CAMPOS ESCALARES EM RELATIVIDADE

Tese de Doutoramento

por

ANTONIO F. DA F. TEIXEIRA

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AGRADECIMENTOS

Ao Centro Brasileiro de Pesquisas Físicas, pelo ambiente que me proporcionou e que tornou possível esta tese;

ao professor Murari M. Som, por sua orientação e ensinamentos;

ao professor Idel Wolk por sua amizade e úteis discussões;

aos professores Erasmo M. Ferreira, Nicim Zagury, Colber Gonçalves de Oliveira e Alfredo Marques de Oliveira pela camaradagem e apoio quando deles precisei;

a todos os que me acompanharam, minha gratidão.

Rio de Janeiro, Junho de 1975

Antonio F. da F. Teixeira

RESUMO

Inicialmente (I) é feito um retrospecto sucinto das principais soluções conhecidas das equações de Einstein, e que têm repercussão sobre o tema desta tese.

A seguir (II) é apresentada uma classe de soluções estáticas com simetria cilíndrica das equações de Einstein-Maxwell com um campo escalar (J. Math. Phys. 15, 1756 (1974)).

Segue-se a reprodução de três outros trabalhos, que resultaram da busca, ainda não completada, de um modelo clássico estático e sem singularidade para as partículas elementares.

O primeiro (III) trata de sistemas contendo apenas campos estáticos de longo alcance, inclusive escalares das classes atrativa e repulsiva (submetido recentemente ao J. Phys. A).

O segundo (IV) considera um sistema constituido por poeira escalarmente carregada, em equilíbrio, sem qualquer simetria espacial (Nuovo Cimento Lett. 12, 319 (1975)).

No terceiro (V) é apresentado um modelo constituido de poeira em equilíbrio estático, com densidade de carga escalar repulsiva de curto alcance (aceito para publicação pelo Phys. Rev. D (1975)).

Completam a tese algumas indicações sobre pesquisas nossas em andamento, na mesma linha dos trabalhos anteriores, e uma bibliografia sobre campos escalares em Relatividade Geral.

CAPÍTULO I

INTRODUÇÃO

Após o sucesso da Teoria Especial da Relatividade [1], muitas foram as tentativas de inclusão dos fenômenos gravitacionais em um formalismo covariante relativista. Foi o próprio criador da T.E.R., Einstein, quem elaborou a melhor teoria de gravitação conhecida no momento, a Teoria Geral da Relatividade. Desde a sua formulação final [2] até os dias de hoje, a T.G.R. tem sido considerada uma das maiores conquistas da criatividade humana; ela conseguiu aliar a um formalismo espaço-temporal de profundo significado uma precisão numérica que vem resistindo às mais rigorosas e recentes medidas [3].

Segundo a T.G.R. o potencial gravitacional é representado pelas dez componentes do tensor métrico $g_{\mu\nu}$, funções das coordenadas espaço-temporais. Esse tensor deve satisfazer às equações do campo, equações diferenciais parciais não lineares, de segunda ordem. O problema central da T.G.R. consiste na obtenção de soluções dessas equações, e na interpretação física dessas soluções.

No caso de vácuo, ou espaço vazio (aquele no qual se acha presente apenas campo gravitacional), as equações da T.G.R. são $R_{\mu\nu} = 0$ [4]. Dada a alta complexidade destas equações, a quase totalidade dos sistemas físicos estudados dispõe de alguma simetria. As soluções estáticas mais conhecidas destas equações são as de Schwarzschild [5], Weyl [6], Levi-Civita [7], e Curzon [8]; e a solução de Kerr [9] é a estacionária de maior relevância. A busca de novas soluções para o vácuo não está encerrada, nós mesmos tendo obtido recentemente alguns resultados [10, 11].

Para espaços não vazios a Relatividade Geral prescreve $G_{\mu\nu} = \kappa T_{\mu\nu}$ onde $T_{\mu\nu}$ é o tensor energia-momentum, cuja forma explícita depende do sistema físico em consideração.

Para matéria fluida, por exemplo, usa-se como $T^{\mu\nu}$ o tensor $M^{\mu\nu} = (\rho c^2 + p)\, u^\mu u^\nu - p\, g^{\mu\nu}$ onde ρ, p e u^μ são respectivamente a densidade, pressão e velocidade da matéria; dentre as soluções estáticas conhecidas para tais sistemas destaca-se a de Schwarzschild [12]; e dentre as soluções estacionárias salientam-se as dos aglomerados de Einstein [13, 14, 15].

Para campos electromagnéticos usa-se, como $T^\mu_\nu, E^\mu_\nu = \varepsilon_0 (F^\mu_\alpha F^\alpha_\nu - \delta^\mu_\nu F^\alpha_\beta F^\beta_\alpha / 4)$ onde F^μ_ν é o tensor do campo electromagnético. Muitas são as soluções exatas de campos electromagnéticos já obtidas [16], tais como as de Reissner [17], Weyl [6], Kar [18], Mukherji [19], Bonnor [20] e nossa [21].

Sistemas físicos constituidos por matéria e campos eletromagnéticos são representados por tensores energia-momentum que são a soma $M^\mu_\nu + E^\mu_\nu$; frequentemente as equações dos campos para tais sistemas apresentam uma complexidade de tal ordem que apenas soluções aproximadas são obtidas [22].

Outros campos podem ser introduzidos na Relatividade Geral [23,24,25]. Usualmente faz-se essa introdução segundo o princípio do acoplamento mínimo [26], do qual resulta a ausência de derivadas de $g_{\mu\nu}$ no tensor energia-momentum dos campos. Esta tese versa fundamentalmente sobre campos escalares em Relatividade Geral, à analise dos quais passamos a dirigir agora a nossa atenção.

A primeira questão a abordar, a respeito do campo escalar, concerne ao papel singular representado por esse campo. Com exceção do que poderia ser considerado um "campo cosmológico" constante e uniforme Λ, o campo escalar é o campo covariante estruturalmente mais simples, dispondo de apenas uma componente $S(x^{\mu})$. Devido a essa simplicidade estrutural, é de esperar que ele ainda venha a representar um papel fundamental nas descrições tanto dos macrossistemas (sob forma estatística) quanto dos microssistemas (sob forma elementar, ou primária).

Um outro ponto a comentar diz respeito à "filosofia" que possa reger a introdução de um campo escalar em uma teoria de gravitação. A teoria Geral de Relatividade permite a coexistência de um campo escalar e de um campo gravitacional, porém nega a impossibilidade da existência de um campo gravitacional independente da presença de um campo escalar. Em outras palavras, a T.G.R. prescinde de um campo escalar, apenas aceitando a sua presença. Outras teorias gravitacionais, entretanto, consideram a presença de um campo escalar como imprescindível para a própria existência da gravitação. Nessas teorias, chamadas teorias escalar-tensoriais, o campo escalar tem a sua presença assegurada nas equações do campo gravitacional, qualquer que seja o sistema físico em consideração. Tais são as teorias de Szekeres [24], Yilmaz [26], Brans e Dicke [27] e a teoria pentadimencional de Jordan [28], entre outras.

Destas teorias, a que recebeu a maior atenção dos pesquisadores, tanto teóricos como experimentais, foi a de Brans-Dicke; ela representa primordialmente uma tentativa de incorporação do princípio de Mach a uma teoria relativista de gravitação. Entretanto, resultados experimentais bastante recentes [3] tendem a não comprovar as predições das teorias gravitacionais escalar-tensoriais; no caso específico da de Brans-Dicke, o valor 7.5 sugerido por Dicke [29] para sua constante adimensional difere muito do valor mínimo (23) requerido por precisas observações recentes de Fomalont e Sramek [30]. Alguns inconvenientes adicionais das teorias de Szekeres e de Yilmaz foram apontados pelo próprio Yilmaz [31], e a teoria pentadimensional de Jordan visou sobretudo à unificação dos campos gravitacionais e eletromagnéticos sob um formalismo penta-covariante especial.

Como a Teoria Geral da Relatividade vem satisfazendo, como teoria gravitacional, a todos os resultados observados dentro dos erros experimentais, é na sua "filosofia" que serão considerados os campos escalares nesta tese.

CAPÍTULO II

CAMPOS ELETROMAGNÉTICOS GENERALIZADOS

Devido à diferença de tratamento matemático, convém distinguir de início dois tipos de campos escalares, os de curto alcance (também chamados de alcance finito, ou ainda massivos) e os de longo alcance (alcance infinito ou de massa nula), muito embora os campos escalares de longo alcance sejam apenas um caso particular dos de curto alcance.

