Brazilian Center for Research in Physics

Ph.D. Thesis

Worldline Superconformal mechanics

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Abstract

A general framework for superconformal mechanics is presented. Starting from d-module reps of 1-D SUSY, superconformal mechanics is developed in systematics steps. We first classify the d-module reps of the $\mathcal{N}$-extend superconformal algebras in one dimension and discuss the parabolic and hyperbolic/trigonometric d-module reps. This is followed by the explicit construction of their corresponding σ- models. The generality of this framework is further attested by the construction of topological models in worldline superconformal mechanics based on pseudo-SUSY. Finally, the framework is concluded with a discussion on its canonical quantization.

Keywords: Conformal Mechanics, Supersymmetry

Resumo

Neste trabalho apresentamos um formalismo geral para a mecânica superconforme em uma dimensão. Partindo das representações d-modulares da álgebra de supersimetria em 1D, a mecânica superconforme é desenvolvida em passos. Começamos classificando as representações d-modulares das álgebras superconformes $\mathcal{N}$-estendidas em 1D e discutindo as representações parabólicas, hiperbólicas e trigonométricas. Em seguida mostramos como os seus respectivos modelos-σ são construídos. A generalidade deste método é atestada ainda pela construção de modelos de mecânica superconforme topológica em 1D baseados em pseudo-supersimetria. Finalmente, concluímos com uma discussão sobre a quantização canônica dos modelos apresentados.

Palavras-chave: Mecânica Conforme, Supersimetria

Introduction

The origins of conformal mechanics can be traced to Calogero's late sixties and early seventies papers [1, 2, 3]. These papers dealt chiefly with n-body problems in one dimension with quadratic and inverse quadratic potentials. Although they may appear to be interesting exercises in quantum mechanics with unimportant theoretical consequences, an ever growing field of research in theoretical physics was born from those papers.

The reason conformal mechanics' importance in theoretical physics grew so quickly in the years after the publication of Calogero's inaugural papers is scale invariance. In their classical paper from 1976 [4], de Alfaro, Fubini and Furlan studied general properties of systems with conformal invariance in one dimension. Their goal was to shed light on quantum field theories near the zero mass regime. Such theories have no dimensional parameters and thus are scale invariant. In one dimension, their symmetry algebra is $sl(2)$.

Fast forward to a couple of decades later. Conformal field theories have been put at the edge of theoretical physics research ever since Maldacena conjectured the AdS/CFT correspondence to be a realization of 't Hooft's holographic principle [5, 6, 7]. Far from being restricted to the realms of string theory and high energy physics, it has seen interesting applications appearing in subjects as diverse as black hole physics [8] and condensed matter. The increasing interest in bulk-edge phenomena as seen in the quantum Hall effect [9, 10, 11] come to mind as an example. As geometrical and topological methods disseminate through every branch of physics, conformal field theories are assured to remain an active research topic.

The present text is intended to provide a general framework to superconformal field theories in one dimension. Although we follow the general lines established by de Alfaro, Fubini and Furlan's gold standard in 1D (super)conformal mechanics, our approach is quite different. It is a hands-on, get-it-done algebraic and algorithmic approach rather than a formal one. The advantages of such an approach become clear in the kinder learning curve required to produce and generalize many of the well known results of 1D conformal mechanics, such as Calogero's and de Alfaro, Fubini and Furlan's. Those with computer-oriented, algorithmic minds will also quickly realize that the methods here presented are a perfect fit to computer-algebraic methods. And indeed algebraic packages were extensively used in the construction of the d-module reps and σ-models here presented.

The first chapter is an overture of the methods and techniques used throughout the rest of the text. It serves as a smooth, quick and self-contained introduction to Clifford and SUSY algebras in one dimension, as well as the d-module representations of the latter. The same techniques are then applied in the last half of the chapter to build a similar machinery for 1D superconformal algebras, both in the parabolic and hyperbolic/trigonometric d-module representations.

The second chapter uses the techniques of the first chapter to construct d-module representations of the superWitt algebra. In this chapter supersymmetric extensions of the results derived in [12] are presented, together with some of their superconformally invariant σ-models. An important point to note is the introduction of the scaling dimension parameter, which had been neglected in the aforementioned paper and allows for the construction of inequivalent, albeit possessing the same simmetries, σ-models. As becomes clear in this chapter, it is related to many interesting aspects of superconformal theories such as criticality of the symmetry algebra. It consists of the results published in the paper *Four types of superconformal mechanics*, by N.L. Holanda and F. Toppan [13].

The third chapter constructs topological conformal field theories in one dimension from pseudo-SUSY. As in the second chapter d-module representations are presented together with topological σ-models. This chapter draws on the paper *A world-line framework for 1D topological conformal σ-models*, by L. Baulieu, N.L. Holanda and F. Toppan [14].

Finally, the fourth chapter completes our framework with the canonical quantization of some of the theories presented. The purpose of this chapter is to show how the worldline theories constructed in the previous chapters can be sistematicaly quantized a la Dirac. New results including the relation between the vacuum energy of de-Alfaro-Fubini-Furlan oscillators and the $D(2,1;\alpha)$ superalgebras are presented. It is, therefore, the most physics-related chapter, albeit the algebraic and algorithmic techniques are still there. The contents of this chapter are in the paper *From worldline to quantum superconformal mechanics with/without oscillatorial terms: $D(2,1;$Îś$)$ and $sl(2|1)$ models*, by I.E. Cunha, N.L. Holanda and F. Toppan [15].

One last point to note is that this work intends to be pedagogical and expository, instead of encyclopedic. Anyone with an undergraduate education in physics should be able to follow most of the topics here presented. Indeed, this should be the hallmark of these techniques: by focusing on the algebraic aspects, important physical results are put on the main stage and derived smoothly, leaving formal complications involving geometry and analysis to the specialists in these fields.

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Chapter 1

Worldline superconformal mechanics

1.1 Introduction

This chapter is intended to be an introduction to the techniques used throughout this work. It also serves the purpose of establishing notations and conventions. Although not strictly necessary for a full comprehension of the following chapters, it will surely be helpful for those unfamiliar with the topics here presented.

The chapter starts with a discussion on 1D SUSY. In sections **2** and **3** we introduce the SUSY algebra in one dimension and Clifford algebras. The latter are the backbone of d-module representations of extended SUSY algebras, as shown in section **4**. Section **5** builds on the root representations constructed in section **4** to generate extra, inequivalent d-module representations of the 1D SUSY algebra using the dressing operator. Our discussion on 1D SUSY ends in section **6** with a brief comment on SUSY-invariant σ-models. A good and quick reference to 1D SUSY is [1].

Superconformal algebras are the topic of sections **7-9**. D-module representations of the conformal algebra in one dimension are introduced in section **7** with an emphasis on the parabolic and hyperbolic types. In section **8**, the methods used in the manufacturing of d-module representations of the SUSY algebra are employed to furnish the conformal algebra with fermionic generators, thus yielding parabolic and hyperbolic d-module representations of the superconformal algebra. We end the chapter with a quick discussion of superconformal σ-models in section **9**. For a general overview of (super)conformal mechanics, see [2][3][4][5].

1.2 The Supersymmetry algebra in one dimension

The $N = (p, q)$-extended pseudo-supersymmetry algebra is defined by

$$\{Q_i, Q_j\} = 2\eta_{ij} H, \tag{1.1}$$

where the $Q_i's$ $(i, j = 1, ..., p + q)$ are the fermionic generators of the algebra, H is the bosonic generator and η is a diagonal signature matrix with the first p entries equal to $+1$ and the last q entries equal to -1. We use here the terminology "pseudoSUSY" to denote algebras that have both $p, q \neq 0$. In case either p or q are zero, we shall refer to the corresponding algebra simply as "SUSY", and write "$N = p$ SUSY algebra" as an equivalent to "$N = (p, 0)$ SUSY algebra".

In physics applications, the bosonic generator H is commonly identified with the generator of time translations (the Hamiltonian) of the system. As we shall see when dealing with D-module representations of the superconformal algebras in one dimension, this needs not be true in general. Nevertheless, it will be useful for us in this chapter to consider H as the generator of time translations. With this caveat in mind, equation (1.1) may be written in a way that explicitly says what H is,

$$\{Q_i, Q_j\} = 2\eta_{ij}\mathbb{I}\partial_t. \tag{1.2}$$

Equation (1.2) entices us to search for the explicit Q_i's that realize it. It is clear from it that they, too, must be differential operators. Furthermore, they cannot be written as simple complex number multiples of ∂_t. Let us illustrate this point with the simplest case, that of $N = 1$ SUSY algebra. Take the operators Q and H to be

$$Q = \begin{pmatrix} 0 & 1 \\ \partial_t & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} \partial_t & 0 \\ 0 & \partial_t \end{pmatrix}. \tag{1.3}$$

This clearly is a representation of the algebra (1.2), with $\mathbb{I}$ taken as the 2x2 identity matrix. Such representations consisting of differential operators in the entries of matrices are called D-module representations, or D-module reps for short. A task thus presents itself to us: to construct in a systematic way all the irreducible D-module reps of the algebra (1.2). This task is accomplished via Clifford algebras, the basics of which we introduce next.

1.3 Clifford algebras

The defining relations of Clifford algebras are much like those in equation (1.1):

$$\{\gamma_i, \gamma_j\} = 2\eta_{ij}\mathbb{I}. \tag{1.4}$$

Just as before, if the diagonal signature matrix η has its first p entries equal to $+1$ and the remaining q entries equal to -1, we refer to the corresponding Clifford algebra as $\mathcal{C}\ell(p,q)$. A trivial example of an irreducible representation (abbreviated to "irrep" when convenience demands) of a Clifford algebra is obtained by setting

$$\gamma = 1, \qquad \eta = 1. \tag{1.5}$$

This is evidently $\mathcal{C}\ell(1,0)$, and it turns out to be pretty useful in spite of its triviality. The reason is that given an inital $D = p+q$ spacetime dimensional signature η and an irreducible representation of its $\mathcal{C}\ell(p,q)$ Clifford algebra, two algorithms exist that allow us to obtain the irreducible representations of the $\mathcal{C}\ell(p+1,q+1)$ and $\mathcal{C}\ell(q+2,p)$ Clifford algebras in $D+2$ spacetime dimensions. These algorithms are described below.

Algorithm i): $\mathcal{C}\ell(p,q) \to \mathcal{C}\ell(p+1,q+1)$.

Define the Clifford matrices Γ of $\mathcal{C}\ell(p+1,q+1)$ as

$$\Gamma_j = \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & \mathbb{I}_d \\ -\mathbb{I}_d & 0 \end{pmatrix}, \qquad \begin{pmatrix} \mathbb{I}_d & 0 \\ 0 & -\mathbb{I}_d \end{pmatrix}. \tag{1.6}$$

Algorithm ii): $\mathcal{C}\ell(p,q) \to \mathcal{C}\ell(q+2,p)$.

$$\Gamma_j = \begin{pmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & \mathbb{I}_d \\ \mathbb{I}_d & 0 \end{pmatrix}, \qquad \begin{pmatrix} \mathbb{I}_d & 0 \\ 0 & -\mathbb{I}_d \end{pmatrix}. \tag{1.7}$$

As an example, successive applications of either algorithms i) or ii) to the irrep of $\mathcal{C}\ell(1,0)$ given in (1.5) generate irreps of the so called real series of Clifford algebras,

$$\mathcal{C}\ell(1,0) \to \mathcal{C}\ell(2,1) \to \mathcal{C}\ell(3,2) \to \mathcal{C}\ell(5,4) \to ... \to \mathcal{C}\ell(p,p-1) \to ... \,. \tag{1.8}$$

It should be noted that all algebras in the above series are maximal: a tentative inclusion of another generator to enlarge the Clifford algebra would require doubling the size of the matrices in the representation. The first algebra in a series, which is $\mathcal{C}\ell(1,0)$ here, is the primitive maximal Clifford algebra of that series. To generate irreps for a series of maximal Clifford algebras, one thus only needs to know the irrep of the primitive maximal Clifford algebra and apply algorithms i) or ii) repeatedly. Representations of non-maximal Clifford algebras are obtained by simply discarding some generators of the corresponding maximal algebra.

Let us write down an explicit example to illustrate these concepts. Applying algorithm i) to the irrep of $\mathcal{C}\ell(1,0)$ given in (1.5) we get an irrep of $\mathcal{C}\ell(2,1)$,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.9}$$

Clearly, it is impossible to enlarge the Clifford algebra $\mathcal{C}\ell(2,1)$ without doubling the size of the matrices. Thus, $\mathcal{C}\ell(2,1)$ is a maximal Clifford algebra. Now let us exclude from the algebra the elements τ_1 and τ_3. We end up with only τ_2, which is a generator of the non-maximal Clifford algebra $\mathcal{C}\ell(0,1)$.

A point worth emphasizing is that, as long as the irrep of the primitive maximal Clifford algebra is real (meaning that all the entries of the matrices representing the generators are real), algorithms i) and ii) ensure that all the irreps of the maximal Clifford algebras in the series will be real too. This is very convenient for classification purposes. It should be noted, however, that nothing prohibits the construction of complex representations of Clifford algebras. For instance, the Clifford algebra $\mathcal{C}\ell(0,1)$ could have been constructed as

Table 1.1: Real irreps of Clifford algebras

1	*	2	*	4	*	8	*	16	*	32	*	64	*	128	*	256	...
(1,0)	⇒	(2,1)	⇒	(3,2)	⇒	(4,3)	⇒	(5,4)	⇒	(6,5)	⇒	(7,6)	⇒	(8,7)	⇒	(9,8)	...
						(1,4)	→	(2,5)	→	(3,6)	→	(4,7)	→	(5,8)	→	(6,9)	...
					↗												
				(0,3)													
					↘												
						(5,0)	→	(6,1)	→	(7,2)	→	(8,3)	→	(9,4)	→	(10,5)	...
								(1,8)	→	(2,9)	→	(3,10)	→	(4,11)	→	(5,12)	...
							↗										
						(0,7)											
							↘										
								(9,0)	→	(10,1)	→	(11,2)	→	(12,3)	→	(13,4)	...
									⋮								

$$\gamma = i, \qquad \eta = -1, \tag{1.10}$$

and this algebra would be maximal. Since the remaining of this work uses only the real representations of Clifford algebras, it will suffice for us to present a table with a sketch of the classification of the real irreps. The table is organized so that the first line specifies the size of the matrices entering each representation. The remaining lines are the real, quaternionic and octonionic series of Clifford algebras. The table continues indefinetely, as indicated by the ellipses.

With the basic tools of Clifford algebras at our disposal, we can now proceed to the construction of D-module representations of SUSY algebras in one dimension.

1.4 D-module representations of SUSY algebras

With the benefit of hindsight, it is now easy to generalize the construction of the D-module rep of the $N = 1$ SUSY algebra given in equation (1.3). We started from $\mathcal{C}\ell(1,0)$ and applied algorithm i) or ii) in a slightly modified form convenient for SUSY, which we present below.

SUSY algorithm i): $\mathcal{C}\ell(p,q) \to N = (p, q+1)$ SUSY.

$$Q_j = \begin{pmatrix} 0 & \gamma_i \\ \gamma_i \partial_t & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & \mathbb{I}_d \\ -\mathbb{I}_d \partial_t & 0 \end{pmatrix} \qquad \text{and} \qquad H = \begin{pmatrix} \mathbb{I}_d \partial_t & 0 \\ 0 & \mathbb{I}_d \partial_t \end{pmatrix}. \tag{1.11}$$

SUSY algorithm ii): $\mathcal{C}\ell(p,q) \to N = (q+1, p)$ SUSY.

$$Q_j = \begin{pmatrix} 0 & \gamma_i \\ -\gamma_i \partial_t & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & \mathbb{I}_d \\ \mathbb{I}_d \partial_t & 0 \end{pmatrix} \qquad \text{and} \qquad H = \begin{pmatrix} \mathbb{I}_d \partial_t & 0 \\ 0 & \mathbb{I}_d \partial_t \end{pmatrix}. \tag{1.12}$$

A first thing to notice in both algorithms above is that the diagonal generators appearing in equations (1.6) and (1.7) have been eliminated. This is the result of an implicit choice that we have made for the representation of the super vector space uppon which these generators will act. This is a good moment to clarify this choice.

A superalgebra is an algebra with a $\mathbb{Z}_2$-grading, meaning that it is a direct sum of two subspaces,

$$A = A_0 \oplus A_1. \tag{1.13}$$

The subspace A_0 is the even or bosonic subspace, while A_1 is the odd or fermionic subspace. Under the binary composition rule $(,) : A \times A \to A$ of the algebra, these subspaces respect the $\mathbb{Z}_2$ grading, meaning that

$$(A_0, A_0) = A_0, \qquad (A_0, A_1) = (A_1, A_0) = A_1 \qquad \text{and} \qquad (A_1, A_1) = A_0. \tag{1.14}$$

Just the same, the super vector space has a $\mathbb{Z}_2$-grading too, written as

$$V = V_0 \oplus V_1. \tag{1.15}$$

Again, the grading of the superalgebra and that of the super vector space respect the $\mathbb{Z}_2$-grading,

$$A_0 V_0 = V_0, \qquad A_0 V_1 = A_1 V_0 = V_1 \qquad \text{and} \qquad A_1 V_1 = V_0. \tag{1.16}$$

Given that the Qs are fermionic generators, they take the bosonic subspace to the fermionic subspace and vice versa. The bosonic generator H on the other hand takes the bosonic subspace to itself, and does the same to the fermionic subspace. The anti-diagonal form of the $Q's$ and the diagonal form of H in equations (1.11) and (1.12) imply that we must represent the supervectors as column vectors with the first d entries representing the bosonic subspace and the last d entries representing the fermionic one. As an example, we show the first Q_j in equation (1.11) acting on the super vector space:

$$\begin{pmatrix} 0 & \gamma_i \\ \gamma_i \partial_t & 0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} \gamma_i V_1 \\ \gamma_i \partial_t V_0 \end{pmatrix}. \tag{1.17}$$

These conventions shall become more clear after working out some examples.

Example 1.1: the $N = 1$ $(1, 1, 0)$ root supermultiplet.
This is just the SUSY rep given in equation (1.3),

$$Q = \begin{pmatrix} 0 & 1 \\ \partial_t & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} \partial_t & 0 \\ 0 & \partial_t \end{pmatrix}.$$

These operators act on the supermultiplet $(x; \psi; 0)$ giving the following linear SUSY transformations:

$$\begin{aligned} Qx &= \psi, \qquad & Q\psi &= \dot{x}, \\ Hx &= \dot{x}, \qquad & H\psi &= \dot{\psi}. \end{aligned} \tag{1.18}$$

Note that we have used the field content of the supermultiplet to label the representation. Thus, $N = 1$ means that we have one generator of SUSY (which is Q here) and $(1, 1, 0)$ means that the generators in this representation act on a supermultiplet consisting of one propagating boson (x), one fermionic field (ψ) and zero auxiliary bosonic fields.

Example 1.2: the $N = 2$ $(2, 2, 0)$ root supermultiplet.
We can construct this rep from application of the SUSY algorithm ii) to the Clifford algebra $\mathcal{C}\ell(2, 1)$ given in equation (1.9). Indeed, define the SUSY generators

$$Q_0 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 \partial_t & 0 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 0 & \tau_2 \\ -\tau_2 \partial_t & 0 \end{pmatrix} \qquad \text{and} \qquad H = \begin{pmatrix} \mathbb{I}_2 \partial_t & 0 \\ 0 & \mathbb{I}_2 \partial_t \end{pmatrix}. \tag{1.19}$$

The action of these generators on the supermultiplet $(x_0, x_1; \psi_0, \psi_1; 0)$ is written below.

$$\begin{aligned} Q_0 x_\alpha &= \psi_\alpha, & Q_0 \psi_\alpha &= \dot{x}_\alpha, \\ Q_1 x_\alpha &= \delta_{0\alpha} \psi_1 - \epsilon_{0\alpha} \psi_0, & Q_1 \psi_\alpha &= -\delta_{0\alpha} \dot{x}_1 + \epsilon_{0\alpha} \dot{x}_0, \\ H x_\alpha &= \dot{x}_\alpha, & H \psi_\alpha &= \dot{\psi}_\alpha. \end{aligned} \tag{1.20}$$

Here $\alpha = 0, 1$ and the antisymmetric symbol $\epsilon_{\alpha\beta}$ is defined with $\epsilon_{01} = 1$.

Example 1.3: the $N = 4$ $(4, 4, 0)$ root supermultiplet.
This rep is constructed from $\mathcal{C}\ell(0, 3)$, the primitive maximal Clifford algebra of the quaternionic series, which comprises the Clifford algebras $\mathcal{C}\ell(0, 3 + 8n)$. The algebra $\mathcal{C}\ell(0, 3)$ has the following generators:

$$q_1 = \tau_3 \otimes \tau_2, \qquad q_2 = \tau_2 \otimes \mathbb{I}_2, \qquad q_3 = \tau_1 \otimes \tau_2. \tag{1.21}$$

The matrices q_i above together with the identity $\mathbb{I}_4$ form a matrix representation of the quaternionic algebra. In this representation, a general quaternion is written

$$q = x_0\mathbb{I}_4 + x_i q_i \tag{1.22}$$

and the quaternionic product is obtained via matrix multiplication,

$$q_i q_j = \epsilon_{ijk} q_k, \tag{1.23}$$

where summation is understood over repeated indices and ϵ_{ijk} is the Levi-Civita symbol in three dimensions. It is therefore clear why this series of maximal Clifford algebras is called quaternionic. It also becomes apparent that the SUSY generators of the $N = 4$ $(4, 4, 0)$ rep should bear the quaternionic structure inside them somehow. This is indeed the case. Applying the SUSY algorithm ii), the generators we get are

$$Q_0 = \begin{pmatrix} 0 & \mathbb{I}_4 \\ \mathbb{I}_4\partial_t & 0 \end{pmatrix}, \qquad Q_i = \begin{pmatrix} 0 & q_i \\ -q_i\partial_t & 0 \end{pmatrix} \qquad \text{and} \qquad H = \begin{pmatrix} \mathbb{I}_4\partial_t & 0 \\ 0 & \mathbb{I}_4\partial_t \end{pmatrix}. \tag{1.24}$$

These generators act on the supermultiplet $(x_0, x_i; \psi_0, \psi_i; 0)$, $i = 1, 2, 3$, as shown below.

$$\begin{aligned} Q_0 x_\alpha &= \psi_\alpha, & Q_0\psi_\alpha &= \dot{x}_\alpha, \\ Q_i x_\alpha &= \delta_{0\alpha}\psi_i - \delta_{i\alpha}\psi_0 - \delta_{j\alpha}\epsilon_{ijk}\psi_k, & Q_i\psi_\alpha &= -\delta_{0\alpha}\dot{x}_i + \delta_{i\alpha}\dot{x}_0 + \delta_{j\alpha}\epsilon_{ijk}\dot{x}_k, \\ H x_\alpha &= \dot{x}_\alpha, & H\psi_\alpha &= \dot{\psi}_\alpha. \end{aligned} \tag{1.25}$$

Here $\alpha = 0, 1, 2, 3$ and we used the summation convention again. This way of doing computations will be standard for us: whenever an index takes values starting at zero, we shall use greek letters. If an index only takes values greater than zero, we use italic letters. The summation convention is understood throughout, unless stated otherwise.

The method we have used so far to construct D-module reps of the SUSY algebra, albeit very effective, does not exhaust all possibilities. As should be clear by now, we can only generate "root" representations this way: no auxiliary fields exist in the supermultiplets. We now present a tool that complements the techniques we have developed, allowing for the construction of new, inequivalent irreps from the root irreps. This tool is the dressing operator.

1.5 The dressing operator

Consider again the $N = 1$ $(1, 1, 0)$ D-module rep of the SUSY algebra in example (1.1). Given that the only requirement imposed on a representation of SUSY is that it satisfies the same relations as the SUSY algebra, one might wonder about writing another representation with a simple trick:

$$Q = \begin{pmatrix} 0 & \partial_t \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} \partial_t & 0 \\ 0 & \partial_t \end{pmatrix}. \tag{1.26}$$

At first sight, this may look like cheating. However if we keep in mind that we have fixed the conventions on our supervector space, with bosons at the top and fermions at the bottom, we will realize that equations (1.26) are indeed a different representation. The supervector space now includes one auxiliary bosonic field b and so is written $(b; \psi)$.

The key property of auxiliary fields in linear representations of SUSY is that they are taken by the SUSY generators to time derivatives of the fermionic fields. In one dimension, the time derivative is a total divergence. This feature of one-dimensional SUSY can be used to show that all finite linear irreps of the SUSY algebra in one dimension descend from a root representation (see [6]). This means that, knowing how to construct the root irreps as we do, one should be able to generate all others. All that is needed is to generalize the trick used to obtain the $N = 1$ $(0, 1, 1)$ irrep. This generalization is the dressing operation.

An intuitive way to look at the dressing operation is through permutation matrices. In fact, the dressing of $N = 1$ $(1, 1, 0)$ may be written explicitly as

$$\begin{pmatrix} 0 & \partial_t \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \partial_t & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which is the same as

$$Q_{(0,1,1)} = P_{12}^{(2)^{-1}} Q_{(1,1,0)} P_{12}^{(2)} \tag{1.27}$$

where $P_{12}^{(2)}$ is the 2x2 permutation matrix that permutes the first line with the second one. The dressing is then simply a conjugation with permutation matrices.

Another approach to the dressing operation in one dimension is obtained by observing that the auxiliary fields have the same dimensions as the time derivatives of propagating bosons. Thus, the equation below is dimensionaly correct:

$$\begin{pmatrix} b \\ \psi \end{pmatrix} = \begin{pmatrix} \partial_t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ \psi \end{pmatrix} \tag{1.28}$$

Formally, the equation above tells us that the supermultiplet $(b;\psi)$ is obtained from the root supermultiplet $(x;\psi)$ by applying the dressing operator

$$S = \begin{pmatrix} \partial_t & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.29}$$

An analogy with linear algebra can be readily made. Let M represent the supermultiplet with a particular field content. For example, for $\mathcal{N}=1$ $(1,1,0)$, $M=(x;\psi)$. Equation (1.28) can be compactly written as

$$M_{(0,1,1)} = SM_{(1,1,0)}. \tag{1.30}$$

Looking at this last equation as a change of basis, we are led to construct the SUSY generator $Q_{(0,1,1)}$ from $Q_{(1,1,0)}$ as

$$Q_{(0,1,1)} = SQ_{(1,1,0)}S^{-1} \tag{1.31}$$

where S is given by equation (1.29). Some remarks should be made about this construct. The first thing to note is that the dressing operation as defined in equation (1.31) is a formal operation in the sense that we take

$$S^{-1} = \begin{pmatrix} \frac{1}{\partial_t} & 0 \\ 0 & 1 \end{pmatrix}.$$

The terms $\frac{1}{\partial_t}$, which are just formal devices useful for algebraic manipulations, will be canceled out by corresponding ∂_t terms in the numerator whenever a valid dressing operation is performed. Therefore, a lawful dressed representation ends up having no $\frac{1}{\partial_t}$ terms. Any dressing operation that results in operators with these illegal terms should be viewed as not producing irreps of the SUSY algebra. A second point to note is that this procedure is fairly general. We have used the simplest $\mathcal{N}=1$ case here as an example, but an analogous procedure will work for higher SUSY algebras too. As an example, to dress the $\mathcal{N}=2$ $(2,2,0)$ supermultiplet to the $\mathcal{N}=2$ $(0,2,2)$, the dressing operator is

$$S = \begin{pmatrix} \mathbb{I}_2\partial_t & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}.$$

Note that in this case the two propagating bosons are dressed to auxiliary fields and therefore the dressing operator has two time derivatives in its bosonic subspace.

Example 1.4: The $\mathcal{N}=4$ $(1,4,3)$ irrep.

Let us write explicitly the SUSY transformations for the $\mathcal{N}=4$ $(1,4,3)$ irrep. Starting from the root generators of the $\mathcal{N}=4$ $(4,4,0)$ irrep, we write the dressing operator as

$$S = diag(1,\partial_t,\partial_t,\partial_t,1,1,1,1).$$

We then use the relation $Q_{\alpha,(1,4,3)} = SQ_{\alpha,(4,4,0)}S^{-}1$ where $\alpha=0,...,3$ labels the SUSY generators. The generators act on the supermultiplet $(x_0,b_i;\psi_\alpha)$. We find the following SUSY transformations.

$$\begin{aligned}
Q_0x_0 &= \psi_0, & Q_0\psi_0 &= \dot{x}_0,\\
Q_0b_i &= \dot{\psi}_i, & Q_0\psi_i &= b_i,\\
Q_ix_0 &= \psi_i, & Q_i\psi_0 &= -b_i,\\
Q_ib_j &= -\delta_{ij}\dot{\psi}_0 - \epsilon_{ijk}\dot{\psi}_k, & Q_i\psi_j &= \delta_{ij}\dot{x}_0 + \epsilon_{ijk}b_k,\\
Hx_0 &= \dot{x}_0, & H\psi_0 &= \dot{\psi}_0,\\
Hb_i &= \dot{b}_i, & H\psi_i &= \dot{\psi}_i.
\end{aligned} \tag{1.32}$$

1.6 SUSY-invariant σ-models

Having outlined the construction of irreps of $\mathcal{N}$-extended SUSY algebras, we now move on the their σ-models. As is well known, an invariant action under the SUSY algebra can be constructed from a Lagrangian $\mathcal{L}$,

$$\mathcal{S} = \int \mathcal{L}dt, \tag{1.33}$$

provided that the action variation under the SUSY generators vanishes. This is accomplished by a Lagrangian that yields a total derivative term under variations of the SUSY generators. We can use the fact that the SUSY generators are square roots of the Hamiltonian to construct manifestly invariant actions. Let us use the $\mathcal{N}=1$ $(1,1,0)$ case again to illustrate this procedure. We apply the SUSY generator Q to the quantity $F(x)\dot{\psi}$. We get

$$Q(F\dot{\psi})=(QF)\dot{\psi}+F\frac{d}{dt}(Q\psi)=F_x(Qx)\dot{\psi}+F\ddot{x}=F_x\psi\dot{\psi}-F_x\dot{x}^2$$

where the last term results from integration by parts (remember that total derivative terms can be neglected). Since the prepotential function $F(x)$ is arbitrary, let us substitute $-F_x$ by another arbitrary function $A(x)$. We thus have constructed a manifestly invariant Lagrangian for the $\mathcal{N}=1$ $(1,1,0)$ SUSY irrep:

$$\mathcal{L}=A(\dot{x}^2+\dot{\psi}\psi), \tag{1.34}$$

with $A=A(x)$ arbitrary. We note that the lagrangian written above is SUSY-invariant by construction, since acting again on it with Q gives

$$Q\mathcal{L}=-Q^2(F(x)\dot{\psi}))=-\frac{d}{dt}(F(x)\dot{\psi})=\frac{d}{dt}(A\dot{x}\psi)$$

which is a total derivative. The last equation is again obtained by integration by parts. As the previous computation should have made clear, the operators of the SUSY algebra act like a Leibniz graded derivation,

$$D(ab)=(Da)b+(-1)^{deg(a)deg(D)}(Db)$$

where deg denotes the grade of a field or operator. The equation above reduces the usual Leibniz derivation rule when either the operator D of the field a are bosonic. When both of them are fermionic (like Q and ψ in our case), the sign change must be observed to produce the correct results.

Although the function $A(x)$ appearing in equation (1.34) is arbitrary for global invariance, we can use dimensional analysis to have an ideia of how it should actually look like. Let us assume that the mass dimension of ∂_t is unity ($[\partial_t]=1$) and let $\lambda=[x]$. Since $Q^2=\partial_t$ and $\psi=Qx$, we find that $[\psi]=\lambda+\frac{1}{2}$. From the definition of the action (1.33) we also have that $[\mathcal{S}]=[\mathcal{L}]+[dt]=[\mathcal{L}]-1$. A dimensionless action is thus recovered for $[\mathcal{L}]=1$. Computing the dimension of $A(x)$ from equation (1.34) we find

$$[A]=[\mathcal{L}]-2(\lambda+1)=-(2\lambda+1).$$

This last equation suggests that $A(x)$ should be given up to a dimensionless constant factor by

$$A(x)=x^{-\frac{2\lambda+1}{\lambda}}. \tag{1.35}$$

Since conformal field theories have scale invariance their σ-models should have dimensionless actions. We thus expect that lagrangian (1.34) with $A(x)$ given by (1.35) should be the $N=1$ $(1,1,0)$ superconformal lagrangian. Later we show this to be true indeed.

Example 1.5: The σ-model for the $\mathcal{N}=2$ $(2,2,0)$ irrep.

Using the SUSY transformations written in example 1.2, the SUSY-invariant lagrangian will be given by $\mathcal{L}=Q_1Q_0(A\psi_0\psi_1)$. Expanding this expression we find

$$\mathcal{L}=A(\dot{x}_\alpha\dot{x}_\alpha+\dot{\psi}_\alpha\psi_\alpha)+A_\beta\dot{x}_\alpha\psi_\beta\psi_\alpha. \tag{1.36}$$

Clearly, applying either Q_0 or Q_1 to this lagrangian will give a total time derivative, since these operators anti-commute. Also worthwhile to mention is that again we have $[A]=-(1+2\lambda)$. In this case a choice for $A(x_0,x_1)$ that is symmetric in the propagating bosons and has the correct dimension is

$$A(x_0,x_1)=r^{-\frac{2\lambda+1}{\lambda}},\qquad r=(x_0^2+x_1^2)^{\frac{1}{2}}. \tag{1.37}$$

Example 1.6: The σ-model for the $\mathcal{N}=4$ $(4,4,0)$ irrep.

For the SUSY transformations given in example 1.3, we write the lagrangian as $\mathcal{L}=Q_0Q_1Q_2Q_3F$, where $F=F(x_0,x_1,x_2,x_3)$ is an arbitrary function. We find

$$\begin{aligned}\mathcal{L}=&A(\dot{x}_\alpha\dot{x}_\alpha+\dot{\psi}_\alpha\psi_\alpha)+A_\beta\dot{x}_\alpha\psi_\beta\psi_\alpha-\frac{1}{2}\epsilon_{ijk}A_i\dot{x}_0\psi_j\psi_k\\&+\epsilon_{ijk}A_i\dot{x}_j\psi_0\psi_k+\frac{1}{2}\epsilon_{ijk}A_0\dot{x}_i\psi_j\psi_k+\frac{1}{6}\epsilon_{ijk}\Box A\psi_i\psi_j\psi_k,\end{aligned} \tag{1.38}$$

where $A=\Box F$ and $\Box=\sum\frac{\partial^2}{\partial x_\alpha^2}$ is the d'Alembertian. Note that once again a dimensionless action is obtained for

$$A(x_0,x_1,x_2,x_3)=r^{-\frac{2\lambda+1}{\lambda}},\qquad r=(x_0^2+x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}. \tag{1.39}$$

1.7 Conformal algebra

A conformal transformation between two manifolds is a transformation that preserves the angle between two lines in the domain space. Although the angle between two lines seems to assume that the manifolds involved have at least two dimensions, it is possible to restrict the set of conformal transformations in higher dimensions to one-dimensional spaces. Since our goal here is to introduce conformal mechanics in one dimension, we are not going to discuss conformal field theories in higher-dimensional spaces. Instead, we shall start from the set of infinitesimal transformations in the line that generate conformal transformations in 1D. We refer the reader to [7][8] for detailed introductions to conformal field theories in two or more dimensions.

The algebra of infinitesimal generators of conformal transformations in one dimension is the $sl(2)$ algebra. As has been done in the previous discussion about SUSY, we use the notation $sl(2)$ to denote the real algebra $sl(2,\mathbb{R})$. A Cartan-Weyl basis of $sl(2)$ is comprised of the generators H, D and K satisfying the commutation relations below:

$$[H,D]=H; \qquad [K,D]=-K; \qquad [H,K]=2D \tag{1.40}$$

We can realize the algebra above with differential operators. Take

$$H=a_H\partial_t+b_H; \qquad D=a_D\partial_t+b_D; \qquad K=a_K\partial_t+b_K \tag{1.41}$$

where $a_H=a_H(t)$, $b_H=b_H(t)$ and so on. Equations (1.40) and (1.41) together lead to the following system of differential equations:

$$\begin{aligned}
a_H\dot{a}_D-a_D\dot{a}_H&=a_H,\\
a_K\dot{a}_D-a_D\dot{a}_K&=-a_K,\\
a_H\dot{a}_K-a_K\dot{a}_H&=2a_D,\\
a_H\dot{b}_D-a_D\dot{b}_H&=b_H,\\
a_K\dot{b}_D-a_D\dot{b}_K&=-b_K,\\
a_H\dot{b}_K-a_K\dot{b}_H&=2b_D.
\end{aligned} \tag{1.42}$$

The equations above can be used to write the functions a_H, b_H and a_K, b_K in terms of arbitrary input functions $a_D=a$ and $b_D=b$. We thus obtain the general d-module representation written below.

$$\begin{aligned}
H&=e^{-\int^t\frac{dt'}{a(t')}}(a(t)\partial_t+b(t)-\lambda),\\
D&=a(t)\partial_t+b(t),\\
K&=e^{\int^t\frac{dt'}{a(t')}}(a(t)\partial_t+b(t)+\lambda).
\end{aligned} \tag{1.43}$$

The solution written above is very general and account for the fact that any linear combination of the generators of $sl(2)$ can be made equal to the generator of time translations ∂_t. Indeed, in systems with conformal invariance, any linear combination of the generators H, D and K will be a conserved charge, and thus can be used to study the evolution of the system. As pointed out in [3] though, the choice of a particular combination of operators carries many subtleties that range from the acceptable domain of time evolution to non-normalizability of eigenstates. Of the solutions contained in equations (1.42), two will be of particular interest to us:

i) The parabolic d-module representation:
This d-module rep takes $a(t)=t$, $b(t)=\lambda$, so that the generator of time translations ($=H$) is a root of the algebra. The parabolic $sl(2)$ generators are

$$H=\partial_t, \qquad D=t\partial_t+\lambda, \qquad K=t^2\partial_t+2\lambda t. \tag{1.44}$$

ii) The hyperbolic/trigonometric d-module representation:
This d-module rep takes $a(t)=1$, $b(t)=0$. The generator of time translations ($=D$) is now the Cartan generator. The hyperbolic $sl(2)$ d-module rep is

$$H=e^{-t}(\partial_t-\lambda), \qquad D=\partial_t, \qquad K=e^{t}(\partial_t+\lambda). \tag{1.45}$$

A variation of the hyperbolic d-module representation can also be constructed by complexification of the time variable, $t\to -it$. We shall call this the trigonometric d-module representation:

$$H=e^{-it}(-i\partial_t-\lambda), \qquad D=-i\partial_t, \qquad K=e^{it}(i\partial_t+\lambda). \tag{1.46}$$

We can view the operators of the d-module reps above as acting on a scalar field $x(t)$. The dimensionless parameter λ in these equations is the scaling dimension. It is a measure of the mass dimension of the field. In the equations above, the d-module representations close the $sl(2)$ algebra for any value of the scaling dimension. We thus say that there is no criticality, meaning that no particular value of the mass dimension of the fields is necessary to ensure conformal symmetry.

