J Eng Math (2010) 67:233–240
DOI 10.1007/s10665-009-9351-6

Some results for an $\mathcal{N}$-dimensional nonlinear diffusion equation with radial symmetry

E. K. Lenzi · M. K. Lenzi · T. M. Gimenez · L. R. da Silva

Received: 30 June 2008 / Accepted: 26 October 2009 / Published online: 10 November 2009

Abstract The solutions of a nonlinear diffusion equation by considering the radially symmetric $\mathcal{N}$-dimensional case are investigated. This equation has the nonlinearity present in the diffusive term and external force. The solutions are obtained by using a similarity method and connected to the q-exponential and q-logarithmic functions which emerge from the Tsallis formalism. In addition, the results obtained here may be useful to investigate a rich class of situations related to anomalous diffusion.

Keywords Anomalous diffusion · Nonlinear diffusion equation · Power-law diffusion coefficient · Similarity · Tsallis formalism

1 Introduction

Extensions of the usual diffusion equation (i.e., the classic constant-coefficient linear diffusion equation) by incorporating nonlinear terms in the diffusive term [1, Chap. 7], external force [2,3] or reaction term [4] have been used to investigate several scenarios such as turbulent diffusion [5], flow of gases through porous media [6, Chap. 11], anomalous diffusion, chemotaxis of biological populations [7], hydraulics problems [8], flow of water in unsaturated soils [9, Chap. 6], simultaneous diffusion and adsorption in porous samples where the adsorption isotherm is of power-law type [10, Chap. 14], catalytic processes in regular, heterogeneous or disordered systems [11–13], diffusion of dissolved solutes into immobile water zones of various sizes, a standard solid-on-solid model for surface growth, thin liquid films spreading under gravity and nonlinear diffusion in hard and soft superconductors [14–16].

E. K. Lenzi (✉)
Departamento de Física, Universidade Estadual de Maringá , Av. Colombo 5790, Maringá, PR 87020-900, Brazil
e-mail: eklenzi@dfi.uem.br

M. K. Lenzi
Departamento de Engenharia Química, Universidade Federal do Paraná, Setor de Tecnologia - Jardim das Américas,
Caixa Postal 19011, Curitiba, PR 81531-990, Brazil

T. M. Gimenez
Departamento de Engenharia Química, Universidade Estadual de Maringá, Av. Colombo 5790, Maringá, PR 87020-900, Brazil

L. R. da Silva
Departamento de Física, Universidade Federal do Rio Grande do Norte, Natal, RN 59072-970, Brazil

One of these equations is the porous media equation which has nonlinearity present in the diffusive term, i.e.,

$$\frac{\partial}{\partial t}\rho(r,t)=\frac{\mathcal{D}}{r^{\mathcal{N}-1}}\frac{\partial}{\partial r}\left\{r^{\mathcal{N}-1}\frac{\partial}{\partial r}\left[\rho(r,t)\right]^{\nu}\right\}. \tag{1}$$

The applications and formal aspects of this equation have been analyzed by taking several scenarios into account such as external forces [17,18], spatial time-dependent diffusion coefficient [19–21], and reaction–diffusion terms [22–24]. Equation 1 has also motivated a generalization of the Arrhenius law [25] and found in Tsallis' formalism [26], A regularly updated bibliography on the subject is accessible at http://tsallis.cat.cbpf.br/biblio.htm] a thermostatics context [27,28]. In addition, Eq. 1 has a corresponding Langevin equation with a multiplicative noise as shown in [29]. Another important equation is the Burgers equation which may be related to several physical contexts such as turbulence [30, Chap. 6], [31, Chap. 7], sound waves in a viscous medium, shock waves in a viscous medium, waves in fluid-filled viscous elastic tubes and magnetohydrodynamics [32]. In particular, it should be mentioned that many other nonlinear extensions of the usual diffusion equation can be found in the literature depending on the physical context analyzed [33–36]. Thus, the comprehension of formal aspects of these equations, which extend the usual one, becomes relevant in order to provide the physical context where each of them can play an important role. In this direction, we devote this work to investigate the following $\mathcal{N}$-dimensional nonlinear diffusion equation

$$\frac{\partial}{\partial t}\rho(r,t)=\frac{1}{r^{\mathcal{N}-1}}\frac{\partial}{\partial r}\left\{r^{\mathcal{N}-1}\mathcal{D}(r,t;\rho)\frac{\partial}{\partial r}\rho(r,t)\right\}-\frac{1}{r^{\mathcal{N}-1}}\frac{\partial}{\partial r}\left[r^{\mathcal{N}-1}F(r,t;\rho)\rho(r,t)\right] \tag{2}$$