Os de longo alcance tem um comportamento um tanto semelhante ao do potencial gravitacional de Newton, e ao do potencial eletrostático de Coulomb. Devido a essa semelhança, muitos foram os trabalhos publicados envolvendo simultaneamente campos gravitacionais, eletrostáticos e escalares de longo alcance.

Os campos gravitacional e eletromagnéticos vem sendo chamados de "generalizados" na literatura, quando acompanhados por um campo escalar de longo alcance. Soluções de tais campos generalizados foram apresentadas, entre outras, por Szekeres [24], Yilmaz [26], Bergmann e Leipnik [32] (ver também Treder [33]), Buchdahl [34], Janis, Newman e Winicour [35], Penney [36], Gautreau [37], Misra e Pandey [38], Tupper [39], Rao, Roy e Tiwari [40], Datta e Rao [41], e por nós [21].

Este último trabalho surgiu com o nosso exame das objeções levantadas por Bonnor [20] (1954) às soluções de campo eletromagnético apresentadas por Mukherji [19]. As soluções eletromagnéticas generalizadas apresentadas em nosso trabalho são estáticas e de simetria cilíndrica, e tendem à solução de vácuo de Marder [43], no caso de ausência dos campos escalar e eletromagnético. A conveniência das nossas soluções é que a solução de Marder resiste às objeções de Bonnor, permitindo a interpretação do seu parâmetro C como densidade linear de matéria em unidades adequadas. A seguir, uma reprodução desse nosso trabalho.

GENERALIZED STATIC ELECTROMAGNETIC FIELDS IN RELATIVITY

A. F. da F. Teixeira, Idel Wolk and M. M. Som
Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro, Brazil

ABSTRACT

A general class of solutions is obtained for static cylindrically symmetric (radial, azimuthal and longitudinal) electromagnetic fields coupled with massless scalar field. These are presented all in a same reference system such that one can have Marder's solution immediately in absence of electromagnetic and scalar fields.

1. INTRODUCTION

In literature there are some interesting static solutions of Einstein-Maxwell equations. In a space-time region symmetric under rotations about a spatial axis as well as under translations about the same axis, these solutions correspond to three types of fields: azimuthal fields (Mukherji[1]), radial fields (Mukherji[1], Bonnor[2]) and longitudinal fields (Bonnor[2], Melvin [3], Ghosh and Sengupta [4]). Mukherji obtained a class of solutions of the field equations corresponding to an infinite straight wire carrying current, using pseudo -cylindrical coordinates. Bonnor [5] observed that with this solution for azimuthal electromagnetic field one has difficulty to interprete the constants the way Mukherji interpreted them, as parameters representing mass, current and radius. Indeed one can eliminate the constant representing mass in his solution by a suitable coordinate transformation. Further for vanishing electromagnetic field his solution goes over to Marder's [6] solution only for a particular value of the parameter associated with gravitational mass in Marder's solution.

In the case of already known solutions of field equations corresponding to an infinite line-charge, Som [7] showed that if there are no singularities in the field, then one must allow negative values of R^0_0 in the source region; negative values of R^0_0 would require then $T^0_0 - T/2 < 0$, which demands a very unusual property of matter. Thus one is forced to infer that no solution seems to exist for a line-charge with positive mass.

Similar is the situation with the solution corresponding to longitudinal fields. The only solution known so far which does not give rise to this situation is that of Melvin. However Melvin's solution corresponds to a magnetic universe free of any source.

In this paper we have studied all these fields coupled with zero-rest mass scalar fields. Though inclusion of scalar field does not remove the previously mentioned difficulties, some interesting results are obtained. Furthermore we have obtained a new class of solutions of the field equations corresponding to an infinite wire carrying current. For vanishing electromagnetic and scalar fields all our solutions go over immediately to Marder's solution.

2.1 - BASIC EQUATIONS

The field equations of space-time containing electromagnetic fields and zero-rest mass scalar field, but no matter are

$$R_{\mu\nu} - g_{\mu\nu} R/2 = - K(E_{\mu\nu} + S_{\mu\nu}) , \tag{2.1}$$

with

$$E_{\mu\nu} = \varepsilon_0 (F_{\mu}^{\ \alpha} F_{\alpha\nu} - g_{\mu\nu} F_{\alpha}^{\ \beta} F_{\beta}^{\ \alpha}/4) \tag{2.2}$$

and

$$S_{\mu\nu} = S_{,\mu} S_{,\nu} - g_{\mu\nu} g^{\rho\sigma} S_{,\rho} S_{,\sigma}/2 , \tag{2.3}$$

where $F_{\mu\nu}$ is the skew-symmetric electromagnetic field tensor which satisfies Maxwell's equations for empty space,

$$F_{[\mu\nu;\alpha]} = 0 \tag{2.4}$$

and

$$F^{\mu\nu}_{\ \ ;\nu} = 0 , \tag{2.5}$$

the semicolon denoting covariant differentiation; and S is the zero-rest mass scalar field, which satisfies

$$S^{;\mu}_{;\mu} = 0 . \tag{2.6}$$

2.2 - SURVIVING COMPONENTS OF ELECTROMAGNETIC FIELDS

We shall first obtain the surviving components of electromagnetic fields in space-time regions where the metric tensor $g_{\mu\nu}$, the Ricci tensor $R_{\mu\nu}$ and the tensor $S_{\mu\nu}$ are all diagonal; then from Eqs. (2.1) and (2.2) one has $E_{\mu\nu}$ diagonal. The vanishing of E_{01} and E_{23} implies respectively

$$g^{22} F_{20} F_{21} + g^{33} F_{30} F_{31} = 0 \tag{2.7}$$

and

$$g^{00} F_{02} F_{03} + g^{11} F_{12} F_{13} = 0 \ . \tag{2.8}$$

One has then

$$F_{20} F_{21} (F_{30} F_{31})^{-1} < 0 \text{ except when } F_{20} F_{21} = F_{30} F_{31} = 0 \tag{2.9}$$

and

$$F_{02} F_{03} (F_{12} F_{13})^{-1} > 0 \text{ except when } F_{02} F_{03} = F_{12} F_{13} = 0 \ . \tag{2.10}$$

Equations (2.7) and (2.8) are then only compatible when either the pair of components (F_{02}, F_{31}) or the pair of components (F_{03}, F_{12}) vanishes. Similar considerations about vanishing of E_{02} and E_{13} show that at least one of the pairs of components (F_{03}, F_{12}) and (F_{01}, F_{23}) must vanish; and vanishing of E_{03} and E_{12} imply vanishing of at least one of the two pairs (F_{01}, F_{23}) and (F_{02}, F_{31}). Consequently of these three pairs of components (F_{01}, F_{23}), (F_{02}, F_{31}) and (F_{03}, F_{12}) always only one pair of components survivies, for nonvanishing electromagnetic fields.

2.3 - FIELD EQUATIONS IN STATIC METRICS WITH CYLINDRICAL SYMMETRY

The line element for a static system with cylindrical symmetry is

$$ds^2 = e^{2\eta} dt^2 - e^{2\lambda} dr^2 - r^2 e^{2\beta} d\phi^2 - e^{2\gamma} dz^2 \ , \tag{2.11}$$

with η, λ, β and γ functions of r only. It is known [8, Chap. 8] that for any static cylindrically symmetric system, the metric components can be described by only three functions of radial coordinate. So we choose a relation

$$\lambda = \eta + \beta + \gamma \ . \tag{2.12}$$

Then the surviving components of Ricci tensor are (a subscript 1 means d/dr)

$$R^0_0 = - e^{-2(\eta+\beta+\gamma)}(\eta_{11} + \eta_1/r) \ , \tag{2.13}$$

$$R^1_1 = - e^{-2(\eta+\beta+\gamma)}[\eta_{11}+\beta_{11}+\gamma_{11}+ (\beta_1-\eta_1-\gamma_1)/r-2(\eta_1\beta_1+\eta_1\gamma_1+\beta_1\gamma_1)] \ , \tag{2.14}$$

$$R^2_2 = - e^{-2(\eta+\beta+\gamma)} \ (\beta_{11} + \beta_1/r) \tag{2.15}$$

and

$$R^3_3 = - e^{-2(\eta+\beta+\gamma)}(\gamma_{11} + \gamma_1/r) \ . \tag{2.16}$$

Since the electromagnetic field $F_{\mu\nu}$ depends only on r, we have from (2.4)

$$F_{02} = \mathcal{E}_\phi, \tag{2.17}$$

$$F_{03} = \mathcal{E}_z \tag{2.18}$$

and

$$F_{23} = c\ \mathcal{B}_r \ , \tag{2.19}$$

where $\mathcal{E}_\phi$, $\mathcal{E}_z$ and $\mathcal{B}_r$ are constants; and from (2.5) one has

$$F_{01} = \mathcal{E}_r \ r^{-1} \ e^{2\eta} \ , \tag{2.20}$$

$$F_{12} = c\ \mathcal{B}_z \ r\ e^{2\beta} \tag{2.21}$$

and

$$F_{13} = c\ \mathcal{B}_\phi \ r^{-1} \ e^{2\gamma} \ , \tag{2.22}$$

where $\mathcal{E}_r$, $\mathcal{B}_z$ and $\mathcal{B}_\phi$ are constants. Also the massless scalar field depends only on r, so from (2.6)

$$S_1 = \mathcal{J} r^{-1} \ , \tag{2.23}$$

where $\mathcal{J}$ is a constant.