1.8 Superconformal algebra

The methods we have described for d-module representations of the SUSY algebra can be directly applied with little modifications to the construction of d-module representations of the superconformal algebra. Indeed that was done in [9] [10]. We again use the example of the $N = 1$ theory as a proxy to illustrate the procedure. As happened with SUSY, the operators act on the supermultiplet (x, ψ). Thus, we represent them with 2x2 matrices. The $sl(2)$ algebra is given by

$$\begin{aligned}
H &= e^{-\int^t \frac{dt'}{a(t')}}\big(a(t)\partial_t \mathbb{I}_2 + B(t) - \Lambda\big),\\
D &= a(t)\partial_t \mathbb{I}_2 + B(t),\\
K &= e^{\int^t \frac{dt'}{a(t')}}\big(a(t)\partial_t \mathbb{I}_2 + B(t) + \Lambda\big),
\end{aligned} \tag{1.47}$$

where $\Lambda = diag(\lambda, \lambda + \frac{1}{2})$. The 2x2 matrix $B(t)$ is zero in the hyperbolic case and equal to Λ in the parabolic one. Note that the scaling dimension parameter for the fermionic fields is shifted by $\frac{1}{2}$. Equation (1.3) now suggests us to take the square root of H as

$$Q = \begin{pmatrix} 0 & 1 \\ e^{-\int^t \frac{dt'}{a(t')}}(a(t)\partial_t + b(t) - \lambda) & 0 \end{pmatrix}. \tag{1.48}$$

And indeed the operator Q in (1.3) parabolic case ($a(t) = t$, $b(t) = \lambda$) is recovered. For the hyperbolic case ($a(t) = 1$, $b(t) = 0$), we perform a similarity transformation on the operator Q. Take $R = diag(1, e^{-\int^t \frac{dt'}{2a(t')}})$ and perform the transformation

$$Q \to R^{-1} Q R.$$

We thus end up with the following operators Q for the parabolic and hyperbolic case:

$$Q_{par} = \begin{pmatrix} 0 & 1 \\ \partial_t & 0 \end{pmatrix}, \qquad Q_{hyp} = e^{-t/2}\begin{pmatrix} 0 & 1 \\ \partial_t - \lambda & 0 \end{pmatrix}. \tag{1.49}$$

To conclude the construction of the $N = 1$ superconformal algebra, all that is left is to find the superconformal partner of Q, which we shall denote by $\bar{Q}$. It will be given by the commutator of Q and K:

$$\bar{Q} = [Q, K]. \tag{1.50}$$

The resulting superalgebra is $osp(1|2)$. Its non-vanishing (anti-)commutation relations are written below.

$$\begin{gathered}
[H, D] = H, \qquad [K, D] = -K, \qquad [H, K] = 2D,\\
[H, \bar{Q}] = Q, \qquad [Q, D] = \frac{Q}{2}, \qquad [\bar{Q}, D] = -\frac{\bar{Q}}{2}, \qquad [K, Q] = -\bar{Q}.\\
\{Q, Q\} = 2H, \qquad \{Q, \bar{Q}\} = 2D, \qquad \{\bar{Q}, \bar{Q}\} = 2K.
\end{gathered} \tag{1.51}$$

The procedure we have just sketched for the $N = 1$ superconformal algebra is general and applies to higher superconformal algebras. We start from the d-module representation of the $sl(2)$ algebra and extend it to include the square roots of H (labeled by Q) and the square roots of K (labeled here by $\bar{Q}$). As illustrated above, knowing the d-module representations of the $sl(2)$ and the $Q_i's$ together with the non-vanishing (anti)commutation relations of the proper $\mathcal{N}$-extended superconformal algebra suffices to construct the whole algebra.

Example 1.7: The $\mathcal{N} = 2$ (2,2,0) d-module reps of the superconformal algebra.

The $\mathcal{N} = 2$ extension of the $sl(2)$ algebra is the $sl(1|2)$ superalgebra. Its bosonic sector consists of the $sl(2)$ algebra H, D, K together with the $u(1)$ generator J, and thus is the direct sum $sl(2) \oplus u(1)$. The fermionic sector is generated by the square roots of H (Q_α) and K ($\bar{Q}_\alpha$). The non-vanishing superbrackets of $sl(1|2)$ are written below.

$$[H, D] = H, \qquad [K, D] = -K, \qquad [H, K] = 2D,$$

$$[H,\bar{Q}_\alpha]=Q_\alpha,\qquad [Q_\alpha,D]=\frac{Q_\alpha}{2},\qquad [\bar{Q}_\alpha,D]=-\frac{\bar{Q}_\alpha}{2},\qquad [K,Q_\alpha]=-\bar{Q}_\alpha.$$
$$[J,Q_\alpha]=\epsilon_{\alpha\beta}Q_\beta,\qquad [J,\bar{Q}_\alpha]=\epsilon_{\alpha\beta}\bar{Q}_\beta,$$
$$\{Q_\alpha,Q_\beta\}=2\delta_{\alpha\beta}H,\qquad \{Q_\alpha,\bar{Q}_\beta\}=2\delta_{\alpha\beta}D+2(1-\delta_{\alpha\beta})J,\qquad \{\bar{Q}_\alpha,\bar{Q}_\beta\}=2\delta_{\alpha\beta}K. \tag{1.52}$$

As should be clear from the superbrackets above, a d-module representation of $sl(1|2)$ can be recovered from Q_0, Q_1 and K. In the parabolic d-module rep, they are

$$Q_0=\begin{pmatrix}0 & \mathbb{I}_2\\ \mathbb{I}_2\partial_t & 0\end{pmatrix},\qquad Q_1=\begin{pmatrix}0 & \tau_2\\ -\tau_2\partial_t & 0\end{pmatrix},\qquad K=t^2\partial_t\mathbb{I}_4+2\Lambda t.$$

where $\Lambda=diag(\lambda,\lambda,\lambda+\frac{1}{2},\lambda+\frac{1}{2})$. In the hyperbolic d-module rep, these generators are given by

$$Q_0=e^{-\frac{t}{2}}\begin{pmatrix}0 & \mathbb{I}_2\\ \mathbb{I}_2(\partial_t-\lambda) & 0\end{pmatrix},\qquad Q_1=e^{-\frac{t}{2}}\begin{pmatrix}0 & \tau_2\\ -\tau_2(\partial_t-\lambda) & 0\end{pmatrix},\qquad K=e^t(\partial_t\mathbb{I}_4+\Lambda).$$

Example 1.8: The $\mathcal{N}=4$ (1,4,3) d-module reps of the superconformal algebra.

The $\mathcal{N}=4$ extension of the $sl(2)$ algebra is the family $D(2,1;\alpha)$ of exceptional Lie superalgebras. The real parameter $\alpha\neq 0,-1$ appears in the structure constants of this family of superalgebras, and thus each value of α labels a different symmetry superalgebra up to isomorphisms that relate the superalgebras with the following values of α:

$$\alpha^{\pm1},\qquad -(1+\alpha)^{\pm1},\qquad -(\frac{\alpha}{1+\alpha})^{\pm1}.$$

The limits $\alpha\to 0,-1$ recover the $A(1,1)$ superalgebra. At the special value $\alpha=1$, the superalgebra $D(2,1;\alpha)$ becomes the superalgebra $D(2,1)$. Thus, we may look at $D(2,1;\alpha)$ as a deformation of $D(2,1)$. More detailed discussions on Lie superalgebras and the $D(2,1;\alpha)$ superalgebra in particular are to be found in [11][12].

The even sector of $D(2,1;\alpha)$ is the Lie algebra $sl(2)\oplus sl(2)\oplus sl(2)$. The R-symmetry therefore has six generators, which we shall call J_i, G_i, $i=1,2,3$. The non-vanishing superbrackets of $D(2,1;\alpha)$ are written below.

$$[H,D]=H,\qquad [K,D]=-K,\qquad [H,K]=2D,$$
$$[J_i,J_j]=-\epsilon_{ijk}((1+2\alpha)G_k+J_k),\qquad [J_i,G_j]=-\epsilon_{ijk}(G_k+(1+2\alpha)J_k),\qquad [G_i,G_j]=-\epsilon_{ijk}((1+2\alpha)G_k+J_k),$$
$$[H,\bar{Q}_\beta]=Q_\beta,\qquad [Q_\beta,D]=\frac{Q_\beta}{2},\qquad [\bar{Q}_\beta,D]=-\frac{\bar{Q}_\beta}{2},\qquad [K,Q_\beta]=-\bar{Q}_\beta,$$
$$[J_i,Q_\beta]=(1+2\alpha)\delta_{0\beta}Q_i-(1+2\alpha)\delta_{i\beta}Q_0-\delta_{\beta j}\epsilon_{ijk}Q_k,\qquad [G_i,Q_\beta]=\delta_{0\beta}Q_i-\delta_{i\beta}Q_0-\delta_{\beta j}\epsilon_{ijk}(1+2\alpha)Q_k,$$
$$[J_i,\bar{Q}_\beta]=(1+2\alpha)\delta_{0\beta}\bar{Q}_i-(1+2\alpha)\delta_{i\beta}\bar{Q}_0-\delta_{\beta j}\epsilon_{ijk}\bar{Q}_k,\qquad [G_i,\bar{Q}_\beta]=\delta_{0\beta}\bar{Q}_i-\delta_{i\beta}\bar{Q}_0-\delta_{\beta j}\epsilon_{ijk}(1+2\alpha)\bar{Q}_k,$$
$$\{Q_\beta,Q_\gamma\}=2\delta_{\beta\gamma}H,\qquad \{Q_\beta,\bar{Q}_\gamma\}=2\delta_{\beta\gamma}D+(\delta_{0\beta}\delta_{\gamma i}+\delta_{0\gamma}\delta_{\beta i})G_i+\delta_{\beta i}\delta_{\gamma j}\epsilon_{ijk}J_k,\qquad \{\bar{Q}_\beta,\bar{Q}_\gamma\}=2\delta_{\alpha\beta}K. \tag{1.53}$$

In the parabolic d-module rep of the $\mathcal{N}=4$ (1,4,3) superconformal algebra, the fermionic generators Q_α are given by (1.32). The generator K is

$$K=t^2\partial_t\mathbb{I}_8+2\Lambda t,$$

where $\Lambda=diag(\lambda,\lambda+1,\lambda+1,\lambda+1,\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2})$. Note that the entries of Λ corresponding to the auxiliary fields are shifted by one with respect to the propagating bosons.

In the hyperbolic d-module rep of the $\mathcal{N}=4$ (1,4,3) superconformal algebra, we have the following sufficient set of generators that recover the complete superalgebra.

$$\begin{aligned}
&Q_0x_0=e^{-\frac{t}{2}}\psi_0, && Q_0\psi_0=e^{-\frac{t}{2}}(\dot{x}_0-\lambda x_0),\\
&Q_0b_i=e^{-\frac{t}{2}}(\dot{\psi}_i-(\lambda+\frac{1}{2})\psi_i), && Q_0\psi_i=e^{-\frac{t}{2}}b_i,\\
&Q_ix_0=e^{-\frac{t}{2}}\psi_i, && Q_i\psi_0=-e^{-\frac{t}{2}}b_i,\\
&Q_ib_j=-e^{-\frac{t}{2}}(\delta_{ij}(\dot{\psi}_0-(\lambda+\frac{1}{2})\psi_0)+\epsilon_{ijk}(\dot{\psi}_k-(\lambda+\frac{1}{2})\psi_k)), && Q_i\psi_j=e^{-\frac{t}{2}}(\delta_{ij}(\dot{x}_0-\lambda x_0)+\epsilon_{ijk}b_k),
\end{aligned}$$

together with $K=e^t(\partial_t+\Lambda)$, where again $\Lambda=diag(\lambda,\lambda+1,\lambda+1,\lambda+1,\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2})$.

Table 1.2: D-module reps of 1D superconformal algebras

SUSY	Supermultiplet	Superconformal algebra	Criticality
$\mathcal{N}=1$	(1,1,0) (0,1,1)	$osp(1\|2)$	–
$\mathcal{N}=2$	(2,2,0) (1,2,1) (0,2,2)	$sl(2\|1)$	–
$\mathcal{N}=4$	(4,4,0) (3,4,1) (2,4,2) (1,4,3) (0,4,4)	$D(2,1;\alpha)$ $D(2,1;\alpha)$ $A(1,1)$ $D(2,1;\alpha)$ $D(2,1;\alpha)$	$\alpha=-2\lambda$ $\alpha=-\lambda$ $\lambda=0$ $\alpha=\lambda$ $\alpha=2\lambda$
$\mathcal{N}=8$	(8,8,0) (7,8,1) (6,8,2) (5,8,3) (4,8,4) (3,8,5) (2,8,6) (1,8,7) (0,8,8)	$D(4,1)$ $F(4)$ $A(3,1)$ $D(2,2)$ - $D(2,2)$ $A(3,1)$ $F(4)$ $D(4,1)$	$\lambda=1/4$ $\lambda=1/3$ $\lambda=1/2$ $\lambda=1$ - $\lambda=-1$ $\lambda=-1/2$ $\lambda=-1/3$ $\lambda=-1/4$

An important point to note here is the relation that exists between the parameter α that labels the family $D(2,1;\alpha)$ of $\mathcal{N}=4$ exceptional Lie superalgebras and the scaling dimension λ. For the $(1,4,3)$ supermultiplet, we have $\alpha=\lambda$. In general, the relation between α and λ in $\mathcal{N}=4$ $(k,4,4-k)$ super multiplets is $\alpha=(2-k)\lambda$. In the $\mathcal{N}=4$ d-module reps the scaling dimension parameter is said to be critical, since it determines the actual symmetry algebra of the d-module representation. Next we present a table with a quick classification of the $1D$ superconformal algebras. Although not complete, it should be useful as a quick reference.

1.9 1D superconformal σ-models

In this last section of this chapter, we present a quick overview of the superconformal σ-models associated to the superalgebras we have described. As has been done with the previous sections, the purpose is mainly to illustrate the techniques that are used throught this work.

Let us start with the $osp(1|2)$-invariant action. In section **1.5** we have shown that the globally invariant action for the $\mathcal{N}=1$ $(1,1,0)$ supermultiplet is given by the Lagrangian (1.34),

$$\mathcal{L}=A(\dot{x}^2+\dot{\psi}\psi).$$

This means that the action is invariant under H and Q in the parabolic rep. Given that all generators of $osp(1|2)$ can be obtained from Q and K using the brackets (1.51), we must only find which function $A(x)$ renders the action invariant under K. This is enough to ascertain that invariance follows for the whole $osp(1|2)$ superalgebra. Thus, the procedure in the parabolic case is fairly straight: given a globally SUSY-invariant action with arbitrary prepotential, find the correct prepotential that leads to an action invariant under K. Since

$$K=\begin{pmatrix} t^2\partial_t+2\lambda t & 0\\ 0 & t^2\partial_t+2(\lambda+\frac{1}{2})t\end{pmatrix},$$

we can straightforwardly compute the variation of the action due to K, bearing in mind that K acts as a Lebniz derivative on the fields. We thus get

$$K(A\dot{x}^2+A\dot{\psi}\psi)=2t(\lambda A_x x+(1+2\lambda)A)(\dot{x}^2+\dot{\psi}\psi)+\frac{d}{dt}(t^2A\dot{x}^2+t^2A\dot{\psi}\psi+4\lambda\int^x A(x')x'dx').$$

As far as conformal invariance of the action is concerned, the equation above should equal a total derivative. This will be true provided that

$$\lambda A_x x+(1+2\lambda)A=0. \tag{1.54}$$

The solution of this differential equation up to constant factors is

$$A(x) = x^{-\frac{2\lambda+1}{\lambda}}.$$

which is exactly what we have found from dimensional analysis in equation (1.35).

Now let us switch to the hyperbolic case. We follow the lines of [3] here. The action in the parabolic case is given by

$$\mathcal{S} = \int dt A(\dot{x}^2 + \dot{\psi}\psi), \tag{1.55}$$

and we know that it will be conformally invariant with the correct choice of $A(x)$. Write the following change of variables and fields:

$$t = e^{\tau}, \qquad x = f(\tau)u(\tau) \qquad \psi = g(\tau)\chi(\tau). \tag{1.56}$$

In words, we are substituting the time variable t by τ in the integration, and the fields $x(t)$, $\psi(t)$ by the new fields $u(\tau)$, $\chi(\tau)$. It is straightforward to verify that the action (1.55) becomes

$$\mathcal{S} = \int d\tau B[e^{-\tau}(f'^2u^2 + 2f'fuu' + f^2u'^2) + g^2\chi'\chi],$$

where the primes denote derivatives with respect to the new time variable τ and $B = B(\tau, u) = A(x)$. Now if the action is conformally invariant, we have

$$A(x) = x^{-\frac{1+2\lambda}{\lambda}} = f^{-\frac{1+2\lambda}{\lambda}}u^{-\frac{1+2\lambda}{\lambda}} = f^{-\frac{1+2\lambda}{\lambda}}A(u) = B(\tau, u),$$

and we may rewrite the action as

$$\mathcal{S} = \int d\tau A(u) f^{-\frac{1+2\lambda}{\lambda}}[e^{-\tau}(f'^2u^2 + 2f'fuu' + f^2u'^2) + g^2\chi'\chi],$$

It is clear from this last equation that we can eliminate any explicitly dependence on τ by setting

$$f(\tau) = e^{-\lambda\tau}, \qquad g(\tau) = e^{-(\lambda+\frac{1}{2})\tau}. \tag{1.57}$$

This will lead us to the following action:

$$\mathcal{S} = \int d\tau A(u)(u'^2 + \chi'\chi + \lambda^2u^2), \tag{1.58}$$

where we have eliminated from it the total derivative term $-2\lambda \int d\tau A(u)uu'$.

The action in (1.58) is the conformally invariant action of the hyperbolic rep. To see that this is so, consider the operator D in the parabolic and hyperbolic reps:

$$D_{par} = t\partial_t\mathbb{I}_2 + \Lambda, \qquad D_{hyp} = \partial_\tau. \tag{1.59}$$

The last equation suggests that to change from the parabolic to the hyperbolic d-module rep we need to perform the change of variable $t = e^{\tau}$, since this leads us to the relation

$$\partial_\tau\mathbb{I}_2 = t\partial_t\mathbb{I}_2.$$

Under this change of variable, D_{par} becomes $\partial_\tau\mathbb{I}_2 + \Lambda$. We can eliminate Λ through a similarity transformation. Indeed, take the 2x2 matrix $R(\tau) = diag(f(\tau), g(\tau))$. A similarity transformation of D_{par} by $R(\tau)$ gives

$$R^{-1}D_{par}R = \partial_\tau\mathbb{I}_2 + \Lambda + diag(\frac{f'}{f}, \frac{g'}{g})$$

and therefore Λ will vanish if we take $diag(\frac{f'}{f}, \frac{g'}{g}) = -\Lambda = diag(-\lambda, -\lambda - \frac{1}{2})$. We thus arrive at the results in (1.57). From these arguments, it is clear that the set of transformations (1.56),(1.57) we have performed to arrive at the action (1.58) are simply the mathematical steps required to change from a parabolic to a hyperbolic conformally invariant action. The hyperbolic Lagrangian is then given by

$$\mathcal{L}_{hyp} = A(u)(u'^2 + \chi'\chi + \lambda^2u^2) = \mathcal{L}_{par} + \lambda^2A(u)u^2. \tag{1.60}$$

We emphasize here that this result, first obtained for conformal mechanics in $1D$ in [3], holds its generality for superconformal mechhanics. In particular, the hyperbolic Lagrangians will be identical to the parabolic ones plus an extra potential term and, for supermultiplets of the form $(k, \mathcal{N}, \mathcal{N}-k)$ with $\mathcal{N} = 1, 2, 4, 8$, the conformal factor $A(x)$ multiplying the kinetic terms in the Lagrangians will be the solution to the differential equation

$$\lambda(\vec{r} \cdot \nabla)A + (1 + 2\lambda)A = 0 \tag{1.61}$$

which will depend only on the modulus of $\vec{r} = (x_1, ..., x_k)$:

$$A(\vec{r}) = r^{-\frac{1+2\lambda}{\lambda}}, \qquad r = (\sum_i^k x_i^2)^{\frac{1}{2}}. \tag{1.62}$$

Having introduced the basic ideas and computations that are performed through this work, we may now move on to its core results.

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Chapter 2

Four types of superconformal mechanics

2.1 Introduction

In this paper we prove the existence of four types of conformally invariant actions for one-dimensional mechanical systems. In [1] Papadopoulos realized that under hyperbolic/trigonometric transformations of the fields, extra potential terms entered the conformal Lagrangians (these extra potential terms are not present when the standard, *parabolic*, realization of the conformal transformations is considered).

We generalize here the results of [1] in two distinct ways. At first we point out that a scaling dimension λ is associated with the parabolic and hyperbolic/trigonometric D-module reps of the conformal algebra $sl(2)$. In [1] λ was only taken at the given fixed value which produces constant kinetic terms.

The scaling parameter λ, however, cannot be so easily dismissed. In the supersymmetric generalizations (starting from the $\mathcal{N}=4$ supersymmetric extension) it acquires a critical property. Depending on the given supermultiplet, see e.g. formula (2.36), it specifies under which of the exceptional $D(2,1;\alpha)$ (parametrized by α) supersymmetry algebras, the system under consideration is superconformally invariant.

Our second extension concerns the generalization to the *inhomogeneous* parabolic and hyperbolic/trigonometric conformal transformations of the fields (in [1] only homogeneous transformations were considered).

We point out, see Appendix **A**, the existence of two inequivalent classes of linear one-dimensional conformal transformations (and their supersymmetric extensions), the homogeneous ones, depending on the critical scaling λ, and the inhomogeneous ones, which are parametrized by the constant ρ.

Hyperbolic versus trigonometric transformations are mutually recovered via an analytic continuation. The passage from parabolic to hyperbolic transformations, see e.g. formula (2.12), requires a singular change of variable. Under this change of variable the properties of their respective D-module reps (the scaling λ or, in the inhomogeneous case, the parameter ρ) are easily recovered. The singularity of the change of variable is, on the other hand, responsible for the appearance in the Lagrangians of the extra potential terms that we mentioned before.

On algebraic grounds the crucial difference between the hyperbolic and the parabolic $sl(2)$ transformations is the following. In the parabolic case the operator proportional to a time-derivative (the "Hamiltonian") is given by the (positive or negative) $sl(2)$ root, while in the hyperbolic case this Hamiltonian operator is associated with the $sl(2)$ Cartan generator. This is the reason why, when we consider superalgebra extensions, the parabolic systems are supersymmetric in the ordinary sense, while the hyperbolic systems, despite being superconformally invariant, are not ordinary supersymmetric theories (see the discussion in Appendix **D**).

We end up, for one-dimensional conformal systems and their supersymmetric extensions, with four types of D-module reps and their associated (super)conformally invariant actions, namely the *homogeneous parabolic*, *inhomogeneous parabolic*, *homogeneous hyperbolic/trigonometric* and *inhomogeneous hyperbolic/trigonometric* cases.

Only for the very special homogeneous parabolic case the conformally invariant actions are based on power-law and contain no dimensional parameter. In all remaining cases we have at disposal at least one dimensional constant to play with.

In the following we present all four types of (super)conformal actions in various exemplifying $d=1$ situations: the $sl(2)$-invariant actions of a single boson, the $osp(1|2)$-invariant ($sl(2|1)$-invariant) actions of an $\mathcal{N}=1$ ($\mathcal{N}=2$) supermultiplet. For a given set of $\mathcal{N}=4$ supermultiplets, the actions are $D(2,1;\alpha)$-invariant in the two homogeneous cases and $A(1,1)$-invariant in the two inhomogeneous cases. The non-trivial relation between homogeneous and inhomogeneous actions (discussed in Appendix **C**) can be appreciated in a different basis obtained through non-linear field redefinitions. In the new basis the actions possess a constant kinetic term (plus interacting potentials), while the superconformal algebra is realized non-linearly.

The construction is also applied, in the Lagrangian setting, to (super)conformal actions in $d=2$ dimensions.

The invariance in this case is under a single copy (for the chiral models) or the direct sum of two copies (the full conformal invariance) of the Witt algebra (the centerless Virasoro algebra) and its supersymmetric extensions.

Contrary to the $d = 1$ case, in $d = 2$, hyperbolic and parabolic active transformations of the field(s) produce the same conformally invariant output. As an example, the inhomogeneous transformations applied on a single boson induce, see (2.32), the conformally invariant Liouville action, while the homogeneous transformations induce a power-law conformally invariant action, see formula (2.31). These two-dimensional actions are related by a non-linear field redefinition.

From the point of view of representation theory we extend here in two directions the results of [2] and [3] on D-module reps of finite $d = 1$ superconformal algebras. We enlarge the D-module reps of the $osp(1|2)$, $sl(2|1)$, $B(1,1) = osp(3|2)$ and $A(1,1) = sl(2|2)/\mathbb{Z}$ superalgebras to the class of *inhomogeneous* (ρ-dependent) D-module reps, see (2.42).

We further construct the D-module reps of the $\mathcal{N} = 1,2,3,4$ centerless superVirasoro algebras, both in the homogeneous case (they are summarized in (2.43)) and, for $\mathcal{N} = 1,2,3$, inhomogeneous case (these results are summarized in (2.44)). The explicit construction of these D-module reps is presented in Appendix **B**.

Conformal mechanics based on the $sl(2)$ algebra has been investigated since the work of de Alfaro, Fubini and Furlan [4]. Models of superconformal mechanics have been presented in [5]–[13] (for an updated review on superconformal mechanics and a list of recent references see, e.g., [14]). For superconformal actions with oscillator potentials see also [15] and [16]. For non-linear realizations see [17]. There are several reasons to study one-dimensional superconformal mechanics (more on that in the Conclusions). Here it is sufficient to mention the applications to test particles moving in the proximity of the horizon of certain black holes, see [12]. In [2] and [3] it was advocated the point of view that superconformal mechanics, in the Lagrangian setting, could be derived from the D-module reps of superconformal algebras. In most of the papers in the literature and all works cited in the [14] review, the superconformal actions are based on power laws, being dependent only on dimensionless constants (apart the optional addition of oscillatorial terms, what is known as the DFF trick [4]). This is what to be expected for the homogeneous parabolic D-module reps. The possibilities offered by the three remaining types of D-module reps (presenting dimensional constants), on the other hand, greatly enlarge the class of available superconformal systems. One should confront, for instance, the power law $\mathcal{N} = 4$ superconformal systems with $A(1,1)$ or $D(2,1;\alpha)$ invariance investigated in [18, 19] and [20]–[25], respectively, with the $\mathcal{N} = 4$ actions presented in Section **9**.

The scheme of the paper is as follows: in Section **2** we introduce the *homogeneous* parabolic and hyperbolic D-module reps of the Witt algebra and its $sl(2)$ subalgebra. The *inhomogeneous* D-module reps of the $sl(2)$ and Witt algebras are discussed in Section **3**. In Section **4** we derive the different types of conformally invariant actions for a single boson. In Section **5** we extend the analysis to the bosonic, conformally invariant actions in $d = 2$ dimensions. In Section **6** we collect the main properties of the finite $d = 1$ superconformal algebras (together with their known D-module reps) and of the $\mathcal{N} = 1,2,3,4$ centerless superVirasoro algebras. The new results on D-module reps for the finite $d = 1$ superconformal algebras and for the centerless superVirasoro algebras are summarized in Section **7**. Different types of $\mathcal{N} = 1,2$ superconformal actions in $d = 1$ and supersymmetric chiral actions in $d = 2$ are given in Section **8**. In Section **9** we present the four types of $\mathcal{N} = 4$ superconformally invariant actions associated with the class of $(1,4,3)$ supermultiplets. In the Conclusions we point out the possible applications of our results and the future lines of investigations. The paper is complemented by four Appendices. A discussion about homogeneous versus inhomogeneous D-module reps is given in Appendix **A**. In Appendix **B** we present the explicit construction of the new supersymmetric D-module reps discussed in the main text. In Appendix **C** the $\mathcal{N} = 4$ superconformally invariant actions of Section **9** are presented in a new basis. In this basis the kinetic term is constant, while the superconformal algebra is realized non-linearly. In Appendix **D** we point out that the one-dimensional, superconformally invariant, hyperbolic and trigonometric systems are not ordinary supersymmetric theories.

2.2 The bosonic case: homogeneous D-module reps of the $sl(2)$ and $Witt$ algebras

The $sl(2)$ algebra is the conformal algebra in $d = 1$ dimensions. Its three generators D, H, K satisfy the commutation relations

$$
\begin{array}{rcl}
[D,H] &=& H,\\
[D,K] &=& -K,\\
[H,K] &=& 2D.
\end{array}
\qquad (2.1)
$$

The Cartan generator D is the dilatation operator.

A (parabolic) D-module representation of (2.1) is given by the differential operators (depending on a single variable t which, in application to physics, plays the role of time)

$$\begin{array}{rcl} H &=& \partial_t, \\ D &=& -t\partial_t - \lambda, \\ K &=& -t^2\partial_t - 2\lambda t. \end{array} \tag{2.2}$$

The constant λ is the scaling parameter. The above D-module rep is non-critical because the commutators (2.1) close for any value of λ.

The Virasoro algebra Vir is the central extension of the algebra of one-dimensional diffeomorphisms (known as "Witt algebra"). Its infinite generators L_n ($n \in \mathbb{Z}$) satisfy the commutation relations

$$[L_m, L_n] \quad = \quad (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}. \tag{2.3}$$

The Virasoro algebra contains $sl(2)$ as a subalgebra. It is obtained by restricting $n = \pm 1, 0$.

In the centerless case the Witt algebra admits a parabolic D-module rep. Indeed

$$L_n^{par.} \quad = \quad -t^{n+1}\partial_t - \lambda_n t^n \tag{2.4}$$

give the commutators (2.3) with $c = 0$ provided that the λ_m's satisfy the set of equations

$$m\lambda_m - n\lambda_n \quad = \quad (m-n)\lambda_{m+n}. \tag{2.5}$$

A solution is recovered for

$$\lambda_n \quad = \quad n\tilde{\lambda} + \tilde{\gamma}, \tag{2.6}$$

with $\tilde{\lambda}, \tilde{\gamma}$ arbitrary constants.

For the $sl(2)$ generators we obtain, in particular,

$$\begin{array}{rcl} L_{-1}^{par.} &=& -\partial_t + (\tilde{\lambda} - \tilde{\gamma})\frac{1}{t}, \\ L_0^{par.} &=& -t\partial_t - \tilde{\gamma}, \\ L_1^{par.} &=& -t^2\partial_t - (\tilde{\lambda} + \tilde{\gamma})t. \end{array} \tag{2.7}$$

The special value $\tilde{\gamma} = \tilde{\lambda}$ allows us to identify, for $\lambda = \tilde{\lambda}$,

$$\begin{array}{rcl} L_{-1}^{par.} &\equiv& -H, \\ L_0^{par.} &\equiv& D, \\ L_1^{par.} &\equiv& K. \end{array} \tag{2.8}$$

At this special value of $\tilde{\gamma}$ one of the root generators of $sl(2)$ is proportional to a time-derivative and, in physics, can be identified with the Hamiltonian.

The constant $\tilde{\gamma}$ is arbitrary and can be changed via a similarity transformation. Indeed, for $f(t) = sgn(t)\hat{\gamma}ln|t|$, we have

$$L_n^{par.} \quad \mapsto \quad L_n^f = e^f L_n^{par.} e^{-f} = L_n^{par.} + \hat{\gamma}t^n.$$

Therefore, $\tilde{\gamma} \mapsto \tilde{\gamma} - \hat{\gamma}$.

For the special choice $\tilde{\gamma} = \tilde{\lambda}$, the parabolic D-module rep of the Witt algebra is

$$L_n^{par.} \quad = \quad -t^{n+1}\partial_t - (n+1)\tilde{\lambda}t^n. \tag{2.9}$$

By using hyperbolic/trigonometric functions, a hyperbolic/trigonometric D-module rep of the Witt algebra can be given. In the hyperbolic case the $c = 0$ commutators (2.3) are satisfied for

$$L_n^{hyp.} \quad = \quad -\frac{1}{\mu}e^{n\mu\tau}(\partial_\tau + \overline{\lambda}_n), \tag{2.10}$$

if $\overline{\lambda}_n = n\overline{\lambda} + \overline{\gamma}$. The dimensional constant μ has been introduced here for dimensional reasons. Without loss of generality we can fix it at the $\mu = 1$ value. In most of the cases, nevertheless, it is convenient to explicitly keep it in the equations in order to facilitate a dimensional analysis.

The $sl(2)$ generators read now

$$
\begin{aligned}
L_1^{hyp.} &= -\frac{1}{\mu}e^{\mu\tau}(\partial_\tau+\overline{\lambda}+\overline{\gamma}),\\
L_0^{hyp.} &= -\frac{1}{\mu}(\partial_\tau+\overline{\gamma}),\\
L_{-1}^{hyp.} &= -\frac{1}{\mu}e^{-\mu\tau}(\partial_\tau-\overline{\lambda}+\overline{\gamma}).
\end{aligned}
\tag{2.11}
$$

In the hyperbolic case the generator proportional to the time-derivative (the "Hamiltonian") coincides, for $\overline{\gamma}=0$, with the $sl(2)$ Cartan generator $L_0^{hyp.}$.
Just like the parabolic case, the constant $\overline{\gamma}$ can be shifted by a similarity transformation.

At this point it is important to stress that the parabolic and the hyperbolic D-module reps of the Witt algebras are singled out, among the most general class of D-module reps, by the aforementioned very special property. Namely, that for a specific value of the constant parameter (either $\tilde{\gamma}$ or $\overline{\gamma}$), one of the $sl(2)$ generators is proportional to the Hamiltonian. The mathematical difference between the parabolic and the hyperbolic D-module reps can be stated as follows. In the parabolic case, the Hamiltonian is identified with the (positive or negative) $sl(2)$ root generator while, in the hyperbolic case, the Hamiltonian is identified with the $sl(2)$ Cartan generator. This difference proves to be crucial in the construction of conformally invariant actions.

From an algebraic point of view the hyperbolic D-module rep can be recovered from the parabolic D-module rep via a singular transformation. Let us call, for simplicity, $\overline{L}_n=L_n^{hyp.}$ when we fix the values $\mu=1$ and $\overline{\gamma}=0$. Therefore $\overline{L}_n=-e^{n\tau}(\partial_\tau+n\overline{\lambda})$. For $t>0$ the change of variable

$$
t\mapsto\tau(t)=ln(t)
\tag{2.12}
$$

allows to recover the parabolic rep $\overline{L}_n=-t^{n+1}\partial_t-n\overline{\lambda}t^n$ at the specific values, for its constants, $\tilde{\lambda}=\overline{\lambda}$ and $\tilde{\gamma}=0$.

The

$$
\tilde{\lambda} = \overline{\lambda}
\tag{2.13}
$$

relation is of particular importance. Extended to superconformal algebras with $\mathcal{N}\geq 4$ (the ones, as discussed in the Introduction, where the criticality of the scale parameter plays a role), it implies that the same critical scaling is recovered in both parabolic and hyperbolic cases (we will see this property at work in the following of the paper).

The singularity of the transformation connecting parabolic and hyperbolic D-module reps has the consequence, for the respective conformal invariant actions, that they are not (at least trivially) related. With respect to the parabolic case, in the hyperbolic case extra potential terms appear due to the presence of the dimensional constant μ (and due to the different identification of the Hamiltonian operator with the given $sl(2)$ generator).

The connection of the trigonometric case (that we do not need here to write down explicitly) with the hyperbolic case is simply given by an analytic continuation. One can perform a Wick rotation of the time coordinate τ by identifying a new periodic variable θ ($\tau\equiv i\theta$). Alternatively, the analytic continuation can also be obtained by performing a Wick rotation of the dimensional constant μ, mapping $\mu\mapsto i\mu$. It will be shown in the following that the extra potential terms entering the conformally invariant actions in the hyperbolic case are not bounded below, due to a "wrong" sign. Since they are proportional to μ^2, the correct sign can be recovered through the latter Wick rotation. The conformally invariant actions based on the trigonometric D-module transformations have therefore well-defined, bounded from below, potentials.

As recalled in [1], the group of diffeomorphisms $Diff(\mathbb{R})$ of the real line induced by the hyperbolic D-module rep is promoted, in the trigonometric case, to the group of diffeomorphisms $Diff(\mathbf{S}^1)$ of the $\mathbf{S}^1$ circle.

2.3 Inhomogeneous D-module reps of the $sl(2)$ and Witt algebras

Besides distinguishing Witt algebra's D-module reps into the two classes of *parabolic* versus *hyperbolic/trigonometric* representations, another discrimination can be introduced. It concerns the *homogeneous* versus the *inhomogeneous* representations.

Let $\varphi(t)$ be a time-dependent field. In the homogeneous case, the action of the Witt generators is written down as

$$
L_n(\varphi) = a_n\dot{\varphi}+b_n\varphi.
\tag{2.14}
$$

In the inhomogeneous case the generators act as

$$
L_n(\varphi) = a_n\dot{\varphi}+d_n.
\tag{2.15}
$$

In both cases the closure of the $c = 0$ (2.3) commutators is guaranteed, provided that the coefficients a_n, b_n and a_n, d_n are fixed to proper values (the coefficients b_n, d_n coincide; for clarity reasons in application to conformal actions, it will be however convenient to denote with different letters their respective normalization constants).

The parabolic subcase requires

$$a_n = -t^{n+1}, \qquad b_n = \tilde{\lambda}\dot{a}_n, \qquad d_n = \tilde{\rho}\dot{a}_n. \tag{2.16}$$

The hyperbolic subcase requires

$$a_n = -\frac{1}{\mu}e^{n\mu\tau}, \qquad b_n = \overline{\lambda}\dot{a}_n, \qquad d_n = \overline{\rho}\dot{a}_n. \tag{2.17}$$

Taking into account the discussion in Appendix **A**, the overall result is the existence of four types of D-module representations of the Witt algebra, labelled as follows:

I (*Hom. par.*) - the *homogeneous parabolic* rep,
II (*Inh. par.*) - the *inhomogeneous parabolic* rep,
III (*Hom. hyp.*) - the *homogeneous hyperbolic* rep,
IV (*Inh. hyp.*) - the *inhomogeneous hyperbolic* rep.

Let $[t] = [\tau] = -1$ be the scaling dimension of the time coordinate(s) (therefore $[\mu] = 1$). Let us furthemore set the scaling dimension of the field φ being given by $[\varphi] = s$.

For consistency, in the respective cases, the scaling dimensions of the $\tilde{\lambda}$, $\tilde{\rho}$, $\overline{\lambda}$, $\overline{\rho}$ parameters are

$$I : [\tilde{\lambda}] = 0, \qquad II : [\tilde{\rho}] = s, \qquad III : [\overline{\lambda}] = 0, \qquad IV : [\overline{\rho}] = s. \tag{2.18}$$

For $s \neq 0$ the *Hom. par.* rep contains no dimensional parameter, while one dimensional parameter ($\tilde{\rho}$) is found in the *Inh. par.* rep, one dimensional parameter (μ) in the *Hom. hyp.* rep and two dimensional parameters ($\mu, \overline{\rho}$) in the *Inh. hyp.* rep.

Similarly to the homogeneous case, the change of variable (2.12) allows to connect the inhomogeneous parabolic and hyperbolic D-module reps. Under this transformation the relation

$$\tilde{\rho} \quad = \quad \overline{\rho} \tag{2.19}$$

is verified.

Since no confusion will arise, in both parabolic and hyperbolic cases, we denote in the following, for simplicity, the scaling parameter of the homogeneous D-module rep as "λ" and the parameter of the inhomogeneous D-module rep as "ρ".

2.4 Conformal actions in $d = 1$

We are looking at first for conformally invariant actions depending on a single field $\varphi(t)$. The Lagrangian has the form

$$\mathcal{L} \quad = \quad g(\varphi)\dot{\varphi}^2 + h(\varphi), \tag{2.20}$$

where $g(\varphi)$ is a (one-dimensional) metric and $h(\varphi)$ is a potential term. The conformal invariance puts restrictions on both g and h.

We present here the general results for the four types of conformal transformations (homogeneous/inhomogeneous and parabolic/hyperbolic) introduced in Sections **2** and **3**.

In the homogeneous parabolic case, the invariance under the L_n transformations (2.14) requires solving the system of equations

$$\begin{aligned} \dot{a}_n[(1+2\lambda)g + \lambda g_\varphi \varphi] &= 0, \\ 2\lambda g \ddot{a}_n \varphi + h_\varphi a_n + N_\varphi^{(n)} &= 0, \\ \lambda h_\varphi \varphi \dot{a}_n + N_t^{(n)} &= 0, \end{aligned} \tag{2.21}$$

with a_n given in (2.16). The set of functions $N^{(n)}(\varphi, t)$ has to be determined; it reflects the arbitrariness of the invariance of the Lagrangian up to a total time-derivative.