which interpolates several nonlinear extensions of the diffusion equation. The diffusion coefficient considered here is given by $\mathcal{D}(r,t;\rho)=\nu\mathcal{D}r^{-\theta}\left(\rho(r,t)\right)^{\nu-1}$ and the external force (drift or convective term) is $F(r,t;\rho)=-k(t)r+\left(\mathcal{K}/r^{\gamma+\theta}\right)+k_{\alpha}r^{\alpha}\left(\rho(r,t)\right)^{\eta-1}$. Equation 2 is also subjected to the boundary condition $\lim_{r\to\infty}\rho(r,t)=0$. Note that the spatial dependence on the diffusion coefficient has been used to investigate diffusion on fractals [37], turbulence [38], and fast electrons in a hot plasma in the presence of an electric field [39]. The external force extends the one which emerges from the logarithmic potential used, for instance, to establish the connection between the fractional diffusion coefficient and the generalized mobility [40] and may find applications in cell–cell adhesion. From this equation, it can be verified that $\int_0^{\infty}\rho(r,t)r^{\mathcal{N}-1}\mathrm{d}r$ is time-independent (hence, if $\rho(r,t)$ is normalized at $t=0$, it will remain so forever). In order to verify the last statement, we may write (2) in the form $\partial_t\rho(r,t)+r^{1-\mathcal{N}}\partial_r\left(r^{\mathcal{N}-1}\mathcal{J}(r,t)\right)=0$, with

$$\mathcal{J}(r,t)=-\mathcal{D}(r,t;\rho)\frac{\partial}{\partial r}\rho(r,t)+F(r,t;\rho)\rho(r,t), \tag{3}$$

and assume the boundary condition $\lim_{r\to\infty}\mathcal{J}(r,t)=0$. Equation 2 may also be connected with the Langevin equation and consequently stochastic processes. In fact, by using the procedure proposed by Frank in [41] it is possible to find the corresponding Langevin equation of (2) and the stochastic process related to the choice of parameters present in (2). In this context, it is interesting to note that Eq. 2 has as particular case in the Ornstein–Uhlenbeck process [42, Chap. 5] and the Rayleigh process [43, Chap. 5]. From the above discussion, we would like to emphasize that our analysis based on (2) with diffusion coefficient and external force previously mentioned extends the situations discussed above and, in particular, recover the one-dimensional form of the Burgers equation for a suitable choice of parameters present in (2).

The plan of this work is to investigate the solutions of (2) by using the concept of the similarity method with suitable parameters γ, α and η. The solutions obtained by using this procedure are expressed in terms of the q-exponential and q-logarithmic functions which emerge from the Tsallis formalism by maximizing the entropy with suitable constraints [26]. This connection between the solutions of (2) and the mathematical structure present in the Tsallis formalism suggests that (2) may find, in this approach, a thermostatistic foundation, similar to the usual diffusion equation and the Boltzmann–Gibbs statistics. Another interesting point about the solutions of (2) is their compact or long-tailed behavior depending on the choice of the parameters γ, θ, ν, α, and η, as in the next section. In particular, for the long-tailed behavior it is possible to relate the solution in the asymptotic limit with the Lévy distributions.

2 Nonlinear diffusion equation

Let us start by considering the case characterized by $\nu = 1$, $\gamma = 1$, $\eta = 2$ and $\alpha = \mathcal{N} - 1 - \theta$. This case leads us to a diffusion equation with a nonlinear external force which presents a nonlinear term similar to the Burgers equation. To obtain the solution for this case, we may use, for example, the concept of the similarity method [44, Chap. 8], [45, Chap. 5], [46, Sect. 2.14], [47, Chap. 5], [48, Chap. 3] in order to reduce this nonlinear diffusion equation with external forces to ordinary differential equations. It is also possible to apply other approaches such as the presented in [49] and [50] to investigate the solutions of (2). Particularly, we work out Eq. 2 by restricting our analysis to solutions which have the form

$$\rho(r,t) = \frac{1}{(\Phi(t))^{\mathcal{N}}}\mathcal{P}\left(\frac{r}{\Phi(t)}\right), \tag{4}$$

which represents a particular class of solutions for (2). Note that $\Phi(t)$ and $\mathcal{P}(z)$ $(z = r/\Phi(t))$ are determined by solving the differential equations which emerge by substituting (4) in (2). Therefore, we extend the usual power-law dependence for the time-dependent function [44, Chap. 8], [45, Chap. 5], employed to solve particular forms of (2), to the function $\Phi(t)$ in order to cover others situations due to the complex structure of (2) and to satisfy the normalization condition, i.e.,

$$\int_{-\infty}^{\infty} \rho(r,t) r^{\mathcal{N}-1} \mathrm{d}r = \int_{-\infty}^{\infty} \mathcal{P}(z) z^{\mathcal{N}-1} \mathrm{d}z = 1. \tag{5}$$