If we now define the constants

$$\mathcal{C}_r = \mathcal{K}\, \varepsilon_0 (\mathcal{E}_r^2 + c^2 \mathcal{B}_r^2)/2 \ , \tag{2.24}$$

$$\mathcal{C}_\phi = \mathcal{K}\, \varepsilon_0 (\mathcal{E}_\phi^2 + c^2 \mathcal{B}_\phi^2)/2 \tag{2.25}$$

and

$$\mathcal{C}_z = \mathcal{K}\, \varepsilon_0 (\mathcal{E}_z^2 + c^2 \mathcal{E}_z^2)/2 \ , \tag{2.26}$$

then simplest from of Einstein equations is

$$r^2 \eta_{11} + r \eta_1 = \{ \mathcal{C}_r e^{2\eta}, \mathcal{C}_\phi e^{2\gamma}, \mathcal{C}_z r^2 e^{2\beta} \} \ , \tag{2.27}$$

$$r^2 \beta_{11} + r \beta_1 = \{ -\mathcal{C}_r e^{2\eta}, \mathcal{C}_\phi e^{2\gamma}, -\mathcal{C}_z r^2 e^{2\beta} \}, \tag{2.28}$$

$$r^2 \gamma_{11} + r \gamma_1 = \{ -\mathcal{C}_r e^{2\eta}, -\mathcal{C}_\phi e^{2\gamma}, \mathcal{C}_z r^2 e^{2\beta} \} \tag{2.29}$$

and

$$r^2 \left[\eta_1 \gamma_1 + (\eta_1 + \gamma_1)(\beta_1 + 1/r) \right] = \mathcal{K} \mathcal{J}^2/2 +$$

$$+ \{ -\mathcal{C}_r e^{2\eta}, \mathcal{C}_\phi e^{2\gamma}, \mathcal{C}_z r^2 e^{2\beta} \} \ , \tag{2.30}$$

where in each bracket { } only one of the terms is to be considered, since in each problem only one of the constants $\mathcal{C}_r$, $\mathcal{C}_\phi$, $\mathcal{C}_z$ can be different from zero.

3. SOLUTIONS OF FIELD EQUATIONS

In all three problems (radial, azimuthal and longitudinal) the method for obtaining the solution is identical; we use the first three equations for obtaining the expressions of η, β and γ (λ is then got from (2.12)), with a total of six constants of integration; then last equation (2.30) gives a relation which reduces these six constants to five. After that, one can easily reduce these five constants to only three essential ones, by suitable coordinate transformations. Let us consider the azimuthal case in details.

In this case $\mathcal{C}_r = \mathcal{C}_z = 0$; then (2.29) gives

$$\gamma = -\log\left[(r/a)^b + \mathcal{C}_\phi (2b)^{-2} (r/a)^{-b}\right] = -\log \mathcal{B}, \tag{3.1}$$

with a and b constants of integration; sum of (2.27) and (2.29) gives

$$\eta = -\gamma + h \log(r/d) , \tag{3.2}$$

with h and d constants of integration; and sum of (2.28) and (2.29) gives

$$\beta = -\gamma + p \log(r/f) , \tag{3.3}$$

with p and f constants of integration; finally substitution of (3.1) to (3.3) into (2.30) gives

$$p\,h - b^2 - k\mathcal{J}^2/2 = 0 . \tag{3.4}$$

One thus obtains

$$g_{00} = (r/d)^{2h} \mathcal{B}^2, \tag{3.5}$$

$$g_{11} = -(r/d)^{2h} (r/f)^{(2b^2 + k\mathcal{J}^2/2)/h} \mathcal{B}^2, \tag{3.6}$$

$$g_{22} = -r^2 (r/f)^{(2b^2 + k\mathcal{J}^2/2)/h} \mathcal{B}^2 \tag{3.7}$$

and

$$g_{33} = -\mathcal{B}^{-2} ; \tag{3.8}$$

with a suitable coordinate transformation one obtains

$$g_{00} = (r/r_0)^{2(b+h)} \mathcal{F}_\phi \ , \tag{3.9}$$

$$g_{11} = - (r/r_0)^{-2+2(b+h) + 2b^2/h + \kappa \mathcal{S}^2/h} \mathcal{F}_\phi \ , \tag{3.10}$$

$$g_{22} = - r^2 (r/r_0)^{-2+2b^2/h+2b+\kappa \mathcal{S}^2/h} \mathcal{F}_\phi \tag{3.11}$$

and

$$g_{33} = - (r/r_0)^{-2b} \mathcal{F}_\phi^{-1} \ , \tag{3.12}$$

where

$$r_0 = a(a/d)^h (a/f)^{(b^2+\kappa \mathcal{S}^2/2)/h} \tag{3.13}$$

and

$$\mathcal{F}_\phi = \left[1 + \mathcal{C}_\phi (2b)^{-2} (r/r_0)^{-2b}\right]^2 \ . \tag{3.14}$$

Mukherji [1, Sec. 2] is a special case of this solution, for vanishing scalar field.

Following similar steps one obtains for the radial case ($\mathcal{C}_\phi = \mathcal{C}_z = 0$)

$$g_{00} = (r/r_0)^{2(b+h)} \mathcal{F}_r^{-1} \ , \tag{3.15}$$

$$g_{11} = - (r/r_0)^{-2+2(b+h)+2b^2/h+\kappa \mathcal{S}^2/h} \mathcal{F}_r \ , \tag{3.16}$$

$$g_{22} = - r^2 (r/r_0)^{-2+2b+2b^2/h+\kappa \mathcal{S}^2/h} \mathcal{F}_r \tag{3.17}$$

and

$$g_{33} = -(r/r_0)^{-2b} \mathcal{F}_r \ , \tag{3.18}$$

where

$$\mathcal{F}_r = \{1 - \mathcal{C}_r \left[2(b+h)\right]^{-2} (r/r_0)^{2(b+h)}\}^2 \ . \tag{3.19}$$

Mukherji [1, Sec. 3] and Bonnor [2, Sec. 2.b] are special cases of this solution.

And for the longitudinal case ($\mathcal{E}_r = \mathcal{E}_\phi = 0$) one obtains

$$g_{00} = (r/r_0)^{2(b+h)} \mathcal{F}_z , \qquad (3.20)$$

$$g_{11} = - (r/r_0)^{-2+2(b+h)+2b^2/h+K\delta^2/h} \mathcal{F}_z , \qquad (3.21)$$

$$g_{22} = - r^2 (r/r_0)^{-2+2b+2b^2/h+K\delta^2/h} \mathcal{F}_z^{-1} \qquad (3.22)$$

and

$$g_{33} = - (r/r_0)^{-2b} \mathcal{F}_z , \qquad (3.23)$$

where

$$\mathcal{F}_z = \{1 + \mathcal{E}_z r_0^2 \left[- 2(b+h) + 2(b+h)^2/h + K\delta^2/h\right]^{-2} \times$$
$$\times (r/r_0)^{-2(b+h)+2(b+h)^2/h+K\delta^2/h}\}^2 . \qquad (3.24)$$

Bonnor [2, Sec. 2.a], Bonnor [5, Sec. 3], Gosh and Sengupta [4] are special cases of this solution. In absence of electromagnetic fields ($\mathcal{F}_r = \mathcal{F}_\phi = \mathcal{F}_z = 1$) solutions for radial, azimuthal and longitudinal problems become identical.

