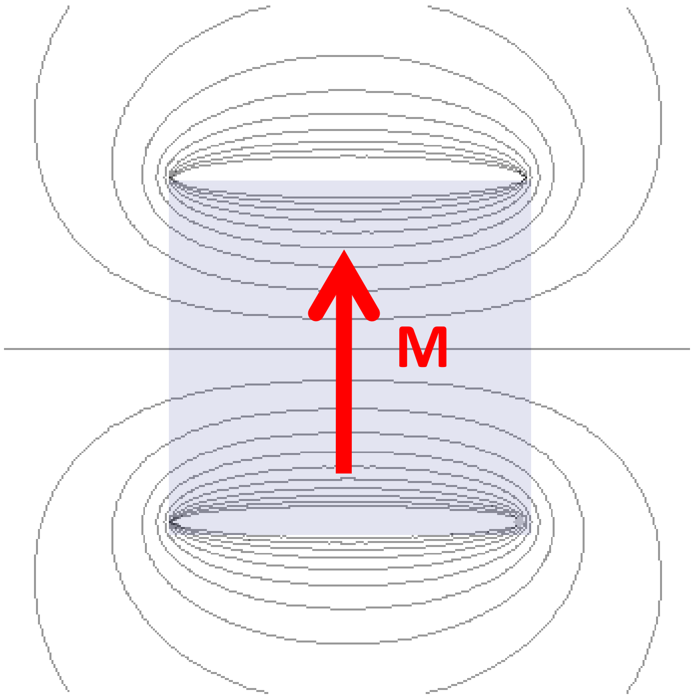
Discrete Permanent Magnet



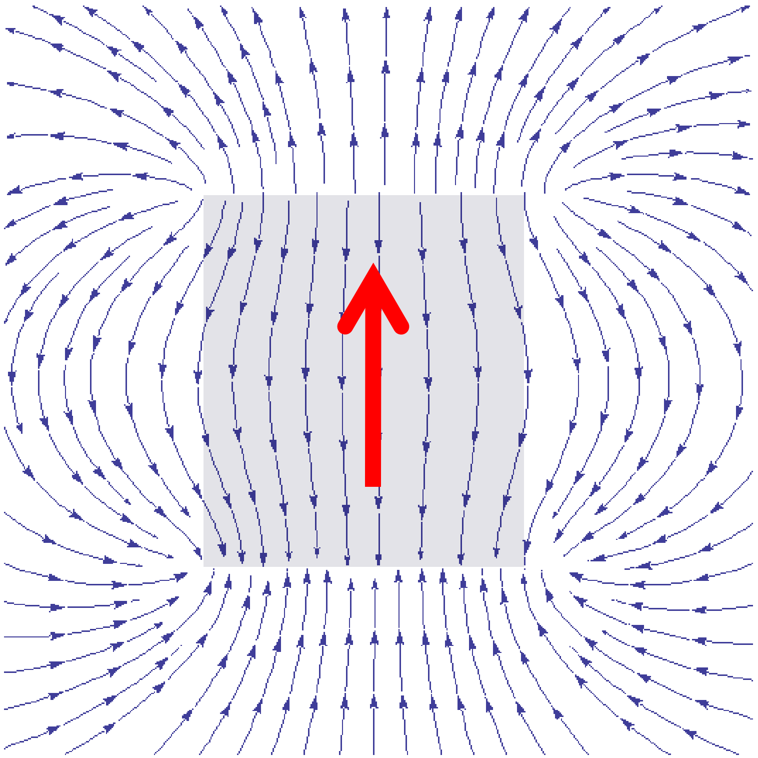
$$\Phi_m = \frac{M}{4\pi}\ln\left[\frac{z-a/2+\sqrt{(z-a/2)^2+(y-b/2)^2}}{z+a/2+\sqrt{(z+a/2)^2+(y-b/2)^2}}\right]$$

$$-\frac{M}{4\pi}\ln\left[\frac{z-a/2+\sqrt{(z-a/2)^2+(y+b/2)^2}}{z+a/2+\sqrt{(z+a/2)^2+(y+b/2)^2}}\right]$$