By substituting (4) in (2), after some calculations it is possible to show that Eq. 2 may be simplified to two ordinary differential equations, one for $\Phi(t)$ and the other for $\mathcal{P}(z)$ as follows:

$$(\Phi(t))^{\xi-2}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) + k(t)\,(\Phi(t))^{\xi-1} = 1, \tag{6}$$

$$-\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{\mathcal{N}}\mathcal{P}(z)\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left\{\mathcal{D}\left[z^{\mathcal{N}-1-\theta}\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{P}(z)\right] - z^{\mathcal{N}-1}\left(\frac{\mathcal{K}}{z^{1+\theta}}\right)\mathcal{P}(z) - k_\alpha z^{\mathcal{N}-1+\alpha}\left(\mathcal{P}(z)\right)^2\right\}, \tag{7}$$

with $z = r/\Phi(t)$ and $\xi = 3 + \theta$. The solution for the time-dependent equation is given by

$$\frac{\Phi(t)}{\Phi(0)} = \left(1 + \frac{\xi-1}{(\Phi(0))^{\xi-1}}\int_0^t e^{(\xi-1)\int_0^{t'} k(\tilde{t})\mathrm{d}\tilde{t}}\,\mathrm{d}t'\right)^{\frac{1}{\xi-1}} \mathrm{e}^{-\int_0^t k(\tilde{t})\mathrm{d}\tilde{t}} \tag{8}$$

where $\Phi(0)$ is related to an initial width of the solution. It is also interesting to note that similar solutions for the time-dependent function have been obtained for other nonlinear or nonlinear fractional diffusion equations [51–53]. This fact implies that these different diffusion equations have a similar spreading of the solution independent of the type of spatial dependence on the solution. This characteristic may be verified, for example, from the second moment, i.e., $\langle r^2\rangle \propto (\Phi(t))^2$, which presents the same time behavior. The second moment may also tell us how diffusion takes place and the conditions for the system exhibit a stationary process, i.e., for long times $\rho(r,t)$ becomes time-independent as found, for example, for the Ornstein–Uhlenbeck [42, Chap. 5] and Rayleigh processes [43, Chap. 5]. It also is possible to describe a subdiffusive, normal or superdiffusive process depending on the choice of parameters present in (2). For Eq. 7, it can be verified that the solution which satisfies the boundary condition $\lim_{r\to\infty}\rho(r,t) = 0$ is given by

$$\mathcal{P}(z) = \frac{\Omega\, z^{\mathcal{K}/\mathcal{D}}\mathrm{e}^{-\frac{z^{2+\theta}}{(2+\theta)\mathcal{D}}}}{1 - \Omega\frac{k_\alpha}{\mathcal{D}}\int_0^z \tilde{z}^{\alpha+\theta+\mathcal{K}/\mathcal{D}}\mathrm{e}^{-\frac{\tilde{z}^{2+\theta}}{(2+\theta)\mathcal{D}}}\mathrm{d}\tilde{z}}, \tag{9}$$

where Ω is obtained from the normalization condition. In particular, for this case it is given by

$$\Omega = \frac{(2+\theta)\,\mathcal{D}\left(1 - \mathrm{e}^{-k_\alpha/\mathcal{D}}\right)}{k_\alpha\left[(2+\theta)\,\mathcal{D}\right]^{\frac{\mathcal{N}+\mathcal{K}/\mathcal{D}}{2+\theta}}\Gamma\left(\frac{\mathcal{N}+\mathcal{K}/\mathcal{D}}{2+\theta}\right)}. \tag{10}$$

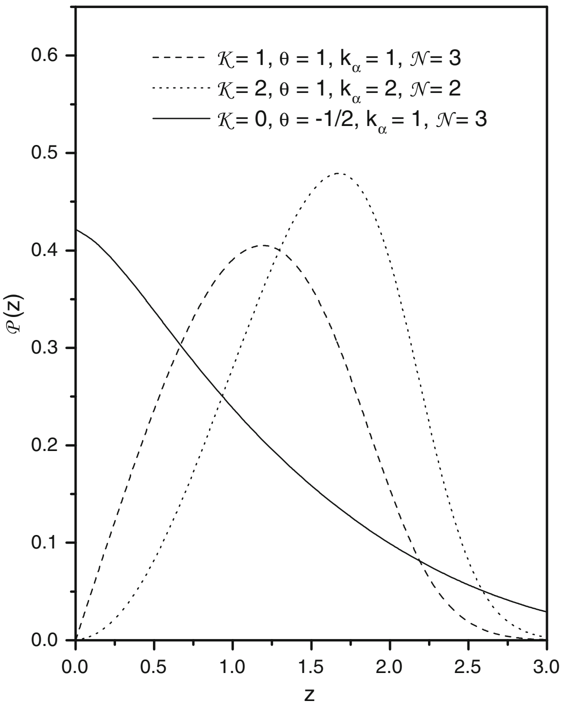

Fig. 1 This figure shows the behavior of (9) versus z for typical values of $\mathcal{K}$, θ, k_α and $\mathcal{N}$ by considering, for simplicity, $\mathcal{D} = 1$