The same system is derived in the homogeneous hyperbolic case with a_n given in (2.17). In the hyperbolic case we have the relation

$$\ddot{a}_n = n^2\mu^2 a_n, \tag{2.22}$$

which is not present in the parabolic case.

Under the inhomogeneous transformations (2.15) the system of equations

$$\begin{aligned}\dot{a}_n[g+\rho g_\varphi] &= 0,\\ 2\rho g\ddot{a}_n + h_\varphi a_n + N_\varphi^{(n)} &= 0,\\ \rho h_\varphi \dot{a}_n + N_t^{(n)} &= 0\end{aligned} \tag{2.23}$$

is derived for both parabolic and hyperbolic cases; a_n is given, respectively, by (2.16) and (2.17).

Solving the above systems for all four cases is straightforward.

In the *Hom. par.* case, for instance, the first set of equations in (2.21) gives for the metric the solution $g = C_1\varphi^{-\frac{(1+2\lambda)}{\lambda}}$ (C_1 is a normalization constant). The third set of (2.21) equations allows to write $N^{(n)} = -\lambda h_\varphi \varphi a_n + M^{(n)}$, where $M^{(n)}(\varphi)$ are arbitrary functions of φ which do not explicitly depend on the time coordinate t. By plugging this result into the second set of equations, together with the (2.16) position for a_n, we end up with the following system: $-2\lambda(n+1)nt^{n-1}g\varphi - t^{n+1}[(1-\lambda)h_\varphi - \lambda h_{\varphi\varphi}\varphi] + M_\varphi^{(n)} = 0$.

The vanishing of the term inside square brackets gives the solution $h = C_2\varphi^{\frac{1}{\lambda}}$ (C_2 is the normalization constant). The first term in the left hand side is vanishing for $n = 0, -1$, while it can be reabsorbed by a suitable choice of $M^{(1)}(\varphi)$ for $n = 1$.

Therefore, the (2.21) system of equations cannot be nontrivially solved, simultaneously, for all $n \in \mathbb{Z}$, but at most for the $sl(2)$ subalgebra.

Deriving the solution for the three remaining cases proceeds along similar lines. In the two hyperbolic cases, the (2.22) relation for the a_n's induces an extra term in the potential, proportional to the metric normalization constant C_1, which is not present in the parabolic cases.

The overall results can be summarized as follows. We obtain four $d = 1$ conformal actions, invariant under different realizations of the $sl(2)$ active transformations of the single bosonic field $\varphi(t)$. Their respective Lagrangians are given by

I - Homogeneous parabolic case:

$$\mathcal{L} = C_1\varphi^{-\frac{(1+2\lambda)}{\lambda}}\dot{\varphi}^2 + C_2\varphi^{\frac{1}{\lambda}}. \tag{2.24}$$

II - Inhomogeneous parabolic case:

$$\mathcal{L} = C_1 e^{-\frac{1}{\rho}\varphi}\dot{\varphi}^2 + C_2 e^{\frac{1}{\rho}\varphi}. \tag{2.25}$$

III - Homogeneous hyperbolic case:

$$\mathcal{L} = C_1[\varphi^{-\frac{(1+2\lambda)}{\lambda}}\dot{\varphi}^2 + \mu^2\lambda^2\varphi^{-\frac{1}{\lambda}}] + C_2\varphi^{\frac{1}{\lambda}}. \tag{2.26}$$

IV - Inhomogeneous hyperbolic case:

$$\mathcal{L} = C_1 e^{-\frac{1}{\rho}\varphi}[\dot{\varphi}^2 + \mu^2\rho^2] + C_2 e^{\frac{1}{\rho}\varphi}. \tag{2.27}$$

In order to have a dimensionless action $\mathcal{S}$ ($[\mathcal{S}] = 0$), the scaling dimension of the Lagrangian is $[\mathcal{L}] = 1$, if we assign the time coordinate to have scaling dimension -1. Taking into account $[\mu] = 1$ and the relations (2.18), we end up with the following dimensional analysis:
- in both homogeneous cases (*I* and *III*), $[C_1] = [C_2] = 0$, provided that $[\varphi] = \lambda$;
- in both inhomogeneous cases (*II* and *IV*), $[C_1] = -1-2s$, $[C_2] = 1$, $[\varphi] = [\rho] = s$, with s arbitrary.

The homogeneous parabolic case is the only one not containing dimensional constants in the conformal action (in the *Hom. hyp.* case the constant μ is present).

The C_1, C_2 constants are arbitrary. On the other hand, in the two hyperbolic cases, extra terms for the potential appear with respect to the parabolic cases. Their normalization constant ($C_1\mu^2$) is linked with the metric normalization constant. Since μ^2 is positive, these potential terms have a “wrong" sign and are not bounded below.

A consistent action with a correct, bounded below, potential is obtained by allowing μ to be a complex variable and performing the $\mu \mapsto i\mu$ Wick rotation. As recalled in Section **2**, this is tantamount to pass from the hyperbolic to the trigonometric version of the D-module representation. We find convenient to derive the trigonometric actions in terms of a complex field φ (the real case can be recovered by setting its imaginary part equal to zero). A simple inspection shows that conformal, $sl(2)$-invariant actions based on the trigonometric D-module reps are given by

i - Homogeneous trigonometric case:

$$\mathcal{L} = C_1(|\varphi|^{-\frac{(1+2\lambda)}{\lambda}}\dot{\varphi}^*\dot{\varphi} - \mu^2\lambda^2|\varphi|^{-\frac{1}{\lambda}}) + C_2|\varphi|^{\frac{1}{\lambda}}; \tag{2.28}$$

ii - Inhomogeneous trigonometric case:

$$\mathcal{L} = C_1 e^{-\frac{\varphi+\varphi^*}{2\rho}}(\dot{\varphi}^*\dot{\varphi} - \mu^2\rho^2) + C_2 e^{\frac{\varphi+\varphi^*}{2\rho}}. \tag{2.29}$$

In both cases the correct, bounded below potentials are obtained by choosing $C_1 > 0$ and $C_2 \leq 0$.

Our results should be compared with the ones derived by Papadopoulos in [1]. In that paper only homogeneous transformations were considered. Furthermore, only constant metrics were discussed. This amounts to set $\lambda = -\frac{1}{2}$ in the homogeneous parabolic Lagrangian (2.24) and in the homogeneous hyperbolic Lagrangian (2.26) (the results of [1] are recovered, as it should be, in these special cases). These restrictions, however, can no longer be justified for the $\mathcal{N}$-extended superconformal actions with $\mathcal{N} \geq 4$. As already pointed out in Section **2**, in the $\mathcal{N} \geq 4$ cases, the parameter λ becomes critical. It specifies under which of the inequivalent superconformal algebras the mechanical system is invariant. We postpone to Appendix **C** (after the introduction of $\mathcal{N} = 4$ superconformal actions in Section **9**) a discussion of the subtle issues concerning the relation of the inhomogenous versus homogeneous actions.

2.5 Conformal actions in $d = 2$

It is instructive to extend the previous analysis to $d = 2$ conformally invariant actions. In the Lagrangian framework and classical case, the infinite-dimensional conformal algebra is $\mathfrak{witt} \oplus \mathfrak{witt}$, the direct sum of two copies of the Witt algebra $\mathfrak{witt}$.

Let $x_{1,2}$ be the coordinates of the plane. The $L_n^\pm$ generators of a D-module rep of $\mathfrak{witt} \oplus \mathfrak{witt}$ can be written in terms of the chiral/antichiral coordinates $z_\pm = x_1 \pm x_2$. The $L_n^\pm$ generators can be recovered from the $d = 1$ L_n generators introduced in Section **2** after replacing either t or τ (in the respective cases) with $z_\pm$. The chiral/antichiral decomposition implies the vanishing of the commutators $[L_n^+, L_m^-] = 0$ for any $n, m \in \mathbb{Z}$.

The two-dimensional conformal actions have a Lagrangian of the form

$$\mathcal{L} = g\varphi_+\varphi_- + h, \tag{2.30}$$

where the $\pm$ suffix denotes the partial derivative with respect to $z_\pm$.
Looking for conformal invariance under the assumption that homogeneous/homogeneous or inhomogeneous/inhomogeneous active D-module transformations of $\varphi(z_\pm)$ apply on both chiral/antichiral sectors, we are led to the following results. Contrary to the $d = 1$ case, the parabolic and hyperbolic D-module reps produce the same output for the Lagrangians, while the actions are invariant under the whole infinite set of $L_n^\pm$ generators.

The two surviving cases correspond to the Homogeneous or the Inhomogeneous transformations, respectively.

In the Homogeneous case g, h are restricted so that the conformally invariant Lagrangian is

$$\mathcal{L} = \frac{C_1}{\varphi^2}\varphi_+\varphi_- + C_2\varphi^{\frac{1}{\lambda}}, \tag{2.31}$$

with C_1, C_2 arbitrary constants.

The resulting action is invariant under the $\delta_n^\pm(\varphi) = -z_\pm^{n+1}\varphi_\pm - \lambda(n+1)z_\pm^n\varphi$ transformations.

In the Inhomogeneous case we recover the Liouville action. The metric needs to be a constant, while the potential is the exponential Liouville potential. We have

$$\mathcal{L} = C_1\phi_+\phi_- + C_2 e^{\frac{\phi}{\rho}}. \tag{2.32}$$

The corresponding action is invariant under the $\delta_n^\pm(\phi) = -z_\pm^{n+1}\phi_\pm - \rho(n+1)z_\pm^n$ transformations.

By requiring the action being dimensionless, and assuming $[z_\pm] = -1$, we obtain the Lagrangian scaling dimension $[\mathcal{L}] = 2$.

In the Homogeneous case the scaling dimensions are fixed to be

$$[\varphi] = 2\lambda, \quad [C_1] = [C_2] = [\lambda] = 0 \tag{2.33}$$

(therefore, no dimensional parameter is present in the theory).
In the Inhomogenous (Liouville) case, for an arbitrary value s, the scaling dimensions are

$$[\phi] = [\rho] = s, \quad [C_1] = -2s, \quad [C_2] = 2. \tag{2.34}$$

In this two-dimensional case the homogeneous action (2.31) is recovered from the Liouville action (2.32) through the non-linear field redefinition $\varphi = e^{\phi}$ and by performing the identification $\lambda = \rho$ (as discussed in Appendix **C** the latter identification is not possible for systems that, unlike (2.31) and (2.32), possess a critical value of the scaling dimension λ).

One should note that the classical Liouville action is invariant under two separate copies of the *centerless* Virasoro algebra. Even in this case, on the other hand, the associated Noether charges, endowed with a Poisson brackets structure, necessarily close the centrally extended version of the algebra, the full $Vir \oplus Vir$ algebra. It is a consequence of a non-equivariant moment map applied to the Liouville theory (see [26] for details).

2.6 On superconformal algebras

We discuss here two types of superconformal algebras, the supersymmetric extensions of the $d = 1$ conformal algebra $sl(2)$ and the supersymmetric extensions of the Virasoro algebra.

The finite one-dimensional superconformal algebras belong to the simple Lie superalgebras classified in [27, 28, 29] and satisfy special properties. A $d = 1$ superconformal algebra $\mathcal{G}$ admits a grading [30] $\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_{-\frac{1}{2}} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_1$. Its even sector $\mathcal{G}_{even} = \mathcal{G}_0 \oplus \mathcal{G}_{-1} \oplus \mathcal{G}_1$ is isomorphic to $sl(2) \oplus R$, where the subalgebra R is known as R-symmetry. The odd sector $(\mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_{-\frac{1}{2}})$ is spanned by $2\mathcal{N}$ generators ($\mathcal{N}$ is the number of extended supersymmetries).
At fixed $\mathcal{N}$ the positive sector $\mathcal{G}_{>0}$ is isomorphic to the $d = 1$ superPoincaré algebra (the algebra of the $\mathcal{N}$-extended supersymmetric quantum mechanics [31]).

If we denote, see (2.1), the $sl(2)$ generators as D, H, K, we have that $\mathcal{G}_1$ ($\mathcal{G}_{-1}$) is spanned by the positive (negative) root H (K), while $\mathcal{G}_0 = D\mathbb{C} \oplus R$.

We are especially interested in the $\mathcal{N} = 1, 2, 4, 8$ extensions. The corresponding list of $d = 1$ superconformal algebras is given by $osp(1|2)$ for $\mathcal{N} = 1$ and $sl(2|1)$ for $\mathcal{N} = 2$. For $\mathcal{N} = 4$ we have the exceptional superalgebras $D(2, 1; \alpha)$, depending on the complex parameter $\alpha \neq 0, -1$ and $A(1, 1) = sl(2|2)/\mathbb{Z}$ (it can be recovered for $\alpha = 0, -1$). Four distinct simple Lie superalgebras exist for $\mathcal{N} = 8$: $A(3, 1)$, $D(4, 1)$, $D(2, 2)$ and the exceptional superalgebra $F(4)$.

The exceptional superalgebras $D(2, 1; \alpha)$, $D(2, 1; \alpha')$ are isomorphic iff α' belongs to an S_3-group orbit generated by the moves $\alpha \mapsto \frac{1}{\alpha}$ and $\alpha \mapsto -(1 + \alpha)$, i.e. if α' takes one of the six values

$$\alpha, \quad \tfrac{1}{\alpha}, \quad -(1+\alpha), \quad -\tfrac{1}{(1+\alpha)}, \quad -\tfrac{(1+\alpha)}{\alpha}, \quad -\tfrac{\alpha}{(1+\alpha)}. \tag{2.35}$$

The (homogeneous and parabolic) D-module reps of the above $d = 1$ superconformal algebras have been constructed in [2] (the $\mathcal{N} = 1, 2, 4$ cases and one $\mathcal{N} = 8$ example) and [3] (the remaining $\mathcal{N} = 8$ cases). The construction relies upon the classification, presented in [32] and [33], of the $d = 1$ superPoincaré (the $\mathcal{G}_{>0}$ subalgebra) D-module reps.

Concerning the $d = 1$ superPoincaré D-module reps for $\mathcal{N} = 1, 2, 4, 8$, the results can be summarized as follows. The differential operators act on $\mathcal{N}$ bosonic and $\mathcal{N}$ fermionic fields (the supermultiplet). For any $k = 0, 1, \ldots, \mathcal{N}$, we have k fields with scaling dimension λ (they are known as the "propagating bosons"), $\mathcal{N}$ fields (the fermions) with scaling dimension $\lambda + \frac{1}{2}$ and the remaining $\mathcal{N} - k$ fields (the so-called "auxiliary bosons") with scaling dimension $\lambda + 1$. Both a supermultiplet and the associated $d = 1$ superPoincaré D-module rep will be denoted with the symbol "$(k, \mathcal{N}, \mathcal{N} - k)_\lambda$".

The extension to a $d = 1$ superconformal algebra D-module rep requires introducing (in compatible way, so that to close the (anti)commutation relations) the extra differential operators associated to the $\mathcal{G}_{\leq 0}$ generators.

The [2] and [3] results can be summarized as follows:
i) for $\mathcal{N} = 1, 2$ and any value of the scaling dimension λ (no criticality), the $(k, \mathcal{N}, \mathcal{N} - k)_\lambda$ supermultiplet induces a D-module rep for $osp(1|2)$ and $sl(2|1)$, respectively;
ii) for $\mathcal{N} = 4$ the $(k, \mathcal{N}, \mathcal{N} - k)_\lambda$ supermultiplet induces a D-module rep for the $D(2, 1; \alpha)$ superalgebra with the identification

$$\alpha = (2 - k)\lambda \tag{2.36}$$

(since α, up to the (2.35) relations, parametrizes inequivalent superalgebras, we already encounter here the criticality of the scaling dimension);
iii) for $\mathcal{N} = 8$ the $(k, \mathcal{N}, \mathcal{N} - k)_\lambda$ supermultiplet induces a D-module rep for a superconformal algebra only for $k \neq 4$ and at the critical scaling dimensions

$$\lambda \equiv \lambda_k = \tfrac{1}{k-4}; \tag{2.37}$$

the given superalgebras are $D(4,1)$for $k=0,8$, $F(4)$ for $k=1,7$, $A(3,1)$ for $k=2,6$ and $D(2,2)$ for $k=3,5$.

The D-module reps for the $\mathcal{N}=4$ $d=1$ superconformal algebra $A(1,1)$ (recovered from the $\alpha=0,-1$ values) are, in particular, obtained at the critical values

$$\lambda=0 \qquad \text{and} \qquad \lambda=\frac{1}{k-2} \quad (k\neq 2) \tag{2.38}$$

for, respectively, the supermultiplets

$$(k,4,4-k)_{\lambda=0} \quad \forall k=0,1,2,3,4 \quad \text{and} \quad (k,4,4-k)_{\lambda=\frac{1}{k-2}}, \quad k\neq 2. \tag{2.39}$$

The singular transformation, discussed in Section **2**, which relates the parabolic and hyperbolic D-module reps of $sl(2)$ (producing, in particular, the (2.13) equality between the respective scaling dimensions) is applicable in all supersymmetric cases. As a consequence, homogeneous hyperbolic D-module reps, with the same criticality properties of the corresponding parabolic cases, are immediately obtained for all the above listed superalgebras.

For what concerns the supersymmetric extensions of the Virasoro algebras, it is known [34] that non-trivial central charges can only exist up to $\mathcal{N}=4$. Since in the following we are dealing with D-module reps, here we only need to consider the *centerless* ($c=0$) $\mathcal{N}=1,2,4$ superVirasoro algebras which generalize the Witt algebra.

The centerless $\mathcal{N}=4$ superVirasoro algebra is spanned by the even generators L_n, J^i_n and by the odd generators Q^I_r, where $I=0,1,2,3$ and $i=1,2,3$. The centerless $\mathcal{N}=1,2$ superVirasoro algebras are its subalgebras, obtained by restricting $I=0,1$ and $i=1$ for $\mathcal{N}=2$ and $I=0$ for $\mathcal{N}=1$ (the latter case includes only the L_n, Q^0_r generators). Two variants of the superalgebras exist [35], the Ramond (R) and the Neveu-Schwarz (NS) versions. In both cases n is an integer ($n\in\mathbb{Z}$); in the Ramond version r is also an integer ($r\in\mathbb{Z}$), while in the Neveu-Schwarz version r is a half-integer number ($r\in\frac{1}{2}+\mathbb{Z}$).

The (anti)commutators of the centerless $\mathcal{N}=4$ superVirasoro algebra are explicitly given by

$$\begin{aligned}
[L_m,L_n] &= (m-n)L_{m+n},\\
[L_n,Q^I_r] &= (\frac{n}{2}-r)Q^I_{n+r},\\
[L_n,J^i_m] &= -mJ^i_{n+m},\\
\{Q^0_r,Q^0_s\} &= 2L_{r+s},\\
\{Q^0_r,Q^i_s\} &= 2(r-s)J^i_{r+s},\\
\{Q^i_r,Q^j_s\} &= 2\delta^{ij}L_{r+s}+2\epsilon^{ijk}(r-s)J^k_{r+s},\\
[Q^0_r,J^i_n] &= \frac{1}{2}Q^i_{n+r},\\
[Q^i_r,J^j_n] &= -\frac{1}{2}\delta^{ij}Q^0_{n+r}-\frac{1}{2}\epsilon^{ijk}Q^k_{n+r},\\
[J^i_n,J^j_m] &= -\epsilon^{ijk}J^k_{n+m}.
\end{aligned} \tag{2.40}$$

The finite $d=1$ $\mathcal{N}=4$ superconformal algebra $A(1,1)$ is recovered as a subalgebra.

In the Ramond version the $A(1,1)$ generators are $L_0, L_{\pm 2}, Q^I_{\pm 1}, J^i_0$;

in the Neveu-Schwarz version they are $L_{\pm 1}, L_0, Q^I_{\pm\frac{1}{2}}, J^i_0$.

The $osp(1|2)$ subalgebra is given by the generators

$L_0, L_{\pm 1}, Q^0_{\pm\frac{1}{2}}$ (NS) or $L_0, L_{\pm 2}, Q^0_{\pm 1}$ (R).

The $sl(2|1)$ subalgebra is given by the generators

$L_0, L_{\pm 1}, Q^0_{\pm\frac{1}{2}}, Q^1_{\pm\frac{1}{2}}, J^1_0$ (NS) or $L_0, L_{\pm 2}, Q^0_{\pm 1}, Q^1_{\pm 1}, J^1_0$ (R).

An extra centerless superVirasoro case which does not fit into the above scheme and contains an extra set of odd generators (W_r) is given by the $\mathcal{N}=3$ extension. The even generators of the $\mathcal{N}=3$ centerless superVirasoro are L_n

and J^i_n, while the odd generators are Q^i_r and W_r $(i=1,2,3)$. The (anti)commutators are explicitly given by

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n},\\
[L_n, Q^i_r] &= (\frac{n}{2}-r)Q^i_{n+r},\\
[L_n, J^i_m] &= -mJ^i_{n+m},\\
[L_n, W_r] &= -(\frac{n}{2}+r)W_{n+r},\\
\{Q^i_r, Q^j_s\} &= 2\delta^{ij}L_{r+s}+2\epsilon^{ijk}(r-s)J^k_{r+s},\\
[Q^i_r, J^j_n] &= -\frac{1}{2}\epsilon^{ijk}Q^k_{n+r}-\frac{n}{2}\delta^{ij}W_{n+r},\\
\{Q^i_r, W_s\} &= 2J^i_{r+s},\\
[J^i_n, J^j_m] &= -\frac{1}{2}\epsilon^{ijk}J^k_{n+m},\\
[J^i_n, W_r] &= 0,\\
\{W_r, W_s\} &= 0.
\end{aligned} \tag{2.41}$$

The finite subalgebra consisting of the twelve generators $L_0, L_{\pm1}, J^i_0, Q^i_{\pm\frac{1}{2}}$ (please note the absence of the W_r's generators) is the $d=1$ $\mathcal{N}=3$ superconformal algebra $B(1,1)=osp(3|2)$.

In [3] a D-module rep for $B(1,1)$ was constructed. It acts on the $(1,3,3,1)$ supermultiplet which contains one bosonic field of scaling dimension λ, three fermionic fields of scaling dimension $\lambda+\frac{1}{2}$, three bosonic fields of scaling dimension $\lambda+1$ and one fermionic field of scaling dimension $\lambda+\frac{3}{2}$. This D-module rep (existing for an arbitrary λ) is non-critical.

2.7 New results for superconformal D-module reps

In this Section we concentrate all new results concerning D-module reps of superconformal algebras. Our analysis heavily used algebraic computations with *Mathematica*.

Two classes of results are presented. At first we extend the construction presented in [2] and [3] of the *homogeneous* D-module reps to the case of the *inhomogeneous* D-module reps of the $d=1$ finite superconformal algebras (the ones we introduced in Section **6**). Next, we extend the [2] and [3] results to the case of D-module reps (both *homogeneous* and *inhomogeneous*) of the centerless $\mathcal{N}=1,2,3,4$ superVirasoro algebras.

The explicit presentation of these D-module reps is given in Appendix **B**.

As discussed in Appendix **A**, the new class of inhomogeneous D-module reps are obtained for $\lambda=0$ and $\rho\neq0$.

Since the presence of at least a propagating boson is required to construct the inhomogeneous term, the inhomogeneous supermultiplets $(k,\mathcal{N},\mathcal{N}-k)_{\lambda=0,\rho\neq0}$ can only exist for $k\geq1$.

The list of inhomogeneous D-module reps for the finite $d=1$ superconformal algebras of Section **6** is given by

$$\begin{array}{lcll}
\mathcal{N}=1 & : & osp(1|2) & \text{with} \quad (1,1)_{0,\rho},\\
\mathcal{N}=2 & : & sl(2|1) & \text{with} \quad (1,2,1)_{0,\rho}, \quad (2,2)_{0,\rho},\\
\mathcal{N}=3 & : & B(1,1) & \text{with} \quad (1,3,3,1)_{0,\rho},\\
\mathcal{N}=4 & : & A(1,1) & \text{with} \quad (1,4,3)_{0,\rho}, \quad (2,4,2)_{0,\rho}, \quad (3,4,1)_{0,\rho}, \quad (4,4,0)_{0,\rho},\\
\mathcal{N}=8 & : & \text{none} &
\end{array} \tag{2.42}$$

(the last result is a consequence of the fact that, for $\mathcal{N}=8$, the $d=1$ finite superconformal algebras have critical scalings $\lambda\neq0$).

Concerning the centerless superVirasoro algebras, the homogeneous supermultiplets are encountered for

$$\begin{array}{lll}
\mathcal{N}=1 \quad \text{SVir}: & (k,1,1-k)_\lambda, & k=0,1 \quad \text{with} \quad \lambda \quad \text{arbitrary},\\
\mathcal{N}=2 \quad \text{SVir}: & (k,2,2-k)_\lambda, & k=0,1,2 \quad \text{with} \quad \lambda \quad \text{arbitrary},\\
\mathcal{N}=3 \quad \text{SVir}: & (1,3,3,1)_\lambda, & \quad \text{with} \quad \lambda \quad \text{arbitrary},\\
\mathcal{N}=4 \quad \text{SVir}: & (k,4,4-k)_\lambda, & k=0,1,2,3,4 \quad \text{with} \quad \lambda=0 \quad \text{or} \quad \lambda=\dfrac{1}{k-2}(k\neq2).
\end{array} \tag{2.43}$$

The inhomogeneous D-module reps of the centerless superVirasoro algebras are only encountered for $\mathcal{N}=1,2,3$ but not for $\mathcal{N}=4$:

$$\begin{array}{lll}\mathcal{N}=1 & \text{SVir}: & (1,1)_{0,\rho},\\ \mathcal{N}=2 & \text{SVir}: & (2,2,0)_{0,\rho} \quad \text{and} \quad (1,2,1)_{0,\rho},\\ \mathcal{N}=3 & \text{SVir}: & (1,3,3,1)_{0,\rho},\\ \mathcal{N}=4 & \text{SVir}: & \text{none.}\end{array} \tag{2.44}$$

It is instructive to show the reason for the absence of the inhomogeneous D-module reps for the centerless $\mathcal{N}=4$ superVirasoro. It is due to the fact that, in particular, the closure of the algebra requires the commutators $[J_n^3, Q_r^3]$ to be proportional to the Q_{n+r}^4 generators. Let's take, as an example, the $(4,4,0)_{\lambda,\rho}$ supermultiplet. We are led to a system of equations to be solved:

$$\begin{array}{rcl} 2\lambda-\frac{1}{2} &=& A,\\ (\frac{1}{2}-2\lambda)\partial_t+\lambda(-n+r(1-4\lambda)) &=& A(-\partial_t-2(n+r)\lambda),\\ -(n+r(4\lambda-1))\rho &=& -2A(n+r)\rho, \end{array} \tag{2.45}$$

where A is a proportionality constant.

To solve the system for all n,r, either one has to set $\lambda=\frac{1}{2}$ and ρ arbitrary (which is equivalent to the homogeneous representation $(4,4,0)_{\frac{1}{2}}$) or $\lambda=\rho=0$.

By restricting the conditions to $n=0$ and $r=\pm\overline{r}$ (the case of the $A(1,1)$ subalgebra), the system is solved for arbitrary values of λ and ρ.

The inspection of the consistency conditions induced by all (anti)commutators leads to the results that we have presented in this Section.

It is worth pointing out, as a last comment, that the inhomogeneous D-module reps discussed here consist of a different and inequivalent class of linear transformations with respect to the inhomogenous D-module reps discussed in [2] and [3]. There, the $sl(2)$ generators act homogeneously and the representations are only obtained at the critical value $\lambda=-1$.

2.8 Some examples of $\mathcal{N}=1,2$ superconformal actions in $d=1,2$ dimensions

We illustrate here an application of supersymmetry with the construction of some $\mathcal{N}=1,2$ superconformal actions in $d=1,2$ dimensions.

In $d=1$ we obtain the following $osp(1|2)$-invariant actions for the $\mathcal{N}=1$ supermultiplet $(1,1)$ (a single boson φ and a single fermion ψ):

I - for the *homogeneous parabolic* case the Lagrangian is

$$\mathcal{L} = C\varphi^{-\frac{1+2\lambda}{\lambda}}(\dot{\varphi}^2+\psi\dot{\psi}), \tag{2.46}$$

with dimensions $[\varphi]=\lambda$, $[\psi]=\lambda+\frac{1}{2}$, $[C]=[\lambda]=0$;
II - for the *inhomogeneous parabolic* case the Lagrangian is

$$\mathcal{L} = Ce^{-\frac{\varphi}{\rho}}(\dot{\varphi}^2+\psi\dot{\psi}), \tag{2.47}$$

with dimensions $[\varphi]=[\rho]=s$, $[\psi]=s+\frac{1}{2}$, $[C]=-1-2s$;
III - for the *homogeneous hyperbolic* case the Lagrangian is

$$\mathcal{L} = C[\varphi^{-\frac{1+2\lambda}{\lambda}}(\dot{\varphi}^2+\mu\psi\dot{\psi})+\mu^2\lambda^2\varphi^{-\frac{1}{\lambda}}], \tag{2.48}$$

with dimensions $[\varphi]=[\psi]=\lambda$, $[\mu]=1$, $[C]=[\lambda]=0$;
IV - for the *inhomogeneous hyperbolic* case the Lagrangian is

$$\mathcal{L} = Ce^{-\frac{\varphi}{\rho}}(\dot{\varphi}^2+\mu\psi\dot{\psi}+\mu^2\rho^2), \tag{2.49}$$

with dimensions $[\varphi]=[\psi]=[\rho]=s$, $[\mu]=1$, $[C]=-1-2s$.

We note a similarity and a difference with respect to the purely bosonic results. Just like the bosonic actions, the hyperbolic cases present a potential term, proportional to $C_1\mu^2$, which is absent in the parabolic cases. On the other hand, the potential terms proportional to C_2 and appearing in (2.24-2.27) are now excluded due to the supersymmetry constraint.

In $d=2$, for a single boson φ and a single fermion ψ, we obtain $\mathcal{N}=1$ chiral (antichiral) actions, invariant under a *single* (*chiral/antichiral*) *copy* of the centerless superVirasoro algebra. The Lagrangians are given by

$$\mathcal{L}^{\pm} = \frac{C}{\varphi^2}(\varphi_+\varphi_- + \psi\psi_{\mp}), \tag{2.50}$$

for the homogeneous case and

$$\mathcal{L}^{\pm} = C(\varphi_+\varphi_- + \psi\psi_{\mp}) \tag{2.51}$$

(the constant kinetic term) for the inhomogeneous case.

It should be pointed out that, in order to get the superLiouville extension, a second fermion needs to be added, see [36] and [37].

As an $\mathcal{N}=2$ example in $d=1$ we present the $sl(2|1)$-invariant actions for the supermultiplet $(1,2,1)$ (a propagating boson φ, two fermions ψ_1,ψ_2 and an auxiliary bosonic field g). The Lagrangians are:

I - for the *homogeneous parabolic* case

$$\mathcal{L} = A(\dot\varphi^2 + \psi_i\dot\psi_i + g^2) - \frac{1}{2}A_\varphi\epsilon^{ij}\psi_i\psi_j g, \qquad A = C\varphi^{-\frac{1+2\lambda}{\lambda}}; \tag{2.52}$$

II - for the *inhomogeneous parabolic* case

$$\mathcal{L} = A(\dot\varphi^2 + \psi_i\dot\psi_i + g^2) - \frac{1}{2}A_\varphi\epsilon^{ij}\psi_i\psi_j g, \qquad A = Ce^{-\frac{\varphi}{\rho}}; \tag{2.53}$$

III - for the *homogeneous hyperbolic* case

$$\mathcal{L} = A(\dot\varphi^2 + \mu\psi_i\dot\psi_i + \mu^2g^2) - \frac{1}{2}\mu^2A_\varphi\epsilon^{ij}\psi_i\psi_j g + \mu^2\lambda^2A\varphi^2, \qquad A = C\varphi^{-\frac{1+2\lambda}{\lambda}}; \tag{2.54}$$

IV - for the *inhomogeneous hyperbolic* case

$$\mathcal{L} = A(\dot\varphi^2 + \mu\psi_i\dot\psi_i + \mu^2g^2) - \frac{1}{2}\mu^2A_\varphi\epsilon^{ij}\psi_i\psi_j g + \mu^2\rho^2A, \qquad A = Ce^{-\frac{\varphi}{\rho}}. \tag{2.55}$$

2.9 On $\mathcal{N}=4$ $d=1$ superconformal actions with exceptional $D(2,1;\alpha)$ invariance

The supermultiplet $(1,4,3)_\lambda$ consists of a single propagating boson φ, four fermions ψ_0,ψ_i and three auxiliary bosons g_i (here $i=1,2,3$; in the formulas below we also use the index $I=0,1,2,3$). For this supermultiplet, see formula (2.36), we have $\alpha=\lambda$. We list here its superconformally invariant actions.
For homogeneous transformations the Lagrangians of the $D(2,1;\alpha)$-invariant actions are,
in the *homogeneous parabolic* case,

$$\begin{aligned}\mathcal{L} &= A(\dot\varphi^2 + \psi_I\dot\psi_I + g_i^2) + A_\varphi(\psi_0\psi_i g_i + \frac{1}{2}\epsilon^{ijk}\psi_i\psi_j g_k) + \frac{1}{6}A_{\varphi\varphi}\epsilon^{ijk}\psi_0\psi_i\psi_j\psi_k,\\ &\text{with}\quad A = C\varphi^{-\frac{1+2\alpha}{\alpha}}\end{aligned} \tag{2.56}$$

and, in the *homogeneous hyperbolic* case,

$$\begin{aligned}\mathcal{L} &= A(\dot\varphi^2 + \mu\psi_I\dot\psi_I + \mu^2g_i^2) + \mu^2A_\varphi(\psi_0\psi_i g_i + \frac{1}{2}\epsilon^{ijk}\psi_i\psi_j g_k) +\\ &\quad \frac{1}{6}\mu^2A_{\varphi\varphi}\epsilon^{ijk}\psi_0\psi_i\psi_j\psi_k + \mu^2\alpha^2A\varphi^2,\\ &\text{with}\quad A = C\varphi^{-\frac{1+2\alpha}{\alpha}}.\end{aligned} \tag{2.57}$$

For inhomogeneous transformations, the requirement that $\rho\neq 0$ with $\lambda=0$ implies that the superconformal actions are only invariant under the $A(1,1)$ superalgebra.

For inhomogeneous transformations the Lagrangians of the $A(1,1)$-invariant actions are,
in the *inhomogeneous parabolic* case,

$$\begin{aligned}\mathcal{L} &= A(\dot{\varphi}^2+\psi_I\dot{\psi}_I+g_i^2)+A_\varphi(\psi_0\psi_i g_i+\frac{1}{2}\epsilon^{ijk}\psi_i\psi_j g_k)+\frac{1}{6}A_{\varphi\varphi}\epsilon^{ijk}\psi_0\psi_i\psi_j\psi_k,\\ &\text{with}\quad A=Ce^{-\frac{\varphi}{\rho}}\end{aligned}\tag{2.58}$$

and, in the *inhomogeneous hyperbolic* case,

$$\begin{aligned}\mathcal{L} &= A(\dot{\varphi}^2+\mu\psi_I\dot{\psi}_I+\mu^2g_i^2)+\mu^2A_\varphi(\psi_0\psi_i g_i+\frac{1}{2}\epsilon^{ijk}\psi_i\psi_j g_k)+\\ &\quad\frac{1}{6}\mu^2A_{\varphi\varphi}\epsilon^{ijk}\psi_0\psi_i\psi_j\psi_k+\mu^2\rho^2A,\\ &\text{with}\quad A=Ce^{-\frac{\varphi}{\rho}}.\end{aligned}\tag{2.59}$$

2.10 Conclusions

We summarize the results of the paper. For what concerns representations, besides the results on the bosonic $sl(2)$ and Witt algebras, we introduced the new class of linear inhomogeneous D-module reps for the finite simple Lie superalgebras $osp(1|2)$, $sl(2|1)$, $B(1,1)$ and $A(1,1)$. These new reps, at the scaling dimension $\lambda=0$, depend on the parameter ρ which measures the inhomogeneity.

D-module reps have also been constructed for the centerless superVirasoro algebras: homogeneous reps for the $\mathcal{N}=1,2,3,4$ extensions and inhomogeneous reps for the $\mathcal{N}=1,2,3$ extensions. They are based on the $(k,\mathcal{N},\mathcal{N}-k)$ supermultiplets (for $\mathcal{N}=1,2,4$) and on the $(1,3,3,1)$ supermultiplet (for $\mathcal{N}=3$).

We pointed out that, for both homogeneous and inhomogeneous reps, two variants of the D-module reps can be presented: parabolic and hyperbolic/trigonometric. They are mutually related by a singular transformation.

We ended up with four different types of active (super)conformal transformations (*hom. par.*, *inhom. par.*, *hom. hyp.* and *inhom. hyp.*) that can be used, in the Lagrangian setting, to construct (super)conformal actions in $d=1$ and $d=2$ dimensions.

For systems with $\mathcal{N}=0,1,2,4$ we presented the four types of $d=1$ actions invariant under their respective finite (super)conformal algebra.

In $d=2$ a non-linear field redefinition relates the two classes of homogeneous and inhomogeneous conformal actions exemplified by (2.31) and by the Liouville action (2.32).

The $d=1$ (super)conformal actions contain no dimensional constants only in the homogeneous parabolic case. This is the class of theories discussed in the [14] review. New classes of superconformally invariant theories can therefore be constructed from the three remaining types of transformations.

It is rather straightforward to extend the results here presented to more complicated cases. For $\mathcal{N}=4$, for instance, one can investigate multi-particle systems by applying our construction to a certain number of supermultiplets in interactions. The provision is that the supermultiplets should carry a representation of the same superconformal algebra (either $A(1,1)$ or $D(2,1;\alpha)$ for a fixed α).

One of the possible interesting applications of our work concerns the investigations on the $CFT(1)/AdS(2)$ correspondence (see [38] and [39]). Our results shed a new light on the left side (the conformal side) of the correspondence.

A very promising field of investigation concerns the extension to non-relativistic conformal Galilei or conformal Newton-Hooke systems (see [40, 41]). Recently, a lot of activity in constructing models for this kind of theories has been motivated by the CFT/AdS correspondence applied to non-relativistic systems like the ones appearing in condensed matter (see, e.g., [42, 43]). Unlike the $(1+0)$-dimensional theories considered here, these conformal systems live in $(1+d)$-dimension, d being the number of space coordinates. A recent paper [44] proved how the (homogeneous parabolic) $\mathcal{N}=2$ superconformal D-module reps in $(1+0)$ can be enlarged to induce $\mathcal{N}=2$ ℓ-conformal Galilei superalgebras in $(1+d)$ dimensions. It is tempting to extend the new class of one-dimensional superconformal D-module reps discussed here to the $(1+d)$-dimensional case.

Appendix A: On D-module reps and the interpolation of *Hom* and *Inhom* conformal actions

In principle one can “mix" the homogeneous and inhomogeneous D-module reps of the Witt algebra by allowing the couple of parameters (λ,ρ) being simultaneously non-vanishing. In the parabolic case, for instance, the general Witt algebra transformations applied on the field φ are

$$L_m^{par}(\varphi) = -t^{m+1}\dot{\varphi}-\lambda(m+\gamma)t^m\varphi-\rho(m+\beta)t^m.\tag{A.1}$$

L_{-1}^{par} is proportional to the Hamiltonian if we set $\gamma = 1$ and $\beta = 1$.

For $\lambda \neq 0$ we can write

$$L_m^{par}(\varphi) \quad = \quad -t^{m+1}\dot{\varphi} - \lambda(m+1)t^m(\varphi + \frac{\rho}{\lambda}), \tag{A.2}$$

so that the action of the homogeneous transformation with scaling dimension $\lambda \neq 0$ is recovered for the shifted field $\overline{\varphi} = \varphi + \frac{\rho}{\lambda}$. Therefore the (λ, ρ) transformations with $\lambda \neq 0$ are equivalent to the pure homogeneous transformations with scaling parameter λ and $\rho = 0$. The same is true in the hyperbolic case.

