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Entanglement Extraction and Stability in Relativistic Quantum Information: Finite-time response function of uniformly accelerated entangled atoms

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Abstract

In this thesis we present a collection of research results about the Unruh effect, the phenomena that we can find at the interface between quantum physics and general relativity and its application in the field of relativistic quantum information. One motivation behind our research work is to find other ways to study the problem of a quantum theory of gravitation. Another motivation is to develop some new techniques to tackle the problem of the quantum information processing in realistic situations and formulate experimental devices for this purpose. In order to identify a conceptual approach for our objective, a conceptual discussion about the two physical theories that conform the background for our physical system, is presented throughout the different chapters in the text.

Finally we present the main results of this research work where we examine the entanglement generation between uniformly accelerated two-level atoms weakly coupled with a massless scalar field in Minkowski vacuum. We investigate this phenomenon in the framework of time-dependent perturbation theory. We evaluate a finite-time response function and we identify the mutual influence of atoms via the quantum field as a coherence agent in each response function terms. The associated thermal spectrum perceived by the atoms is found for a long observational time interval. In addition, we study the mean life of entangled states for different accelerations. The possible relevance of our results is discussed.

À luta do povo brasileiro pela sua emancipação e libertação. Porque temos que queimar o céu se é necessário por viver, obrigado pela inspiração. À educação, com o teu posso e com o meu quero vamos juntos companheira, a crise vai passar. Ao professor Nami Fux Svaiter agradeço a sua orientação, toda uma linha de pensamento que vem se construindo com o tempo desde antes de ele se faz presente em cada discussão. Ao professor Gabriel Menezes pela ajuda e colaboração na pesquisa. Ao professor Helayël-Neto por ensinar ter a mente aberta em frente a qualquer caminho da unificação. Os ensinos, da desordem vital e as noites sem dormir, dos amigos e amigas da escola e universidade, dos professores do passado, seguem presentes. Um abraço especial ao grupo Tecendo Consciência Ambiental pelo processo da ajuda à nossa mãe natureza. Para aquela planta que cresce sempre à luz, obrigado pela sempre constante companhia e calor.

Finalmente à minha família que me entregam o seu vital sorriso todos os dias. Às minhas avós que percorreram e prepararam o caminho que eu seguirei; as donas do futuro, dos abraços e a sabedoria são elas, poucas pessoas conseguem aquele sonho. Junto com elas, Mãe e Pai, o seu exemplo de valentia, entrega, persistência e amor, me acompanham todos os dias, com cada sol e cada lua sento o seu calor no meu andar. A sua preseņ na minha vida, como eu disse em um outro tempo, serve para não perder o sentido do que foi alcançado e o curso do que resta a ser alcançado. Os pais não são uma escada para alcaņar o céu em vão, mas uma ponte para o amanhã.

Sentí que me sumergía en esa agua fresca y supe que el viaje a través del dolor
terminaba en un vacío absoluto. Al diluirme tuve la revelación de que ese vacío
está lleno de todo lo que contiene el universo. Es nada y todo a la vez.
Luz sacramental y oscuridad insondable. Soy el vacío, soy todo lo que existe,
estoy en cada hoja del bosque, en cada gota de rocío,
en cada partícula de ceniza que el agua arrastra,
soy nada y todo lo demás en esta vida y en otras vidas,
inmortal.
-Isabel Allende.

Contents

Chapter 1

Introduction

The vacuum holds the key to a full understanding
of the forces of nature.
P. C. W. Davies.

The vacuum has been a concept of great interest for scientist, philosophers, artists throughout the history. The evolution of this concept has become focus of interest accompanied with other concepts as the nothing, the emptiness, the To Be and Not To Be and the concept of the abstract zero. Discussions about this set of concepts show that there is a considerable deepness and ampleness in the contemplation of the vacuum and the nothing, show that the vacuum has been a topic that has fascinated the human mind and has served as the basis in the search for answers for the fundamental questions such as why are we here? Where do we come from? Why is the universe made in that way? Could those questions be the wrong questions?. As we shall see, each aspect in the evolution of the knowledge contributes with a new significance of the vacuum, especially after the quantum revolution of the 1900. According to the present ideas there is no vacuum in the ordinary sense of tranquil nothingness. There is instead a fluctuating quantum vacuum [1].

Early studies of the complexity of the vacuum were made by theology searching if we come from it and if we had the risk of coming back to the vacuum. The ancient Greek philosophy found certain of contradictions from the process of application of pure logic to the concept of vacuum. On the other hand, while the occidental philosophical traditions tried to scape from the vacuum and the Not To Be, the Buddhist and Hindus meditation exercises, actively searched the zero and the Not To Be to achieve the unity with the cosmos [2]. This would be one of the firsts inklings where we can observe the contrast between the classical structure of physics and the new form of thinking given by quantum mechanics and relativity and their convergence in quantum field theory.

Classical atomism considered space as logically previous to its material content. Classical physics and mechanistic philosophy proclaimed that the material substance is the only true reality, however, for Democritus if the To Be was the eternal and indestructible atoms, the Not To Be would be the space and then, how could the Not To Be be logically previous to the To Be? Absurd conclusions were expressed, it is necessary to take in to account that several texts highlighted the confusion between logical, ontological and temporal antecedence.

This confusion became apparent when Henry Moore on 1671 indicated that attributes of space become the same as the scholastics generally assigned to the Supreme: *One, Simple, Stationary, Eternal, Complete, Independent, Existing within itself, subsisting per sé, Incorruptible, necessary, immense, uncreated, incompehensible, omnipresent, formless, all-pervading, to be in essence, to be in act, pure act.* This deification of space exerted much influence on Newton's nature philosophy. The logical and temporary antecedence of space above its physical content was a dogma that few people dared to question. Another characteristic that was not made so explicit, but was the source of the other two important ones (independence and immutability) was the homogeneity.

The homogeneity comes from Greek atomism once they separated the space and its physical content. The space supposed a non-qualitative differentiation principle, because the space allows to distinguish two feelings qualitatively identical that could be different thanks to the space; these two feelings are different because of its position; in words of Locke (contemporary to Newton) *Two simultaneously perceived objects only can be numerically distinct if they are in two different places* [3].

Then, if we had a new differentiation principle that contrasted to the qualitative one, all the positions in the space would be equivalent. The difference between all these positions was its juxtaposition or coexistence relation. *All the points are qualitatively similar, and are distinguished by the mere fact to be situated ones out of others* [4].

Now, if the homogeneity was establishing the equivalence of all points, those points that configure an extreme or a boundary would disappear. The finitude of space contradicts the idea of its homogeneity, since if there is an extreme, an end, a boundary, these points that belong to this extreme have a special character, are different to the rest, breaking down any inkling of homogeneity, since, there will be a set of different points to the rest. Then, the homogeneity must imply infinitude. As Bertran Rusell say *How can some line, or some surface, form an impenetrable barrier for the space, or have a different mobility in gender of the others lines or surfaces? In philosophy, this notion can not be allow for one moment, because it destroys the most fundamental of all the axioms, the homogeneity of the space* [4]

When it is showed that homogeneity implies infinitude, automatically emerges the property of universality of the coexistence of points. The coexistence relates any pair of points as close as they are. To affirm that some space intervals are indivisible means that it is impossible to discover parts coexisting between them; since the coexistence is the pure essence of the spatiality, it would mean that such intervals are lacking of spatiality [5]. In other words, if there is an interval where the process of infinite division cannot be performed, there is not coexistent points between this interval and this implies that there is not spatiality. So, the indivisibility denies the basic structure of mathematics, denies zero longitudes, denies the point.

Classical physics before 1900 was consolidated as the final and complete framework of the human cognitive faculties in the understanding of the objective world. This *completeness* was achieved by the Newtonian and Lagrangian mechanics, supported by the euclidean geometry. From those structures, were obtained certain results that altered philosophically concepts such as space, time, matter, movement, energy and causality. Despite this character, classical physics at the end of 19th century were fighting against some conflicts.

A wide set of intellectual and experimental efforts were performed to shield the classical physics in the face of those conflicts. However, failures were countless. A solid and logic structure was not achieved because there was not one that imply a contradiction inside whole theory.

It is so, in front of the arrival of Einstein's theory of relativity (special and general) and quantum theory of matter, and its precise adjust and exact predictions to experimental results, the classical concepts (space, time, matter, movement, energy and causality) went to be radically transformed and with them, the different philosophical lines that had its basis in these. Philosophy had lost its scientific base and science had lost its philosophical base (built mutually with the physics for more than 200 years) in the 20th century by the research about kinematic properties of light, black body radiation, atomic stability, etcetera. We shall deal with a discussions about the classical concepts in order to understand the revolutionary character of the events that would consolidate modern physics and its impact in philosophy, understanding in what sense they differ from classical concepts.

Following [6], to be situated in a boundary produces, in an unavoidable way, a go and comeback like a pendulum, between the alternating surroundings located to each side of the boundary. The remarkable fact of situating in a limit, forces the reason to live in an incessant dialectic between its inner territories and an unattachable outer edge. It is clear, that our intuitive thinking is tightly related with the structure of thinking of the classical physics. A great work in physics must be to identify that go and comeback of our reason between the classical and modern concepts; find out how the classical concepts and our intuitive thinking are immersed in the new theories and made this, surpass them to be able to advance in the new forms of thinking. The new theories would be our battle field, where the language must be improved, adjusted and rebuilt in order to advance in our understanding of the physical nature.

Being language the projection of our thinking in the world, we have to reformulate it; in order to keep away classic and intuitive structures of the roots of new theories, impeding the development of the same. Our interest is formulate a background for the concept of vacuum and elucidate its capability of unifying the quantum and relativistic aspects of nature and in this way trace a path to a new theory of physical nature.

The general theory of relativity predicts the existence of a physical system which separates causally two regions of space-time. This physical system is called black hole [7]. A black hole is the final stage of a star gravitational collapse [8]. An important characteristic of these systems consists in the formation of a singularity (mathematical point where is supposed to be the whole collapse matter) which is enclosed by a boundary called event horizon (known as a *no return point*). The events that occur inside this boundary do not affect an outsider observer, in other words, this surface is the boundary of two regions causally disconnected of space-time [9]. Furthermore, inside event horizon and more specifically in the singularity, the classical description of space-time becomes not valid. These kind of anomalous prediction at the singularity are in the small distance scale and high energy domain. This fact shows that is required the use of concepts of quantum mechanics for the consistency of the general theory of relativity [10].

The Quantum Field Theory consolidates a convergence space of quantum mechanics and relativity in flat space-time [11]. When we extend that formalism to curved space-time, we are talking about a semi-classical approach of gravity [12]. In quantum field theory in curved space-time we have two transcendental results where elements of general relativity and quantum mechanics are combined to form thermodynamic phenomena associated with the concept of vacuum. These results are the Unruh [13] and Hawking effect [14].

The Unruh effect consists in an accelerated observer in a flat space-time, coupled to a quantum field in its vacuum state, will detect thermal radiation of this quantum field, being the perceived temperature proportional to the proper acceleration of the observer [15, 16]. We shall see that in the background of this phenomena is the fact that in quantum field theory in curved space-time there is not a unique quantisation scheme because there is not a unique choosing of the time coordinate, this will give us that the notion of particle (understood as an excitation of a quantum field) is an observer's dependent quantity. In other words, in a quantum state where certain observers do not perceive particles of the field, another observers with different motion state will perceive a non-null content of particles.

We can highlight two kind of techniques to approach the problem of perception of thermal radiation by different observers in quantum field theory in curve space-time: the first one are the *Bogoliubov transformations* that allow us express the vacuum state of a quantisation scheme in terms of excitations of the field quantised in other set of coordinates [17, 18, 19]. The other one is examination of the problem by particle detectors, in particular the Unruh-DeWitt model of detectors which are devices that experiment an interaction with a quantum field and will be excited in presence of particles of the field [13, 20].

In this work, we use the Unruh-DeWitt detector model to study the Unruh effect. When we work with this formalism we have a tool for extract the information of the detector's response in the particle-number measurement. When we work in first-order perturbation theory, this tool is the so-called *response function* and allows obtain information about the physical mechanisms of excitation and relaxation processes of the detector [12, 13, 20, 21].

We can prepare a two-level atom as an Unruh-DeWitt detector in order to obtain measurable information in the laboratory [22]. When we work with an individual atom, the Unruh effect can be a phenomena that induces quantum decoherence but when we have two accelerated atoms interacting with the same quantum field, the Unruh effect, beside the mutual influences due to the quantum field, can generate entanglement [23]. The entanglement is the most non-classical feature discovered in quantum mechanics [24]. The entanglement is the principal tool for quantum communication [25, 26]. This entanglement creation constitutes the principal topic in this work that is guided to show the existence of a non-null probability of entanglement in accelerated atomic systems and subsequently the possible entanglement extraction from gravitational systems. That is to say, we shall show we are able to extract quantum properties from purely gravitational systems.

This framework invites us to build the content of this work around a conceptual discussion of the two paradigmatic theories of modern physics in order to understand the convergence and divergence points and explore the deep relation between quantum mechanics, gravity, electromagnetism and thermodynamics that this kind of phenomena are showing us. Although the contents of the firsts chapters can be found in diverse books of the topic, they are writing with the objective of explore the relation before mentioned, based on the concepts, the change of language and important results. Exploring this relation of that form we would be opening the way to discussion of the validity of construction of a full quantum theory of gravity.

This text is organized in the following form: In chapter 2 we review crucial aspects of the theory of relativity both special and general. We discuss its origin in a conceptual framework. We study certain aspects of the non-euclidean geometry and we derive the Einstein Field Equations, where we can find the first inkling of the vacuum existence. We deduce the Rindler metric, that corresponds to the physical situation of the accelerated observer, because this physical situation is the central topic in this work. We extend our treatment and derive the Schwarzschild metric, that corresponds to a gravitational field of a spherical body which under certain conditions can be a black hole. We study the relation between the Rindler and Schwarzschild metric.

With this mathematical framework in chapter 3 we study the theory that would join the two paradigms of modern physics, that is, quantum field theory. As we will observe, thanks to quantum mechanics and special relativity, the concept of particle was completely changed and quantum field theory will deal with this problem in a new and solid way. Although in principle each one (quantum mechanics and special relativity) contributed with a certain different aspect in the construction of the new concept (and in a superficial view these may appear contradictory), each aspect would be complementary to the other one. The theory that achieved bringing together all these aspects in a consistent way was quantum field theory. We can observe that the concept of field can be more fundamental that the one of particle, thus, we study the process of the second quantization for the fields of interest. We deduce from that formalism the necessity of a state of minimum energy, that is the so-called *vacuum state*. Then we observe how quantum field theory becomes the adequate place where we can stablish the concept of the vacuum. We study the first physical implication of this state exploring the Casimir effect with a briefly discuss about the vacuum polarization and its implication in an unification scheme. The existence of these effects is a direct consequence of the *reality* of the quantum vacuum and its fluctuations sea that is conforming it. Its existence is surrounding all the forces of nature. It attaches the gravity with the quantum character of the energy and we will see that the vacuum polarization influences the strength of the electro-weak and strong forces. Finally we study the quantization in the Minkowski space where we develop our treatment in the following chapters.

In chapter 4 we study the quantum field theory in curved spaces. As we shall see in chapter 2 choosing an unique set of coordinates is not the best option. This situation leads to the fact that since we do not have a unique coordinate time, we wont have a unique set of frequency modes. This give us problem of the non-uniqueness of quantum vacuum due to the different quantization frames. With the Bogoliubov transformations we find the relation between these different quantization frames. Finally, following these ideas, we remark some different vacuum states of physical interest.

Chapter 5 is the heart of this text. Here we explore another manifestation of the quantum vacuum and the consequences of the problem of the non-uniqueness of the quantum vacuum. As we shall see, an accelerated observer perceives the vacuum as a thermal bath, with a well-defined temperature proportional to its proper acceleration. This is the Unruh effect. If we perform a conformal transformation, an accelerated observer can be an observer in a gravitational field. We can choose, for example, a black hole's gravitational field, in this case we should refer to the Hawking effect, where we have a thermal radiation process from the event horizon of a black hole. The Unruh effect beside the Hawking effect are of great importance for the comprehension of the form in which the fundamental laws of the nature are correlated. As we will explain briefly, physics has deal with a series of unifying processes. The Unruh and Hawking effect, are a marvellous examples of a processes that are at the same time relativistic, quantum, gravitational and thermodynamic.

In this chapter The Unruh-DeWitt (UD) model detector is reviewed and we define the finite-time response function that is the way how we can extract information about the particles (or excitations) of a quantum field. We study the case of an inertial detector and one detector that is in a heat bath in order to stablish some analogues. Afterwards we go to the accelerated detector and we explore the thermodynamics that are emerging in this problem. In order to study the boundary effects as in the Casimir effect, we study the situation of the accelerated detector in the presence of one an two infinite reflecting planes.

In chapter 6 we review some important aspects of quantum information processing, such as the qubit, the entanglement and the quantum teleportation. We study the Unruh effect with the formalism of the Bogoliubov transformations and with this we explore how that effect will affect those important aspects of the quantum information processing. This chapter give us an introduction to the main results of this work, because here we explore how the thermodynamics that are emerging in this problem can be used to extract properties uniquely quantum, such as the quantum entanglement.

Finally in chapter 7 we present the main results of this work. We study radiative processes and entanglement extraction and stability of uniformly accelerated two-level atoms interacting with a quantum massless scalar field. We evaluate the transition rates within time-dependent perturbation theory in a finite time interval. We discuss the Hamiltonian that describes the system of two identical two-level atoms weakly coupled with a massless scalar field in Minkowski vacuum. We present the eigenstates and respective energies of the atomic Hamiltonian. We evaluate the associated response functions. We mainly focus our attention in the so-called crossed response functions and we compute the general expression for different accelerations and time intervals for the symmetric state transition. We also present the total transition rate for equal accelerations of the atoms. We study the mean life of the symmetric entangled state. We present additional results where a infinite reflecting plane is present. Discussions, conclusions and extensions are presented in chapter 8. The main results can be found in [77]. We work with units such that $\hbar = c = k_B = 1$ and the Minkowski metric we use is given by $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. We denote the coordinate time by $x^0 = t$ and the proper time by τ.

Chapter 2

Causal Structure and Horizons in the Space-Time

In this chapter we present a review of results from the special and general theory of relativity which serve as the mathematical background where our field dynamics will develop. We focus our attention in two physical situations: accelerated observer and the black hole. As we shall see these situations are generating a topological structure that delimits two causally disconnected regions of the space-time. This topological structure are known as *horizons* and has an intrinsic relation with the thermal effects that we will explore in the next chapters. Let us first discuss some conceptual aspects of the theory in order to clarify this relation.

The classical structure before the 1900 (even after) was considered as the maximum theoretical construction and effort of the human mind. It is easy to evidence, for instance, in the fact that those classic concepts are yet in some modern and new physical theories. As we will see, all aspects of this structure would be drastically altered or erased.

Thanks to this scientific and philosophical construction of centuries, scientists and philosophers said that experimental results that could be in contradiction with all this, only were unknown appearances of the structure consolidated and that they were serve to strengthen it.

One of these experimental results that showed unexpected results, was Michelson's interferometer experiment in 1881 and that several times was repeated by him and other physicists. The experiment wanted to confirm the idea of absolute space, basing in the hypothesis of the existence of the ether.

The negative results of experiment of Michelson, did bring explanations that although valid in some moment, confronted serious philosophical and scientific difficulties that made that the existence of an interplanetary medium that had dynamic and cinematic properties comparable to the ones of the ordinary bodies had to be re-evaluated.

Several books, from technical to divulging, appoint the contraction of Lorentz-Fitzgerald as what initiates all the changes in physics that would modify in a definitive form the study of nature. They seem to forget that the objective of this contraction was the contrary. The failure of the experiment of Michelson showed the constancy of speed of light. The hypothesis of Lorentz-Fitzgerald supposed that the dimensions of moving bodies contracted in a defined proportion, restoring the equality of two optical ways in the Michelson interferometer, and the rays reflected arrive to the same phase; therefore, it does not produce any change.

According to the hypothesis, a body that moves in an absolute manner, contracts; analogously, if it is in absolute rest, it will keep its properties without alteration. This enhances the absolutist theory of space, since, being it real, has to react in a verifiable physical way against the movements that are producing in it. This type of assertions began to give some ideas on the dynamic difference between rest and movement, even if this was rectilinear and uniform.

One can say that this hypothesis would affect the causal inaction characteristic of space, but the contraction of Lorentz-Fitzgerald takes place with respect to the ether full space (and not with respect to empty space). In these terms, since the basic elements of the matter were considered like an aggregated of ethereal atoms, what was happening was that the appointed contraction changed the configuration of these aggregated.

The contraction affected also time measurements. In terms of considering the absolute movement like a movement through the ethereal medium, they said that there was an braking effect and such resistance effect took place inside the moving system. Fortunately, this was not the way that followed the physics at beginning of 20th century. Although, this pair of expressions of contraction keep still up to now and served as theoretical base for the development of the relativity, the triumph consisted in Einstein interpreted this contraction as an affirmation that the constancy of light speed, was one of the fundamental and irreducible facets of the physical reality; whereas Lorentz and Fitzgerald expected that the constancy of light speed were derived from the unmodified laws of the classical mechanics.

With the arrival of the theory of relativity any concept that implied simultaneity, would have to be debated. The classical physics considered all the history of physical world like a continuous succession of instantaneous material configurations. In the classical three-dimensional model, in which it takes in to account time, we have an Euclidean plane and the time is symbolised as a straight line perpendicular to the plane. In this way there would be an infinity of successive instantaneous spaces, represented by parallel planes (perpendicular all to the time) or transversal cuts.

These transversal cuts represented the present state of universe. The points contained in the instantaneous transversal sections were simultaneous in an absolute sense. Roughly speaking, taking the four dimensions, each transversal cut would mean a three-dimensional Euclidean space. This concept of transversal cut goes in total concordance with the one of absolute space; since, the absolute space is defined like a juxtaposition of points, that is equivalent, in this we have the notion of coexistence or simultaneous existence.

Accepting relativity, we would have to refuse the notion of transversal cut as it had been stipulated, because in this frame, the transversal sections are different for the different inertial systems. With the relativization of the simultaneity (to not say the negation) the present inferred by an observer will be different for other moving observers.

These three-dimensional cuts where all the events 'truly' instantaneous were located, in front of light of relativity, left to exist, furthermore the character of true, came ambiguous. Then such operation of separation of space and time so present and known in classical physics realized impossible in the relativity, yield here to the fusion between space and time.

The fusion of space with time, from early dates had an erroneous representation. Initiating with Minkowski that indicated the *continuum* of four dimensions as an operation in which, the temporal component was absorbed by the space. The time was considered like an additional space dimension, but this confronted a first difficulty, because the inherent characteristic to the space was the coexistence. Although Einstein, in the frame of causal structure, argued that the time had its asymmetry, in several contexts the people began to call the process as a spatialization of time.

If the time was spatialized, we would kept the idea of the present like a point moving along the axis, from the past to the future. Any spatial character observed in a given moment found complete, in other words, all its parts are given at the same time, simultaneously. The spatial scheme suggested that the successive moments, actually coexist, and that its character of past and future is not authentic. From this point of view, future events exist right now. The true reality was timeless and any concept of succession was an illusion, from our (or one) consciousness that is being moved to the future; arising a duality between the timeless physical world and the temporal consciousness. Definitively the spacialization of time and the strict determinism carried to several contradictions in the light of relativity.

Now, although the juxtaposition became relative, the succession of distant events did not. The succession of events causally connected is defined as an invariant; that is, it retains its character of succession for all possible observers. These types of succession are represented by causal series (world lines); contrarily to the spatial coexistence, the irreversibility of world lines has an absolute meaning, possessing an authentic and objective independent reality of election of reference system. *The ontological priority of the time on the space hardly could have a more convincing illustration.*

As we saw special theory of relativity makes the process of fusion of space and time, give us a new structure, but also makes the fusion between matter and energy. General relativity goes one steep beyond making the fusion between space-time and matter-energy. In classic terms, the space and its physical content are right now an identity.

When the distinction between space-time and matter-energy disappears, duality between natural and forced movements tends to disappear too, because now all movement is natural, since it is produced through a geodesic. Any movement, accelerated or not accelerated, is a consequence of the local spatio-temporal structure.

In general theory of relativity, the gravitational action remains reduced to a local deformation of the four dimensional non-Euclidean *continuum*; the matter remains reduced to the presence of these deformations, since, the matter is manifesting by its gravitational field from which it cannot be separated. It would to be an error say that matter causes the corresponding curvature in the space-time, as it is used to say in divulging and technician books. The relation between curvature and matter is the one of identity: the matter and the local curvature of the space-time are an identical reality.

We see how the classical positivist rational structure is falling; it must be decision of the scientist, be refugee in the rise of the reason that did in the illustration and then never understand in a clear way the new emergent physics or abandon all this and begin the process of exploration beyond the structured reason.

To clarify those new concepts, in 1911, Langevin proposes the space travel paradox; in which a subject depart from the earth in a spaceship with a speed near to light speed. After a year of voyage, the traveller invert its direction and returns to Earth. As a result, they obtain that for who was in the spaceship, passed two years, while when traveller arrives to the earth, sees it two centuries older. It is clear the drastic acceleration that suffered the spaceship at the moment to come back and in the frame of the equivalence principle, this means a large gravitational field. Hence, with this mental experiment, some reciprocity present in the special theory was broken. We extract out here two interesting results.

The first one is the fact that temporal dilatation is a physical fact, it is not a perception question. With the space travel the temporal dilatation acquires the character of an authentic modification of the respective proper time of a system, without any symmetric replica. The second one is the concretion of an universal time. Following Langevin's idea, the subject that stays in the Earth and the traveller, not only they share the departure event, but also the arrival event. We have that the two temporal intervals are limited by the same successive moments (that well be an invariant) and then we have the concept of contemporaneity.

Then despite of metric diversity, the temporary series, are contemporary ones with other and this relation of topological contemporaneity describes the own essence of the relativistic space-time.

2.1 Review of Differential Geometry

Any ordered set of n independent variables (real or complex) x^μ with $\mu = 1, ..., n$ can be considered as the coordinates of a space $\mathcal{M}$ of dimension n. Each set $\mathbf{x}$ define a point on $\mathcal{M}$. The space $\mathcal{M}$ can be considered as a manifold. The properties of this space and the relations between distinct geometric objects can be formulated without the use of coordinates [27, 28]

There are two ways in the study of these spaces. The first one is based on the development without the use of coordinates, because in principle, the physical results do not depend of the coordinates. The second one, is the contrary, here we use the coordinates in order to extract certain paths to characterize the results with experimental devices. This last approach is widely use in the special and general theory of relativity.

Following this line, if y^α is defining a point on $\mathcal{M}$ and $y^\alpha \equiv y^\alpha(\mathbf{x})$, we obtain a coordinate transformation on $\mathcal{M}$. The Jacobian transformation matrix is given by

$$\Lambda^\alpha_\mu = \frac{\partial y^\alpha}{\partial x^\mu}. \tag{2.1.1}$$

Clearly, the coordinate transform from $\mathbf{x}$ to $\mathbf{y}$ must be invertible, then

$$det\left(\Lambda^\alpha_\mu\right) \neq 0. \tag{2.1.2}$$

With this, the inverse Jacobian matrix is

$$(\Lambda^{-1})^{\mu}_{\alpha} = \frac{\partial x^{\mu}}{\partial y^{\alpha}}. \tag{2.1.3}$$

This matrix satisfies

$$\Lambda^{\alpha}_{\mu}(\Lambda^{-1})^{\mu}_{\beta} = \delta^{\alpha}_{\beta}. \tag{2.1.4}$$

Now, let us introduce functions of the form $f : \mathcal{M} \to \mathcal{F}$ on $\mathcal{M}$, where $\mathcal{F}$ is a space of dimension N. In an analogous way that we introduce coordinates $\mathbf{x}$ on $\mathcal{M}$, we can introduce coordinates $\mathbf{f}$ on $\mathcal{F}$. The coordinates of $\mathbf{f}$ in the coordinate system $\mathbf{x}$ are $\mathbf{f}(\mathbf{x})$ with components $f^i(\mathbf{x})$ with $i = 1, ..., N$. In a same form, the coordinates of $\mathbf{f}$ in the coordinate system of $\mathbf{y}$ are $\tilde{\mathbf{f}}(\mathbf{y})$ with components $\tilde{f}^a$ with $a = 1, ..., N$. The functions $\mathbf{f}$ are classified with its components number N and the behaviour under transformations of its components f^i [28].

Such functions that have only one component are called scalar fields and transform as follows

$$\tilde{\phi}(\mathbf{y}) = \phi(\mathbf{x}), \tag{2.1.5}$$

therefore its value is independent of the coordinate system.

The following functions are the vectors, that transform

$$dy^{\alpha} = \Lambda^{\alpha}_{\mu} dx^{\mu}. \tag{2.1.6}$$

If we define a contravariant vector, it will transform as

$$\tilde{v}^{\alpha}(\mathbf{y}) = \Lambda^{\alpha}_{\mu} v^{\mu}(\mathbf{x}). \tag{2.1.7}$$

We can form vectors from scalar fields and these quantities transform as

$$\frac{\partial \tilde{\phi}}{\partial y^{\alpha}} = (\Lambda^{-1})^{\mu}_{\alpha} \frac{\partial \phi}{\partial x^{\mu}}. \tag{2.1.8}$$

Analogously, the covariant vectors transform as follows

$$\tilde{v}_{\alpha}(\mathbf{y}) = (\Lambda^{-1})^{\mu}_{\alpha} v_{\mu}(\mathbf{x}). \tag{2.1.9}$$

The following functions are the tensors which are defined based on its behaviour under the transformations. For covariant second rank tensor, with n^2 components, the transformation rule is

$$\tilde{g}_{\alpha\beta}(\mathbf{y}) = (\Lambda^{-1})^{\mu}_{\alpha}(\Lambda^{-1})^{\nu}_{\beta} g_{\mu\nu}(\mathbf{x}). \tag{2.1.10}$$

For a contravariant second rank tensor yields

$$\tilde{g}^{\alpha\beta}(\mathbf{y}) = \Lambda^{\alpha}_{\mu}\Lambda^{\beta}_{\nu} g^{\mu\nu}(\mathbf{x}). \tag{2.1.11}$$

For a tensor with mixed components,

$$\tilde{g}^{\alpha}_{\beta}(\mathbf{y}) = \Lambda^{\alpha}_{\mu}(\Lambda^{-1})^{\nu}_{\beta} g^{\mu}_{\nu}. \tag{2.1.12}$$

The contraction of these objects reads

$$\tilde{g}^{\alpha}_{\alpha}(\mathbf{y}) = \Lambda^{\alpha}_{\mu}(\Lambda^{-1})^{\nu}_{\alpha} g^{\mu}_{\nu}(\mathbf{x}) = \delta^{\nu}_{\mu} g^{\mu}_{\nu}(\mathbf{x}) = g^{\mu}_{\mu}(\mathbf{x}), \tag{2.1.13}$$

thus is a scalar.

We say that a second rank tensor is symmetric if

$$S_{\mu\nu} = S_{\nu\mu}, \tag{2.1.14}$$

and antisymmetric if

$$A_{\mu\nu} = -A_{\nu\mu}. \tag{2.1.15}$$

However, the object $\partial\tilde{v}_\alpha/\partial y^\beta(\mathbf{y})$ does not transform as a tensor [27, 28]

$$\frac{\partial\tilde{v}_\alpha}{\partial y^\beta}(\mathbf{y}) = \frac{\partial x^\nu}{\partial y^\beta}\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial v_\mu}{\partial x^\nu}(\mathbf{x}) + \frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\beta}v_\mu(\mathbf{x}). \tag{2.1.16}$$

We must define a derivative process that transforms as a tensor. This problem can be solved studying the problem of the geodesic lines in curved spaces.

2.1.1 Geodesic lines in curved spaces

Let us consider the shortest path between two points p and q on a manifold. The line element in a curved manifold is given by [29]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \tag{2.1.17}$$

We can parametrize the curves on the manifold as

$$s = \int_{\tau_1}^{\tau_2}\left[g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]^{\frac{1}{2}} d\tau. \tag{2.1.18}$$

Since we are searching the shortest distance between the given points, we must perform $\delta s \to 0$. Then

$$S = \int_{\lambda_1}^{\lambda_2} L(x^\mu, \dot{x}^\mu)d\lambda. \tag{2.1.19}$$

The functional derivative is defined as

$$\delta S \equiv \frac{d}{d\lambda}\frac{\partial}{\partial\dot{x}^\alpha}L - \frac{\partial}{\partial x^\alpha}L, \tag{2.1.20}$$

applying this to the expression (2.1.18), we obtain

$$\frac{d}{d\tau}\frac{\partial}{\partial\dot{x}^\alpha}\left[g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]^{\frac{1}{2}} - \frac{\partial}{\partial x^\alpha}\left[g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]^{\frac{1}{2}} = 0. \tag{2.1.21}$$

The first term of the equation yields

$$\begin{aligned}\frac{1}{2}\left[g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]^{-\frac{1}{2}}\frac{\partial}{\partial\dot{x}^\alpha}\left[g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right] &= \frac{1}{2}\left[g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]^{-\frac{1}{2}} g_{\mu\nu}\left[\frac{dx^\nu}{d\tau}\delta^\mu_\alpha + \frac{dx^\mu}{d\tau}\delta^\nu_\alpha\right] \\ &= \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right]^{-\frac{1}{2}} g_{\alpha\beta}2\dot{x}^\beta,\end{aligned} \tag{2.1.22}$$

where we have supposed the metric does not depend of $\dot{x}^\gamma$ and we have used the notation $\frac{dx^\gamma}{d\tau} = \dot{x}^\gamma$. On the other hand, the second term of the equation reads

$$\frac{\partial}{\partial x^\alpha}\left[g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]^{\frac{1}{2}} = \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right]^{-\frac{1}{2}}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu\dot{x}^\nu. \tag{2.1.23}$$

Therefore, the expression (2.1.21) is now

$$\frac{d}{d\tau}\left[\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{-\frac{1}{2}}g_{\alpha\beta}\dot{x}^{\beta}\right]-\frac{1}{2}\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{-\frac{1}{2}}\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\dot{x}^{\mu}\dot{x}^{\nu}=0, \tag{2.1.24}$$

expanding the first derivative in the before expression

$$\frac{d}{d\tau}\left[\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{-\frac{1}{2}}g_{\alpha\beta}\dot{x}^{\beta}\right]=\frac{d}{d\tau}\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{-\frac{1}{2}}g_{\alpha\beta}\dot{x}^{\beta}+\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{-\frac{1}{2}}\frac{dg_{\alpha\beta}}{d\tau}\dot{x}^{\beta}+\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{-\frac{1}{2}}g_{\alpha\beta}\ddot{x}^{\beta}. \tag{2.1.25}$$

We can express the total differential of the metric as $dg_{\alpha\beta}=(\partial g_{\alpha\beta}/\partial x^{\mu})dx^{\mu}$, thus

$$\frac{dg_{\alpha\beta}}{d\tau}=\frac{\partial g_{\alpha\beta}}{\partial x^{\mu}}\dot{x}^{\mu}, \tag{2.1.26}$$

therefore,

$$g_{\alpha\beta}\ddot{x}^{\beta}-\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\dot{x}^{\mu}\dot{x}^{\nu}+\frac{\partial g_{\alpha\beta}}{\partial x^{\mu}}\dot{x}^{\mu}\dot{x}^{\beta}=g_{\alpha\beta}\dot{x}^{\beta}\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{\frac{1}{2}}\frac{d}{d\tau}\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{-\frac{1}{2}}. \tag{2.1.27}$$

Expressing

$$\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}\dot{x}^{\beta}\dot{x}^{\gamma}=\frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}+\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}\right)\dot{x}^{\beta}\dot{x}^{\gamma}=\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}\dot{x}^{\beta}\dot{x}^{\gamma}+\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}\dot{x}^{\beta}\dot{x}^{\gamma},$$

and as β and γ are dummy indices,

$$\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}\dot{x}^{\beta}\dot{x}^{\gamma}=\frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}}+\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}\right)\dot{x}^{\beta}\dot{x}^{\gamma}, \tag{2.1.28}$$

then, from (2.1.27) and (2.1.28) we obtain

$$g_{\alpha\beta}\ddot{x}^{\beta}+\frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}}+\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}-\frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)\dot{x}^{\beta}\dot{x}^{\gamma}=g_{\alpha\beta}\dot{x}^{\beta}\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{-\frac{1}{2}}\frac{d}{d\tau}\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{\frac{1}{2}}. \tag{2.1.29}$$

Now, if $s\equiv\tau$, $\dot{s}=\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{\frac{1}{2}}$, $\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{-\frac{1}{2}}\frac{d}{d\tau}\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{\frac{1}{2}}=\ddot{s}/\dot{s}$ and $\ddot{s}=0$, the before expression yields,

$$g^{\delta\alpha}g_{\beta\alpha}\ddot{x}^{\beta}+g^{\delta\alpha}\frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}}+\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}-\frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)\dot{x}^{\beta}\dot{x}^{\gamma}=0, \tag{2.1.30}$$

that is to say,

$$\frac{d^{2}x^{\delta}}{ds^{2}}+\Gamma^{\delta}_{\alpha\gamma}\frac{dx^{\beta}}{ds}\frac{dx^{\gamma}}{ds}=0, \tag{2.1.31}$$

being

$$\Gamma^{\delta}_{\alpha\gamma}:=\frac{1}{2}g^{\delta\rho}\left(\frac{\partial g_{\gamma\rho}}{\partial x^{\alpha}}+\frac{\partial g_{\alpha\rho}}{\partial x^{\gamma}}-\frac{\partial g_{\alpha\gamma}}{\partial x^{\rho}}\right). \tag{2.1.32}$$

The problem found from the expression (2.1.16) is solved thanks to this object. Examining its transformation properties we have

$$\tilde{\Gamma}^{\lambda}_{\mu\nu}(\mathbf{y})=\frac{\partial x^{\alpha}}{\partial y^{\mu}}\frac{\partial x^{\gamma}}{\partial y^{\nu}}\left[\Gamma^{\delta}_{\alpha\gamma}(\mathbf{x})\frac{\partial y^{\lambda}}{\partial x^{\delta}}-\frac{\partial^{2}y^{\lambda}}{\partial x^{\alpha}\partial x^{\gamma}}\right]=\frac{\partial y^{\lambda}}{\partial x^{\delta}}\left[\Gamma^{\delta}_{\alpha\gamma}(\mathbf{x})\frac{\partial x^{\alpha}}{\partial y^{\mu}}\frac{\partial x^{\gamma}}{\partial y^{\nu}}+\frac{\partial^{2}x^{\delta}}{\partial y^{\mu}\partial y^{\nu}}\right]. \tag{2.1.33}$$

Then, defining the covariant derivative as

$$\nabla_{\alpha}v_{\gamma}=\frac{\partial v_{\gamma}}{\partial x^{\alpha}}-\Gamma^{\delta}_{\alpha\gamma}v_{\delta}, \tag{2.1.34}$$

it transforms as follows

$$\tilde{\nabla}_{\mu}\tilde{v}_{\nu}(\mathbf{y})=\frac{\partial x^{\gamma}}{\partial y^{\nu}}\frac{\partial x^{\alpha}}{\partial y^{\mu}}\nabla_{\alpha}v_{\gamma}(\mathbf{x}). \tag{2.1.35}$$

Therefore this new operation transforms as a tensor, thanks to the object (2.1.32). If the components of an object transform as (2.1.33) we say that this object is a connection.