Equipotential and H lines



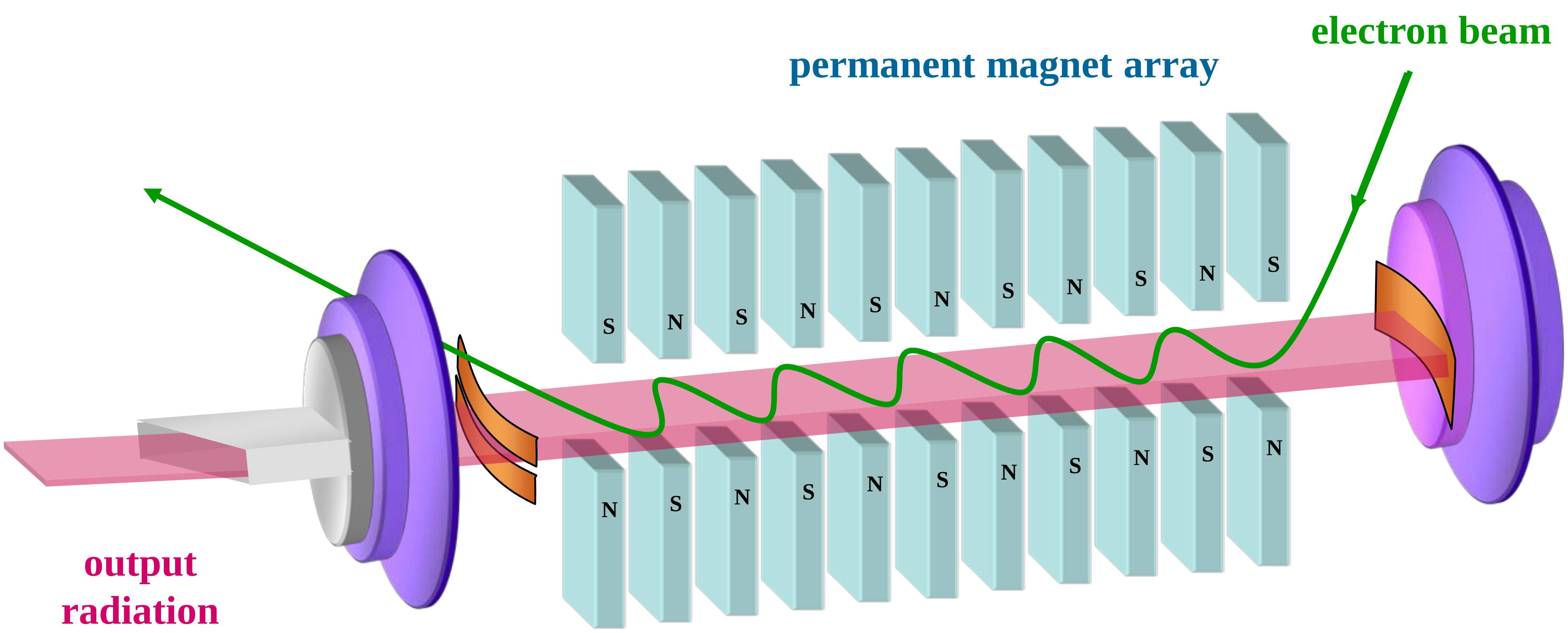
Equipotential lines



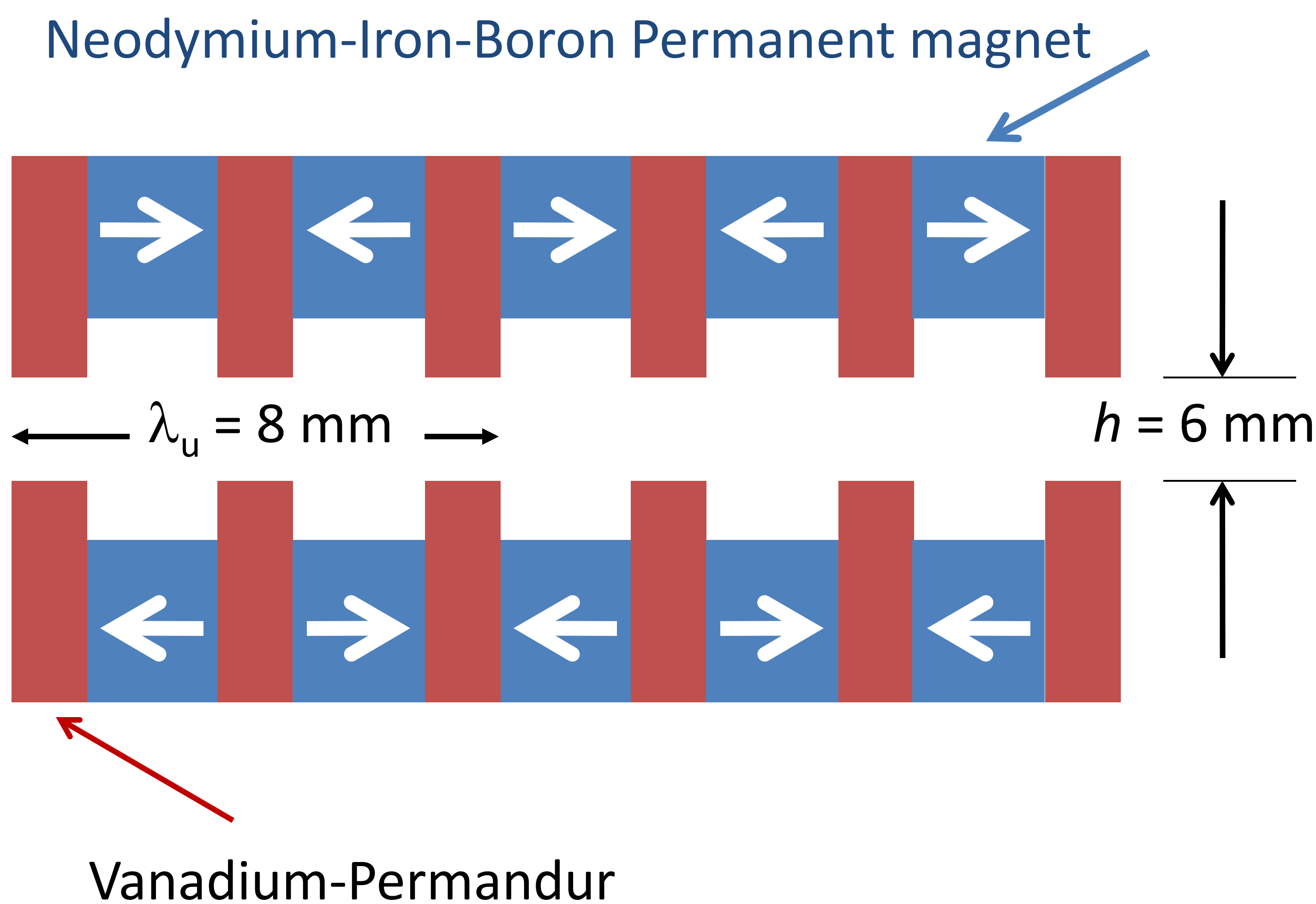
H-field lines

UH/CBPF Undulator

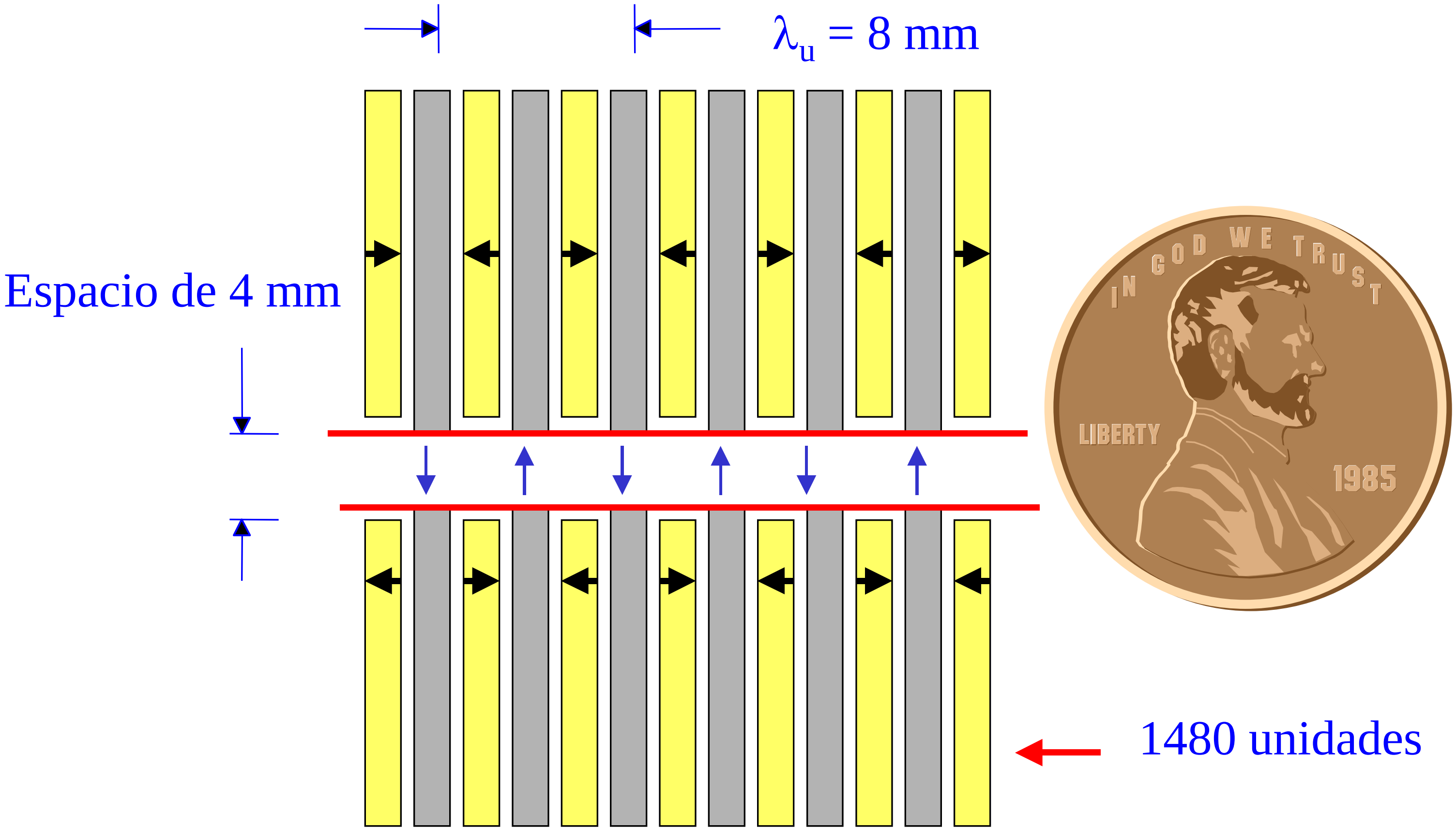