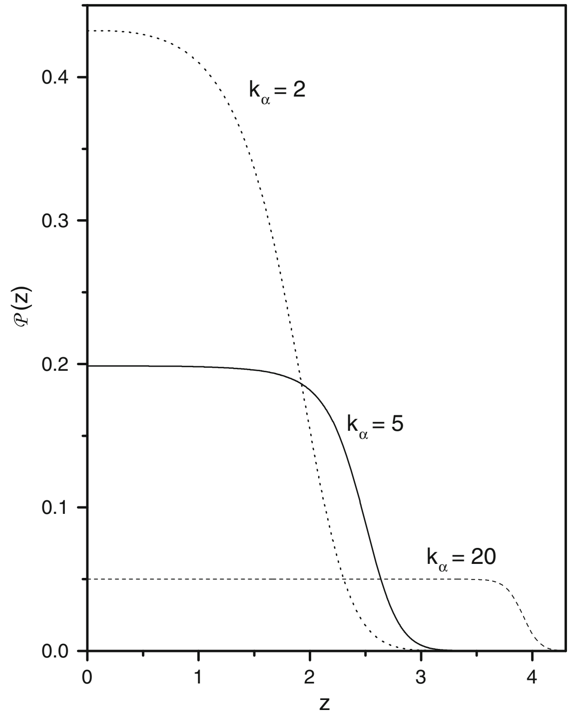

Fig. 2 This figure shows the behavior of (9) versus z for typical values of k_α by considering, for simplicity, $\mathcal{K} = 0$, $\mathcal{D} = 1$, $\theta = 1$ and $\mathcal{N} = 3$. Note that for large k_α is evidenced the shock formation by (9)

In Fig. 1, we illustrate (9) for typical values of the parameters $\mathcal{K}$, θ, k_α and $\mathcal{N}$ in order to show some corresponding trends which may be obtained from the solution. This solution extends results presented in [2] [44, Chap. 8], [45, Chap. 5] for the Burgers equation; depending on the quantity $k_\alpha/\mathcal{D}$ it may exhibit a diffusive or shock behavior similar to the solutions obtained from the classical form of the Burgers equation [44, Chap. 8], [45, Chap. 5]. Indeed, the quantity $k_\alpha/\mathcal{D}$ may be interpreted as a Reynolds number ($\mathcal{R}$) [54, Chap. 4] and the solution indicates that for small $\mathcal{R}(\mathcal{R} << 1,\ \text{i.e.}, k_\alpha << \mathcal{D})$, the effect of the diffusive term in (9) is more significant than the nonlinear term, i.e., the solution is dominated by the diffusive behavior and for large $\mathcal{R}(\mathcal{R} >> 1,\ \text{i.e.}, k_\alpha >> \mathcal{D})$, the presence of shocks [44, Chap. 8], [45, Chap. 5] as the ones present in the usual form of the Burgers equation is expected, i.e., the solution is dominated by the nonlinear term (see Fig. 2). This behavior is evidenced in Fig. 2 for increasing k_α. At this point the presence of a stationary solution for (9) depending on the behavior of $k(t)$ is also interesting. In particular, for this case $\rho(r, t)$ is given by

$$\rho(r,t) = \frac{\Omega}{(\Phi(t))^{\mathcal{N}}}\left(\frac{r}{\Phi(t)}\right)^{\frac{\mathcal{K}}{\mathcal{D}}} \frac{e^{-\frac{r^{2+\theta}}{(2+\theta)\mathcal{D}(\Phi(t))^{2+\theta}}}}{1-\Omega\frac{k_\alpha}{\mathcal{D}}\int_0^{r/\Phi(t)} \tilde{z}^{\alpha+\theta+\mathcal{K}/\mathcal{D}} e^{-\frac{\tilde{z}^{2+\theta}}{(2+\theta)\mathcal{D}}} d\tilde{z}}. \tag{11}$$