2.1.2 Riemann curvature tensor

The covariant derivative satisfies the Leibniz rule, then

$$\nabla_\mu(\phi v_\nu) = \frac{\partial(\phi v_\nu)}{\partial x^\mu} - \Gamma^\gamma_{\mu\nu}\phi v_\gamma$$

$$\phi\nabla_\mu v_\nu + v_\nu\nabla_\mu\phi = \phi\nabla_\mu v_\nu + v_\nu\frac{\partial\phi}{\partial x^\mu}. \tag{2.1.36}$$

Thus,

$$\nabla_\mu\phi = \frac{\partial\phi}{\partial x^\mu}. \tag{2.1.37}$$

If we define $\phi \equiv v_\delta w^\delta$, we have

$$\nabla_\mu(v_\delta w^\delta) = \frac{\partial(v_\delta w^\delta)}{\partial x^\mu}$$

$$v_\delta\nabla_\mu w^\delta + w^\delta\nabla_\mu v_\delta = v_\delta\frac{\partial w^\delta}{\partial x^\mu} + w^\delta\frac{\partial v_\delta}{\partial x^\mu}. \tag{2.1.38}$$

If the vector is contravariant

$$\nabla_\mu v^\nu = \frac{\partial v^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\gamma}v^\gamma. \tag{2.1.39}$$

Generally the second covariant derivatives of any object do not commute. The difference of two covariant derivatives is a tensor. This difference is given by the Ricci identity

$$\nabla_\mu\nabla_\nu v_\gamma - \nabla_\nu\nabla_\mu v_\gamma = -R^\sigma_{\gamma\mu\nu}(\mathbf{\Gamma})v_\sigma - T^\sigma_{\mu\nu}(\mathbf{\Gamma})\nabla_\sigma v_\gamma, \tag{2.1.40}$$

where

$$R^\sigma_{\gamma\mu\nu}(\mathbf{\Gamma}) = \partial_\mu\Gamma^\sigma_{\nu\gamma} - \partial_\nu\Gamma^\sigma_{\mu\gamma} + \Gamma^\sigma_{\mu\delta}\Gamma^\delta_{\nu\gamma} - \Gamma^\sigma_{\nu\delta}\Gamma^\delta_{\mu\gamma}, \tag{2.1.41}$$

is the Riemann curvature tensor and

$$T^\sigma_{\mu\nu}(\mathbf{\Gamma}) = \Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu}, \tag{2.1.42}$$

is the torsion.

If we contract the Riemann tensor, we obtain the Riemann-Ricci tensor

$$H_{\mu\nu}(\mathbf{\Gamma}) = R^\sigma_{\sigma\mu\nu}(\mathbf{\Gamma}) = \partial_\mu\Gamma_\nu - \partial_\nu\Gamma_\mu, \tag{2.1.43}$$

being $\Gamma_\mu = \Gamma^\sigma_{\mu\sigma}$. Another contraction give the Ricci tensor

$$R_{\mu\nu}(\mathbf{\Gamma}) = R^\sigma_{\mu\sigma\nu}(\mathbf{\Gamma}) = \partial_\sigma\Gamma^\sigma_{\mu\nu} - \partial_\nu\Gamma^\sigma_{\sigma\mu} + \Gamma^\sigma_{\sigma\gamma}\Gamma^\gamma_{\nu\mu} - \Gamma^\sigma_{\nu\gamma}\Gamma^\gamma_{\sigma\mu}. \tag{2.1.44}$$

2.1.3 Einstein Field Equations

Consider an action of the form

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(2\Lambda - R\right). \tag{2.1.45}$$

where R is the Ricci scalar, Λ is a constant and $g = detg_{\mu\nu}$. Rewriting the before expression

$$S = \int d^4x\sqrt{-g}2\Lambda - \int d^4x\sqrt{-g}R.$$

We make a variation over the action in the following form

$$\delta S = \int d^4x\left[2\sqrt{-g}\delta\Lambda + 2\Lambda\delta\sqrt{-g}\right] - \int d^4x\left[R\delta\sqrt{-g} + \sqrt{-g}\delta R\right]. \tag{2.1.46}$$

As Λ is a constant

$$\delta\Lambda = 0. \tag{2.1.47}$$

Since $g^{\mu\lambda}g_{\lambda\nu} = \delta^{\mu}_{\nu}$, the variation of δ is

$$g_{\lambda\nu}\delta g^{\mu\lambda} + g^{\mu\lambda}\delta g_{\lambda\nu} = 0. \tag{2.1.48}$$

Thus

$$\delta g_{\rho\nu} = -g_{\mu\rho}g_{\lambda\nu}\delta g^{\mu\lambda}, \tag{2.1.49}$$

then

$$\frac{1}{g}\delta g = g^{\mu\nu}\delta g_{\mu\nu}. \tag{2.1.50}$$

In the same form

$$\delta g = -g g_{\mu\nu}\delta g^{\mu\nu}. \tag{2.1.51}$$

With this in mind, we obtain

$$\delta(\sqrt{-g}) = -\frac{1}{2\sqrt{-g}}\delta g = \frac{g}{2\sqrt{-g}}g_{\mu\nu}\delta g^{\mu\nu}. \tag{2.1.52}$$

As we see before, the Riemann tensor is defined by

$$R^{\alpha}_{\mu\beta\nu} = \partial_\beta \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\nu} + \Gamma^{\alpha}_{\sigma\beta}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\sigma}_{\mu\beta}, \tag{2.1.53}$$

make a variation of the tensor, we have

$$\delta R^{\alpha}_{\mu\beta\nu} = \nabla_\beta\left(\delta\Gamma^{\alpha}_{\mu\nu}\right) - \nabla_\nu\left(\delta\Gamma^{\alpha}_{\beta\mu}\right). \tag{2.1.54}$$

Since $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ the variation of the Ricci tensor is

$$\delta R_{\mu\nu} = \nabla_\alpha\left(\delta\Gamma^{\alpha}_{\mu\nu}\right) - \nabla_\nu\left(\delta\Gamma^{\alpha}_{\alpha\mu}\right). \tag{2.1.55}$$

In the same form, the variation of the Ricci scalar R yields

$$\delta R = \delta\left(g^{\mu\nu}R_{\mu\nu}\right) = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}, \tag{2.1.56}$$

that is to say,

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\left[\nabla_\alpha\left(\delta\Gamma^{\alpha}_{\mu\nu}\right) - \nabla_\nu\left(\delta\Gamma^{\alpha}_{\alpha\mu}\right)\right]. \tag{2.1.57}$$

Therefore the expression (2.1.46) is now

$$\delta S = \int d^4x\left[2\Lambda\left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\right)\right]$$

$$-\int d^4x\left[-\frac{1}{2}\sqrt{-g}g_{\mu\nu}R\delta g^{\mu\nu} + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}\nabla_\alpha\left(g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu}\right) - \sqrt{-g}\nabla_\nu\left(g^{\mu\nu}\delta\Gamma^{\alpha}_{\alpha\mu}\right)\right]. \tag{2.1.58}$$

Since we have fixed extremes $\delta\Gamma = 0$ then

$$\delta S = \int d^4x\left[-\Lambda g_{\mu\nu} - R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R\right]\sqrt{-g}\delta g^{\mu\nu}. \tag{2.1.59}$$

As we are imposing $\delta S = 0$ over an arbitrary variation, the integrand must be zero in order to satisfy the imposition, then we obtain the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0. \tag{2.1.60}$$

Finally, if we add to the action (2.1.45) one term of the form

$$S_m = \int d^4x \mathcal{L}_m \sqrt{-g}, \tag{2.1.61}$$

the Einstein field equations have the form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{2.1.62}$$

where

$$T_{\mu\nu} = \frac{1}{2}\frac{1}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}}. \tag{2.1.63}$$

This term is known as the stress–energy tensor and has information about the matter fields that we will explore later. In the framework of the quantization, that tensor will be a mean value of quantum objects. The general relativity remains as a classical theory and when we are taking quantum objects influencing over the geometry as mean values, we work in a semi-classical world.

From the Einstein field equations in the vacuum (2.1.60) note that if we rewrite the expression as follows

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\Lambda g_{\mu\nu}, \tag{2.1.64}$$

the tensor $\Lambda g_{\mu\nu}$ is working as a stress-energy tensor. This is one of the firsts inklings indicating the physical reality of a vacuum energy that permeates every fibber of the universe. Currently, the cosmological implications of this kind of tensor are showing interesting results in the explanation of the accelerated expansion of universe, big-bang and galactic structures formation from primordial effects in early universe.

2.2 Rindler Metric

The Rindler metric emerges from the physical situation of the accelerated observer [30]. Let us explore the physics of the accelerated observer in the framework of special relativity in order to stablish the Rindler metric. We define the 4-vector of position as

$$x^\mu = (t, \mathbf{r}), \tag{2.2.1}$$

where $\mathbf{r}$ are the spatial coordinates of position and t the measured time in any inertial reference system. The physical velocity of a particle is defined by

$$\mathbf{u} = \frac{d\mathbf{r}}{dt}, \tag{2.2.2}$$

thus the 4-velocity is defined as

$$U^\alpha = \frac{dx^\alpha}{d\tau}, \tag{2.2.3}$$

being τ the proper time. The relation between proper time and coordinate time is given by

$$\frac{dt}{d\tau} = \gamma(\mathbf{u}) = \frac{1}{\sqrt{1 - |\mathbf{u}|^2}}. \tag{2.2.4}$$

Such that, the 4-velocity is now

$$U^\alpha = \gamma(\mathbf{u})(1, \mathbf{u}). \tag{2.2.5}$$

In a similar way, we define the 4-acceleration as

$$A^\mu = \frac{dU^\mu}{d\tau}, \tag{2.2.6}$$

the physical acceleration is

$$\mathbf{a} = \frac{d\mathbf{u}}{dt}. \tag{2.2.7}$$

Therefore the 4-acceleration yields [31]

$$A^\alpha = \gamma(\mathbf{u}) \left(\frac{d\gamma(\mathbf{u})}{dt}, \frac{d\gamma(\mathbf{u})}{dt}\mathbf{u} + \gamma(\mathbf{u})\mathbf{a} \right). \tag{2.2.8}$$

Let us suppose the 4-velocity and the 4-acceleration components are given by

$$U^\mu = (U^0, U_x, 0, 0) \tag{2.2.9}$$

$$A^\mu = (A^0, A_x, 0, 0). \tag{2.2.10}$$

Each one satisfy the following relations

$$U^2 = -1 = -(U^0)^2 + (U_x)^2, \tag{2.2.11}$$

$$U \cdot A = 0 = -U^0 A^0 - U_x A_x, \tag{2.2.12}$$

$$A^2 = \alpha^2 = -(A^0)^2 - (A_x)^2. \tag{2.2.13}$$

From the expression (2.2.12) we obtain

$$A^0 = \frac{A_x B_x}{U^0} \tag{2.2.14}$$

and replacing in the expression (2.2.13) we have

$$\alpha^2 = (A_x)^2 \left[1 - \left(\frac{U_x}{U_0} \right)^2 \right], \tag{2.2.15}$$

where using the expression (2.2.11) we have

$$A_x = \alpha U^0, \tag{2.2.16}$$

therefore

$$A^0 = \alpha U_x. \tag{2.2.17}$$

Differentiating respect the proper time the expression (2.2.16) and taking into account the expression (2.2.17) we obtain a differential equation for U_x [31]

$$\frac{d^2 U_x}{d\tau^2} = \alpha \frac{dU^0}{d\tau} = \alpha A^0 = \alpha^2 U_x, \tag{2.2.18}$$

then

$$\frac{d^2 U_x}{d\tau^2} - \alpha^2 U_x = 0. \tag{2.2.19}$$

The solutions of (2.2.19) are given by hyperbolic functions

$$U_x = C_1 \sinh(\alpha\tau) + C_2 \cosh(\alpha\tau), \tag{2.2.20}$$

where C_1 and C_2 are integration constants which we can obtain imposing initial conditions, such as

$$U_x(\tau = 0) = 0, \tag{2.2.21}$$

$$\left. \frac{dU_x}{d\tau} \right|_{\tau=0} = \alpha. \tag{2.2.22}$$

From the condition (2.2.21) we obtain that $C_2 = 0$ and from (2.2.22) we have $C_1 = 1$. Therefore we can express the quantities as follows

$$U_x = \frac{dx}{d\tau} = \sinh(\alpha\tau), \tag{2.2.23}$$

$$U^0 = \frac{dx^0}{d\tau} = \cosh(\alpha\tau). \tag{2.2.24}$$

Integrating the above expressions

$$x^0 = \alpha^{-1}\sinh(\alpha\tau) + D_1, \tag{2.2.25}$$

$$x^1 = \alpha^{-1}\cosh(\alpha\tau) + D_2. \tag{2.2.26}$$

From this we have that the trajectories of an accelerated observer in Minkowski space, are hyperbolic trajectories. From this we can stablish the formulation of the Rindler coordinates.

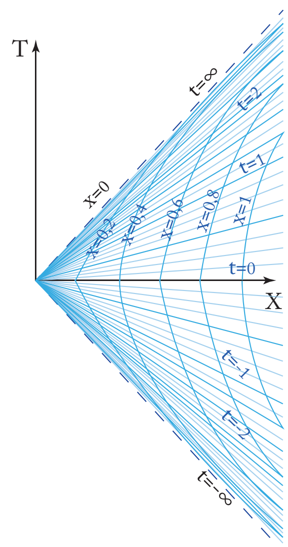

Figure 2.1: Hyperbolic trajectories of a Rindler observer.

Let us therefore, consider two-dimensional Minkowski space

$$ds^2 = dt^2 - dx^2. \tag{2.2.27}$$

Going to null coordinates defined by

$$u = t - x, \quad v = t + x, \tag{2.2.28}$$

the line element (2.2.27) becomes

$$ds^2 = dudv \tag{2.2.29}$$

so that

$$g_{\mu\nu} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.2.30}$$

Performing the following coordinate transformation

$$t = a^{-1}e^{a\xi}\sinh a\eta \tag{2.2.31}$$

$$x = a^{-1}e^{a\xi}\cosh a\eta, \tag{2.2.32}$$

with $a =$ constant and $-\infty < \eta,\ \xi < \infty$, or

$$u = -a^{-1}e^{-a\bar{u}} \tag{2.2.33}$$

$$v = a^{-1}e^{a\bar{v}}, \tag{2.2.34}$$

where

$$\bar{u} = \eta - \xi, \tag{2.2.35}$$

$$\bar{v} = \eta + \xi, \tag{2.2.36}$$

the line elements (2.2.27) and (2.2.29) becomes

$$ds^2 = e^{2a\xi}d\bar{u}d\bar{v} = e^{2a\xi}(d\eta^2 - d\xi^2). \tag{2.2.37}$$

The coordinates (η, ξ) cover only a quadrant of Minkowski space; the wedge $x > |t|$. Lines of constant ξ are hyperbolas

$$x^2 - t^2 = a^{-2}e^{2a\xi} = \text{constant}. \tag{2.2.38}$$

These hyperbolas represent the world lines of uniformly accelerated observers. In comparison, we have that

$$ae^{-a\xi} = \alpha^{-1} = \text{proper acceleration}. \tag{2.2.39}$$

All the hyperbolas are asymptotic to the null rays $u = 0$, $v = 0$ (or $\bar{u} = \infty$, $\bar{v} = \infty$). Then the accelerated observers approach the speed of light $\eta \pm \infty$. The proper time of the accelerated observers is related to ξ and η by

$$\tau = e^{a\xi}\eta. \tag{2.2.40}$$

The system (η, ξ) is known as the Rindler coordinate system [30].

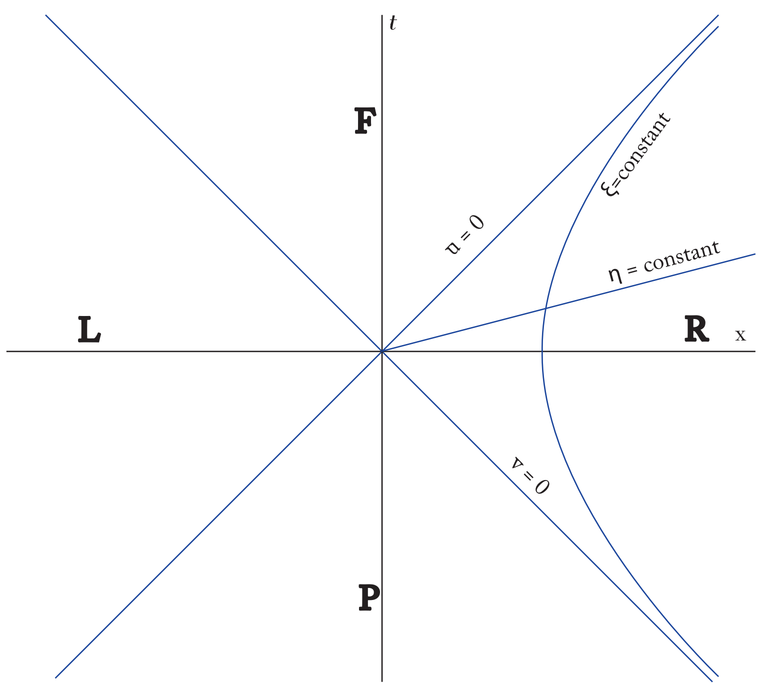

Figure 2.2: Regions in Rindler space.

2.2.1 Radiation from uniformly accelerated motion

The Unruh effect can be considered as a back-reaction problem as we shall see later. In this section we discuss the situation in which an accelerated charge experiments a radiation reaction force. This situation would bring some interesting physical aspects in the understanding of the thermal radiation perceived by an accelerated observer in the vacuum.

The non-relativistic equation of motion including the radiation reaction is

$$m\dot{\mathbf{v}} = \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{react}}, \tag{2.2.41}$$

where $\mathbf{F}_{ext}$ is an external force and $\mathbf{F}_{\text{react}}$

$$\mathbf{F}_{\text{react}} = \frac{2e^2}{3}\ddot{\mathbf{v}} + \mathcal{O}(\mathbf{v}) \tag{2.2.42}$$

is the radiation reaction force, $\mathbf{v}$ is the velocity of the electron and the dot indicates differentiation with respect to time. Using covariant notation, the relativistic version of the before expression is [32]

$$m\frac{du^\mu}{d\tau} = F^\mu_{\text{ext}} + F^\mu_{\text{react}}, \tag{2.2.43}$$

with external 4-force F^μ_{ext} and radiation-reaction 4-force given by

$$F^\mu_{\text{react}} = \frac{2e^2}{3}\frac{d^2u^\mu}{d\tau^2} - Ru^\mu, \tag{2.2.44}$$

where

$$R = -\frac{2e^2}{3}\frac{du_\nu}{d\tau}\frac{du^\nu}{d\tau} = \frac{2e^2\gamma^6}{3}\left[\dot{\mathbf{v}}^2 - (\mathbf{v}\times\dot{\mathbf{v}})^2\right] \geq 0 \tag{2.2.45}$$

is the invariant rate of radiation of energy of an accelerated charge and $u^\mu = \gamma(1, \mathbf{v})$ is the 4-velocity.

The time component of the equation (2.2.43) can be written as

$$\frac{d\gamma m}{dt} = \mathbf{F}_{\text{ext}}\cdot\mathbf{v} + \frac{dQ}{dt} - R, \tag{2.2.46}$$

being

$$Q = \frac{2\gamma^4e^2\mathbf{v}\cdot\dot{\mathbf{v}}}{3}, \tag{2.2.47}$$

is an energy being stored in the electron in virtue of its acceleration [33, 34]. The space components are

$$\frac{d\gamma m\mathbf{v}}{dt} = \mathbf{F}_{\text{ext}} + \frac{2e^2\gamma^2}{3}\left[\ddot{\mathbf{v}} + 3\gamma^2(\mathbf{v}\cdot\dot{\mathbf{v}})\dot{\mathbf{v}} + \gamma^2(\mathbf{v}\cdot\ddot{\mathbf{v}})\mathbf{v} + 3\gamma^4(\mathbf{v}\cdot\dot{\mathbf{v}})^2\mathbf{v}\right] \tag{2.2.48}$$

2.3 Schwarzschild Metric

The Schwarzschild solution of the Einstein field equations, is the successive solution of three problems: isotropic field, stationary isotropic field and static isotropic field [35]

The isotropic field implies physically that ds^2 must be an scalar under the rotations group in the three-dimensional space. The scalars that can be constructed under this conditions are: t, dt, $\mathbf{r}\cdot\mathbf{r}$, $\mathbf{r}\cdot d\mathbf{r}$ y $d\mathbf{r}\cdot d\mathbf{r}$. Then, the more general isotropic interval can be expressed as

$$ds^2 = a(\mathbf{r}\cdot\mathbf{r},t)(dt)^2 + b(\mathbf{r}\cdot\mathbf{r},t)\mathbf{r}\cdot d\mathbf{r}dt + c(\mathbf{r}\cdot\mathbf{r},t)(\mathbf{r}\cdot d\mathbf{r})^2 + f(\mathbf{r}\cdot\mathbf{r},t)d\mathbf{r}\cdot d\mathbf{r}. \tag{2.3.1}$$

Now, the fundamental characteristic of a stationary isotropic field is that the tensor metric components are time independent, then, the interval is now

$$ds^2 = a(\mathbf{r}\cdot\mathbf{r})(dt)^2 + b(\mathbf{r}\cdot\mathbf{r})\mathbf{r}\cdot d\mathbf{r}dt + c(\mathbf{r}\cdot\mathbf{r})(\mathbf{r}\cdot d\mathbf{r})^2 + f(\mathbf{r}\cdot\mathbf{r})d\mathbf{r}\cdot d\mathbf{r}. \tag{2.3.2}$$

However, the static isotropic field impose that the following condition: the components g_{a0} must vanish. Then the interval yields,

$$ds^2 = a(\mathbf{r}\cdot\mathbf{r})(dt)^2 + c(\mathbf{r}\cdot\mathbf{r})(\mathbf{r}\cdot d\mathbf{r})^2 + f(\mathbf{r}\cdot\mathbf{r})d\mathbf{r}\cdot d\mathbf{r}. \tag{2.3.3}$$

Working with the spherical coordinates, we have $\mathbf{r} = r\hat{r}$, $d\mathbf{r} = dr\hat{r} + rd\theta\hat{\theta} + rsin\theta d\varphi\hat{\varphi}$, and the scalar products reads,

$$\mathbf{r}\cdot\mathbf{r} = r^2,$$

$$\mathbf{r}\cdot d\mathbf{r} = rdr,$$

$$d\mathbf{r}\cdot d\mathbf{r} = (dr)^2 + r^2\left[(d\theta)^2 + sin^2\theta(d\varphi)^2\right].$$

Then we can express the interval as follows,

$$ds^2 = -a(r)dt^2 + c(r)dr^2 + f(r)r^2\left(d\theta^2 + sin^2\theta d\varphi^2\right). \tag{2.3.4}$$

Now changing the coordinates $(ct, r, \theta, \varphi) \to (ct, \rho, \theta, \varphi)$, where $\rho \equiv f(r)r^2$, we have,

$$ds^2 = -A(\rho)c^2dt^2 + B(\rho)d\rho^2 + \rho^2(d\theta^2 + sin^2\theta d\varphi^2). \tag{2.3.5}$$

With this the non-null components of the Ricci tensor reads,

$$R_{00} = -\frac{1}{4\rho AB^2}\left[2\rho AB\frac{d^2A}{d\rho^2} - \rho B\left(\frac{dA}{d\rho}\right)^2 - \rho A\frac{dA}{d\rho}\frac{dB}{d\rho} + 4AB\frac{dA}{d\rho}\right], \tag{2.3.6}$$

$$R_{11} = \frac{1}{4\rho A^2B}\left[2\rho AB\frac{d^2A}{d\rho^2} - \rho B\left(\frac{dA}{d\rho}\right)^2 - \rho A\frac{dA}{d\rho}\frac{dB}{d\rho} - 4A^2\frac{dB}{d\rho}\right], \tag{2.3.7}$$

$$R_{22} = -\frac{1}{2AB^2}\left[-\rho B\frac{dA}{d\rho} + \rho A\frac{dB}{d\rho} + 2AB^2 - 2AB\right], \tag{2.3.8}$$

$$R_{33} = R_{22}sin^2\theta. \tag{2.3.9}$$

The system that stablish the spatio-temporal curvature is a matter-energy distribution concentrated in a finite region of the space-time, in principle with spherical symmetry. Then this distribution has a defined radius R. With this solution we want study the exterior of this distribution so we can assume that $T_{\mu\nu} = 0$. The Einstein field equations yields

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}, \tag{2.3.10}$$

if we up an index,

$$R^{\mu}_{\nu} - \frac{1}{2}g^{\mu}_{\nu}R = \kappa T^{\mu}_{\nu},$$

and contracting making $\mu = \nu$, we have

$$R^{\mu}_{\mu} - \frac{1}{2}g^{\mu}_{\mu}R = \kappa T^{\mu}_{\mu}. \tag{2.3.11}$$

Then we obtain

$$R - 2R = \kappa T^{\mu}_{\mu} \Longrightarrow R = -\kappa T^{\mu}_{\mu}, \tag{2.3.12}$$

replacing this in the Einstein field equations

$$R_{\mu\nu} = \kappa \left(g^{\alpha}_{\mu} g^{\beta}_{\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \right) T_{\alpha\beta}. \tag{2.3.13}$$

Then from this last expression we can conclude that $T_{\mu\nu} = 0$ implies $R_{\mu\nu} = 0$. With this result we have that $R_{00} = R_{11} = R_{22} = 0$ and then, the components of the Ricci tensor conform now the following set of coupled differential equations [36, 37]

$$2\rho AB \frac{d^2A}{d\rho^2} - \rho B \left(\frac{dA}{d\rho} \right)^2 - \rho A \frac{dA}{d\rho} \frac{dB}{d\rho} = -4AB \frac{dA}{d\rho}, \tag{2.3.14}$$

$$2\rho AB \frac{d^2A}{d\rho^2} - \rho B \left(\frac{dA}{d\rho} \right)^2 - \rho A \frac{dA}{d\rho} \frac{dB}{d\rho} = 4A^2 \frac{dB}{d\rho}, \tag{2.3.15}$$

$$-\rho B \frac{dA}{d\rho} + \rho A \frac{dB}{d\rho} + 2AB^2 - 2AB = 0. \tag{2.3.16}$$

We can observe that the left side of the expressions (2.3.14) and (2.3.15) are equal, therefore

$$-4AB \frac{dA}{d\rho} = 4A^2 \frac{dB}{d\rho}, \tag{2.3.17}$$

in other words,

$$A \frac{dB}{d\rho} + \frac{dA}{d\rho} B = 0, \tag{2.3.18}$$

rewriting this

$$\frac{d}{d\rho} (AB) = 0. \tag{2.3.19}$$

This is indicating that these functions product is a constant, then we can relate both functions as follows

$$A(\rho)B(\rho) = c_1 \Longrightarrow B(\rho) = \frac{c_1}{A(\rho)}, \tag{2.3.20}$$

replacing this in the expression (2.3.16), we obtain

$$\rho \frac{dA}{d\rho} + A = c_1 \Longrightarrow \frac{d}{d\rho}(\rho A) = c_1. \tag{2.3.21}$$

The solution of the last equation is

$$A(\rho) = c_1 \left(1 + \frac{c_2}{\rho} \right). \tag{2.3.22}$$

We can find these constants imposing the Newtonian limit to these solutions. In the weak field approximation we have

$$g_{00} = 1 + \frac{2\Phi}{c^2}, \tag{2.3.23}$$

comparing with (2.3.22), we can evidence that $c_1 = 1$ and

$$\frac{c_2}{\rho} = -\frac{2GM}{c^2 r}. \tag{2.3.24}$$

Since these two variables are overdetermined we can choose $\rho = r$, thus

$$c_2 = -\frac{2GM}{c^2}, \tag{2.3.25}$$

that is to say,

$$A = \frac{1}{B} = 1 - \frac{2GM}{c^2 r}, \tag{2.3.26}$$

then, the interval is,

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.3.27}$$

In the matrix representation the metric tensor takes the form

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2GM}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2GM}{c^2 r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix}. \tag{2.3.28}$$

This solution is valid for the interval $r > R$, being R the radius of the spherical distribution. If we want this solution cover all space, we can collapse the whole distribution in a point that would be the origin of the coordinate system. This is the Schwarzschild problem. For this situation the metric (2.3.27) will be valid for the whole space except in the point $r = 0$. This situation is describing a black hole.

2.3.1 Singularities and causal structure

From the metric (2.3.27) we evidence a divergence when $r = 2M$. We can show that this divergence has a coordinate origin. Then under a coordinate transform, this divergence will disappear. Therefore we must construct a coordinate system where we can study the physical properties of the metric in this point.

In order to study the behaviour of the usual coordinates close to $r = 2M$, we consider a probe element in this field. Its trajectory is given by the following Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \tag{2.3.29}$$

which acquire the following form in terms of the Schwarzschild metric [9]

$$2\mathcal{L} = -\left(1 - \frac{2M}{r}\right) \dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + r^2(\dot{\theta}^2 + sin^2\theta\dot{\varphi}^2). \tag{2.3.30}$$

By definition of the proper time, we have that throughout the world line $x^\mu(\tau)$ we obtain

$$2\mathcal{L} = -1. \tag{2.3.31}$$

Taking $\theta = \pi/2$, without loss of generality

$$2\mathcal{L} = -\left(1 - \frac{2M}{r}\right) \dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + r^2\dot{\varphi}^2. \tag{2.3.32}$$

The variables φ and t are cyclic, then

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = r^2 \dot{\varphi} \equiv cte := L, \tag{2.3.33}$$

$$-\frac{\partial \mathcal{L}}{\partial \dot{t}} = \dot{t}\left(1 - \frac{2M}{r}\right) \equiv cte := E. \tag{2.3.34}$$

Introducing (2.3.33) and (2.3.34) in (2.3.31)

$$\left(1-\frac{2M}{r}\right)^{-1}E^2-\left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2-\frac{L^2}{r^2}=1. \tag{2.3.35}$$

We consider the physical situation where the probe element are following time-like radial geodesics close to $r=2M$. For $L=0$, from the last expression we have

$$\dot{r}^2=\frac{2M}{r}+E^2-1. \tag{2.3.36}$$

Supposing the probe element is in rest at $r=R$ such that

$$\frac{2M}{r}=1-E^2, \tag{2.3.37}$$

from (2.3.36) we have

$$d\tau=\left(\frac{2M}{r}-\frac{2M}{R}\right)^{-1/2}dr. \tag{2.3.38}$$

From here we obtain the functional relation $x^\mu\equiv x^\mu(\tau)$. Furthermore from the last expression we can evidence that nothing strange occurs in $r=2M$. On the other hand, if we consider r as a function of the coordinate time t, from (2.3.34) we obtain

$$\dot{r}=\frac{dr}{dt}\dot{t}=\frac{dr}{dt}\left(1-\frac{2M}{r}\right)^{-1}E. \tag{2.3.39}$$

Introducing the following radial coordinate transformation

$$r^*=r+2M\ln\left(\frac{r}{2M}-1\right), \tag{2.3.40}$$

we have that

$$\frac{dr^*}{dt}=\left(1-\frac{2M}{r}\right)^{-1}\frac{dr}{dt}, \tag{2.3.41}$$

from this expression, the equation (2.3.39) yields

$$\dot{r}=E\frac{dr^*}{dt}. \tag{2.3.42}$$

Introducing this last expression in (2.3.36), we obtain

$$\left(E\frac{dr^*}{dt}\right)^2=E^2-1+\frac{2M}{r}. \tag{2.3.43}$$

Then, when $r<2M$, $r^*\to-\infty$ and the right side of the last equation tends to E^2, thus for $r\simeq 2M$

$$\frac{dr^*}{dt}\simeq -1, \tag{2.3.44}$$

and

$$r^*\simeq -t+c_1, \tag{2.3.45}$$

in other words,

$$r\simeq 2M+c_1e^{-t/2M}. \tag{2.3.46}$$

This is indicating that the probe element will reach the surface $r = 2M$ only when has passed an infinite coordinate time.

Considering null radial directions for the metric, in other words, considering $ds = 0$, we obtain an expression of the form

$$\frac{dr}{dt} = \pm\left(1 - \frac{2M}{r}\right). \tag{2.3.47}$$

That is indicating for $r < 2M$ the opening angle of the light cones will increase and change its direction. Then the simultaneous use of the coordinates r and t is limited. Another indicator that show the problem generated by $r = 2M$ is a coordinate problem, is the Kretschmann scalar. This invariant can be obtained by the contraction of the Riemann tensor with itself,

$$K = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}, \tag{2.3.48}$$

where

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\mu}R^{\mu}_{\beta\gamma\delta}, \tag{2.3.49}$$

and

$$R^{\alpha\beta\gamma\delta} = g^{\beta\mu}g^{\gamma\nu}g^{\delta\xi}R^{\alpha}_{\mu\nu\xi}. \tag{2.3.50}$$

From the Schwarzschild solution, in the usual coordinates, the non null components of the Riemann tensor are: $R^0_{101}, R^0_{202}, R^0_{303}, R^1_{212}, R^1_{313}, R^2_{323}$. Therefore,

$$K = 4\left(\frac{4M^2}{6} + \frac{M^2}{6} + \frac{M^2}{6} + \frac{M^2}{6} + \frac{M^2}{6} + \frac{4M^2}{6}\right)$$

$$K = \frac{48M^2}{r^6}. \tag{2.3.51}$$

Again, nothing strange occurs in $r = 2M$. Note that in $r = 0$ there is a divergence; $r = 0$ is known as a *essential singularity.*

With these results, we can find a coordinate transformation such that the metric acquires the following form

$$ds^2 = -f^2(u,v)\left(dv^2 - du^2\right) + r^2\left(d\theta^2 + sin^2\theta d\varphi^2\right). \tag{2.3.52}$$

We use that choose in order the for radially emitted light rays we have

$$\frac{du}{dv} = \pm 1.$$

Additionally, the two dimensional sub-manifold, when $\theta, \varphi \equiv cte$, is conformally equivalent to the Minkowski metric $dv^2 - du^2$. The coordinates that satisfy these conditions are the Kruskal coordinates defined by [8]

$$u = \sqrt{\frac{r}{2M} - 1}e^{r/4M}\cosh\left(\frac{t}{4M}\right), \tag{2.3.53}$$

$$v = \sqrt{\frac{r}{2M} - 1}e^{r/4M}\sinh\left(\frac{t}{4M}\right). \tag{2.3.54}$$

With

$$f^2 = \frac{32M^3}{r}e^{-r/2M}. \tag{2.3.55}$$

Applying the usual hyperbolic identities we have that

$$u^2 - v^2 = \left(\frac{r}{2M} - 1\right)e^{r/2M}, \tag{2.3.56}$$

$$\frac{v}{u} = \tanh\left(\frac{t}{4M}\right). \tag{2.3.57}$$

Then, similar to Rindler, we have that the lines corresponding to $r \equiv const$ are hyperbolas; in the limit $r \to 2M$ these hyperbolas approach to lines with an inclination of 45^o. Furthermore, observing (2.3.57) we have that the lines corresponding to $t \equiv const$ are radial lines that pass through the origin.

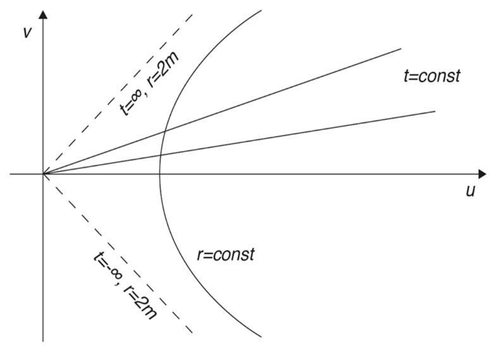

Figure 2.3: Constant time and radius lines in the Kruskal plane.

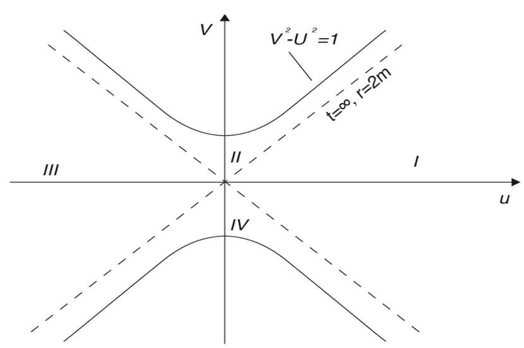

Figure 2.4: Regions in the Kruskal plane.

In the expression (2.3.55) we conclude that the metric becomes singular when $r \to 0$; from (2.3.56) this situation is given in regions where $v^2 - u^2 = 1$. If we perform the change $(u, v) \to (-u, -v)$ we can conclude that the regions I and III are isometric (See Figure 2.4).