UH/CBPF FEL Oscillator



CBPF undulator



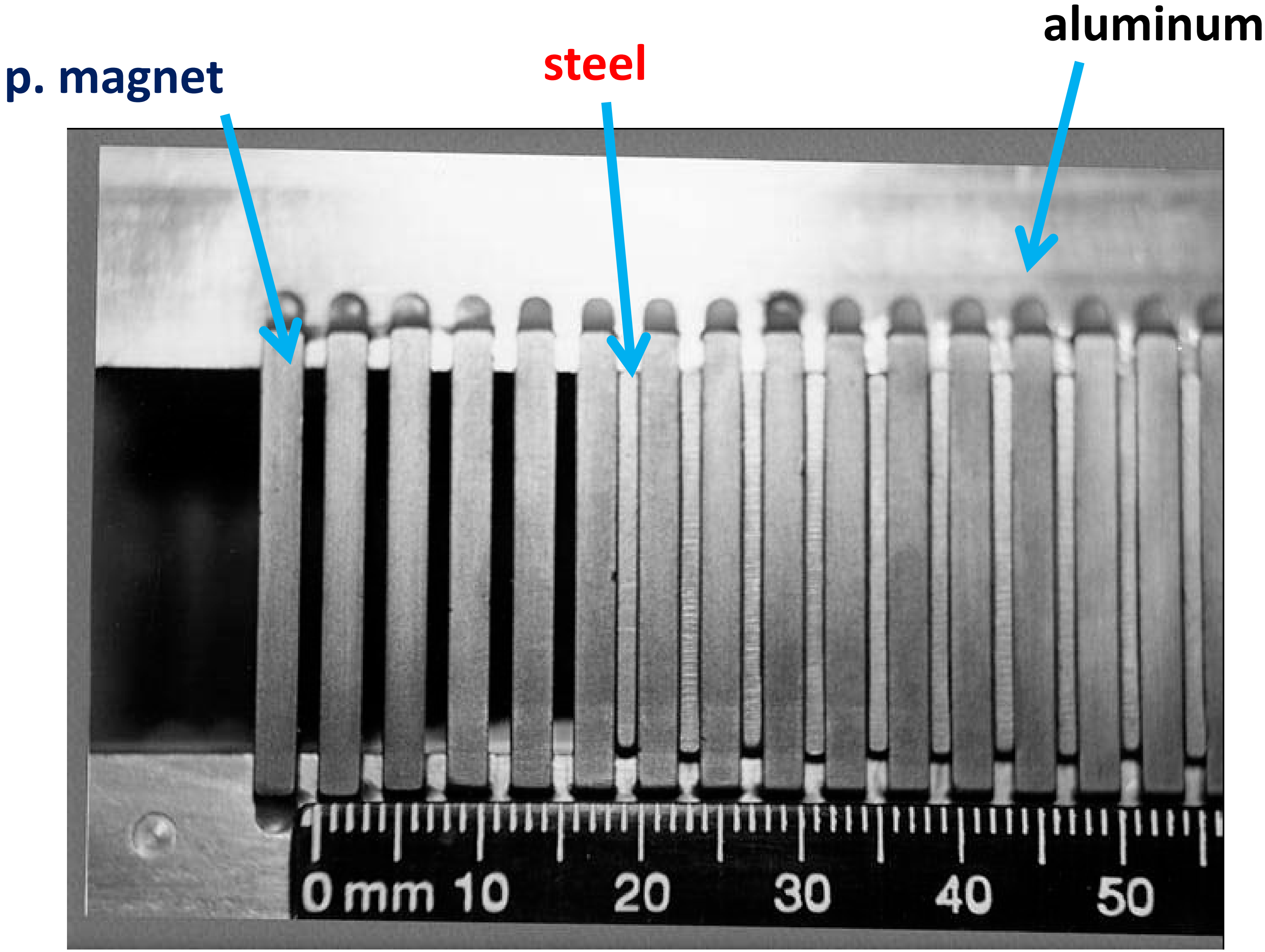
Magnetic Structure



Undulator parameters

Undulator period	*8 millimeters*
Number of complete periods	*185*
Undulator length	*1.5 meters*
Undulator gap	*6 millimeters*
Peak undulator field	*0.2 Tesla*
Root mean square peak field error	*0.3%*
Permanent magnet material	*NdFeB (NeoMax 33SH)*
Magnet remnant field	*1.17 Tesla (+/- 1%)*
Permanent magnet size	*2.7 by 13.0 by 30.0 millimeter*^{3}
Magnetization direction	*Parallel to 2.7 mm dimension (+/-1.5 degrees)*
Pole material	*1010 steel*
Pole size	*1.28 by 28 by 50 millimeter*^{3}