The case $\nu = 1$, $\gamma = 1$, $\eta = 2+\theta$ and $\alpha = (\mathcal{N}-1)(1+\theta)$ leads us to a generalization of the usual form of the Burgers equation which for $\theta = 0$, $k(t) = 0$, $\mathcal{K} = 0$ and $\mathcal{N} = 1$ is recovered. For these parameters, after applying the procedure of the previous case, Eq. 2 may be simplified to two ordinary differential equations. One of these is equal to (6) and the other is given by

$$-\frac{d}{dz}\left(z^{\mathcal{N}}\mathcal{P}(z)\right) = \frac{d}{dz}\left\{\mathcal{D}\left[z^{\mathcal{N}-1-\theta}\frac{d}{dz}\mathcal{P}(z)\right] - z^{\mathcal{N}-1}\left(\frac{\mathcal{K}}{z^{1+\theta}}\right)\mathcal{P}(z) - k_\alpha z^{\mathcal{N}-1+\alpha}\left(\mathcal{P}(z)\right)^{2+\theta}\right\} \tag{12}$$

which has the solution

$$\mathcal{P}(z)=\frac{\bar{\Omega}\, z^{\mathcal{K}/\mathcal{D}}\mathrm{e}^{-\frac{z^{2+\theta}}{(2+\theta)\mathcal{D}}}}{\left(1-(1+\theta)\,\frac{k_\alpha}{\mathcal{D}}\bar{\Omega}^{1+\theta}\int_0^z\,\tilde{z}^{\alpha+\theta+(1+\theta)\mathcal{K}/\mathcal{D}}\mathrm{e}^{-\frac{(1+\theta)\tilde{z}^{2+\theta}}{(2+\theta)\mathcal{D}}}\mathrm{d}\tilde{z}\right)^{\frac{1}{1+\theta}}} \tag{13}$$

with $\bar{\Omega}$ defined by the normalization. Similar to (9), depending on the quantity $k_\alpha/\mathcal{D}$, (13) is dominated by the diffusive term or the nonlinear term present in the external force. In particular, for $k_\alpha/\mathcal{D}\ll 1$, i.e., in the diffusive limit, Eq. 13 and 9 are very close and, for $\theta=0$ (13) is equal to (9) as expected. By substituting (13) in (5), we obtain

$$\rho(r,t)=\frac{\bar{\Omega}}{(\Phi(t))^{\mathcal{N}}}\left(\frac{r}{\Phi(t)}\right)^{\frac{\mathcal{K}}{\mathcal{D}}}\frac{\mathrm{e}^{-\frac{r^{2+\theta}}{(2+\theta)\mathcal{D}(\Phi(t))^{2+\theta}}}}{\left(1-(1+\theta)\,\frac{k_\alpha}{\mathcal{D}}\bar{\Omega}^{1+\theta}\int_0^{r/\Phi(t)}\,\tilde{z}^{\alpha+\theta+(1+\theta)\mathcal{K}/\mathcal{D}}\mathrm{e}^{-\frac{(1+\theta)\tilde{z}^{2+\theta}}{(2+\theta)\mathcal{D}}}\mathrm{d}\tilde{z}\right)^{\frac{1}{1+\theta}}}. \tag{14}$$

Now, we consider the case characterized by ν as arbitrary parameter, $\gamma=1+(\nu-1)\mathcal{N}$, $\eta=\nu$ and $\alpha+1+\theta=0$. By employing the previous procedure based on (4), we also obtain ordinary differential equations to be solved. One of these is the same as that of (6). However, with $\xi=3+\theta+(\nu-1)\mathcal{N}$ and the others, for z ($z=r/\Phi(t)$), it is given by

$$-\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{\mathcal{N}}\mathcal{P}(z)\right)=\frac{\mathrm{d}}{\mathrm{d}z}\left\{\mathcal{D}z^{\mathcal{N}-1-\theta}\frac{\mathrm{d}}{\mathrm{d}z}\left(\mathcal{P}(z)\right)^{\nu}\;-z^{\mathcal{N}-1}\left(\frac{K}{z^{\gamma+\theta}}+k_\alpha z^{\alpha}\left(\mathcal{P}(z)\right)^{\nu-1}\right)\mathcal{P}(z)\right\}. \tag{15}$$

The solution of this equation, by taking the previous boundary condition into account, is given by

$$\mathcal{P}(z)=\hat{\Omega}z^{\frac{k_\alpha}{\nu\mathcal{D}}}\exp_q\left[-\frac{1}{\nu\mathcal{D}\hat{\Omega}^{\nu-1}}\left(\frac{z^{2+\theta-\frac{k_\alpha}{\nu\mathcal{D}}(\nu-1)}}{2+\theta-\frac{k_\alpha}{\nu\mathcal{D}}(\nu-1)}-\mathcal{K}\log_{\tilde{q}}z\right)\right], \tag{16}$$

where $\hat{\Omega}$ is defined by the normalization condition, $q=2-\nu$ and $\tilde{q}=[k_\alpha/(\nu\mathcal{D})]\,(\nu-1)+\gamma$, where $\exp_q(x)$ is the q-exponential function and $\log_q(x)$ is the q-logarithmic function. In particular, the q-exponential is defined as