(A.1) fails to interpolate the two cases, leaving us with the two inequivalent classes of

i) $(\lambda, 0)$ homogeneous and

ii) $(0, \rho)$ inhomogeneous transformations.

The (A.1) transformations, on the other hand, are useful to interpolate the conformally invariant actions. An $sl(2)$-invariant action (for $m = \pm 1, 0$) based on (A.1) is given by the Lagrangian

$$\mathcal{L} \quad = \quad K_1(\lambda\varphi + \rho)^{-\frac{1+2\lambda}{\lambda}}\dot{\varphi}^2 + K_2(\lambda\varphi + \rho)^{\frac{1}{\lambda}}, \tag{A.3}$$

with K_1, K_2 arbitrary constants.

The homogeneous Lagrangian (2.24) is recovered for $\rho = 0$.

The inhomogeneous Lagrangian (2.25) is recovered in the $\lambda \to 0$ limit by suitably rescaling the constants K_1, K_2. This is accomplished by expressing the (A.3) Lagrangian as

$$\mathcal{L} = K_1\rho^{-\frac{1+2\lambda}{\lambda}}\left(1 + \frac{\lambda\varphi}{\rho}\right)^{-\frac{1}{\lambda}-2}\dot{\varphi}^2 + K_2\rho^{\frac{1}{\lambda}}\left(1 + \frac{\lambda\varphi}{\rho}\right)^{\frac{1}{\lambda}}$$

and by taking $K_1 = C_1\rho^{\frac{1+2\lambda}{\lambda}}$ and $K_2 = C_2\rho^{-\frac{1}{\lambda}}$.

The possibility offered by the interpolation allows simplifying the constructions of the homogeneous and inhomogeneous conformally invariant actions, since both actions can be derived at one stroke.

The extension of the properties here discussed to the supersymmetric cases is immediate.

Appendix B: Explicit presentation of the new supersymmetric D-module reps

We present here, for completeness, the explicit constructions of the new D-module reps introduced in Section **7**. They are

i) the *inhomogeneous* D-module reps of the finite $d = 1$ superconformal algebras $osp(1|2)$, $sl(2|1)$, $B(1,1)$, $A(1,1)$ and

ii) the (both *homogeneous* and *inhomogeneous*) D-module reps of the centerless $\mathcal{N} = 1, 2, 3, 4$ superVirasoro algebras.

The D-module reps with $\mathcal{N} = 1, 2, 4$ act on the $(\mathcal{N}+1|\mathcal{N})$ supermultiplets m, $m^T = (\varphi_1, \ldots, \varphi_k, g_1, \ldots, g_{\mathcal{N}-k}, 1|\psi_1, \ldots, \psi_{\mathcal{N}})$, with component fields $\varphi_a, g_i, \psi_\alpha$ and constant entry 1 in the $(\mathcal{N}+1)$-th position. The $\mathcal{N} = 3$ D-module rep acts on a $(5|4)$ supermultiplet with 1 in the 5-th position. The homogeneous D-module reps are recovered by deleting the row and the column associated with the constant entry 1 in the supermultiplet.

For $\mathcal{N} = 1$, in matrix form and in the hyperbolic presentation, we can write for the centerless superVirasoro generators

$$\begin{aligned} Q_r^0 \quad &= \quad e^{rt}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -\partial_t - 2r\lambda & -2r\rho & 0 \end{pmatrix}, \\ L_n \quad &= \quad e^{nt}\begin{pmatrix} -\partial_t - n\lambda & -n\rho & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\partial_t - \frac{1}{2}n(1+2\lambda) \end{pmatrix}. \end{aligned} \tag{B.1}$$

The inhomogeneous D-module rep of $osp(1|2)$ is recovered for $n = 0, \pm 1$, $r = \pm\frac{1}{2}$ and by setting $\lambda = 0$.

To save space, in the remaining cases we limit to present here the odd generators (the even generators are recovered, see (2.40), from their anticommutators) and write them in terms of the E_{ij} matrices, whose entries are 1 in the i-th row, j-th column and vanishing otherwise. We have,

for $\mathcal{N}=2$:
the $(2,2,0)$ rep is constructed from

$$\begin{array}{rcl} Q_r^0 &=& e^{rt}[E_{14}+E_{25}-2(E_{43}+E_{53})r\rho-(E_{41}+E_{52})(\partial_t+2r\lambda)],\\ Q_r^1 &=& e^{rt}[E_{15}-E_{24}+2(E_{43}-E_{53})r\rho+(E_{42}-E_{51})(\partial_t+2r\lambda)]; \end{array} \quad (B.2)$$

the $(1,2,1)$ rep is constructed from

$$\begin{array}{rcl} Q_r^0 &=& e^{rt}[E_{14}+E_{52}-2E_{43}r\rho-E_{25}r-(E_{25}+E_{41})(\partial_t+2r\lambda)],\\ Q_r^1 &=& e^{rt}[E_{15}-E_{42}-2E_{53}r\rho+E_{24}r+(E_{24}-E_{51})(\partial_t+2r\lambda)]; \end{array} \quad (B.3)$$

the $(0,2,2)$ rep is constructed from

$$\begin{array}{rcl} Q_r^0 &=& e^{rt}[E_{41}+E_{52}-(E_{14}+E_{25})(\partial_t+r+2r\lambda)],\\ Q_r^1 &=& e^{rt}[E_{51}-E_{42}+(E_{24}-E_{15})(\partial_t+r+2r\lambda)]; \end{array} \quad (B.4)$$

for $\mathcal{N}=4$:
the $(4,4,0)$ rep is constructed from

$$\begin{array}{rcl} Q_r^0 &=& e^{rt}[E_{16}+E_{27}+E_{38}+E_{49}-2(E_{65}+E_{75}+E_{85}+E_{95})r\rho\\ && -(E_{61}+E_{72}+E_{83}+E_{94})(\partial_t+2r\lambda)],\\ Q_r^1 &=& e^{rt}[E_{17}-E_{26}-E_{39}+E_{48}+2(E_{65}-E_{75}-E_{85}+E_{95})r\rho\\ && +(E_{62}-E_{71}-E_{84}+E_{93})(\partial_t+2r\lambda)],\\ Q_r^2 &=& e^{rt}[E_{18}+E_{29}-E_{36}-E_{47}+2(E_{65}+E_{75}-E_{85}-E_{95})r\rho\\ && +(E_{63}+E_{74}-E_{81}-E_{92})(\partial_t+2r\lambda)],\\ Q_r^3 &=& e^{rt}[E_{19}-E_{28}+E_{37}-E_{46}+2(E_{65}-E_{75}+E_{85}-E_{95})r\rho\\ && +(E_{64}-E_{73}+E_{82}-E_{91})(\partial_t+2r\lambda)]; \end{array} \quad (B.5)$$

the $(3,4,1)$ rep is constructed from

$$\begin{array}{rcl} Q_r^0 &=& e^{rt}[E_{16}+E_{27}+E_{38}+E_{94}-2(E_{65}+E_{75}+E_{85})r\rho-E_{49}r\\ && -(E_{49}+E_{61}+E_{72}+E_{83})(\partial_t+2r\lambda)],\\ Q_r^1 &=& e^{rt}[E_{17}-E_{26}-E_{39}+E_{84}+2(E_{65}-E_{75}+E_{95})r\rho-E_{48}r\\ && +(E_{62}-E_{48}-E_{71}+E_{93})(\partial_t+2r\lambda)],\\ Q_r^2 &=& e^{rt}[E_{18}+E_{29}-E_{36}-E_{74}+2(E_{65}-E_{85}-E_{95})r\rho+E_{47}r\\ && +(E_{47}+E_{63}-E_{81}-E_{92})(\partial_t+2r\lambda)],\\ Q_r^3 &=& e^{rt}[E_{19}-E_{28}+E_{37}-E_{64}+2(E_{85}-E_{75}-E_{95})r\rho+E_{46}r\\ && +(E_{46}-E_{73}+E_{82}-E_{91})(\partial_t+2r\lambda)]; \end{array} \quad (B.6)$$

the $(2,4,2)$ rep is constructed from

$$\begin{array}{rcl} Q_r^0 &=& e^{rt}[E_{16}+E_{27}+E_{83}+E_{94}-2(E_{65}+E_{75})r\rho-(E_{38}+E_{49})r\\ && -(E_{38}+E_{49}+E_{61}+E_{72})(\partial_t+2r\lambda)],\\ Q_r^1 &=& e^{rt}[E_{17}-E_{26}+E_{84}-E_{93}+2(E_{65}-E_{75})r\rho+(E_{39}-E_{48})r\\ && +(E_{39}-E_{48}+E_{62}-E_{71})(\partial_t+2r\lambda)],\\ Q_r^2 &=& e^{rt}[E_{18}+E_{29}-E_{63}-E_{74}-2(E_{85}+E_{95})r\rho+(E_{36}+E_{47})r\\ && +(E_{36}+E_{47}-E_{81}-E_{92})(\partial_t+2r\lambda)],\\ Q_r^3 &=& e^{rt}[E_{19}-E_{28}-E_{64}+E_{73}+2(E_{85}-E_{95})r\rho+(E_{46}-E_{37})r\\ && +(E_{46}-E_{37}+E_{82}-E_{91})(\partial_t+2r\lambda)]; \end{array} \quad (B.7)$$

the $(1,4,3)$ rep is constructed from

$$\begin{array}{rcl}
Q_r^0 &=& e^{rt}[E_{16}+E_{72}+E_{83}+E_{94}-2E_{65}r\rho\\
&& -(E_{27}+E_{38}+E_{49})r-(E_{27}+E_{38}+E_{49}+E_{61})(\partial_t+2r\lambda)],\\
Q_r^1 &=& e^{rt}[E_{17}-E_{62}+E_{84}-E_{93}-2E_{75}r\rho+(E_{26}+E_{39}-E_{48})r\\
&& +(E_{26}+E_{39}-E_{48}-E_{71})(\partial_t+2r\lambda)],\\
Q_r^2 &=& e^{rt}[E_{18}-E_{63}-E_{74}+E_{92}-2E_{85}r\rho+(E_{36}-E_{29}+E_{47})r\\
&& +(E_{36}-E_{29}+E_{47}-E_{81})(\partial_t+2r\lambda)],\\
Q_r^3 &=& e^{rt}[E_{19}-E_{64}+E_{73}-E_{82}-2E_{95}r\rho+(E_{28}-E_{37}+E_{46})r\\
&& +(E_{28}-E_{37}+E_{46}-E_{91})(\partial_t+2r\lambda)];
\end{array} \tag{B.8}$$

the $(0,4,4)$ rep is constructed from

$$\begin{array}{rcl}
Q_r^0 &=& e^{rt}[E_{61}+E_{72}+E_{83}+E_{94}-(E_{16}+E_{27}+E_{38}\\
&& +E_{49})(\partial_t+r+2r\lambda)],\\
Q_r^1 &=& e^{rt}[E_{71}-E_{62}+E_{84}-E_{93}\\
&& +(E_{26}-E_{17}+E_{39}-E_{48})(\partial_t+r+2r\lambda)],\\
Q_r^2 &=& e^{rt}[E_{81}-E_{63}-E_{74}+E_{92}\\
&& +(E_{36}-E_{18}-E_{29}+E_{47})(\partial_t+r+2r\lambda)],\\
Q_r^3 &=& e^{rt}[E_{73}-E_{64}-E_{82}+E_{91}\\
&& +(E_{28}-E_{19}-E_{37}+E_{46})(\partial_t+r+2r\lambda)].
\end{array} \tag{B.9}$$

The above operators produce D-module reps for the respective superalgebras only at the critical values, which have been presented in Section **7**, for λ and ρ. The inhomogeneous D-module reps of the finite $d=1$ superconformal algebras are obtained for $r=\pm\frac{1}{2}$ and $\lambda=0$.

The $\mathcal{N}=3$ D-module rep of the centerless superVirasoro algebra (2.41) exists for arbitrary values of λ and ρ. At $\lambda=0$ the restriction to the $B(1,1)$ subalgebra generators produces the $(1,3,3,1)_{0,\rho}$ inhomogeneous D-module rep of $B(1,1)$.

To reconstruct the full D-module rep is sufficient to present the three Q^i's operators.

The $\mathcal{N}=3$ SuperVirasoro $(1,3,3,1)$ rep is obtained from

$$\begin{array}{rcl}
Q_r^1 &=& e^{rt}[E_{17}-E_{39}-E_{62}+E_{84}-2E_{75}r\rho\\
&& +(E_{26}-E_{48}+2E_{93})r+(E_{26}-E_{48}-E_{71}+E_{93})(\partial_t+2r\lambda)],\\
Q_r^2 &=& e^{rt}[E_{18}+E_{29}-E_{63}-E_{74}-2E_{85}r\rho\\
&& +(E_{36}+E_{47}-2E_{92})r+(E_{36}+E_{47}-E_{81}-E_{92})(\partial_t+2r\lambda)],\\
Q_r^3 &=& e^{rt}[E_{16}+E_{49}+E_{72}+E_{83}-2E_{65}r\rho-(E_{27}+E_{38}+2E_{94})r\\
&& -(E_{27}+E_{38}+E_{61}+E_{94})(\partial_t+2r\lambda)].
\end{array} \tag{B.10}$$

Appendix C: Homogeneous versus Inhomogeneous actions: the $\mathcal{N}=4$ cases revisited

The subtle issues of the relation of the homogeneous versus inhomogeneous actions is already encountered in systems with $\mathcal{N}=0,1,2$ supersymmetries. It is, however, of particular relevance for $\mathcal{N}=4$ models due to the criticality of the scaling dimension λ. Indeed, for the $(1,4,3)$ superconformal actions of Section **9**, λ coincides with α, the parameter specifying the superconformal symmetry algebra $D(2,1;\alpha)$.

One should note that the homogeneous actions (2.56) and (2.57) are not defined at $\alpha=0$ and that even their $\alpha\to 0$ limit produces a trivial vanishing Lagrangian ($\lim_{\alpha\to 0}\mathcal{L}(\alpha)=0$) if the normalization factor C is kept constant. On the other hand, the inhomogeneous actions (2.58) and (2.59) are defined at $\lambda=0$ (therefore, at $\alpha=0$).

In order to facilitate the comparison of the different actions it is convenient to bring them into a standard form, with the help of fields redefinitions. For this purpose we require the kinetic part of the action being expressed, in terms of the new fields, (denoted as $\overline{\varphi},\overline{\psi}_I,\overline{g}_i$, with $i=1,2,3$ and $I=0,1,2,3$) as a constant kinetic term. For this reason the new basis of fields will be called the "constant kinetic basis".

For the two homogeneous (both parabolic and hyperbolic) actions, the fields redefinition is accomplished by the transformations

$$
\begin{aligned}
\bar{\phi} &= -2\alpha\phi^{-\frac{1}{2\alpha}}, \\
\bar{\psi}_I &= \phi^{-\frac{1+2\alpha}{2\alpha}}\psi_I, \\
\bar{g}_i &= \phi^{-\frac{1+2\alpha}{2\alpha}}g_i.
\end{aligned}
\tag{C.1}
$$

For the two inhomogeneous (both parabolic and hyperbolic) actions, the fields redefinition is given by the transformations

$$
\begin{aligned}
\bar{\phi} &= -2\rho e^{-\frac{\phi}{2\rho}}, \\
\bar{\psi}_I &= e^{-\frac{\phi}{2\rho}}\psi_I, \\
\bar{g}_i &= e^{-\frac{\phi}{2\rho}}g_i.
\end{aligned}
\tag{C.2}
$$

The actions (2.56), (2.57), (2.58), (2.59), in their respective "constant kinetic basis", are expressed by

i) *Homogeneous parabolic case:*

$$
\begin{aligned}
\mathcal{L} \;=\;& C(\dot{\bar{\phi}}^2 + \bar{\psi}_I\dot{\bar{\psi}}_I + \bar{g}_i{}^2) + \frac{2(1+2\alpha)C}{\bar{\phi}}\left(\bar{\psi}_0\bar{\psi}_i\bar{g}_i + \frac{1}{2}\epsilon^{ijk}\bar{\psi}_i\bar{\psi}_j\bar{g}_k\right) + \\
& \frac{2(1+2\alpha)(1+3\alpha)C}{3\bar{\phi}^2}\epsilon^{ijk}\bar{\psi}_0\bar{\psi}_i\bar{\psi}_j\bar{\psi}_k,
\end{aligned}
\tag{C.3}
$$

ii) *Homogeneous hyperbolic case:*

$$
\begin{aligned}
\mathcal{L} \;=\;& C(\dot{\bar{\phi}}^2 + \mu\bar{\psi}_I\dot{\bar{\psi}}_I + \mu^2\bar{g}_i{}^2) + \frac{2(1+2\alpha)\mu^2 C}{\bar{\phi}}\left(\bar{\psi}_0\bar{\psi}_i\bar{g}_i + \frac{1}{2}\epsilon^{ijk}\bar{\psi}_i\bar{\psi}_j\bar{g}_k\right) + \\
& \frac{2(1+2\alpha)(1+3\alpha)\mu^2 C}{3\bar{\phi}^2}\epsilon^{ijk}\bar{\psi}_0\bar{\psi}_i\bar{\psi}_j\bar{\psi}_k + \frac{\mu^2 C}{4}\bar{\phi}^2,
\end{aligned}
\tag{C.4}
$$

iii) *Inhomogeneous parabolic case:*

$$
\begin{aligned}
\mathcal{L} \;=\;& C(\dot{\bar{\phi}}^2 + \bar{\psi}_I\dot{\bar{\psi}}_I + \bar{g}_i{}^2) + \frac{2C}{\bar{\phi}}\left(\bar{\psi}_0\bar{\psi}_i\bar{g}_i + \frac{1}{2}\epsilon^{ijk}\bar{\psi}_i\bar{\psi}_j\bar{g}_k\right) + \\
& \frac{2C}{3\bar{\phi}^2}\epsilon^{ijk}\bar{\psi}_0\bar{\psi}_i\bar{\psi}_j\bar{\psi}_k,
\end{aligned}
\tag{C.5}
$$

iv) *Inhomogeneous hyperbolic case:*

$$
\begin{aligned}
\mathcal{L} \;=\;& C(\dot{\bar{\phi}}^2 + \mu\bar{\psi}_I\dot{\bar{\psi}}_I + \mu^2\bar{g}_i{}^2) + \frac{2\mu^2 C}{\bar{\phi}}\left(\bar{\psi}_0\bar{\psi}_i\bar{g}_i + \frac{1}{2}\epsilon^{ijk}\bar{\psi}_i\bar{\psi}_j\bar{g}_k\right) + \\
& \frac{2\mu^2 C}{3\bar{\phi}^2}\epsilon^{ijk}\bar{\psi}_0\bar{\psi}_i\bar{\psi}_j\bar{\psi}_k + \frac{\mu^2 C}{4}\bar{\phi}^2.
\end{aligned}
\tag{C.6}
$$

We see that, in this new basis, the $\alpha \to 0$ limit of the homogeneous actions are well-defined and coincide with the actions obtained from the inhomogeneous transformations. The extra potential term of the hyperbolic case is an oscillator potential.

In the constant kinetic basis the superconformal transformations are realized non-linearly. The actions of the eight odd generators (the whole superconformal algebra is recovered through their repeated anticommutators) are given by (here $r = \pm\frac{1}{2}$; in the parabolic case $r = -\frac{1}{2}$ corresponds to the four supersymmetry transformations, while $r = \frac{1}{2}$ corresponds to their four superconformal partners):

- in the parabolic case (here $f_r = t^{r-\frac{1}{2}}$):

$$
\begin{aligned}
Q^i_r\bar\phi &= tf_r\bar\psi_i,\\
Q^i_r\bar g_j &= f_r\epsilon^{ijk}[t\dot{\bar\psi}_k + (2r+1)(\alpha+\frac{1}{2})\bar\psi_k] + f_r\delta^{ij}[t\dot{\bar\psi} + (2r+1)(\alpha+\frac{1}{2})\bar\psi] +\\
&\quad (1+2\alpha)\frac{tf_r}{\bar\phi}[\bar\psi_i\bar g_j - \epsilon^{ijk}\dot{\bar\phi}\bar\psi_k - \delta^{ij}\dot{\bar\phi}\bar\psi],\\
Q^i_r\bar\psi &= -tf_r\bar g_i - (1+2\alpha)tf_r\frac{\bar\psi\bar\psi_i}{\bar\phi},\\
Q^i_r\bar\psi_j &= tf_r\epsilon^{ijk}\bar g_k - f_r\delta^{ij}\left[t\dot{\bar\phi} + (2r+1)\alpha\bar\phi\right] + (1+2\alpha)f_r[t\frac{\bar\psi_i\bar\psi_j}{\bar\phi} + \delta^{ij}(r+\frac{1}{2})\bar\phi],\\
Q^0_r\bar\phi &= tf_r\bar\psi,\\
Q^0_r\bar g_i &= -f_r[t\dot{\bar\psi}_i + (2r+1)(\alpha+\frac{1}{2})\bar\psi_i] + (1+2\alpha)\frac{tf_r}{\bar\phi}[\dot{\bar\phi}\bar\psi_i + \bar\psi\bar g_i],\\
Q^0_r\bar\psi &= -f_r[t\dot{\bar\phi} + (2r+1)\alpha\bar\phi] + (1+2\alpha)f_r(r+\frac{1}{2})\bar\phi,\\
Q^0_r\bar\psi_i &= tf_r\bar g_i + (1+2\alpha)tf_r\frac{\bar\psi\bar\psi_i}{\bar\phi}.
\end{aligned}
\tag{C.7}
$$

- in the hyperbolic case (here $f_r = \frac{e^{\mu rt}}{\mu}$):

$$
\begin{aligned}
Q^i_r\bar\phi &= \mu f_r\bar\psi_i,\\
Q^i_r\bar g_j &= f_r\epsilon^{ijk}[\dot{\bar\psi}_k + 2r\mu(\alpha+\frac{1}{2})\bar\psi_k] + f_r\delta^{ij}[\dot{\bar\psi} + 2r\mu(\alpha+\frac{1}{2})\bar\psi] +\\
&\quad (1+2\alpha)\frac{f_r}{\bar\phi}[\mu\bar\psi_i\bar g_j - \epsilon^{ijk}\dot{\bar\phi}\bar\psi_k - \delta^{ij}\dot{\bar\phi}\bar\psi],\\
Q^i_r\bar\psi &= -\mu f_r\bar g_i - (1+2\alpha)\mu f_r\frac{\bar\psi\bar\psi_i}{\bar\phi},\\
Q^i_r\bar\psi_j &= \mu f_r\epsilon^{ijk}\bar g_k - f_r\delta^{ij}[\dot{\bar\phi} + 2r\mu\alpha\bar\phi] + (1+2\alpha)\mu f_r[\frac{\bar\psi_i\bar\psi_j}{\bar\phi} + \delta^{ij}r\bar\phi],\\
Q^0_r\bar\phi &= \mu f_r\bar\psi,\\
Q^0_r\bar g_i &= -f_r[\dot{\bar\psi}_i + 2r\mu(\alpha+\frac{1}{2})\bar\psi_i] + (1+2\alpha)\frac{f_r}{\bar\phi}[\dot{\bar\phi}\bar\psi_i + \mu\bar\psi\bar g_i],\\
Q^0_r\bar\psi &= -f_r[\dot{\bar\phi} + 2r\mu\alpha\bar\phi] + (1+2\alpha)\mu f_r r\bar\phi,\\
Q^0_r\bar\psi_i &= \mu f_r\bar g_i + (1+2\alpha)\mu f_r\frac{\bar\psi\bar\psi_i}{\bar\phi}.
\end{aligned}
\tag{C.8}
$$

The superconformal transformations corresponding to the inhomogeneous cases are recovered by setting $\alpha = 0$.

Some comments are in order. Contrary to the "linear supersymmetry basis of fields" discussed in Section **9**, the inhomogeneous actions are now recovered as a non-singular $\alpha \to 0$ limit. On the other hand, the linearization of the supersymmetry transformations (given by the inverse of formula (C.2)), is not recovered from the inverse of (C.1) in the $\alpha \to 0$ limit. In other words, the linearization of the $\alpha = 0$ supersymmetry (C.7,C.8) requires the new notion of inhomogeneous superconformal transformations.

The special $\alpha = -\frac{1}{2}$ point implies that the Lagrangian coincides with a free kinetic term (with the addition of the oscillator potential in the hyperbolic case). At this special point the "constant kinetic basis" coincides with the "linear supersymmetry basis".

This makes clear why we had to discuss the generalization of the Papadopoulos construction for arbitrary values of λ in the first place. To obtain the actions (C.3, C.4,C.5,C.6) based on the non-linear superconformal transformations (C.7,C.8) we formulated at first the linear superconformal problem (for arbitrary λ's). In the linear case the situation is under control and, for low values of $\mathcal{N}$, the linear superconformal transformations are classified.

Appendix D: Note on the hyperbolic/trigonometric superconformally invariant one-dimensional models

It is worth pointing out explicitly that the one-dimensional superconformally invariant hyperbolic or trigonometric theories (contrary to their parabolic counterparts) discussed in this paper are not supersymmetric theories (at least if we assume the ordinary sense of the word “supersymmetry").

The ordinary supersymmetry requires, for a given $\mathcal{N}$, that a set of $\mathcal{N}$ fermionic symmetry generators Q_i closes the supersymmetry algebra $\{Q_i, Q_j\} = 2\delta_{ij}H$, $[H, Q_i] = 0$ $(i, j = 1, \ldots, \mathcal{N})$, where H is the time-derivative operator (the “Hamiltonian").

In the hyperbolic/trigonometric cases, $\mathcal{N}$ fermionic symmetry generators can be found. They are the square roots of a symmetry generator (let's call it Z), which does not coincide with the Hamiltonian H. As a matter of fact, in the hyperbolic/trigonometric cases, two independent symmetry subalgebras $\{Q_i^\pm, Q_j^\pm\} = 2\delta_{ij}Z^\pm$, $[Z^\pm, Q_i^\pm] = 0$ (with $Z^+ \neq H$ and $Z^- \neq H$) are encountered. In the parabolic cases two independent symmetry subalgebras are also encountered and one of them can be identified with the ordinary supersymmetry ($Z^- = H$, $Z^+ \neq H$).

In the hyperbolic/trigonometric cases the Hamiltonian H continues to be a symmetry operator. It belongs, however, to the 0-grading sector of the superconformal algebra and *is not* the square of any fermionic symmetry operator (contrary to the operators $Z^\pm$, which belong to the ± 1 grading sectors, respectively).

These points can be illustrated with the simplest hyperbolic example, the $\mathcal{N} = 1$ theory based on the $(1, 1)$ supermultiplet (a single bosonic field φ and a single fermionic field ψ) admitting constant kinetic term and $osp(1|2)$ invariance. This hyperbolic action can be written as

$$\mathcal{S} = \int dt(\dot{\varphi}^2 - \psi\dot{\psi} + \varphi^2). \tag{D.1}$$

The five invariant operators (closing the $osp(1|2)$ algebra) are given by

$$\begin{aligned} Q^\pm\varphi &= e^{\pm t}\psi, \qquad & Q^\pm\psi &= e^{\pm t}(\dot{\varphi} \mp \varphi), \\ Z^\pm\varphi &= e^{\pm 2t}(\dot{\varphi} \mp \varphi), \qquad & Z^\pm\psi &= e^{\pm 2t}\dot{\psi}, \\ H\varphi &= \dot{\varphi}, \qquad & H\psi &= \dot{\psi}. \end{aligned} \tag{D.2}$$

One shoulde note that $Z^\pm = (Q^\pm)^2$.

No change of time variable $t \mapsto \tau(t)$ allows to represent either Z^+ or Z^- as a time-derivative operator with respect to the new time τ.

The same considerations immediately apply to the other hyperbolic cases with $\mathcal{N} > 1$ and to all the trigonometric cases (possessing a well-defined, bounded below, potential) which, as discussed in Section **2**, can be recovered from their hyperbolic counterparts.

It could be perhaps useful to employ the notion of *weak supersymmetry* to describe the nature of systems that, like (D.1), possess a symmetry invariance Z which is the square of fermionic symmetry operators and such that (in contrast with the ordinary, *strong* supersymmetry) $Z \neq H$, H being the Hamiltonian. The notion of “weak supersymmetry”, in a different but related context, has already been introduced in [45] (see also [46, 47]).

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Chapter 3

Worldline topological conformal mechanics

3.1 Introduction

In this paper, using constraints from world-line pseudo-supersymmetry and some of the results in [1] and [2], we generalize the class of one-dimensional $sl(2|1)$-invariant conformal topological σ-models of [3]. Specifically, we construct invariant actions for any real value λ of the scaling dimension, for both chiral and real supermultiplets and in both parabolic and hyperbolic/trigonometric cases of the D-module representations of the $sl(2|1)$ superalgebra. For $\lambda \neq -\frac{1}{2}$ the invariant actions possess an interacting term (only the free actions were discussed in [3]).

A further result of this paper is the extension of the criticality condition of the D-module representations of one-dimensional superconformal algebras from the case of positive supersymmetry (analyzed in [4, 5, 2]) to the case of pseudo-supersymmetry $\mathcal{N} = (p, q)$ with both $p, q > 0$. The extension to pseudo-supersymmetry has non-trivial consequences. Unlike the positive supersymmetry, criticality of D-module representation is already encountered for supermultiplets with two bosonic and two fermionic component fields, the $D(2, 1; \alpha)$ superalgebras being induced by $\mathcal{N} = (2, 2)$ acting on the chiral supermultiplet. Furthermore, free parameters enter the construction of topological charges, which are nilpotent operators obtained from linear combinations of the pseudo-supersymmetry generators. We will see in which cases these parameters have to be fixed in order to guarantee the existence of superconformally invariant actions. Some other interesting features appear w.r.t. the positive superconformal case. The boost operator K (the $sl(2)$ conformal partner of the Hamiltonian H), for instance, can be non-diagonal (a non-diagonal K operator is required by superconformal invariance of the real supermultiplet actions).

The framework of our paper is that of (one-dimensional) topological quantum field theory, realized as an $N = 2$ supersymmetric quantum mechanics with twisted superalgebra, see [6, 7, 8, 9]. Different from our worldline approach, a construction based on target manifold restrictions [10, 11, 12, 13, 14] has been discussed for (n, n) superconformal mechanics in [15].

There is a vast literature on (positive, i.e. untwisted) superconformal mechanics [16, 17], its model building and its applications (which range from AdS_2/CFT_1 correspondence, near-horizon geometry of extremal black holes, etc.). One can consult the two review papers [18, 19] and the references therein.

It is useful to present our terminology and notations.

The $\mathcal{N} = (p, q)$ pseudo-supersymmetry algebra is given by $p + q$ fermionic operators Q_I and a single bosonic operator H (the Hamiltonian), as

$$\{\mathrm{Q}_I, \mathrm{Q}_J\} = \eta_{IJ} H, \qquad [H, \mathrm{Q}_I] = 0, \tag{D.1}$$

where p entries of the diagonal matrix η_{IJ} are $+1$ and q entries are -1 (we write, accordingly, $\{\mathrm{Q}_I\} \equiv \{Q_i^+, Q_j^-\}$ for $i = 1, \ldots, p$, $j = 1, \ldots q$, so that $Q_\star^{\pm 2} = \pm H$). For $q = 0$ we recover the (p-extended) superalgebra of the Supersymmetric Quantum Mechanics [20]. The $q = 0$ case is called the "positive" or "untwisted" supersymmetry.

The D-module representations of (D.1), in terms of differential operators in one variable (the "time" coordinate) have been constructed in [21, 22]. They act on supermultiplets of time-dependent component fields with different scaling dimensions. For the case of interest here the supermultiplets are of the form $(k, n, n - k)$, namely possessing k bosons of scaling dimension λ, n fermions of scaling dimension $\lambda + \frac{1}{2}$, $n - k$ bosons of scaling dimension $\lambda + 1$. As customary, with a slight abuse of language (this is only true when invariant actions are constructed), the first set of bosons will be referred to as the "propagating bosons", the remaining ones are referred to as the "auxiliary fields".

For $\mathcal{N} = (2, 2)$ the irreducible supermultiplets are $(2, 2, 0)$ (also known as the "chiral supermultiplet") and $(1, 2, 1)$ (also known as the "real supermultiplet").

The finite one-dimensional superconformal algebras are the simple Lie superalgebras [23, 24] $\mathcal{G}$ which can be decomposed according to the grading $\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_{-\frac{1}{2}} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_1$ and whose even sector is $\mathcal{G}_{even} = sl(2) \oplus R$ (the subalgebra R is known as R-symmetry). The odd sector $(\mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_{-\frac{1}{2}})$ is spanned by $2m$ generators. The positive sector $\mathcal{G}_{>0}$ is isomorphic to the (D.1) superalgebra for $m = p + q$. If D, H, K are the $sl(2)$ generators, with D being the Cartan element, $\mathcal{G}_1$ ($\mathcal{G}_{-1}$) is spanned by H (K), while $\mathcal{G}_0 = D\mathbb{C} \oplus R$.

Since (D.1) is the $\mathcal{G}_{>0}$ subalgebra, under certain (critical) conditions, the D-module representations of (D.1) can be extended to a differential realization for an associated superconformal algebra, see [4, 5, 2] for the $q = 0$ case. Following [2], the "parabolic" D-module reps of [4, 5] can be mapped into "hyperbolic/trigonometric" D-module reps via a similarity transformation and a change of the time coordinate. These D-module reps are written in terms of hyperbolic/trigonometric functions (hence, a dimensional parameter μ is introduced). The presence of μ allows for extra potential terms (absent in the parabolic case) in the invariant actions [1, 2]. What characterizes the parabolic versus the hyperbolic/trigonometric D-module reps is the fact that, in the latter case, the Cartan generator D is proportional to the time-derivative operator, while in the parabolic case the time-derivative operator is proportional to H, the $sl(2)$ positive root. Since trigonometric D-module reps are recovered from the hyperbolic D-module reps via an analityc continuation of μ [2], from now on, for simplicity, we just call "hyperbolic" this class of D-module reps.

The twisted supersymmetry is the presentation of a superalgebra in terms of nilpotent fermionic generators. Following [9] a nilpotent operator Q can be constructed as a linear combination of the pseudo-supersymmetry generators Q_I in (D.1) as $Q = \sum_I c_I \mathrm{Q}_I$. The coefficients c_I have to be chosen to guarantee $Q^2 = 0$. All c_I's have to be real to respect the reality condition of the superalgebra. Therefore, the nilpotency of Q requires a pseudo-supersymmetry with both $p, q > 0$. In the case of the $sl(2|1)$ superalgebra its twisted version is expressed with two nilpotent operators Q_1, Q_2 ($Q_1^2 = Q_2^2 = 0$, $\{Q_1, Q_2\} = H$) and their two superconformal partners.

In the functional integral approach, via path-integral, to twisted supersymmetry, the passage from a $2n$ positive supersymmetry to a (n, n) pseudo-supersymmetry is formally realized via complex transformations. When dealing, as here, with representation theory, this passage is not allowed because we need to respect the real form of the superalgebras. The connection between twisted and untwisted versions of supersymmetric theories has been discussed in [9]. Getting nilpotent supersymmetry generators provides a good viewpoint for investigating the properties of supersymmetry since many questions can be reduced to cohomological questions, with Q-invariance often realized as Q-exactness. Depending on which problem one considers, various interesting properties of supersymmetric theories can be better understood either in the twisted or in the untwisted formulation.

Starting from the (D.1) pseudo-supersymmetry operators acting on a given supermultiplet, the following steps have to be fulfilled in order to obtain a one-dimensional, topological, conformally invariant, σ-model:
i) at first a set of nilpotent operators Q_i's has to be chosen by properly selecting, for any given i, the c_I's coefficients (a certain arbitrariness can be reflected in the appearance of free parameters);
ii) next, the most general boost operator K which, together with the Q_i's operators, allow to construct a superconformal algebra, must be determined;
iii) finally, the superconformal algebra is implemented as a symmetry by constructing an action manifestly invariant under the Q_i's transformations (for positive supersymmetry this procedure is discussed in [22]) and by imposing (following [5, 4, 2]), the invariance under the K transformation as a constraint.

These three steps are presented in Sections **2** and **3**. We postpone to the Conclusions the discussion of the results here obtained.

The scheme of the paper is as follows. In Section **2** we discuss the D-module representations of the $sl(2|1)$ superconformal algebra realized on supermultiplets with two bosons and two fermions. Even if it is not the largest superalgebra acting on these supermultiplets, $sl(2|1)$ is the largest symmetry superalgebra of the invariant actions. We present in Section **3** the construction of the $sl(2|1)$-invariant σ-models. In Section **4** we extend the results of [4, 5, 2], presenting the list of D-module reps of superconformal algebras induced by the supermultiplets associated with the pseudo-supersymmetry. In Appendix **A** the smallest topological conformal algebra and its invariant actions are introduced. In Appendix **B** the D-module reps discussed in Section **4** are explicitly constructed. In the Conclusions we comment our results and point out the directions of future development.

3.2 Symmetry algebra of the topological conformal σ-models

For the models under consideration the superconformal symmetry algebra contains in the bosonic sector 4 generators, H, D, K and G, which close the $sl(2) \oplus u(1)$ algebra. Its non-vanishing commutators are

$$[H, D] = H, \quad [K, D] = -K, \quad [H, K] = 2D. \tag{D.2}$$

H, D, K generates the one-dimensional conformal algebra $sl(2)$, while the $u(1)$ charge G is the R-symmetry operator.

The topological conformal extension of $sl(2) \oplus u(1)$ requires the introduction of 4 extra nilpotent fermionic generators (the twisted supersymmetry operators Q_1, Q_2 and their superconformal partners $\bar{Q}_1$ and $\bar{Q}_2$, so that $Q_1^2 = Q_2^2 = \bar{Q}_1^2 = \bar{Q}_2^2 = 0$). The consistency condition, provided by the graded-Jacobi identities [24], determines the structure constants of the superalgebra. The extra non-vanishing (anti)commutators are

$$\{Q_k, Q_j\} = (\sigma_1)_{kj} H, \quad \{Q_k, \bar{Q}_j\} = -(\sigma_1)_{kj} D + i(\sigma_2)_{kj} G, \quad \{\bar{Q}_k, \bar{Q}_j\} = (\sigma_1)_{kj} K, \tag{D.3}$$

together with

$$\begin{gathered} [K, Q_k] = \bar{Q}_k, \quad [H, \bar{Q}_k] = -Q_k, \\ [Q_k, D] = \frac{Q_k}{2}, \quad [\bar{Q}_k, D] = -\frac{\bar{Q}_k}{2}, \\ [Q_k, G] = (\sigma_3)_{kj} \frac{Q_j}{2}, \quad [\bar{Q}_k, G] = (\sigma_3)_{kj} \frac{\bar{Q}_j}{2}. \end{gathered} \tag{D.4}$$

In the r.h.s. the Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ have been used. One should note that the structure constants of the superalgebra are real.

The commutator with D defines the standard conformal weights: $+1$ (H), 0 (D, G), -1 (K) and $+\frac{1}{2}$ (Q_1, Q_2), $-\frac{1}{2}$ $(\bar{Q}_1, \bar{Q}_2)$.

Interestingly enough, the topological superalgebra also possesses a conserved bigrading, defined for each operator as follows

$$H^{1,-1}, \quad D^{0,0}, \quad G^{0,0}, \quad K^{-1,1}, \quad Q_1^{1,0}, \quad Q_2^{0,-1}, \quad \bar{Q}_1^{-1,0}, \quad \bar{Q}_2^{0,1}.$$

The sum of both values of the indices of the bigrading can be called the *ghost number*. Operators and fields with even (odd) ghost number are commuting (anticommuting). If the bigrading is enforced, the operation of *untwisting* the (D.2-D.4) algebra (namely, presenting the fermionic generators in a pseudo-supersymmetry diagonal basis) is not allowed, because linear combinations of the fermionic generators do not preserve the bigrading. On the other hand, the untwisting operation is allowed if we just impose the conservation of the ghost number (modulo two). The duality relating H and K, Q_k and $\bar{Q}_k$, corresponds to a parity transformation which preserves the bigrading.