If we redefine the transformation as follows

$$u = \sqrt{1 - \frac{r}{2M}} e^{r/4M} \sinh\left(\frac{t}{4M}\right), \tag{2.3.58}$$

$$v = \sqrt{1 \frac{r}{2M}} e^{r/4M} \cosh\left(\frac{t}{4M}\right), \tag{2.3.59}$$

we have that the image of $0 < r < 2M$ under this transformation is the region II,

$$v^2 - u^2 = \left(1 - \frac{r}{2M}\right) e^{r/2M}, \tag{2.3.60}$$

$$\frac{u}{v} = \tanh\left(\frac{t}{4M}\right). \tag{2.3.61}$$

Then the regions II and IV are isometric.

The causal structure of the metric described by the Kruskal coordinates is described by the light rays that forms the lines with 45^o of inclination, analogous to the case in Minkowski manifold. The observers in the regions I and III can receive signals that come from the region IV and after send to the region II. Any particle that get in the region II, inevitably will go to the singularity in $r = 0$, with a finite proper time; the causal structure indicates that there is not another possible trajectory. Any particle that is perceived in the region IV must come from the singularity in a previus proper time. There is not causal connection between the regions I and III.

The future singularity is perceived (by distant observers in the region I and III) cover by a surface called *event horizons*. By definition this is the boundary of the region that is casually disconnected with distant observers in the regions I and III and in this case is given by $r = 2M$. As we shall see later, this topological characteristic is fundamental in the appearing of thermal radiation. Situations or physical objects that present this regions separation will show similarities with the effects that are perceived on the event horizons of the Schwarzschild problem.

Chapter 3

Field Quantization

In this chapter we present the formalism of the canonical quantization in quantum field theory. From the results of the primitive quantum mechanics, we have that the electromagnetic field can be interpreted as a physical system composed by many particles, called photons. On the other hand, the Schrödinger equation represents an individual particle. As we shall see the concept of field is more fundamental that the particle and in this frame, the wave function that is solution of the Schrödinger equation, will be an operator that represents a particles field instead representing an individual particle. The Schrödinger equation wave equation turns into a classic field equation that must be quantized by the commutation canonical rules of quantum mechanics. Let us review some conceptual aspects in order to clarify these ideas.

The concept of field comes from the classical physics in the sense that we needed stablish laws of Nature that were local. The laws of Coulomb and Newton that involved the so-called action at a distance became unsatisfactory from the experimental point of view. The field theories of Maxwell and Einstein remedy the situation with all interaction mediated in a local fashion by the field.

Another experiment, which results would accompany relativity in the breakdown and change of classical paradigm, was made by Max Karl Ernst Ludwig Planck. This experiment studied the black-body radiation. This situation was in front of a problem known as the 'ultraviolet catastrophe'. There were a lot of experiments till Planck could present the final version of his research in December 14th of 1900. This final version, was based on Boltzmann's statistical interpretation of the second law of thermodynamics.

His final work, fundamentally postulated that electromagnetic energy could be emitted only in quantized form. The mathematical expression where the Planck's h constant appeared, was proved a lot o times and its exactitude was incomparable, however, this result gave us a more transcendent result: The quantization of the action. At the same time, that sentence was the first step towards the disappearing of the barrier between particles and waves.

Another experiment that showed the evidence of corpuscular character of light was the photoelectric effect, which consisted in emission of electrons by a material when electromagnetic radiation illuminated it. The theoretical explanation was done by Albert Einstein, who published in 1905 the revolutionary article "heuristic of the generation and conversion of the light", basing his formulation of the photo-electricity as an extension of the work of Max Planck's quanta.

From special theory of relativity, mass was equivalent with the energy, result that is expressed in the famous equation $E = mc^2$, on the other hand from quantization of action, the Planck's formula for the energy was $E = h\nu$ (being ν the frequency of the electromagnetic waves). De Broglie, proposed that if waves behaved like particles, particles would have to behave like waves. With a combination of two expressions of the energy De Broglie arrives to

$$\lambda = \frac{h}{mv} \tag{3.0.1}$$

This revolutionary expression, relates the wavelength with the mass, erasing the classical dichotomy between wave and particle. Then, What was the light?, electromagnetic waves or an agglomeration of small corpuscles?. Quantum mechanics says with security that light is both. Another problem of the classical physics was the stability of the atoms, for which Bohr proposed that similar to other properties of matter, electronic orbits were also quantized $L = n\hbar$.

As a *final* result of this quantization road, In January 1926, Erwin Schrödinger shows to the world an equation that would change history

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi. \tag{3.0.2}$$

The letter ψ is continuously related with the mind. Schrödinger said that the solution of this equations live in the mind and that the real part was the square of this function, that must be interpreted as a probability. When the scientists brought the character of probability the dynamics of the fundamental particles, they were giving us back the freedom snatched by determinism, since trough light of quantum mechanics, a physical state is a superposition of a lot of events, that evolves tending to the most probable one, however, in other circumstances another event with less probability can be the one that evolves.

In 1927, Heisenberg developed his uncertainty principle, based on a equivalent formulation of wave quantum mechanics of Schrödinger. The uncertainty principle reads

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \tag{3.0.3}$$

Now, from the quantization of the action and the principle of uncertainty, we obtain two results that were latent in the general theory and come to be complemented by the quantum theory. In first instance the notion of spatial dinamization implies the notion of novelty and transformation. When we are denying the juxtaposition or coexistence and, with this, the spatial character that was being saw in the time, we are denying the coexistence of present and future, denying any determinism. Such notion of novelty and transformation, in terms of the abolition of determinism, is strengthened by the principle of uncertainty.

Being the principle of uncertainty and the quantization inherent properties of nature, space-time would have to possess a pulse character. Why pulse character? Remembering that quantization carries to the most terrible absurd, for example, if we think in a chrono-geometric quantum, it would consist of edges and the edges would imply points, or instants, entities without length, entities that in the frame of relativity would not exist, but, can we talk about chrono-geometric pulse? As we will see, the quantum field theory would bring us fundamental ideas to advance in this way.

The pulse character sees strengthened by the uncertainty principle. Being The uncertainty principle something inherent to nature (not a question of observation as some people ensure) it denies the existence of a concrete value; it would be a wrong interpretation that such deltas are fluctuations, since when we are speaking of fluctuations, we affirm implicitly a concrete value which is the center of the fluctuations. If we affirm the uncertainty as the intrinsic reality of nature, the pulse character of space-time would remain established.

Precisely in the framework of these quantum concepts, the notion of particle also remains insufficient. In the relativistic frame we have that particles decay once and once more in other so many times, or in virtue of the mass-energy equivalence, we have creation and annihilation processes. In terms of observation the same particle cannot be seen twice, moreover in terms of the uncertainty principle we cannot observe it not even once. The word that would describe better these physical entities, would be the one of *event*, and each decay or process of annihilation or destruction would be contiguous events to the previous, taking into account that the matter-energy is an identity with the space-time. Furthermore, the notion of movement would be reformed to the notion of change (of the structure of the space-time).

As we will see, quantum field theory would re-affirm these concepts about the event-particles in a logical and plausible structure and would bring us more tools in the discussion about the identity between matter-energy and space-time and its implications with the particle-event concept and the vacuum concept to construct a new paradigm.

The first consolidated field theory was developed by James Clerk Maxwell in 1865, known since then as electromagnetism. With the Maxwell's equations, the theory introduced a *new form of matter*, the electromagnetic field [11]. The failure of the mechanical model applied to electromagnetic waves gave us relativity. On the other hand, Schrödinger's equation is non relativistic and its solutions would be considered as a field in the sense that it is a continuous dynamical system, a system with infinite degrees of freedom or a dynamical variable characterizing an aspect of a system.

As we just showed, the concepts developed by quantum mechanics and relativity not only were not contradictory but they are complementary. Dirac unificated special relativity and non relativistic quantum mechanics and gave origin to quantum field theory. At principle, the infinities surrounded all the results and there were not physical significance for these. The solution to this problem was the renormalization, proposed by Freeman Dyson, Richard Feynman, Julian Schwinger and Sinitiro Tomonaga.

In classical physics we have a lot of examples of fields: the string that is a one dimensional field $\psi(z,t)$ where z is a continuous parameter indicating position on the string and t the time; we have the displacement of the membrane $\psi(x_1,x_2,t)$, being x_1,x_2 parameters indicating positions on the drum and t the time, but those can be not only spatial aspects but color or temperature changing. The field variable can be a scalar, vector, tensor or spinor.

For example we have the electromagnetic field $F_{\mu\nu}(x)$, where x is the four dimensional parameter that refers to space-time $(ct,\mathbf{x})$. One can say that in general, a field $\varphi(\mathbf{x},t)$ is a dynamical variable for a continuous system whose points are indexed by the parameters t and $\mathbf{x}$. This, conceptually talking, generates a great advantage in front of non relativistic quantum mechanics, since it has a clear separation between space and time, being the space an operator and time a simple parameter.

With the relativistic quantum equation of motions like Klein-Gordon equation

$$(\Box + m^2)\psi = 0, \tag{3.0.4}$$

for spin 0 systems, and Dirac equation

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0, \tag{3.0.5}$$

for spin 1/2 systems, we have the problem of negative energy states. In order to solve the problem of a possible (minus) infinite decay of the energy, Dirac postulated that the negative energy states are completely filled by other particles, then we would have the prediction of a *hole* in the filled sea, such is known as an *antiparticle* and it was a great triumph and will found a new paradigm into making physics.

The Dirac postulate presented the difficulty that it required a many-particle picture in contradiction to the original single-particle interpretation, this problem disappears if the variables ψ of the relativistic equations are not interpreted as single particle wave functions but as dynamical variables for continuous systems, or simply, fields [11].

3.1 Classic Lagrangian and Hamiltonian Formulation

A classical physical system composed by a finite number of particles is characterized by a Lagrangian function $L \equiv L(q_i,\dot{q}_i,t)$ which is a function of the generalized coordinates of the system q_i, its temporal derivatives $\dot{q}_i = dq_i/dt$ and the time t. In order to obtain the classic trajectory of the system between the points $q_i(t_1)$ at the time t_1 unto the points $q_i(t_2)$ at time t_2 (for $i = 1,2,...$ covering all of freedom degrees), we have to define the classic action

$$S = \int_{t_1}^{t_2} dt L(q_i,\dot{q}_i,t), \tag{3.1.1}$$

and impose that this object must be stationary; in other words $\delta S = 0$ in a trajectory where $\delta q_i(t_1) = \delta q_i(t_2) = 0$, such that,

$$\begin{aligned}
\delta S &= \delta \int_{t_1}^{t_2} dt L(q_i, \dot{q}_i, t) \\
&= \sum_i \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) \\
&= \sum_i \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d\delta q_i}{dt} \right) \\
&= \sum_i \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right] \\
&= \sum_i \left\{ \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta q_i \bigg|_{t_1}^{t_2} \right\} \\
&= 0.
\end{aligned}$$

The last term in the above expression is null due to the boundary condition $\delta q_i(t_1) = \delta q_i(t_2) = 0$. Since the variation of δq_i is arbitrary, the above expression will be valid, for any δq_i, when

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0, \quad i = 1, 2, 3, ... \tag{3.1.2}$$

such equations are known as Euler-Lagrange equations. Those equations must describe the motion of the physical system, as we can see with the following general Lagrangian

$$L(q_i, \dot{q}_i, t) = \sum_i \left[\frac{1}{2} m \dot{q}_i^2 - V(q_i, t) \right], \tag{3.1.3}$$

where $V(q_i, t)$ is the potential energy of the system. If we replace this in the equation (3.1.2) we have

$$\frac{d}{dt}(m\dot{q}_i) = -\frac{\partial V(q_i, t)}{\partial q_i}, \quad i = 1, 2, 3, ... \tag{3.1.4}$$

that is the Newton's Second Law.

Now, we can define $p_i \equiv \partial L / (\partial \dot{q}_i)$ as the canonically conjugated variable to q_i, such that, we can define the Hamiltonian function as the following Legendre transform of $L(q_i, \dot{q}_i, t)$,

$$H(q_i, \dot{q}_i, t) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t) = \sum_i \left[\frac{p_i^2}{2m} + V(q_i, t) \right]. \tag{3.1.5}$$

Then, the equations of motion takes the form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2, 3, ... \tag{3.1.6}$$

3.2 Lagrangian and Hamiltonian Formulation of Fields

In this case we will have an analogous formulation this time for a classic field described by a real function $\varphi(\mathbf{r}, t)$ with an infinite degrees of freedom. Let us introduce the Lagrangian density $\mathcal{L} \equiv \mathcal{L}(\varphi, \nabla\varphi, \dot{\varphi}, t)$ that will be used to define the Lagrangian of the system in the following form

$$L = \int d^3r \mathcal{L}(\varphi, \nabla\varphi, \dot{\varphi}, t). \tag{3.2.1}$$

Where the $\nabla\varphi$ dependence is telling us the continuous variation of the field $\varphi(\mathbf{r},t)$ in the position variable in the vicinity of any point in the space. The classic action takes the form

$$S = \int_{t_1}^{t_2} dtL = \int_{t_1}^{t_2} dt \int d^3r\mathcal{L}(\varphi, \nabla\varphi, \dot{\varphi}, t) = \int d^4r\mathcal{L}(\varphi, \nabla\varphi, \dot{\varphi}, t). \tag{3.2.2}$$

Imposing that the action must be stationary in the time interval between t_1 and t_2, as above, and the boundary condition $\delta\varphi(t_1) = \delta\varphi(t_2) = 0$, we have

$$\begin{aligned}
\delta S &= \delta \int d^4r\mathcal{L} \\
&= \int d^4r \left(\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\delta\dot{\varphi} + \frac{\partial\mathcal{L}}{\partial\nabla\varphi}\delta\nabla\varphi \right) \\
&= \int d^4r \left(\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\frac{d}{dt}\delta\varphi + \frac{\partial\mathcal{L}}{\partial\nabla\varphi}\nabla\delta\varphi \right) \\
&= \int d^4r \left[\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\delta\varphi \right) + \nabla\left(\frac{\partial\mathcal{L}}{\partial\nabla\varphi}\delta\varphi \right) - \left(\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} \right)\delta\varphi - \left(\nabla\frac{\partial\mathcal{L}}{\partial\nabla\varphi} \right)\delta\varphi \right] \\
&= \int d^4r \left[\frac{\partial\mathcal{L}}{\partial\varphi} - \left(\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} \right) - \left(\nabla\frac{\partial\mathcal{L}}{\partial\nabla\varphi} \right) \right]\delta\varphi,
\end{aligned}$$

where the other terms are zero because of

$$\int d^4r\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} \right)\delta\varphi = \int d^3r\ \frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\delta\varphi\Big|_{t_1}^{t_2} = 0,$$

due to the boundary condition $\delta\varphi(t_1) = \delta\varphi(t_2) = 0$ and, using the Gauss theorem,

$$\int d^4r\nabla\left(\frac{\partial\mathcal{L}}{\partial\nabla\varphi}\delta\varphi \right) = \int dt \oint d\mathbf{S}\cdot\left(\frac{\partial\mathcal{L}}{\partial\nabla\varphi}\delta\varphi \right),$$

we have the closed integral over a sphere surface with infinite radius, where the fields must be zero, therefore

$$\lim_{\mathbf{r}\to\infty}\delta\varphi(\mathbf{r}) = 0. \tag{3.2.3}$$

Since the action is stationary for any $\delta\varphi$, the Euler-Lagrange equations, take the form

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} - \nabla\frac{\partial\mathcal{L}}{\partial\nabla\varphi} = 0. \tag{3.2.4}$$

If the field is a complex field, both ψ and ψ^* are independent degrees, such that

$$\frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\psi}} - \nabla\frac{\partial\mathcal{L}}{\partial\nabla\psi} = 0, \tag{3.2.5}$$

$$\frac{\partial\mathcal{L}}{\partial\psi^*} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\psi}^*} - \nabla\frac{\partial\mathcal{L}}{\partial\nabla\psi^*} = 0. \tag{3.2.6}$$

Now, in order to find the Hamiltonian density, we can express the Lagrangian density as follows

$$\mathcal{L} = i\psi^*\psi - \frac{1}{2m}\nabla\psi^*\nabla\psi - V(\mathbf{r},t)\psi^*\psi. \tag{3.2.7}$$

Therefore, the conjugate variable of $\psi(\mathbf{r},t)$ is

$$\pi(\mathbf{r},t) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}^*} = i\psi^*(\mathbf{r},t). \tag{3.2.8}$$

Then, the Hamiltonian density will be

$$\begin{aligned}
\mathcal{H} &= \pi\dot{\psi} - \mathcal{L} \\
&= -\frac{i}{2m}\nabla\pi\nabla\psi - iV(\mathbf{r},t)\pi\psi \\
&= \frac{1}{2m}\nabla\psi^*\nabla\psi + V\psi^*\psi.
\end{aligned} \tag{3.2.9}$$

We can take the following derivatives

$$\frac{\partial \mathcal{L}}{\partial \psi^*} = i\dot{\psi} - V\psi, \quad \frac{\partial \mathcal{L}}{\partial \dot{\psi}^*} = 0, \quad \frac{\partial \mathcal{L}}{\partial \nabla\psi^*} = \frac{1}{2m}\nabla\psi, \tag{3.2.10}$$

and replace in the expression (3.2.6) to show that the Lagrangian density (3.2.7) is describing the Schrödinger wave equation and taking the derivatives

$$\frac{\partial \mathcal{L}}{\partial \psi} = -V\psi^*, \quad \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = 0, \quad \frac{\partial \mathcal{L}}{\partial \nabla\psi} = \frac{1}{2m}\nabla\psi^*, \tag{3.2.11}$$

and replacing in (3.2.5) we obtain the complex conjugate of the Schrödinger wave equation. We can observe that the Lagrangian density (3.2.7) is showing an asymmetry between the equations (3.2.10) and (3.2.11). It can be found another Lagrangian density where this asymmetry would disappear but it will not be simpler than (3.2.7).

Now, we can define the Hamiltonian from the Hamiltonian density (3.2.9) as

$$\begin{aligned}
H &= \int d^3r\mathcal{H} \\
&= \int d^3r \left(\frac{1}{2m}\nabla\psi^*\nabla\psi + V\psi^*\psi\right) \\
&= \int d^3r\psi^* \left(-\frac{1}{2m}\nabla^2 + V\right)\psi,
\end{aligned} \tag{3.2.12}$$

where the last steep is obtained by integration by parts and despise the following term

$$\int d^3r\nabla(\psi^*\nabla\psi) = \oint d\mathbf{s}(\psi^*\nabla\psi). \tag{3.2.13}$$

The equations of motion in the Hamiltonian form yields

$$\dot{\psi}(\mathbf{r},t) = \frac{\partial \mathcal{H}(\psi,\dot{\psi},\nabla\psi,t)}{\partial \pi(\mathbf{r},t)}, \tag{3.2.14}$$

$$\dot{\pi}(\mathbf{r},t) = -\frac{\partial \mathcal{H}(\psi,\dot{\psi},\nabla\psi,t)}{\partial \psi(\mathbf{r},t)}. \tag{3.2.15}$$

3.3 Second Quantization: Why Fields?

The fundamental hypothesis in this process is suppose that $\psi(\mathbf{r},t)$ represents a field operator instead a wave function and beside its conjugated operator $\pi = i\psi^\dagger$, satisfy the commutation rules imposed by the fundamental quantum postulates to the conjugated variables, in other words, these new operators satisfy

$$[\psi(\mathbf{r}_1,t),\pi(\mathbf{r}_2,t)] = i\delta^3(\mathbf{r}_1-\mathbf{r}_2), \tag{3.3.1}$$

$$[\psi(\mathbf{r}_1,t),\psi(\mathbf{r}_2,t)] = 0, \tag{3.3.2}$$

$$[\pi(\mathbf{r}_1,t),\pi(\mathbf{r}_2,t)] = 0. \tag{3.3.3}$$

Is clear that for a Hilbert space from a physical situation described by the Schrödinger equation, there is a complete set of orthonormal functions $\varphi_n(\mathbf{r},t)$, $n=0,1,2,...$ ($\int d\zeta\varphi_n^*\varphi_l = \delta_{nl}$) which are eigenfunctions of an observable associated with a given physical magnitude (being $n=0$ the fundamental state of the system). This fact allow us expand the wave function $\psi(\mathbf{r},t)$ as

$$\psi(\mathbf{r},t) = \sum_n a_n\varphi_n(\mathbf{r},t), \tag{3.3.4}$$

where the coefficients of the expansion are given by $a_n = \int d\zeta\varphi_n^*\psi$. When the wave functions solutions to the Schrödinger equation becomes field operators, the above expansion turns to a sum over a complete and infinite operators set $\{a_n, a_n^\dagger, n=0,1,2,...\}$, that is,

$$\psi(\mathbf{r},t) = \sum_n a_n\varphi_n(\mathbf{r},t), \quad \psi^\dagger(\mathbf{r},t) = \sum_n a_n^\dagger\varphi_n^*(\mathbf{r},t), \tag{3.3.5}$$

where the coefficients are right now the functions $\varphi_n(\mathbf{r},t)$ and the complete operators set is defined by $a_n = \int d\zeta\varphi_n\psi$ and its adjoin hermitian.

The algebra that the new operators $a_n, a_n^\dagger$ must satisfy, have to be compatible with the algebra expressed in (3.3.1), (3.3.2) and (3.3.3), that is,

$$\left[\psi(\mathbf{r}_1,t),\psi^\dagger(\mathbf{r}_2,t)\right] = \sum_n\sum_l \varphi_n(\mathbf{r}_1,t)\varphi_l^*(\mathbf{r}_2,t)[a_n,a_l^\dagger] = \delta^3(\mathbf{r}_1-\mathbf{r}_2). \tag{3.3.6}$$

Here, we can use the completeness of the eigenfunctions (sufficient and necessary condition to obtain a complete set of functions in the Hilbert space),

$$\sum_n \varphi_n(\mathbf{r}_1,t)\varphi_n^*(\mathbf{r}_2,t) = \delta^3(\mathbf{r}_1-\mathbf{r}_2), \tag{3.3.7}$$

such that, this condition will be satisfied if

$$[a_n,a_l^\dagger] = \delta_{n,l}, \quad n,l = 0,1,2,... \tag{3.3.8}$$

Using the other two commutations relations, we obtain

$$[a_n,a_l] = [a_n^\dagger,a_l^\dagger] = 0, \quad n,l = 0,1,2,... \tag{3.3.9}$$

With this in mind, from the fundamental operators ψ and $\psi^\dagger$ we can construct other operators with physical relevance. For example, taking the expression (3.2.12), its operator form will be

$$\begin{aligned} H &= \int d^3r\psi^\dagger \left(-\frac{1}{2m}\nabla^2 + V\right)\psi \\ &= -i\int d^3r\pi H^{(0)}\psi \\ &= \sum_l\sum_n a_l^\dagger a_n \int d^3r\varphi_l^*(\mathbf{r},t)H^{(0)}\varphi_n(\mathbf{r},t) \\ &= \sum_l\sum_n a_l^\dagger a_n (H^{(0)})_{ln}, \end{aligned} \tag{3.3.10}$$

where

$$H^{(0)} = -\frac{1}{2m}\nabla^2 + V, \tag{3.3.11}$$

is the Hamiltonian related to the Schrödinger equation, that describe an individual particle of the field and $(H^{(0)})_{ln}$ is the matrix element l, n of such Hamiltonian.

The above expression, let us think in the following operator

$$N = \int d^3r\psi^\dagger\psi = \sum_l\sum_n a_l^\dagger a_n \int d^3r\varphi_l^*\varphi_n = \sum_l a_l^\dagger a_l = \sum_l n_l, \tag{3.3.12}$$

that is showing us the total number of particles. Beside this, we define the occupation number operator $n_l = a_l^\dagger a_l$ that will be an important object in the theory.

The algebra of these operators comes defined by the algebra of the field operators. An interesting operator is $[N, H]$, that is

$$\begin{aligned} [N, H] &= \int d^3r_1 d^3r_2 [\psi^\dagger(\mathbf{r}_1,t)\psi(\mathbf{r}_1,t), \psi^\dagger(\mathbf{r}_2,t)H^{(0)}_{r_2}\psi(\mathbf{r}_2,t)] \\ &= \int d^3r_1 d^3r_2 \psi^\dagger(\mathbf{r}_1,t)[\psi(\mathbf{r}_1,t), \psi^\dagger(\mathbf{r}_2,t)]H^{(0)}_{r_2}\psi(\mathbf{r}_2,t) \\ &\quad + \int d^3r_1 d^3r_2 [\psi^\dagger(\mathbf{r}_1,t), \psi^\dagger(\mathbf{r}_2,t)]\psi(\mathbf{r}_1,t)H^{(0)}_{r_2}\psi(\mathbf{r}_2,t) \\ &\quad + \int d^3r_1 d^3r_2 \psi^\dagger(\mathbf{r}_2,t)H^{(0)}_{r_2}[\psi^\dagger(\mathbf{r}_1,t), \psi(\mathbf{r}_2,t)]\psi(\mathbf{r}_1,t) \\ &\quad + \int d^3r_1 d^3r_2 \psi^\dagger(\mathbf{r}_1,t)\psi^\dagger(\mathbf{r}_2,t)H^{(0)}_{r_2}[\psi(\mathbf{r}_1,t), \psi(\mathbf{r}_2,t)] \\ &= \int d^3r_1 d^3r_2 \psi^\dagger(\mathbf{r}_1,t)\delta^3(\mathbf{r}_1 - \mathbf{r}_2)H^{(0)}_{r_2}\psi(\mathbf{r}_2,t) \\ &\quad - \int d^3r_1 d^3r_2 \psi^\dagger(\mathbf{r}_2,t)H^{(0)}_{r_2}\delta^3(\mathbf{r}_1 - \mathbf{r}_2)\psi(\mathbf{r}_1,t) \\ &= 0. \end{aligned} \tag{3.3.13}$$

In a similar form we have

$$\begin{aligned} [n_l, n_k] &= [a_l^\dagger a_l, a_k^\dagger a_k] \\ &= a_l^\dagger[a_l, a_k^\dagger a_k] + [a_l^\dagger, a_k^\dagger a_k]a_l \\ &= a_l^\dagger[a_l, a_k^\dagger]a_k + a_l^\dagger a_k^\dagger[a_l, a_k] + a_k^\dagger[a_l^\dagger, a_k]a_l + [a_l^\dagger, a_k^\dagger]a_k a_l \\ &= a_l^\dagger a_k\delta_{lk} - a_k^\dagger a_l\delta_{kl} \\ &= (a_l^\dagger a_k + a_k^\dagger a_l)\delta_{kl} \\ &= 0. \end{aligned} \tag{3.3.14}$$

Using the above result

$$[N, n_l] = [\sum_k n_k, n_l] = \sum_k [n_k, n_l] = 0. \tag{3.3.15}$$

3.3.1 Equations of motion

Within the Hamiltonian operator (3.3.10) and the algebra of the field operators, we can obtain the equations of motion of the physical fields, at the same time, we can define certain conservation laws. In particular we may be interested in the commutators $[\psi, H]$, $[\psi^\dagger, H]$, $[N, H]$. We have,

$$\begin{aligned}
[\psi(\mathbf{r},t), H] &= \left[\psi(\mathbf{r},t), -i\int d^3r_1 \pi(\mathbf{r}_1,t) H^{(0)}_{r_1} \psi(\mathbf{r}_1,t)\right] \\
&= -i\int d^3r_1 \left[\psi(\mathbf{r},t), \pi(\mathbf{r}_1,t) H^{(0)}_{r_1} \psi(\mathbf{r}_1,t)\right] \\
&= -i\int d^3r_1 \left[\psi(\mathbf{r},t), \pi(\mathbf{r}_1,t)\right] H^{(0)}_{r_1} \psi(\mathbf{r}_1,t) \\
&\quad - i\int d^3r_1 \pi(\mathbf{r}_1,t) H^{(0)}_{r_1} \left[\psi(\mathbf{r},t), \psi(\mathbf{r}_1,t)\right] \\
&= -i\int d^3r_1 i\delta^3(\mathbf{r}-\mathbf{r}_1) H^{(0)}_{r_1} \psi(\mathbf{r}_1,t) \\
&= H^{(0)}_r \psi(\mathbf{r},t) \\
&= \left(-\frac{1}{2m}\nabla^2_r + V(\mathbf{r},t)\right)\psi(\mathbf{r},t) \\
&= i\dot{\psi}(\mathbf{r},t) \\
&= i\frac{d\psi(\mathbf{r},t)}{dt}.
\end{aligned}$$

Therefore,

$$\dot{\psi}(\mathbf{r},t) = -i[\psi(\mathbf{r},t), H], \tag{3.3.16}$$

that is nothing that the time evolution of an operator in the Heisenberg picture. In other words, the Heisenberg equation for the time evolution of an arbitrary operator O,

$$\dot{O} = -i[O, H], \tag{3.3.17}$$

is bringing us the Schrödinger equation when it is applied to a field operator ψ. It is a clear proof of the consistency of the Quantum Field Theory.

In order to confirm this consistency, we can obtain the time evolution of the operator $\pi(\mathbf{r},t)$ as follows

$$i\frac{\partial\psi^\dagger}{\partial t} = \dot{\pi}(\mathbf{r},t) = -i[\pi(\mathbf{r},t), H] = [\psi^\dagger, H]$$

$$
\begin{aligned}
[\psi^\dagger, H] &= -i\left[\pi(\mathbf{r},t), -i\int d^3r_1 \pi(\mathbf{r}_1,t) H^{(0)}_{r_1}\psi(\mathbf{r}_1,t)\right] \\
&= -\int d^3r_1 \left[\pi(\mathbf{r},t), \pi(\mathbf{r}_1,t) H^{(0)}_{r_1}\psi(\mathbf{r}_1,t)\right] \\
&= -\int d^3r_1 \left[\pi(\mathbf{r},t), \pi(\mathbf{r}_1,t)\right] H^{(0)}_{r_1}\psi(\mathbf{r}_1,t) \\
&\quad -\int d^3r_1 \pi(\mathbf{r}_1,t) H^{(0)}_{r_1}\left[\pi(\mathbf{r},t), \psi(\mathbf{r}_1,t)\right] \\
&= -\int d^3r_1 \psi^\dagger(\mathbf{r}_1,t) H^{(0)}_{r_1}\left[\pi(\mathbf{r},t), \psi(\mathbf{r}_1,t)\right] \\
&= -\int d^3r_1 \left(H^{(0)}_{r_1}\psi^\dagger(\mathbf{r}_1,t)\right)^\dagger (-i)\delta^3(\mathbf{r}-\mathbf{r}_1) \\
&= -\left(H^{(0)}_{r}\psi^\dagger(\mathbf{r},t)\right)^\dagger \\
&= -\left(-\frac{1}{2m}\nabla^2_r + V(\mathbf{r},t)\right)\psi^\dagger(\mathbf{r},t).
\end{aligned}
\tag{3.3.18}
$$

The above expression is obtained from the complex conjugate of the Schrödinger equation before make the second quantization and impose the equality between the field operators.

Taking the before results, we can calculate the quantity $[N, H]$, we have

$$
\dot{N} = -i[N,H] = -i\int d^3r [\psi^\dagger\psi, H]
$$

$$
\begin{aligned}
\dot{N} &= -i\int d^3r \left(\psi^\dagger[\psi,H] + [\psi^\dagger,H]\psi\right) \\
&= -i\int d^3r \left(\psi^\dagger H^{(0)}\psi - (H^{(0)}\psi)^\dagger\psi\right) \\
&= -i\int d^3r \left(\psi^\dagger H^{(0)}\psi - \psi^\dagger H^{(0)}\psi\right) \\
&= 0.
\end{aligned}
\tag{3.3.19}
$$

Then, the associated physical magnitude to the operator N is a constant of motion.

3.3.2 Particle number representation

We just have shown that, with the algebra postulated for the field operators,

$$
[N,H] = [n_l, n_k] = [N, n_l] = 0, \quad l,k = 0,1,2,... \tag{3.3.20}
$$

Now, although the expansion (3.3.5) is thought to be the complete set of eigenfunctions of any operator, in the practice, the formalism of the second quantization will reach its maximum power when that operator is the Hamiltonian of the individual particle of the system. That is to say, when we have a complete set of eigenfunctions that satisfy

$$
H^{(0)}\varphi^E_n(\mathbf{r},t) = E^{(0)}_n\varphi^E_n(\mathbf{r},t), \tag{3.3.21}
$$

being $H^{(0)} = -\nabla^2/2m + V(\mathbf{r},t)$ the Hamiltonian of an individual particle of the field.

When we use the eigenfunctions of $H^{(0)}$ as the complete set for the expansions, we have the field $\psi(\mathbf{r},t)$ in the so-called energy representation. This representation bring us the matrix elements $(H^{(0)})_{ln}$ being diagonals in the form $E_n^0\delta_{nl}$ and the Hamiltonian operator takes the form

$$H = \sum_l a_l^\dagger a_l E_l^{(0)} = \sum_l n_l E_l^{(0)}, \tag{3.3.22}$$

where n_l is the operator defined in (3.3.12). Within this representation is easy to show that $[H, n_l] = 0$ for $l = 0, 1, 2, ...$ Then,

$$[N, H] = [n_j, H] = [N, n_j] = [n_l, n_j] = 0, \quad l, j = 0, 1, 2, ... \tag{3.3.23}$$

This imply that there exist a complete set of eigenvectors that can diagonalize simultaneously the operators

$$\{H, N, n_j, \quad j = 0, 1, 2, ...\}, \tag{3.3.24}$$

that have a simply and important physical significance. This complete set of eigenvectors are wide known as the particle number representation.

In order to find the set of eigenvalues and eigenvectors of the operators $n_l = a_l^\dagger a_l$, let us firs stablish the algebra between the operators n_k, $a_l^\dagger$, a_l,

$$[n_k, a_l] = [a_k^\dagger a_k, a_l] = a_k^\dagger[a_k, a_l] + [a_k^\dagger, a_l]a_k = [a_k^\dagger, a_l]a_k = -a_k\delta_{kl}, \tag{3.3.25}$$

$$[n_k, a_l^\dagger] = [a_k^\dagger a_k, a_l^\dagger] = a_k^\dagger[a_k, a_l^\dagger] + [a_k^\dagger, a_l^\dagger]a_k = a_k^\dagger[a_k, a_l^\dagger] = a_k^\dagger\delta_{kl}. \tag{3.3.26}$$

Now we can define $|n_l\rangle$ the eigenvector of n_l with eigenvalue n_l. Then, we can say that $|0_l\rangle$ is an eigenvalue with zero particles with quantum number l; which corresponds to the vacuum of the state l, which has a dual $(|0_l\rangle)^\dagger = \langle 0_l|$. A collection of these states will define a special space which describes systematically the state vectors that describe one, two, three and more particles in well-defined spacetime positions. This space is known as the Fock Space.

3.4 Fock Space

Let us consider a collection of particles in quantum mechanics, we would have $\mathcal{H}_N$ a Hilbert space of a system of N identical particles. The union of all $\mathcal{H}_N$ is called the Fock space

$$\bigcup_{N=0}^{\infty} \mathcal{H}_N. \tag{3.4.1}$$

The subspace $N = 0$ contains the vacuum state as its only member. We have to introduce operators on Fock space that connect subspaces of different N. An elementary operator of this kind creates or annihilates one particle at a point space; such an operator is a quantum field operator, since it is a spatial function. This is why a quantum-mechanical many-particle system automatically rises to a quantum field [38].

If we stay in the Schrödinger picture we can define an operator $\psi(\mathbf{r})$ that annihilates one particle at $\mathbf{r}$, its hermitian conjugate $\psi^\dagger(\mathbf{r})$ will create one particle at $\mathbf{r}$. We impose the commutation relations

$$\begin{gathered}[\psi(\mathbf{r}_1, t), \psi^\dagger(\mathbf{r}_2, t)]_\pm = \delta^3(\mathbf{r}_1 - \mathbf{r}_2), \\ [\psi(\mathbf{r}_1, t), \psi(\mathbf{r}_2, t)]_\pm = 0,\end{gathered} \tag{3.4.2}$$

where $[A, B]_\pm = AB \pm BA$ and the plus sign corresponds to bosons and the minus sign corresponds to fermions.

Now, a general N-particle Hamiltonian has the structure

$$H = \sum_i f(\mathbf{r}_i, t) + \sum_{i<j} g(\mathbf{r}_i, \mathbf{r}_j, t) + \sum_{i<j<k} h(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k, t) + \cdots. \tag{3.4.3}$$

We can construct that Hamiltonian on Fock space with the following procedure

$$\begin{aligned}\sum_i f(\mathbf{r}_i, t) &\to \int d^3r \psi^\dagger(\mathbf{r}, t) f(\mathbf{r}, t) \psi(\mathbf{r}, t) \\ \sum_{i<j} g(\mathbf{r}_i, \mathbf{r}_j, t) &\to \frac{1}{2} \int d^3r_1 d^3r_2 \psi^\dagger(\mathbf{r}_1, t) \psi^\dagger(\mathbf{r}_2, t) g_{12} \psi(\mathbf{r}_2, t) \psi(\mathbf{r}_1, t) \\ &\vdots\end{aligned} \tag{3.4.4}$$

where $g_{12} = g(\mathbf{r}_1, \mathbf{r}_2, t)$. The significance is that the action of $\psi(\mathbf{r}, t)$ on an eigenstate of N is to decrease its eigenvalue by 1, while the action of $\psi^\dagger(\mathbf{r}, t)$ is to increase it by 1. The vacuum state $|0\rangle$ is defined as the eigenstate of N with eigenvalue zero. It is annihilated by all annihilation operators

$$\psi(\mathbf{r}, t) |0\rangle = 0. \tag{3.4.5}$$

Now, we suppose that we have states $|E, n\rangle$ such that

$$\begin{aligned}H |E, n\rangle &= E |E, n\rangle \\ N |E, n\rangle &= n |E, n\rangle,\end{aligned} \tag{3.4.6}$$

then we define the n-particle wave function corresponding to $|E, n\rangle$ as (taking out the temporal dependence without losing generality)

$$\Psi_E(\mathbf{r}_1, ..., \mathbf{r}_n) \equiv \frac{1}{\sqrt{n!}} \langle 0|\psi(\mathbf{r}_1) \cdots \psi(\mathbf{r}_n)|E, n\rangle. \tag{3.4.7}$$

This is showing us that the probability amplitude of finding n particles at the positions $\mathbf{r}_1, ..., \mathbf{r}_n$ can be found by annihilating the particles at the respective positions from the state $|E, n\rangle$, and evaluating the overlap between the resulting state and the vacuum function. It is easy to show that this wave function satisfies the n-particle Schrödinger equation, this is how the quantum field is equivalent to the many-particle system.