$$\exp_q(x)=\begin{cases}(1+(1-q)x)^{\frac{1}{1-q}} & \text{for } 1+(1-q)x\geq 0\\ 0 & \text{for } 1+(1-q)x<0\end{cases} \tag{17}$$

and the q-logarithmic is given by $\log_q(x)=(x^{1-q}-1)/(1-q)$. For $q\to 1$, we recover the usual expressions for the exponential and logarithmic functions from the q-exponential and the q-logarithmic. These functions emerge from the Tsallis formalism when the entropy [26] is maximized with suitable constraints. In Fig. 3, we illustrate (16) for typical values of the parameters $\mathcal{K}$, ν and k_α in order to show some corresponding trends which may be obtained from the solution. From the definition of the q-exponential function, it is interesting to note that (16) may exhibit a compact or long-tailed behavior depending on the values of q. For the last case, it is possible to relate (16) with the Lévy distributions for a suitable choice of ν, θ and $\mathcal{N}$. In fact, for $\nu<1$ with $2+\theta>(1-\nu)\mathcal{N}$, the asymptotic behavior of (16) for large argument is governed by the power law $\mathcal{P}(z)\sim 1/z^{(2+\theta)/(q-1)}$ which compares with the asymptotic behavior of the Lévy distribution, i.e., $\mathcal{P}(z)\sim 1/z^{1+\mu}$, results in $q=(3+\theta+\mu)\,/\,(1+\mu)$. This result connects the parameter q (or ν) to the Lévy index μ. Similar results have been obtained for nonlinear fractional diffusion equations [55]. In order to obtain $\rho(r,t)$ for this case as done for the previous cases, we insert (16) into (5) yielding

$$\rho(r,t)=\frac{\hat{\Omega}}{(\Phi(t))^{\mathcal{N}}}\left(\frac{r}{\Phi(t)}\right)^{\frac{k_\alpha}{\nu\mathcal{D}}}\exp_q\left[-\frac{1}{\nu\mathcal{D}\hat{\Omega}^{\nu-1}}\left(\frac{1}{\varepsilon}\left(\frac{r}{\Phi(t)}\right)^{\varepsilon}-\mathcal{K}\log_{\tilde{q}}\left(\frac{r}{\Phi(t)}\right)\right)\right] \tag{18}$$

with $\varepsilon=2+\theta-k_\alpha(\nu-1)/(\nu\mathcal{D})$.

3 Discussion and conclusions

We have investigated a nonlinear diffusion equation by taking several situations into account. We started by considering the case characterized by $\nu=1$, $\gamma=1$, $\eta=2$ and $\alpha=\mathcal{N}-1-\theta$, which corresponds to an extension of the

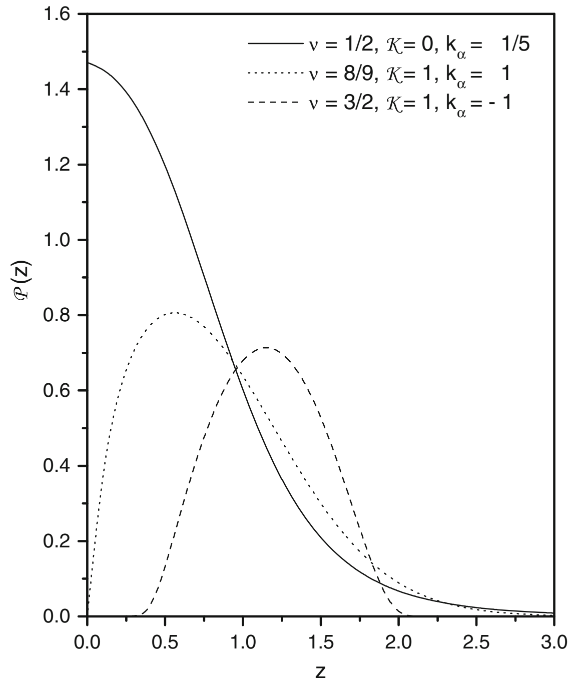

Fig. 3 This figure illustrates the behavior of (16) versus z for typical values $\mathcal{K}$, ν and k_α, by considering for simplicity $\mathcal{D} = 1, \theta = 1$ and $\mathcal{N} = 3$