One should note that, once the conformal boost operator K and the Q_k's operators have been determined in a given representation, the conformal partners $\bar{Q}_k$'s are obtained from the relation $[K, Q_k] = \bar{Q}_k$.

When computing, in Section **3**, invariant actions we assume for the Lagrangians the vanishing of the ghost number. Other invariants (such as anomalies and cocycles of higher degrees), possessing different values of the ghost number, exist. These invariants can be called the "topological observables".

The topological conformal algebra (D.2-D.4) is a real form of the $A(1,0) \approx sl(2|1)$ superalgebra. $A(1,0)$ is contained as a subalgebra in the superalgebras $D(2,1;\alpha)$ and $A(1,1)$ with its central extension. In appendix **B** it is shown that the irreducible D-module representations of these superalgebras are realized on supermultiplets with 2 bosonic and 2 fermionic component fields. Clearly, on the same set of fields, a D-module rep of $A(1,0)$ is induced. The key point is that $A(1,0)$ is the largest subalgebra which can emerge as a symmetry algebra of a topological action constructed with this basic set of $2+2$ component fields.

In this Section we pave the way for the construction of the topological conformal σ-models of Section **3** by discussing the properties of the induced $A(1,0)$ D-module representations. As mentioned in the Introduction, the representations can be either parabolic or hyperbolic; the basic supermultiplets in this case are either $(2,2,0)$ (the chiral supermultiplet) or $(1,2,1)$ (the real one).

3.2.1 The $(2,2,0)$ D-module reps of the $sl(2|1)$ superalgebra

The operators are assumed to act upon the $(2,2,0)$ supermultiplet $(x, \bar{x}; \psi, \bar{\psi})$, with x, $\bar{x}$ propagating bosons. In the parabolic case the representation is constructed as follows. One starts with the four fermionic operators, see Appendix **B**, $Q_1^{\pm}$, $Q_2^{\pm}$ of the $\mathcal{N} = (2,2)$ superconformal algebra $(2,2,0)$ D-module rep. Explicitly,

$$Q_1^+ = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \partial_t & 0 & 0 \\ \partial_t & 0 & 0 & 0 \end{pmatrix}, \quad Q_1^- = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & \partial_t & 0 & 0 \\ -\partial_t & 0 & 0 & 0 \end{pmatrix},$$
$$Q_2^+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \partial_t & 0 & 0 & 0 \\ 0 & -\partial_t & 0 & 0 \end{pmatrix}, \quad Q_2^- = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\partial_t & 0 & 0 & 0 \\ 0 & -\partial_t & 0 & 0 \end{pmatrix}.$$

They are the square roots of the $\pm H$ operator. The following two nilpotent operators can be constructed for an arbitrary real value of the parameter β,

$$Q_1 = \frac{Q_1^+ + Q_1^-}{2}, \qquad Q_2^{(\beta)} = \frac{Q_1^+ - Q_1^-}{2} + \beta \frac{Q_2^+ + Q_2^-}{2}. \tag{D.5}$$

The next step is considering the most general realization of the $sl(2)$ algebra which is compatible with the scaling dimension of the fields. Let $\mathbb{I}_4$ be the 4×4 identity matrix and Λ the matrix $\Lambda = diag(\lambda, \lambda, \lambda+\frac{1}{2}, \lambda+\frac{1}{2})$, expressed in terms of the real scaling dimension parameter λ. We can assume without loss of generality that the Borel generators H, D are realized in a diagonal form:

$$H = \mathbb{I}_4 \cdot \partial_t, \qquad D = \mathbb{I}_4 \cdot t\partial_t + \Lambda. \tag{D.6}$$

The most general form of the operator K which preserves the dimensionality of the fields is

$$K = \begin{pmatrix} t^2\partial_t + 2\lambda t & a_{12}t^2\partial_t + b_{12}t & 0 & 0 \\ a_{21}t^2\partial_t + b_{21}t & t^2\partial_t + 2\lambda t & 0 & 0 \\ 0 & 0 & t^2\partial_t + (2\lambda+1)t & a_{34}t^2\partial_t + b_{34}t \\ 0 & 0 & a_{43}t^2\partial_t + b_{43}t & t^2\partial_t + (2\lambda+1)t \end{pmatrix}.$$

The dimensionless parameters a_{ij}'s and b_{ij}'s are allowed by power counting. Nevertheless, the closure of the $sl(2)$ algebra, and particularly the condition $[H, K] = 2D$, forces us to set all the parameters a_{ij}'s, b_{ij}'s equal to zero.

Therefore, the $(2, 2, 0)$ D-module rep requires a diagonal K, given by

$$K = \mathbb{I}_4 \cdot t^2\partial_t + 2\Lambda t. \tag{D.7}$$

The two extra fermionic operators $\bar{Q}_1, \bar{Q}_2$ are obtained from the commutators $\bar{Q}_1 = [K, Q_1]$, $\bar{Q}_2 = [K, Q_2]$, while the R-symmetry generator G is recovered from, e.g., the anticommutator $\{Q_1, \bar{Q}_2\}$, see (D.3) and (D.4). It is a straightforward exercise to verify, for any real λ, the closure of the $sl(2|1)$ superalgebra on this set of operators.

Following [1] and [2] a hyperbolic D-module representation can be constructed by
i) performing a similarity transformation on the operators of the parabolic D-module rep and
ii) performing a change of the time coordinate.
What characterizes the hyperbolic rep with respect to the parabolic rep is the fact that in this case the time-derivative operator is H, the positive root of $sl(2)$. In the hyperbolic case the time-derivative operator (for the new time coordinate) is D, the Cartan generator of $sl(2)$ (as a consequence, the energy spectrum of the invariant theories is continuous in the parabolic case and discrete in the hyperbolic case).

We present here a set of operators which allows to reconstruct the full set of $(2, 2, 0)$ D-module rep generators in the hyperbolic case. For arbitrary λ and at the special value $\beta = 0$ (in Section **3** we prove that this is the unique value of β which produces $sl(2|1)$-invariant actions for the $(2, 2, 0)$ supermultiplet) we have

$$Q_1 = e^{-\frac{\mu t}{2}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\mu}\partial_t - \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = e^{-\frac{\mu t}{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\mu}\partial_t - \lambda & 0 & 0 & 0 \end{pmatrix}. \tag{D.8}$$

The $sl(2)$ algebra operators are

$$H = e^{-\mu t}(\mathbb{I}_4 \cdot \frac{1}{\mu}\partial_t - \Lambda), \quad D = \mathbb{I}_4 \cdot \frac{1}{\mu}\partial_t, \quad K = e^{\mu t}(\mathbb{I}_4 \cdot \frac{1}{\mu}\partial_t + \Lambda). \tag{D.9}$$

Here again $\Lambda = diag(\lambda, \lambda, \lambda + \frac{1}{2}, \lambda + \frac{1}{2})$. The remaining operators are obtained from the (anti)commutators in (D.3,D.4). The parameter μ is a dimensional parameter which can be set equal to 1 without loss of generality.

3.2.2 The $(1, 2, 1)$ D-module reps of the $sl(2|1)$ superalgebra

The construction is similar to the previous case. The operators now act on the supermultiplet $(x, b; \psi, \bar{\psi})$, where x is a propagating boson and b an auxiliary field. In the parabolic case we start with the four fermionic operators given in Appendix **B**,

$$Q_1^+ = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \partial_t & 0 \\ 0 & 1 & 0 & 0 \\ \partial_t & 0 & 0 & 0 \end{pmatrix}, \quad Q_1^- = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\partial_t & 0 \\ 0 & 1 & 0 & 0 \\ -\partial_t & 0 & 0 & 0 \end{pmatrix},$$

$$Q_2^+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\partial_t \\ \partial_t & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad Q_2^- = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \partial_t \\ -\partial_t & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

These are the square roots of $\pm H$ in the $(1, 2, 1)$ D-module rep. Two nilpotent operators (the second one depending on an arbitrary parameter γ) are constructed as follows

$$Q_1 = \frac{Q_1^+ + Q_1^-}{2}, \qquad Q_2^{(\gamma)} = \frac{Q_1^+ - Q_1^-}{2} - \gamma\frac{Q_2^+ + Q_2^-}{2}. \tag{D.10}$$

As in the previous case and without loss of generality the H, D generators of the $sl(2)$ Borel subalgebra can be assumed to be diagonal

$$H = \mathbb{I}_4 \cdot \partial_t, \quad D = \mathbb{I}_4 \cdot t\partial_t + \Lambda. \tag{D.11}$$

The matrix Λ is now $\Lambda = diag(\lambda, \lambda+1, \lambda+\frac{1}{2}, \lambda+\frac{1}{2})$.

The most general form of the operator K, preserving the dimensionality of the fields, is now

$$K = \begin{pmatrix} t^2\partial_t + 2\lambda t & a_{12}t^3\partial_t + b_{12}t^2 & 0 & 0 \\ a_{21}t\partial_t + b_{21} & t^2\partial_t + 2(\lambda+1)t & 0 & 0 \\ 0 & 0 & t^2\partial_t + (2\lambda+1)t & a_{34}t^2\partial_t + b_{34}t \\ 0 & 0 & a_{43}t^2\partial_t + b_{43}t & t^2\partial_t + (2\lambda+1)t \end{pmatrix}.$$

Contrary to the previous case, the closure of the $sl(2)$ algebra allows a non-diagonal form for the operator K. Indeed, its most general solution is

$$K = \mathbb{I}_4 \cdot t^2\partial_t + 2\Lambda t + \delta E_{21}, \tag{D.12}$$

where E_{21} denotes the matrix whose only non-vanishing entry is 1 in the $(2,1)$ position. The parameter δ is arbitrary. At $\delta = 0$ we obtain a diagonal form for K.

As in the previous case, the remaining generators are computed from the (anti)commutators given in (D.3,D.4). The $(1,2,1)$ D-module rep of the $sl(2|1)$ superalgebra depends on three arbitrary real parameters, the scaling dimension λ, the parameter γ entering (D.10) and the parameter δ entering (D.12). In Section **3** we prove that the $sl(2|1)$-invariant actions require $\delta \neq 0$ (therefore, a non-diagonal form of K). In the invariant actions the parameter γ can be rescaled so that, without loss of generality, one can set $\gamma = 1$. In its turn δ has to be fixed at $\delta = -\lambda$.

For these special values, the operators which induce the hyperbolic $(1,2,1)$ D-module rep of $sl(2|1)$ are the following. The two nilpotent operators Q_1, Q_2 are

$$Q_1 = e^{-\frac{\mu t}{2}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = e^{-\frac{\mu t}{2}} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{1}{\mu}\partial_t - \lambda - \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\mu}\partial_t - \lambda & 1 & 0 & 0 \end{pmatrix}. \tag{D.13}$$

The $sl(2)$ algebra generators are

$$H = e^{-\mu t}(\mathbb{I}_4 \cdot \frac{1}{\mu}\partial_t - \Lambda), \quad D = \mathbb{I}_4 \cdot \frac{1}{\mu}\partial_t, \quad K = e^{\mu t}(\mathbb{I}_4 \cdot \frac{1}{\mu}\partial_t + \Lambda - \lambda E_{21}), \tag{D.14}$$

with $\Lambda = diag(\lambda, \lambda+1, \lambda+\frac{1}{2}, \lambda+\frac{1}{2})$. The remaining $sl(2|1)$ generators are derived from the (anti)commutators (D.3,D.4).

3.3 One-dimensional topological conformal σ-models

We present here the conformal topological actions which are $sl(2|1)$-invariant. We require global (twisted) supersymmetry and impose the conformal invariance as a constraint. The procedure repeats the one adopted in [2] and [3] for deriving one-dimensional superconformal σ-models from the supermultiplets of the untwisted supersymmetry. Contrary to the untwisted case, besides the scaling dimension λ, extra free parameters enter and label the twisted superconformal representations. As shown in Section **2**, one extra parameter, β, enters the $(2,2,0)$ D-module representation. Two extra parameters, γ and δ, enter the $(1,2,1)$ D-module representation. The requirement of the existence of superconformally invariant actions implies assigning a fixed value to the parameters β and δ while, without loss of generality, γ can be rescaled to 1.

3.3.1 Superconformal models for the $(2,2,0)$ supermultiplet

By construction, a global twisted supersymmetry invariance is obtained in terms of the Lagrangian

$$\mathcal{L} = Q_1 Q_2^{(\beta)}(F\bar{\psi}\psi)$$

where $F = F(x, \bar{x})$ and $Q_1, Q_2^{(\beta)}$ are defined in (D.5). This yields, up to a total time derivative,

$$\mathcal{L} = F\dot{x}\dot{\bar{x}} + F\dot{\psi}\dot{\bar{\psi}} + F_{\bar{x}}\dot{\bar{x}}\psi\bar{\psi} - \beta(F\dot{\bar{x}}^2 + F_x\dot{\bar{x}}\bar{\psi}\psi). \tag{D.15}$$

The superconformal invariance requires that the action of the operator K on $\mathcal{L}$ produces a total derivative. The bosonic and the fermionic parts of the actions can be treated separately and generate the same constraint. As an example, we have for the bosonic sector

$$K(F\dot{x}\dot{\bar{x}} - \beta F\dot{\bar{x}}^2) = 2t[\lambda F_x x + \lambda F_{\bar{x}}\bar{x} + (1+2\lambda)F](\dot{x}\dot{\bar{x}} + \beta\dot{\bar{x}}^2) + 2\lambda F x\dot{\bar{x}} + 2\lambda F\bar{x}\dot{x} + 4\beta\lambda F\bar{x}\dot{\bar{x}}.$$

We are led to simultaneously satisfy the system of equations

$$\begin{array}{rcl}\lambda F_x x + \lambda F_{\bar{x}} \bar{x} + (1+2\lambda) F & = & 0, \\ 2\lambda F x \dot{\bar{x}} + 2\lambda F \bar{x} \dot{x} + 4\beta\lambda F \bar{x} \dot{\bar{x}} & = & \frac{d}{dt}(\ldots). \end{array} \quad \text{(D.16)}$$

The only solution for the above system is obtained for $\beta = 0$, with

$$F(x, \bar{x}) = C(x\bar{x})^{-\frac{1+2\lambda}{2\lambda}}. \quad \text{(D.17)}$$

Therefore, the Lagrangian producing the twisted superconformally invariant σ-model,

$$\mathcal{L} = F(\dot{x}\dot{\bar{x}} + \dot{\psi}\bar{\psi}) + F_{\bar{x}} \dot{\bar{x}} \psi \bar{\psi}, \quad \text{(D.18)}$$

with F given in (D.17), is only recovered at the special critical value $\beta = 0$. It is worth pointing out that, for this value, the $(2,2,0)$ D-module rep obtained from (D.5, D.6, D.7) is reducible.

The action derived from (D.17,D.18) is only defined for $\lambda \neq 0$. It was pointed out in [2] that a consistent $\lambda \to 0$ limit exists if the propagating bosons are suitably shifted. In the present case this corresponds to replace $x, \bar{x}$ with the shifted fields $x + \frac{\rho}{\lambda}, \bar{x} + \frac{\bar{\rho}}{\lambda}$, for $\rho, \bar{\rho}$ constants with the same dimensionality as $x, \bar{x}$. In the $\lambda \to 0$ limit, a representation of the superconformal algebra which only depends on $\rho, \bar{\rho}$ emerges. The corresponding supermultiplet which emerges in this limit has been called the *inhomogeneous supermultiplet* in [2] (see also [25]). The construction of the inhomogeneous superconformal actions has been discussed at length in that paper; since it can be straightforwardly applied to the present twisted case, it is sufficient to present here the final results. The inhomogeneous invariant superconformal action is expressed by the same Lagrangian (D.18), but with the identification of F given by

$$F(x, \bar{x}) = Ce^{-\frac{x+\bar{x}}{2\rho}}. \quad \text{(D.19)}$$

The construction of the twisted superconformal action, invariant under the hyperbolic representation, proceeds on similar lines. In the homogeneous $\lambda \neq 0$ case the Lagrangian is

$$\mathcal{L} = F(\dot{x}\dot{\bar{x}} + \mu\dot{\psi}\bar{\psi} + \mu^2\lambda^2 x\bar{x}) + \mu F_{\bar{x}} \psi\bar{\psi}\dot{\bar{x}}, \quad \text{(D.20)}$$

with the same identification of F as in (D.17). In the inhomogeneous $\lambda = 0$ case the invariant Lagrangian is

$$\mathcal{L} = F(\dot{x}\dot{\bar{x}} + \mu\dot{\psi}\bar{\psi} + \mu^2\rho^2) + \mu F_{\bar{x}} \psi\bar{\psi}\dot{\bar{x}}, \quad \text{(D.21)}$$

with F given by (D.19).

As it happens for the untwisted superconformal case [1, 2], the difference between parabolic versus hyperbolic superconformal actions consists in the presence, in the latter case, of an extra potential term.

3.3.2 Superconformal models for the $(1,2,1)$ supermultiplet

The construction of the twisted superconformal actions for the $(1,2,1)$ supermultiplet proceeds along similar lines. A global supersymmetric action is obtained from the Lagrangian $\mathcal{L} = Q_1 Q_2^{(\gamma)}(F\bar{\psi}\psi)$, with $Q_1, Q_2^{(\gamma)}$ given in (D.10). We get

$$\mathcal{L} = Fb\dot{x} + F\dot{\psi}\bar{\psi} + \gamma F b^2 + \gamma F_x \bar{\psi}\psi b, \quad \text{(D.22)}$$

where $F = F(x)$.

The superconformal invariance is guaranteed if the constraint generated by K is satisfied. In the $(1,2,1)$ case the generator K carries a dependence on a off-diagonal parameter δ, see (D.12) (for clarity reason, it is therefore convenient to write $K \equiv K^{(\delta)}$). The action of $K^{(\delta)}$ on the purely bosonic part of the Lagrangian reads as follows

$$K^{(\delta)}(Fb\dot{x} + \gamma F b^2) = 2t[\lambda F_x x + (1+2\lambda)F](\gamma b^2 + b\dot{x}) + 2(\lambda + \gamma\delta)Fxb.$$

The superconformal invariance requires simultaneously satisfying the system (no further constraint is obtained from the fermionic part of the Lagrangian)

$$\begin{array}{rcl}\lambda F_x x + (1+2\lambda)F & = & 0, \\ \gamma\delta & = & -\lambda. \end{array} \quad \text{(D.23)}$$

The homogeneous case $\lambda \neq 0$ implies that both γ and δ are non-vanishing. This means, in particular, that the $(1,2,1)$ representation of the twisted superconformal algebra entering the invariant action is irreducible.

Without loss of generality γ can be set to 1 ($\gamma = 1$) via the rescaling of the fields and of the prepotential F: $\gamma b \to b$, $\gamma\bar{\psi} \to \bar{\psi}$, $\frac{F}{\gamma} \to F$. The Lagrangian can indeed be written as

$$\mathcal{L} = \frac{1}{\gamma}[F(\gamma b)^2 + F(\gamma b)\dot{x} + F\dot{\psi}(\gamma\bar{\psi}) + F_x(\gamma\bar{\psi})\psi(\gamma b)]. \tag{D.24}$$

With the choice $\gamma = 1$, $\delta = -\lambda$, the Lagrangian of the parabolic homogeneous case is

$$\mathcal{L} = F(b^2 + b\dot{x} + \dot{\psi}\bar{\psi}) + F_x\bar{\psi}\psi b, \quad \text{with} \quad F(x) = Cx^{-\frac{1+2\lambda}{\lambda}}. \tag{D.25}$$

The parabolic inhomogeneous case is obtained in the $\lambda \to 0$ limit for operators acting on the shifted supermultiplet $(x + \frac{\rho}{\lambda}, b; \psi, \bar{\psi})$. It produces an invariant action whose Lagrangian is

$$\mathcal{L} = F(b^2 + b\dot{x} + \dot{\psi}\bar{\psi}) + F_x\bar{\psi}\psi b, \quad \text{with} \quad F(x) = Ce^{-\frac{x}{\rho}}. \tag{D.26}$$

In the hyperbolic homogeneous case the invariant action is given by the Lagrangian

$$\mathcal{L} = F(\mu^2 b^2 + \mu b\dot{x} + \mu\dot{\psi}\bar{\psi} - \mu^2\lambda x b) + F_x(\mu^2\bar{\psi}\psi b + \frac{1}{2}\mu^2\lambda x\psi\bar{\psi}), \quad \text{with} \quad F(x) = Cx^{-\frac{1+2\lambda}{\lambda}}. \tag{D.27}$$

In the hyperbolic inhomogeneous case the invariant action is given by the Lagrangian

$$\mathcal{L} = F(\mu^2 b^2 + \mu b\dot{x} + \mu\dot{\psi}\bar{\psi} - \mu^2\rho b) + F_x(\mu^2\bar{\psi}\psi b + \frac{1}{2}\mu^2\rho\psi\bar{\psi}), \quad \text{with} \quad F(x) = Ce^{-\frac{x}{\rho}}. \tag{D.28}$$

3.3.3 Superconformal models in the constant kinetic basis

It is convenient to compare the four inequivalent superconformal actions introduced above by making field redefinitions. Under such redefinitions the superconformal symmetry is realized non-linearly. The Lagrangians, on the other hand, have a simpler form, being recast as a constant kinetic term plus (for $\lambda \neq -\frac{1}{2}$) an interaction.

For the $(2,2,0)$ supermultiplet, in both parabolic and hyperbolic cases, the following field redefinitions are made:
i) in the homogeneous case

$$y = -2\lambda x^{-\frac{1}{2\lambda}}, \qquad \bar{y} = -2\lambda\bar{x}^{-\frac{1}{2\lambda}}, \qquad \chi = \psi, \qquad \bar{\chi} = (x\bar{x})^{-\frac{1+2\lambda}{2\lambda}}\bar{\psi}; \tag{D.29}$$

ii) in the inhomogeneous case

$$y = -2\rho e^{-\frac{x}{2\rho}}, \qquad \bar{y} = -2\rho e^{-\frac{\bar{x}}{2\rho}}, \qquad \chi = \psi, \qquad \bar{\chi} = e^{-\frac{x+\bar{x}}{2\rho}}\bar{\psi}. \tag{D.30}$$

For the $(1,2,1)$ supermultiplet, in both parabolic and hyperbolic cases, the field redefinitions are
i) in the homogeneous case

$$y = -2\lambda x^{-\frac{1}{2\lambda}}, \qquad a = x^{-\frac{1+2\lambda}{2\lambda}}b, \qquad \chi = \psi, \qquad \bar{\chi} = x^{-\frac{1+2\lambda}{\lambda}}\bar{\psi}; \tag{D.31}$$

ii) in the inhomogeneous case

$$y = -2\rho e^{-\frac{x}{2\rho}}, \qquad a = e^{-\frac{x}{2\rho}}b, \qquad \chi = \psi, \qquad \bar{\chi} = e^{-\frac{x}{\rho}}\bar{\psi}. \tag{D.32}$$

In terms of the redefined fields the Lagrangians of the superconformally invariant σ-models are very compactly written. Without loss of generality, in the hyperbolic case the dimensional parameter μ can be set $\mu = 1$ (if needed, we can restore the μ-dependence by dimensional analysis).

Associated with the $(2,2,0)$ supermultiplet we have the Lagrangian

$$\mathcal{L} = \dot{y}\dot{\bar{y}} + \dot{\chi}\bar{\chi} + \frac{\epsilon}{4}y\bar{y} + (1+2\lambda)\frac{\chi\bar{\chi}\dot{\bar{y}}}{\bar{y}}. \tag{D.33}$$

Associated with the $(1,2,1)$ supermultiplet we have the Lagrangian

$$\mathcal{L} = a^2 + a\dot{y} + \dot{\chi}\bar{\chi} + \frac{\epsilon}{2}ya + \frac{\epsilon}{2}(1+2\lambda)\bar{\chi}\chi + 2(1+2\lambda)\frac{\bar{\chi}\chi a}{y}. \tag{D.34}$$

In the above Lagrangians the parameter ϵ takes the value $\epsilon = 0$ in the parabolic case and $\epsilon = 1$ in the hyperbolic case. The inhomogeneous case corresponds to the special (non-singular) value $\lambda = 0$. Another special value, $\lambda = -\frac{1}{2}$, corresponds to the non-interacting theory. It is interesting to note that the same twisted superconformal invariance is possessed by actions, even in the presence of an interaction (for $\lambda \neq -\frac{1}{2}$). The coupling constant labels a class of superconformally invariant σ-models and is expressed as $2\lambda + 1$.

3.4 SCA D-module reps from pseudo-supersymmetry

In this Section we summarize the results about inducing D-module representations of semi-simple finite superconformal algebras, recovered from pseudo-supersymmetry supermultiplets. This Section extends the results of [4, 5, 2]. Some of the D-module representations below were used to construct the topological conformal algebras discussed in Section **2**. In the Appendix **B** the D-module representations are explicitly given (in terms of operators which allow to reconstruct the complete set of superconformal algebra generators). We have

- The $\mathcal{N} = (1,1)$-induced superconformal algebra

The superalgebra is $sl(2|1)$. The D-module rep is recovered from the $(1,1,0)$ supermultiplet for any value of the scaling dimension λ.

- The $\mathcal{N} = (2,2)$-induced superconformal algebras

The D-module reps are recovered from the $(k,2,2-k)$ supermultiplets, with $k = 0,1,2$, for any value of the scaling dimension λ.

The corresponding superconformal algebras are
- $D(2,1;\alpha)$, for $(2,2,0)$ and $(0,2,2)$. The relation between α and λ is $\alpha = 2(1-k)\lambda$;
- $A(1,1)$ for $(1,2,1)$ and at the critical value $\lambda = 0$;
- the centrally extended algebra $\widehat{A}(1,1)$ for $(1,2,1)$ and $\lambda \neq 0$.

- The $\mathcal{N} = (3,3)$-induced superconformal algebras

These superconformal algebras are induced by the supermultiplets $(k,4,4-k)$, for $k \neq 2$, at the critical values of the scaling dimension λ given by $\lambda_{cr} = \frac{1}{2(k-2)}$.

The identification of the superconformal algebras goes as follows:
- $D(3,1)$, recovered from the supermultiplets $(4,4,0)$ and $(0,4,4)$;
- $A(2,1)$, recovered from the supermultiplets $(3,4,1)$ and $(1,4,3)$.

One should note that in the Kac list of semisimple Lie superalgebras [23] besides $D(3,1)$ and $A(2,1)$ there exists an extra superalgebra, $B(1,2)$, which has the property of being one-dimensionally superconformal (see [26]) with $6+6$ fermionic generators. This superalgebra, however, is not recovered from the $(k,4,4-k)$-induced D-module reps.

- The $\mathcal{N} = (4,4)$-induced superconformal algebra

The only supermultiplets inducing a D-module rep of a superconformal algebra are $(8,8,0)$ and $(0,8,8)$. The other values of k ($k = 1,2,\ldots,7$) fail to induce a D-module rep of a superconformal algebra. In both cases, the induced superalgebra is $D(4,1)$ and is only recovered at the critical values of the scaling dimension λ, $\lambda_{cr} = \frac{1}{4}$ for $(8,8,0)$ and $\lambda_{cr} = -\frac{1}{4}$ for $(0,8,8)$. This case should be compared with the induced D-module reps from positive $\mathcal{N} = (8,0)$ supersymmetry. In that case, see [5], the four superconformal algebras $D(4,1)$, $D(2,2)$, $A(3,1)$ and $F(4)$ are induced from D-module reps of the $(k,8,8-k)$ supermultiplets for different values of $k \neq 4$ at the given critical scaling dimensions.

- The $\mathcal{N} = (5,1)$-induced superconformal algebra

In this case the only supermultiplets inducing a D-module rep of a superconformal algebra are $(8,8,0)$ and $(0,8,8)$. The other values of k ($k = 1,2,\ldots,7$) fail to induce a D-module rep of a superconformal algebra. In both cases, the induced superalgebra is $D(3,1)$, recovered at the critical values of the scaling dimension λ, $\lambda_{cr} = \frac{1}{4}$ for $(8,8,0)$ and $\lambda_{cr} = -\frac{1}{4}$ for $(0,8,8)$.

3.5 Conclusions

The $sl(2|1)$-invariant topological actions are presented in (D.33) and (D.34) for the $(2,2,0)$ chiral and the $(1,2,1)$ real supermultiplet, respectively. The parameter $\epsilon = 0,1$ discriminates the parabolic versus the hyperbolic case. A coupling constant is given by $2\lambda + 1$; at the $\lambda = -\frac{1}{2}$ value the actions are free.

The construction of topological $sl(2|1)$-invariant actions for several supermultiplets in interactions is straightforward. It can be done by adapting to the twisted supersymmetry the construction discussed in [5] for the untwisted superconformal invariance.

At least two approaches can be used to construct topological actions invariant under a larger superalgebra, let's say $D(2,1;\alpha)$. One can start with $sl(2|1)$-invariant actions from supermultiplets in interactions and then search which values in the space of parameters characterize the enhanced superconformal invariance. To give an example, the $(4,4,0)$ supermultiplet is expected to possess $D(2,1;\alpha)$-invariant actions, with α linked to its scaling dimension. Under $sl(2|1)$, the $(4,4,0)$ supermultiplet is decomposed into the direct sum of two $(2,2,0)$ supermultiplets: $(4,4,0) = (2,2,0) \oplus (2,2,0)$.

The other approach is more direct. It requires the construction of the nilpotent operators (which, together with the boost operator K, induce a $D(2,1;\alpha)$ superalgebra) from the linear combination of the $3+3$ pseudo-supersymmetry operators acting (see Appendix **B**) on four bosons and four fermions. We leave this construction for a future work.

In this paper we proved the flexibility of the world-line framework to investigate one-dimensional topological conformal σ-models [27]. A natural question, which goes under the name of "oxidation" [28] or "enhancement" [29], is whether data from one-dimensional (twisted) supersymmetry can encode information of higher dimensional theories and whether they allow to reconstruct them. From the [30] paper it is known that two-dimensional topological σ-models are at the core of the geometric Langlands program. An approach to enhance world-line supersymmetry to world-sheet supersymmetry has been presented in [31]. It looks promising investigating its application to topological (i.e. twisted supersymmetric) theories.

Appendix A: the smallest topological conformal algebra

One can easily recognize that the smallest invariant topological conformal algebra is a subalgebra of the $\mathcal{N}=(1,1)$ superconformal algebra $sl(2|1)$. This invariant algebra, $sl(2)\niplus gr(2)$, is realized on the $(1,1)$ supermultiplet. It consists of the bosonic $sl(2)$ subalgebra acting on the Grassmann algebra $gr(2)$ (see [24]) of two anticommuting nilpotent operators $Q,\bar{Q}$. In the parabolic realization the operator Q can be constructed from the positive and negative square roots of H,

$$Q^+ = \begin{pmatrix} 0 & 1 \\ \partial_t & 0 \end{pmatrix}, \qquad Q^- = \begin{pmatrix} 0 & 1 \\ -\partial_t & 0 \end{pmatrix}, \tag{A.1}$$

either as

$$i) \quad Q = \frac{Q^+ + Q^-}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{or} \qquad ii) \quad Q = \frac{Q^+ - Q^-}{2} = \begin{pmatrix} 0 & 0 \\ \partial_t & 0 \end{pmatrix}. \tag{A.2}$$

The $sl(2)$ generators are

$$H = \mathbb{I}_2 \cdot \partial_t, \quad D = \mathbb{I}_2 \cdot t\partial_t + \Lambda, \quad K = \mathbb{I}_2 \cdot t^2\partial_t + 2t\Lambda, \tag{A.3}$$

where $\Lambda = diag(\lambda, \lambda+\frac{1}{2})$. In both cases, the nilpotent partner $\bar{Q}$ is given by $\bar{Q} = [K, Q]$. The invariant action is given by $\mathcal{S} = \int dt \cdot \mathcal{L}$, where the Lagrangian is $\mathcal{L} = Q(G(x)\dot{\psi})$. In the following it is convenient to introduce $G(x) = \int^x dx' F(x')$. We have

i) for the first choice,

$$\mathcal{L} = F\psi\dot{\psi}, \tag{A.4}$$

where for the *homogeneous/inhomogeneous* cases discussed in Section **3** we have, respectively, $F(x) = Cx^{-\frac{1+2\lambda}{\lambda}}$ and $F(x) = Ce^{-\frac{x}{\rho}}$;

ii) for the second choice,

$$\mathcal{L} = F\dot{x}^2 + V(x), \tag{A.5}$$

with $F(x) = Cx^{-\frac{1+2\lambda}{\lambda}}$, $V(x) = kx^{\frac{1}{\lambda}}$ in the homogeneous case and $F(x) = Ce^{-\frac{x}{\rho}}$, $V(x) = ke^{\frac{x}{\rho}}$ in the inhomogeneous case .

In the second case the potential term $V(x)$ is automatically invariant, up to a time derivative, so that the parameter k is arbitrary.

The construction can be repeated for the hyperbolic realization. We obtain in this case

$$i) \quad Q = e^{-\frac{\mu t}{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{or} \qquad ii) \quad Q = e^{-\frac{\mu t}{2}} \begin{pmatrix} 0 & 0 \\ \frac{1}{\mu}\partial_t - \lambda & 0 \end{pmatrix}. \tag{A.6}$$

The invariant actions are given by the Lagrangians

i) for the first choice,

$$\mathcal{L} = \mu F\psi\dot{\psi}, \tag{A.7}$$

with $F(x) = Cx^{-\frac{1+2\lambda}{\lambda}}$ in the homogeneous case and $F(x) = Ce^{-\frac{x}{\rho}}$ in the inhomogeneous case;

ii) for the second choice,

$$\mathcal{L} = F[\dot{x}^2 + \mu^2(\lambda x + \rho)^2] + V(x), \tag{A.8}$$

with $F(x) = Cx^{-\frac{1+2\lambda}{\lambda}}$, $V(x) = kx^{\frac{1}{\lambda}}$ and $\rho = 0$ in the homogeneous case and $F(x) = Ce^{-\frac{x}{\rho}}$, $V(x) = ke^{\frac{x}{\rho}}$ and $\lambda = 0$ in the inhomogeneous case.

Appendix B: explicit D-module reps of the topological superconformal algebras

We explicitly give here, for each representation discussed in Section **4**, a set of operators which allows to recover (via repeated mutual commutators/anticommutators) the complete set of superconformal algebra generators. It is sufficient to present the parabolic case. In the following $\mathbb{I}_n$ denotes the $n\times n$ identity matrix, while E_{ij} stands for the matrix whose only non-vanishing entry is 1 at the crossing of the i-th row with the j-th column.

For all the D-module representations below the generators H, D of the $sl(2)$ Borel subalgebra have the form

$$\begin{aligned} H &= \mathbb{I}_n \cdot \partial_t, \\ D &= \mathbb{I}_n \cdot t\partial_t + \Lambda, \end{aligned} \tag{B.1}$$

for a given integer n and a given diagonal matrix Λ.

One can prove that the remaining $sl(2)$ generator, K, is in diagonal form for the $(1,1,0)$ D-module reps of the $\mathcal{N}=(1,1)$ superconformal algebra and the $(2,2,0)$, $(0,2,2)$ D-module reps of the $\mathcal{N}=(2,2)$ superconformal algebras. K admits a consistent non-diagonal form (parametrized by the free, off-diagonal parameter δ, the diagonal case being recovered at $\delta=0$) for the $(1,2,1)$ D-module reps of the $\mathcal{N}=(2,2)$ superconformal algebras. In this case K is given by formula (D.12), that is $K=\mathbb{I}_4\cdot t^2\partial_t+2\Lambda t+\delta E_{21}$, with $\Lambda=diag(\lambda,\lambda+1,\lambda+\frac{1}{2},\lambda+\frac{1}{2})$.

In all remaining cases with extended number of supersymmetries, $\mathcal{N}=(3,3)$, $\mathcal{N}=(4,4)$ and $\mathcal{N}=(5,1)$, we investigated the closure of the representations for diagonal K. Therefore, in the following, apart the $(1,2,1)$ D-module rep of the $\mathcal{N}=(2,2)$ superconformal algebra with K given by (D.12), we have, in all other cases,

$$K = \mathbb{I}_n\cdot t^2\partial_t+2\Lambda t. \tag{B.2}$$

The $\mathcal{N}=(1,1)$ superconformal algebra

In this case the unique supermultiplet is $(1,1,0)$, with component fields $(x;\psi)$.
We have $n=2$, while the matrix Λ is given, for an arbitrary λ, by $\Lambda=diag(\lambda,\lambda+\frac{1}{2})$.
The square roots of $\pm H$ are the operators
$Q^+=E_{12}+E_{21}\partial_t$,
$Q^-=E_{12}-E_{21}\partial_t$.

The $\mathcal{N}=(2,2)$ superconformal algebras

In this case $n=4$. Different superconformal algebras, see Section **4**, are associated with different values of the scaling dimension λ. The representations are labeled by the supermultiplets

- $(2,2,0)$**:**
Here $\Lambda=diag(\lambda,\lambda,\lambda+\frac{1}{2},\lambda+\frac{1}{2})$. The component fields are $(x,\bar{x};\psi,\bar{\psi})$, where x, $\bar{x}$ are propagating bosons.
The square roots of $\pm H$ are the operators
$Q_1^+=E_{14}+E_{23}+(E_{32}+E_{41})\partial_t$,
$Q_2^+=E_{13}-E_{24}+(E_{31}-E_{42})\partial_t$,
$Q_1^-=E_{14}-E_{23}+(E_{32}-E_{41})\partial_t$,
$Q_2^-=E_{13}+E_{24}-(E_{31}+E_{42})\partial_t$.

- $(1,2,1)$**:**
Here $\Lambda=diag(\lambda,\lambda+1,\lambda+\frac{1}{2},\lambda+\frac{1}{2})$. The component fields are $(x,b;\psi,\bar{\psi})$, where x is the propagating boson.
The square roots of $\pm H$ are the operators
$Q_1^+=E_{14}+E_{32}+(E_{23}+E_{41})\partial_t$,
$Q_2^+=E_{13}-E_{42}+(-E_{24}+E_{31})\partial_t$,
$Q_1^-=E_{14}+E_{32}-(E_{23}+E_{41})\partial_t$,
$Q_2^-=E_{13}-E_{42}+(E_{24}-E_{31})\partial_t$.

- $(0,2,2)$**:**
Here $\Lambda=diag(\lambda+1,\lambda+1,\lambda+\frac{1}{2},\lambda+\frac{1}{2})$. This supermultiplet has no propagating bosons. The component fields are $(b,\bar{b};\psi,\bar{\psi})$.
The square roots of $\pm H$ are the operators
$Q_1^+=(E_{14}+E_{23})\partial_t+E_{32}+E_{41}$,
$Q_2^+=(E_{13}-E_{24})\partial_t+E_{31}-E_{42}$,
$Q_1^-=(E_{14}-E_{23})\partial_t+E_{32}-E_{41}$,
$Q_2^-=(E_{13}+E_{24})\partial_t-E_{31}-E_{42}$.