It is important to recall that many quantum fields have no classical counterparts, because as we see before, many interpretations of modern theories are interfered by classical concepts. The notion of quantum field or continuous systems with quantum characteristics begins to be clear of certain classical analogues.

Now, if a field operator $\psi(\mathbf{r})$ annihilates a particle at $\mathbf{r}$, it is annihilating a particle which wave function is a δ function. Then it can be written as a linear superposition of a complete set of wave functions, such that

$$\begin{aligned}\psi(\mathbf{r}) &= \sum_k u_k(\mathbf{r}) a_k \\ \psi^\dagger(\mathbf{r}) &= \sum_k u_k^*(\mathbf{r}) a_k^\dagger,\end{aligned} \tag{3.4.8}$$

where

$$\begin{aligned}\int d^3r u_k^*(\mathbf{r}) u_{k'}(\mathbf{r}) &= \delta_{kk'} \\ \sum_k u_k(\mathbf{r}) u_k^*(\mathbf{r}') &= \delta^3(\mathbf{r} - \mathbf{r}').\end{aligned} \tag{3.4.9}$$

The coefficient a_k and $a_k^\dagger$ satisfy

$$[a_k, a_{k'}^\dagger]_\pm = \delta_{kk'}$$

$$[a_k, a_{k'}]_\pm = 0. \tag{3.4.10}$$

The actions of a and $a^\dagger$ are

$$a\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle$$

$$a^\dagger\left|n\right\rangle = \sqrt{1\pm n}\left|n+1\right\rangle. \tag{3.4.11}$$

It is showing that a annihilates a particle in the state with wave function $u(\mathbf{r})$ and $a^\dagger$ creates such a particle.

3.5 Scalar Field Quantization

We can obtain the Schrödinger equation from a classical Hamiltonian of a physical system where we make the following assignments

$$H \to i\frac{\partial}{\partial t}, \quad \mathbf{p} \to -i\nabla, \quad \mathbf{r} \to \mathbf{r}, \tag{3.5.1}$$

thus the equation for the classical Hamiltonian

$$H = \frac{p^2}{2m} + V, \tag{3.5.2}$$

can be written as the following operator relation

$$i\frac{\partial}{\partial t} = -\frac{1}{2m}\nabla^2 + V, \tag{3.5.3}$$

which operating on a wave function ψ give us immediately the Schrödinger equation.

For a free particle, we have $V = 0$, then the Schrödinger equation yields

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi, \tag{3.5.4}$$

which has the time in a first derivative and the space as a second derivative, this evidently cannot be a Lorentz invariant, where under these structure, the space and time constitute an only concept.

The relativistic expression for the energy of a free particle with mass m and linear momentum $\mathbf{p}$ is

$$E = \sqrt{p^2 + m^2}, \tag{3.5.5}$$

and making the substitutions (3.5.1), we obtain a Hamiltonian operator of the form

$$H = \sqrt{-\nabla^2 + m^2}, \tag{3.5.6}$$

that has the inconvenient that do not have a unique definition. If we try a MacLaurin series expansion, we would have a infinite number of terms with partial derivatives in all even order and then we would have a non local theory; this without the analysis of the convergence problem.

3.5.1 Klein-Gordon Field Quantization

One form to surpass the problem is obtain the square

$$H^2 \to p^2 + m^2 = E^2. \tag{3.5.7}$$

and using (3.5.1) we obtain the following differential equation

$$-\frac{\partial^2 \varphi}{\partial t^2} = -\nabla^2 \varphi + m^2 \varphi, \tag{3.5.8}$$

which we can reexpress

$$\frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi + +m^2 \varphi = 0. \tag{3.5.9}$$

The above expression is known as the Klein-Gordon equation. We can define the d'alembertian operator $\Box$ as

$$\Box \equiv \frac{\partial^2}{\partial t^2} - \nabla^2, \tag{3.5.10}$$

such that the equation (3.5.9) takes the form

$$(\Box + m^2)\varphi(\mathbf{r}, t) = 0. \tag{3.5.11}$$

Is easy to show that the associated Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial t}\right)^2 - (\nabla \varphi)^2 - m^2 \varphi^2\right] = \frac{1}{2}\left[\partial^\mu \varphi \partial_\mu \varphi - -m^2 \varphi^2\right], \tag{3.5.12}$$

and the Hamiltonian density yields

$$\mathcal{H} = \frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial t}\right)^2 + (\nabla \varphi)^2 + m^2 \varphi^2\right]. \tag{3.5.13}$$

We can see that the Lagrangian density is Lorentz-invariant, while the Hamiltonian density is describing an energy density that cannot be invariant. For this reason, relativistic theories are usually specified via the Lagrangian density.

Real Scalar Field

In order to quantize the fields solutions of the Klein-Gordon equation, we first consider a Lorentz invariant real scalar field enclosed in a large periodic box of volume Ω and expanded in a Fourier series

$$\varphi(\mathbf{r}, t) = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k}} q_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{3.5.14}$$

where

$$q^*_{\mathbf{k}}(t) = q_{-\mathbf{k}}(t) \tag{3.5.15}$$

and assuming that this field satisfies the Klein-Gordon equation, we have

$$\ddot{q}_{\mathbf{k}} + \omega_k^2 q_{\mathbf{k}} = 0, \tag{3.5.16}$$

where

$$\omega_k^2 = \mathbf{k}^2 + m^2. \tag{3.5.17}$$

We have that this system is equivalent to have a collection of harmonic oscillators, and we can quantize it imposing the commutation relations

$$\begin{aligned} i[\dot{q}^{\dagger}_{\mathbf{k}}(t), q_{\mathbf{k}'}(t)] &= \delta_{\mathbf{k}\mathbf{k}'} \\ [q_{\mathbf{k}}(t), q_{\mathbf{k}'}(t)] &= 0 \end{aligned} \tag{3.5.18}$$

that imply

$$\begin{aligned} i[\dot{\varphi}(\mathbf{r},t), \varphi(\mathbf{r}',t)] &= \delta^3(\mathbf{r}-\mathbf{r}') \\ [\varphi(\mathbf{r},t), \varphi(\mathbf{r}',t)] &= 0. \end{aligned} \tag{3.5.19}$$

We can write the solution in the form

$$q_{\mathbf{k}}(t) = \frac{1}{\sqrt{2\omega_k}}[a_{\mathbf{k}}e^{-i\omega_k t} + a^{\dagger}_{-\mathbf{k}}e^{i\omega_k t}]. \tag{3.5.20}$$

We see that (3.5.20) will imply

$$\begin{aligned} [a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}'}] &= \delta_{\mathbf{k}\mathbf{k}'} \\ [a_{\mathbf{k}}, a_{\mathbf{k}'}] &= 0. \end{aligned} \tag{3.5.21}$$

Then we can identify that $a_{\mathbf{k}}$ is an annihilation operator and $a^{\dagger}_{\mathbf{k}'}$ a creation operator for a boson. Hence, the time-dependent quantized-field operator can be represented in the form

$$\varphi(\mathbf{r},t) = \frac{1}{\sqrt{\Omega}}\sum_{\mathbf{k}}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}[a_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + a^{\dagger}_{\mathbf{k}}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}]. \tag{3.5.22}$$

Here we can see that the notion of particle in field theories is related with normal modes or quanta of excitation of the field. We can decompose a wave in a superposition of normal modes and the independence of these, in a first approximation, give us tools to make a simpler analysis; this independence is the source of the name *particle*. The mode $\mathbf{k}$ has frequency ω_k, and the energy $\hbar\omega_k$ is called the energy quantum of the mode.

Now in words of Sunny Y. Auyang: Normal modes, field quanta, and particles are good concepts for describing continuous system only when the coupling between them is negligible. This condition is not always satisfied. For instance, the modes of a string cannot be regarded as independent of each other when the vibration is violent enough to become anharmonic. Similarly, when quantum fields interact, quanta can be excited and deexcited easily so that the static picture of free fields depicted above no longer applies. That is why field theorists say particles are epiphenomena and the concept of particle is not central to the description of fields.

If we replace the last expression (3.5.22) in the Hamiltonian density (3.5.13) we obtain a diagonalized form for the Hamiltonian

$$H = \sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}} + \frac{1}{2}\right). \tag{3.5.23}$$

Here we have encountered a zero-point energy divergence, that can be extracted out with a cutoff procedure; at the end, the cutoff has no physical relevance, since the energy of any state relative to the vacuum is independent of it.

The energy of a particle is given by

$$\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2} \tag{3.5.24}$$

where $\mathbf{k}$ is its momentum and m is the rest mass. In this frame, the total momentum operator reads

$$\mathbf{p} = \sum_{\mathbf{k}}\mathbf{k}a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}}. \tag{3.5.25}$$

As we have seen above, the time evolution of the fields is given by the Heisenberg equation

$$\frac{\partial\varphi(\mathbf{r},t)}{\partial t} = -i[\varphi(\mathbf{r},t), H]. \tag{3.5.26}$$

The formal solution is

$$\varphi(\mathbf{r},t) = e^{iHt}\varphi(\mathbf{r},0)e^{-iHt}, \tag{3.5.27}$$

substituting this in the expansion (3.5.22) we have

$$\varphi(\mathbf{r},t) = \frac{1}{\sqrt{\Omega}}\sum_{\mathbf{k}}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}e^{iHt}[a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} + a^\dagger_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{r}}]e^{-iHt}. \tag{3.5.28}$$

Using the identity for two operators A and B

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots, \tag{3.5.29}$$

and for the free Hamiltonian we have

$$e^{iHt}a_{\mathbf{k}}e^{-iHt} = a_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}t}. \tag{3.5.30}$$

Following this line, we have that the 4-momentum is defined as

$$p^\mu = \sum_{\mathbf{k}} k^\mu a^\dagger_{\mathbf{k}}a_{\mathbf{k}}, \tag{3.5.31}$$

then, using the same identity,

$$e^{ipx}a_{\mathbf{k}}e^{-ipx} = a_{\mathbf{k}}e^{-ikx} \tag{3.5.32}$$

and the commutator $[p^\mu, a_{\mathbf{k}}]$ is

$$[p^\mu, a_{\mathbf{k}}] = -k^\mu a_{\mathbf{k}}. \tag{3.5.33}$$

Therefore,

$$[p^\mu, \varphi(x)] = i\partial^\mu\varphi(x). \tag{3.5.34}$$

That is showing us that p^μ is the generator of spacetime translations.

Now, in the limit $\Omega\to\infty$, the values of $\mathbf{k}$ approach a continuum, thus we can make the replacements

$$\frac{1}{\Omega}\sum_{\mathbf{k}} \to \int\frac{d^3k}{(2\pi)^3}, \tag{3.5.35}$$

$$\Omega\delta_{\mathbf{k}\mathbf{k}'} \to (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}'). \tag{3.5.36}$$

Following this line, we define continuum versions of the annihilation and creation operators as

$$a(\mathbf{k}) \equiv \Omega a_{\mathbf{k}} \tag{3.5.37}$$

$$a^\dagger(\mathbf{k}) \equiv \Omega a^\dagger_{\mathbf{k}}. \tag{3.5.38}$$

Then the commutators will have the form

$$[a(\mathbf{k}), a^\dagger(\mathbf{k})] = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}'), \tag{3.5.39}$$

$$[a(\mathbf{k}), a(\mathbf{k}')] = 0. \tag{3.5.40}$$

Within these forms, we are able to represent the field as a Fourier integral

$$\varphi(\mathbf{r},t) = \int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}[a(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + a^\dagger(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}]. \tag{3.5.41}$$

As before, we define the vacuum state $|0\rangle$ as $a(\mathbf{k})|0\rangle = 0$ with $\langle 0|0\rangle$, then the one-particle state is defined by

$$|1_{\mathbf{k}}\rangle = a^\dagger(\mathbf{k})|0\rangle. \tag{3.5.42}$$

With the commutations relations we have

$$a(\mathbf{k})\left|1_{\mathbf{p}}\right\rangle = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{p})\left|0\right\rangle, \tag{3.5.43}$$

and with this, the single-particle wave function is

$$\langle 0|\varphi(x)|1_{\mathbf{k}}\rangle = \frac{e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}}{\sqrt{2\omega_{\mathbf{k}}}}. \tag{3.5.44}$$

In the same form, the Hamiltonian and total momentum take the forms

$$H = \int \frac{d^3k}{(2\pi)^3}\omega_{\mathbf{k}} a^\dagger(\mathbf{k})a(\mathbf{k}), \tag{3.5.45}$$

$$\mathbf{p} = \int \frac{d^3k}{(2\pi)^3}\mathbf{k}a^\dagger(\mathbf{k})a(\mathbf{k}) \tag{3.5.46}$$

Complex Scalar Field

We define a complex scalar field as field where the real and imaginary parts are real scalar fields. In this form, we can find a new symmetry between the tow fields, it bring us a conserved current that can be interpreted as electric charge. That is to say, the complex field can have electric charge while the real field must be neutral.

We express the complex scalar field as

$$\psi(\mathbf{r},t) = \frac{1}{\sqrt{2}}(\varphi_1(\mathbf{r},t)+i\varphi_2(\mathbf{r},t)). \tag{3.5.47}$$

The Lagrangian density has the form [38]

$$\begin{aligned}\mathcal{L}(x) &= \partial^\mu\psi^\dagger(x)\partial_\mu\psi(x) - m^2\psi^\dagger(x)\psi(x)\\ &= \frac{1}{2}\sum_{i=1}^{2}[\partial^\mu\varphi_i(x)\partial_\mu\varphi_i(x) - m^2\varphi_i(x)\varphi_i(x)].\end{aligned} \tag{3.5.48}$$

Within this Lagrangian density, we quantize the system imposing the following commutation relations

$$i[\dot{\varphi}_i(\mathbf{r}_1,t),\varphi_j(\mathbf{r}_2,t)] = \delta_{ij}\delta^3(\mathbf{r}_1-\mathbf{r}_2), \tag{3.5.49}$$

$$[\varphi_i(\mathbf{r}_2,t),\varphi_j(\mathbf{r}_2,t)] = 0. \tag{3.5.50}$$

In this case the complex field $\psi(\mathbf{r},t)$ becomes a non-Hermitian operator satisfying

$$i[\dot{\psi}^\dagger(\mathbf{r}_1,t),\psi(\mathbf{r}_2,t)] = \delta^3(\mathbf{r}_1-\mathbf{r}_2), \tag{3.5.51}$$

$$[\dot{\psi}(\mathbf{r}_1,t),\psi(\mathbf{r}_2,t)] = [\psi^\dagger(\mathbf{r}_1,t),\psi(\mathbf{r}_2,t)] = [\psi(\mathbf{r}_1,t),\psi(\mathbf{r}_2,t)] = 0. \tag{3.5.52}$$

Here, the canonical conjugate to ψ is $\dot{\psi}$. Thus, we can expand φ_j in terms of annihilation and creation operators

$$\varphi_j(\mathbf{r},t) = \frac{1}{\sqrt{\Omega}}\sum_{\mathbf{k}}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}[a_{j\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + a^\dagger_{j\mathbf{k}}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}], \quad j=1,2, \tag{3.5.53}$$

where such operators satisfy the commutation relations

$$[a_{i\mathbf{k}},a^\dagger_{j\mathbf{k}'}] = \delta_{ij}\delta_{\mathbf{k}\mathbf{k}'}, \tag{3.5.54}$$

$$[a_{i\mathbf{k}}, a_{j\mathbf{k}'}] = 0. \tag{3.5.55}$$

Therefore, the complex scalar field takes the form

$$\psi(\mathbf{r}, t) = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} [b_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t)} + c_{\mathbf{k}}^\dagger e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t)}] \tag{3.5.56}$$

and

$$\psi(\mathbf{r}, t) = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} [b_{\mathbf{k}}^\dagger e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t)} + c_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t)}] \tag{3.5.57}$$

being

$$b_{\mathbf{k}} = \frac{1}{\sqrt{2}}(a_{1\mathbf{k}} + ia_{2\mathbf{k}}) \tag{3.5.58}$$

$$c_{\mathbf{k}} = \frac{1}{\sqrt{2}}(a_{1\mathbf{k}} - ia_{2\mathbf{k}}) \tag{3.5.59}$$

with commutation relations

$$[b_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'} \tag{3.5.60}$$

$$[c_{\mathbf{k}}, c_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'} \tag{3.5.61}$$

$$[b_{\mathbf{k}}, b_{\mathbf{k}'}] = [c_{\mathbf{k}}, c_{\mathbf{k}'}] = [b_{\mathbf{k}}, c_{\mathbf{k}'}] = 0. \tag{3.5.62}$$

As above, making the necessary replacements, the Hamiltonian and the momentum in terms of these operators, yield

$$H = \sum_{\mathbf{k}} \omega_{\mathbf{k}}(a_{1\mathbf{k}}^\dagger a_{1\mathbf{k}} + a_{2\mathbf{k}}^\dagger a_{2\mathbf{k}}) = \sum_{\mathbf{k}} \omega_{\mathbf{k}}(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + c_{\mathbf{k}}^\dagger c_{\mathbf{k}}) \tag{3.5.63}$$

$$\mathbf{p} = \sum_{\mathbf{k}} \omega_{\mathbf{k}}\mathbf{k}(a_{1\mathbf{k}}^\dagger a_{1\mathbf{k}} + a_{2\mathbf{k}}^\dagger a_{2\mathbf{k}}) = \sum_{\mathbf{k}} \omega_{\mathbf{k}}\mathbf{k}(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + c_{\mathbf{k}}^\dagger c_{\mathbf{k}}). \tag{3.5.64}$$

Thus we have here two type of quanta which can be designated either as a_1 and a_2 quanta or b and c quanta. Studying the conserved current we shall see that only the b and c quanta have definite charge.

The current density for the complex scalar field is defined by

$$j^\mu = \psi\partial^\mu\psi^* - \psi^*\partial^\mu\psi = \frac{1}{2}(\varphi_2\partial^\mu\varphi_1 - \varphi_1\partial^\mu\varphi_2), \tag{3.5.65}$$

which satisfies the conservation law $\partial_\mu j^\mu = 0$, expanding

$$\frac{\partial j^0}{\partial t} + \nabla \cdot \mathbf{j} = 0. \tag{3.5.66}$$

If we integrating both sides over a spatial volume, we obtain

$$\int d^3x \frac{\partial j^0}{\partial t} = \frac{d}{dt}\int d^3x j^0 = 0$$

$$\frac{dQ}{dt} = 0 \tag{3.5.67}$$

being Q the total charge operator

$$\begin{aligned} Q &= \frac{d}{dt}\int d^3x j^0 \\ &= \sum_k (a_{1\mathbf{k}}^\dagger a_{2\mathbf{k}}^\dagger - a_{2\mathbf{k}}^\dagger a_{1\mathbf{k}}) \\ &= \sum_k (b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - c_{\mathbf{k}}^\dagger c_{\mathbf{k}}). \end{aligned} \tag{3.5.68}$$

Then, the b quantum carries one unit of positive charge and the c quantum carries one unit of negative charge. Since a_1 and a_2 quanta are linear combination of b and c, would not have definite charge. By convention, c quanta is taken as an antiparticle. With this frame, the positive-frequency part of ψ annihilates a particle and its negative-frequency part creates an antiparticle.

3.6 The Vacuum

As we stablish before, we have defined the vacuum state $|0\rangle$ as

$$a(\mathbf{k})\,|0\rangle = 0, \quad \forall \mathbf{k}. \tag{3.6.1}$$

Equally if we evaluate the expression (3.5.23) in the vacuum state, we have a divergence. Let us express the Hamiltonian in the continuum limit in order to clarify this situation. In the Hamiltonian

$$H = \frac{1}{2}\int d^3r \left[\pi^2 + (\nabla\varphi)^2 + m^2\varphi^2\right],$$

we replace the Fourier integral representation of the fields, such that, the before expression yields

$$\begin{aligned}
H = \frac{1}{2}\int \frac{d^3r d^3k d^3q}{(2\pi)^6} &\left[-\frac{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{q}}}}{2}\left(a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t} - a^\dagger(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}\right)\left(a(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}-\omega_{\mathbf{q}}t} - a^\dagger(\mathbf{q})e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega_{\mathbf{q}}t)}\right)\right. \\
&\frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{q}}}}\left(i\mathbf{k}a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t} - i\mathbf{k}a^\dagger(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}\right)\left(i\mathbf{q}a(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}-\omega_{\mathbf{q}}t} - i\mathbf{q}a^\dagger(\mathbf{q})e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega_{\mathbf{q}}t)}\right) \\
&\left.+\frac{m^2}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{q}}}}\left(a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t} + a^\dagger(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}\right)\left(a(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}-\omega_{\mathbf{q}}t} + a^\dagger(\mathbf{q})e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega_{\mathbf{q}}t)}\right)\right] \\
= \frac{1}{4}\int \frac{d^3k}{(2\pi)^3\omega_{\mathbf{k}}} &\left[\left(-\omega_{\mathbf{k}}^2 + \mathbf{k}^2 + m^2\right)\left(a(\mathbf{k})a(-\mathbf{k}) + a^\dagger(\mathbf{k})a^\dagger(-\mathbf{k})\right) + \left(\omega_{\mathbf{k}}^2 + \mathbf{k}^2 + m^2\right)\left(a(\mathbf{k})a^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k})\right)\right],
\end{aligned} \tag{3.6.2}$$

where we have integrated over d^3r to get delta functions $\delta^3(\mathbf{k}\pm\mathbf{q})$, which allow us to perform the d^3q integral. We know that $\omega_{\mathbf{k}}^2 = \mathbf{k}^2 + m^2$, then

$$\begin{aligned}
H &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\omega_{\mathbf{k}}\left[a(\mathbf{k})a^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k})\right] \\
&= \int \frac{d^3k}{(2\pi)^3}\omega_{\mathbf{k}}\left[a^\dagger(\mathbf{k})a(\mathbf{k}) + \frac{1}{2}(2\pi)^3\delta^3(0)\right].
\end{aligned} \tag{3.6.3}$$

The last expression is showing us a delta function evaluated at zero, where it has its infinite spike. Furthermore, the integral over $\omega_{\mathbf{k}}$ diverges at large $\mathbf{k}$. Moreover,

$$H\,|0\rangle \equiv E_0\,|0\rangle = \left[\int d^3k \frac{1}{2}\omega_{\mathbf{k}}\delta^3(0)\right]|0\rangle = \infty\,|0\rangle\,. \tag{3.6.4}$$

This infinite is telling us that we have two situations here, either we are doing something wrong or asking the wrong question. Furthermore, we have two kinds of infinites here. The first is showing us information about the infiniteness of the space (this kind of infinite is referred as infra-red divergences). To extract out this infinite, we can put the theory in a box with sides of length L and we impose periodic boundary conditions on the field. Then we have

$$(2\pi)^3\delta^3(0) = \lim_{L\to\infty}\int_{-L/2}^{L/2} d^3r e^{i\mathbf{k}\cdot\mathbf{r}}\bigg|_{\mathbf{k}=0} = \lim_{L\to\infty}\int_{-L/2}^{L/2} d^3r = V, \tag{3.6.5}$$

being V the volume of the box. Therefore, the divergence due to $\delta(0)$ is obtained because we are computing the total energy, instead the energy density $\mathcal{E}_0$. To find $\mathcal{E}_0$ we divide by the volume

$$\mathcal{E}_0 = \frac{E_0}{V} = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\omega_{\mathbf{k}}, \tag{3.6.6}$$

which is still infinite. In this expression we can evidence the sum of ground state energies for each mode of oscillation. It is clear that $\mathcal{E}_0 \to \infty$ due to the $|\mathbf{k}| \to \infty$ limit of the integral. This is a high frequency (or short distance) infinity known as an ultra-violet divergence. This result allows the physical situation where arbitrarily short distance scales has access to arbitrarily high energies. This situation makes no sense in the real world. Then, the integral should be cut-off at high momentum in order to reflect the fact that the theory is likely to break down in some way.

3.6.1 The Casimir Effect

When we use the normal ordering, we are setting $E_0 = 0$, taking into account that we are measuring only energies differences. However, there are situations where we can measure differences in the energy of vacuum fluctuations; information that is hold out with the normal ordering.

The Casimir effect is one of these situations where the vacuum fluctuations are explicitly manifest and its effects can be direct measure.

The study of this effect is a great example where we apply the cut-off technique to regulate the ultra-violet divergences. To regulate the infra-red divergences, we shall make one direction periodic with size L and impose the following boundary conditions

$$\varphi(\mathbf{r}) = \varphi(\mathbf{r} + L\hat{\mathbf{n}}), \tag{3.6.7}$$

being $\hat{\mathbf{n}} = (1, 0, 0)$. Now we insert two reflecting plates, separated by a distance $d \ll L$ in the x direction. The plates give us the condition $\varphi(x) = 0$ at the position of the plates. Then, the presence of these plates affects the Fourier decomposition of the field, and bring us a quantized momentum of the field inside the plates

$$\mathbf{p} = \left(\frac{n\pi}{a}, p_y, p_z\right). \tag{3.6.8}$$

For a massless scalar field, the fundamental energy state between the plates is

$$\frac{E(a)}{A} = \sum_{n=1}^{\infty}\int \frac{dp_y dp_z}{(2\pi)^2}\frac{1}{2}\sqrt{\left(\frac{n\pi}{a}\right)^2 + p_y^2 + p_z^2}, \tag{3.6.9}$$

and the energy outside the plates is $E(L-a)$. The total energy is

$$E = E(a) + E(L-a) \tag{3.6.10}$$

Physically we could argue that any real plate cannot reflect waves of arbitrarily high frequency. Mathematically, we want to find a way to neglect modes of momentum $p \gg \lambda^{-1}$ for some distance scale $\lambda \ll a$, known as the ultra-violet cut-off.

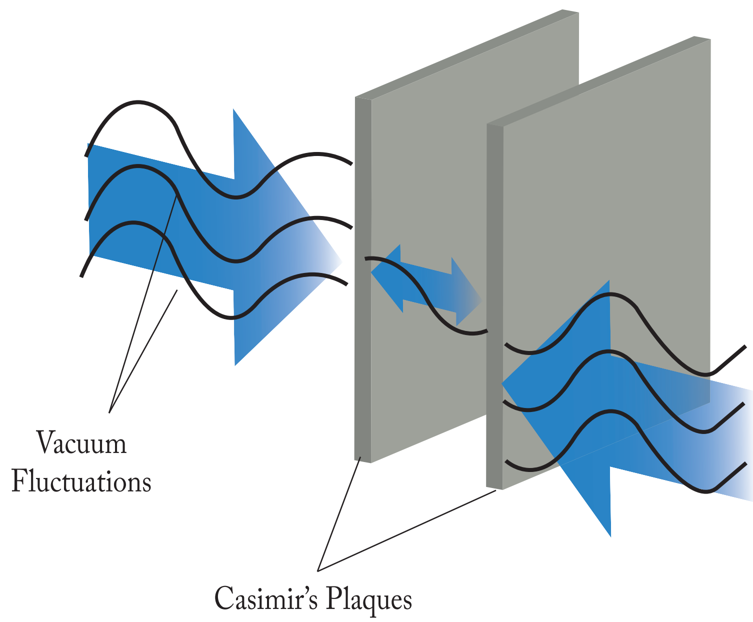

Figure 3.1: Schematic diagram of the Casimir effect and modification of vacuum fluctuations.

One way to do this, is changing the integral (3.6.9) to

$$\frac{E(a)}{A} = \sum_{n=1}^{\infty} \int \frac{dp_y dp_z}{(2\pi)^2} \frac{1}{2} \sqrt{\left(\frac{n\pi}{a}\right)^2 + p_y^2 + p_z^2}\, e^{-\lambda\sqrt{\left(\frac{n\pi}{a}\right)^2 + p_y^2 + p_z^2}}, \tag{3.6.11}$$

where we can evidence that as $\lambda \to 0$ we recuperate the integral (3.6.9). In order to explain the procedure we may look the problem in $d = 1+1$ dimensions. In this case, the energy is given by

$$E(a) = \frac{\pi}{2a} \sum_{n=1}^{\infty} n. \tag{3.6.12}$$

We now regulate this sum by introducing the UV cut-off λ. Then we have

$$\begin{aligned} E(a,\lambda) &= \frac{\pi}{2a} \sum_{n=1}^{\infty} n e^{-\lambda n\pi/a} \\ &= -\frac{1}{2} \frac{\partial}{\partial \lambda} \sum_{n=1}^{\infty} e^{-\lambda n\pi/a} \\ &= -\frac{1}{2} \frac{\partial}{\partial \lambda} \frac{1}{1 - e^{-\lambda\pi/a}} \\ &= \frac{\pi}{2a} \frac{e^{\lambda\pi/a}}{(e^{\lambda\pi/a} - 1)^2}, \end{aligned} \tag{3.6.13}$$

using the expansion

$$\frac{e^x}{(e^x - 1)^2} = \frac{1}{x^2} - \frac{1}{12} + \frac{x^2}{240} + O\left(x^4\right), \tag{3.6.14}$$

we obtain

$$E(a,\lambda) = \frac{a}{2\pi\lambda^2} - \frac{\pi}{24a} + O(\lambda^2). \tag{3.6.15}$$

Thus the full energy takes de form

$$E = E(a,\lambda) + E(L-a,\lambda) = \frac{L}{2\pi\lambda^2} - \frac{\pi}{24}\left(\frac{1}{a} + \frac{1}{L-a}\right) + O(\lambda^2). \tag{3.6.16}$$

The expression of force is given by

$$\frac{\partial E}{\partial a} = \frac{\pi}{24a^2} + \cdots. \tag{3.6.17}$$

Then as we remove both regulators, and take $\lambda \to 0$ and $L \to \infty$, the force between the plates remains finite. This is the Casimir force

3.7 Causality and Wave Equation Solutions

We have that events at two space-time points lying outside of each other's light cone cannot influence each other

The condition of microcausality tell us that two fields operators at points separated by a spacelike interval must commute

$$[\varphi(x), \varphi(x')] = 0$$

if $(x-x')^2 < 0$. This commutator at fixed x' satisfies the Klein-Gordon equation, because $\varphi(x)$ does. We can equate it with its vacuum expectation value

$$[\varphi(x), \varphi(y)] = \langle 0|[\varphi(x), \varphi(y)]|0\rangle \equiv i\Delta(x-y). \tag{3.7.1}$$

This object is a Lorentz-invariant correlation function $\Delta(x-y)$, which depends on $x-y$, and not on x and y separately, because of the translational invariance of the vacuum state. We use the expansion (3.5.22) to obtain

$$\langle 0|\varphi(\mathbf{r},t)\varphi(0)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}}t)} \tag{3.7.2}$$

$$\langle 0|\varphi(0)\varphi(\mathbf{r},t)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}}t)}. \tag{3.7.3}$$

Subtracting one from the other, we have

$$\Delta(x) = -i\,\langle 0|[\varphi(\mathbf{r},t), \varphi(0)]|0\rangle = -\int \frac{d^3k}{(2\pi)^3} \frac{\sin(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}}t)}{\omega_{\mathbf{k}}} \tag{3.7.4}$$

The propagation of a field particle in the vacuum is described by the correlation function

$$\Delta^{(+)}(x_2 - x_1) = -i\,\langle 0|\,\psi(x_2)\psi^\dagger(x_1)\,|0\rangle \tag{3.7.5}$$

in which $\psi^\dagger(x_1)$ creates a particle from the vacuum at x_1, which is annihilated by $\psi(x_2)$ at x_2. This makes sense physically when $t_2 > t_1$. Analogously, the correlation function

$$\Delta^{(-)}(x_2 - x_1) = -i\,\langle 0|\,\psi^\dagger(x_1)\psi(x_2)\,|0\rangle \tag{3.7.6}$$

describes the propagation of an antiparticle from x_2 to x_1. Equally, this makes sense physically when $t_1 > t_2$. We can use either $\Delta^{(+)}$ or $\Delta^{(-)}$ to obtain a correlation function that has physical meaning. This propagator is the Feynman propagator or causal propagator

$$\Delta_F(x_2 - x_1) = -i\,\langle 0|\,T\psi(x_2)\psi^\dagger(x_1)\,|0\rangle\,, \tag{3.7.7}$$

where the time-ordering operator T rearranges the operators, such that the operators stand in such order that time increases from right to left

$$TA(t_2)B(t_1) = \begin{cases} A(t_2)B(t_1) & \text{for } t_2 > t_1 \\ B(t_1)A(t_2) & \text{for } t_1 > t_2 \end{cases} \tag{3.7.8}$$

This is showing us that for $t_2 > t_1$, the Feynman propagator is describing the propagation of a particle from x_1 to x_2; when $t_1 > t_2$, the propagator is describing the propagation of an antiparticle from x_2 to x_1.

In order to calculate the propagator, as before, we start with

$$\Delta_F(x) = -i \begin{cases} \langle 0|\, \psi(x)\psi^\dagger(0)\, |0\rangle & \text{for } x^0 > 0 \\ \langle 0|\, \psi^\dagger(0)\psi(x)\, |0\rangle & \text{for } x^0 < 0 \end{cases} \tag{3.7.9}$$

now, we insert an identity operator with a complete set of states

$$\Delta_F(x) = -i \int \frac{d^3k}{(2\pi)^3} \begin{cases} \langle 0|\, \psi(x)\, |\mathbf{k}\rangle \langle \mathbf{k}|\, \psi^\dagger(0)\, |0\rangle & \text{for } x^0 > 0 \\ \langle 0|\, \psi^\dagger(0)\, |\mathbf{k}\rangle \langle \mathbf{k}|\, \psi(x)\, |0\rangle & \text{for } x^0 < 0 \end{cases}. \tag{3.7.10}$$

Using $\psi(x) = e^{ipx}\psi(0)e^{-ipx}$, and changing the integration variable from $\mathbf{k}$ to $-\mathbf{k}$, we obtain

$$\Delta_F(x) = -i \int \frac{d^3k}{(2\pi)^3} |\langle 0|\, \psi(0)\, |0\rangle\, |^2 e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\omega_\mathbf{k}|t|}. \tag{3.7.11}$$

Using the integral representation

$$e^{-i\omega_\mathbf{k}|t|} = \frac{i\omega}{\pi} \int_{-\infty}^{\infty} dk_0 \frac{e^{ik_0 t}}{k_0^2 - \omega_\mathbf{k}^2 + i\eta}, \quad \eta \to 0^+, \tag{3.7.12}$$

we obtain

$$\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4} 2\omega_\mathbf{k} |\langle 0|\, \psi(0)\, |0\rangle\, |^2 \frac{e^{ikx}}{k^2 - m^2 + i\eta}, \quad \eta \to 0^+, \tag{3.7.13}$$

where from (3.5.44) we have

$$|\langle 0|\, \psi(0)\, |0\rangle\, |^2 = \frac{1}{2\omega_\mathbf{k}}. \tag{3.7.14}$$

Thus,

$$\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ikx}}{k^2 - m^2 + i\eta}, \quad \eta \to 0^+. \tag{3.7.15}$$

If we operate both sides of the last expression by $\Box + m^2$ we finally obtain

$$(\Box + m^2)\Delta_F(x) = -\delta^4(x), \tag{3.7.16}$$

this is showing us that the Feynman propagator is a Green's Function of the Klein-Gordon equation.

3.8 Field Quantization in Minkowski Space

In this chapter we consider the quantization of a scalar field $\varphi(\mathbf{r}, t)$ defined in all points $(\mathbf{r}, t)$ of an n-dimensional Minkowski spacetime. We adopt the notation where a spacetime point $(\mathbf{r}, t)$ is referred as x. As before, to quantize the theory, we need the Lagrangian density, in this case reads

$$\mathcal{L}(x) = \frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - m^2\varphi^2\right). \tag{3.8.1}$$

The canonically conjugate variable to the field $\varphi(\mathbf{r}, t)$ is

$$\pi(\mathbf{r}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} = \partial_t \varphi. \tag{3.8.2}$$

With that object, we construct the following algebra, in order to quantize the field $\varphi(\mathbf{r}, t)$,

$$[\varphi(\mathbf{r}, t), \varphi(\mathbf{r}', t)] = 0, \tag{3.8.3}$$

$$[\pi(\mathbf{r}, t), \pi(\mathbf{r}', t)] = 0, \tag{3.8.4}$$

$$[\varphi(\mathbf{r}, t), \pi(\mathbf{r}', t)] = i\delta^{n-1}(\mathbf{r} - \mathbf{r}'). \tag{3.8.5}$$

As we see before, using the Heisenberg equation we can show that the field satisfies the field equation

$$(\Box + m^2)\varphi = 0, \tag{3.8.6}$$

where $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$, being $\eta^{\mu\nu}$ the Minkowskian metric tensor, while the quantity m is interpreted as the mass of the field quanta when the theory is quantized.