We next impose that our coordinates be Weyl canonical ones, in absence of electromagnetic fields; this means $g_{00}\, g_{22} = - r^2$, and g_{11} and g_{33} become identical as a consequence of (2.12). This imposition relates the three constants b, h, δ by

$$h - (b + h)^2 - K\delta^2/2 = 0 . \qquad (3.25)$$

If we further relabel the combination b + h,

$$b + h = 2m \qquad (3.26)$$

then we get in absence of electromagnetic field

$$g_{00} = (r/r_0)^{4m} , \tag{3.27}$$

$$g_{11} = g_{33} = - (r/r_0)^{-4m(1-2m) + K\mathcal{S}^2} \tag{3.28}$$

and

$$g_{22} = - r^2 (r/r_0)^{-4m} . \tag{3.29}$$

For the factors corresponding to each problem with electromagnetic field we get in this notation

$$\mathcal{F}_r = \left[1 - \mathcal{E}_r (4m)^{-2} (r/r_0)^{4m} \right]^2 , \tag{3.30}$$

$$\mathcal{F}_\phi = \left[1 + \mathcal{E}_\phi (- 4m + 8m^2 + K\mathcal{S}^2)^{-2} (r/r_0)^{-4m + 8m^2 + K\mathcal{S}^2} \right] \tag{3.31}$$

and

$$\mathcal{F}_z = \left[1 + \mathcal{E}_z r_0^2 (2 - 4m)^{-2} (r/r_0)^{2-4m} \right]^2 . \tag{3.32}$$

We have thus obtained a set of solutions which go over to the solution given by Marder [6] in absence of electromagnetic and massless scalar fields. And in absence of scalar field and making parameter m vanish in the longitudinal solution gives the electromagnetic geon of Melvin [3]. It should be noticed that our radial and longitudinal solutions are expressed in Weyl canonical coordinates, while azimuthal solution is not.

4. DISCUSSION OF THE RESULTS

In Sec. 2.2 we showed that static cylindrically symmetric systems may contain exclusively radial, or azimuthal or longitudinal electromagnetic field, not combination of these fields. Choice of coordinates $r^2 g_{rr} = - g_{00} g_{\phi\phi} g_{zz}$ proved to give equations easier to solve than others more used in literature, such as

$g_{rr} = g_{zz}$ or $g_{\phi\phi} = r^2\, g_{rr}$; our choice of coordinates and an appropriate labeling of constants of integration allowed solutions of the three independent systems to become identical in absence of electromagnetic fields; and further these tend to Marder's solution in case of vanishing massless scalar field, when one of the two constants b and h is fixed by the condition (3.25).

It is evident from Eqs. (3.30) to (3.32) that one cannot have m = 0 for radial field and m = 1/2 for longitudinal field, independently of scalar field. With azimuthal field the situation is different: only in absence of scalar field the solution corresponding to azimuthal field does not allow both m = 0 and m = = 1/2. It is further interesting to note that for vanishing electromagnetic field the expression for Kretschman scalar takes the form

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \left[64m^2(1-2m)^2(1-2m+4m^2) - 16m(1-2m)(1-4m+8m^2)\kappa\mathcal{S}^2 + (3-8m+16m^2)\kappa^2\mathcal{S}^4\right] r_0^{-4}(r/r_0)^{-4+8m-16m^2-2\kappa\mathcal{S}^2};$$

when $\mathcal{S} \neq 0$ this is always positive and tends to zero at infinity; however, when $\mathcal{S} = 0$ it tends to zero everywhere as m tends to either the value zero or to one-half.

Janis et al.[9] prescribed a method of obtaining some generalized electromagnetic fields from the vacuum field solutions irrespective of any symmetry, but their prescriptions do not admit combination of electric and magnetic fields. However our solutions allow combinations of electric and magnetic fields.

ACKNOWLEDGEMENTS

One of the author (M. M. S.) acknowledges financial support from Conselho Nacional de Pesquisas (*Brazil*).

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CAPÍTULO III

ELETROVÁCUOS GENERALIZADOS

Uma outra linha de pesquisa em campos escalares de longo alcance consiste na obtenção de vácuos generalizados e de eletrovácuos generalizados a partir de uma solução conhecida de vácuo. Dá-se o nome de eletrovácuo a um sistema contendo apenas campo eletromagnético, e de uma forma tal que, com o desaparecimento desse campo, o novo sistema se veja despojado de campo gravitacional ($R_{\mu\nu\rho\sigma} = 0$). Entre outros, Buchdahl [34], Das [44], Janis, Robinson e Winicour [45], Misra e Pandey [38], Datta e Rao [41] obtiveram soluções parciais do problema.

Uma generalização dessas soluções foi recentemente obtida por nós [46], e considera duas classes distintas de campos escalares de longo alcance: a do tipo atrativo (como o campo gravitacional, que provoca atração entre duas massas) e a do tipo repulsivo (como o eletrostático, que provoca repulsão entre duas cargas de mesmo sinal). Segue-se uma reprodução do trabalho em apreço.

ON GENERALIZED STATIC ELECTROVACS

A.F. da F. Teixeira, Idel Wolk and M.M. Som
Centro Brasileiro de Pesquisas Físicas, ZC-82 Rio de Janeiro, Brazil

ABSTRACT

Generalized static electrovac solutions of Einstein's equations are obtained from known vacuum solutions, in which the specific electric charge as well as the specific scalar charge may assume arbitrary values. In the process of generalization two types of long range scalar fields were considered: one is repulsive, like Coulomb field, the other is attractive, like gravity field. The effect of the superposition of these two types of fields has also been studied. The solutions presented tend straightforwardly to the corresponding vacuum solutions when the constants associated to the electromagnetic and scalar strengths tend to zero.

1. INTRODUCTION

In a pioneer work Weyl (1917) obtained the class of solutions of Einstein-Maxwell's equations for static systems with cylindrical symmetry, when there exists a functional relationship between the electrostatic potential ϕ and the metric component g_{oo}. Later Majumdar (1947) generalized these results for electrostatic systems with or without spatial symmetry, showing that $g_{oo}=1+A\phi+\phi^2$; however the interpretation of the constant A was not given. At the same time Papapetrou (1947) presented a special class of solutions involving magnetostatic potentials ϕ^* for which $g_{oo}=(1+|\phi^*|)^2$. After that, Bonnor (1954) showed how to generate a purely magnetostatic solution from a purely electrostatic one. Soon after, Ehlers (1955) in a remarkable work presented a method for developing a purely electrostatic or a purely magnetostatic solution out from a given vacuum solution; in the same year he extended (Ehlers 1955a) his results to include the constant A of Majumdar, and also introduced a one-parameter class of long range scalar fields; however in his solutions one finds difficulty in interpreting the constants associated with the electromagnetic and scalar fields, and also the original vacuum solutions cannot be obtained simply by putting the constants associated with these fields equal to zero. These difficulties are also encountered in later works of Bonnor (1961) and of Janis et al. (1969).

In this paper we present an extended solution of the problem proposed by Ehlers (1955, 1955a); using the recent results of Parker (1975) we were able to introduce simultaneously electros tatic and magnetostatic fields. Differently from previous works, all the constants which appear in our solutions have easy interpretations at least in the weak field approximation. Since scalar fields seem to play an interesting role in the structure of elementary particles (Teixeira et al. 1975), we consider here simultaneously two possible classes of one parameter scalar fields that can be fitted in Einstein's theory. An interesting feature of our solution is the straightforward recovering of the original vacuum solutions when one puts the constants associated to the strengths of the electromagnetic and scalar fields equal to zero.

2. FIELD EQUATIONS

We consider the general static line element

$$ds^2 = e^{2\psi}(dx^0)^2 - e^{-2\psi}h_{ij}\, dx^i dx^j \quad , \tag{1}$$

with ψ and h_{ij} functions of the space coordinates x^k only. Then the nonvanishing components of the Ricci tensor $R_{\mu\nu}$ are

$$R_{oo} = -e^{4\psi}\Delta\psi \quad , \tag{2}$$

$$R_{ij} = H_{ij} + 2\psi_{,i}\psi_{,j} - h_{ij}\Delta\psi \quad , \tag{3}$$

where H_{ij} and Δ are the Ricci tensor and the laplacian operator built up from h_{ij}, and a comma denotes ordinary derivative.