classical Burgers equation for the $\mathcal{N}$-dimensional case with radial symmetry and a spatially dependent diffusion coefficient. This solution depending on the quantity $k_\alpha/\mathcal{D}$ is dominated by the diffusive term or the nonlinear term present in the external force. In this context, we analyzed another extension of the Burgers equation by choosing $\nu = 1$, $\gamma = 1$, $\eta = 2 + \theta$ and $\alpha = (\mathcal{N} - 1)(1 + \theta)$. For these cases, which extend the Burgers equation, $\theta = 0$ leads us to the same results. Following this, we consider the case obtained for ν arbitrary, $\gamma = 1 + (\nu - 1)\mathcal{N}$, $\eta = \nu$ and $\alpha + 1 + \theta = 0$. The solution for this case is given in terms of q-exponential and q-logarithmic functions. The presence of these functions establishes a connection to the Tsallis formalism suggesting a different thermostatistic context for (2) when subjected to these parameter values. In this direction, for a different choice of parameters in (2), we may apply the procedure in [56] to find the corresponding entropy and analyze the corresponding H-theorem. Another interesting aspect of this solution is the relation to the Lévy distribution as discussed above and the compact behavior which is obtained for q less than unity (or ν greater than unity). Finally, we hope that the results presented here may be useful to investigate situations where Eq. 2 or particular cases may be considered.

Acknowledgment We would like to thank CNPq/INCT-SC, CAPES and Fundação Araucária for partial financial support.

References

1. Frank TD (2005) Nonlinear Fokker–Planck equations. Springer-Verlag, Heidelberg
2. Zola RS, Lenzi MK, Evangelista LR, Lenzi EK, Lucena LS, da Silva LR (2008) Exact solutions for a diffusion equation with a nonlinear external force. Phys Lett A 372:2359–2363
3. Assis PC Jr, da Silva PC, da Silva LR, Lenzi EK, Lenzi MK (2006) Nonlinear diffusion equation and nonlinear external force: exact solution. J Math Phys 47:103302
4. Assis PC Jr, da Silva LR, Lenzi EK, Malacarne LC, Mendes RS (2005) Nonlinear diffusion equation. Tsallis formalism and exact solutions. J Math Phys 46:123303
5. Grarilov VP, Klepikova NV, Rodean HC (1995) Trial of a nonlinear diffusion equation as a model of turbulent-diffusion. Atmos Environ 29:2317–2322
6. Muskat M (1937) The flow of homogeneous fluids through porous media. McGraw-Hill, Michigan