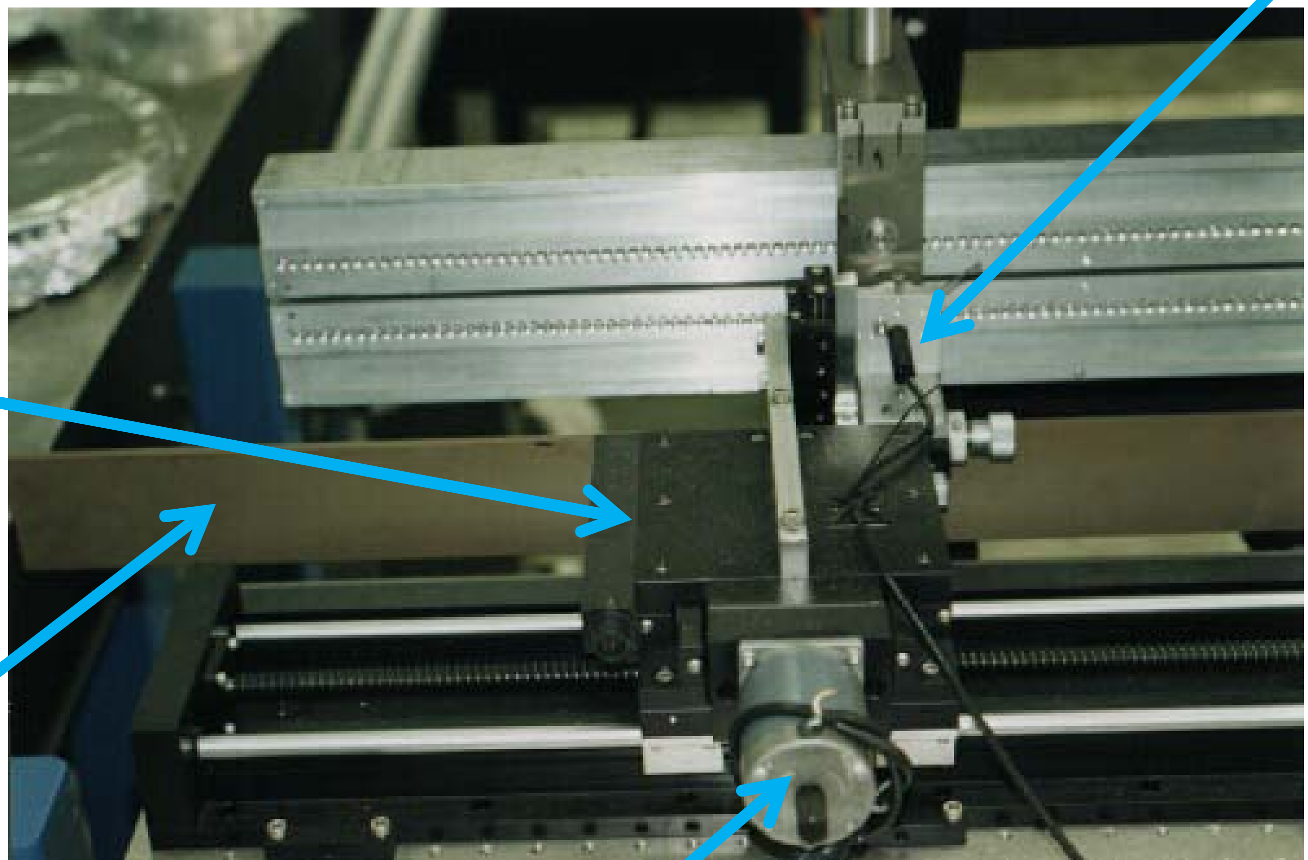
Undulator section



Undulator field measurement apparatus

Hall probe

Precision linear motor stage



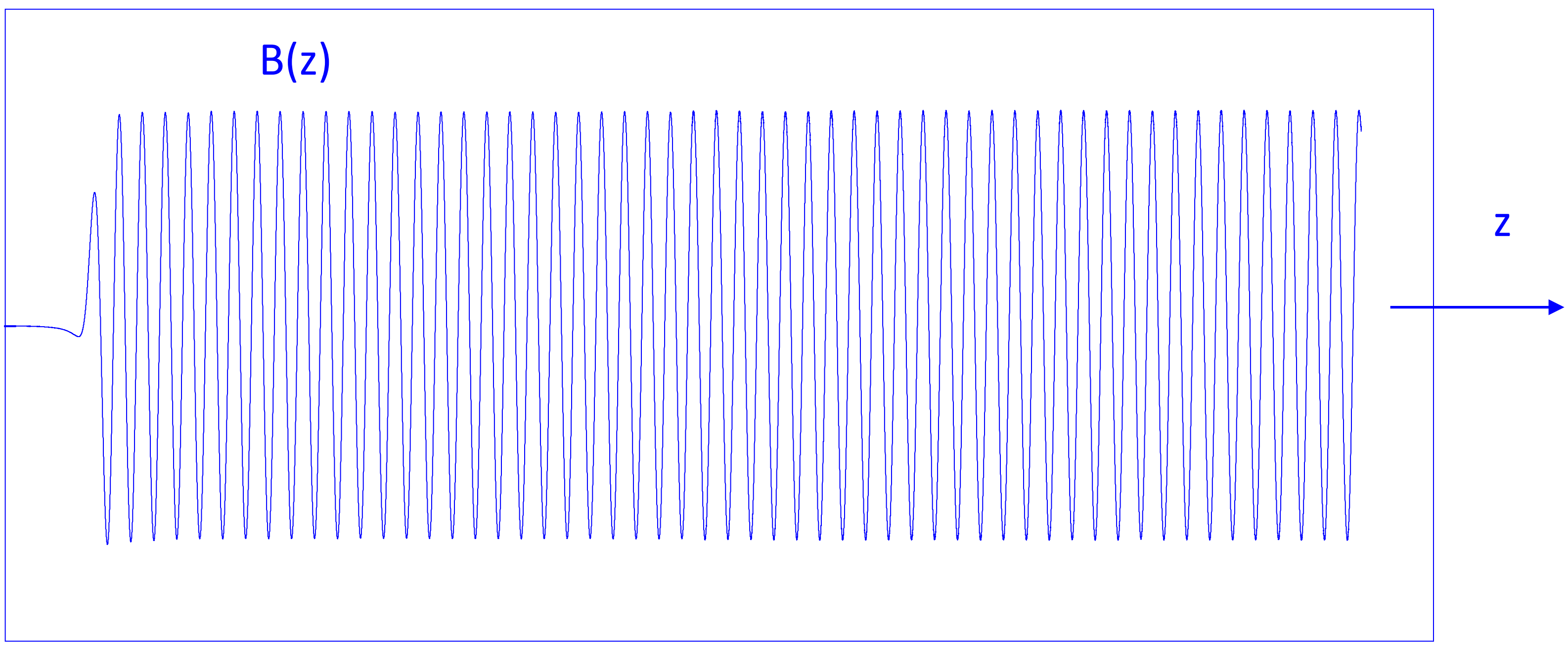
Straight edge

Stepping motor

Measured undulator magnetic field

N = 180

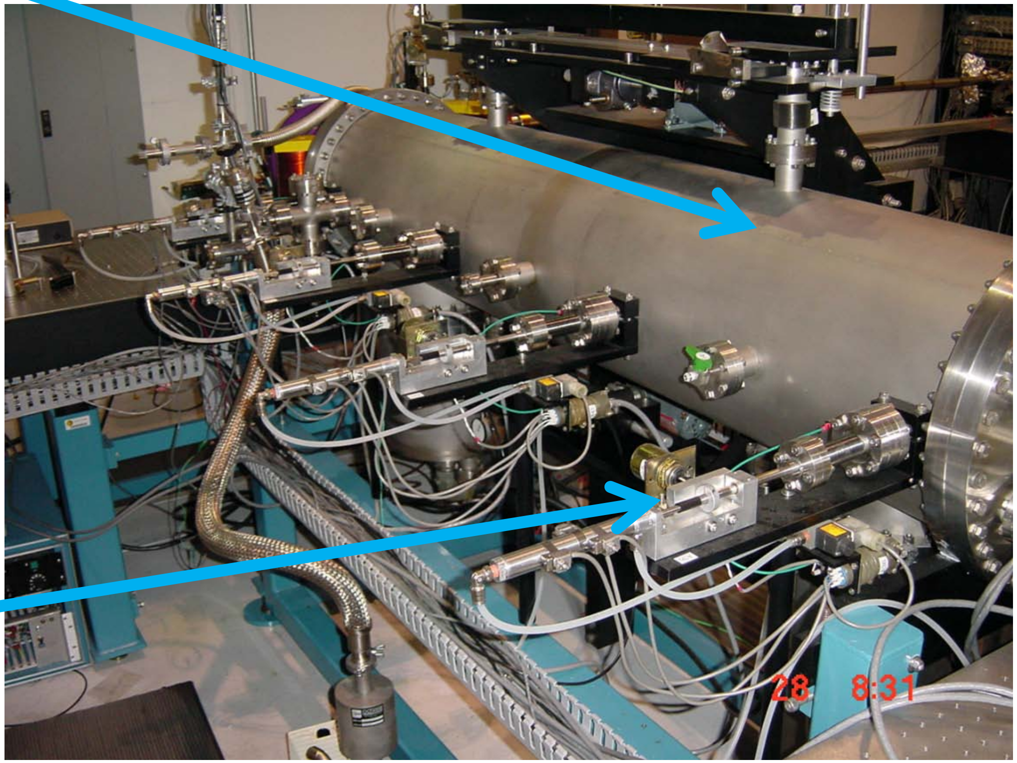