Maxwell's equations in empty space are

$$F^{\mu\nu}{}_{;\mu} = 0 \quad , \tag{4}$$

$$\varepsilon^{\mu\nu\rho\sigma}\, F_{\mu\nu;\rho} = 0 \quad , \tag{5}$$

where a semicolon denotes covariant derivative and $\varepsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric unitary contravariant tensor density of weight + 1; for stationary systems we can write, from (5)

$$F_{oi} = -\phi_{,i} \tag{6}$$

where from (4) and (1) the electrostatic potential ϕ satisfies

$$(h^{1/2}\, e^{-2\psi} h^{ij} \phi_{,i})_{,j} = 0 \quad , \tag{7}$$

with $h = \det h_{ij}$ and $h^{ij}h_{jk} = \delta^i_k$. The energy momentum tensor

$$E_{\mu\nu} = (F_{\mu\alpha}\, F^{\alpha}{}_{\nu} - g_{\mu\nu} F_{\alpha\beta} F^{\beta\alpha}/4)/4\pi \quad , \tag{8}$$

corresponding to an electromagnetic field whose only antisymmetric components are given by (6) has the surviving components

$$8\pi E_{oo} = e^{2\psi}\, h^{ij}\, \phi_{,i}\, \phi_{,j} \quad , \tag{9}$$

$$8\pi E_{ij} = (h_{ij}\, h^{km}\, \phi_{,k}\, \phi_{,m} - 2\phi_{,i}\, \phi_{,j}) e^{-2\psi} \quad . \tag{10}$$

Another long range field which we will consider is the "massless" real scalar field S, which satisfies in regions free of scalar sources (Idel Wolk et al. 1975) the equation $S^{;\mu}_{;\mu} = 0$, which in static systems (1) takes the form

$$(h^{1/2}\, h^{ij}\, S_{,i})_{,j} = 0 \quad . \tag{11}$$

There are indeed two classes of such fields: the first class is responsible for an attraction between two static sources of same sign, so we call attractive such gravity-like fields; sources of opposite sign would repel each other. The second class is a Coulomb-like field, in the sense that two static sources of same sign repel each other, while sources of opposite sign feel a mutual attraction; we call such fields repulsive. The energy momentum tensor of a long range scalar field is

$$S_{\mu\nu} = \pm(S_{,\mu} S_{,\nu} - g_{\mu\nu} S'^{\alpha} S_{,\alpha}/2)/4\pi \quad , \qquad (12)$$

where upper and lower sign correspond to attractive and repulsive scalar fields, respectively; the demonstration of this assertion is given in the Appendix. Throughout this paper we follow this sign convention.

Let us consider now the Ehlers (1955a) problem: given a static solution (V, h_{ij}) of Einstein's vacuum equations

$$ds^2 = e^{2V}(dx^o)^2 - e^{-2V} h_{ij} dx^i dx^j \quad , \qquad (13)$$

$$R_{\mu\nu} = 0 \quad , \qquad (14)$$

we want a static solution (ψ, h_{ij}, ϕ, S) of Einstein-Maxwell-scalar equations

$$ds^2 = e^{2\psi}(dx^o)^2 - e^{-2\psi} h_{ij} \, dx^i dx^j \quad , \qquad (15)$$

$$R_{\mu\nu} - g_{\mu\nu} R/2 = - 8\pi(E_{\mu\nu} + S_{\mu\nu}) \quad , \qquad (16)$$

with the electrostatic potential ϕ functionally related to the gravitational potential ψ, and also the scalar potential S functionally related to ψ; also this ψ is to bear a functional relationship with the vacuum gravitational potential V.

That is to say, given the functions V and h_{ij} of the space coordinates satisfying

$$(h^{1/2}\, h^{ij}\, V_{,i})_{,j} = 0 \quad , \tag{17}$$

$$H_{ij} + 2V_{,i}\, V_{,j} = 0 \quad , \tag{18}$$

we want ψ, ϕ and S satisfying Maxwell and scalar equations

$$(e^{-2\psi} h^{1/2} h^{ij} \phi_{,i})_{,j} = 0 \quad , \tag{19}$$

$$(h^{1/2}\, h^{ij}\, S_{,i})_{,j} = 0 \quad , \tag{20}$$

and also Einstein's equations

$$e^{2\psi}\, h^{-1/2} (h^{1/2}\, h^{ij} \psi_{,i})_{,j} = h^{ij}\, \phi_{,i}\, \phi_{,j} \quad , \tag{21}$$

$$H_{ij} + 2\psi_{,i}\, \psi_{,j} - h^{-1/2} h_{ij}\, (h^{1/2} h^{km} \psi_{,k})_{,m} = -(h_{ij} h^{km} \phi_{,k} \phi_{,m} - 2\phi_{,i} \phi_{,j}) e^{-2\psi} \mp$$

$$\mp 2S_{,i} S_{,j} \quad , \tag{22}$$

each of the four functions (ψ, ϕ, S, V) having to be functionally related to the remaining three.

3. SOLUTIONS OF THE EQUATIONS

From (17) and (19) and considering that both ϕ and ψ are functionally related with V, we see that

$$e^{-2\psi}\, \phi_{,i} = -\, a\, V_{,i} \quad , \quad a = \text{const} \quad ; \tag{23}$$

similarly from (17) and (20) we get

$$S_{,i} = \pm\, c\, V_{,i} \quad , \quad c = \text{const} \quad . \tag{24}$$

Substituting now (18) and (21) into (22), and considering (23) and (24) gives us

$$V_{,i} = (C^2+a^2e^{2\psi})^{-1/2}\ \psi_{,i} \tag{25}$$

$$C = (1 \mp c^2)^{1/2} \quad , \tag{26}$$

which yields on integration

$$e^{-\psi} = \cosh CV - (1+a^2/C^2)^{1/2} \sinh CV\ . \tag{27}$$

Then the electrostatic potential ϕ and the electrostatic field are, from (23), (27) and (6),

$$\phi = -(a/C)e^{\psi} \sinh CV, \tag{28}$$

$$F_{oi} = a\ e^{2\psi}\ V_{,i} \quad , \tag{29}$$

while the scalar field is, from (24),

$$S = \pm\ c\ V\ . \tag{30}$$

4. <u>CONCLUSIONS</u>

In the case of vanishing electrostatic and scalar fields ($a = c = 0$), the two line elements (13) and (15) coincide, since from (27) and (26) we get $C=1$ and $\psi=V$. This easiness in recovering the vacumm solution is an interesting feature of our solutions.

In absence of scalar fields one gets from (27) and (28) th Majumdar's relation

$$e^{2\psi} = 1-2(1+a^2)^{1/2}\ \phi/a + \phi^2 \quad , \qquad c = 0\ ; \tag{31}$$

for weakly charged systems (small values of a) we have from (28) the ratio ϕ/a finite.

We have imposed a functional relationship between the electrostatic potential ϕ and the gravitational potential ψ ; in the weak field approximation we have both $\psi \simeq 0$ and $V \simeq 0$, so from (28) $\phi \simeq - aV$. Except in some particular space-symmetric systems, the proportionallity of these potentials can only be achieved when the sources of gravity and electrostatics bear a ratio independent on position. Then the parameter a can be interpreted in this approximation as a constant ratio between the electrostatic charge density and the original mass density. Similarly the parameter c in (30) would be the constant ratio between a scalar source density and the original mass density.

It is known (Parker 1975) that the same energy momentum tensor $E_{\mu\nu}$ corresponds to two different electromagnetic fields $F_{\mu\nu}$ and $F'_{\mu\nu}$ when these two fields are related by a duality rotation,

$$F'_{\mu\nu} = F_{\mu\nu} \cos\theta + {}^{*}F_{\mu\nu} \sin\theta \quad , \tag{32}$$

where θ is an arbitrary real constant and ${}^{*}F_{\mu\nu}$ is the dual of $F_{\mu\nu}$,

$${}^{*}F^{\mu\nu} = (-g)^{-1/2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}/2 \quad ; \tag{33}$$

as a consequence, we can generalize our solution (27) to

$$e^{-\psi} = \cosh CV - \left[1+(a^2+b^2)/C^2\right]^{1/2} \sinh CV , \tag{34}$$

by including a magnetostatic field

$$F^{ij} = (b/a)(-g)^{-1/2} \varepsilon^{oijk} \phi_{,k} \quad , \tag{35}$$

where b is a constant related to the angle θ of the duality rotation by $\tan\theta = -b/a$; this field would be that produced by a magnetic mo nopole density bearing a constant ratio b to the original mass density, in the weak field approximation.