One set of solutions of (3.8.6) is

$$u_{\mathbf{k}}(\mathbf{r}, t) \sim e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t)}, \tag{3.8.7}$$

where

$$\omega_{\mathbf{k}} = \sqrt{k^2 + m^2}, \tag{3.8.8}$$

being

$$k \equiv |\mathbf{k}| = \left(\sum_{i=1}^{n-1} k_i^2\right)^{1/2}, \tag{3.8.9}$$

and the Cartesian components of $\mathbf{k}$ can take the values $-\infty < k_i < \infty$, $i = 1, ..., n-1$. Furthermore, we define the modes (3.8.7) as positive-frequency modes with respect to t, being eigenfunctions of the operator $\partial/\partial t$

$$\frac{\partial}{\partial t} u_{\mathbf{k}}(\mathbf{r}, t) = -i\omega_{\mathbf{k}} u_{\mathbf{k}}(\mathbf{r}, t), \quad \omega_{\mathbf{k}} > \infty. \tag{3.8.10}$$

We define the scalar product

$$\begin{aligned} (\varphi_1, \varphi_2) &= -i \int \{\varphi_1(x)\partial_t \varphi_2^*(x) - [\partial_t \varphi_1(x)]\varphi_2^*(x)\}\, d^{n-1}x \\ &= -i \int_t \varphi_1(x) \overleftrightarrow{\partial}_t \varphi_2^*(x) d^{n-1}x, \end{aligned} \tag{3.8.11}$$

where t denotes a spacelike hyperplane of simultaneity at instant t. Then, under this scalar product,

$$(u_{\mathbf{k}}, u_{\mathbf{k}'}) = 0, \quad \mathbf{k} \neq \mathbf{k}'. \tag{3.8.12}$$

Choosing an appropriate normalization factor, such that,

$$u_{\mathbf{k}}(\mathbf{r}, t) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}(2\pi)^{n-1}}} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t)}, \tag{3.8.13}$$

the expression (3.8.12) yields

$$(u_{\mathbf{k}}, u_{\mathbf{k}'}) = \delta^{n-1}(\mathbf{k} - \mathbf{k}'). \tag{3.8.14}$$

As we can observe, with the before expressions, the field modes and their respective complex conjugates form a complete orthonormal basis under the scalar product (3.8.11), then $\varphi(\mathbf{r},t)$ can be expanded as

$$\varphi(\mathbf{r},t)=\sum_{\mathbf{k}}[a_{\mathbf{k}}u_{\mathbf{k}}(\mathbf{r},t)+a_{\mathbf{k}}^{\dagger}u_{\mathbf{k}}^{*}(\mathbf{r},t)], \tag{3.8.15}$$

where from the commutation relations, we can obtain an algebra for the operators $a_{\mathbf{k}}, a_{\mathbf{k}}^{\dagger}$

$$[a_{\mathbf{k}},a_{\mathbf{k}'}^{\dagger}]=0, \tag{3.8.16}$$

$$[a_{\mathbf{k}}^{\dagger},a_{\mathbf{k}'}^{\dagger}]=0, \tag{3.8.17}$$

$$[a_{\mathbf{k}},a_{\mathbf{k}'}^{\dagger}]=\delta_{\mathbf{k}\mathbf{k}'}. \tag{3.8.18}$$

As before, the vectors constructed from the vector $|0\rangle$, defined by

$$a_{\mathbf{k}}\left|0\right\rangle=0,\quad \forall\mathbf{k}, \tag{3.8.19}$$

span a Fock space in the sense that one-particle state is

$$\left|1_{\mathbf{k}}\right\rangle=a_{\mathbf{k}}^{\dagger}\left|0\right\rangle \tag{3.8.20}$$

and the many-particle state is

$$\left|1_{\mathbf{k}_1},1_{\mathbf{k}_2},...,1_{\mathbf{k}_j}\right\rangle=a_{\mathbf{k}_1}^{\dagger}a_{\mathbf{k}_2}^{\dagger}\cdots a_{\mathbf{k}_j}^{\dagger}\left|0\right\rangle, \tag{3.8.21}$$

if all $\mathbf{k}_1,\mathbf{k}_2,...,\mathbf{k}_j$ are distinct. When any $a_{\mathbf{k}}^{\dagger}$ is repeated, we have

$$\left|{}^1n_{\mathbf{k}_1},{}^2n_{\mathbf{k}_2},...,{}^jn_{\mathbf{k}_j}\right\rangle=({}^1n!{}^2n!...{}^jn!)^{-1/2}(a_{\mathbf{k}_1}^{\dagger})^{{}^1n}(a_{\mathbf{k}_2}^{\dagger})^{{}^2n}\cdots(a_{\mathbf{k}_j}^{\dagger})^{{}^jn}\left|0\right\rangle, \tag{3.8.22}$$

where the $n!$ terms are indicating the Bose statistics of identical scalar particles. Now, as before, for each $\mathbf{k}$

$$a_{\mathbf{k}}^{\dagger}\left|n_{\mathbf{k}}\right\rangle=\sqrt{n+1}\left|(n+1)_{\mathbf{k}}\right\rangle, \tag{3.8.23}$$

$$a_{\mathbf{k}}\left|n_{\mathbf{k}}\right\rangle=\sqrt{n}\left|(n-1)_{\mathbf{k}}\right\rangle. \tag{3.8.24}$$

These basis vectors are normalized according to

$$\left\langle{}^1n_{\mathbf{k}_1},{}^2n_{\mathbf{k}_2},...,{}^rn_{\mathbf{k}_r}\right|\left.{}^1m_{\mathbf{k}_1'},{}^2m_{\mathbf{k}_2'},...,{}^sm_{\mathbf{k}_s'}\right\rangle=\delta_{rs}\sum_{\alpha}\delta_{{}^1n^{\alpha(1)}m}\cdots\delta_{{}^rn^{\alpha(s)}m}\delta_{\mathbf{k}_1\mathbf{k}_{\alpha(1)}'}\cdots\delta_{\mathbf{r}\mathbf{k}_{\alpha(s)}'} \tag{3.8.25}$$

where the sum is over all permutations α of the integers $1...s$.

We define the stress tensor $T_{\mu\nu}$ as

$$T_{\mu\nu}=\partial_\mu\varphi\partial_\nu\varphi-\frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi+\frac{1}{2}m^2\varphi^2\eta_{\mu\nu}, \tag{3.8.26}$$

from which we obtain the Hamiltonian density

$$T_{tt}=\frac{1}{2}\left[(\partial_t\varphi)^2+\sum_{i=1}^{n-1}(\partial_i\varphi)^2+m^2\varphi^2\right] \tag{3.8.27}$$

and the momentum density

$$T_{ti}=\partial_t\varphi\partial_i\varphi,\quad i=1,...,n-1, \tag{3.8.28}$$

in terms of Minkowski coordinates. Following the standard procedure, we substitute the expansion of the field (3.8.15) in the Hamiltonian density (3.8.26) and integrating over all space, we have

$$H = \int_t T_{tt} d^{n-1}x = \frac{1}{2}\sum_{\mathbf{k}} \omega_{\mathbf{k}} (a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{\mathbf{k}} a_{\mathbf{k}}^\dagger) \tag{3.8.29}$$

$$p_i = \int_t T_{ti} d^{n-1}x = \sum_{\mathbf{k}} \mathbf{k}_i a_{\mathbf{k}}^\dagger a_{\mathbf{k}}. \tag{3.8.30}$$

Using the commutation relations, we could found that the vacuum energy divergence is in the quantum field theory in Minkowski space too

$$H = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2} \right). \tag{3.8.31}$$

This divergence can be extracted out with the renormalization procedure or with the normal ordering operation.

3.8.1 Green Functions

As we see before, vacuum expectation values of varius products of free field operators can be identified with various Green functions of the wave equation. Here we define another set of Green functions from the expectation values of commutators and anticommutators of the field. We define the Pauli-Jordan or Schwinger function as

$$iG(x, x') = \langle 0| \left[\varphi(x), \varphi(x')\right] |0\rangle , \tag{3.8.32}$$

and the Hadamard's elementary function is given by

$$G^{(1)}(x, x') = \langle 0| \{\varphi(x), \varphi(x')\} |0\rangle . \tag{3.8.33}$$

Expanding the commutator and anticommutator, these Green functions can be split into their positive and negative frequency parts as

$$iG(x, x') = G^+(x, x') - G^-(x, x'), \tag{3.8.34}$$

$$G^{(1)}(x, x') = G^+(x, x') + G^-(x, x'), \tag{3.8.35}$$

where $G^\pm$ are known as the Wightman functions, given by

$$G^+(x, x') = \langle 0| \varphi(x)\varphi(x') |0\rangle , \tag{3.8.36}$$

$$G^-(x, x') = \langle 0| \varphi(x')\varphi(x) |0\rangle . \tag{3.8.37}$$

As before, in the Minkowski space, we can define the Feynman propagator G_F as a time ordered product of the fields and as a combination of Wightman functions as follows,

$$\begin{aligned} iG_F(x, x') &= \langle 0| T(\varphi(x)\varphi(x')) |0\rangle \\ &= \Theta(t - t')G^+(x, x') + \Theta(t + -t)G^-(x, x'). \end{aligned} \tag{3.8.38}$$

The retarded Green function is defined as

$$G_R(x, x') = -\Theta(t - t')G(x, x'), \tag{3.8.39}$$

while the advanced Green function is defined by

$$G_A(x, x') = \Theta(t' - t)G(x, x') \tag{3.8.40}$$

and their averaged is

$$\bar{G}(x,x') = \frac{1}{2}[G_R(x,x') + G_A(x,x')], \tag{3.8.41}$$

which can be used to define the Feynman propagator as follows

$$G_F(x,x') = -\bar{G}(x,x') - \frac{i}{2}G^{(1)}(x,x'). \tag{3.8.42}$$

It is easy to show that G, $G^{(1)}$, $G^{\pm}$ all satisfy the homogeneous equation

$$(\Box_x + m^2)\mathcal{G}(x,x') = 0. \tag{3.8.43}$$

The same form, using $\partial_t\Theta(t-t') = \delta(t-t')$, we obtain that the Feynman propagator and the retarded and advanced Green functions satisfy

$$(\Box_x + m^2)G_F(x,x') = -\delta^n(x,x'), \tag{3.8.44}$$

$$(\Box_x + m^2)G_{R,A}(x,x') = \delta^n(x,x'). \tag{3.8.45}$$

The Green functions $G_{F,R,A}$ describe the propagation of field disturbances subject to certain boundary conditions [12]. If we write explicitly the expansion (3.8.15) in the definitions of the Green functions as vacuum expectation values, we find that all the Green functions can be represented as

$$\mathcal{G} = \frac{1}{(2\pi)^n}\int \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-ik^0(t-t')}}{(k^0)^2 - |\mathbf{k}|^2 - m^2} d^n k. \tag{3.8.46}$$

The integral has poles at $k^0 = \pm\sqrt{|\mathbf{k}|^2+m^2}$. Using complex analysis tools, we can perform a contour integral for k^0. The distinct choose of the contours that enclose the poles are referring the boundary conditions of the fields and will determine which of the various Green functions is obtained from (3.8.46).

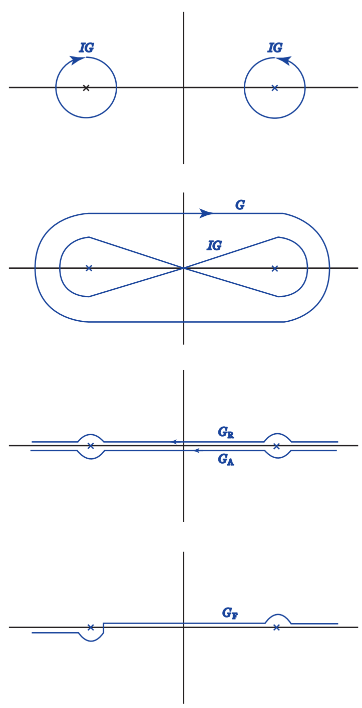

Figure 3.2: Contours of integration.

Chapter 4

Field Quantization in Curved Space

In this chapter we present some generalizations of results from the previous chapter. We present the canonical quantization of fields in curved space-times. Here we have a quantum field propagating in curved space-times, the basic background for general relativity. With this formalism we can study the effects of quantum field on geometry and vice versa. When we have these quantum fields propagating in a classical curved background produced by a stress-energy tensor, we are working a semi-classical regime. Since the semi-classical treatment of electrodynamics, where the electromagnetic field was a classic entity interacting with quantum systems, gave results that consolidated the path from a full quantum theory of electrodynamics (quantum electrodynamics), is expected that this semi-classical treatment of gravity bring some aspects in order to advance forwards a full quantum theory of the gravitational interaction.

4.1 Real scalar field quantization

Let us consider a n-dimensional manifold, globally hyperbolic $(\mathcal{M}, g_{\mu\nu})$ equipped with a metric $g_{\mu\nu}$ and a quantum scalar field φ which is propagating in this manifold. The generalization of the expression of the classic action of a free real scalar field, of mass m, which propagates in the Minkowski space to curved space is

$$S = -\frac{1}{2}\int d^n x\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + (m^2+\xi R)\varphi^2\right], \tag{4.1.1}$$

where n is the dimension, $g = \det g_{\mu\nu}$ is the determinant of the metric $g_{\mu\nu}$, ξ is a constant and R is the Ricci scalar of curvature. When the coupling between the field and the geometry is minimum, we have $\xi = 0$. In the case of a massless field, we have $\xi = \xi_n = (n-2)/(4n-4)$. Within this ξ_n the action is an invariant under conformal transformations

$$\tilde{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}, \quad \varphi = \Omega^{(2-n)/2}\varphi. \tag{4.1.2}$$

Following the standard procedure, from the action (4.1.1) we obtain the equation of motion for the field

$$\left(-\Box + m^2 + \xi R\right)\varphi = 0, \tag{4.1.3}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu = (-g)^{-1/2}\partial_\mu[(-g)^{1/2}g^{\mu\nu}\partial_\nu\varphi]$, being ∇_μ the covariant derivative.

As we did in the Minkowski space, in order to quantize the theory, we generalize the scalar product (3.8.11) as

$$(\varphi_1,\varphi_2) = -i\int_\Sigma \sqrt{-g}[\varphi_1\partial_\mu\varphi_2^* - (\partial_\mu\varphi_1)\varphi_2^*]d\Sigma^\mu, \tag{4.1.4}$$

where $d\Sigma^\mu = \hat{n}^\mu d\Sigma$ denotes the volume element, being $\hat{n}^\mu$ an unitary vector oriented to the future, orthogonal to the space-like hypersurface Σ. From Gauss's theorem, on can show that if φ_1 and φ_2 are solutions of the field equation (4.1.3) that decay such that are null at the spatial infinite, the product (φ_1, φ_2) is independent of the space-like hypersurface Σ. The product (4.1.4) satisfy

$$(\varphi_1, \varphi_2)^* = -(\varphi_1^*, \varphi_2^*) = (\varphi_2, \varphi_1), \tag{4.1.5}$$

$$(\varphi_1, \varphi_1^*) = 0. \tag{4.1.6}$$

The solutions of the field equation (4.1.3) can be expanded as

$$\varphi(x) = \sum_{\mathbf{k}} [a_{\mathbf{k}} u_{\mathbf{k}}(x) + a_{\mathbf{k}}^\dagger u_{\mathbf{k}}^*(x)], \tag{4.1.7}$$

where the complete set of functions $u_{\mathbf{k}}(x)$ under the product (4.1.4) satisfy

$$(u_{\mathbf{k}}(x), u_{\mathbf{k}'}(x)) = \delta_{\mathbf{k}\mathbf{k}'} \tag{4.1.8}$$

$$(u_{\mathbf{k}}(x), u_{\mathbf{k}'}^*(x)) = 0. \tag{4.1.9}$$

Therefore, from the usual commutation relations, we have the usual commutation relations (3.8.16), (3.8.17), (3.8.18) for the coefficients $a_{\mathbf{k}} = (\varphi(x), u_{\mathbf{k}}(x))$, now operators. Then, we define the vacuum state $|0\rangle$ as $a_{\mathbf{k}} |0\rangle = 0$ $\forall \mathbf{k}$ and construct the corresponding Fock space. We could make a different choice in the functions for expand the solutions of (4.1.3), then the definition of vacuum and particles have a inherent ambiguity. For instance, we can use a set of functions $\bar{u}_{\mathbf{p}}(x)$ instead $u_{\mathbf{k}}(x)$ such that $(\bar{u}_{\mathbf{p}}^*, u_{\mathbf{k}}) \neq 0$ (for any $\mathbf{k}$ and $\mathbf{p}$) and we define a new vacuum state as $\bar{a}_{\mathbf{k}} |\bar{0}\rangle = 0$, being $\bar{a}_{\mathbf{k}} = (\varphi, \bar{u}_{\mathbf{k}})$, we obtain that $a_{\mathbf{k}} |\bar{0}\rangle = (\varphi, u_{\mathbf{k}}) |\bar{0}\rangle = \sum_{\mathbf{p}} (\bar{u}_{\mathbf{p}}^*, u_{\mathbf{k}}) \bar{a}_{\mathbf{p}}^\dagger |\bar{0}\rangle \neq 0$. Then, the two Fock spaces spanned from the distinct choices $u_{\mathbf{k}}$ and $\bar{u}_{\mathbf{p}}$ are different. The physical implications of this situation will be discussed later.

We construct the Green's functions in the usual form. We first define the elementary Hadamard function

$$G(x, x') = \langle 0| \{\varphi(x), \varphi(x')\} |0\rangle \tag{4.1.10}$$

and the Feynman propagator

$$G_F(x, x') = i \langle 0| T[\varphi(x), \varphi(x')] |0\rangle . \tag{4.1.11}$$

These functions satisfy

$$[\Box_x - m^2 - \xi R] G(x, x') = 0, \tag{4.1.12}$$

$$[\Box_x - m^2 - \xi R] G_F(x, x') = -(-g)^{-1/2} \delta(x, x'). \tag{4.1.13}$$

The elementary Hadamard function can be obtained from the Feynman propagator as follows

$$G(x, x') = 2\mathrm{Im}[G_F(x, x')]. \tag{4.1.14}$$

We can evidence that the mean value of square of the field $\langle \varphi^2 \rangle$ may be expressed as

$$\langle \varphi^2 \rangle = \frac{1}{2} \lim_{x \to x'} G(x, x') = \frac{1}{2} \lim_{x \to x'} \mathrm{Im}[G_F(x, x')]. \tag{4.1.15}$$

As we see before, the stress-energy tensor can be obtained from the action by the definition

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}. \tag{4.1.16}$$

From the action (4.1.1) we have

$$\begin{aligned} T_{\mu\nu} = (1-2\xi)\partial_\mu\varphi\partial_\nu\varphi - 2\xi\varphi\nabla_\mu\nabla_\nu\varphi + 2\xi g_{\mu\nu}\varphi\Box\varphi \\ + \xi\varphi^2 G_{\mu\nu} + \left(2\xi - \frac{1}{2}\right) g_{\mu\nu}\partial_\gamma\partial^\gamma\varphi - \frac{m^2}{2} g_{\mu\nu}\varphi^2, \end{aligned} \tag{4.1.17}$$

where $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}R/2$ is the Einstein tensor.

We could here define the corresponding quantum operator after adopt a convention for a normal operator ordering and after would replace in that expression the quantum field operator in the place of the classic field. This procedure will take the product of two distributions valued at the same point, which is not well defined. Therefore, is necessary apply a regularization method. If we calculate the mean value $\langle T_{\mu\nu}\rangle$ for a given quantum state it will be divergent. These divergences are ultra-violet, since they appear when we take the product of two distributions valued at the same point. Thus, in order to characterize those objects we must study the behaviour of such products for distinct but too close points. We can adopt the Weyl prescription for the operator ordering where, for instance, the term $\varphi\nabla_\mu\nabla_\nu\varphi$ is replace by $\{\varphi, \nabla_\mu\nabla_\nu\varphi\}/2$, where $\{,\}$ denotes the anticommutator.

4.2 Construction of $\left\langle\varphi^2\right\rangle_{ren}$ and $\langle T_{\mu\nu}\rangle_{ren}$

In this section we resume some methods used for obtain the renormalized quantities related to $\left\langle\varphi^2\right\rangle$ and $\langle T_{\mu\nu}\rangle$. The mena value $\langle T_{\mu\nu}\rangle$ is an important quantity because it is a fundamental object when we are studying the effects of the quantum fields on the space-time geometry. we can perceive the it self is appearing as font in the semi-classical Einstein field equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi\left\langle T_{\mu\nu}(x)\right\rangle. \tag{4.2.1}$$

For the above equation becomes meaningful, we can use a regularization method, expressing $\langle T_{\mu\nu}(x)\rangle$ as a sum of a divergent part and a finite part

$$\langle T_{\mu\nu}(x)\rangle = \langle T_{\mu\nu}(x)\rangle_{\text{div}} + \langle T_{\mu\nu}(x)\rangle_{\text{ren}}. \tag{4.2.2}$$

This separation of the divergent terms is so far to be trivial. The divergent part must be cancellated with an adequate counter-terms, on the other hand, the left side of the equation (4.2.2) satisfies the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$, then we must have $\nabla^\mu\left\langle T_{\mu\nu}(x)\right\rangle_{\text{ren}} = 0$, which is guaranteed if we use a covariant regularization method.

One adequate regularization method is the covariant separation of points which consists in conceive the momentum-energy tensor evaluate in any x point as the limit when $x \to x'$ of a bi-tensor $\tau_{\mu\nu'}(x, x')$ which transform as a tensor product in x to other one in x'. With this procedure, we can express the momentum-energy tensor in terms of the Hadamard propagator as follows

$$\begin{aligned} \langle T_{\mu\nu}(x)\rangle = \lim_{x\to x'} \Bigg\{ & \frac{1}{4}(1-2\xi)(G_{;\mu'\nu}(x,x') + G_{;\mu\nu'}(x,x')) \\ & - \frac{1}{2}\xi(G_{;\mu\nu}(x,x') + G_{;\mu'\nu'}(x,x')) + \frac{1}{8}\xi g_{\mu\nu}(G^{;\sigma}_{;\sigma}(x,x') + G^{;\sigma'}_{;\sigma'}(x,x')) \\ & + \frac{n-1}{n}\xi(\xi R + m^2)G(x,x')g_{\mu\nu} + \frac{1}{2}\xi G_{\mu\nu}G(x,x') - \frac{m^2}{4}g_{\mu\nu}G(x,x') \\ & + \frac{1}{n}\left(2\xi - \frac{1}{2}\right) g_{\mu\nu}(G^{;\gamma}_{;\gamma'}(x,x') + G^{;\gamma'}_{;\gamma}(x,x'))\Bigg\}. \end{aligned} \tag{4.2.3}$$

This expression for $\langle T_{\mu\nu}(x)\rangle$ is merely formal, since it is divergent and we have quantity that are transforming as tensors, but in different space-time points. For give a solution to that situation, we can define methods of parallel transport that bring a certain significance to this expression.

In order to separate in a covariant form the divergent part, we use a covariant development for the propagator G_1, that has the form known as the Hadamard elementary solution

$$G_H(x,x') = \frac{u(x,x')}{\sigma(x,x')} + v(x,x')\ln\sigma(x,x') + w(x,x'), \tag{4.2.4}$$

where u, v and w are non singular and symmetric functions and $\sigma(x,x') = s^2(x,x')/2$ is a square of the one half of the geodesic distance between the points x and x' if they are content in a normal coordinate environment.

The most used procedure is the so-called Schwinger-De Witt method, that is given by

$$G^{SD}(x,x') = -2\mathrm{Im}\Delta^{1/2}(x,x')\int_0^\infty \frac{ds}{(4i\pi s)}^{n/2}\exp\left(\frac{\sigma(x,x')}{2s} - im^2 s\right)\sum_{k\geq 0}(is)^k a_k(x,x'), \tag{4.2.5}$$

being $\Delta^{1/2}(x,x') = -\mathtt{det}[\partial_\mu\partial_\nu\sigma(x,x')][g(x)g(x')]^{-1/2}$ the Van Vleck's determinant. The functions $a_k(x,x')$ are defined by recurrence relations from $a_0(x,x') = 1$, which guarantee that $G^{SD}(x,x')$ is a solution of the equation (4.1.12). It can be showed that this expression has the Hadamard form (4.2.4).

In the limit $x \to x'$ the functions $a_k(x,x')$ are scalars composed by the metric and its derivatives. As k increases, a_k contents more quantity of derivatives of the metric, i. e.,

$$a_1(x,x) = -\left(\xi - \frac{1}{6}\right)R, \tag{4.2.6}$$

$$a_2(x,x) = \frac{1}{180}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu}) + \frac{1}{6}\left(\frac{1}{5} - \xi\right)\Box R + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2, \tag{4.2.7}$$

$$\vdots$$

In general, the coefficient a_k has $2k$ derivatives of the metric. In this form, we obtain a adiabatic expression for G, where the adiabatic order is given by the number of derivatives of the metric that are appearing in the coefficients. Following this line, a_1 has an adiabatic order of two and a_2 is of fourth adiabatic order.

Introducing the propagator G^{SD} in the equations (4.1.15) and (4.2.3) we obtain a adiabatic expansion for $\left\langle\varphi^2(x)\right\rangle$ and $\langle T_{\mu\nu}(x)\rangle$. Using methods of parallel transport, we can separate in a covariant form the divergent part of each one. The limit $x \to x'$ is taken at the final of calculus. On the other hand, one can work directly with $x = x'$ and use the dimensional regularization method.

For $x = x'$ is easy perform the integral (4.2.5). Introducing a mass scale μ in order to maintain the correct units of G^{DS} in an space-time of $\bar{n}$ dimensions, we obtain

$$G^{SD}(x,x) = 2\left(\frac{\mu}{m}\right)^{\bar{n}-n}\sum_{k\geq 0}\frac{a_k(x,x)}{(4\pi)^n}m^{\bar{n}-2(k+1)}\Gamma\left(1 + k - \frac{n}{2}\right). \tag{4.2.8}$$

We can observe that for $\bar{n}$ dimensions there is a finite quantity of the series terms that are divergent for $n \to \bar{n}$, for instance, the firsts terms with $k \leq \mathtt{int}(n/2 - 1)$, where $\mathtt{int}(x)$ is the integer part of x. With this, using the expression (4.1.15) we can characterize the divergent that will appear in the adiabatic expansion of $\left\langle\varphi^2(x)\right\rangle$.

With n dimensions, one can show that de divergences that are appearing in the adiabatic expansion of $\left\langle\varphi^2(x)\right\rangle$ and $\langle T_{\mu\nu}(x)\rangle$ have coefficients of adiabatic order $2i \leq 2i^u_{max}$ and $2j \leq 2j^u_{max}$, respectively, as

$$2i^u_{max} = 2\mathtt{int}\left(\frac{n}{2} - 1\right), \tag{4.2.9}$$

$$2j^u_{max} = 2\mathtt{int}\left(\frac{n}{2}\right), \tag{4.2.10}$$

where the index u (for usual) is for differentiate these results that belong to the case of the usual dimensional regularization.

Therefore, we define the renormalized mean values by the subtraction

$$\left\langle \varphi^2(x) \right\rangle_{ren} = \left\langle \varphi^2(x) \right\rangle - \left\langle \varphi^2(x) \right\rangle^{(0)} - \cdots - \left\langle \varphi^2(x) \right\rangle^{2i_{max}}, \tag{4.2.11}$$

$$\left\langle T_{\mu\nu}(x) \right\rangle_{ren} = \left\langle T_{\mu\nu}(x) \right\rangle - \left\langle T_{\mu\nu}(x) \right\rangle^{(0)} - \cdots - \left\langle T_{\mu\nu}(x) \right\rangle^{(2j_{max})}, \tag{4.2.12}$$

where the index $2l$ is denoting the term of the adiabatic order of $2l$ that is contributing to the adiabatic expansion of the corresponding object.

Here is important to note that the mean values $\left\langle \varphi^2(x) \right\rangle$ and $\left\langle T_{\mu\nu}(x) \right\rangle$ are strongly dependent of the election of the state for which we take these mean values in contrast to the terms that are subtracting which are independents of such election. A consequence of this is that not all states will bring us a finite $\left\langle T_{\mu\nu}(x) \right\rangle_{ren}$. This is due by the fact that the Hadamard propagator constructed from such states and the derivatives of it self would not have the same singular structure of the propagator G^{SD} and its derivatives. If the elected state does not bring us a finite $\left\langle T_{\mu\nu}(x) \right\rangle_{ren}$ we can say that such state is not a physical state.

Another important fact of this construction is, because of the method is covariant, we obtain automatically $\nabla_\mu \left\langle T^\mu_\nu(x) \right\rangle_{ren} = 0$. It guarantee the consistency with the Einstein's field equations.

Finally, we know that for the massless and conformal coupling case the momentum-energy tensor is traceless. From this construction we have a trace anomaly for even dimensions. For n dimensions, this anomaly is given by the coefficient $a_{n/2}$ in the Schwinger-De Witt expansion. For two and four dimensions we have

$$\left\langle T^\mu_\mu(x) \right\rangle_{ren} = \frac{a_1}{4\pi} = \frac{R}{24\pi}, \quad n = 2, \tag{4.2.13}$$

$$\left\langle T^\mu_\mu(x) \right\rangle_{ren} = \frac{a_2}{16\pi^2} = \frac{1}{2880}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu} + \Box R), \quad n = 4. \tag{4.2.14}$$

4.3 Bogoliubov transformations: A first view

As we can perceive, the field expansion is completely arbitrary and the orthonormal set of functions cannot be unique. The Bogoliubov transformations will stablish a relationship between the different orthonormal sets of functions that expand a field. Since each set of functions define a Fock space and then a vacuum state, the Bogoliubov transformations show that the vacuum state can be expressed in terms of many-particle states from another quantization scheme. The physical implications of these results results will be discussed later.

Therefore we can also expand the field φ in terms of a second complete orthonormal set of modes $\bar{u}_{\mathbf{p}}(x)$ as

$$\varphi(x) = \sum_{\mathbf{p}} [\bar{a}_{\mathbf{p}} \bar{u}_{\mathbf{p}}(x) + \bar{a}^\dagger_{\mathbf{p}} \bar{u}^*_{\mathbf{p}}(x)], \tag{4.3.1}$$

as before, this decomposition will define a new vacuum state $|\bar{0}\rangle$

$$\bar{a}_{\mathbf{p}} |\bar{0}\rangle = 0, \quad \forall \mathbf{p} \tag{4.3.2}$$

and a new Fock space.

Since both sets are complete, we can expand the new modes $\bar{u}_{\mathbf{p}}(x)$ in terms of the old

$$\bar{u}_{\mathbf{p}} = \sum_{\mathbf{k}} (\alpha_{\mathbf{pk}} u_{\mathbf{k}} + \beta_{\mathbf{pk}} u^*_{\mathbf{k}}). \tag{4.3.3}$$

Conversely

$$u_{\mathbf{k}} = \sum_{\mathbf{p}} (\alpha^*_{\mathbf{pk}} \bar{u}_{\mathbf{p}} - \beta_{\mathbf{pk}} \bar{u}^*_{\mathbf{p}}). \tag{4.3.4}$$

The relations presented above are widely known as Bogoliuvob transformations. The matrices $\alpha_{\mathbf{pk}}$ and $\beta_{\mathbf{pk}}$ are called Bogoliuvob coefficients and by using (4.1.8) and (4.3.3) they can be evaluated as

$$\alpha_{\mathbf{pk}} = (\bar{u}_{\mathbf{p}}, u_{\mathbf{k}}), \quad \beta_{\mathbf{pk}} = -(\bar{u}_{\mathbf{p}}, u^*_{\mathbf{k}}). \tag{4.3.5}$$

In an analogous way, we can express the old annihilation and creation operators in terms of the new by equating the expansions (4.1.7) and (4.3.1) and using the relations (4.3.3), (4.3.4) and the orthonormality of the modes, we have

$$a_{\mathbf{k}} = \sum_{\mathbf{p}} (\alpha_{\mathbf{pk}} \bar{a}_{\mathbf{p}} + \beta^*_{\mathbf{pk}} \bar{a}^\dagger_{\mathbf{p}}) \tag{4.3.6}$$

and

$$\bar{a}_{\mathbf{p}} = \sum_{\mathbf{p}} (\alpha^*_{\mathbf{pk}} a_{\mathbf{k}} - \beta^*_{\mathbf{pk}} a^\dagger_{\mathbf{k}}). \tag{4.3.7}$$

Note the state defined by (4.3.2) will not be annihilated by the operator $a_{\mathbf{k}}$,

$$a_{\mathbf{k}} \left|\bar{0}\right\rangle = \sum_{\mathbf{p}} \beta^*_{\mathbf{pk}} \left|\bar{1}_{\mathbf{p}}\right\rangle \neq 0. \tag{4.3.8}$$

From this, we can establish the following properties of the Bogoliubov coefficients

$$\sum_{\mathbf{q}} (\alpha_{\mathbf{kq}} \alpha^*_{\mathbf{pq}} - \beta_{\mathbf{kq}} \beta^*_{\mathbf{pq}}) = \delta_{\mathbf{kp}} \tag{4.3.9}$$

$$\sum_{\mathbf{q}} (\alpha_{\mathbf{kq}} \beta_{\mathbf{pq}} - \beta_{\mathbf{kq}} \alpha_{\mathbf{pq}}) = 0. \tag{4.3.10}$$

Following this line, the expectation value of the operator $N_{\mathbf{k}} = a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$ for the number if $u_{\mathbf{k}}$-mode particles in the state $\left|\bar{0}\right\rangle$ is

$$\left\langle\bar{0}\right| N_{\mathbf{k}} \left|\bar{0}\right\rangle = \sum_{\mathbf{p}} |\beta_{\mathbf{pk}}|^2, \tag{4.3.11}$$

that fundamental expression is saying us that the vacuum of the $\bar{u}_{\mathbf{p}}$ modes contains $\sum_{\mathbf{p}} |\beta_{\mathbf{pk}}|^2$ particles in the $u_{\mathbf{k}}$ mode.

If $u_{\mathbf{p}}$ satisfies

$$\mathcal{L}_\xi u_{\mathbf{p}} = -i\omega u_{\mathbf{p}}, \quad \omega > 0, \tag{4.3.12}$$

which is to say that $u_{\mathbf{p}}$ are positive frequency modes with respect to some time-like Killing vector field ξ, and $\bar{u}_{\mathbf{k}}$ contains only positive frequencies with respect to ξ, then $\beta_{\mathbf{pk}} = 0$. Therefore, $\bar{a}_{\mathbf{k}} \left|0\right\rangle = 0$ as well $a_{\mathbf{p}} \left|\bar{0}\right\rangle = 0$. Then, the two sets of modes $u_{\mathbf{p}}$ and $\bar{u}_{\mathbf{k}}$ share a common vacuum state. Now, if any $\beta_{\mathbf{pk}} \neq 0$, the $\bar{u}_{\mathbf{k}}$ will contain a mixture of positive-$u_{\mathbf{p}}$ and negative-$u^*_{\mathbf{p}}$ frequency modes, and particles will be present.

The Fock space based on $\left|0\right\rangle$ can be related to that based on $\left|\bar{0}\right\rangle$ using the completeness of the Fock space basis elements

$$\left|{}^1 n_{\mathbf{k}_1}, {}^2 n_{\mathbf{k}_2}, ...\right\rangle = \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{\mathbf{p}_1 \cdots \mathbf{p}_l} \left|\bar{1}_{\mathbf{p}_1}, \bar{1}_{\mathbf{p}_2}, ..., \bar{1}_{\mathbf{p}_l}\right\rangle \left\langle \bar{1}_{\mathbf{p}_1}, \bar{1}_{\mathbf{p}_2}, ..., \bar{1}_{\mathbf{p}_l} \right| \left. {}^1 n_{\mathbf{k}_1}, {}^2 n_{\mathbf{k}_2}, ...\right\rangle, \tag{4.3.13}$$

where we are following the notation used in [12]

$$\left|{}^1 n_{\mathbf{k}_1}\right\rangle = \left|1_{\mathbf{k}_1}, 1_{\mathbf{k}_1}, ..., 1_{\mathbf{k}_1}\right\rangle / ({}^1 n_{\mathbf{k}}!)^{1/2}$$

being the $1_{\mathbf{k}_1}$ repeated ${}^1 n_{\mathbf{k}}$ times.

With these expressions, one can write the vacuum to many-particle amplitudes in terms of the Bogoliubov coefficients

$$\langle \bar{0}\,|1_{\mathbf{p}_1}, 1_{\mathbf{p}_2}, ..., 1_{\mathbf{p}_l}\rangle = \begin{cases} i^{l/2}\,\langle \bar{0}\,|0\rangle \sum_\rho \Lambda_{\rho_1\rho_2}\cdots\Lambda_{\rho_{l-1}\rho_l} & \text{for } k \quad \text{even} \\ 0 & \text{for } k \quad \text{odd} \end{cases} \tag{4.3.14}$$

$$\langle \bar{1}_{\mathbf{p}_1}, \bar{1}_{\mathbf{p}_2}, ..., \bar{1}_{\mathbf{p}_l}|\,0\rangle = \begin{cases} i^{l/2}\,\langle \bar{0}\,|0\rangle \sum_\rho V_{\rho_1\rho_2}\cdots V_{\rho_{l-1}\rho_l} & \text{for } k \quad \text{even} \\ 0 & \text{for } k \quad \text{odd} \end{cases} \tag{4.3.15}$$

where ρ represents all distinct permutations of $\{\mathbf{p}_1, ..., \mathbf{p}_l\}$ and

$$\Lambda_{\mathbf{kp}} = -i\sum_{\mathbf{q}} \beta_{\mathbf{qp}}\alpha^{-1}_{\mathbf{kq}} \tag{4.3.16}$$

$$V_{\mathbf{kp}} = i\sum_{\mathbf{q}} \beta^*_{\mathbf{pq}}\alpha^{-1}_{\mathbf{qk}}. \tag{4.3.17}$$

4.4 Vacuum states

From the results obtained in the study of the Schwarzschild problem, we can affirm that a unique choosing of the system coordinate is completely useless. If we want to stablish the different physical implications of this solution we must be moving between different coordinates. Since different time choosing will impose different definitions of the positive frequency modes, we will have different vacuum states.

One of the objectives is calculate the mean value of the energy in these vacuum states. Therefore we shall stablish a stress-energy tensor that describes the physical situation corresponding to the vacuum choosing. Considering a non stationary space-time

$$ds^2 = -e^{-2\xi(u,v)}dudv = e^{2\xi(z,t)}(dz^2 - dt^2), \tag{4.4.1}$$

the curvature is given by

$$R = -2\Box\xi = 8e^{-2\xi}\partial_u\partial_v\xi, \tag{4.4.2}$$

where the operator $\Box$ is defined as

$$\Box \equiv 4e^{-2\xi}\partial_u\partial_v, \tag{4.4.3}$$

and the non null Christoffel symbols are $\Gamma^u_{uu} = 2\partial_u\xi$ and $\Gamma^v_{vv} = 2\partial_v\xi$.