7. Chavanis PH (2008) Nonlinear mean field Fokker-Planck equations. Application to the chemotaxis of biological populations. Eur Phys J B 62:179–208
8. Daly E, Porporato A (2004) Similarity solutions of nonlinear diffusion problems related to mathematical hydraulics and the Fokker-Planck equation. Phys Rev E 70:056303
9. De Wiest JM (ed) (1969) Flow through porous media. Academic Press, New York
10. Crank J (1956) The mathematics of diffusion. Oxford University Press, London
11. Havlin S, Ben-Avraham D (1987) Diffusion in disordered media. Adv Phys 36:695–798
12. Lee BP (1994) Renormalization-group calculation for the reaction kA $\rightarrow$ 0. J Phys A 27:2633–2652
13. Alemany PA, Zanette DH, Wio HS (1994) Time-dependent reactivity for diffusion-controlled annihilation and coagulation in 2 dimensions. Phys Rev E 50:3646–3655
14. Gilchrist J, Van der Beek CJ (1994) Nonlinear diffusion in hard and soft superconductors. Physica C 231:147–156
15. Vinokur VM, Feigel'man MV, Geshkenbein VB (1991) Exact solution for flux creep with logarithmic u(j) dependence—self-organized critical state in high-tc superconductors. Phys Rev Lett 67:915–918
16. Mayergoyz I (1998) Nonlinear diffusion of electromagnetic fields. Academic Press, New York
17. Tsallis C, Bukman DJ (1996) Anomalous diffusion in the presence of external forces: exact time-dependent solutions and their thermostatistical basis. Phys Rev E 54:R2197–R2200
18. Borland L, Pennini F, Plastino AR, Plastino A (1999) The nonlinear Fokker-Planck equation with state-dependent diffusion—a nonextensive maximum entropy approach. Eur Phys J B 12:285–297
19. Malacarne LC, Mendes RS, Pedron IT, Lenzi EK (2001) Nonlinear equation for anomalous diffusion: unified power-law and stretched exponential exact solution. Phys Rev E 63:030101
20. Malacarne LC, Mendes RS, Pedron IT, Lenzi EK (2002) N-dimensional nonlinear Fokker-Planck equation with time-dependent coefficients. Phys Rev E 65:052101
21. Giordano C, Plastino AR, Casas M, Plastino A (2001) Nonlinear diffusion under a time dependent external force: q-maximum entropy solutions. Eur Phys J B 22:361–368
22. Drazer G, Wio HS, Tsallis C (2000) Anomalous diffusion with absorption: exact time-dependent solutions. Phys Rev E 61:1417–1422
23. Rigo A, Plastino AR, Casas M, Plastino A (2000) Anomalous diffusion coupled with Verhulst-like growth dynamics: exact time-dependent solutions. Phys Lett A 276:97–102
24. Plastino AR, Casas M, Plastino A (2000) A nonextensive maximum entropy approach to a family of nonlinear reaction-diffusion equations. Physica A 280:289–303
25. Lenzi EK, Anteneodo C, Borland L (2001) Escape time in anomalous diffusive media. Phys Rev E 63:051109
26. Tsallis C (1988) Possible generalization of Boltzmann-Gibbs statistics. J Stat Phys 52:479–487
27. Plastino AR, Plastino A (1995) Non-extensive statistical mechanics and generalized Fokker-Planck equation. Physica A 222:347–354
28. Frank TD (2002) On a general link between anomalous diffusion and nonextensivity. J Math Phys 43:344–350
29. Borland L (1998) Microscopic dynamics of the nonlinear Fokker-Planck equation: a phenomenological model. Phys Rev E 57:6634–6642
30. Woyczynski WA (1999) Burgers-KPZ turbulence: Gottingen lectures. In: Lecture notes in mathematics. Springer-Verlag, Telos
31. Frish U, Bec J (2000) Burgulence. In: Lesieur M, David F, Yaglom AM (eds) Proceedings of the Les Houches Summer School, New trends in turbulence. Springer EDP-Sciences, Berlin
32. Olesen P (2003) Integrable version of Burgers equation in magnetohydrodynamics. Phys Rev E 68:016307
33. Witelski TP (2003) Intermediate asymptotics for Richards' equation in a finite layer. J Eng Math 45:379–399
34. Grundy RE (1983) Asymptotic solution of a model nonlinear convective diffusion equation. IMA J Appl Math 31:121–137
35. Dawson CN, van Duijn CJ, Grundy RE (1996) Large time asymptotics in contaminant transport in porous media. SIAM J Appl Math 56:965–993
36. van Duyn CJ, de Graaf JM (1987) Limiting profiles in contaminant transport through porous media. SIAM J Math Anal 18:728–743
37. Shaughnessy B, Procaccia I (1985) Analytical solutions for diffusion on fractal objects. Phys Rev Lett 54:455–458
38. Sokolov IM, Klafter J, Blumen A (2000) Ballistic versus diffusive pair dispersion in the Richardson regime. Phys Rev E 61:2717–2722
39. Vedenov AA (1967) Theory of a weakly turbulent plasma. Rev Plasma Phys 3:229
40. Vlad MO (1994) Fractional diffusion equation on fractals: self-similar stationary solutions in a force field derived from a logarithmic potential. Chaos Solitons Fractals 4:191–199
41. Frank TD (2002) Generalized Fokker-Planck equations derived from generalized linear nonequilibrium thermodynamics. Physica A 310:397–412
42. Risken H (1984) The Fokker–Planck equation. Springer-Verlag, New York
43. Gardiner CW (1996) Handbook of stochastic methods: for physics, chemistry and the natural sciences, springer series in synergetics. Springer-Verlag, New York
44. Debnath L (1997) Nonlinear partial differential equations for scientists and engineers. Birkhäuser, Boston
45. Logan JD (1994) An introduction to nonlinear partial differential equations. Wiley, New York
46. Bluman GW, Cole JD (1974) Similarity methods for differential equations. Springer, Berlin
47. Ovsiannikov LV (1980) The group analysis of differential equations. Academic Press, New York

48. Olver P (1986) Applications of Lie groups to differential equations. Springer, Berlin
49. Cherniha R (1998) New non-Lie ansätze and exact solutions of nonlinear reaction-diffusion-convection equations. J Phys A 31:8179–8198
50. Cherniha R, King JR (2005) Non-linear reaction–diffusion systems with variable diffusivities: Lie symmetries, ansätze and exact solutions. J Math Anal Appl 308:11–35
51. Silva AT, Lenzi EK, Evangelista LR, Lenzi MK, da Silva LR (2007) Fractional nonlinear diffusion equation, solutions and anomalous diffusion. Physica A 375:65–71
52. Wang J, Zhang W-J, Liang J-R, Zhang P, Ren F-Y (2008) Fractional nonlinear diffusion equation and first passage time. Physica A 387:764–772
53. Liang J-R, Ren F-Y, Qiu W-Y, Xiao J-B (2007) Exact solutions for nonlinear fractional anomalous diffusion equations. Physica A 385:80–94
54. Whitham GB (1974) Linear and nonlinear waves. Wiley, New York
55. Tsallis C, Lenzi EK (2002) Anomalous diffusion: nonlinear fractional Fokker-Planck equation. Chem Phys 284:341–347 (Erratum, Chem Phys 287:295, 2003)
56. Schwämmle V, Curado EMF, Nobre FD (2007) A general nonlinear Fokker-Planck equation and its associated entropy. Eur Phys J B 58:159–165