One sees from (26) and (27) a fundamental difference between the introduction of an attractive scalar field and that of a repulsive one; the difference manifests itself in the square root $(1 \mp c^2)^{1/2}$. While a repulsive scalar field (lower sign) can be introduced with arbitrary strength c, the strength of an attractive scalar field is bounded to values of $c^2 < 1$. We can try a classical picture to see the origins of this difference. We can obtain a repulsive scalar field solution of the equations by introducing in the sources of the original static vacuum solution some amount of source of repulsive scalar field; in order to restore the equilibrium of the system we can introduce some additional mass (attractive); for strong repulsive scalar fields, large amounts of mass should be added. The situation is not the same, however, when one wants an attractive scalar field; if one starts from a static system and introduces in it attractive scalar source, the staticity can be mantained by taking off some quantity of the original mass; since the mass of the original system is limited, one should expect a limit also on the amount of attractive scalar source which could be introduced without destroying the equilibrium.

One can have a superposition of a scalar field of the attractive class with one of the repulsive class; these are incoherent, and then the constant C in (27) is given by

$$C = (1-c_a^2 + c_r^2)^{1/2} \quad , \tag{36}$$

where c_a and c_r are the coefficients in (30) corresponding to the attractive and repulsive scalar fields, respectively.

One can still have a superposition of two or more scalar fields of the same class. That superposition can be either coherent or incoherent. A coherent superposition of two scalar fields of the same class occurs similarly to the superposition of two electrostatic potentials corresponding to two densities of electric charge, say; the net coefficient c in (30) of an attractive scalar field which is a coherent superposition of two attractive scalar fields of coefficients c_1 and c_2 is $c = c_1 + c_2$, and the constant C in (27) becomes

$$C = \left[1 - (c_1+c_2)^2\right]^{1/2} \quad ; \tag{37}$$

these c_1 and c_2 can be of the same or of opposite sign. An incoherent superposition of two scalar fields of the same class occurs similarly to the superposition of an electrostatic potential (repulsive) and a scalar magnetostatic potential (also repulsive) originated by magnetic monopoles; neither do the corresponding sources add up, nor the mathematical addition of the two scalar fields has then any physical sense; the constant C in (27) becomes in this case

$$C = \left[1 \mp (c_1^2 + c_2^2)\right]^{1/2} \quad . \tag{38}$$

APPENDIX

Consider the energy momentum tensor

$$T^{\mu\nu} = \rho u^{\mu} u^{\nu} + S^{\mu\nu} \quad , \tag{A.1}$$

where ρ is a mass density, $u^{\mu} = dx^{\mu}/ds$ is the corresponding four-velocity vector field, and $S^{\mu\nu}$ is the energy momentum tensor associated to a scalar field S according to

$$S_{\mu\nu} = \pm(S_{,\mu} S_{,\nu} - g_{\mu\nu} S^{;\alpha} S_{,\alpha}/2)/4\pi \; . \tag{A.2}$$

The conservation equation $T^{\mu\nu}{}_{;\nu} = 0$ gives

$$\rho w^{\mu} = \mp S^{;\mu} S^{;\alpha}{}_{;\alpha}/4\pi \quad , \tag{A.3}$$

where w^{μ} is the four-acceleration $u^{\mu}{}_{;\nu} u^{\nu}$.

In a locally Minkowskian coordinate system with signature -2 we get

$$\rho w^{i} = - (\mp S^{,\alpha}{}_{,\alpha}/4\pi) S_{,i} \quad ; \tag{A.4}$$

the right hand side is then the i^{th} component of the density of force (of non-gravitational nature) which is acting upon the dust; the expression of this force contains a scalar quantity times the gradient $S_{,i}$ of the scalar field. Following the usual definition of value of a source as the negative ratio of the force experienced by that source by the gradient of the (potential) field, we see that the density σ of scalar source is given by

$$\sigma = \mp S'^{,\alpha}_{,\alpha}/4\pi \quad , \tag{A.5}$$

or in static systems

$$\Delta S = \pm\, 4\pi\sigma \quad . \tag{A.6}$$

By analogy with the equation for the attractive Newton potential $\Delta V = 4\pi\rho$ and that for the repulsive Coulomb potential $\Delta\phi = -4\pi\lambda$ we are now able to relate the upper and lower sign in (A.6) with the attractive and repulsive scalar fields, respectively (see Section 2).

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CAPÍTULO IV

FONTE ESCALAR DE LONGO ALCANCE

Vários foram os trabalhos dedicados ao estudo da densidade de fonte produtora de campo escalar de longo alcance (densidade de massa escalar, ou de carga escalar, ou ainda de fonte escalar); dentre todos, destacou-se o de Das [44], que demonstrou que em certos casos particulares uma poeira escalarmente carregada só pode ficar em equilíbrio estático se a densidade de fonte escalar tiver em cada ponto valor absoluto igual à densidade de massa nesse ponto.

Foi-nos possível demonstrar [47], utilizando mais extensamente as equações do campo, que essa condição para o equilíbrio não se restringia a algumas das situações particulares consideradas por Das, sendo de ordem bastante mais geral.

A seguir, uma reprodução desse nosso trabalho.

STATIC SCALARLY CHARGED DUST

Idel Wolk, A.F.F. Teixeira and M.M. Som

Centro Brasileiro de Pesquisas Físicas, ZC-82 Rio de Janeiro, Brazil

ABSTRACT - In this note the necessary condition $g_{oo}=\exp|2S|$ between the metric coefficient g_{oo} and the massless scalar field S is proved when these fields are due to a static scalarly charged dust distribution

In an earlier work Das (1962) studied the static coupled gravitational and electrostatic as well as scalar fields due to incoherent dust charged in electric or scalar sense; he found some very general results for such systems. However in his derivation a functional relationship between the metric coefficient g_{oo} and the potential Φ is assumed, and an equality of matter and charge densities is imposed, this imposition being justified by Newtonian mechanics. Later De and Raychaudhuri (1968) found that a functional relationship between the three functions g_{oo}, Φ and σ/ρ (ratio of electric charge and mass densities) follows directly from the field equations; then the equality of matter and electric charge densities is found to be the only possibl solution in regions free of singularity.

Our purpose is to demonstrate that $g_{oo}=\exp|2S|$ follows directly from the field equations in the case S is the long range repulsive scalar field produced by static scalarly charged dust; and that the equality of densities of matter and scalar charge is also a consequence of the Einstein - scalar equations.

In the general static line element

$$ds^2 = \exp(2\eta)(dx^0)^2 + g_{ij}dx^i dx^j \qquad (1)$$

we consider $\eta(x^i)$ and $g_{ij}(x^k)$; and in Einstein equations we use

$$T_{\mu\nu} = \rho u_\mu u_\nu - (S_{,\mu}S_{,\nu} - g_{\mu\nu}\, g^{\alpha\beta}\, S_{,\alpha}S_{,\beta}/2)/4\pi \quad , \qquad (2)$$

with $\rho(x^i)$ the mass density, $u_\mu = \delta^0_\mu \exp\eta$ the four-velocity field of ρ, and $S(x^i)$ the long range repulsive scalar field satisfying

$$S^{;\mu}_{;\mu} = 4\pi\alpha \quad , \qquad (3)$$

where α is the source density of S. Comma and semicolon denote partial and covariant derivatives, respectively.

The contracted Bianchi identities give, using (3),

$$\rho\eta_{,i} + \alpha S_{,i} = 0 \quad . \qquad (4)$$

From this equation one finds that all Jacobians $\partial(\eta,S)/\partial(x^i,x^j) = 0$, then one obtains that $\eta(x^i)$ and $S(x^i)$ are functionally related; that is, there exists a relation $f(\eta,S) = 0$ independent of the point x^i and of the ratio α/ρ. We thus define $\Sigma = ds/d\eta$ and get from (4)

$$\rho = -\alpha\Sigma \quad . \qquad (5)$$

Now (3) and $R^0_0 + 8\pi(T^0_0 - T/2) = 0$ give respectively

$$(g^{1/2} S^{,i})_{,i} - 4\pi\, g^{1/2}\, \alpha = 0 \quad , \tag{6}$$

$$(g^{1/2} \eta^{,i})_{,i} + 4\pi\, g^{1/2}\, \rho = 0 \quad , \tag{7}$$

where $g = -\det g_{\mu\nu}$; the substitution of (5) into (7) and use of (6) give

$$\left[g^{1/2} (\Sigma^2 - 1)^{1/2} \eta^{,i}\right]_{,i} = 0 \quad . \tag{8}$$

An analysis similar to that of De and Raychaudhuri (1968) shows that this equation can only be satisfied, in regions free of singularities, when $\Sigma^2 = 1$, so from (5) and (4)

$$\alpha = \pm \rho \quad , \tag{9}$$

$$S = \mp \eta \quad . \tag{10}$$

These equalities are thus not only sufficient for the equilibrium, as usually considered, but also necessary.