In this situation we have that the quantum effects, such that fluctuations, will induce a stress-energy tensor due to the curvature [29]. For a massless scalar field, that stress-energy tensor is given by

$$T^\alpha_\alpha = \frac{\hbar}{24\pi}R, \tag{4.4.4}$$

which satisfies the relation

$$\nabla_\beta T^\alpha_\beta = 0. \tag{4.4.5}$$

The solution of the equation given by (4.4.4) for any quantum state is given by

$$T_{\alpha\beta} = \Theta_{\alpha\beta}[\xi] + F^{out}(u)\partial_\alpha u\partial_\beta u + F^{in}(v)\partial_\alpha v\partial_\beta v, \tag{4.4.6}$$

where,

$$\Theta_{\alpha\beta}[\xi] = \frac{\hbar}{12\pi}\left\{\nabla_\beta\nabla_\alpha\xi + \partial_\alpha\xi\partial_\beta\xi - g_{\alpha\beta}\left[\Box\xi + \frac{1}{2}\left(\partial_\alpha\xi\partial^\alpha\xi\right)\right]\right\}, \tag{4.4.7}$$

and ξ, F^{in}, F^{out}, are arbitrary functions. With this in mind, we can define the useful vacuum states

4.4.1 Boulware state $|0\rangle_B$

The Boulware state $|0\rangle_B$ is a state that has zero positive frequency modes with respect to the temporal Killing parameter t of a static space-time. In other words, a static observer wont perceive particles in this state.

Furthermore, if this static space-time is asymptotically flat, the Boulware vacuum in the infinite is indistinguishable with the Minkowski vacuum. Additionally if ${}_B\langle 0\,|T_{\alpha\beta}|\,0\rangle_B \to 0$ at the infinite, we make the choose $F^{in} = F^{out} = 0$, thus,

$$ {}_B\langle 0\,|T_{\alpha\beta}|\,0\rangle_B = \Theta_{\alpha\beta}[\xi]. \tag{4.4.8}$$

In the horizon, the variable $\xi \to -\infty$, then, in this region, the stress-energy tensor evaluated in the Boulware vacuum is singular. Therefore, work with the Boulware vacuum in a black hole is not recommended because it is unstable. This vacuum corresponds to the zero point temperature for the inner space, as is to say, the surrounding of a static star.

4.4.2 Hartle-Hawking state $|0\rangle_H$

Considering an eternal static black hole, the Hartle-Hawking vacuum state $|0\rangle_H$ has zero positive frequency modes with respect the Krustal times U and V. For this case, a free falling observed wont perceive particles in the horizon.

The stress-energy tensor in this situation is bounded by the future and past horizons H^+, H^-, respectively. In the Kruskal coordinates the metric (4.4.1) takes the form

$$ds^2 = -e^{2\zeta(U,V)} dU dV, \tag{4.4.9}$$

where,

$$\zeta = \xi - \frac{1}{2}\ln[U'(u)V'(v)], \tag{4.4.10}$$

is regular in the horizons. The logarithms are related with the generalized surface gravities as

$$[\ln U'(u)]' = \kappa^{out}(u, v = -\infty), \tag{4.4.11}$$

$$\ln[V'(v)]' = \kappa^{in}(u = \infty, v). \tag{4.4.12}$$

Thus,

$$\Theta_{\alpha\beta}[\zeta] = \Theta_{\alpha\beta}[\xi] + H^{out}(u)\partial_\alpha u \partial_\beta u + H^{in}(v)\partial_\alpha v \partial_\beta v, \tag{4.4.13}$$

where

$$H^{out}(u) = \frac{\hbar}{48\pi}\left\{([\ln U'(u)]')^2 - \frac{1}{2}R(u, v = -\infty)\right\}, \tag{4.4.14}$$

$$H^{in}(v) = \frac{\hbar}{48\pi}\left\{([\ln V'(v)]')^2 - \frac{1}{2}R(u = \infty, v)\right\}. \tag{4.4.15}$$

Since ${}_H\langle 0\,|T_{\alpha\beta}|\,0\rangle_H$ must satisfy the boundary condition imposed by the future and past horizons, it must have the form

$$ {}_H\langle 0\,|T_{\alpha\beta}|\,0\rangle_H = \Theta_{\alpha\beta}[\zeta]. \tag{4.4.16}$$

A free falling observer will measure the energy value given by (4.4.16) in the horizon, using local lorentzian coordinates to define its proper notion of positive frequency. Furthermore, from (4.4.13), (4.4.8) and (4.4.16), we have

$$ {}_H\langle 0\,|T_{\alpha\beta}|\,0\rangle_H = {}_B\langle 0\,|T_{\alpha\beta}|\,0\rangle_B + H^{out}(u)\partial_\alpha u \partial_\beta u + H^{in}(v)\partial_\alpha v \partial_\beta v. \tag{4.4.17}$$

From the expression (4.4.17) we evidence that the energy measured by the Hartle-Hawking state is given in terms of the energy measured by the Boulware state that presents divergences in the horizon. Physically, the terms that are divergent in the Boulware stress-energy tensor, can be understood as a light-like negative energy current infinitely shifted to the blue, radially incoming and outcome, in the past and future horizons, respectively. Those currents can be neutralized by fluxes of positive energy from the infinite towards the infinite. Then corrected divergences can be neutralized too. Therefore, the state $|0\rangle_H$ corresponds to a black hole in thermal equilibrium with its confined radiation. The same equilibrium that we can obtain in a typical thermodynamic system isolated in a perfectly reflecting cavity.

4.4.3 Unruh state $|0\rangle_U$

Over a Schwarzschild maximally extended, we can construct a vacuum state that may reproduce the effects of a collapsing mass. That vacuum state is the Unruh state $|0\rangle_U$.

The Unruh vacuum has zero positive frequency modes with respect to the advanced time v neither respect to the retarded Kruskal time U, in the manifold that represents the maximally extended black hole. In other words, this vacuum is defined in terms of incoming modes from the infinite with positive frequency respect to $\partial/\partial t$. Furthermore, the modes that come from the past horizon are taken as positive frequency modes with respect to U.

We have that the Unruh state is vacuum in the past infinite, then the term $F^{in}(v)$ in (4.4.6) disappear and the expression (4.4.17) is regular in the past horizon, then

$$_U\langle 0\,|T_{\alpha\beta}|\,0\rangle_U =_B \langle 0\,|T_{\alpha\beta}|\,0\rangle_B + H^{out}(u)\partial_\alpha u\partial_\beta u. \tag{4.4.18}$$

Since in the future infinite $_B\langle 0\,|T_{\alpha\beta}|\,0\rangle_B \to 0$, the second term represents the characteristic thermal flux of an evaporating black hole. From (4.4.17) and (4.4.18) we obtain,

$$_U\langle 0\,|T_{\alpha\beta}|\,0\rangle_U =_H \langle 0\,|T_{\alpha\beta}|\,0\rangle_H - H^{in}(v)\partial_\alpha v\partial_\beta v. \tag{4.4.19}$$

Here we can evidence that in this last expression there is a incoming negative energy flux through the future horizon. From the above discussion, we can affirm that the Unruh state is useful for express the vacuum state in a gravitational collapse.

As we shall see later, the difference between the Bogoliubov coefficients between two different quantization frames, has a thermal form, the same thermal form that has the difference between the stress-energy tensors of the Hartle-Hawking and Boulware states.

Chapter 5

The Unruh Effect

An observer moving with uniform acceleration a perceives the Minkowski vacuum state of the quantum field as a thermal bath with temperature [13, 15]

$$T = \frac{\hbar a}{2\pi c k_B}, \tag{5.0.1}$$

where $\hbar, k_B, c$ are the Planck ad Boltzmann's constants, and the speed of light respectively. Using first-order perturbation theory it can be shown that the transition rates of the Unruh-DeWitt detector [16] interacting with a scalar field in the Minkowski vacuum is given by the Fourier transform of the positive frequency Wightman function evaluated on the world line of the detector [40]. In the case of an uniformly accelerated detector, it is found that, if it is initially prepared in its ground state, it will be excited by the thermal radiation perceived by it [21].

In this chapter we present the formalism of particle detectors, that bring the mathematical structure and the objects of interest in the scenario where we can extract entanglement from these physical implications. For the accelerated detector we show the Unruh effect and its thermal character. In order to achieve realistic situations, our formalism is presented generalizing the before studies placing all the physical systems in a finite observational time. In this chapter we present some main results of this work.

5.1 Particle Detectors

The physical situation of interest is composed by a background space-time (which in principle is unaltered); a free real scalar quantum field φ, which its excitations are the object of the measurement (as we see before, the mass and curvature coupling parameter of this field can be arbitrary) and one particle detector which is coupled to φ and is following a smooth trajectory $x^\mu(\tau)$.

We define the overall Hilbert space as a direct product of the Hilbert spaces corresponding to Hilbert space of the quantum scalar field $\mathcal{H}_{Field}$ and the Hilbert space of the detector's inner degrees of freedom $\mathcal{H}_D$,

$$\mathcal{H} = \mathcal{H}_D \otimes \mathcal{H}_{Field}. \tag{5.1.1}$$

A common base which we can expand the detector's Hilbert space is given by the well defined energy states $|\omega_k\rangle$, $k \in \mathbb{N}$. Furthermore, we specify the initial state of the field as $|\psi_0\rangle$.

We work in the so-called interaction picture, where we have that evolution of both the quantum field and the detector's inner degrees of freedom, is absorbed by the operators. The state of the system will evolve with an interaction Hamiltonian which is describing the interaction between the field and the detector. Denoting the quantum field as φ, a general interaction Hamiltonian is given by

$$H_{int}(\tau) = g\chi(\tau)m_{\mu\nu\ldots}(\tau)F[\varphi]^{\mu\nu\cdots}(\tau), \tag{5.1.2}$$

where τ is the proper time of the detector; g is the coupling constant which modules the interaction intensity; χ specifies how the interaction is turned on and off, which allows a coupling that varies with the time; $m_{\mu\nu\ldots}(\tau)$ is the detector's moment (monopole, dipole,...); and $F[\varphi]^{\mu\nu\cdots}(\tau)$ is a functional of the field. In this work we consider a massless scalar field φ. Depending on the physical situation we can add a spatial profile p to the detector in order to regularize the divergences obtained from the point-like treatment of the detector [41, 42]. If the detector is coupled to a components of a non-scalar field, as the electromagnetic field, we will have multipolar detectors. Those multipolar detectors can be coupled to directional derivatives of a scalar field [43, 44]. For instance, in [45] bring out the following examples: If we have a scalar real field with a spatial profile p, the most common coupling is the linear coupling

$$H_{int}(\tau) = g\chi(\tau)m(\tau)\int_{\mathbb{R}^n} p(\mathbf{x}(\tau), \mathbf{y})\varphi(\mathbf{y})d\mathbf{y}; \tag{5.1.3}$$

if we work with a complex field φ defined as real linear combination of fields $\varphi = 1/\sqrt{2}(\varphi_1 + i\varphi_2)$ we must work with a quadratic coupling in order to preserve certain symmetries

$$H_{int}(\tau) = g\chi(\tau)m(\tau)\int_{\mathbb{R}^n} p(\mathbf{x}(\tau), \mathbf{y})\varphi^\dagger(\mathbf{y})\varphi(\mathbf{y})d\mathbf{y}. \tag{5.1.4}$$

It can be viewed more clearly if the field is a quantized spinor field ψ, where the simplest self-adjoint Lorentz scalar is $\bar{\psi}\psi = \psi^\dagger\gamma^0\psi$, the interaction is given by

$$H_{int}(\tau) = g\chi(\tau)m(\tau)\int_{\mathbb{R}^n} p(\mathbf{x}(\tau), \mathbf{y})\bar{\psi}(\mathbf{y})\psi(\mathbf{y}), \tag{5.1.5}$$

this form is ensures that the detector can only pair create or annihilate fermion antifermion pairs. Another interaction is given by $H_{int} \sim \varphi^2$.

Along to this work, in order to study some properties of the quantum vacuum and its capability to enhance the entanglement, we will extract the majority of the features of the Unruh-DeWitt model, which is a point-like detector with two energy eigenstates with the monopole matrix elements between these states different from zero. The interaction Hamiltonian in this model is given by

$$H_{int} = g\chi(\tau)m(\tau)\varphi(x^\mu(\tau)), \tag{5.1.6}$$

where g is defined as before and m is the detector's monopole moment operator. We assume the trajectory $x^\mu(\tau)$ to be smooth. Furthermore, we specify the initial state of the field as $|\psi_0\rangle$ and the detector initially prepared in the state $|\omega_i\rangle$.

As is named in [12] if we prepare the system choosing the initial state of the field to be the vacuum state, there are some physical situations where a detector may register particles in this 'vacuum state'. For further purposes in this text we examine those particle detections for finite observational times. As we will show later, although an inertial detector switched on for a long time will not perceive any particle, during finite observational times, it will register an excitation; it due to the uncertainty principle. We study the inertial Unruh-DeWitt detector immersed in a thermal bath too and with this we are able to generalize the situation going to the accelerated UD detector and stablish the Unruh Effect. Since we have a thermal nature in this result, the finite-observational-times study will bring us some aspects of the non-equilibrium regime.

A natural extension of the accelerated systems is such that the detector is in a gravitational field. That is explored by Parker in [19] showing the particle creation in an expanding universe. Before that result, there were some discussions of the possibility of similar particle production due to space-time curvature. This might enable us some feature of the close relation between the gravitational and electromagnetic interaction. At the same time, those studies were extracting out a thermodynamic parallel with the fundamental discovery of Hawking radiation in black holes [14].

Finally, since the presence of boundaries will affect the vacuum behaviour, we will study the boundary effects of the presence one and two infinite reflecting planes and the survivor of the thermal nature in this phenomena.

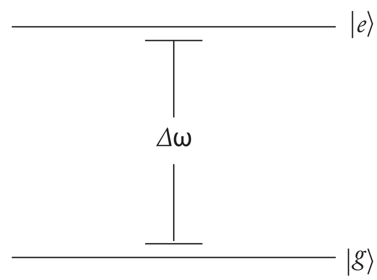

Figure 5.1: Schematic diagram of the degrees of freedom of a Unruh-DeWitt detector.

In order to study the detection of particles and radiative process of the UD detector, we must stablish the amplitude for a general transition of states. Considering one UD detector, the overall Hilbert space is defined by (5.1.1), such that the Hamiltonian H of the system with respect to the coordinate time t is

$$H = H_F + H_D + H_{int}, \tag{5.1.7}$$

where H_F is the Hamiltonian of the massless real scalar field, H_D is the detector's Hamiltonian and H_{int} us the interaction Hamiltonian, respectively defined by

$$H_F = \frac{1}{2}\int d^3x \left[(\dot{\varphi}(x))^2 + (\nabla\varphi(x))^2\right], \tag{5.1.8}$$

$$H_D = \left[(E_i + \Delta\omega)\left|\omega_f\right\rangle\left\langle\omega_f\right| + E_i\left|\omega_i\right\rangle\left\langle\omega_i\right|\right], \tag{5.1.9}$$

$$H_{int}(\tau) = g\left(\left|\omega_f\right\rangle\left\langle\omega_i\right| e^{i\Delta\omega\tau} + \left|\omega_i\right\rangle\left\langle\omega_f\right| e^{-i\Delta\omega\tau}\right)\varphi(x^\mu(\tau)). \tag{5.1.10}$$

As is mentioned above g is the coupling constant of the detector and $\varphi(x^\mu(\tau))$ is the field at the point of the detector. We define the monopole matrix as

$$m(\tau) := \left|\omega_f\right\rangle\left\langle\omega_i\right| e^{i\Delta\omega\tau} + \left|\omega_i\right\rangle\left\langle\omega_f\right| e^{-i\Delta\omega\tau}. \tag{5.1.11}$$

The states $\left|\omega_i\right\rangle$ and $\left|\omega_f\right\rangle$ represent the initial and final state, with energy E_i and $E_i + \Delta\omega$, respectively. The proper time of the detector is τ.

The time evolution operator is given by

$$U = T\exp\left[-ig\int d\tau m(\tau)\varphi(x^\mu(\tau))\right], \tag{5.1.12}$$

where T is the time ordering operator. The detector is moving along the world line $x^\mu(\tau)$ in a four-dimensional Minkowski space.

We suppose the field φ is in its Minkowski vacuum state $\left|0_M\right\rangle$ and the detector is in its ground state $\left|\omega_i\right\rangle$. We work with a weak coupling between the field and the detector, then the amplitude for a general transition is given by first order perturbation theory

$$A_{\left|\omega_i;0_M\right\rangle\to\left|\omega_f;\varphi_f\right\rangle} = ig\left\langle\omega_f;\varphi_f\right|\left[\int_{\tau_0}^{\tau_f} d\tau m(\tau)\varphi(x^\mu(\tau))\right]\left|\omega_i;0_M\right\rangle, \tag{5.1.13}$$

being the integral's interval interpreted as the measurement time that initiates at τ_0 and ends at τ_f. Using the equation for the time evolution for $m(\tau(t))$,

$$m(\tau) = e^{iH_0\tau} m(0) e^{-iH_0\tau}, \tag{5.1.14}$$

where $H_0 \left|\omega_k\right\rangle = E_k \left|\omega_k\right\rangle$. Therefore the transition probability to all possible detector and field states in first-order approximation is given by

$$\Gamma_{|\omega'\rangle \to |\omega\rangle}(\Delta\omega, \tau_0, \tau_f) = g^2 \sum_{\omega} \left[|m_{\omega\omega'}|^2 F(\Delta\omega, \tau_0, \tau_f) \right], \tag{5.1.15}$$

where $\Delta\omega = \omega - \omega'$, the matrix elements are given by

$$m_{\omega\omega'} = \left\langle \omega \right| m(0) \left| \omega' \right\rangle, \tag{5.1.16}$$

and

$$F(\Delta\omega, \tau_0, \tau_f) = \int_{\tau_0}^{\tau_f} d\tau \int_{\tau_0}^{\tau_f} d\tau' e^{-i\Delta\omega(\tau-\tau')} G^+(x(\tau), x(\tau')) \tag{5.1.17}$$

is the detector *response function* which is independent of the details of the detector, and is determined by the positive frequency Wightman Green function

$$G^+ \left(x(\tau), x(\tau')\right) = \left\langle 0_M \right| \varphi\left(x^\mu(\tau)\right) \varphi\left(x^\mu(\tau')\right) \left|0_M\right\rangle \tag{5.1.18}$$

that represents the bath of particles that the detector effectively experiences as a result of its motion [12]. The matrix elements are the *selectivity* of the detector that depends on the internal structure of the detector itself. From the integral representation (3.8.46) and with an appropriated contour, the positive frequency Wightman function for a massless scalar field, where the world lines are parametrized by the proper time, is given by

$$D^+ \left(x(\tau), x(\tau')\right) = -\frac{1}{4\pi^2} \frac{1}{(t(\tau) - t(\tau') - i\epsilon)^2 - |\mathbf{x}(\tau) - \mathbf{x}(\tau')|^2}. \tag{5.1.19}$$

5.2 Inertial detectors

In this section we evaluate, for a finite time interval $[\tau_0, \tau_f]$, the response function of the inertial UD detector interacting with the scalar field in its vacuum state $|0_M\rangle$. Multiplying this quantity by the selectivity of the detector we obtain the probability of the transition that starts in τ_0 and ends in τ_f.

5.2.1 Finite-time response function of a free inertial detector

The trajectory for an inertial detector is given by

$$\mathbf{x}(\tau) = \mathbf{x}_0 + \mathbf{v}t(\tau) = \mathbf{x}_0 + \mathbf{v}\tau(1 - v^2)^{-1/2}, \tag{5.2.1}$$

where $\mathbf{x}_0$ and $\mathbf{v}$ are constants and $|\mathbf{v}| < 1$.

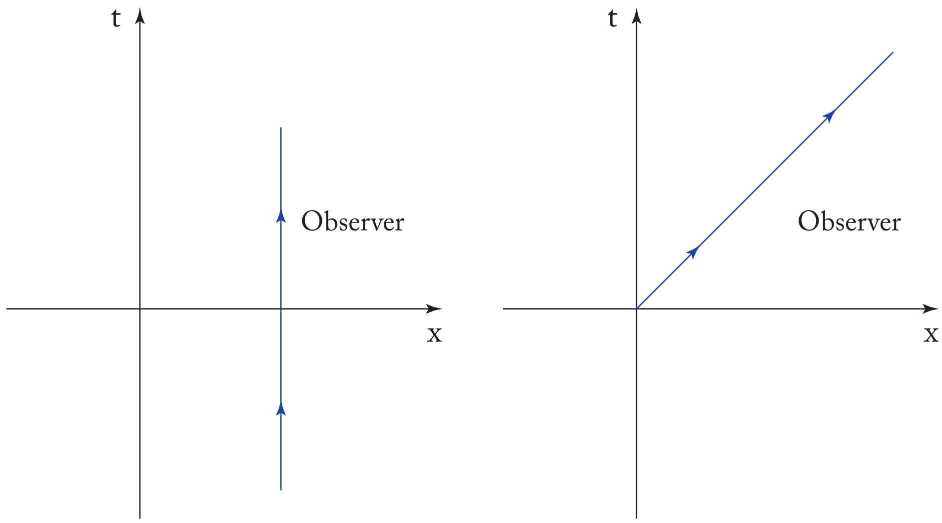

Figure 5.2: Trajectories of an inertial observer.

The Wightman function in this case yields

$$D^{+}_{iner}\left(x(\tau), x(\tau')\right) = -\frac{1}{4\pi^2}\frac{1}{(\tau-\tau'-i\epsilon)^2}, \tag{5.2.2}$$

where the infinitesimal quantity ϵ is introduced in order to specify correctly the singularities of the Wightman function, and the factor $(1-v^2)^{-1/2}$ is absorbed in the $i\epsilon$ term. Then, changing variables to $\psi = \tau_1 - \tau_1', \quad \eta = \tau_1 + \tau_1'$ and being $\Delta t = \tau_f - \tau_0$, the response function reads,

$$F_{iner}(\Delta\omega, \Delta t) = -\frac{1}{4\pi^2}\int_{-\Delta t}^{\Delta t} d\psi(\Delta t - |\psi|)e^{-i\Delta\omega\psi}\frac{1}{(\psi-i\epsilon)^2}. \tag{5.2.3}$$

Thus

$$F_{iner}(\Delta\omega, \Delta t) = F_{iner1}(\Delta\omega, \Delta t) + F_{iner2}(\Delta\omega, \Delta t), \tag{5.2.4}$$

where

$$F_{iner1}(\Delta\omega, \Delta t) = -\frac{\Delta t}{4\pi^2}\int_{-\Delta t}^{\Delta t} d\psi\frac{e^{-i\Delta\omega\psi}}{(\psi-i\epsilon)^2} \tag{5.2.5}$$

and

$$F_{iner2}(\Delta\omega, \Delta t) = \frac{1}{4\pi^2}\int_{-\Delta t}^{\Delta t} d\psi\frac{|\psi|e^{-i\Delta\omega\psi}}{(\psi-i\epsilon)^2}. \tag{5.2.6}$$

The before integrals can be performed using complex variable methods. We use the contours of the Figure 5.3. We divide the integral in a part covering all space and the other one cover the information of the before and after measurement time. Using the Cauchy theorem and taking the limit $\epsilon \to 0$, we have

$$\begin{aligned}\int_{-\Delta t}^{\Delta t} d\psi\frac{e^{-i\Delta\omega\psi}}{(\psi-i\epsilon)^2} &= 2\pi i \mathrm{Res}\left[\frac{e^{-i\Delta\omega\psi}}{(\psi-i\epsilon)^2}; i\epsilon\right]\Theta(-\Delta\omega) - \int_{\mathbb{R}/[\tau_0,\tau_f]} d\psi\frac{e^{-i\Delta\omega\psi}}{(\psi-i\epsilon)^2} \\ &= 2\pi\Delta\omega\Theta(-\Delta\omega) - 2\int_{\Delta t}^{\infty} d\psi\frac{\cos(\Delta\omega\psi)}{\psi^2}. \end{aligned} \tag{5.2.7}$$

After some algebraic steeps, the expression (5.2.5) yields

$$F_{iner1}(\Delta\omega, \Delta t) = \frac{\Delta t}{2\pi}\left[-\Delta\omega\Theta(-\Delta\omega) + \frac{\cos(\Delta\omega\Delta t)}{\pi\Delta t} + \frac{|\Delta\omega|}{\pi}\left(\mathrm{Si}(|\Delta\omega|\Delta t) - \frac{\pi}{2}\right)\right], \tag{5.2.8}$$

where $\mathrm{Si}(z)$ is defined by

$$\mathrm{Si}(z) = \int_0^z \frac{\sin t}{t}dt. \tag{5.2.9}$$

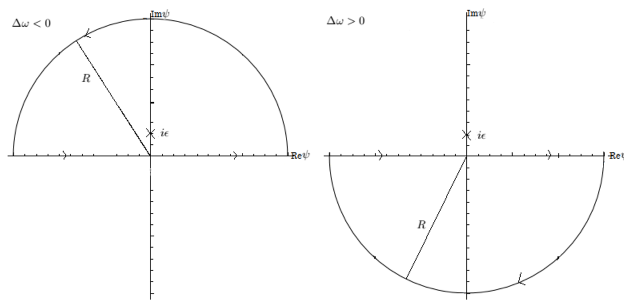

Figure 5.3: Contour used for perform the integral of $F_{iner1}(\Delta\omega, \Delta t)$.

The expression (5.2.6) yields [46]

$$F_{iner2}(\Delta\omega, \Delta t) = \frac{1}{2\pi^2}\left[-\gamma + \mathrm{Ci}(|\Delta\omega|\Delta t) - \log(|\Delta\omega|\epsilon) - 1\right] \tag{5.2.10}$$

where $\mathrm{Ci}(z)$ is

$$\mathrm{Ci}(z) = \gamma + \log z + \int_0^z \frac{\cos t - 1}{t}dt \tag{5.2.11}$$

and γ is the Euler constant. Then the response function $F_{iner}(\Delta\omega, \Delta t)$ finally reads

$$\begin{aligned} F_{iner}(\Delta\omega, \Delta t) &= \frac{1}{2\pi^2}\left\{|\Delta\omega|\Delta t\left[\pi\Theta(-\Delta\omega) + \mathrm{Si}(|\Delta\omega|\Delta t) - \frac{\pi}{2}\right]\right. \\ &+ \left.\log\left(\frac{\Delta t}{\epsilon}\right) + \cos(\Delta\omega\Delta t) - 1 + \int_0^{\Delta t} d\psi \frac{\cos(\Delta\omega\psi) - 1}{\psi}\right\}. \end{aligned} \tag{5.2.12}$$

In the last expression we can observe two divergences: one is given by $\log\Delta t$ as $\Delta t \to 0^+$ and the other one is given by $\log\epsilon$. The $\log\Delta t$ divergence is expected to occur. This problem can be solved defining a renormalized response function $F_{iner}^{(ren)}(\Delta\omega, \Delta t)$

$$F_{iner}^{(ren)}(\Delta\omega, \Delta t) = F_{iner}(\Delta\omega, \Delta t) - \frac{1}{2\pi^2}\log\left(\frac{\Delta t}{\epsilon}\right). \tag{5.2.13}$$

Another form to solve the problem of the divergence is defining the rate

$$R(\Delta\omega, \Delta t) = \frac{dF(\Delta\omega, \Delta t)}{d(\Delta t)} \tag{5.2.14}$$

which multiplied by the selectivity gives the probability of transition per unity proper time. The inverse of this quantity is the mean life of the state. That define an important quantity in this text which we will discuss later.

In this particular case the transition rate is given by

$$R_{iner}(\Delta\omega, \Delta t) = \frac{dF_{iner}(\Delta\omega, \Delta t)}{d(\Delta t)} = \frac{1}{2\pi}\left[-\Delta\omega\Theta(-\Delta\omega) + \frac{\cos(\Delta\omega\Delta t)}{\pi\Delta t} + \frac{|\Delta\omega|}{\pi}\left(\mathrm{Si}(|\Delta\omega|\Delta t) - \frac{\pi}{2}\right)\right]. \quad (5.2.15)$$

The term $\Theta(-\Delta\omega)$ is a spontaneous emission contribution. The other terms are absorption and emission terms induced by the vacuum fluctuations. The behaviour of this transition rate is depicted in the Figure 5.6. In the limit $\Delta t \to \infty$, we obtain

$$\lim_{\Delta t\to\infty} R_{iner}(\Delta\omega, \Delta t) = \frac{|\Delta\omega|}{2\pi}\Theta(-\Delta\omega) \quad (5.2.16)$$

that is the result reported in [40]. This expression shows that for a long observational times, we obtain the classical result where an inertial detector in the vacuum will not perceive any particle. On the other hand, if a detector is in its excited state, it will radiate due to the vacuum fluctuations. Furthermore, for a finite time interval, the final state of the system is a state of N quanta of the field and the detector in its excited state.

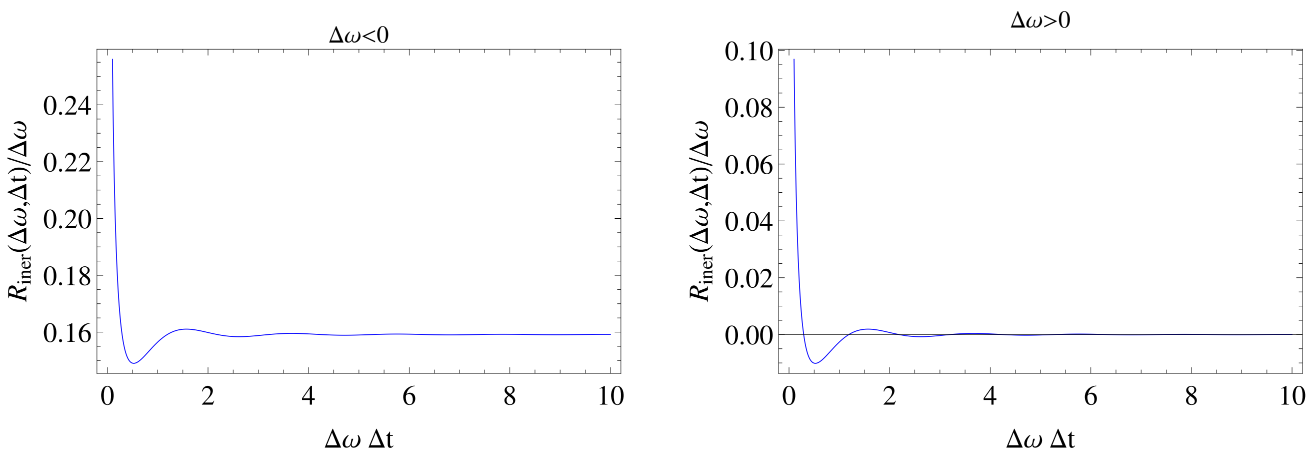

Figure 5.4: Behaviour of the rate $R_{iner}(\Delta\omega, \Delta t)$ in the time for a radiative process (left) and for a detection (right).

5.2.2 Finite-time response function of an inertial detector in a thermal bath

In this case the thermal Green function can be written as an infinite imaginary-time image sum of the corresponding zero-temperature Green function

$$D^{(1)}_{beta}(x(\tau), x(\tau')) = -\frac{1}{4\pi^2}\sum_{k=-\infty}^{\infty}(\tau - \tau' - i\epsilon + i\beta k)^{-2}. \quad (5.2.17)$$

Using the same coordinate transformation as before, the response function in this case is

$$F_\beta = -\frac{1}{4\pi^2}\sum_{k=-\infty}^{\infty}\int_{-\Delta t}^{\Delta t} d\psi(\Delta t - |\psi|)e^{-i\Delta\omega\psi}\frac{1}{(\psi - i\epsilon + i\beta k)^2}. \quad (5.2.18)$$

Then

$$F_{\beta}(\Delta\omega,\Delta t) = F_{\beta 1}(\Delta\omega,\Delta t) + F_{\beta 2}(\Delta\omega,\Delta t), \tag{5.2.19}$$

where

$$F_{\beta 1}(\Delta\omega,\Delta t) = -\frac{\Delta t}{4\pi^2}\sum_{k=-\infty}^{\infty}\int_{-\Delta t}^{\Delta t} d\psi \frac{e^{-i\Delta\omega\psi}}{(\psi - i\epsilon + i\beta k)^2} \tag{5.2.20}$$

and

$$F_{\beta 2}(\Delta\omega,\Delta t) = \frac{1}{4\pi^2}\sum_{k=-\infty}^{\infty}\int_{-\Delta t}^{\Delta t} d\psi \frac{|\psi| e^{-i\Delta\omega\psi}}{(\psi - i\epsilon + i\beta k)^2}. \tag{5.2.21}$$

For (5.2.20) the term k gives exactly (5.2.5), then we have

$$F_{\beta 1}(\Delta\omega,\Delta t) = F_{iner1}(\Delta\omega,\Delta t) + F_{\beta s}(\Delta\omega,\Delta t) \tag{5.2.22}$$

where

$$F_{\beta s}(\Delta\omega,\Delta t) = -\frac{\Delta t}{4\pi^2}\sum_{k'=-\infty}^{\infty}\int_{-\Delta t}^{\Delta t} d\psi \frac{e^{-i\Delta\omega\psi}}{(\psi - i\epsilon + i\beta k)^2} \tag{5.2.23}$$

and the $'$ in the sum means that the term $k = 0$ is to be excluded. In the evaluation of the last expression, the limit $\epsilon \to 0$ can be taken directly. Using the same procedures as in the before section, we obtain [46]

$$F_{\beta s}(\Delta\omega,\Delta t) = \frac{1}{2\pi}\frac{|\Delta\omega|\Delta t}{e^{\beta\Delta\omega} - 1} + \frac{\Delta t}{2\pi^2}\int_{\Delta t}^{\infty} d\psi \cos(\Delta\omega\psi)\left(\frac{(\pi/\beta)^2}{\sinh^2(\psi(\pi/\beta)^2)} - \frac{1}{\psi^2}\right). \tag{5.2.24}$$

For the evaluation of (5.2.21), the terms $k \neq 0$ and $k = 0$ may be dealt separately. The term $k = 0$ is similar to (5.2.6). For the term $k \neq 0$ we can take the limit $\epsilon \to 0$ directly; after some manipulations we get

$$F_{\beta 2}(\Delta\omega,\Delta t) = \frac{1}{2\pi^2}\left[-\gamma + \mathrm{Ci}(|\Delta\omega|\Delta t) - \log(|\Delta\omega|\epsilon) - 1\right] + \mathrm{C}(\Delta\omega,\Delta t), \tag{5.2.25}$$

where

$$\mathrm{C}(\Delta\omega,\Delta t) = \frac{1}{2\pi^2}\int_{0}^{\Delta t} d\psi\psi \cos(\Delta\omega\psi)\left(\frac{(\pi/\beta)^2}{\sinh^2(\psi(\pi/\beta)^2)} - \frac{1}{\psi^2}\right). \tag{5.2.26}$$

The transition rate is given by

$$\begin{aligned} R_{\beta}(\Delta\omega,\Delta t) \quad &= \quad \frac{|\Delta\omega|}{2\pi}\left\{\Theta(-\Delta\omega) + \frac{1}{e^{\beta|\Delta\omega|} - 1} + \frac{\cos(\Delta\omega\Delta t)}{\pi\Delta\omega\Delta t} + \frac{\mathrm{Si}\Delta\omega\Delta t}{\pi} - \frac{1}{2}\right\} \\ &+ \quad \frac{1}{2\pi^2}\int_{\Delta t}^{\infty} d\psi \cos(\Delta\omega\psi)\left(\frac{(\pi/\beta)^2}{\sinh^2(\psi(\pi/\beta)^2)} - \frac{1}{\psi^2}\right). \end{aligned} \tag{5.2.27}$$

Taking the limit $\Delta t \to \infty$ we have

$$\lim_{\Delta t\to\infty} R_{\beta}(\Delta\omega,\Delta t) = \frac{|\Delta\omega|}{2\pi}\left\{\Theta(-\Delta\omega)\left[1 + \frac{1}{e^{\beta|\Delta\omega|} - 1}\right] + \Theta(\Delta\omega)\frac{1}{e^{\beta\Delta\omega} - 1}\right\}. \tag{5.2.28}$$

As is expected the appearance of the Planck factor show that for a long observational times, the detector immersed in a thermal bath will reach the equilibrium with a temperature β^{-1}.

5.3 Accelerated detectors

5.3.1 Finite-time response function of an accelerated detector

In this case the for response function $F_\alpha(\Delta\omega, \Delta t)$, we assume that the detector accelerates uniformly with acceleration α^{-1} and is moving along a trajectory in the (t, z) plane, such that it is describing a hyperbola where the parameter is its proper time [12],

$$x = 0 \qquad y = 0 \qquad z(\tau) = \left(t(\tau)^2 + \alpha^2\right)^{1/2}, \tag{5.3.1}$$

with

$$t(\tau) = \alpha \sinh\left(\frac{\tau}{\alpha}\right). \tag{5.3.2}$$

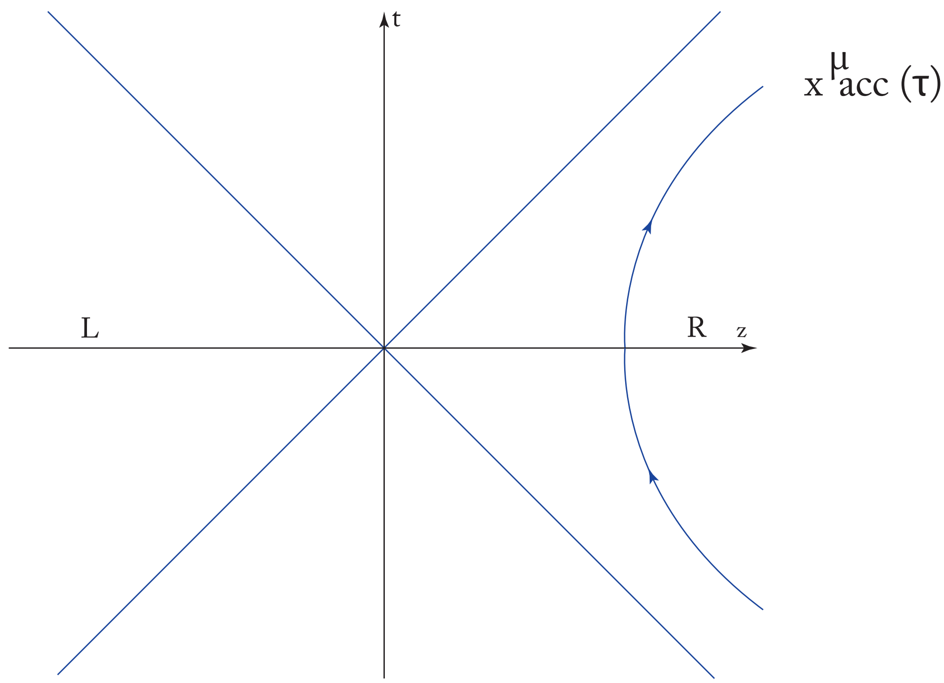

Figure 5.5: Trajectories of an accelerated observer.