The $\mathcal{N}=(3,3)$ superconformal algebras

In this case $n=8$. Different superconformal algebras, see Section **4**, are associated with different critical values of the scaling dimension λ. The representations are labeled by the supermultiplets

- $(4,4,0)$**:**
This superalgebra is critical at $\lambda=\frac{1}{4}$. We have $\Lambda=diag(\lambda,\lambda,\lambda,\lambda,\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2})$.
The square roots of $\pm H$ are the operators

$Q_1^+ = E_{18} - E_{27} - E_{36} + E_{45} + (E_{54} - E_{63} - E_{72} + E_{81})\partial_t,$
$Q_2^+ = E_{17} + E_{28} + E_{35} + E_{46} + (E_{53} + E_{64} + E_{71} + E_{82})\partial_t,$
$Q_3^+ = E_{15} + E_{26} - E_{37} - E_{48} + (E_{51} + E_{62} - E_{73} - E_{84})\partial_t,$
$Q_1^- = E_{18} + E_{27} - E_{36} - E_{45} + (E_{54} + E_{63} - E_{72} - E_{81})\partial_t,$
$Q_2^- = E_{17} - E_{28} - E_{35} + E_{46} + (E_{53} - E_{64} - E_{71} + E_{82})\partial_t,$
$Q_3^- = E_{15} + E_{26} + E_{37} + E_{48} - (E_{51} + E_{62} + E_{73} + E_{84})\partial_t.$

- $(3,4,1)$**:**
This superalgebra is critical at $\lambda = \frac{1}{2}$. We have $\Lambda = diag(\lambda,\lambda,\lambda,\lambda+1,\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2})$.
The square roots of $\pm H$ are the operators
$Q_1^+ = E_{18} - E_{27} - E_{36} + E_{54} + (E_{45} - E_{63} - E_{72} + E_{81})\partial_t,$
$Q_2^+ = E_{17} + E_{28} + E_{35} + E_{64} + (E_{46} + E_{53} + E_{71} + E_{82})\partial_t,$
$Q_3^+ = E_{15} + E_{26} - E_{37} - E_{84} + (-E_{48} + E_{51} + E_{62} - E_{73})\partial_t,$
$Q_1^- = E_{18} + E_{27} - E_{36} + E_{54} + (-E_{45} + E_{63} - E_{72} - E_{81})\partial_t,$
$Q_2^- = E_{17} - E_{28} - E_{35} - E_{64} + (E_{46} + E_{53} - E_{71} + E_{82})\partial_t,$
$Q_3^- = E_{15} + E_{26} + E_{37} - E_{84} + (E_{48} - E_{51} - E_{62} - E_{73})\partial_t.$

- $(1,4,3)$**:**
This superalgebra is critical at $\lambda = -\frac{1}{2}$. We have $\Lambda = diag(\lambda,\lambda+1,\lambda+1,\lambda+1,\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2})$.
The square roots of $\pm H$ are the operators
$Q_1^+ = (-E_{27} - E_{36} + E_{45} + E_{81})\partial_t + E_{18} + E_{54} - E_{63} - E_{72},$
$Q_2^+ = (E_{28} + E_{35} + E_{46} + E_{71})\partial_t + E_{17} + E_{53} + E_{64} + E_{82},$
$Q_3^+ = (E_{26} - E_{37} - E_{48} + E_{51})\partial_t + E_{15} + E_{62} - E_{73} - E_{84},$
$Q_1^- = (E_{27} - E_{36} - E_{45} - E_{81})\partial_t + E_{18} + E_{54} + E_{63} - E_{72},$
$Q_2^- = (-E_{28} - E_{35} + E_{46} - E_{71})\partial_t + E_{17} + E_{53} - E_{64} + E_{82},$
$Q_3^- = (E_{26} + E_{37} + E_{48} - E_{51})\partial_t + E_{15} - E_{62} - E_{73} - E_{84}.$

- $(0,4,4)$**:**
The superalgebra is critical at $\lambda = -\frac{1}{4}$. We have $\Lambda = diag(\lambda+1,\lambda+1,\lambda+1,\lambda+1,\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2})$.
The square roots of $\pm H$ are the operators
$Q_1^+ = (E_{18} - E_{27} - E_{36} + E_{45})\partial_t + E_{54} - E_{63} - E_{72} + E_{81},$
$Q_2^+ = (E_{17} + E_{28} + E_{35} + E_{46})\partial_t + E_{53} + E_{64} + E_{71} + E_{82},$
$Q_3^+ = (E_{15} + E_{26} - E_{37} - E_{48})\partial_t + E_{51} + E_{62} - E_{73} - E_{84},$
$Q_1^- = (E_{18} + E_{27} - E_{36} - E_{45})\partial_t + E_{54} + E_{63} - E_{72} - E_{81},$
$Q_2^- = (E_{17} - E_{28} - E_{35} + E_{46})\partial_t + E_{53} - E_{64} - E_{71} + E_{82},$
$Q_3^- = (E_{15} + E_{26} + E_{37} + E_{48})\partial_t - E_{51} - E_{62} - E_{73} - E_{84}.$

The $\mathcal{N} = (4,4)$ superconformal algebra

In this case $n = 16$. There is a unique superconformal algebra, see Section **4**. Its two representations are given by the supermultiplets

- $(8,8,0)$**:**
We have $\Lambda = diag(\lambda,\lambda,\lambda,\lambda,\lambda,\lambda,\lambda,\lambda,\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2})$, at the critical value, due to the closure conditions, $\lambda = \frac{1}{4}$.
The square roots of $\pm H$ are the operators
$Q_1^+ = E_{1,16} + E_{2,15} + E_{3,14} + E_{4,13} + E_{5,12} + E_{6,11} + E_{7,10} + E_{8,9} + (E_{9,8} + E_{10,7} + E_{11,6} + E_{12,5} + E_{13,4} + E_{14,3} + E_{15,2} + E_{16,1})\partial_t,$
$Q_2^+ = E_{1,15} - E_{2,16} + E_{3,13} - E_{4,14} + E_{5,11} - E_{6,12} + E_{7,9} - E_{8,10} + (E_{9,7} - E_{10,8} + E_{11,5} - E_{12,6} + E_{13,3} - E_{14,4} + E_{15,1} - E_{16,2})\partial_t,$
$Q_3^+ = E_{1,13} + E_{2,14} - E_{3,15} - E_{4,16} + E_{5,9} + E_{6,10} - E_{7,11} - E_{8,12} + (E_{9,5} + E_{10,6} - E_{11,7} - E_{12,8} + E_{13,1} + E_{14,2} - E_{15,3} - E_{16,4})\partial_t,$
$Q_4^+ = E_{1,9} + E_{2,10} + E_{3,11} + E_{4,12} - E_{5,13} - E_{6,14} - E_{7,15} - E_{8,16} + (E_{9,1} + E_{10,2} + E_{11,3} + E_{12,4} - E_{13,5} - E_{14,6} - E_{15,7} - E_{16,8})\partial_t,$
$Q_1^- = E_{1,16} - E_{2,15} + E_{3,14} - E_{4,13} + E_{5,12} - E_{6,11} + E_{7,10} - E_{8,9} + (E_{9,8} - E_{10,7} + E_{11,6} - E_{12,5} + E_{13,4} - E_{14,3} + E_{15,2} - E_{16,1})\partial_t,$
$Q_2^- = E_{1,15} + E_{2,16} - E_{3,13} - E_{4,14} + E_{5,11} + E_{6,12} - E_{7,9} - E_{8,10} + (E_{9,7} + E_{10,8} - E_{11,5} - E_{12,6} + E_{13,3} + E_{14,4} - E_{15,1} - E_{16,2})\partial_t,$

$Q_3^- = E_{1,13} + E_{2,14} + E_{3,15} + E_{4,16} - E_{5,9} - E_{6,10} - E_{7,11} - E_{8,12} + (E_{9,5} + E_{10,6} + E_{11,7} + E_{12,8} - E_{13,1} - E_{14,2} - E_{15,3} - E_{16,4})\partial_t$,
$Q_4^- = E_{1,9} + E_{2,10} + E_{3,11} + E_{4,12} + E_{5,13} + E_{6,14} + E_{7,15} + E_{8,16} - (E_{9,1} + E_{10,2} + E_{11,3} + E_{12,4} + E_{13,5} + E_{14,6} + E_{15,7} + E_{16,8})\partial_t$.

- $(0,8,8)$**:**
We have $\Lambda = diag(\lambda+1,\lambda+1,\lambda+1,\lambda+1,\lambda+1,\lambda+1,\lambda+1,\lambda+1,\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2})$ at the critical value $\lambda = -\frac{1}{4}$.
The square roots of $\pm H$ are the operators
$Q_1^+ = (E_{1,16} + E_{2,15} + E_{3,14} + E_{4,13} + E_{5,12} + E_{6,11} + E_{7,10} + E_{8,9})\partial_t + E_{9,8} + E_{10,7} + E_{11,6} + E_{12,5} + E_{13,4} + E_{14,3} + E_{15,2} + E_{16,1}$,
$Q_2^+ = (E_{1,15} - E_{2,16} + E_{3,13} - E_{4,14} + E_{5,11} - E_{6,12} + E_{7,9} - E_{8,10})\partial_t + E_{9,7} - E_{10,8} + E_{11,5} - E_{12,6} + E_{13,3} - E_{14,4} + E_{15,1} - E_{16,2}$,
$Q_3^+ = (E_{1,13} + E_{2,14} - E_{3,15} - E_{4,16} + E_{5,9} + E_{6,10} - E_{7,11} - E_{8,12})\partial_t + E_{9,5} + E_{10,6} - E_{11,7} - E_{12,8} + E_{13,1} + E_{14,2} - E_{15,3} - E_{16,4}$,
$Q_4^+ = (E_{1,9} + E_{2,10} + E_{3,11} + E_{4,12} - E_{5,13} - E_{6,14} - E_{7,15} - E_{8,16})\partial_t + E_{9,1} + E_{10,2} + E_{11,3} + E_{12,4} - E_{13,5} - E_{14,6} - E_{15,7} - E_{16,8}$,
$Q_1^- = (E_{1,16} - E_{2,15} + E_{3,14} - E_{4,13} + E_{5,12} - E_{6,11} + E_{7,10} - E_{8,9})\partial_t + E_{9,8} - E_{10,7} + E_{11,6} - E_{12,5} + E_{13,4} - E_{14,3} + E_{15,2} - E_{16,1}$,
$Q_2^- = (E_{1,15} + E_{2,16} - E_{3,13} - E_{4,14} + E_{5,11} + E_{6,12} - E_{7,9} - E_{8,10})\partial_t + E_{9,7} + E_{10,8} - E_{11,5} - E_{12,6} + E_{13,3} + E_{14,4} - E_{15,1} - E_{16,2}$,
$Q_3^- = (E_{1,13} + E_{2,14} + E_{3,15} + E_{4,16} - E_{5,9} - E_{6,10} - E_{7,11} - E_{8,12})\partial_t + E_{9,5} + E_{10,6} + E_{11,7} + E_{12,8} - E_{13,1} - E_{14,2} - E_{15,3} - E_{16,4}$,
$Q_4^- = (E_{1,9} + E_{2,10} + E_{3,11} + E_{4,12} + E_{5,13} + E_{6,14} + E_{7,15} + E_{8,16})\partial_t - (E_{9,1} + E_{10,2} + E_{11,3} + E_{12,4} + E_{13,5} + E_{14,6} + E_{15,7} + E_{16,8})$.

The $\mathcal{N} = (5,1)$ superconformal algebra

In this case $n = 16$. There is a unique superconformal algebra, see Section **4**. Its two representations are given by the supermultiplets

- $(8,8,0)$**:**
We have $\Lambda = diag(\lambda,\lambda,\lambda,\lambda,\lambda,\lambda,\lambda,\lambda,\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2})$ at the critical value $\lambda = \frac{1}{4}$.
The square roots of $\pm H$ are the operators
$Q_1^+ = E_{1,9} + E_{2,10} + E_{3,11} + E_{4,12} - E_{5,13} - E_{6,14} - E_{7,15} - E_{8,16} + (E_{9,1} + E_{10,2} + E_{11,3} + E_{12,4} - E_{13,5} - E_{14,6} - E_{15,7} - E_{16,8})\partial_t$,
$Q_2^+ = E_{1,16} - E_{2,15} + E_{3,14} - E_{4,13} - E_{5,12} + E_{6,11} - E_{7,10} + E_{8,9} + (E_{9,8} - E_{10,7} + E_{11,6} - E_{12,5} - E_{13,4} + E_{14,3} - E_{15,2} + E_{16,1})\partial_t$,
$Q_3^+ = E_{1,15} + E_{2,16} - E_{3,13} - E_{4,14} - E_{5,11} - E_{6,12} + E_{7,9} + E_{8,10} + (E_{9,7} + E_{10,8} - E_{11,5} - E_{12,6} - E_{13,3} - E_{14,4} + E_{15,1} + E_{16,2})\partial_t$,
$Q_4^+ = E_{1,14} - E_{2,13} - E_{3,16} + E_{4,15} - E_{5,10} + E_{6,9} + E_{7,12} - E_{8,11} + (E_{9,6} - E_{10,5} - E_{11,8} + E_{12,7} - E_{13,2} + E_{14,1} + E_{15,4} - E_{16,3})\partial_t$,
$Q_5^+ = E_{1,13} + E_{2,14} + E_{3,15} + E_{4,16} + E_{5,9} + E_{6,10} + E_{7,11} + E_{8,12} + (E_{9,5} + E_{10,6} + E_{11,7} + E_{12,8} + E_{13,1} + E_{14,2} + E_{15,3} + E_{16,4})\partial_t$,
$Q_1^- = E_{1,9} + E_{2,10} + E_{3,11} + E_{4,12} + E_{5,13} + E_{6,14} + E_{7,15} + E_{8,16} - (E_{9,1} + E_{10,2} + E_{11,3} + E_{12,4} + E_{13,5} + E_{14,6} + E_{15,7} + E_{16,8})\partial_t$.

- $(0,8,8)$**:**
We have $\Lambda = diag(\lambda+1,\lambda+1,\lambda+1,\lambda+1,\lambda+1,\lambda+1,\lambda+1,\lambda+1,\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2})$ at the critical value $\lambda = -\frac{1}{4}$.
The square roots of $\pm H$ are the operators
$Q_1^+ = (E_{1,9} + E_{2,10} + E_{3,11} + E_{4,12} - E_{5,13} - E_{6,14} - E_{7,15} - E_{8,16})\partial_t + E_{9,1} + E_{10,2} + E_{11,3} + E_{12,4} - E_{13,5} - E_{14,6} - E_{15,7} - E_{16,8}$,
$Q_2^+ = (E_{1,16} - E_{2,15} + E_{3,14} - E_{4,13} - E_{5,12} + E_{6,11} - E_{7,10} + E_{8,9})\partial_t + E_{9,8} - E_{10,7} + E_{11,6} - E_{12,5} - E_{13,4} + E_{14,3} - E_{15,2} + E_{16,1}$,
$Q_3^+ = (E_{1,15} + E_{2,16} - E_{3,13} - E_{4,14} - E_{5,11} - E_{6,12} + E_{7,9} + E_{8,10})\partial_t + E_{9,7} + E_{10,8} - E_{11,5} - E_{12,6} - E_{13,3} - E_{14,4} + E_{15,1} + E_{16,2}$,
$Q_4^+ = (E_{1,14} - E_{2,13} - E_{3,16} + E_{4,15} - E_{5,10} + E_{6,9} + E_{7,12} - E_{8,11})\partial_t + E_{9,6} - E_{10,5} - E_{11,8} + E_{12,7} - E_{13,2} + E_{14,1} +$

$E_{15,4} - E_{16,3}$,
$Q_5^+ = (E_{1,13} + E_{2,14} + E_{3,15} + E_{4,16} + E_{5,9} + E_{6,10} + E_{7,11} + E_{8,12})\partial_t + E_{9,5} + E_{10,6} + E_{11,7} + E_{12,8} + E_{13,1} + E_{14,2} + E_{15,3} + E_{16,4}$,
$Q_1^- = (E_{1,9} + E_{2,10} + E_{3,11} + E_{4,12} + E_{5,13} + E_{6,14} + E_{7,15} + E_{8,16})\partial_t - (E_{9,1} + E_{10,2} + E_{11,3} + E_{12,4} + E_{13,5} + E_{14,6} + E_{15,7} + E_{16,8})$.

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Chapter 4

From worldline to quantum SUSY

4.1 Introduction

In this paper we quantize superconformal σ-models defined by worldline supermultiplets. We consider two types of superconformal mechanics, parabolic or trigonometric [1], namely in the absence or, respectively, in the presence of an oscillatorial DFF term [2].

In the absence of a DFF term the systems under consideration possess only a Calogero potential [3]; they are supersymmetric and with a continuous spectrum. In the presence of a DFF term they correspond to deformed (if the Calogero potential is present) or undeformed oscillators with a discrete, bounded from below, spectrum. For these (un)deformed oscillators the classical invariant superconformal algebra acts as a spectrum-generating algebra of the quantum theory.

We illustrate at first our method with two $osp(1|2)$-invariant examples, the ordinary one-dimensional harmonic oscillator being recovered in the trigonometric case. Later we explicitly quantize the superconformally-invariant worldine σ-models defined by
i) the $\mathcal{N}=4$ $(1,4,3)$ supermultiplet with scaling dimension $\alpha\neq 0,-1$ (these models are classically invariant under the exceptional $D(2,1;\alpha)$ Lie superalgebra) and
ii) the $\mathcal{N}=2$ $(2,2,0)$ supermultiplet of scaling dimension λ (these models present a two-dimensional target and classical $sl(2|1)$-invariance).

For the $(1,4,3)$ supermultiplet, at the special $\alpha=-\frac{1}{2}$ value, the Calogero potential terms are vanishing. For this value the invariant superalgebra is $D(2,1;-\frac{1}{2})=D(2,1)\approx osp(4|2)$.
An interesting result, in the $(1,4,3)$ trigonometric case, consists in the direct and simple interpretation of α as a vacuum energy (if α is regarded as an external control parameter, it determines the Casimir energy of the system).

For the $sl(2|1)$ models the scaling dimension λ is quantized (either $\lambda=\frac{1}{2}+\mathbb{Z}$ or $\lambda=\mathbb{Z}$). In the trigonometric case the ordinary two-dimensional oscillator (without Calogero potential terms) is recovered from the special $\lambda=-\frac{1}{2}$ value after a superselection of the spectrum, defined by a projection operator, is imposed. The spectrum of the theory turns out to be decomposed into an infinite set of lowest weight representations of $sl(2|1)$. By construction, the role of $sl(2|1)$ as a spectrum-generating algebra is expected. What is quite unexpected and surprising is the further result that extra fermionic raising operators, not belonging to the $sl(2|1)$ superalgebra, allow to construct the whole spectrum from the single $\lambda=\frac{1}{2}+\mathbb{Z}$ bosonic vacuum (in Appendix **A** this action is visualized in diagrams).

Models of superconformal mechanics have been investigated in [4]–[12] (see, e.g., the review [13] and references therein). For superconformal actions with oscillator potentials see [14, 15, 1]. (Super)conformal mechanics is currently a very active area of research; among the motivations for this interest one can mention the AdS_2/CFT_1 correspondence [16, 17], or the possibility to apply it to test particles moving in the proximity of the horizon of certain black holes, see [11].

$\mathcal{N}=4$ superconformal models based on the exceptional (see [18]) Lie superalgebra $D(2,1;\alpha)$ were investigated in [19]–[26]. The models considered in those works, mostly classical, are supersymmetric; for that reason they do not allow the presence of the oscillatorial DFF terms (in Appendix **C** we comment about the "soft" supersymmetry property of the oscillatorial models). The recognition in [28] that conformal mechanics could allow new potentials, permitted the introduction in [1] of the trigonometric (read, oscillatorial) classical $D(2,1;\alpha)$ models.

The scheme of the paper is the following.

Sections **2**, **3**, **4** are propaedeutic. In Section **2** we discuss the change of coordinates from linear to non-linear realizations of the superconformal algebras (the "constant kinetic basis") which allows us to present the worldline superconformal σ-models in the Hamiltonian framework. A detailed description of the passage from classical Lagrangians to Hamiltonians is given in Section **3**. In Section **4** the quantization procedure and the construction of the

Noether charges is explained for two examples, the parabolic and trigonometric $osp(1|2)$-invariant σ-models. Section **5** contains the main results for the quantization of the parabolic (i.e. both superconformal and supersymmetric) quantum models with $D(2,1;\alpha)$-invariance, based on the $\mathcal{N}=4$ worldline supermultiplet $(1,4,3)$, and $sl(2,1)$-invariance, based on the $\mathcal{N}=(2,2,0)$ worldline supermultiplet. In Section **6** the main results of their quantum trigonometric versions are derived. These systems contain DFF terms and are "softly supersymmetric". They correspond to (un)deformed oscillators. The main results are the derivation of the vacuum energy in terms of the α scaling dimension for the $(1,4,3)$ supermultiplet and the derivation of the spectrum-generating superalgebra for the (un)deformed two-dimensional oscillator with quantized scaling dimension λ. In Appendix **A** diagrams are presented illustrating the decomposition of the two-dimensional oscillators in terms of the $sl(2|1)$ lowest weight representations, interconnected by the puzzling extra fermionic raising and lowering operators introduced in Section **6**. For completeness in Appendix **B** the classical version of the trigonometric $\mathcal{N}=2$ $(2,2,0)$ superconformal σ-model is presented. Finally, in Appendix **C** we discuss the "soft supersymmetry" of the (un)deformed oscillators and the role, for these theories, of the spectrum generating superalgebras. In the Conclusions we present the open questions raised by our analysis.

4.2 Worldline (super)conformal σ-models in constant kinetic basis

A convenient approach, in constructing one-dimensional superconformal σ-models, consists in starting from a linear D-module representation of the superconformal algebra. Once such a representation is known, the Lagrangian defining the superconformally invariant action can be systematically constructed by applying fermionic generators to a prepotential function which depends only on the propagating bosons. The requirement of superconformal invariance, imposed as a constraint, determines the specific form of the prepotential. This method (and its applications) has been discussed in [1].

The kinetic term $\Phi(\vec{x})\frac{1}{2}\delta_{ij}(\dot{x}_i\dot{x}_j+\ldots)$ of the derived Lagrangian is an ordinary constant kinetic term multiplied by a conformal factor $\Phi(\vec{x})$ which is a function of the propagating bosons. In order to apply the standard methods of quantization we need to reabsorb the conformal factor. One way to do this consists in introducing a new set of fields. In the new basis of fields the kinetic term is expressed as a constant coefficient (hence the name "constant kinetic basis" given in [1]); the superalgebra, on the other hand, is realized non-linearly.

In [1] the procedure of changing the basis (from the "linear" to the "constant kinetic" basis) was sketched for certain D-module representations acting on supermultiplets consisting of a single propagating boson. We discuss it here in a more general framework.

Let us consider a D-module irrep of a $\mathcal{N}$-extended superconformal algebra (for our purposes $\mathcal{N}=1,2,4,8$) acting on a $(k,\mathcal{N},\mathcal{N}-k)$ supermultiplet [29, 30, 31, 32] (namely, k propagating bosons, $\mathcal{N}$ fermions and $\mathcal{N}-k$ bosonic auxiliary fields). In the linear basis the propagating bosons are labeled as x_1, ..., x_k, the fermions as ψ_1, ..., $\psi_{\mathcal{N}}$ and the auxiliary bosons as b_1, ..., $b_{\mathcal{N}-k}$. The kinetic term in the Lagrangian is given by

$$\frac{1}{2}r^{-\frac{1+2\lambda}{\lambda}}(\dot{x}_m\dot{x}_m+i\omega\psi_\beta\dot{\psi}_\beta-\omega^2b_nb_n). \tag{4.1}$$

In the above equation the convention over repeated indices is used. The constant ω is dimensionless (and can be set equal to unity) in the parabolic case, while it is dimensional, see [1], in the hyperbolic/trigonometric case. The function r is $r=(x_mx_m)^{\frac{1}{2}}$ and the parameter λ is the scaling dimension of the supermultiplet. At $\lambda=-\frac{1}{2}$ the kinetic term is constant. For the remaining $\lambda\neq-\frac{1}{2}$ values a change to a constant kinetic basis is required in order to present a kinetic term with constant coefficients. Let us denote the propagating bosons in the constant kinetic basis as y_1, ..., y_k, the fermions as χ_1, ..., $\chi_{\mathcal{N}}$ and the auxiliary bosons as a_1, ..., $a_{\mathcal{N}-k}$. The transformations passing from the "linear" to the "constant kinetic" basis are given by:
i) for the $(1,\mathcal{N},\mathcal{N}-1)$ supermultiplets we have

$$y=-2\lambda x^{-\frac{1}{2\lambda}},\qquad \chi_\beta=x^{-\frac{1+2\lambda}{2\lambda}}\psi_\beta,\qquad a_n=x^{-\frac{1+2\lambda}{2\lambda}}b_n; \tag{4.2}$$

in terms of the new fields equation (4.1) is expressed as

$$\frac{1}{2}(\dot{y}\dot{y}+i\omega\chi_\beta\dot{\chi}_\beta-\omega^2a_na_n); \tag{4.3}$$

ii) when $\mathcal{N}\geq 2$, for the $(2,\mathcal{N},\mathcal{N}-2)$ supermultiplets it is convenient to use a complex notation for the propagating bosons and set

$$\begin{aligned} y&=-2\lambda(x_1+ix_2)^{-\frac{1}{2\lambda}}, & y^*&=-2\lambda(x_1-ix_2)^{-\frac{1}{2\lambda}},\\ \chi_\beta&=r^{-\frac{1+2\lambda}{2\lambda}}\psi_\beta, & a_n&=r^{-\frac{1+2\lambda}{2\lambda}}b_n, \end{aligned} \tag{4.4}$$

so that the kinetic term can be expressed as

$$\frac{1}{2}(\dot{y}\dot{y}^* + i\omega\chi_\beta\dot{\chi}_\beta - \omega^2 a_n a_n); \tag{4.5}$$

iii) when $\mathcal{N} = 4, 8$ it is possible to construct a constant kinetic basis for any $(k, \mathcal{N}, \mathcal{N} - k)$ supermultiplet at the specific $\lambda = 1/2$ value of the scaling dimension via the transformations

$$y_m = \frac{x_m}{r^2}, \qquad \chi_\beta = \frac{\psi_\beta}{r^2}, \qquad a_n = \frac{b_n}{r^2}, \tag{4.6}$$

leading to the kinetic term

$$\frac{1}{2}(\dot{y}_m\dot{y}_m + i\omega\chi_\beta\dot{\chi}_\beta - \omega^2 a_n a_n). \tag{4.7}$$

For $\mathcal{N} = 4$ and $k \neq 2$, irreps of the exceptional superalgebras $D(2, 1; \alpha)$ are recovered, see [25, 26, 1], from the $(k, 4, 4 - k)$ supermultiplets according to the relation

$$\alpha = (2 - k)\lambda. \tag{4.8}$$

At the special $\lambda = \frac{1}{2}$ value the associated superalgebra is $A(1, 1)$ for the $(4, 4, 0)$ supermultiplet and $D(2, 1)$ for the $(3, 4, 1)$ supermultiplet.

For $\mathcal{N} = 8$ and $k \neq 4$, irreps of superconformal algebras are recovered for each supermultiplet $(k, 8, 8 - k)$ at the critical values of the scaling dimension given by

$$\lambda_k = \frac{1}{k - 4}. \tag{4.9}$$

The special value $\lambda = \frac{1}{2}$ yields an irrep of $A(3, 1)$ acting on the supermultiplet $(6, 8, 2)$. The reader is referred to [25, 26] for a detailed discussions on the criticality of the scaling dimension of the $\mathcal{N} = 4, 8$ superconformal algebras.

4.3 From Lagrangians to classical Hamiltonians: an application to the $osp(1|2)$-invariant σ-models

The quantization of the 1D superconformal σ-models follows the canonical procedure formalized by Dirac and based on the classical Hamiltonian formalism. Since these σ-models have fermionic degrees of freedom, the passage from the Lagrangian to the classical Hamiltonian formalism requires the use of Dirac brackets (see, e.g., [33]). The need for Dirac brackets becomes clear after inspecting equations (4.3), (4.5) and (4.7); it is due to the fact that the linear dependence on the fermionic velocities $\dot{\chi}_\beta$ forces us to extend the phase space of the system and treat the fermionic canonical momenta as constraints in this extended phase space. In Dirac's language these constraints are both *primary* (they hold even without using the equations of motion) and *second class* (namely, a constraint that has a non-vanishing Poisson brackets with at least one of the constraints).

This procedure, used throughout the paper, will be illustrated in detail for the simplest possibility given by the $osp(1|2)$-invariant σ-models (their two variants, *parabolic* and *hyperbolic/trigonometric*, see [1]). In the parabolic case the Hamiltonian is identified with a bosonic root of the superconformal algebra, while in the hyperbolic/trigonometric case it is associated with a Cartan element. The parabolic D-module reps describe systems which are supersymmetric, while the hyperbolic/trigonometric reps furnish only a *soft* version of supersymmetry, see the discussion in the Introduction. The hyperbolic and trigonometric models are interrelated via a Wick rotation of the dimensional parameter ω. The trigonometric case is here emphasized with respect to the hyperbolic one because it yields a bounded from below Hamiltonian.

In the rest of this Section we discuss in detail the Hamiltonian formulation of both parabolic and trigonometric $osp(1|2)$-invariant σ-models. The method, notations and conventions here presented are later applied to models with larger superconformal symmetry.

4.3.1 The $osp(1|2)$-invariant parabolic σ-model

In the constant kinetic basis the generators of the $osp(1|2)$ parabolic D-module rep read as

$$H = \begin{pmatrix} \partial_t & 0 \\ 0 & \partial_t \end{pmatrix}, \quad D = \begin{pmatrix} t\partial_t - \frac{1}{2} & 0 \\ 0 & t\partial_t \end{pmatrix}, \quad K = \begin{pmatrix} t^2\partial_t - t & 0 \\ 0 & t^2\partial_t \end{pmatrix},$$
$$Q = \begin{pmatrix} 0 & 1 \\ i\partial_t & 0 \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} 0 & t \\ it\partial_t - i & 0 \end{pmatrix}. \tag{4.10}$$

The above generators act on the column vector supermultiplet $(y,\chi)^T$ possessing the scaling dimension $\lambda=-\frac{1}{2}$.

The bosonic generators H, D, K span the $sl(2)$ Lie subalgebra, while the fermionic generators Q, $\bar{Q}$ span the odd sector of $osp(1|2)$.

The associated $osp(1|2)$-invariant action is simply

$$\mathcal{S}=\int dt\mathcal{L}=\int dt\tfrac{1}{2}(\dot{y}^2+i\chi\dot{\chi}). \tag{4.11}$$

Unlike the $\mathcal{N}\geq 2$ superconformal algebras discussed in the following, for $osp(1|2)$ the same action is recovered by starting from a generic D-module rep with scaling dimension $\lambda\neq-\frac{1}{2}$ and applying the (4.2) change of basis.

For a theory possessing bosons and fermions a conserved Noether charge is expressed, for a symmetry generator O, as

$$C_O=(\delta_O\phi_I)\frac{\partial\mathcal{L}}{\partial\dot{\phi}_I}-J_O, \tag{4.12}$$

where J_O stems from the variation $\delta_O\mathcal{L}=\frac{dJ_O}{dt}$; the sum over the repeated index I labeling the fields is understood. The given ordering of the right hand side of (4.12) is essential in dealing with Grassmann variables and derivatives.

For the case at hand the classical Noether charges are

$$C_H=\tfrac{\dot{y}^2}{2},\quad C_D=\tfrac{t\dot{y}^2}{2}-\tfrac{y\dot{y}}{2},\quad C_K=\tfrac{t^2\dot{y}^2}{2}-ty\dot{y}+\tfrac{y^2}{2},\quad C_Q=\dot{y}\chi,\quad C_{\bar{Q}}=t\dot{y}\chi+y\chi. \tag{4.13}$$

The Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\phi}=\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}) \tag{4.14}$$

lead to the equations of motion

$$\ddot{y}=0,\qquad \dot{\chi}=0. \tag{4.15}$$

The Grassmann variable in the classical $osp(1|2)$ model is a constant and plays essentially no physical role besides ensuring the $osp(1|2)$ invariance.

To introduce the Hamiltonian formalism we have to compute the conjugate momenta given by

$$p=\tfrac{\partial\mathcal{L}}{\partial\dot{y}}=\dot{y},\quad \pi=\tfrac{\partial\mathcal{L}}{\partial\dot{\chi}}=-\tfrac{i\chi}{2}. \tag{4.16}$$

In the Hamiltonian framework the classical charges (4.13) are rewritten as

$$C_H=\frac{p^2}{2},\quad C_D=\frac{tp^2}{2}-\frac{yp}{2},\quad C_K=\frac{t^2p^2}{2}-typ+\frac{y^2}{2},\quad C_Q=p\chi,\quad C_{\bar{Q}}=tp\chi+y\chi. \tag{4.17}$$

The last step requires defining the Dirac brackets. The second equation in (4.16) makes clear why Dirac brackets need to be introduced. The conjugate momentum π to the Grassmann variable χ is not an invertible function of the velocity $\dot{\chi}$. The second equation in (4.16) should therefore be viewed as a second class constraint on the phase space,

$$u=\pi+\frac{i\chi}{2}. \tag{4.18}$$

The super-Poisson bracket involving even or odd f, g functions is given by

$$\{f,g\}_P=\sum_I(-1)^{deg(f)\cdot deg(g)}\frac{\partial f}{\partial\phi_I}\frac{\partial g}{\partial\pi_I}-\frac{\partial f}{\partial\pi_I}\frac{\partial g}{\partial\phi_I}, \tag{4.19}$$

where the degree function deg is 0 if evaluated on bosons and 1 on fermions.

Denoting with u_i the set of all second class contraints, the Dirac bracket reads as

$$\{f,g\}_D=\{f,g\}_P-\sum_{k,l}\{f,u_k\}_PU^{-1}_{kl}\{u_l,g\}_P, \tag{4.20}$$

where $U_{kl}=\{u_k,u_l\}_P$ is a matrix constructed from the super-Poisson brackets of all second class constraints.

u entering (4.18) is a second class constraint, since it satisfies

$$\{u,u\}_P=-i.$$

A straightforward computation gives the non-vanishing Dirac brackets

$$\{y, p\}_D = 1, \qquad \{\chi, \chi\}_D = -i. \tag{4.21}$$

We can derive, with the use of the Dirac brackets, the equations of motion in the Hamiltonian formalism and compute (recovering $osp(1|2)$) the superalgebra satisfied by the (4.17) conserved charges.

In terms of Dirac brackets the Hamilton's equations are

$$\dot{\phi} = \frac{\partial \phi}{\partial t} + \{\phi, C_H\}_D. \tag{4.22}$$

For the case at hand we get

$$\dot{p} = 0, \qquad \dot{\chi} = 0, \tag{4.23}$$

which, together with the $p = \dot{y}$ position, allow to recover (4.15).

4.3.2 The $osp(1|2)$-invariant trigonometric σ-model

In the trigonometric case the passage from the Lagrangian to the Hamiltonian formalism follows the same steps as before. We therefore skip unnecessary comments.

In the constant kinetic basis the generators of the $osp(1|2)$ trigonometric D-module rep are

$$\begin{aligned} H &= e^{i\omega t}\begin{pmatrix} \frac{1}{\omega}\partial_t - \frac{i}{2}x & 0 \\ 0 & \frac{1}{\omega}\partial_t \end{pmatrix}, \quad D = \begin{pmatrix} \frac{1}{\omega}\partial_t & 0 \\ 0 & \frac{1}{\omega}\partial_t \end{pmatrix}, \quad K = e^{-i\omega t}\begin{pmatrix} \frac{1}{\omega}\partial_t + \frac{i}{2}x & 0 \\ 0 & \frac{1}{\omega}\partial_t \end{pmatrix}, \\ Q &= e^{\frac{i\omega t}{2}}\begin{pmatrix} 0 & 1 \\ \frac{i}{\omega}\partial_t + \frac{1}{2}x & 0 \end{pmatrix}, \qquad \bar{Q} = e^{-\frac{i\omega t}{2}}\begin{pmatrix} 0 & 1 \\ \frac{i}{\omega}\partial_t - \frac{1}{2}x & 0 \end{pmatrix}. \end{aligned} \tag{4.24}$$

The $osp(1|2)$-invariant action is

$$\mathcal{S} = \int dt\mathcal{L} = \int dt\frac{1}{2}(\dot{y}^2 + i\omega\chi\dot{\chi}) - \frac{\omega^2}{8}y^2. \tag{4.25}$$

The derived conserved Noether charges are

$$\begin{aligned} C_H &= e^{i\omega t}(\frac{1}{2\omega}\dot{y}^2 - \frac{i}{2}y\dot{y} - \frac{\omega}{8}y^2), \quad C_D = \frac{1}{2\omega}\dot{y}^2 + \frac{\omega}{8}y^2, \quad C_K = e^{-i\omega t}(\frac{1}{2\omega}\dot{y}^2 + \frac{i}{2}y\dot{y} - \frac{\omega}{8}y^2), \\ C_Q &= e^{\frac{i\omega}{2}t}(\dot{y}\chi - \frac{i\omega}{2}y\chi), \quad C_{\bar{Q}} = e^{-\frac{i\omega}{2}t}(\dot{y}\chi + \frac{i\omega}{2}y\chi). \end{aligned} \tag{4.26}$$

The Euler-Lagrange equations of motion are

$$\ddot{y} = -\frac{\omega^2 y}{4}, \qquad \dot{\chi} = 0. \tag{4.27}$$

The conjugate momenta are given by

$$p = \tfrac{\partial \mathcal{L}}{\partial \dot{y}} = \dot{y}, \quad \pi = \tfrac{\partial \mathcal{L}}{\partial \dot{\chi}} = -\tfrac{i\omega\chi}{2}. \tag{4.28}$$

In the Hamiltonian formulation, the (4.26) conserved charges are

$$\begin{aligned} C_H &= e^{i\omega t}(\frac{1}{2\omega}p^2 - \frac{i}{2}yp - \frac{\omega}{8}y^2), \quad C_D = \frac{1}{2\omega}p^2 + \frac{\omega}{8}y^2, \quad C_K = e^{-i\omega t}(\frac{1}{2\omega}p^2 + \frac{i}{2}yp - \frac{\omega}{8}y^2), \\ C_Q &= e^{\frac{i\omega}{2}t}(p\chi - \frac{i\omega}{2}y\chi), \quad C_{\bar{Q}} = e^{-\frac{i\omega}{2}t}(p\chi + \frac{i\omega}{2}y\chi). \end{aligned} \tag{4.29}$$

The second equation in (4.28) gives the constraint in phase space

$$u = \pi + \frac{i\omega\chi}{2}, \tag{4.30}$$

which allows to compute the Dirac brackets as before. The non-vanishing Dirac brackets are

$$\{y, p\}_D = 1, \qquad \{\chi, \chi\}_D = -\frac{i}{\omega}. \tag{4.31}$$

The Hamilton's equations of motion are now written as

$$\dot{\phi} = \omega\{\phi, C_D\}_D + \frac{\partial\phi}{\partial t}. \tag{4.32}$$

One should note that, while in the parabolic σ-model the charge C_H is the physical Hamiltonian and the symmetry operator H is the generator of the time translations, in the trigonometric σ-model the physical hamiltonian is given by ωC_D, the Cartan generator ωD being the generator of the time translations. One can readily check that equation (4.32) leads to

$$\dot{p} = -\frac{\omega^2 y}{4}, \qquad \dot{\chi} = 0, \tag{4.33}$$

which reproduce (4.27) by taking into account that $p = \dot{y}$.

4.4 The quantization. Quantum versus classical Noether charges and the $osp(1|2)$ models

The canonical quantization of the models presented in Section **3** is realized by substituting the Dirac Brackets by the appropriate (based on the superalgebra structure) (anti)commutators, that we will denote with the "$[.,.\}$" symbol:

$$\{A, B\}_D \quad \rightarrow \quad \frac{1}{i\hbar}[A, B\}. \tag{4.34}$$

By applying (4.34) to (4.21) and (4.31) we get, respectively, the parabolic and trigonometric $osp(1|2)$-invariant quantum superconformal models.

We point out that, since the observables must be Hermitian operators, the parabolic and trigonometric quantum models correspond to different real forms (read, conjugations) of the invariant superalgebra. We illustrate in detail this feature, which is also valid for $\mathcal{N} \geq 2$ invariant theories.