ACKNOWLEDGEMENTS

One of the authors (M.M.S.) acknowledges financial support from Conselho Nacional de Pesquisas (Brazil).

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CAPÍTULO V

FONTE ESCALAR DE CURTO ALCANCE

Contrastando com a abundância de soluções exatas para sistemas dispondo de campo escalar de longo alcance, até hoje ainda não foi obtida uma única solução exata envolvendo campo escalar de curto alcance. Entre outros, trataram do assunto Duan'-I-Shi [48], Brahmachary [49], Stephenson [50], Penney [51], Das [52], Roy e Rao [53] e nós mesmos [54].

Neste último trabalho consideramos uma distribuição esférica estática de poeira, cujos constituintes são fontes de campo gravitacional e de um campo escalar repulsivo de curto alcance. Obtivemos uma solução aproximada (linearizada) das equações.

Um aspecto interessante da nossa distribuição é que o equilíbrio estático foi conseguido sem a utilização de qualquer conceito tido usualmente como macroscópico, como pressão ou tensões; ao invés, empregou um campo escalar considerado dos mais "elementares" da natureza. Conquanto a nossa distribuição não sirva, em sua forma atual, como um bom modelo de qualquer partícula elementar, acreditamos que ela contenha um germe que convém seja desenvolvido, esse germe consistindo no conceito de densidade de fonte escalar de alcance finito.

É nossa intenção em futuro próximo dedicar atenção a um modelo eletricamente carregado, e posteriormente tentar incluir um momento angular nesse modelo; as distribuições resultantes já conteriam então um número razoável dos elementos que caracterizam as chamadas partículas elementares, com a propriedade bastante desejável, segundo Einstein e Rosen [55] de não apresentarem singularidades.

Segue-se uma reprodução do nosso trabalho [54]

TENTATIVE RELATIVISTIC MODEL FOR NEUTRAL YUKAWA SYSTEMS

A.F.F. Teixeira, I. Wolk and M.M. Som
Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, ZC-82, Brazil

ABSTRACT

Some structurally important results in the small as well as in the large dimensions are presented considering a spherical distribution of incoherent dust, the constituents of which are simultaneously sources of gravitational as well as repulsive short range scalar fields.

1. INTRODUCTION

In investigating the structure of "elementary" particles, it is believed that the effects of gravitational interaction cannot be neglected [1]. In fifties considerable interest has been focussed on a set of equations in which a real scalar field of long range is coupled with the gravitational field [2,3,4,5]. Since the scalar fields which find applications in physics are usually of short range some other physical interpretation is intended for the scalar potential. In the case of a real scalar field with short range exact solutions are quite difficult to obtain. Duan'-I-Shi [6] first obtained a class of solutions to the equations corresponding to the fields of a point nuclear charge; however his solutions contain several functions whose forms are not explicitly known. Stephenson [7] presented an approximate static spherically symmetric solution of the Einstein-Maxwell-Yukawa field equations which, he claimed, represents the classical fields of a proton.

It seems worthwhile to investigate the effect of a real scalar field coupled with the gravitational field in the formation of elementary structures. In a recent work the present authors [8] have shown that if one considers the distribution of incoherent dust charged in scalar sense, in equilibrium under the influence of its own gravitational and long range repulsive scalar fields, then the only possible solutions are those given by Das [9]; however for such a distribution one has the matter density equal to the scalar charge density, which cannot be attributed to any known elementary particle.

In this paper we propose to consider a spherical distribution of incoherent dust in static equilibrium. The constituents of the dust are supposed to be the sources of gravitational as well as short range repulsive Yukawa-type field. In view of difficulties in obtaining an exact solution, we have studied the approximate solutions and obtained some interesting results which are structurally important in the small as well as in the large dimensions.

2. GENERAL EQUATIONS

The Einstein-scalar field equations are

$$R^{\mu}_{\nu} = -2\varepsilon_c \left(T^{\mu}_{\nu} - \delta^{\mu}_{\nu} T/2\right)/c^2 \quad , \qquad (1)$$

$$S^{;\mu}_{;\mu} + S/\ell^2 = \varepsilon\sigma \quad , \qquad (2)$$

$$T^{\mu}_{\nu} = \rho c^2 u^{\mu} u_{\nu} - c^2(2S'^{\mu}S_{,\nu} - \delta^{\mu}_{\nu} S'^{\alpha}S_{,\alpha} + \delta^{\mu}_{\nu} S^2/\ell^2)/2\varepsilon \quad , \qquad (3)$$

where

$$\varepsilon = 4\pi G/c^2 \quad , \qquad (4)$$

ρ is a matter density, S is a short range (ℓ) repulsive scalar field, σ is the density source of S, and u^{μ} is the velocity field of ρ and σ. Comma and semicolons mean ordinary and covariant derivatives, respectively.

For static spherically symmetric systems we consider the line element

$$ds^2 = e^{2\eta}dx^{o2} - e^{2\alpha}\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \quad , \qquad (5)$$

with all functions ρ, σ, S, η, α depending only on r. Since the static condition implies $u^{\mu} = \delta^{\mu}_{o}\, e^{-\eta}$, the equations (1) and (2) become

$$\left[\eta_{11} + \eta_1(\eta_1-\alpha_1) + 2\eta_1/r \right] e^{-2\alpha} = \varepsilon\rho + S^2/\ell^2 \quad , \qquad (6)$$

$$\left[\eta_{11} + \eta_1(\eta_1 - \alpha_1) - 2\alpha_1/r \right] e^{-2\alpha} = -\varepsilon\rho + S^2/\ell^2 + 2e^{-2\alpha}S_1^2 \quad , \qquad (7)$$

$$r^{-2} - \left[r^{-2} + (\eta_1-\alpha_1)/r \right] e^{-2\alpha} = \varepsilon\rho - S^2/\ell^2 \quad , \qquad (8)$$

$$r^{-2}\, e^{-\eta-\alpha}(r^2\, e^{\eta-\alpha}S_1)_1 - S/\ell^2 = -\varepsilon\sigma \quad , \qquad (9)$$

where a subscript 1 means d/dr. One obtains from the Bianchi identity $T^{\mu}_{\nu;\mu} = 0$ the relation

$$\rho\eta_1 + \sigma S_1 = 0. \qquad (10)$$

We are considering a static spherical distribution of incoherent dust of matter density $\rho(r)$. The gravitational collapse of this dust is supposed to be prevented by a stronger short range repulsion of a Yukawa-type source density $\sigma(r)$ associated to the dust. Since the four equations (6) to (9) contain five unknowns (η, α, ρ, σ, S), one additional equation such as

$$\sigma = f\rho \quad , \quad f = \text{const} \quad , \tag{11}$$

may be added.