$N\lambda_u$



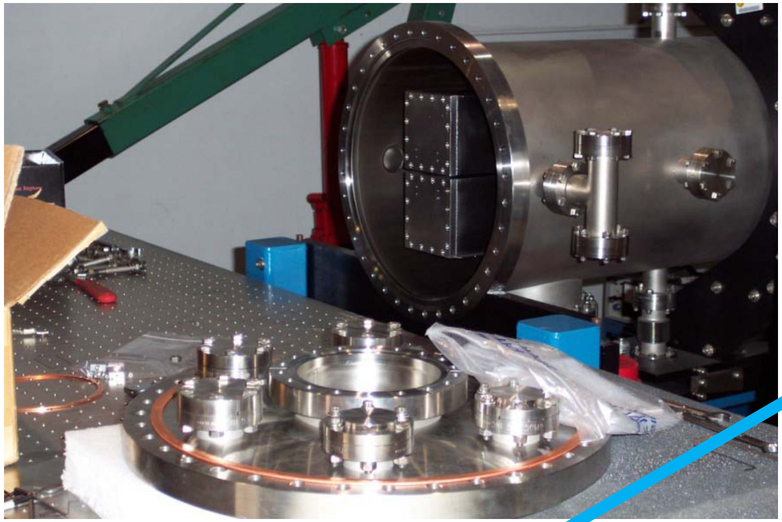
Undulator Apparatus

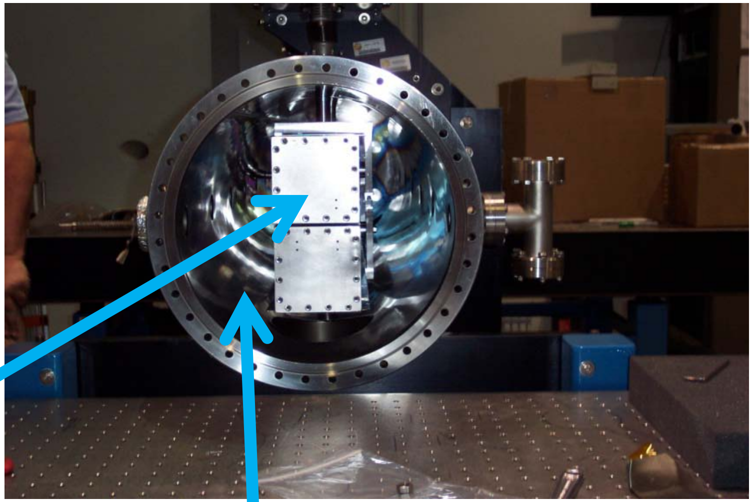
High vacuum chamber

E. Beam diagnostic screen assembly



Undulator vacuum chamber





Undulator boxes

10^{-9} torr vacuum

Electron beam and undulator parameters

Electron beam energy	0.9-1.7 MeV
Electron beam current	0-0.3 Amperes
Radiation wavelength	220-600 microns
Undulator wavevector (ku)	785 meters
Undulator parameter	0.15
Transverse oscillation amplitude	45-74 microns
Betatron period	0.2-0.32 meters
Normalized emittance	2Pi-mm-mrad
Minimum beam envelope	0.3 millimeters