4.4.1 The parabolic $osp(1|2)$-invariant quantum σ-model

The non-vanishing (anti)commutators recovered from (4.21) are

$$[\hat{y}, \hat{p}] = i\hbar, \qquad \{\hat{\chi}, \hat{\chi}\} = \hbar. \tag{4.35}$$

In the position-space representation the above operators are given by

$$\hat{y} = y, \qquad \hat{p} = -i\hbar\partial_y, \qquad \hat{\chi} = \sqrt{\tfrac{\hbar}{2}}. \tag{4.36}$$

The last equation is particularly important because it tells us that the fermionic field χ, classically represented by a Grassmann variable, becomes a Clifford variable $\hat{\chi}$ in the quantum version. The choice in (4.36) of representing $\hat{\chi}$ as a real number (that we can think of as the generator of the $Cl(1,0)$ Clifford algebra), is not unique. An alternative choice, which respects the $\mathbb{Z}_2$-graded structure of the super-vector space acted upon by the operators $\hat{y}, \hat{p}, \hat{\chi}$, consists in picking $\hat{\chi}$ as the 2×2 matrix element corresponding to the antidiagonal Clifford algebra generator $Cl(2,1)$ with positive square. In this $\mathbb{Z}_2$-graded representation, the operators $\hat{y}, \hat{p}, \hat{\chi}$ are

$$\hat{y} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}, \quad \hat{p} = \begin{pmatrix} -i\hbar\partial_y & 0 \\ 0 & -i\hbar\partial_y \end{pmatrix}, \quad \hat{\chi} = \sqrt{\tfrac{\hbar}{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad N_f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.37}$$

while N_f is the Fermion number operator.

The possibility, offered by the $\mathbb{Z}_2$-graded structure, of doubling the vector space, will be used in the following in constructing the trigonometric and $\mathcal{N} = 2, 4$ quantum models.

The parabolic quantum $osp(1|2)$ superalgebra obtained by the (4.34) quantization of the classical counterpart, leads to

$$\begin{aligned}
&[\hat{H}, \hat{D}] = i\hbar\hat{H}, \quad [\hat{H}, \hat{K}] = 2i\hbar\hat{D}, \quad [\hat{K}, \hat{D}] = -i\hbar\hat{K} \\
&[\hat{H}, \hat{\tilde{Q}}] = i\hbar\hat{Q}, \quad [\hat{K}, \hat{Q}] = -i\hbar\hat{\tilde{Q}}, \quad [\hat{Q}, \hat{D}] = \frac{i\hbar}{2}\hat{Q}, \quad [\hat{\tilde{Q}}, \hat{D}] = -\frac{i\hbar}{2}\hat{\tilde{Q}}, \\
&\{\hat{Q}, \hat{Q}\} = 2\hbar\hat{H}, \quad \{\hat{Q}, \hat{\tilde{Q}}\} = 2\hbar D, \quad \{\hat{\tilde{Q}}, \hat{\tilde{Q}}\} = 2\hbar K.
\end{aligned} \tag{4.38}$$

The remaining (anti)commutators are vanishing.

The above superalgebra is realized by the quantum charges

$$\hat{H} = \frac{1}{2}\hat{p}^2, \quad \hat{D} = \frac{t}{2}\hat{p}^2 - \frac{1}{4}(\hat{y}\hat{p} + \hat{p}\hat{y}), \quad \hat{K} = \frac{t^2}{2}\hat{p}^2 - \frac{t}{2}(\hat{y}\hat{p} + \hat{p}\hat{y}) + \frac{1}{2}\hat{y}^2,$$
$$\hat{Q} = \hat{\chi}\hat{p}, \qquad \hat{\tilde{Q}} = t\hat{\chi}\hat{p} - \hat{y}\hat{\chi}. \tag{4.39}$$

They are, up to symmetrization, identical to the classical charges. This is a unique feature of the $\mathcal{N} = 1$ $osp(1|2)$-invariant models. From $\mathcal{N} \geq 2$ the models explicitly depend on the scaling dimension λ. As a result, the quantum versions of these theories require corrections which are traced backed to the mapping of the classical Grassmann variables into quantum Clifford generators.

The Hamiltonian $\hat{H}$ in (4.39) corresponds to the one-dimensional free particle. The operators $\hat{H}$, $\hat{D}$ $\hat{K}$ close the $sl(2)$ bosonic symmetry algebra of the system. $\hat{H}$ and $\hat{Q}$ gives the $\mathcal{N} = 1$ algebra of the Supersymmetric Quantum Mechanics. In terms of the (4.36) realization ($\hat{\chi}$ is a real number) the parabolic $osp(1|2)$-invariant model admits no fermionic degrees of freedom. This is no longer the case (fermions are present) if the model is expressed via the (4.37) realization.

In the parabolic model all charges entering (4.39) are observables. The superalgebra (4.38) can be re-expressed in terms of the canonical $osp(1|2)$ Cartan-Weyl basis $H, F^{\pm}, E^{\pm}$ (such that all the structure constants are real), see [18], through the identifications

$$\hat{H} = -E^-, \quad \hat{D} = iH, \quad \hat{K} = -E^+, \quad \hat{Q} = 2F^-, \quad \hat{\tilde{Q}} = 2iF^+. \tag{4.40}$$

The computation of the $osp(1|2)$ structure constants in the new basis is immediate.

The superalgebra conjugation corresponding to (4.39) reads, in the Cartan-Weyl basis, as

$$(E^{\pm})^\dagger = E^{\pm}, \quad H^\dagger = -H, \quad (F^{\pm})^\dagger = \mp(F^{\pm}). \tag{4.41}$$

Concerning the dimensional analysis of the model we can set, without loss of generality, $[\partial_t] = 1$. If we set the Planck constant $\hbar$ and the action $\mathcal{S}$ to be dimensionless, we therefore get $[\hat{y}] = -\frac{1}{2}$, $[\hat{p}] = \frac{1}{2}$, $[\hat{\chi}] = [\mathcal{S}] = 0$.

4.4.2 The trigonometric $osp(1|2)$-invariant quantum σ-model

The quantization of the trigonometric model follows the same lines of the parabolic one. Without loss of generality we can set $\omega = 1$, reproducing the non-vanishing (anti)commutators (4.35) and the (4.36) and (4.37) position-space representations for the operators $\hat{y}, \hat{p}, \hat{\chi}$.

The quantum trigonometric generators, identical to the classical ones up to symmetrization, are

$$\hat{H} = e^{it}(\frac{1}{2}\hat{p}^2 - \frac{i}{4}(\hat{y}\hat{p} + \hat{p}\hat{y}) - \frac{1}{8}\hat{y}^2), \quad \hat{K} = e^{-it}(\frac{1}{2}\hat{p}^2 + \frac{i}{4}(\hat{y}\hat{p} + \hat{p}\hat{y}) - \frac{1}{8}\hat{y}^2),$$
$$\hat{D} = \frac{1}{2}\hat{p}^2 + \frac{1}{8}\hat{y}^2, \qquad \hat{Q} = e^{\frac{it}{2}}(\hat{\chi}\hat{p} - \frac{i}{2}\hat{\chi}\hat{y}), \qquad \hat{\tilde{Q}} = e^{-\frac{it}{2}}(\hat{\chi}\hat{p} + \frac{i}{2}\hat{\chi}\hat{y}). \tag{4.42}$$

In the (4.42) realization, the $osp(1|2)$ non-vanishing brackets reads as

$$[\hat{H}, \hat{D}] = \hbar\hat{H}, \quad [\hat{H}, \hat{K}] = 2\hbar\hat{D}, \quad [\hat{K}, \hat{D}] = -\hbar\hat{K},$$
$$[\hat{H}, \hat{\tilde{Q}}] = \hbar\hat{Q}, \quad [\hat{K}, \hat{Q}] = -\hbar\hat{\tilde{Q}}, \quad [\hat{Q}, \hat{D}] = \frac{\hbar}{2}\hat{Q}, \quad [\hat{\tilde{Q}}, \hat{D}] = -\frac{\hbar}{2}\hat{\tilde{Q}},$$
$$\{\hat{Q}, \hat{Q}\} = 2\hbar\hat{H}, \quad \{\hat{Q}, \hat{\tilde{Q}}\} = 2\hbar D, \quad \{\hat{\tilde{Q}}, \hat{\tilde{Q}}\} = 2\hbar K. \tag{4.43}$$

The $osp(1|2)$ Cartan-Weyl basis is recovered, from the (4.42) trigonometric charges, via the identifications

$$\hat{H} = E^-, \quad \hat{D} = H, \quad \hat{K} = -E^+, \quad \hat{Q} = 2iF^-, \quad \hat{\tilde{Q}} = -2iF^+. \tag{4.44}$$

We obtain a different conjugation with respect to the parabolic case, given by

$$(E^{\pm})^\dagger = -E^{\mp}, \qquad H^\dagger = H, \qquad (F^{\pm})^\dagger = F^{\mp}. \tag{4.45}$$

In the trigonometric case the Hamiltonian is given by the $osp(1|2)$ Cartan generator $\omega\hat{D}$.

By taking into account the presence of the dimensional parameter ω that we set, for convenience, equal to 1 in the formulas above, the dimensional analysis of the trigonometric model gives us the dimensions $[t] = -1$, $[\hat{y}] = -\frac{1}{2}$, $[\hat{p}] = \frac{1}{2}$, $[\hat{\chi}] = -\frac{1}{2}$, $[\omega] = 1$, $[\mathcal{S}] = 0$.

4.5 Superconformal Quantum Mechanics with Calogero potentials: $1D$ $D(2,1;\alpha)$ and $2D$ $sl(2|1)$ models

In this Section we quantize the worldline superconformal σ-models recovered from the $\mathcal{N}=4$ $(1,4,3)$ (i.e., one-dimensional target) and $\mathcal{N}=2$ $(2,2,0)$ (i.e., two-dimensional target) parabolic supermultiplets. Unlike the $\mathcal{N}=1$ parabolic model analyzed in Section **4**, non-trivial potential terms and non-trivial quantum corrections to the classical Hamiltonians, appear.

The $\mathcal{N}=4$ $(1,4,3)$ parabolic model possesses a $D(2,1;\alpha)$ invariance, where $\alpha\neq 0,-1$ is identified with the scaling dimension of the supermultiplet. The Hamiltonian describes a particle moving on a line under an inverse square potential and includes spin-like degrees of freedom.

The $\mathcal{N}=2$ $(2,2,0)$ parabolic model possesses an $sl(2|1)$ invariance. Its Hamiltonian describes a particle moving on a plane under an inverse square potential and with a spin-orbit coupling.

4.5.1 The $\mathcal{N}=4$ $(1,4,3)$ parabolic model with $D(2,1;\alpha)$ invariance

A discussion of the classical $\mathcal{N}=4$ $(1,4,3)$ superconformal worldline model can be found, e.g., in [1]. We present here the quantization of this model repeating the same steps discussed in Section **4** for the $osp(1|2)$-invariant model.

The non-vanishing (anti)commutators obtained from quantizing the Dirac brackets are

$$[\hat{y},\hat{p}]=i,\qquad \{\hat{\chi}_\alpha,\hat{\chi}_\beta\}=\delta_{\alpha\beta}, \tag{4.46}$$

with $\alpha,\beta=0,...,3$. The above equations define the superalgebra $\mathfrak{h}_1\oplus C_4$, with the one-dimensional Heisenberg algebra $\mathfrak{h}_1$ in its even sector and the four $C\ell(4,0)$ Clifford algebra gamma-matrices in its odd sector. These gamma-matrices can be expressed as 4×4 complex matrices. We choose, to respect the $\mathbb{Z}_2$-graded structure of the superalgebra, block-antidiagonal gamma matrices, while representing the Heisenberg generators as block-diagonal operators.

The position-space representation of (4.46) is

$$\begin{array}{l}\hat{y}=y\mathbb{I}_4,\qquad \hat{p}=-i\partial_y\mathbb{I}_4,\\ \hat{\chi}_0=\frac{1}{\sqrt{2}}\sigma_2\otimes\mathbb{I}_2,\quad \hat{\chi}_1=-\frac{1}{\sqrt{2}}\sigma_1\otimes\sigma_1,\quad \hat{\chi}_2=-\frac{1}{\sqrt{2}}\sigma_1\otimes\sigma_2,\quad \hat{\chi}_3=-\frac{1}{\sqrt{2}}\sigma_1\otimes\sigma_3,\end{array} \tag{4.47}$$

where $\mathbb{I}_n$ is the n x n identity matrix and the σ_i's $(i=1,2,3)$ are the Pauli matrices.

The quantum charges are given by

$$\begin{array}{rcl}\hat{H}&=&(\frac{\hat{p}^2}{2}+\frac{(1+2\alpha)^2}{8\hat{y}^2})\mathbb{I}_4+\frac{1+2\alpha}{4\hat{y}^2}\mathcal{F}_4,\\ \hat{D}&=&(\frac{t\hat{p}^2}{2}-\frac{1}{4}(\hat{y}\hat{p}+\hat{p}\hat{y})+\frac{t(1+2\alpha)^2}{8\hat{y}^2})\mathbb{I}_4+\frac{t(1+2\alpha)}{4\hat{y}^2}\mathcal{F}_4,\\ \hat{K}&=&(\frac{t^2\hat{p}^2}{2}-\frac{t}{2}(\hat{y}\hat{p}+\hat{p}\hat{y})+\frac{\hat{y}^2}{2}+\frac{t^2(1+2\alpha)^2}{8\hat{y}^2})\mathbb{I}_4+\frac{t^2(1+2\alpha)}{4\hat{y}^2}\mathcal{F}_4,\\ \hat{Q}_0&=&\hat{\chi}_0\hat{p}+\frac{i(1+2\alpha)}{6}\epsilon_{ijk}\frac{\hat{\chi}_i\hat{\chi}_j\hat{\chi}_k}{\hat{y}},\\ \hat{Q}_i&=&\hat{\chi}_i\hat{p}-\frac{i(1+2\alpha)}{2}\epsilon_{ijk}\frac{\hat{\chi}_0\hat{\chi}_j\hat{\chi}_k}{\hat{y}},\\ \hat{\tilde{Q}}_0&=&t\hat{\chi}_0\hat{p}-\chi_0\hat{y}+\frac{it(1+2\alpha)}{6}\epsilon_{ijk}\frac{\hat{\chi}_i\hat{\chi}_j\hat{\chi}_k}{\hat{y}},\\ \hat{\tilde{Q}}_i&=&t\hat{\chi}_i\hat{p}-\chi_i\hat{y}-\frac{it(1+2\alpha)}{2}\epsilon_{ijk}\frac{\hat{\chi}_0\hat{\chi}_j\hat{\chi}_k}{\hat{y}},\\ \hat{J}_i&=&-i(\frac{1}{2}\epsilon_{ijk}\hat{\chi}_j\hat{\chi}_k+\hat{\chi}_0\hat{\chi}_i),\\ \hat{L}_i&=&-i(\frac{1}{2}\epsilon_{ijk}\hat{\chi}_j\hat{\chi}_k-\hat{\chi}_0\hat{\chi}_i).\end{array} \tag{4.48}$$

In the above formulas we used the Fermi number operator $\mathcal{F}_4$, defined by $\mathcal{F}_{2n}=\left(\begin{smallmatrix}\mathbb{I}_n & 0\\ 0 & -\mathbb{I}_n\end{smallmatrix}\right)$.

One should note that the quantum operators $\hat{H}$, $\hat{D}$, $\hat{K}$ contain an Ehrenfest quantum correction term, proportional to $\frac{\hbar^2(1+2\alpha)^2}{\hat{y}^2}\mathbb{I}_4$, which is not present in the classical charges. Its appearance can be traced to the change from classical Grassmann to quantum Clifford variables.

At a given value $\alpha\neq 0,-1$, the above operators close the exceptional superalgebra $D(2,1;\alpha)$. The R-symmetry generators $\hat{J}_i$ and $\hat{L}_i$, $i=1,2,3$, close two independent ($[\hat{J}_i,\hat{L}_j]=0$) $su(2)$ subalgebras.

In the Cartan-Weyl basis the non-vanishing $D(2,1;\alpha)$ brackets are given by

$$\begin{aligned}
&[H,E^{\pm}]=\pm E^{\pm},\quad [E^+,E^-]=2H,\quad [H,F^{\pm}_{\beta}]=\pm\frac{1}{2}F^{\pm}_{\beta},\quad [E^{\pm},F^{\mp}_{\beta}]=-F^{\pm}_{\beta},\\
&\{F^{\pm}_0,F^{\mp}_j\}=-\frac{i}{4}(\lambda J_j+(1+\lambda)L_j),\quad \{F^{+}_j,F^{-}_k\}=\epsilon_{jkl}(-\frac{i\lambda}{4}J_l+\frac{i(\lambda+1)}{4}L_l),\\
&\{F^{\pm}_{\beta},F^{\pm}_{\gamma}\}=\pm\frac{1}{2}\delta_{\beta\gamma}E^{\pm},\quad [J_j,F^{\pm}_0]=iF^{\pm}_j,\quad [J_j,F^{\pm}_k]=i(-\delta_{jk}F^{\pm}_0+\epsilon_{jkl}F^{\pm}_l),\\
&[L_j,F^{\pm}_0]=-iF^{\pm}_j,\qquad [L_j,F^{\pm}_k]=i(\delta_{jk}F^{\pm}_0+\epsilon_{jkl}F^{\pm}_l),\\
&[J_j,J_k]=2i\epsilon_{jkl}J_l,\qquad [L_j,L_k]=2i\epsilon_{jkl}L_l.
\end{aligned}\tag{4.49}$$

The above superalgebra is realized by the (4.48) quantum operators via the identifications

$$\hat{H}=-E^-,\quad \hat{D}=iH,\quad \hat{K}=-E^+,\quad \hat{Q}_{\beta}=2F^-_{\beta},\quad \hat{\tilde{Q}}_{\beta}=2iF^+_{\beta},\quad \hat{J}_j=J_j,\quad \hat{K}_j=K_j.\tag{4.50}$$

The Hamiltonian operator $\hat{H}$, explicitly written in 4×4 supermatrix form, is given by

$$\hat{H}=\left(\begin{array}{c|c}(\frac{\hat{p}^2}{2}+\frac{4\alpha^2+8\alpha+3}{8\hat{y}^2})\mathbb{I}_2 & 0\\ \hline 0 & (\frac{\hat{p}^2}{2}+\frac{4\alpha^2-1}{8\hat{y}^2})\mathbb{I}_2\end{array}\right).\tag{4.51}$$

It is the Hamiltonian of the $\mathcal{N}=4$ super-Calogero model with $D(2,1;\alpha)$ invariance.

It contains a (purely bosonic) Calogero Hamiltonian in both its upper and lower diagonal blocks. We recall that the Calogero Hamiltonian $\mathcal{H}_C$ is given by

$$\mathcal{H}_C=\frac{1}{2}\hat{p}^2+\frac{g^2}{\hat{y}^2}.\tag{4.52}$$

The self-adjointness of the Calogero Hamiltonian $\mathcal{H}_C$ depends on the value of the coupling parameter g. We refer to the [34, 35] papers for a thorough discussion of this subtle point.

For our purposes it is important to note here the relation between the coupling constant g and the scaling dimension parameter α. From [34] we know that $\mathcal{H}_C$ is self-adjoint, provided that the inequality $g^2>-\frac{1}{8}$ is satisfied. Under this condition the boundary value problem

$$\mathcal{H}_C\phi_k=E_k\phi_k,\qquad \phi_k(0)=0,$$

gives a continuous positive spectrum, $0\leq E_k<\infty$, the eigenfunctions and eigenvalues being

$$\phi_k(y)=2^{\mu-\frac{1}{2}}\Gamma(\mu+\frac{1}{2})(ky)^{-(\mu-\frac{1}{2})}J_{\mu-\frac{1}{2}}(ky)y^{\mu},\qquad E_k=\frac{1}{2}k^2,$$

for

$$g^2=\frac{1}{2}\mu(\mu-1).\tag{4.53}$$

Let us set

$$g_b^2=\frac{4\alpha^2+8\alpha+3}{8},\qquad g_f^2=\frac{4\alpha^2-1}{8},\tag{4.54}$$

for the Calogero parameters entering, respectively, upper and lower diagonal blocks of the blocks of the (4.51) Hamiltonian. It is quite rewarding that imposing, simultaneously, the $g_b^2,g_f^2>-\frac{1}{8}$ condition, we end up with the $\alpha\neq 0,-1$ inequality for the scaling dimension. The class of exceptional $D(2,1;\alpha)$ superalgebras guarantee the existence of a well-defined Hamiltonian with a continuous positive spectrum bounded from below.

At the special $\alpha=-\frac{1}{2}$ value the Calogero potential terms (in both upper and lower blocks) vanish. Therefore, this special point corresponds to a free theory. At this given value, see [18], we have $D(2,1;-\frac{1}{2})=D(2,1)$, so that the invariant superalgebra coincides with the classical $D(2,1)\approx osp(4|2)$ superalgebra.

We can express, from (4.53), g_b,g_f in term of their respective μ_b,μ_f parameters. From (4.54) μ_b, μ_f can be given in terms of α. The result is the linear relations

$$\mu_b=\tfrac{1}{2}\pm(\alpha+1),\quad \mu_f=\tfrac{1}{2}\pm\alpha.\tag{4.55}$$

In quantum mechanics the continuity conditions are also imposed imposed on the probability currents. Since the zero-energy wave functionis (up to a normalizing factor) $\phi_0(y)=y^{\mu}$, these conditions imply that both μ_b,μ_f must satisfy $\mu_b, \mu_f > \frac{1}{2}$ to ensure continuity at the origin. The (4.55) equations show that any $\alpha \neq 0, -1$ is suitable to fulfill these constraints.

As a final comment we point out that the energy levels of both bosonic (upper) and fermionic (lower) blocks are doubly degenerated. This degeneracy is removed by taking into account the hermitian operators $\hat{J}_3$, $\hat{L}_3$ which commute with $\hat{H}$. Indeed,

$$\hat{J}_3 = \left(\begin{array}{c|c} \sigma_3 & 0 \\ \hline 0 & 0 \end{array}\right), \qquad \hat{L}_3 = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \sigma_3 \end{array}\right), \tag{4.56}$$

are both diagonal and specify spin-like quantum numbers in the bosonic and fermionic sectors, respectively. We can say that the bosonic states have $\frac{1}{2}$ $\hat{J}$-spin and 0 $\hat{L}$-spin, while the fermionic states have 0 $\hat{J}$-spin and $\frac{1}{2}$ $\hat{L}$-spin .

4.5.2 The $\mathcal{N} = 2$ $(2, 2, 0)$ parabolic model with $sl(2|1)$ invariance

The classical $sl(2|1)$-invariant action based on the parabolic D-module rep of the $(2, 2, 0)$ supermultiplet is presented in Appendix **B**. Its quantization is performed with the techniques previously outlined (introduction of the "constant kinetic basis", Dirac brackets, etc.). For this model it is convenient to express the two propagating bosons in terms of a complex field y.

We obtain the non-vanishing (anti)commutators

$$[y^*, p_{y^*}] = [y, p_y] = i\hbar, \quad \{\chi, \chi^\dagger\} = \tfrac{\hbar}{C}. \tag{4.57}$$

The fermions can be expressed as $\chi = \sqrt{\frac{\hbar}{C}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\chi^\dagger = \sqrt{\frac{\hbar}{C}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Let us fix, for simplicity, $\hbar = 1$ and $C = \frac{1}{2}$. The quantum operators $\hat{Q}^1_-$, $\hat{Q}^2_-$ can be written as

$$\hat{Q}^1_- = i\begin{pmatrix} 0 & -A \\ A^\dagger & 0 \end{pmatrix}, \qquad \hat{Q}^2_- = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}, \tag{4.58}$$

where

$$A^\dagger = -\tfrac{i}{\sqrt{2}} e^{-i2\lambda\theta} \left(\partial_r + \tfrac{i}{r}\partial_\theta + \tfrac{2\lambda+1}{2r}\right), \quad A = -\tfrac{i}{\sqrt{2}} e^{i2\lambda\theta} \left(\partial_r - \tfrac{i}{r}\partial_\theta + \tfrac{2\lambda+1}{2r}\right), \tag{4.59}$$

are expressed in polar coordinates ($y = re^{i\theta}$, $y^* = re^{-i\theta}$).

The quantum hamiltonian is

$$\hat{H} = -\tfrac{1}{2}\left(\partial_r^2 + \tfrac{1}{r}\partial_r + \tfrac{1}{r^2}\partial_\theta^2\right) + i\tfrac{(2\lambda+1)}{2r^2}\sigma_z\partial_\theta + \tfrac{(2\lambda+1)^2}{8r^2}, \tag{4.60}$$

with σ_z the diagonal Pauli matrix. $\frac{(2\lambda+1)^2}{8r^2}$ is the Ehrenfest term resulting from quantization.

The remaining quantum Noether charges are

$$\begin{gathered} \hat{L}_0 = t\hat{H} + \tfrac{i}{2}\left(r\partial_r + 1\right), \ \hat{L}_1 = t^2\hat{H} + it\left(r\partial_r + 1\right) + \tfrac{r^2}{2}, \ \hat{J} = -\tfrac{i}{2}\partial_\theta - \tfrac{2\lambda-1}{4}\sigma_z, \\ \hat{Q}^1_+ = t\hat{Q}^1_- - i\tfrac{r}{\sqrt{2}}\begin{pmatrix} 0 & e^{i2\lambda\theta} \\ -e^{-i2\lambda\theta} & 0 \end{pmatrix}, \ \hat{Q}^2_+ = t\hat{Q}^2_- - \tfrac{r}{\sqrt{2}}\begin{pmatrix} 0 & e^{i2\lambda\theta} \\ e^{-i2\lambda\theta} & 0 \end{pmatrix}. \end{gathered} \tag{4.61}$$

The non-vanishing (anti)commutators, closing the $sl(2|1)$ superalgebra are $(m, n = 0, \pm 1)$:

$$\begin{gathered} [\hat{L}_n, \hat{L}_m] = i\left(m-n\right)\hat{L}_{m+n}, \quad [\hat{L}_0, \hat{Q}^a_\pm] = \frac{i}{2}\hat{Q}^a_\pm, \quad [\hat{L}_{\pm1}, \hat{Q}^a_\mp] = \mp i\hat{Q}^a_\pm, \\ [\hat{J}, \hat{Q}^a_\pm] = \frac{i}{2}\epsilon_{ab}\hat{Q}^b_\pm, \quad \{\hat{Q}^a_\pm, \hat{Q}^b_\pm\} = 2\delta_{ab}\hat{L}_{\pm1}, \quad \{\hat{Q}^a_\pm, \hat{Q}^b_\mp\} = 2\delta_{ab}\hat{L}_0, \quad \{\hat{Q}^a_\pm, \hat{Q}^b_\mp\} = \pm 2\epsilon_{ab}\hat{J}, \end{gathered} \tag{4.62}$$

where $\hat{L}_{-1} = \hat{H}$, $a, b = 1, 2$ and $\epsilon_{12} = -\epsilon_{21} = 1$.

The eigenvalue equation $\hat{H}\psi_{E_{m\pm}} = E_{m\pm}\psi_{E_{m\pm}}$, for $E_{m\pm} > 0$, produces a continuum spectrum with eigenfunctions

$$\begin{aligned} \psi_{Em+}(r,\theta) &= J_{|\frac{2\lambda+1}{2}-m|}(\alpha r)e^{im\theta}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \psi_{Em-}(r,\theta) &= J_{|\frac{2\lambda+1}{2}+m|}(\alpha r)e^{im\theta}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \tag{4.63}$$

where $J_{|\frac{2\lambda+1}{2}-m|}(\alpha r)$ and $J_{|\frac{2\lambda+1}{2}+m|}(\alpha r)$ are Bessel functions and $\alpha=\sqrt{2E}$.

To conclude the analysis of this model, we present it as a Supersymmetric Quantum Mechanics. Let us introduce

$$\hat{Q}=\frac{\hat{Q}_{-}^{2}+i\hat{Q}_{-}^{1}}{2}=\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad \hat{Q}^{\dagger}=\frac{\hat{Q}_{-}^{2}-i\hat{Q}_{-}^{1}}{2}=\begin{pmatrix} 0 & 0 \\ A^{\dagger} & 0 \end{pmatrix}. \tag{4.64}$$

We get $\{\hat{Q},\hat{Q}^{\dagger}\}=2\hat{H}$ and $\hat{Q}^{2}=(\hat{Q}^{\dagger})^{2}=0$.

From the expressions (4.59), it follows that $\hat{Q}\psi_{E_{m-}}=\psi_{E_{(m+2\lambda)+}}$ and $\hat{Q}^{\dagger}\psi_{E_{m+}}=\psi_{E_{(m-2\lambda)-}}$. Since $m+2\lambda$ and $m-2\lambda$ need to be intege numbersr, $\hat{Q}\psi_{E_{m-}}$ and $\hat{Q}^{\dagger}\psi_{E_{m+}}$ belong to the Hilbert space only if 2λ is an integer number. A supersymmetric pair is therefore only encountered for the quantized values of the scaling dimension, either $\lambda\in\frac{1}{2}+\mathbb{Z}$ or $\lambda\in\mathbb{Z}$.

4.6 Superconformal Quantum Mechanics with DFF oscillator potential terms: $1D$ $D(2,1;\alpha)$ and $2D$ $sl(2|1)$ models

In this Section we quantize the worldline trigonometric σ-models obtained from the $\mathcal{N}=4$ $(1,4,3)$ and $\mathcal{N}=2$ $(2,2,0)$ supermultiplets (see Appendix **B**). They contain (besides a Calogero potential) an oscillatorial (DFF term) which furnishes a discrete, bounded from below, spectrum. The associated $D(2,1;\alpha)$ and, respectively, $sl(2|1)$superconformal algebras act as spectrum-generating algebras for these models.

The $D(2,1;\alpha)$ $(1,4,3)$ trigonometric σ-models shed some new light on the results of Calogero [3] and de Alfaro, Fubini and Furlan [2]. Indeed, their Casimir energy linearly depends (in two regions) on the scaling dimension parameter α (in contrast with the complicated dependence expressed in terms of the Calogero coupling constant, see [34]).

Interesting features are also presented by the $sl(2|1)$ $(2,2,0)$ trigonometric σ-models. The scaling dimension λ needs to be quantized (either $\lambda=\frac{1}{2}+\mathbb{Z}$ or $\lambda\in\mathbb{Z}$). At the special $\lambda=-\frac{1}{2}$ value the ordinary two-dimensional oscillator (since the Calogero potential vanishes at this special point) can be recovered. The Hilbert space of these class of models is decomposed into an infinite direct sum of $sl(2|1)$ lowest weight representations. An unexpected feature is the existence of fermionic raising operators (not entering the $sl(2|1)$ superalgebra) which allow, together with the $sl(2|1)$ raising operators, for $\lambda=\frac{1}{2}+\mathbb{Z}$ to recover the whole Hilbert space of the theory from the single bosonic vacuum. The existence of these extra fermionic operators is traced to the presence of a discrete symmetry.

4.6.1 The quantum $D(2,1;\alpha)$ trigonometric model from $\mathcal{N}=4$ $(1,4,3)$

The quantization of this model follows the same steps as the quantization of the $osp(1|2)$-invariant trigonometric model described in Section **4**. We end up, just like its $\mathcal{N}=4$ $(1,4,3)$ parabolic counterpart of Section **5**, with (anti)commutators defining the the $\mathfrak{h}_1\oplus C_4$ superalgebra (4.46). We set, for convenience and without loss of generality, the dimensional parameter ω (its presence in the equations can be restored by means of dimensional analysis).

The quantum operators are ($\mathcal{F}_4$ is the Fermion Number operator introduced in (4.48))

$$\begin{aligned}
\hat{H} &= e^{it}(\frac{\hat{p}^2}{2} - \frac{i}{4}(\hat{y}\hat{p}+\hat{p}\hat{y}) - \frac{\hat{y}^2}{8} + \frac{(1+2\alpha)^2}{8\hat{y}^2})\mathbb{I}_4 + e^{it}\frac{1+2\alpha}{4\hat{y}^2}\mathcal{F}_4,\\
\hat{D} &= (\frac{\hat{p}^2}{2} + \frac{\hat{y}^2}{8} + \frac{(1+2\alpha)^2}{8\hat{y}^2})\mathbb{I}_4 + \frac{(1+2\alpha)}{4\hat{y}^2}\mathcal{F}_4,\\
\hat{K} &= e^{-it}(\frac{\hat{p}^2}{2} + \frac{i}{4}(\hat{y}\hat{p}+\hat{p}\hat{y}) - \frac{\hat{y}^2}{8} + \frac{(1+2\alpha)^2}{8\hat{y}^2})\mathbb{I}_4 + e^{-it}\frac{1+2\alpha}{4\hat{y}^2}\mathcal{F}_4,\\
\hat{Q}_0 &= e^{\frac{it}{2}}(\hat{\chi}_0\hat{p} - \frac{i}{2}\hat{\chi}_0\hat{y} + \frac{i(1+2\alpha)}{6}\epsilon_{ijk}\frac{\hat{\chi}_i\hat{\chi}_j\hat{\chi}_k}{\hat{y}}),\\
\hat{Q}_i &= e^{\frac{it}{2}}(\hat{\chi}_i\hat{p} - \frac{i}{2}\hat{\chi}_i\hat{y} - \frac{i(1+2\alpha)}{2}\epsilon_{ijk}\frac{\hat{\chi}_0\hat{\chi}_j\hat{\chi}_k}{\hat{y}}),\\
\hat{\bar{Q}}_0 &= e^{-\frac{it}{2}}(\hat{\chi}_0\hat{p} + \frac{i}{2}\hat{\chi}_0\hat{y} + \frac{i(1+2\alpha)}{6}\epsilon_{ijk}\frac{\hat{\chi}_i\hat{\chi}_j\hat{\chi}_k}{\hat{y}}),\\
\hat{\bar{Q}}_i &= e^{-\frac{it}{2}}(\hat{\chi}_i\hat{p} + \frac{i}{2}\hat{\chi}_i\hat{y} - \frac{i(1+2\alpha)}{2}\epsilon_{ijk}\frac{\hat{\chi}_0\hat{\chi}_j\hat{\chi}_k}{\hat{y}}),\\
\hat{J}_i &= -i(\frac{1}{2}\epsilon_{ijk}\hat{\chi}_j\hat{\chi}_k + \hat{\chi}_0\hat{\chi}_i),\\
\hat{L}_i &= -i(\frac{1}{2}\epsilon_{ijk}\hat{\chi}_j\hat{\chi}_k - \hat{\chi}_0\hat{\chi}_i).
\end{aligned} \tag{4.65}$$

The above operators realize the $D(2,1;\alpha)$ superalgebra (4.49) with the identifications

$$\hat{H} = E^-, \quad \hat{D} = H, \quad \hat{K} = -E^+, \quad \hat{Q}_\beta = 2iF^-_\beta, \quad \hat{\bar{Q}}_\beta = -2iF^+_\beta, \quad \hat{J}_j = J_j, \quad \hat{K}_j = K_j. \tag{4.66}$$

The quantum Hamiltonian $\hat{\mathcal{H}} \equiv \hat{D}$ is, explicitly,

$$\hat{D} = \left(\begin{array}{c|c} (\frac{\hat{p}^2}{2} + \frac{4\alpha^2+8\alpha+3}{8\hat{y}^2} + \frac{\hat{y}^2}{8})\mathbb{I}_2 & 0 \\ \hline 0 & (\frac{\hat{p}^2}{2} + \frac{4\alpha^2-1}{8\hat{y}^2} + \frac{\hat{y}^2}{8})\mathbb{I}_2 \end{array}\right). \tag{4.67}$$

Both upper (bosonic) and lower (fermionic) diagonal blocks of $\hat{D}$ contain a Calogero Hamiltonian with the DFF oscillatorial potential,

$$\hat{\mathcal{H}}_{DFF} = \frac{1}{2}\hat{p}^2 + \frac{g^2}{\hat{y}^2} + \frac{\hat{y}^2}{8}. \tag{4.68}$$

A detailed analysis of this Hamiltonian can be found in [3, 34]. Just like the parabolic case, the inequality $g^2 > -\frac{1}{8}$ guarantees the existence of physically acceptable solutions. The boundary value problem

$$\hat{\mathcal{H}}_{DFF}\phi_n = E_n\phi_n, \quad \phi_n(0) = 0, \qquad n = 0,1,2,\ldots, \tag{4.69}$$

implies the discrete spectrum

$$E_n = \frac{1}{2}(n+\nu+1), \tag{4.70}$$

with eigenfunctions given (up to normalizion) by

$$\phi_n(y) = y^{\nu+\frac{1}{2}}\exp(-\frac{y^2}{4})L_n^\nu(\frac{1}{2}y^2). \tag{4.71}$$

In the right hand side L_n^ν stands for the modified Laguerre polynomials. The parameter ν entering the Casimir energy $\frac{1}{2}(\nu+1)$ is

$$\nu = \frac{1}{2}(1+8g^2)^{\frac{1}{2}}. \tag{4.72}$$

Comparing equations (4.67) and (4.68) we see that g_b, g_f are again given by equations (4.54), so that $\alpha \neq 0, -1$ to ensure that both g_b^2 and g_f^2 are greater than $-\frac{1}{8}$.

Since the Hamiltonian is a Cartan generator of the (4.65) superalgebra, the whole spectrum can be recovered from a lowest weight representation of $D(2,1;\alpha)$, where the Q_β's are the lowering and the $\bar{Q}_\beta$'s are the raising operators. The vacuum $|\Lambda\rangle$ is introduced from

$$Q_\beta|\Lambda\rangle = 0, \qquad \beta = 0,1,2,3. \tag{4.73}$$

From the definition of the Q_β's in (4.65) the four differential equations (4.73) can be recasted into the single differential equation

$$(\hat{p} - \frac{i}{2}\hat{y} - \frac{i(1+2\alpha)}{2\hat{y}}\mathcal{F}_4)\,|\Lambda\rangle = 0. \tag{4.74}$$

In position-space representation, (4.74) splits into two separate equations for the bosonic (+) and respectively fermionic (−) subspaces,

$$\frac{d\phi_{0,\sigma}}{dy} = -\frac{1}{2}(y \pm \frac{1+2\alpha}{y})\phi_{0,\sigma}. \tag{4.75}$$

The label σ accounts, just as in the parabolic case, for the $\hat{J}$,$\hat{L}$-spin degrees of freedom.

Integrating the above equation we get, up to normalization, the vacuum solutions

$$\phi_{0,\sigma} \quad = \quad y^{\mp(\frac{1+2\alpha}{2})}\exp(-\frac{y^2}{4}). \tag{4.76}$$

This result is in agreement with (4.71) provided that we set

$$\nu_b = -(1+\alpha), \qquad \nu_f = \alpha. \tag{4.77}$$

This analysis forces us to conclude that two degenerate lowest energy vacua exist for $\alpha \neq -\frac{1}{2}$. They are bosonic for $\alpha < -\frac{1}{2}$ and fermionic for $\alpha > -\frac{1}{2}$. This is implied by equation (4.71) which tells us that any bosonic (fermionic) vacuum should be such that $\nu_b + \frac{1}{2} > 0$ ($\nu_f + \frac{1}{2} > 0$).

At the special $\alpha = -\frac{1}{2}$ value we have that $D(2,1;-\frac{1}{2}) \equiv D(2,1) \approx osp(4|2)$. The Calogero potential terms vanish both in the upper and lower diagonal blocks. At $\alpha = -\frac{1}{2}$ we recover four undeformed harmonic oscillator equations. All the states of the theory (including the minimal energy states) are four times degenerated, with two bosonic and two fermionic states of same energy.

The energy levels of the system are given by

$$E_{b,n} = \frac{1}{2}(n-\alpha), \qquad E_{f,n} = \frac{1}{2}(n+\alpha+1), \quad n = 0,1,2,\ldots. \tag{4.78}$$

$E_{b,n}$ ($E_{f,n}$) are the energy levels of the bosonic (fermionic) states (they coincide for $\alpha = -\frac{1}{2}$).