In view of difficulty in obtaining exact solutions we make use of an approximate method; we expand our three potentials η, α, S and the two densities ρ, σ in integral powers of some small dimensionless constant κ, to be identified later. In the lowest approximation we have taken the densities independent of κ, and differently from Stephenson (7) we take the three potentials proportional to κ. Then (6) to (10) simplify to

$$\eta_{11} + 2\eta_1/r = \varepsilon\rho \quad , \tag{12}$$

$$\eta_{11} - 2\alpha_1/r = -\varepsilon\rho \ , \tag{13}$$

$$2\alpha/r - \eta_1 + \alpha_1 = \varepsilon\rho r, \tag{14}$$

$$r^{-2}(r^2 S_1)_1 - S/\ell^2 = -f\,\varepsilon\rho \ , \tag{15}$$

$$\rho(\eta_1 + fS_1) = 0. \tag{16}$$

3. INTERNAL SOLUTIONS

With $\rho \neq 0$ we get from (12), (15) and (16)

$$S_1 = \varepsilon(f^2-1)f^{-1}\ell^2\rho \quad , \tag{17}$$

where the subscript i means internal. Substituting (17) into (15) we obtain

$$r^{-2}(r^2\rho_1)_1 + \ell^{-2}(f^2-1)^{-1}\,\rho = 0. \tag{18}$$

A short reflection shows that we must have $|\sigma| > \rho$, since if this were not the case, the sphere would collapse. So with $f^2 > 1$ in (18) we choose the regular solution

$$\rho = A\nu^2\, r^{-1} \sin \nu r \quad , \qquad A = \text{const} \quad , \tag{19}$$

$$\nu = \ell^{-1}(f^2-1)^{-1/2} \quad . \tag{20}$$

Then from (17)

$$S_i = \varepsilon' (R/fr) \sin \nu r \quad , \tag{21}$$

where R is the radius of the distribution, and

$$\varepsilon' = \varepsilon A/R \tag{22}$$

is a dimensionless constant; and from (16) and (14), respectively,

$$\eta_i = -\,\varepsilon'(B + Rr^{-1} \sin \nu r) \quad , \quad B = \text{const} \quad , \tag{23}$$

$$\alpha_i = \varepsilon'\,(Rr^{-1} \sin \nu r - \nu R \cos \nu r) \quad . \tag{24}$$

The equation (13) is identically satisfied; the constant B will be fixed by boundary conditions.

4. EXTERNAL SOLUTIONS; BOUNDARY CONDITIONS

For $\rho = 0$ we easily obtain integrating (12) to (15)

$$\eta_e = -\,\alpha_e = -\,Cr^{-1} \quad , \qquad S_e = Dr^{-1}\, e^{-r/\ell} \quad , \tag{25}$$

where the subscript e means external, and C and D are constants of integrations. These potentials have the usual Schwarzschild and Yukawa behaviour at infinity.

We now impose that at the boundary of the sphere $(r=R)$ the three potentials η, α, S be continuous as well as the radial first derivatives of η and S.

The continuities of S and S_1 not only fix the constant D, but also prescribe to the radius R the discret set of values given by

$$\cot \nu R = - (f^2 - 1)^{1/2} \quad . \tag{26}$$

Since $\rho(r)$ in (19) should always be positive, we get from (26) that

$$\pi/2 < \nu R < \pi \quad . \tag{27}$$

Then with (26) and (27) we have

$$\sin \nu R = |f|^{-1} \quad , \tag{28}$$

so from (21) and (25)

$$S_i = \varepsilon' R f^{-1} r^{-1} \sin \nu r \quad , \quad S_e = \varepsilon' (f|f|r)^{-1} e^{-(r-R)/\ell} . \tag{29}$$

The continuities of η and η_1 at $r=R$ specify the constants B and C, and the continuity of α is identically satisfied giving

$$\eta_i = - \varepsilon' (R r^{-1} \sin \nu r - \nu R \cos \nu R) \quad , \tag{30}$$

$$\alpha_i = \varepsilon' (R r^{-1} \sin \nu r - \nu R \cos \nu r) \quad , \tag{31}$$

$$\eta_e = - \alpha_e = - \varepsilon' (1+R/\ell) R |f|^{-1} r^{-1} . \tag{32}$$

From equations (29) to (32) it is natural to identify the constant ε' with the necessarily small constant κ in terms of which we made our series expansion. This identification implies a restric tion on the parameters of the system. Remembering that in Schwarzschild-like systems the mass m is defined by

$$\eta_e = - Gm/c^2 r \tag{33}$$

we get from (32) that

$$\varepsilon' = |f| \, (1+R/\ell)^{-1}(Gm/c^2 R) \quad ; \tag{34}$$

one can verify from (20) and (28) that when $1 < f^2 < \infty$ we have that $\pi^{-1} < |f|(1+R/\ell)^{-1} < 1$, so from (34) our approximate solution is valid when

$$Gm/c^2 R << 1 \quad . \tag{35}$$

For practical purposes we define the "effective Yukawa charge" Y of our sphere by

$$c^2 S_e = G \, Y \, r^{-1} \, e^{-r/\ell} \; ; \tag{36}$$

then from (29) we obtain

$$Y = mf^{-1}(1 + R/\ell)^{-1} \, e^{R/\ell} \quad , \tag{37}$$

and we can verify from (20), (27), (28) and (37) that $m < Y/f$ for all values of $f^2 > 1$.

A case which deserves a special consideration is that when $f^2 >> 1$; then from (27) and (28) one has $\nu R \simeq \pi - |f|^{-1}$, so from (20) $R/\ell \simeq \pi|f| - 1$, that is, $R >> \ell$. The internal quantities become

$$\rho \simeq (m\pi/4R^3)(\gamma^{-1} \sin\gamma - \ell R^{-1} \cos\gamma) \quad , \tag{38}$$

$$\eta_i \simeq - (Gm/c^2 R)(1 + \gamma^{-1} \sin\gamma) \quad , \tag{39}$$

$$\alpha_i \simeq (Gm/c^2 R)(\gamma^{-1} \sin\gamma - \cos\gamma) \quad , \tag{40}$$

$$S_i \approx \pm \pi (Gm/c^2 R)(\ell/R)(\gamma^{-1} \sin\gamma + \ell/R) \quad , \tag{41}$$

$$\sigma \approx \pm \rho R/\pi\ell \quad , \tag{42}$$

where

$$\gamma = \pi r/R \tag{43}$$

is a radial variable; the external quantities become

$$\eta_e = - \alpha_e = - Gm/c^2 r \quad , \tag{44}$$

$$S_e = \pm \pi (Gm/c^2 r)(\ell/R)^2 \, e^{-(r-R)/\ell} \quad . \tag{45}$$

5. DISCUSSIONS

Our approximate solution represents a static spherically symmetric system whose only far reaching interaction is a gravitational one; we did not use any macroscopic concept like pressure to describe such structure. Since our solution does not show any singularity either in the source densities or in the potentials, it can be accepted as a naive classical model of an uncharged spinless elementary particle.

In our model the mass density $\rho(r)$ has a maximum finite value at the origin, and decreases monotonically to a finite value at the boundary. The condition (35) for the validity of our approximation is usually met both in the very small as well as in the very large physical systems.

A novel feature of our model is the prescription (26) for the radius of the system; all previous solutions [9, 10] based on electromagnetic and long range scalar fields allowed arbitrary values for the radius.

In our approximate solution we have made an expansion in integral powers of the small constant ε', and we have taken only the lowest order term of the fields and sources; as a consequence our model presented all functions ρ, σ, η, α, S linearly proportional to a same constant A or equivalently m. A non-linearity would appear only if higher order terms in ε' were considered.

Our final expressions for the source densities (11) and (19) and for the fields (29) to (32) are not suitable for obtaining the limit of long range scalar field $\ell \to \infty$; indeed it is known [9] that for these fields it must be $f^2 = 1$, and that the internal structure $\rho(r)$ remains undefined in the system (5) to (8).

It would be illustrative to evaluate the constants appearing in our model when applied to an elementary particle. Let us choose the neutral pion, with mass $m = 2.4 \times 10^{-25}$ g. and nuclear range $\ell = 1.5 \times 10^{-13}$cm. Assuming the radius $R = \ell$ we obtain from (26) $|f| = 1.12$, and we verify from (35) that $\varepsilon' \approx 10^{-42}$. The mass density $\rho(r)$ in (19) decreases from 2.8×10^{13}g/cm^3 at the origin to about half of this value on the boundary. For the "effective Yukawa charge" (37) we get $|Y| = 2.9 \times 10^{-25}$g.

The case where the radius of the spherical system is much larger than the range of the Yukawa field ($R \gg \ell$) is particularly interesting: the exterior scalar field S_e (45) is very small (ℓ^2/R^2) relative to the exterior gravitational field η_e (44) on the boundary R, and even smaller at larger distances. If we neglect this S_e, we can replace the two concepts of internal scalar field S_i and its source σ by the single concept of a pressure $p(r)$; since in our lowest order of approximation in ε'

$$p_1 + c^2 \rho n_1 = 0 \quad , \tag{46}$$

we get from (38) and (39) on integration

$$p = 2\varepsilon' c^2 R^3 \rho^2 \quad . \tag{47}$$

Similarly to the density $\rho(r)$ (38) this pressure has a finite value at the origin and decreases monotonically to zero (for $\ell/R \to 0$) on the boundary.

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