Then, the Wightman function $D_\alpha^+\left(x(\tau), x(\tau')\right)$ is

$$D_\alpha^+\left(x(\tau), x(\tau')\right) = -\frac{1}{16\pi^2\alpha^2 \sinh^2\left(\frac{\tau-\tau'}{2\alpha} - \frac{i\epsilon}{\alpha}\right)}. \tag{5.3.3}$$

Using known series identities [47] and since $D_\alpha^+\left(x(\tau), x(\tau')\right) = D_\alpha^+(\tau - \tau')$ we can rewrite (7.2.1) as

$$D_\alpha^+(\tau - \tau') = -\frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\left((\tau - \tau') - 2i\epsilon + 2\pi i\alpha n\right)^{-2}. \tag{5.3.4}$$

Changing variables to

$$\psi = \tau - \tau' \qquad \eta = \tau + \tau', \tag{5.3.5}$$

we have, being $\Delta t = \tau_f - \tau_0$,

$$F_\alpha(\Delta\omega, \Delta t) = -\frac{1}{2}\int_{-\Delta t}^{\Delta t} d\psi\,(-2|\psi|+2\Delta t)\, e^{-i\Delta\omega\psi} D_\alpha^+(\psi). \tag{5.3.6}$$

The resulting expression is [46]

$$\begin{aligned} F_\alpha(\Delta\omega, \Delta t) &= \frac{\Delta t}{2\pi^2}\left\{\pi|\Delta\omega|\Theta(-\Delta\omega) + |\Delta\omega|\Big(\mathrm{Si}\Delta\omega\Delta t - \frac{\pi}{2}\Big) + \frac{\pi|\Delta\omega|}{e^{2\pi\alpha|\Delta\omega|}-1}\right. \\ &+ \left.\int_{\Delta t}^{\infty} d\psi \cos(\Delta\omega\psi)\left(\frac{1/(2\alpha)^2}{\sinh^2\psi/(2\alpha)} - \frac{1}{\psi^2}\right)\right\} + \frac{1}{2\pi^2}\left\{\cos(\Delta\omega\Delta t) + \log\left(\frac{\Delta t}{2\pi\epsilon}\right) - 1\right. \\ &+ \left.\int_0^{\Delta t} d\psi \frac{\cos(\Delta\omega\psi)-1}{\psi} + \int_0^{\Delta t} d\psi\psi\cos\Delta\omega\psi\left(\frac{1/(2\alpha)^2}{\sinh^2\psi/(2\alpha)} - \frac{1}{\psi^2}\right)\right\}. \end{aligned} \tag{5.3.7}$$

See the Appendix (B.2) for the explicit calculus for a finite interval.

5.3.2 Thermal nature of the Unruh effect

We are more interested in the rate

$$R_\alpha(\Delta\omega, \Delta t) = \frac{dF_\alpha(\Delta\omega, \Delta t)}{d(\Delta t)} \tag{5.3.8}$$

which is related to the mean life of states. For this particular case, the transition rate yields

$$\begin{aligned} R_\alpha(\Delta\omega, \Delta t) &= \frac{|\Delta\omega|}{2\pi}\left\{\Theta(-\Delta\omega) + \frac{1}{e^{2\pi\alpha|\Delta\omega|}-1} + \frac{\cos(\Delta\omega\Delta t)}{\pi\Delta\omega\Delta t} + \frac{\mathrm{Si}\Delta\omega\Delta t}{\pi} - \frac{1}{2}\right\} \\ &+ \frac{1}{2\pi^2}\int_{\Delta t}^{\infty} d\psi \cos(\Delta\omega\psi)\left(\frac{1/(2\alpha)^2}{\sinh^2\psi/(2\alpha)} - \frac{1}{\psi^2}\right). \end{aligned} \tag{5.3.9}$$

From the expression (B.2.3) we obtain that for large time intervals we have the following expression

$$\lim_{\Delta t\to\infty} R_\alpha(\Delta\omega, \Delta t) = \frac{|\Delta\omega|}{2\pi}\left\{\Theta(-\Delta\omega)\left[1 + \frac{1}{e^{2\pi\alpha|\Delta\omega|}-1}\right] + \Theta(\Delta\omega)\frac{1}{e^{2\pi\alpha\Delta\omega}-1}\right\}, \tag{5.3.10}$$

that shows us that the equilibrium between the accelerated detector and scalar field in the Minkowski vacuum state $|0_M\rangle$ is the same as that of an inertial detector in equilibrium with a bath of thermal radiation at the temperature $\beta^{-1} = 1/2\pi\alpha$.

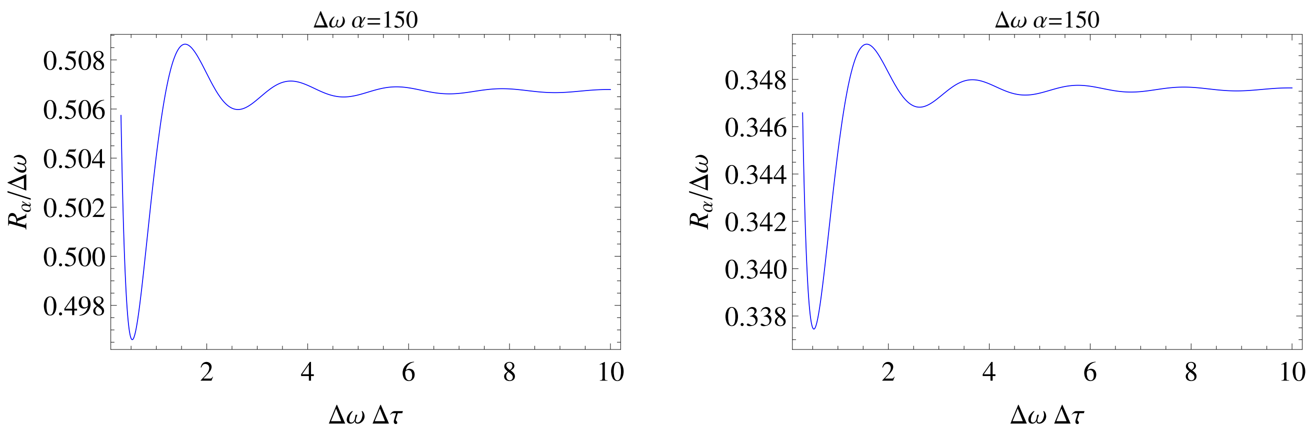

Figure 5.6: Behaviour of the rate $R_\alpha(\Delta\omega, \Delta t)$ in the time for a radiative process (left) and for a detection (right).

5.3.3 Finite-time response function of an accelerated detector in presence of an infinite reflecting plane

Consider the simple case of an infinite plane in unbounded four-dimensional Minkowski space and a massless scalar field constrained to vanish at the plane's surface $x_3 = 0$, i.e., Dirichlet boundary conditions:

$$\varphi(x_3 = 0) = 0. \tag{5.3.11}$$

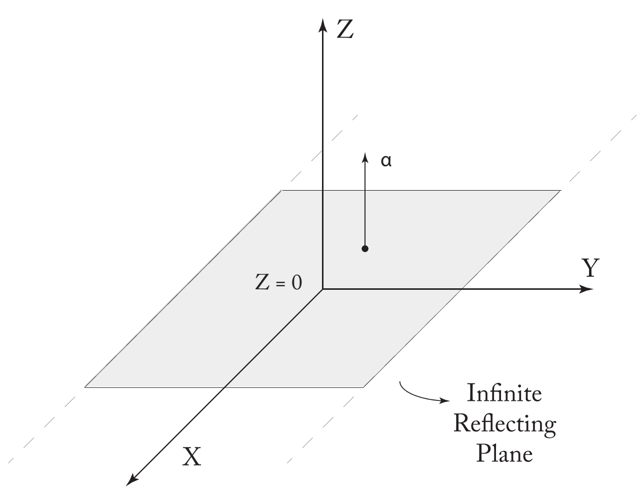

Figure 5.7: Configuration of the accelerated detector in presence of an infinite reflecting plane.

The positive frequency Wightman function will no longer be given by (7.1.14). Its form may be found using the traditional method of images and reads

$$
\begin{aligned}
D_\alpha^{+(b)}(x(\tau), x_j(\tau_j')) &= -\frac{1}{4\pi^2}\{[(t(\tau)-t(\tau)-i\epsilon)^2-(x(\tau)-x(\tau'))^2 \\
&- (y(\tau)-y(\tau'))^2-(z(\tau)-z(\tau'))^2]^{-1} \\
&- [(t(\tau)-t(\tau')-i\epsilon)^2-(x(\tau)-x(\tau'))^2 \\
&- (y(\tau)-y(\tau'))^2-(z(\tau)+z(\tau'))^2]^{-1}\}
\end{aligned} \tag{5.3.12}
$$

From (5.3.12) we can see that, the expression for $D^+_{\alpha b}(x(\tau), x(\tau'))$ has the form

$$
D_\alpha^{+(b)}[x(\tau), x(\tau')] = -\frac{1}{16\pi^2\alpha^2}\left[\frac{1}{\sinh^2\left(\frac{\tau-\tau'}{2\alpha}-\frac{i\epsilon}{\alpha}\right)}+\frac{1}{\cosh^2\left(\frac{\tau+\tau'}{2\alpha}-\frac{i\epsilon}{\alpha}\right)}\right]. \tag{5.3.13}
$$

Such that,

$$
F_\alpha^{(b)}(\Delta\omega, \Delta t) = F_\alpha(\Delta\omega, \Delta t) + F_\alpha^{(b_0)}(\Delta\omega, \Delta t), \tag{5.3.14}
$$

being

$$
F_\alpha^{(b_0)}(\Delta\omega, \Delta t) = -\frac{1}{16\pi^2\alpha^2}\int_{\tau_0}^{\tau_f} d\tau \int_{\tau_0}^{\tau_f} d\tau' e^{-i\Delta\omega(\tau-\tau')}\frac{1}{\cosh^2\left(\frac{\tau+\tau'}{2\alpha}-\frac{i\epsilon}{\alpha}\right)}. \tag{5.3.15}
$$

Using the well know expansion

$$
\sec^2\frac{\pi x}{2} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left\{\frac{1}{(2n-1-x)^2}+\frac{1}{(2n-1+x)^2}\right\}, \tag{5.3.16}
$$

the transformation (5.3.5), and using a symmetric temporal interval $\tau_f = -\tau_0 = T$, after such algebraic steeps, we finally have

$$
F_\alpha^{(b_0)}(\Delta\omega, \Delta t) = \frac{1}{2\pi^2\Delta\omega}\left\{\mathrm{Im}\left[e^{i\Delta\omega\Delta t}\sum_{n=1}^{\infty}\int_0^{\Delta t} d\eta e^{-i\Delta\omega\eta}G_{\alpha,n}^{(b)}(\eta)\right]\right\}, \tag{5.3.17}
$$

where

$$
G_{\alpha,n}^{(b)}(\eta) = \frac{1}{(\eta+2\pi i\alpha n-\pi i\alpha)^2}+\frac{1}{(\eta-2\pi i\alpha n+\pi i\alpha)^2}. \tag{5.3.18}
$$

Following the same line of the last sections, we divide the integral in order to study the thermal characteristic of response function,

$$
\int_0^{\Delta t} d\eta e^{-i\Delta\omega\eta}G_{\alpha,n}^{(b)}(\eta) = \int_0^{\infty} d\eta e^{-i\Delta\omega\eta}G_{\alpha,n}^{(b)}(\eta)| - \int_{\Delta t}^{\infty} d\eta e^{-i\Delta\omega\eta}G_{\alpha,n}^{(b)}(\eta). \tag{5.3.19}
$$

We use the method of residues to perform the integrals. These integrals have two kind of second order poles situated along the imaginary axis,

$$
z_n^{\pm} = \pm\pi i\alpha(2n-1). \tag{5.3.20}
$$

Using the contour showed in Figure (5.8), we have

$$
\sum_{n=0}^{\infty}\int_0^{\infty} d\eta e^{-i\Delta\omega\eta}G_{\alpha,n}^{(b)}(\eta) = \left\{\frac{\pi\Delta\omega e^{\pi\alpha|\Delta\omega|}}{e^{2\pi\alpha|\Delta\omega|}-1}+\zeta_\alpha^{(b)}(\Delta\omega)\right\}\Theta(-\Delta\omega)+\left\{\frac{\pi\Delta\omega e^{\pi\alpha\Delta\omega}}{e^{2\pi\alpha\Delta\omega}-1}+\zeta_\alpha^{(b)}(\Delta\omega)\right\}\Theta(\Delta\omega), \tag{5.3.21}
$$

where

$$
\begin{aligned}
\zeta_\alpha^{(b)}(\Delta\omega) &= \frac{i}{\pi\alpha}\left[\tanh^{-1}\left(e^{\pi\alpha\Delta\omega}\right)-\tanh^{-1}\left(e^{-\pi\alpha\Delta\omega}\right)\right] \\
&- i\Delta\omega\left[e^{\pi\alpha\Delta\omega}\tilde{\Phi}^{(0,1,0)}\left(e^{2\pi\alpha\Delta\omega},0,\frac{1}{2}\right)+e^{-\pi\alpha\Delta\omega}\tilde{\Phi}^{(0,1,0)}\left(e^{-2\pi\alpha\Delta\omega},0,\frac{1}{2}\right)\right],
\end{aligned} \tag{5.3.22}
$$

being $\tilde{\Phi}(z,s,a)$ is the Lerch zeta-function and $\tilde{\Phi}^{(0,1,0)}(z,s,a)$ its first derivative respect its second variable.

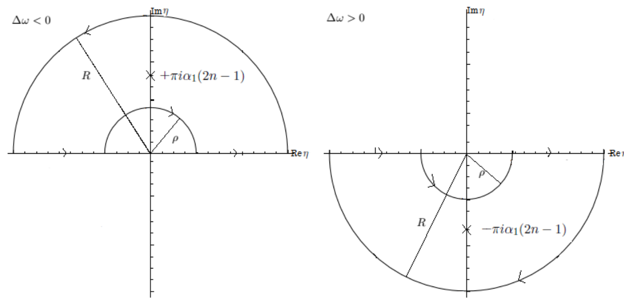

Figure 5.8: Contour used for perform the integral of $F_\alpha^{(b_0)}(\Delta\omega, \Delta t)$.

5.3.4 Finite-time response function of an accelerated detector in presence of two infinite reflecting planes

Let us generalize the above situation and place the accelerated detector between two infinite reflecting planes. The planes are situated in $z = 0$ and $z = L$. The detector has a proper acceleration in the x direction and is placed in a constant $0 < z < L$ coordinate.

The Green function may be computed as an infinite image sum

$$\begin{aligned} D_\alpha^{+(c)}(x, x') &= \frac{1}{2\pi^2} \sum_{n=-\infty}^{\infty} \left[\frac{1}{(x-x')^2 + (y-y')^2 + (z-z'-Ln)^2 - (t-t')^2} \right. \\ &- \left. \frac{1}{(x-x')^2 + (y-y')^2 + (z+z'-Ln)^2 - (t-t')^2} \right]. \end{aligned} \tag{5.3.23}$$

Considering now the acceleration of detector is orthogonal to a normal unitary vector of the planes, the Wightman function $D_\alpha^{+(c)}(x, x')$ yields

$$D_\alpha^{+(c)}(x, x') = \frac{1}{2\pi^2} \sum_{n=-\infty}^{\infty} \left[\frac{1}{-4\alpha^2 \sinh^2\left(\frac{\tau-\tau'}{2\alpha}\right) + (Ln)^2} - \frac{1}{-4\alpha^2 \sinh^2\left(\frac{\tau-\tau'}{2\alpha}\right) + (2z-Ln)^2} \right]. \tag{5.3.24}$$

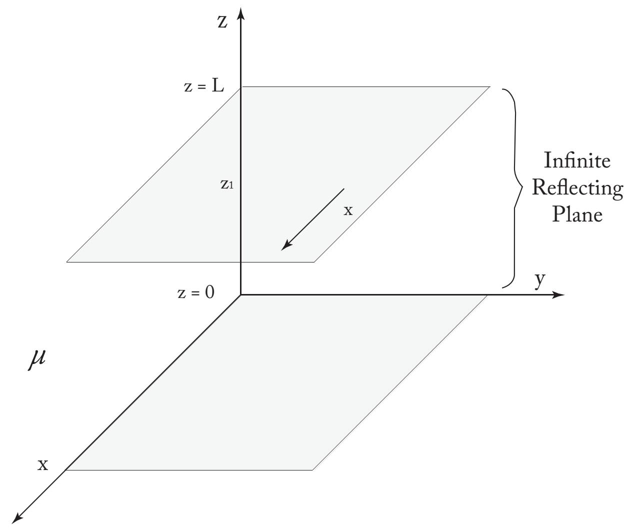

Figure 5.9: Configuration of the accelerated detector in presence of two infinite reflecting plane.

The Wightman function $D_\alpha^{+(c)}(x,x')$ has two sums with the following convergence values

$$\sum_{n=-\infty}^{\infty}\frac{1}{-4\alpha^2\sinh^2\left(\frac{\tau-\tau'}{2\alpha}\right)+(Ln)^2}=\frac{1}{4\alpha^2\sinh^2\left(\frac{\tau-\tau'}{2\alpha}\right)}-\frac{\pi}{2L\alpha\sinh\left(\frac{\tau-\tau'}{2\alpha}\right)}\cot\left(\frac{\sqrt{2}\pi\alpha}{L}\sinh\left(\frac{\tau-\tau'}{2\alpha}\right)\right) \tag{5.3.25}$$

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{1}{-4\alpha^2\sinh^2\left(\frac{\tau-\tau'}{2\alpha}\right)+(Ln-2z)^2} &= \frac{\pi}{4L\alpha\sinh\left(\frac{\tau-\tau'}{2\alpha}\right)}\left\{\cot\left(\frac{2\pi}{L}\left(z-\alpha\sinh\left(\frac{\tau-\tau'}{2\alpha}\right)\right)\right)\right.\\ &- \left.\cot\left(\frac{2\pi}{L}\left(z+\alpha\sinh\left(\frac{\tau-\tau'}{2\alpha}\right)\right)\right)\right\}\\ &- \frac{1}{-4\alpha^2\sinh^2\left(\frac{\tau-\tau'}{2\alpha}\right)+4z^2}.\end{aligned} \tag{5.3.26}$$

Going to $\psi=\tau-\tau'$ and $\eta=\tau+\tau'$ and introducing a small constant ϵ, the response functions yields

$$F_\alpha^{(c)}(\Delta\omega,\Delta t)=\frac{1}{4\pi}\left\{F_\alpha^{(c)1}(\Delta\omega,\Delta t)-F_\alpha^{(c)2}(\Delta\omega,\Delta t)-F_\alpha^{(c)3}(\Delta\omega,\Delta t)-F_\alpha^{(c)4}(\Delta\omega,\Delta t)\right\} \tag{5.3.27}$$

where the functions $F_\alpha^{(c)i}(\Delta\omega,\Delta t)$ are defined by

$$F_\alpha^{(c)1}(\Delta\omega,\Delta t)=\int_{-\Delta t}^{\Delta t}[-|\psi|+\Delta t]\frac{e^{-i\Delta\omega\psi}}{2\alpha^2\sinh^2\left(\frac{\psi}{2\alpha}-\frac{i\epsilon}{\alpha}\right)}, \tag{5.3.28}$$

$$F_\alpha^{(c)2}(\Delta\omega,\Delta t)=\frac{\pi}{L\alpha}\int_{-\Delta t}^{\Delta t}[-|\psi|+\Delta t]\frac{e^{-i\Delta\omega\psi}}{\sinh\left(\frac{\psi}{2\alpha}-\frac{i\epsilon}{\alpha}\right)}\cot\left(\frac{2\pi\alpha}{L}\sinh\left(\frac{\psi}{2\alpha}-\frac{i\epsilon}{\alpha}\right)\right), \tag{5.3.29}$$

$$F_\alpha^{(c)3}(\Delta\omega,\Delta t)=\frac{\pi}{2L\alpha}\int_{-\Delta t}^{\Delta t}[-|\psi|+\Delta t]\frac{e^{-i\Delta\omega\psi}}{\sinh\left(\frac{\psi}{2\alpha}-\frac{i\epsilon}{\alpha}\right)}(f_-(\psi)-f_+(\psi)), \tag{5.3.30}$$

$$F_\alpha^{(c)4}(\Delta\omega,\Delta t) = 2\int_{-\Delta t}^{\Delta t}[-|\psi|+\Delta t]\frac{e^{-i\Delta\omega\psi}}{4z^2 - 4\alpha^2\sinh^2\left(\frac{\psi}{2\alpha}-\frac{i\epsilon}{\alpha}\right)}, \tag{5.3.31}$$

with $f_\pm(\psi)$ defined by

$$f_\pm(\psi) = \cot\left(\frac{2\pi}{L}\left(z \pm \alpha\sinh\left(\frac{\tau-\tau'}{2\alpha}\right)\right)\right). \tag{5.3.32}$$

Note that the expression (5.3.28) is the contribution of the free accelerated detector. Following the same method presented above we divide these integrals in a infinite interval and an interval that give the information after the measurement time.

For the expression (5.3.29) we observe that integrand has poles at

$$\psi = 2\pi i\alpha k + 2i\epsilon, \quad k\in\mathbb{Z} \tag{5.3.33}$$

and

$$\psi = 2\alpha\,\mathrm{arcsinh}\left(\frac{kL}{2\alpha}\right) + 2i\epsilon, \quad k\in\mathbb{Z}. \tag{5.3.34}$$

Then the first part of the expression (5.3.29) yields

$$\frac{\pi\Delta t}{L\alpha}\left\{2\pi i\left[-\frac{4iL\alpha\Delta\omega}{\pi} - \frac{2iL\alpha\Delta\omega}{\pi}\sum_{k=-\infty}^{\infty}e^{2\pi\alpha\Delta\omega k} + \frac{4\alpha^2}{\pi}\sum_{l=-\infty}^{\infty}\frac{e^{-2i\alpha\Delta\omega\,\mathrm{arcsinh}\left(\frac{kL}{2\alpha}\right)}}{l\sqrt{(lL)^2+(2\alpha)^2}}\right].\right\} \tag{5.3.35}$$

If we work with the high accelerations regime $L \ll \alpha$ the total expression (5.3.29) reads

$$\begin{aligned} F_\alpha^{(c)2}(\Delta\omega,\Delta t) &= \frac{\pi\Delta t}{L\alpha}\left\{2\pi i\left[-\frac{4iL\alpha\Delta\omega}{\pi} - \frac{2iL\alpha\Delta\omega}{\pi}\sum_{k=-\infty}^{\infty}e^{2\pi\alpha\Delta\omega k} - \frac{8i\alpha^2}{\pi}\sum_{l=1}^{\infty}\frac{\sin(\Delta\omega Ll)}{l\left(2\alpha+\frac{(lL)^2}{4\alpha}\right)}\right].\right. \\ &+ \left. 2\int_{\Delta t}^{\infty}d\psi\frac{\cos(\Delta\omega\psi)}{\sinh\left(\frac{\psi}{2\alpha}-\frac{i\epsilon}{\alpha}\right)}\cot\left(\frac{2\pi\alpha}{L}\sinh\left(\frac{\psi}{2\alpha}-\frac{i\epsilon}{\alpha}\right)\right)\right\} \\ &- \frac{\pi}{L\alpha}\int_0^{\Delta t}d\psi\frac{\psi\cos(\Delta\omega\psi)}{\sinh\left(\frac{\psi}{2\alpha}-\frac{i\epsilon}{\alpha}\right)}\cot\left(\frac{2\pi\alpha}{L}\sinh\left(\frac{\psi}{2\alpha}-\frac{i\epsilon}{\alpha}\right)\right). \end{aligned} \tag{5.3.36}$$

For the expression (5.3.30) we identify the following poles

$$\psi = 2\pi i\alpha k + 2i\epsilon, \quad k\in\mathbb{Z} \tag{5.3.37}$$

and

$$\psi = 2\alpha\,\mathrm{arcsinh}\left(\frac{\pm kL \mp 2z}{2\alpha}\right) + 2i\epsilon, \quad k\in\mathbb{Z}. \tag{5.3.38}$$

Then the expression (5.3.30) yields

$$\begin{aligned} F_\alpha^{(c)3}(\Delta\omega,\Delta t) &= \frac{\pi\Delta t}{2L\alpha}\left\{\frac{4iL\alpha}{\pi}[f_{s_1}^{(c)}(\Delta\omega)+f_{s_2}^{(c)}(\Delta\omega)] + 2\int_{\Delta t}^{\infty}d\psi\frac{\cos(\Delta\omega\psi)}{\sinh\left(\frac{\psi}{2\alpha}-\frac{i\epsilon}{\alpha}\right)}(f_-(\psi)-f_+(\psi))\right\} \\ &- \frac{\pi}{2L\alpha}\int_0^{\Delta t}d\psi\frac{\psi\cos(\Delta\omega\psi)}{\sinh\left(\frac{\psi}{2\alpha}-\frac{i\epsilon}{\alpha}\right)}(f_-(\psi)-f_+(\psi)), \end{aligned} \tag{5.3.39}$$

where $f^{(c)}_{s_1}(\Delta\omega)$ and $f^{(c)}_{s_2}(\Delta\omega)$ are infinite sums that converge to a real-valued linear combination of Hurwitz-Lerch zeta-function, defined in the appendix B.

Finally the poles of the integrand of the expression (5.3.31) are

$$\psi = 2\pi i\alpha k \pm 2\alpha\,\text{arcsinh}\left(\frac{z}{\alpha}\right) + 2i\epsilon, \quad k \in \mathbb{Z}. \tag{5.3.40}$$

Thus as above, we obtain

$$\begin{aligned} F^{(c)4}_{\alpha}(\Delta\omega, \Delta t) \quad &= \quad 2\Delta t \left\{ \frac{-4\pi\sin(2\Delta\omega z)}{4z + \frac{2z^3}{\alpha}} \sum_{k=-\infty}^{\infty} e^{2\pi\alpha\Delta\omega k} + 2\int_{\Delta t}^{\infty} d\psi \frac{\cos(\Delta\omega\psi)}{4z^2 - 4\alpha^2\sinh^2\left(\frac{\psi}{2\alpha} - \frac{i\epsilon}{\alpha}\right)} \right\} \\ &- \quad 2\int_0^{\Delta t} d\psi \frac{\psi\cos(\Delta\omega\psi)}{4z^2 - 4\alpha^2\sinh^2\left(\frac{\psi}{2\alpha} - \frac{i\epsilon}{\alpha}\right)}, \end{aligned} \tag{5.3.41}$$

where we treat the sums as above obtaining the thermal term and for decay process, beside the thermal term we have the vacuum fluctuations contribution. We have that the oscillating modulation is inversely proportional to the distance z.

Chapter 6

Relativistic Quantum Information

Quantum entanglement may be considered as the most non-classical feature of quantum mechanics; it is one of the properties that would distinguish a quantum system from any classical counterpart. Quantum entanglement is considered as one of the key features of quantum information processing. Several sources of entangled quantum systems have been discussed in literature, for instance, in solid-state physics, quantum optics, and also atoms in cavity quantum electrodynamics [26]. Some examples of generation of entangled systems of two-level atoms interacting with a bosonic field can be found in Refs. [48, 49]. Aside from production of those entangled systems, quantum-information processing requires the presence of a strong coherent coupling between the entities of the system. Nevertheless, under realistic experimental conditions, entanglement is degraded through uncontrolled coupling to environment [50].

However quantum entanglement and quantum information processing have been widely explored in the context of the non-relativistic quantum mechanics. As we see above quantum field theory, as the convergence point of quantum mechanics and relativity, has become a more fundamental framework. A natural extension of the studies of quantum entanglement and quantum information processing can be the treatment of these phenomena with the mathematical and conceptual formalism offered by quantum field theory. This is the focus topic of relativistic quantum information. In recent years the field of relativistic quantum information has emerged as an important research topic and is generating increasing interest within the scientific community. In this framework, the mutual influence of atoms through their interaction with quantum fields is an important stimulating issue in order to analyse decoherence properties [51, 52, 53]. Some works studying quantum entanglement in different setups are given by Refs. [22, 51, 54, 55, 56, 57, 58]. In turn, for a wide set of results in this area and special issues of performing satellite experiments, we refer the reader the Ref. [59]. Many of such works demonstrate that entanglement is an observer-dependent quantity.

Placing quantum information theory in the formalism of quantum field theory in curved space-time we can explore deeper quantum aspects of black holes, cosmology and search a conceptual and experimental guide to quantum gravity. On the other hand, dealing these phenomena in this form, we can obtain technical applications in order to improve the control of entanglement in some realistic contexts in areas such as quantum communication, quantum simulation, quantum computing and quantum metrology. We review these ideas in this chapter

6.1 The Qubit

In 1982 Richard Feynman introduced the idea of the construction of a computer based on the laws of quantum mechanics. The mean idea introduced by Feynman was that only quantum simulators will extract the complete behaviour of a quantum system. The aim in that quantum computation was to take advantage of the principle of superposition that was in the foundations of quantum mechanics. The concept of the superposition has been widely discussed in diverse frameworks, exploring its philosophical implications to the foundations of the physical character of the nature. The quantum computation leaves those discussions and use the results of quantum mechanics in a pragmatic way.

The classic information is stored in bits which can take the logic value of 0 or 1, in quantum computation, the information is stored in a quantum bit or *qubit* that is a state vector of a quantum-mechanical system, e.g. atom, photon or nucleus states. The difference between the bit and the qubit is given by the fact that a bit can be 0 or 1 while a qubit can be 0 and 1 at the same time, due to the superposition principle. The properties of a qubit are governed by the Schrödinger equation. The general state of a qubit can be expressed as [26, 60]

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle \tag{6.1.1}$$

where we impose the normalization condition,

$$|c_0|^2+|c_1|^2= 1. \tag{6.1.2}$$

Furthermore, we impose the following orthogonality condition

$$\langle 0|1\rangle = 0. \tag{6.1.3}$$

We can use different properties of a system in order to stablish a qubit. For example, we can use the ground and excited state of an atom as a *distinguishable* property that serves as a base for expand the qubits. In this case, we can define the qubit as

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle \equiv c_0|g\rangle + c_1|e\rangle \tag{6.1.4}$$

being $|g\rangle$ the atomic ground state and $|e\rangle$ the excited atomic state. A collection of N qubits is called a *quantum register* of size N.

The wave function of an arbitrary system of two qubits is the combination of the possible individual qubit states

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle \tag{6.1.5}$$

where the notation $|ij\rangle$ indicates the qubit 1 is in the state i and the qubit 2 is in the state j. It can be generalized to any number of qubits. For example, for three qubits,

$$\begin{aligned} |\psi\rangle \;&=\; c_{000}|000\rangle + c_{001}|001\rangle + c_{010}|010\rangle + c_{011}|011\rangle \\ &+\; c_{100}|100\rangle + c_{101}|101\rangle + c_{110}|110\rangle + c_{111}|111\rangle. \end{aligned} \tag{6.1.6}$$

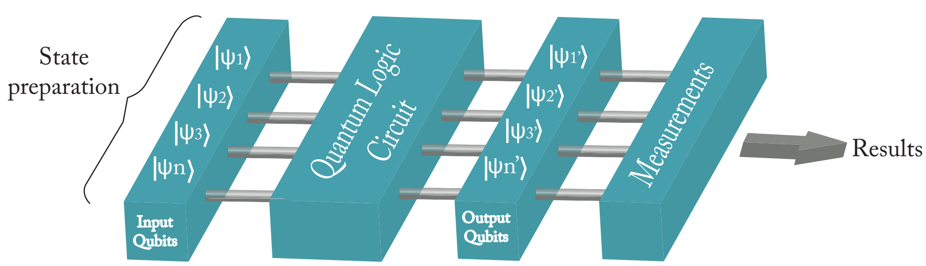

Figure 6.1: Schematic picture of a quantum processor.

The quantum information is stored in the coefficients of each ket. Those coefficients are complex numbers which its modulus varies from 0 to 1. Since for N qubits the wave function is described by 2^N coefficients, we have that the stored information increases exponentially with number of qubits. The quantum information of these coefficients can be obtained by measurements and interactions with other systems.

6.1.1 Quantum logic gates

In classic computation, the processor make operations over a set of stored bits. Depending the procedure given by a programming code, we obtain as a result a new set of bits. The processing operations are made by millions of logic binary gates as NOT and NAND. The idea for a quantum computer is basically the same, the processing tasks are due to the quantum logic gates, that conform a quantum logic circuit. The advantage consists in the fact that even we have N qubits incoming and N qubits outgoing, we have 2^N data of information corresponding to the N qubits [60, 61].

A quantum logic circuit consists in a programmed sequence of quantum logic gates. One of these is the logic gate which have one qubit incoming and only one qubit outgoing. The incoming qubit is

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle \tag{6.1.7}$$

and the outgoing bit is

$$|\psi'\rangle = c_0'|0\rangle + c_1'|1\rangle, \tag{6.1.8}$$

being the coefficients c_0' and c_1' defined as

$$\begin{pmatrix} c_0' \\ c_1' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \tag{6.1.9}$$

where the quantum logic gate is defined by this transformation matrix. This matrix must be unitary

$$\mathbf{M}\mathbf{M}^\dagger = \mathbf{I} \tag{6.1.10}$$

Therefore, the matrix representation of the NOT gate is

$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{6.1.11}$$

the $\mathbf{Z}$ gate is,

$$\mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.1.12}$$

and the Hadamard gate is [60, 61]

$$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{6.1.13}$$

Then when a qubit $|\psi\rangle$ defined in (6.1.7) enter in the $\mathbf{X}$ gate becomes

$$|\psi'\rangle = c_1|0\rangle + c_0|1\rangle. \tag{6.1.14}$$

If the qubit $|\psi\rangle$ enter in the $\mathbf{Z}$ gate will be

$$|\psi'\rangle = c_0|0\rangle - c_1|1\rangle. \tag{6.1.15}$$

Finally, if that qubit enter in the Hadamard gate we obtain

$$|\psi'\rangle = \frac{c_0 + c_1}{\sqrt{2}}|0\rangle + \frac{c_0 - c_1}{\sqrt{2}}|1\rangle. \tag{6.1.16}$$

For example, if the initial qubit is in the state $|0\rangle$ and goes to a Hadamard gate and afterword to a $\mathbf{Z}$, the resulting state will be

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \tag{6.1.17}$$

$$|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle. \tag{6.1.18}$$

Now, passing to the case where we have two incoming qubits, one of the most used gate is the controlled-NOT gate *C-NOT*, which its matrix representation is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{6.1.19}$$

The C-NOT gate has two inputs. One of these, leave one qubit without any modification and the other one apply the NOT operation to the second [60, 61]. The effect of this operation over an arbitrary two qubit state is as follows

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_{00} \\ c_{01} \\ c_{10} \\ c_{11} \end{pmatrix} = \begin{pmatrix} c_{00} \\ c_{01} \\ c_{11} \\ c_{10} \end{pmatrix}. \tag{6.1.20}$$

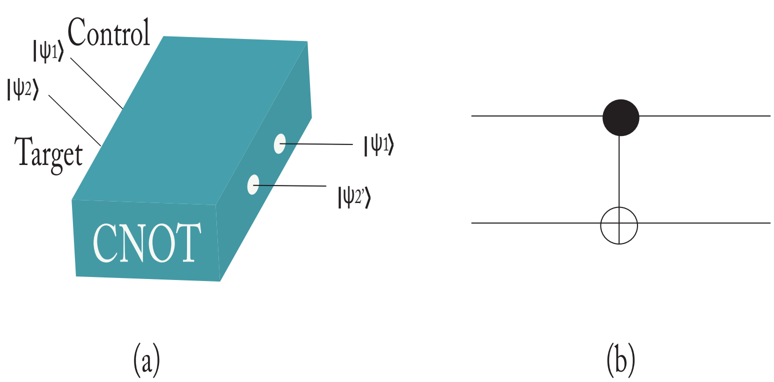

Figure 6.2: Schematic diagram of a C-NOT gate.

Therefore, if we consider a state that is initially in $|0\rangle$ and goes to a Hadamard gate, we obtain

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \tag{6.1.21}$$

$$|\psi_H\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle). \tag{6.1.22}$$

From there, goes to a control input of a C-NOT gate where in the other input there are a state $|0\rangle$, we have that the incoming state is

$$|\psi\rangle = |\psi_H\rangle|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \tag{6.1.23}$$

then the final state is given by,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{6.1.24}$$

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{6.1.25}$$

That kind of states, known as entangled states, will be subject of interest in the one of the branch of quantum information processing that is the quantum teleportation. Quantum teleportation relies heavily on the properties of entangled states.