The energy of the degenerate vacua is

$$E_{b,vac} = -\frac{1}{2}\alpha, \quad (\alpha \le -\frac{1}{2}) \quad ; \quad E_{f,vac} = \frac{1}{2}(\alpha+1), \quad (\alpha \ge -\frac{1}{2}). \tag{4.79}$$

the fermionic degenerate vacua ($\alpha < -\frac{1}{2}$) $E_{b,n}$ applies to the bosonic states, $E_{f,n}$ to the fermionic degenerate vacua.

The scaling dimension α can be regarded as an external control parameter of the theory, so that the vacuum energy can be interpreted as a Casimir energy. The Casimir energy of the $(1,4,3)$ $D(2,1;\alpha)$ (un)deformed oscillator admits a very nice expression in terms of α, being simply given by

$$E_{vac} \quad = \quad \frac{1}{4}(1+|2\alpha+1|). \tag{4.80}$$

This expression should be compared with the much more complicated expression of the vacuum energy in terms of the Calogero coupling constant g and derived from (4.72). This result suggests that the scaling dimension α has a more direct physical interpretation of the Calogero coupling constant g. One should also note that, contrary to g, α directly enters the spectrum-generating superalgebra $D(2,1;\alpha)$.

4.6.2 The $\mathcal{N}=2$ $(2,2,0)$ trigonometric model with $sl(2|1)$ invariance

As in the parabolic case, we obtain from quantization the non-vanishing (anti)commutators

$$[y^*, p_{y*}] = [y, p_y] = i\hbar, \qquad \{\chi, \chi^\dagger\} = \frac{\hbar}{\omega C}, \tag{4.81}$$

with $\chi = \sqrt{\frac{\hbar}{\omega C}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\chi^\dagger = \sqrt{\frac{\hbar}{\omega C}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. We work with $\hbar = 1$, $C = \frac{1}{2}$, $\omega = 2$.

The fermionic operators $\hat{Q}_\pm^{(I)}$, $I = 1,2$, entering $sl(2|1)$ are

$$\hat{Q}_\pm^{(1)} = ie^{\mp it}\begin{pmatrix} 0 & -A_\pm \\ B_\pm & 0 \end{pmatrix}, \quad \hat{Q}_\pm^{(2)} = e^{\mp it}\begin{pmatrix} 0 & A_\pm \\ B_\pm & 0 \end{pmatrix}, \tag{4.82}$$

where, using the polar coordinates as in the parabolic case, we have

$$\begin{aligned} A_{\pm} &= -\frac{i}{2}e^{i2\lambda\theta}\left(\partial_r - \frac{i}{r}\partial_\theta + \frac{2\lambda+1}{2r} \pm r\right), \\ B_{\pm} &= -\frac{i}{2}e^{-i2\lambda\theta}\left(\partial_r + \frac{i}{r}\partial_\theta + \frac{2\lambda+1}{2r} \pm r\right). \end{aligned} \tag{4.83}$$

In the trigonometric case the Hamiltonian $\mathcal{H}$ is the Cartan generator $\hat{D}$, given by

$$\hat{D} = [-\frac{1}{2}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2) + i\frac{(2\lambda+1)}{2r^2}\sigma_z\partial_\theta + \frac{(2\lambda+1)^2}{8r^2} + \frac{r^2}{2}]\mathbb{I}_2. \tag{4.84}$$

In the r.h.s. σ_z is the diagonal Pauli matrix.

The three remaining bosonic symmetry operators which close the $sl(2|1)$ superalgebra are

$$\begin{aligned} \hat{L}_{\pm1} &= i\frac{e^{\mp 2it}}{2}[-\frac{1}{2}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2) + i\frac{(2\lambda+1)}{2r^2}\sigma_z\partial_\theta + \frac{(2\lambda+1)^2}{8r^2} - \frac{r^2}{2} \pm (r\partial_r + 1)]\mathbb{I}_2, \\ \hat{J} &= -\frac{i}{2}\mathbb{I}_2\partial_\theta - \frac{2\lambda-1}{4}\sigma_z. \end{aligned} \tag{4.85}$$

One can easily check that the $sl(2|1)$ superalgebra is recovered from the (anti)commutators of the operators (4.82,4.84,4.85).

The differential equation for the radial part of the eigenfunctions $\psi = e^{im\theta}R_{\pm}(r)e_{\pm}$ of $\hat{D}$, where $e_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, is

$$[-\frac{1}{2}(\partial_r^2 + \frac{1}{r}\partial_r) + \frac{1}{2r^2}(m \mp \frac{2\lambda+1}{2})^2 + \frac{r^2}{2} - E]R_{\pm}(r) \quad = \quad 0. \tag{4.86}$$

E is the energy. In [3] the same equation is found and solved for the problem of three bodies in a line. Furthermore, the issue of selfadjointness of the differential operator acting on $R_{\pm}$ was investigated in [36]; since $\sqrt{\left(m \pm \frac{2\lambda+1}{2}\right)^2} \geq 0$, the existence of a selfadjoint extension for the Halmiltonian (4.84) is ensured.

The requirement of single-valuedness for the operators $\hat{Q}_{\pm}^{(I)}$ on the $\mathbb{R}^2$-plane implies, from the exponents in (4.83), that the constraint $4\lambda\pi = 2k\pi$, with k integer, must be satisfied. Therefore the scaling dimension λ has to be quantized, either $\lambda = \frac{1}{2} + \mathbb{Z}$ or $\lambda = \mathbb{Z}$. We discuss in detail the half-integer case, with side remarks about the models with integer values of λ.

One should note that at $\lambda = -\frac{1}{2}$ one obtains (two copies of) the Hamiltonian of the undeformed two-dimensional bosonic oscillator.

For half-integer λ the $\hat{Q}_{\pm}^{(I)}$ operators act as raising/lowering operators. Let us take, e.g., $\hat{Q}_{\pm}^{(2)}$; it follows, from the commutators $[\hat{D}, \hat{Q}_{\pm}^{(2)}] = \mp\hat{Q}_{\pm}^{(2)}$, that an energy eigenstate ψ with eigenvalue E_n is mapped into an eigenstate $\hat{Q}_{\pm}^{(2)}\psi$ with eigenvalue $E_n \mp 1$ (provided that $E_n \mp 1 \neq 0$):

$\hat{D}\psi = E_n\psi \rightarrow \hat{D}\hat{Q}_{\pm}^{(2)}\psi = (E_n \mp 1)\,\hat{Q}_{\pm}^{(2)}\psi.$

Therefore, starting from a lowest weight state satisfying $\hat{Q}_{+}^{(2)}\psi = 0$, an infinite tower of higher energy eigenstates are constructed by repeatedly applying $\hat{Q}_{-}^{(2)}$. The solutions of the lowest weight equation $\hat{Q}_{+}^{(2)}\psi = 0$ are given by the eigenfunctions

$$\begin{aligned} \psi_{m+}(r,\theta) &= A_m r^{(m-\frac{2\lambda+1}{2})}e^{-r^2}e^{im\theta}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \psi_{m-}(r,\theta) &= B_m r^{-(m+\frac{2\lambda+1}{2})}e^{-r^2}e^{im\theta}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \tag{4.87}$$

where A_m, B_m are normalization constants given by

$$\begin{aligned} A_m &= 2^{\frac{\alpha+1}{2}}\frac{1}{\sqrt{\pi\Gamma(\alpha+1)}}, \quad \alpha = m - \frac{2\lambda+1}{2}, \\ B_m &= 2^{\frac{\beta+1}{2}}\frac{1}{\sqrt{\pi\Gamma(\beta+1)}}, \quad \beta = -(m + \frac{2\lambda+1}{2}) \end{aligned} \tag{4.88}$$

and Γ is the gamma function.

In order to have finite lowest weight eigenfunctions at the origin, the integer m is constrained. From the bosonic states the necessary condition is

$$m \geq \frac{2\lambda+1}{2}, \tag{4.89}$$

while from the fermionic states the necessary condition is

$$m \leq -\frac{2\lambda+1}{2}. \tag{4.90}$$

The energy eigenvalue equation of the bosonic and fermionic lowest weight eigenstates is respectively given by

$$\begin{aligned} \hat{D}\psi_{m+} &= (1+m-\frac{2\lambda+1}{2})\psi_{m+}, \\ \hat{D}\psi_{m-} &= (1-(m+\frac{2\lambda+1}{2}))\psi_{m-}. \end{aligned} \tag{4.91}$$

Two minimal vacua, one bosonic and the other fermionic, are obtained with vacuum energy 1. They are recovered from the "saturated" bosonic and fermionic lowest weight eigenstates with, respectively, $m=\frac{2\lambda+1}{2}$ and $m=-\frac{2\lambda+1}{2}$.

The lowest weight condition obtained from the lowering operator $Q_+^{(1)}$ ($Q_+^{(1)}\psi=0$) produces the same set of lowest weight states (4.87). The application of the raising operator $Q_-^{(1)}$ produces, up to a phase, the higher energy states obtained from the raising operator $Q_-^{(2)}$.

The theory therefore possesses a degenerate vacuum, one vacuum state being bosonic, the other one fermionic. As discussed in Appendix **A** it is possible to impose a superselection rule, imposed by a projector, which selects half of the states being physical. The superselected theory possesses a unique bosonic vacuum and, for $\lambda=-\frac{1}{2}$, its spectrum coincides with the spectrum of the ordinary two-dimensional (undeformed) oscillator, which can therefore be recovered as the superselected, $\lambda=-\frac{1}{2}$, $sl(2|1)$ acting on $(2,2,0)$, quantum trigonometric model.

We conclude this Section with two important remarks. Contrary to the two vacua of the (not superselected) $\lambda=\frac{1}{2}+\mathbb{Z}$ theory, the $\lambda\in\mathbb{Z}$ quantum deformed oscillators possess four vacuum states (two bosonic and two fermionic states). The construction of the Hilbert space follows the same lines as the half-integer λ case. The main difference lies in the fact that the necessary conditions (4.89) and (4.90) for the integer m cannot be satisfied as equalities when $\lambda\in\mathbb{Z}$. It is beyond the scope of this work to present the detailed analysis of the $\lambda\in\mathbb{Z}$ deformed oscillators, which will be presented elsewhere.

The second important remark concerns the fact that, for the superselected $\lambda=\frac{1}{2}+\mathbb{Z}$ theory, the Hilbert space cannot be recovered by repeatedly acting with the $sl(2|1)$ raising operators from the vacuum state. The Hilbert space is decomposed (this point is discussed in Appendix **A**) in a infinite direct sum of the $sl(2|1)$ lowest weight representations. This is in sharp contrast with respect to the one-dimensional harmonic oscillator, whose single irreducible lowest weight representation of the $osp(1|2)$ spectrum-generating superalgebra allows to recover the whole Hilbert space.

One can note, however, that it is possible to construct an extra set of fermionic symmetry operators, $\overline{Q}_\pm^{(I)}$, which also act as raising/lowering operators. The construction goes as follows. At first a discrete symmetry operator $\hat{C}$, playing the role of a charge conjugation operator, is introduced. It is given by

$$\hat{C} = \begin{pmatrix} 0 & e^{i(2\lambda+1)\theta} \\ e^{-i(2\lambda+1)\theta} & 0 \end{pmatrix}. \tag{4.92}$$

One can verify that $[\hat{D},\hat{C}]=0$ and that $\hat{C}^2=I$. This operator also commutes with the $\hat{L}_{\pm1}$ operators in (4.85). It does not commute, however, with $\hat{J}$ and the $sl(2|1)$ fermionic operators.

With the help of $\hat{C}$ we can introduce the new symmetry operators

$$\hat{C}\hat{Q}_\pm^{(1)}\hat{C} = \overline{Q}_\pm^{(1)} = ie^{\mp it}\begin{pmatrix} 0 & C_\pm \\ -D_\pm & 0 \end{pmatrix}, \qquad \hat{C}\hat{Q}_\pm^{(2)}\hat{C} = \overline{Q}_\pm^{(2)} = e^{\mp it}\begin{pmatrix} 0 & C_\pm \\ D_\pm & 0 \end{pmatrix}, \tag{4.93}$$

where

$$\begin{aligned} C_\pm &= -\frac{i}{2}e^{i2(\lambda+1)\theta}(\partial_r - \frac{i}{r}\partial_\theta - \frac{2\lambda+1}{2r} \pm r), \\ D_\pm &= -\frac{i}{2}e^{-i2(\lambda+1)\theta}(\partial_r + \frac{i}{r}\partial_\theta - \frac{2\lambda+1}{2r} \pm r), \end{aligned} \tag{4.94}$$

and

$$\hat{C}\hat{J}\hat{C} = \overline{J} = -\tfrac{i}{2}\partial_\theta - \tfrac{2\lambda+3}{4}\sigma_z. \tag{4.95}$$

As discussed in Appendix **A** (where a schematic presentation in diagrams, the dashed lines, of the action of the $\overline{Q}^I_\pm$ operators is given), $\overline{Q}^I_\pm$ act as raising/lowering operators for the eigenstates of the Hilbert space of the theory. Any given eigenstate can be reached by repeatedly applying to the vacuum both sets of $\hat{Q}^I_+, \overline{Q}^I_+$ raising operators.

In terms of $\hat{C}$ we can also introduce the new quantum operators

$$\mathcal{J} = \hat{J} + \overline{J} = -i\partial_\theta - \tfrac{2\lambda+1}{2}\sigma_z, \quad N_f = \sigma_z = \hat{J} - \overline{J}, \tag{4.96}$$

which allows us to define the new quantum numbers (used in Appendix **A**, see Figure 4):

$$\hat{D}\left|n,j,\epsilon\right\rangle = (n+1)\left|n,j,\epsilon\right\rangle, \quad \mathcal{J}\left|n,j,\epsilon\right\rangle = j\left|n,j,\epsilon\right\rangle, \quad \sigma_z\left|n,j,\epsilon\right\rangle = \epsilon\left|n,j,\epsilon\right\rangle. \tag{4.97}$$

4.7 Conclusions

In this paper we presented a framework for quantizing the large class of classical worldline superconformal σ-models derived from supermultiplets. These systems are defined in [25] (for the parabolic case) and [1] (for the trigonometric case). We applied the quantization prescription to derive explicitly the $\mathcal{N}=4$ $(1,4,3)$ and the $\mathcal{N}=2$ $(2,2,0)$ quantum superconformal mechanics (with $D(2,1;\alpha)$ and $sl(2|1)$ dynamical symmetry, respectively). The parameter $\alpha\neq 0,-1$ is the scaling dimension of the $(1,4,3)$ supermultiplet, while the scaling dimension of the $(2,2,0)$ supermultiplet is quantized and given by $\lambda=\frac{1}{2}+\mathbb{Z}$ or $\lambda\in\mathbb{Z}$.

The results concerning the trigonometric models are particularly relevant. These systems are only “softly supersymmetric", see the discussion in Appendix **C**. As such they have not received much attention like the parabolic models. The trigonometric models correspond to superconformal mechanics in the presence of the DFF damping oscillatorial term; stated otherwise, they are oscillators where Calogero potential terms are possibly present. Their spectrum is discrete and bounded from below.

For the $(1,4,3)$ trigonometric models (i.e., the $D(2,1;\alpha)$ oscillators) we derive the following nice formula for the vacuum energy:

$$E_{vac} \quad = \quad \frac{1}{4}(1+|2\alpha+1|). \tag{4.98}$$

If α is interpreted as a physical external parameter, then (4.98) can be interpreted as a Casimir energy.

Concerning the $(2,2,0)$ trigonometric models, at the special value $\lambda=-\frac{1}{2}$ one recovers, after imposing a superselection rule derived by a projector, see Appendix **A**, the spectrum of the ordinary two-dimensional oscillator.

It has been noted very recently, see [37], that the ordinary two-dimensional quantum oscillator possesses an $sl(2|1)$ dynamical symmetry. As a byproduct of our framework we can further point out that the spectrum of the two-dimensional oscillator is decomposed into an infinite direct sum of $sl(2|1)$ lowest weight representations.

In our approach the existence of $sl(2|1)$ as a dynamical symmetry, not only of the undeformed $\lambda=-\frac{1}{2}$, but also of the deformed ($\lambda\in\frac{1}{2}+\mathbb{Z}$ and $\lambda\in\mathbb{Z}$) two-dimensional oscillators, is a natural consequence of the construction of these models from the $\mathcal{N}=2$ $(2,2,0)$ (trigonometric) supermultiplet. The decomposition of the spectrum in a direct sum of $sl(2|1)$ lowest weight representations comes as a bonus and is not surprising. What is really puzzling and unexpected is another feature, discussed at length in Section **6** and in Appendix **A** and **C**, the presence of the extra fermionic symmetry generators which act as raising and lowering operators. They allow to reach each state belonging to the Hilbert space of the two-dimensional oscillator by repeatedly applying the raising operators to the vacuum state.

This result is very puzzling. It is quite possible that, in order to recover the spectrum of the two-dimensional oscillator from a single, irreducible, lowest weight representation, one needs to extend the concept of superalgebra, possibly by making use of the notion of generalized supersymmetry. In a related context the appearance of a generalized superalgebra as a symmetry of a dynamical system has been noted in [38].

In a forthcoming paper we will present a detailed investigation of the puzzling properties of the deformed $\lambda\in\frac{1}{2}+\mathbb{Z}$ and $\lambda\in\mathbb{Z}$ two-dimensional oscillators.

Appendix A: Diagrams of the spectrum-generating superalgebra for the $\mathcal{N}=2$, $(2,2,0)$, $\lambda=\frac{1}{2}+\mathbb{Z}$ trigonometric cases.

It is convenient, for the two-dimensional cases based on the $\mathcal{N}=2$ $(2,2,0)$ trigonometric reps, to encode in diagrams the action of the raising and lowering operators of the spectrum-generating superalgebra. We explicitly present three such diagrams, Figures 1, 2 and 3, respectively associated with three values of the scaling dimension, $\lambda=\frac{1}{2}$, $\lambda=-\frac{1}{2}$, $\lambda=-\frac{3}{2}$. In a further diagram the general features of the $\lambda=\frac{1}{2}+\mathbb{Z}$ case are presented.

In the diagrams the bosonic (fermionic) states are denoted by white (black) dots. Grey dots denote the presence of both bosonic and fermionic states. The vertical axis represents the energy level, labeled by n, while the horizontal axis represents the angular momentum, labeled by m. We denote with ϵ the eigenvalues of the Fermion Number operator ($\epsilon=+1$ for bosons, $\epsilon=-1$ for fermions). Solid (dashed) lines represent states connected by $\widehat{Q}_{\pm}^{(I)}$ (respectively, $\overline{Q}_{\pm}^{(I)}$) raising and lowering operators with $I=1,2$, see (4.82) and (4.93) (for simplicity we drop here the indices).

The $sl(2|1)$ lowest weight states appear, in the diagrams, as the dots where the solid lines originate (in the upward direction). In Figure 2 and 4 the existence of such lowest weight states is not immediately evident, this is however just a side effect of the condensed notation used (a grey dot being associated with two states).

The operators $\widehat{Q}_{\pm}^{(1)},\widehat{Q}_{\pm}^{(2)}$ (and, similarly, $\overline{Q}_{\pm}^{(1)},\overline{Q}_{\pm}^{(2)}$), applied to a $|n,m,\epsilon\rangle$ state which does not coincide with a lowest weight state produce, apart a normalization factor, the same state. We can write, for $I=1,2$,

$$\begin{aligned}\widehat{Q}_{\pm}^{(I)}\,|n,m,\epsilon\rangle &\propto |n\mp 1,m-\epsilon 2\lambda,-\epsilon\rangle\,,\\ \overline{Q}_{\pm}^{(I)}\,|n,m,\epsilon\rangle &\propto |n\mp 1,m-\epsilon 2\,(\lambda+1)\,,-\epsilon\rangle\,.\end{aligned}\tag{A.1}$$

From the three diagrams, Figures 1, 2 and 3, we can immediately read several important features. In particular, in all three cases, the $n>0$ higher energy states are produced via repeated applications of the $\widehat{Q}$'s, $\overline{Q}$'s raising operators from the two (one bosonic and one fermionic) $n=0$ fundamental level states. As a corollary, we need both types ($\widehat{Q}$'s, $\overline{Q}$'s) of raising operators to recover the Hilbert space of the associated model. This means, stated otherwise, that the Hilbert space is *reducible* with respect to the $sl(2|1)$ superalgebra defined by the $\widehat{Q}_{\pm}^{(I)}$ operators alone. In terms of a $sl(2|1)$ decomposition, an infinite tower (one state at each given integer value n) of lowest weight states need to be introduced to recover the Hilbert space of the theory. Therefore, in order to have an irreducible description, the $\overline{Q}_{+}^{(I)}$ operators need to enter the picture.

One shoud note that the $\lambda=-\frac{1}{2}$ case corresponds to the undeformed (namely, without the extra Calogero potential term) two-dimensional harmonic oscillator. The Hilbert space defined by Figure 2 contains a double degeneracy. Two eigenstates (one bosonic, the other one fermionic) are associated with each n,m pair of eigenvalues. The introduction of a suitable projection allows to remove the double degeneracy and recover the Hilbert space of the ordinary two-dimensional harmonic oscillator. The superselection rule is defined in terms of the projection operator $\hat{P}$ ($\hat{P}^2=\mathbb{I}$), given by

$$\hat{P} \;=\; N_f e^{i\pi\mathcal{H}},\tag{A.2}$$

where N_f is the fermion number operator and $\mathcal{H}=\hat{D}$ is the Hamiltonian (its eigenvalues are the non-negative integers n). The

$$\hat{P}|\Psi\rangle \;=\; |\Psi\rangle\tag{A.3}$$

superselection rule implies that the Hilbert space of the superselected theory is given by bosonic states at even energy eigenvalues ($n=2k$, with $k=0,1,2,\ldots$) and fermionic states at odd energy eigenvalues ($n=2k+1$).

The superselection removes, in particular, the degeneracy of the vacuum, the single vacuum state being now bosonic. The spectrum of the ordinary two-dimensional harmonic oscillator is therefore recovered from the *superselected* $\mathcal{N}=2$ $(2,2,0)$ model at scaling dimension $\lambda=-\frac{1}{2}$.

For any half-integer value $\lambda=\frac{1}{2}+\mathbb{Z}$ the Hilbert space of the two-dimensional deformed (due to the presence, besides the quadratic potential, of a Calogero potential term) harmonic oscillator, can be formally recovered from the $\lambda=-\frac{1}{2}$ Figure 2 diagram, by replacing the angular momentum m with the j eigenvalues of the $\mathcal{J}$ operator introduced in (4.96) (this is also true for the $\lambda=\frac{1}{2},-\frac{3}{2}$ cases explicitly introduced in Figure 1 and 3).

Let us introduce the basis defined by the quantum numbers

$$\hat{D}\,|n,j,\epsilon\rangle=(n+1)\,|n,j,\epsilon\rangle\,;\quad \hat{\mathcal{J}}\,|n,j,\epsilon\rangle=j\,|n,j,\epsilon\rangle\,,(j\in\mathbb{Z});\quad N_f\,|n,j,\epsilon\rangle=\epsilon\,|n,j,\epsilon\rangle\,,(\epsilon=\pm1).$$

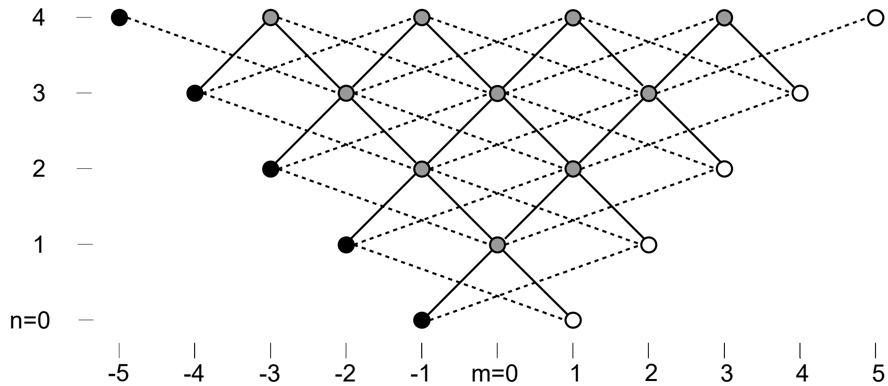

Figure 4.1: $\lambda = \frac{1}{2}$ diagram of $\widehat{Q}$'s, $\overline{Q}$'s raising and lowering operators.

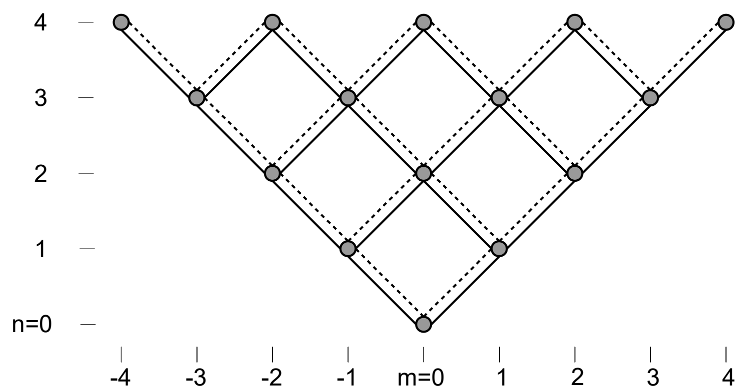

Figure 4.2: $\lambda = -\frac{1}{2}$ diagram of $\widehat{Q}$'s, $\overline{Q}$'s raising and lowering operators..

In this basis the action of $\hat{Q}^{(I)}_{\pm}$, $\overline{Q}^{(I)}_{\pm}$ on a state which does not coincide with a lowest weight state, reads as follows

$$\hat{Q}^{(I)}_{\pm} |n, j, \epsilon\rangle \propto |n \mp 1, j + \epsilon, -\epsilon\rangle\,, \quad \overline{Q}^{(I)}_{\pm} |n, j, \epsilon\rangle \propto |n \mp 1, j - \epsilon, -\epsilon\rangle\,. \tag{A.4}$$

The $\lambda = \frac{1}{2} + \mathbb{Z}$ associated diagrams are presented in Figure 4.

This makes clear that the superselection rule induced by (A.2) can be imposed on any $\lambda = \frac{1}{2} + \mathbb{Z}$ deformed oscillator, guaranteeing in all these cases the existence of a Hilbert space with a single bosonic vacuum.

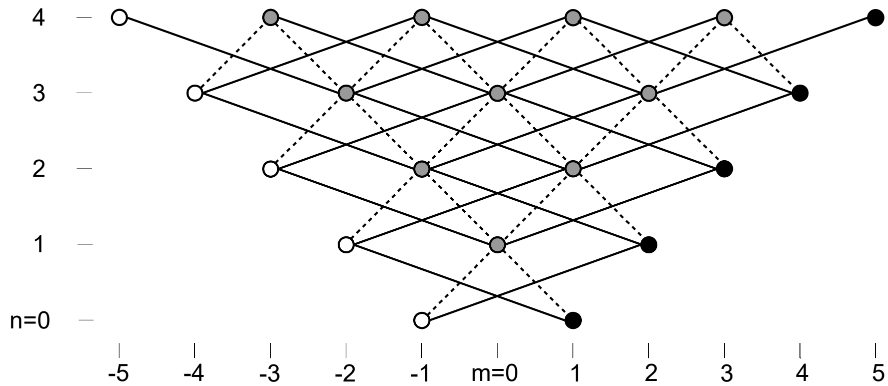

Figure 4.3: $\lambda = -\frac{3}{2}$ diagram of $\widehat{Q}$'s, $\overline{Q}$'s raising and lowering operators.

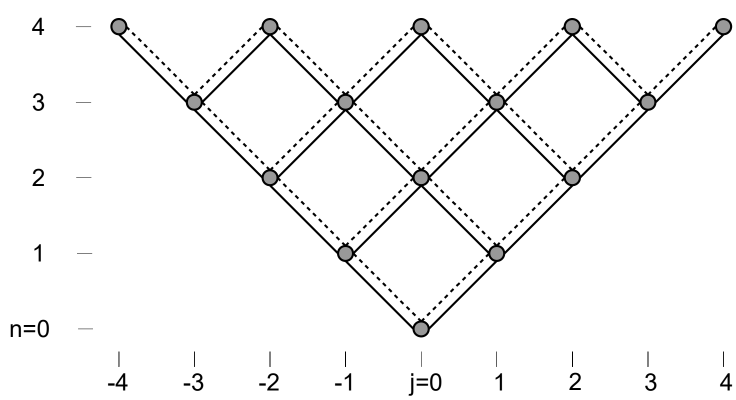

Figure 4.4: the $\lambda = \frac{1}{2} + \mathbb{Z}$ general diagram.

Appendix B: The classical $(2,2,0)$ $sl(2|1)$-invariant models.

We present, for completeness, the construction of the $sl(2|1)$-invariant classical actions obtained from, respectively, the parabolic and the trigonometric D-module reps acting on the $(2,2,0)$ supermultiplet.

The parabolic D-module rep is given by the transformations

$$\begin{aligned}
&L_n x_i = t^n\left(t\dot{x}_i + (n+1)\,\lambda x_i\right), && L_n\psi_i = t^n(t\dot{\psi}_i + (n+1)\,(\frac{2\lambda+1}{2})\psi_i),\ n=0,\pm 1;\\
&Jx_i = -\lambda\epsilon_{ij}x_j, && J\psi_i = -\frac{2\lambda-1}{2}\epsilon_{ij}\psi_j,\\
&Q^1_\pm x_i = t^{\frac{1\pm 1}{2}}\epsilon_{ij}\psi_j, && Q^1_\pm\psi_i = -it^{\frac{1\pm 1}{2}}\epsilon_{ij}\left(t\dot{x}_j + (1\pm 1)\,\lambda x_j\right),\\
&Q^2_\pm x_i = t^{\frac{1\pm 1}{2}}\psi_i, && Q^2_\pm\psi_i = it^{\frac{1\pm 1}{2}}\left(t\dot{x}_i + (1\pm 1)\,\lambda x_i\right),
\end{aligned} \tag{B.1}$$

where the x_i's $(i = 1,2)$ are the propagating bosons and the ψ_i's the fermionic fields.

The above transformations close the $sl\,(2|1)$ superalgebra.

The $sl(2|1)$-invariant action is obtained from the Lagrangian $\mathcal{L} = Q^2_+Q^1_+(\frac{1}{2}F\epsilon_{ij}\psi_i\psi_j)$, with the operators Q^2_+, Q^1_+ acting on the prepotential $F = C(x_ix_i)^{-\frac{2\lambda+1}{2\lambda}}$ (C is a normalization constant). Explicitly, the invariant action of the classical $(2,2,0)$ parabolic model is

$$\mathcal{S} = \int dt\mathcal{L} = \int dt(F(\dot{x}_i\dot{x}_i - i\dot{\psi}_i\psi_i) - iF_i\dot{x}_j\psi_i\psi_j). \tag{B.2}$$

The trigonometric D-module rep is given by the transformations

$$\begin{aligned}
&L_n x_i = \frac{e^{-in\omega t}}{-i\omega}\left(\dot{x}_i - in\lambda\omega x_i\right), && L_n\psi_i = \frac{e^{-in\omega t}}{-i\omega}(\dot{\psi}_i - in(\frac{2\lambda+1}{2})\omega\psi_i),\ n=0,\pm 1;\\
&Jx_i = -\lambda\epsilon_{ij}x_j, && J\psi_i = -\frac{2\lambda-1}{2}\epsilon_{ij}\psi_j,\\
&Q^1_\pm x_i = e^{\mp i\frac{\omega}{2}t}\epsilon_{ij}\psi_j, && Q^1_\pm\psi_i = \frac{e^{\mp i\frac{\omega}{2}t}}{i\omega}\epsilon_{ij}\left(\dot{x}_j \mp i\lambda\omega x_j\right),\\
&Q^2_\pm x_i = e^{\mp i\frac{\omega}{2}t}\psi_i, && Q^2_\pm\psi_i = \frac{e^{\mp i\frac{\omega}{2}t}}{-i\omega}\left(\dot{x}_i \mp i\lambda\omega x_i\right).
\end{aligned} \tag{B.3}$$

Without loss of generality we can set $\omega = 1$. The classical action, $sl(2|1)$-invariant under the (B.3) trigonometric transformations, is therefore given by

$$\mathcal{S} = \int dt\mathcal{L} = \int dt(F(\dot{x}_i\dot{x}_i - i\dot{\psi}_i\psi_i) - iF_i\dot{x}_j\psi_i\psi_j + C\lambda^2\,(x_ix_i)^{-\frac{1}{2\lambda}}). \tag{B.4}$$

Appendix C: On the “soft" supersymmetry of the oscillators.

We make here some comments on the role of superalgebras applied to oscillators (either the ordinary quantum oscillators or the oscillators which are “deformed" by the presence of a Calogero potential term).

The starting point is the famous work of Wigner [39]. In modern terms, after the concept of superalgebra was introduced in mathematics, Wigner's results can be reinterpreted (see [40]) according to the following lines. For the ordinary quantum oscillator, with creation/annihilation operators a, $a^\dagger$ (satisfying $[a,a^\dagger]=1$) and symmetrized Hamiltonian $\mathcal{H} = \{a,a^\dagger\}$, we can assign odd-grading to the operators $a,a^\dagger$, so that they belong to a set of 5 operators, $a, a^\dagger, a^2, (a^\dagger)^2, \mathcal{H} = \{a,a^\dagger\}$, closing the $osp(1|2)$ superalgebra under (anti)commutations. The last three (bosonic) operators close the $sl(2)$ subalgebra. Under this construction we have an alternative point of view for describing the computation of the the spectrum of the ordinary (one-dimensional) harmonic oscillator: we can state that, instead of deriving it from the Fock vacuum $|0\rangle$, annihilated by a ($a|0\rangle = 0$), the spectrum is obtained from a lowest weight representation of $osp(1|2)$, the Hamiltonian being the Cartan element. By adopting this viewpoint the superalgebra $osp(1|2)$ becomes a spectrum-generating superalgebra for the ordinary quantum oscillator, with its Hilbert space being recovered from a single, irreducible, $osp(1|2)$ lowest weight representation.

One should note that the bosonic $sl(2)$ subalgebra also acts as a spectrum-generating algebra for the harmonic oscillator. The Hilbert space of the harmonic oscillator is, however, reducible under the $sl(2)$ decomposition. It is given by the direct sum of two irreducible $sl(2)$ lowest-weight representations. The first lowest state is the vacuum of the theory (proportional to the gaussian e^{-x^2} under proper conventions and normalization). The other lowest state is the first excited state, with eigenfunction proportional to xe^{-x^2} and having odd-parity with respect to the $x \mapsto -x$ transformation. The two $sl(2)$ lowest weight reps correspond to, respectively, the even-parity and the odd-parity energy eigenstates. The role of the fermionic operators in $osp(1|2)$ consists in connecting energy eigenstates of even and odd parity.

After the introduction and the subsequent classification of simple Lie superalgebras [41, 42], the Wigner's approach was advocated in [43], with special emphasis on parastatistics, prompting a series of investigations on lowest weight representations of simple Lie superalgebras (for a recent review see, e.g., [44]).

On a separate development the DFF "trick" of introducing oscillator damping potentials in conformal mechanics relates oscillators (with/without the Calogero potential term) to conformal algebras.

It was recognized in [28] that, due to the DFF "trick", the introduction of new potentials for conformal mechanics became possible. The two aspects, superalgebra versus conformal algebra, were reconciled in [1]. The notion of parabolic versus trigonometric/hyperbolic D-module reps of superconformal algebras was pointed out, with the latter class describing the (deformed or undeformed) oscillators and bounded from below potentials in the trigonometric case.

The main property shared by the two big classes of superconformal theories, parabolic versus trigonometric, is that at the classical level their respective actions are superconformally invariant. Concerning their differences:
i) the parabolic models are, both classically and quantum, superconformal and supersymmetric. The supersymmetry implies the existence of a symmetry operator $\mathcal{Q}$ which is the "square root" of the Hamiltonian $\mathcal{H}$, namely $\mathcal{Q}^2 = \mathcal{H}$;
ii) the trigonometric models, on the other hand, despite being superconformally invariant, are not supersymmetric. In this case symmetry operators $\mathcal{Q}$, $\mathcal{Z}$ exist such that $\mathcal{Q}^2 = \mathcal{Z}$. The key point is that the operator $\mathcal{Z}$ does not coincide with the Hamiltonian: $\mathcal{Z} \neq \mathcal{H}$.

One can easily say that the trigonometric models are "intermediate" between the supersymmetric and the non-supersymmetric theories. This "intermediate notion of supersymmetry", namely $\mathcal{Q}^2 = \mathcal{Z} \neq \mathcal{H}$, has no special name in the literature. In [1] the notion of "weak supersymmetry" was employed, borrowing the term from a construction described in [45] which shares a similar feature. The use of the term "weak supersymmetry", however, could be misleading since the models in [45] are not based on superconformal algebras. In that paper a "weak supersymmetric oscillator" is discussed that has no relation with the oscillators derived from the trigonometric D-module reps of superconformal algebras.

For this reason it seems more appropriate to denote this important class of trigonometric models (which include, as shown in this paper, the ordinary one-dimensional and two-dimensional harmonic oscillators) as "softly supersymmetric". As far as we know the term "soft supersymmetry" has not been employed in a different context, making this term both suitable and available to describe the special properties of the trigonometric superconformal mechanics.

The softly supersymmetric trigonometric models are characterized by
i) classical superconformal invariance of the action;
ii) spontaneous breaking of the superconformal invariance. Indeed, in the simplest application, the Fock vacuum $|0\rangle$ of the harmonic oscillator is annihilated by a and not by the hermitian operator $a+a^\dagger$: $(a+a^\dagger)|0\rangle \neq 0$;
iii) in the quantum case the role of the superconformal algebra is that of a spectrum-generating superalgebra.

Concerning the last point, we indeed proved, see Appendix **A**, that the spectrum of the ordinary two-dimensional oscillator is decomposed into an infinite tower of $sl(2|1)$ irreducible lowest weight representations. The puzzling presence of the extra fermionic generators (4.93) which connect eigenstates belonging to different lowest weight reps reminds the role, just discussed above, played by the $osp(1|2)$ fermionic generators in connecting the two $sl(2)$ lowest weight reps of the one-dimensional oscillator.

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Conclusion

At this point, all the main ideas have had their due time so we feel it is a good moment to call it a day. It is not to be understood, however, that this work is complete. There is indeed a lot left to be done. In this final conclusion, we will sketch some of the interesting issues that were left open or instigated by this work.

As should be clear by now, we have presented a general recipe to conformal mechanics in one dimension: starting from Clifford algebras, construct d-module representations of superconformal algebras and their σ-models, and then quantize them. In principle, this may be viewed as a top-down, abstract approach: it may seem that all we are doing is algebraic manipulations to find σ-models of unknown physical content. Indeed, physicists are more used to a bottom-up approach in which they model known systems *from* a Lagrangian, not *to* a Lagrangian. The important thing to note is that the recipe we presented here is perfectly suited to this bottom-up approach. Given a σ-model, construct its symmetry algebra and extend it using the SUSY techniques that were displayed here. We thus have a procedure that leads to SUSY versions (or weak SUSY versions, as explained in appendix D of chapter 2) of a theory quickly and effortlessly.

A good toy model to put these claims to proof is the quantum Hall effect. A simpler, more direct way to study the quantum Hall effect with these techniques is to view it as a worldline model. It should, at least in principle, be possible to extend its symmetry algebra using our tools. The Hall effect carries in itself the gauge freedom of electromagnetic potential, which adds interest to this pursuit.

A more detailed analysis of the quantum Hall effect should look at it as a model in $2 + 1$ dimensions. Such an analysis would be based on finding the largest symmetry algebra of the Hall effect, similar to what Niederer has done to the free particle and the harmonic oscillator (see references in the introduction). From the point of view of our methods, this would have the added advantage of giving new insights in the oxidation problem, which amounts to developing these techniques in higher dimensions.