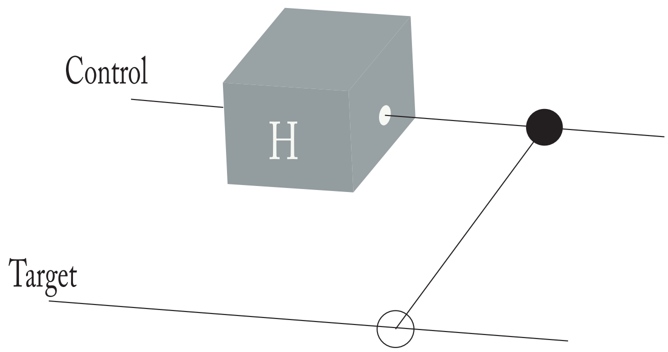

Figure 6.3: Circuit C-NOT-Hardamard.

6.2 Entanglement

An entangled state is such that its wave function cannot be factorized into a product of the wave functions of individual particles. Experimentally we can produce those entangled states by an atom decay or via high-energy lasers guided to non-linear crystals, where the photons obtained from the crystal are sent to different beam-splitter where we can measure the correlated polarization.

For instance, considering the last example, observe the Figure 6.4, suppose that the correlated emitted photons have the following property: if $D_1(0)$ fires, then $D_2(0)$ always fires, and if $D_1(1)$ fires, then $D_2(1)$ always fires. Alternatively, if $D_1(0)$ fires, then $D_2(1)$ and vice versa [61]. The above situation can be described by the following states

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0_1, 0_2\rangle \pm |1_1, 1_2\rangle) \tag{6.2.1}$$

or in the alternative case,

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0_1, 1_2\rangle \pm |1_1, 0_2\rangle). \tag{6.2.2}$$

Those states are known as Bell states and are called as the four maximally entangled two-qubit Bell states. We can obtain these states from a quantum circuit as follows (see the Figure 6.5).

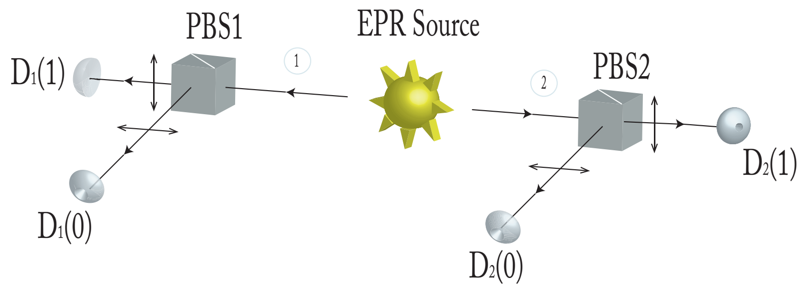

Figure 6.4: EPR states source.

Let us analyse each input, where the first arrow indicates the action of a Hadamard gate over one qubit and the second one indicates the action of a C-NOT gate [62]

- $|00\rangle$

$$|00\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$$\longrightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \beta_{00} \equiv |\Phi^+\rangle \tag{6.2.3}$$

- $|10\rangle$

$$|10\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$$\longrightarrow \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \beta_{10} \equiv |\Phi^-\rangle \tag{6.2.4}$$

- $|01\rangle$

$$|01\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$$

$$\longrightarrow \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \beta_{01} \equiv |\Psi^+\rangle \tag{6.2.5}$$

- $|11\rangle$

$$|11\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|1\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)$$

$$\longrightarrow \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \beta_{11} \equiv |\Psi^-\rangle \tag{6.2.6}$$

These states form a basis in the Hilbert space of dimension 4 and will be the principal tool in the process of the quantum teleportation.

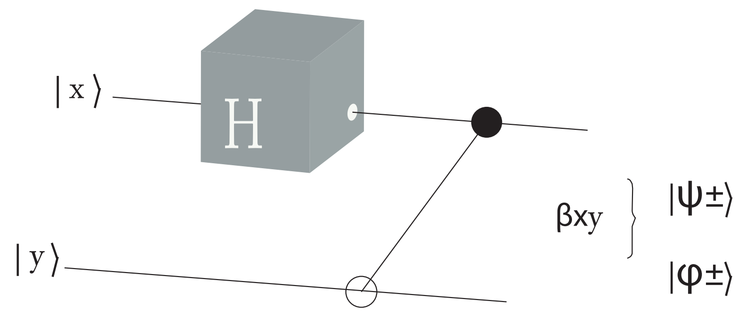

Figure 6.5: Schematic diagram for the production of Bell states.

6.3 Quantum Teleportation

The basic idea of the quantum teleportation is to transfer the quantum state of one qubit to another that is physically separated from it [61]. As we mentioned before, we can use electrons, atoms, nucleus or photons. Following with the use of the quantum circuits, we shall send the information without qubit direct interchange.

The procedure is the following: One qubit is sent to one circuit's input and its state $|\psi\rangle$ is unknown. In the other circuit's input will enter a qubit that comes from an entangled state. The other qubit of this entangled state is in the hands of a receptor. In the laboratory of the transmitter is obtained the information by the Bell state measurement of the pair of photons. This obtained result is classically communicated to a receiver, whom makes a unitary operation obtaining here the state $|\psi\rangle$.

With this in mind, let $|\psi\rangle$ the qubit to teport,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle. \tag{6.3.1}$$

The transmitter and the receptor share an entangled state $|\Phi^+\rangle$, then

$$\begin{aligned} |\psi\rangle \otimes |\Phi^+\rangle &= (\alpha|0\rangle + \beta|1\rangle)\left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right) \\ &= \frac{1}{\sqrt{2}}(\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle|00\rangle + |11\rangle) \end{aligned} \tag{6.3.2}$$

Applying C-NOT to $|\Phi^+\rangle$

$$\longrightarrow \frac{1}{\sqrt{2}}(\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle|10\rangle + |01\rangle) \tag{6.3.3}$$

Now applying Hadamard to $|\psi\rangle$

$$\begin{aligned} &\longrightarrow \frac{1}{\sqrt{2}}(\alpha\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) \\ &\quad + \beta\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)) \end{aligned} \tag{6.3.4}$$

$$\begin{aligned} &= \frac{1}{2}[|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) \\ &\quad + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)] \end{aligned} \tag{6.3.5}$$

$$= \frac{1}{2}\sum_{b_1 b_2 = 0}^{1} |b_1 b_2\rangle \left(\mathbf{X}^{b_2}\mathbf{Z}^{b_1}|\psi\rangle\right) \tag{6.3.6}$$

Thus, applying $\mathbf{X}^{b_2}\mathbf{Z}^{b_1}$ the receptor will obtain the state $|\psi\rangle$ [62].

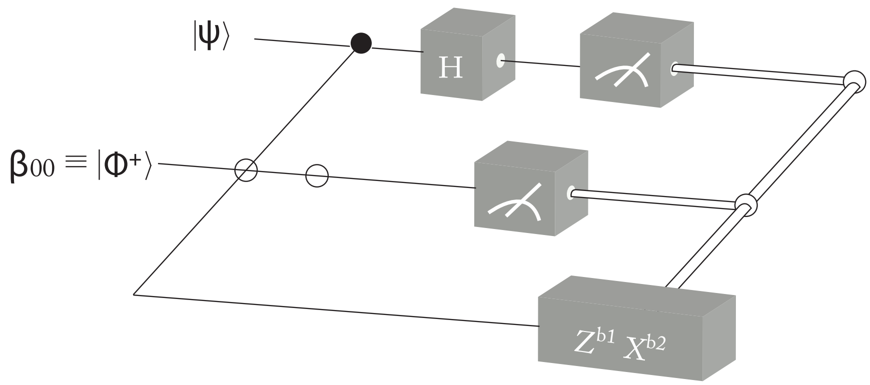

Figure 6.6: Complete quantum circuit for the teleportation process.

In order to give a view more clear, we can put labels for the 3 qubits. The qubit 1 is the qubit to be teleported, the qubits 2 and 3 compose the initial entangled Bell state. Now, supposing that we are working with $|\Psi^-\rangle$, the total wave function is given by

$$|\Psi\rangle_{123} = \frac{1}{\sqrt{2}}(\alpha|0\rangle_1 + \beta|1\rangle_1)(|0\rangle_2|1\rangle_3 - |1\rangle_2|0\rangle_3). \tag{6.3.7}$$

Factoring the Bell states we have

$$\begin{aligned} |\Psi\rangle_{123} &= \frac{1}{2}(|\Phi^+\rangle(\alpha|1\rangle_3 - \beta|0\rangle_3) + |\Phi^-\rangle(\alpha|1\rangle_3 + \beta|0\rangle_3) \\ &+ |\Psi^+\rangle(-\alpha|0\rangle_3 + \beta|1\rangle_3) + |\Psi^-\rangle(\alpha|0\rangle_3 + \beta|1\rangle_3). \end{aligned} \tag{6.3.8}$$

That is to say, if the measurements in the transmitter's laboratory give the result of $|\Phi^-\rangle$, the receptor will know the state that is in its hands is such given by

$$|\psi\rangle_3 = \alpha|1\rangle_3 + \beta|0\rangle_3. \tag{6.3.9}$$

Now, if there are three persons involved in the teleportation process, we have the following: The person A has a qubit entangled with a qubit that belongs with the person B and one person C with qubit that are entangled with another one that belongs with the person B but is not the same that such is entangled with the qubit of A, in other words, we can considerate the following product

$$|\Phi^+\rangle^{(AB)} \otimes |\Phi^+\rangle^{(B'C)}. \tag{6.3.10}$$

We can use with this state, the C-NOT operator applying it to a qubit B' using the qubit B as a control, after we apply the Hadamard operator to B and finally we perform a Bell measurement over B and B'. With this measurement the qubits of A and C are now entangled. What is happening here, is that the measurement involves a wave collapse, then the information of B and B' is classic and this information can be sent trivially to C where a measurement can be performed to obtain the quantum information. The problem is that now B is uncommunicated.

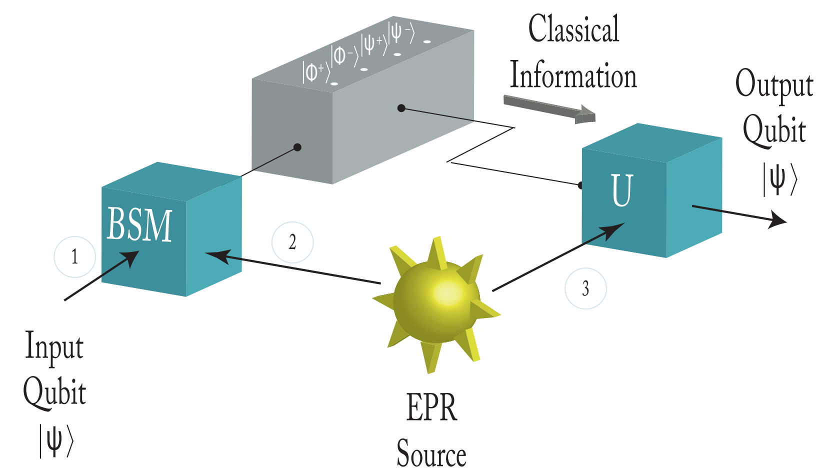

Figure 6.7: Schematic diagram of an arrangement for teleportation and Bell measurement.

6.4 The density operator

In quantum mechanics, the information of a system is given by the state vectors $|\psi\rangle$. The information consists of quantum numbers associated with a set of commuting observables. From the principle of superposition and the properties of the Hilbert space, if $|\psi_1\rangle$ and $|\psi_2\rangle$ are two possible quantum states, its coherent superposition

$$|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle \tag{6.4.1}$$

will be a quantum state too if the coefficients c_1 and c_2 are known. With the condition of orthogonality, $\langle\psi_2|\psi_1\rangle = 0$, and the normalization we can obtain $|c_1|^2+|c_2|^2= 1$. In the most of situations we have that the state vector is not known. If the system of interest is interacting with another system (that can be very large, as a reservoir), both become entangled. In this case, is possible to write state vectors for the multicomponent system but not for the subsystem of interest [60].

The states that are described by state vectors are called *pure* states. Those states that cannot be described by state vectors are defined as *mixed* states. The mixed states are described by the density operator

$$\rho = \sum_i |\psi_i\rangle\, p_i \,\langle\psi_i| = \sum_i p_i \,|\psi_i\rangle\,\langle\psi_i| \,, \tag{6.4.2}$$

where the sum is over an ensemble, p_i is the probability of the system being in the ith state of the ensemble $|\psi_i\rangle$ and $\langle\psi_i|\psi_i\rangle = 1$. The probabilities satisfy

$$0 \leq p_i \leq 1, \quad \sum_i p_i = 1, \quad \sum_i p_i^2 \leq 1. \tag{6.4.3}$$

We obtain the density operator for a pure state when $p_i = \delta_{ij}$ which yields

$$\rho = |\psi_j\rangle\,\langle\psi_j| \,. \tag{6.4.4}$$

Introducing a complete and orthonormal basis $\{\phi_n\}$, the ith member of the ensemble may write as

$$|\psi_i\rangle = \sum_n c_n^{(i)}\, |\phi_n\rangle \tag{6.4.5}$$

where $c_n^{(i)} = \langle\phi_n|\psi_i\rangle$. In this base, the matrix elements are given by

$$\langle\phi_n|\rho|\phi_m\rangle = \sum_i p_i c_n^{(i)} c_m^{(i)*}. \tag{6.4.6}$$

Taking the trace of this matrix we have

$$\begin{aligned}\mathrm{Tr}\rho &= \sum_n \langle\phi_n|\rho|\phi_n\rangle \\ &= \sum_i \sum_n p_i \langle\phi_n|\psi_i\rangle \langle\psi_i|\psi_n\rangle \\ &= \sum_i p_i \\ &= 1.\end{aligned} \tag{6.4.7}$$

Since ρ is Hermitian, the diagonal elements must be real and satisfy $0 \leq \langle\phi_n|\rho|\phi_n\rangle \leq 1$. Considering the square of the density operator for a pure state we obtain

$$\rho^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \rho. \tag{6.4.8}$$

Then the trace of ρ^2 is equal to the trace of ρ for a pure state. For a statistical mixture

$$\rho^2 = \sum_i \sum_j p_i p_j |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|\,. \tag{6.4.9}$$

Taking the trace

$$\begin{aligned}\mathrm{Tr}\rho^2 &= \sum_n \langle\phi_n|\rho^2|\phi_n\rangle \\ &= \sum_n \sum_i \sum_j p_i p_j \langle\phi_n|\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|\phi_n\rangle \\ &= \sum_i \sum_j p_i p_j |\langle\psi_i|\psi_j\rangle|^2 \\ &\leq \left[\sum_i p_i\right]^2 \\ &= 1.\end{aligned} \tag{6.4.10}$$

Therefore we have a criteria for pure and mixed states. The expectation value of an operator will be given by

$$\langle O\rangle = \mathrm{Tr}(\rho O) \tag{6.4.11}$$

6.5 von Neumann Entropy

The concept of entropy is commonly used in thermodynamics and is understood as a measure of disorder. As the disorder increases the entropy is greater. In statistical mechanics and information theory, the entropy indicates the missing information.

The Von Neumann entropy is defined for a density operator as [63]

$$S(\rho) = -\mathrm{Tr}[\rho \ln \rho] \tag{6.5.1}$$

and has a parallel with the entropy defined in the context of statistical mechanics. We have that for a pure state this entropy is zero. Then repeated measurements of that pure state wont bring new information. For a mixed state we obtain $S(\rho_{mixed}) > 0$. The procedure for calculate the entropy is in general difficult. In a basis for which the density operator is diagonal, we can evaluate the entropy as

$$S(\rho) = -\sum_k \rho_{kk} \ln \rho_{kk}. \tag{6.5.2}$$

Because we have that all diagonal matrix elements must be real and $0 \leq \rho_{kk} \leq 1$, from this we say that $S(\rho)$ must be positive semidefinite.

Consider the bipartite state

$$|\psi\rangle = \frac{1}{\sqrt{1+|\zeta|^2}}(|0\rangle_1 |0\rangle_2 + \zeta |1\rangle_1 |1\rangle_2). \tag{6.5.3}$$

The density operators for each subsystem are

$$\rho_1 = \frac{1}{1+|\zeta|^2}[|0\rangle_1 \quad {}_1\langle 0| + |\zeta|^2 |1\rangle_1 \quad {}_1\langle 1|] \tag{6.5.4}$$

$$\rho_2 = \frac{1}{1+|\zeta|^2}[|0\rangle_2 \quad {}_2\langle 0| + |\zeta|^2 |1\rangle_2 \quad {}_2\langle 1|]. \tag{6.5.5}$$

Then the corresponding von Neumann entropies are

$$S(\rho_1) = -\left\{\frac{1}{1+|\zeta|^2}\ln\left[\frac{1}{1+|\zeta|^2}\right] + \frac{|\zeta|^2}{1+|\zeta|^2}\ln\left[\frac{|\zeta|^2}{1+|\zeta|^2}\right]\right\} = S(\rho_2). \tag{6.5.6}$$

When $\zeta = 0$ we have a non-entangled system $|0\rangle_1 |0\rangle_2$ and $S(\rho_1) = S(\rho_2) = 0$, as expected. When we have $|\zeta|= 1$ we obtain $S(\rho_1) = S(\rho_2) = \ln 2$, which represents maximal entanglement for this state indicating maximal quantum correlations. Information about these correlations is destroyed upon tracing over one of the subsystems [60]

6.6 Bogoliubov Transformations Approach to Unruh Effect

The metric (2.2.37) is conformal to the whole of Minkowski space, for under the conformal transformation

$$g_{\mu\nu} \to e^{-2a\xi} g_{\mu\nu}, \tag{6.6.1}$$

reduces to $d\eta^2 - d\xi^2$. Since the wave equation is conformally invariant, we can write it in Rindler coordinates as follows

$$e^{2a\xi}\Box\varphi = \left(\frac{\partial^2}{\partial\eta^2} - \frac{\partial^2}{\partial\xi^2}\right)\varphi = \frac{\partial^2}{\partial\bar{u}\partial\bar{v}} = 0, \tag{6.6.2}$$

which has the following mode solutions

$$\bar{u}_k = \frac{1}{\sqrt{4\pi\omega}} e^{ik\xi \pm i\omega\eta}, \tag{6.6.3}$$

with

$$\omega = |k| > 0, \quad -\infty < k < \infty. \tag{6.6.4}$$

The + sign in the expression (6.6.3) applies in region L and the − sign in region R. These modes are positive frequency with respect to the time-like Killing vector $+\partial_\eta$ in R and $-\partial_\eta$ in L, satisfying

$$\mathcal{L}_{\pm\partial_\eta}\bar{u}_k = -i\omega\bar{u}_k. \tag{6.6.5}$$

We define

$$u_k^{(R)} = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{ik\xi - i\omega\eta} & \text{in R} \\ 0 & \text{in L} \end{cases} \tag{6.6.6}$$

$$u_k^{(L)} = \begin{cases} 0 & \text{in R} \\ \frac{1}{\sqrt{4\pi\omega}} e^{ik\xi + i\omega\eta} & \text{in L} \end{cases} \tag{6.6.7}$$

The set (6.6.6) is complete in the region R, while the set (6.6.7) is complete in L. As we see before, the field can be expanded by the modes solutions of the wave equation in Minkowski space,

$$\varphi = \sum_{k=-\infty}^{\infty} (a_k u_k + a_k^\dagger u_k^*), \tag{6.6.8}$$

or in the Rindler modes

$$\varphi = \sum_{k=-\infty}^{\infty} (b_k^{(1)} u_k^{(L)} + b_k^{(1)\dagger} u_k^{(L)*} + b_k^{(2)} u_k^{(R)} + b_k^{(2)\dagger} u_k^{(R)*}), \tag{6.6.9}$$

where we can identify two alternative Fock spaces, thus two vacuum states; the Minkowski vacuum state $|0_M\rangle$ and the Rindler vacuum state $|0_R\rangle$, defined by

$$a_k |0_M\rangle = 0 \tag{6.6.10}$$

or

$$b_k^{(1)} |0_R\rangle = b_k^{(2)} |0_R\rangle = 0. \tag{6.6.11}$$

By inspection of the Rindler modes, we have that these modes are non-analytic at the point $u = v = 0$. Therefore the Rindler modes cannot be a combination of pure positive frequency Minkowski modes, but must also contain negative frequencies [12]. The mixing of positive and negative frequencies implies that the vacuum states cannot be the same, in other words, the vacuum associated with one set of modes contains particles associated with the other set of modes.

In order to determine what Rindler particles are present in the Minkowski vacuum, we must determine the Bogoliubov transformation between the two sets of modes. From the expression (4.3.5) we can observe that the Bogoliubov coefficients in this situation can be obtained by a Fourier transform of the Rindler modes. In [12] is named an alternative form, referring to a method due to Unruh [13], where is indicated that the combinations

$$u_k^{(R)} + e^{-\pi\omega/a} u_{-k}^{(L)*} \tag{6.6.12}$$

$$u_{-k}^{(R)*} + e^{\pi\omega/a} u_k^{(L)} \tag{6.6.13}$$

are analytic and bounded, both for all real u, v and in any point in the lower half complex u and v planes, in contrast with the separated Rindler modes (6.6.6) (6.6.7).

Since the modes (6.6.12) and (6.6.13) share the positive frequency analyticity propertoes of the Minkowski modes u_k, they must also share a common vacuum state $|0_M\rangle$, thus we can expand the field in a new form

$$\begin{aligned} \varphi = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2\sinh(\pi\omega/a)}} \Big[& d_k^{(1)} (e^{\pi\omega/2a} u_k^{(R)} + e^{-\pi\omega/2a} u_{-k}^{(L)*}) \\ & + d_k^{(2)} (e^{-\pi\omega/2a} u_{-k}^{(R)*} + e^{\pi\omega/2a} u_k^{(L)}) \Big] + \text{h.c.} \end{aligned} \tag{6.6.14}$$

where

$$d_k^{(1)} |0_M\rangle = d_k^{(2)} |0_M\rangle = 0. \tag{6.6.15}$$

The operators $b_k^{(1,2)}$ can be related to $d_k^{(1,2)}$ taking the inner products $(\varphi, u_k^{(R)})$, $(\varphi, u_k^{(L)})$ with the expansion field given by (6.6.9) and then with (6.6.14). After this procedure, we obtain

$$b_k^{(1)} = \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}[e^{\pi\omega/2a}d_k^{(2)} + e^{-\pi\omega/2a}d_k^{(1)\dagger}] \tag{6.6.16}$$

$$b_k^{(2)} = \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}[e^{\pi\omega/2a}d_k^{(1)} + e^{-\pi\omega/2a}d_{-k}^{(2)\dagger}]. \tag{6.6.17}$$

These Bogoliubov transformations give as the required relation between the vacuum states $|0_R\rangle$ and $|0_M\rangle$.

Considering an accelerated observer with a hyperbolic trajectory at $\xi =$ constant. The vacuum state for this accelerated observer $|0_R\rangle$ is the state associated with modes which are positive frequency with respect to η. For instance, an accelerated observed in L will detect particles counted by the number operator $b_k^{(1)}b_k^{(1)\dagger}$. If the field is initially prepared in the Minkowski vacuum state $|0_M\rangle$, then an accelerated observer will detect

$$\langle 0_M|\, b_k^{(1)}b_k^{(1)\dagger}\,|0_R\rangle = \frac{e^{-\pi\omega/a}}{2\sinh(\pi\omega/a)} = \frac{1}{e^{2\pi\omega/a}-1} \tag{6.6.18}$$

particles in the mode k. This is precisely the Planck spectrum for radiation at temperature

$$T = \frac{a}{2\pi} \tag{6.6.19}$$

which is in exact agreement with the result obtained in (7.2.7). Then an accelerated observer perceives the vacuum in the flat space as a set of thermal particles. Under a conformal transformation we could obtain a thermal bath seen by an inertial observer in curved space. This is the so-called Hawking effect.

Finally, one can invert the Bogoliubov transformations (6.6.16) and (6.6.17) in order to express the creation and annihilation operators $d_k^{(1,2)}$ or a_k in terms of the Rindler creation and annihilation operators $b_k^{(1,2)}$ and apply this expansion to the state $|0_M\rangle$. This procedure will give explicitly the form in which Rindler particles are composing the Minkowski vacuum. The Minkowski vacuum in terms of Rindler particles has the following two-mode squeezed states form [64, 65, 66]

$$|0_k\rangle_M \sim \frac{1}{\cosh r}\sum_{n=0}^{\infty} \tanh^n r\, |n_k\rangle_R\, |n_k\rangle_L\,, \tag{6.6.20}$$

where

$$\cosh r = \frac{1}{\sqrt{1-e^{-2\pi\omega/a}}}, \tag{6.6.21}$$

and $|n_k\rangle_R\,, |n_k\rangle_L$ refer to the mode decomposition in region R and L, respectively, of Rindler space and

$$|0_M\rangle = \prod_j |0_j\rangle_M\,. \tag{6.6.22}$$

Tracing out over the degrees of freedom associated with Rindler region L, we obtain the density matrix for the region R given by

$$\rho = \prod_j \frac{1}{1-e^{-2\pi\omega_j/a}} \sum_{n_j} e^{-2\pi n_j\omega_j/a}\, |n_j\rangle_{R\,R}\langle n_j|\,, \tag{6.6.23}$$

that corresponds precisely with a thermal density matrix. Then the Minlowski vacuum corresponds to a thermal state in each Rindler wedge at temperature $T = a/2\pi$.

6.7 Entanglement in Noninertial Frames

Quantum information has been explored widely in the framework of the quantum mechanics formalism. Diverse areas such that quantum optics, atomic physics, solid-state physics, have explored the creation of entangled states and its application as a tool resource for quantum information processing. Nevertheless, under realistic laboratory conditions, entanglement is degrading through uncontrolled coupling to environment.

Exploring the relativistic aspects of the physical systems that are involved in the quantum information processing, may contribute in task of controlling and enhance the entanglement in realistic conditions. The field of relativistic quantum information are interested in the quantum field effects as an environment of the qubits; the quantum decoherence induced by the quantum field and the entanglement dynamics and its observer-dependent nature.

In this section we review the results reported by Fuentes-Schuller and Mann in [54] where they investigated the entanglement between two modes of a noninteracting massless scalar field when one of the observers describing the state is uniformly accelerated. By Unruh effect, they found that the entanglement is degraded. On the other hand, although the quantum field induce decoherence in the qubit, if we place two or more, the mutual influences via the quantum field will enhance the entanglement [23]. With this in mind we discuss the results in the Ref. [57] where is demonstrated that, in the presence of curvature, the amount of entanglement can be increased. We can evidence two kinds of studies in this framework. The first one, is related to study the entanglement of bosonic free modes of a scalar field. The second one, study the entanglement between detectors. This last one is adopted in the next chapter, where we show the entanglement extraction and stability of two UD accelerated detectors in the Minkowski vacuum.

In order to stablish the entanglement degradation due to the Unruh effect, we consider that two modes, a and b, of a free massless scalar field in Minkowski space-time, are maximally entangled form an inertial perspective, that is, the quantum field is initially in the state

$$\frac{1}{\sqrt{2}}(|0_a\rangle_M |0_b\rangle_M + |1_a\rangle_M |1_b\rangle_M). \tag{6.7.1}$$

The states are $|0_j\rangle_M$ and $|1_j\rangle_M$ are the vacuum and single particle excitation states of the mode j in Minkowski space. The inertial observed, called Alice, has a detector which only detects mode a and the accelerated observer, called Rob, has a detector sensitive only to mode b. Since Rob has a uniform acceleration α, the states corresponding to mode b must be expressed in Rindler coordinates in order to describe what Rob sees. We use then the expression (6.6.20).

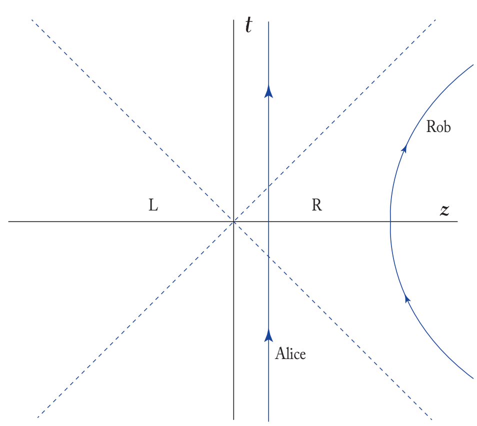

Figure 6.8: Schematic diagram for the entanglement between an inertial and accelerated observers [54].

In this situation is considered that the rest of the modes in the field, apart from a and b, are in the vacuum. Therefore, we can trace over all the modes except for a and b. From the expression (6.6.20) and

$$\left|1_b\right\rangle_M = \frac{1}{\cosh^2 r}\sum_{n=0}^{\infty}\tanh^n r\sqrt{n+1}\left|(n+1)_b\right\rangle_R \left|n_b\right\rangle_L, \tag{6.7.2}$$

we can rewrite the expression (6.7.1) in terms of Minkowski modes and Rindler modes, such that tracing over the region L, we obtain

$$\rho_{AR} = \frac{1}{2\cosh^2 r}\sum_n \tanh^{2n} r\rho_n, \tag{6.7.3}$$

where

$$\rho_n = \left|0,n\right\rangle\left\langle 0,n\right| + \frac{\sqrt{n+1}}{\cosh r}\left|0,n\right\rangle\left\langle 1,n+1\right| + \frac{\sqrt{n+1}}{\cosh r}\left|1,n+1\right\rangle\left\langle 0,n\right| + \frac{n+1}{\cosh^2 r}\left|1,n+1\right\rangle\left\langle 1,n+1\right|, \tag{6.7.4}$$

where $\left|n,m\right\rangle = \left|n_a\right\rangle_M \left|m_B\right\rangle_R$. In order to measure the entanglement degradation, the logarithmic negativity is calculated. It is defined as

$$N(\rho) = \log_2 ||\rho^T||_1 \tag{6.7.5}$$

where $||\rho^T||_1$ is the trace norm of the density matrix. For the density matrix ρ_{AR} the logarithmic negativity reads

$$N(\rho_{AR}) = \log_2\left(\frac{1}{2\cosh^2 r} + \Sigma\right) \tag{6.7.6}$$

where

$$\Sigma = \sum_{n=0}^{\infty}\frac{\tanh^{2n} r}{2\cosh^2 r}\sqrt{\left(\frac{n}{\sinh^2 r} + \tanh^2 r\right)^2 + \frac{4}{\cosh^2 r}}. \tag{6.7.7}$$

For vanishing acceleration, $r = 0$, $N(\rho_{AR}) = 1$ as expected. In the limit $r \to \infty$, the negativity is exactly 0. The wide set of physical implications that these results open will be discussed beside the results of the following chapter.

Chapter 7

Entanglement stability of entangled accelerated detectors

In this chapter are presented the main results of this work. We examine the entanglement stability of a pair of accelerated two-level atoms weakly coupled with a massless scalar field in Minkowski vacuum. We obtain the entanglement by the excitation of the atomic system due to Unruh effect and its mutual influences by the field. We identify the mutual influences of atoms via fields as a coherence agent in each response function terms. The thermal spectrum measured by the accelerated atoms is found for a long observational time interval. In addition, we obtain a general expression for the mean life of those entangled states for different accelerations and on-times switching.

7.1 Two identical atoms coupled with a massless scalar field

Let us consider two identical two-level atoms interacting with a massless scalar field in a four-dimensional Minkowski space-time. Here we consider that the atoms are moving along different hyperbolic trajectories. Let us first establish the dependence between the proper times by the structure of Rindler coordinates [12]

$$t = a^{-1}e^{a\xi}\sinh(a\eta), \qquad z = a^{-1}e^{a\xi}\cosh(a\eta). \tag{7.1.1}$$

The lines of constant η are straight lines ($z \sim t$), whereas the lines of constant ξ are hyperbolae $z^2 - t^2 = a^{-2}e^{2a\xi} \equiv$ constant. The proper acceleration is defined by $ae^{-a\xi} = \alpha^{-1}$, and the proper time of the atoms τ is related to ξ and η by $\tau = e^{a\xi}\eta$. We assume that the j-th atom accelerates uniformly with acceleration α_j^{-1}, $j = 1, 2$. The dependence between the proper times is given by the lines of constant η, such that

$$\tau_2(\tau_1) = \tau_1 e^{a(\xi_2 - \xi_1)}, \tag{7.1.2}$$

with $e^{a(\xi_2-\xi_1)} = \alpha_2/\alpha_1$.

The time evolution of the atom-field quantum system with respect to the coordinate time t is described by the total Hamiltonian H which reads

$$H = H_A + H_F + H_{int},$$

where H_A is the free atomic Hamiltonian, H_F is the free field Hamiltonian and H_{int} describes the interaction between the atoms and the fields.

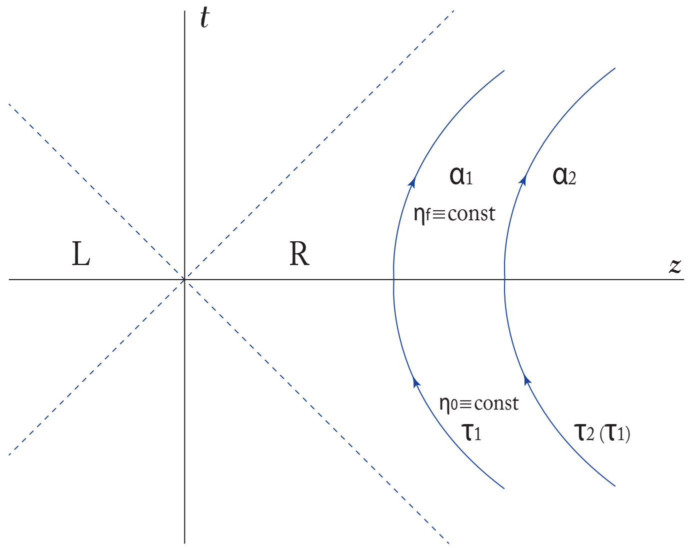

Figure 7.1: Schematic diagram for the entanglement formation between two accelerated observers.

Let us briefly discuss each of such terms. We may express the atomic Hamiltonian in the Dicke notation as [67]

$$H_A = \frac{\omega_0}{2}\left[S_1^z(\tau)\otimes \mathbb{1}_2\,\frac{d\tau_1}{dt} + \mathbb{1}_1\otimes S_2^z(\tau)\,\frac{d\tau_2}{dt}\right] \tag{7.1.3}$$

where $S_i^z = (|e_i\rangle\langle e_i| - |g_i\rangle\langle g_i|)/2$ is associated with the i-th atom and $|g_i\rangle$, $|e_i\rangle$ is the ground and excited state of the i-th atom, respectively. Also, τ_i is the proper time of the i-th atom. The eigenstates and respective energies are given by

$$\begin{aligned} E_{gg} &= -\omega_0 \quad |gg\rangle = |g_1\rangle|g_2\rangle\,, \\ E_{ge} &= 0 \quad |ge\rangle = |g_1\rangle|e_2\rangle\,, \\ E_{eg} &= 0 \quad |eg\rangle = |e_1\rangle|g_2\rangle\,, \\ E_{ee} &= \omega_0 \quad |ee\rangle = |e_1\rangle|e_2\rangle\,, \end{aligned} \tag{7.1.4}$$

where a tensor product is implicit. Another possible choice is the Bell-state basis. The Bell states are known as the four maximally entangled two-qubit states. In terms of the above product states, the Bell states are expressed as

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}\left(|g_1\rangle|e_2\rangle \pm |e_1\rangle|g_2\rangle\right), \tag{7.1.5}$$

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}\left(|g_1\rangle|g_2\rangle \pm |e_1\rangle|e_2\rangle\right). \tag{7.1.6}$$

The Hamiltonian (7.1.3) is showing a degeneracy associated with the eigenstates $|ge\rangle$ and $|eg\rangle$. Any linear combination of these degenerate eigenstates will be an eigenstate of the atomic Hamiltonian and these linear combination must have the same degenerate energy value. Then (7.1.5) are eigenstates of H_A. On the other hand, it is clear that (7.1.6) are not eigenstates of the atomic Hamiltonian.

The free Hamiltonian of the quantum field which governs the field time evolution is given by

$$H_F = \frac{1}{2}\int d^3x\left[(\dot{\varphi}(x))^2 + (\nabla\varphi(x))^2\right], \tag{7.1.7}$$

where the dot represents the derivative with respect to t. Finally we assume that the coupling between the atoms and the field is described by a monopole interaction in the form

$$H_{int}(t) = \sum_{j=1}^{2} g_j m^{(j)}(\tau_j(t))\varphi[x_j(\tau_j(t))]\frac{d\tau_j(t)}{dt}. \tag{7.1.8}$$

The quantity g_j is the coupling constant of the j-th atom, $\varphi[x_j(\tau_j(t))]$ is the field at the point of the j-th atom. Hereafter we set $g_1 = g_2 = g$ and we assume that g is small.

As mentioned above, we consider that the atoms are moving along world lines $x^{\mu}_{1,2}(\tau_{1,2})$ parametrized by the proper times $\tau_{1,2}$. Since in this case the proper times of the atoms do not coincide, we write the time-evolution operator as [57]

$$U = \exp\left[-i\int d\tau_1 g\left[m^{(1)}(\tau_1)\varphi\left(x^{\mu}_1(\tau_1)\right) \;+\; m^{(2)}(\tau_2(\tau_1))\varphi\left(x^{\mu}_2(\tau_2(\tau_1))\right)\frac{d\tau_2(\tau_1)}{d\tau_1}\right]\right]. \tag{7.1.9}$$

Preparing the field φ in the Minkowski vacuum state $|0_M\rangle$ and the atoms in the collective ground state $|gg\rangle$, the amplitude in first-order perturbation theory for a general transition is given by

$$\begin{aligned} A_{|gg;0_M\rangle\to|\omega;\varphi_f\rangle} = ig\,\langle\omega;\varphi_f|\int_{\tau_0}^{\tau_f} d\tau_1 &\left[m^{(1)}(\tau_1)\varphi\left(x^{\mu}_1(\tau_1)\right)\right. \\ &\left. + \; m^{(2)}(\tau_2(\tau_1))\varphi\left(x^{\mu}_2(\tau_2(\tau_1))\right)\frac{d\tau_2(\tau_1)}{d\tau_1}\right]|gg;0_M\rangle\,. \end{aligned} \tag{7.1.10}$$

In the interaction picture, we have

$$m^{(k)}(\tau_j) = e^{iH_A\tau_j} m^{(k)}(0) e^{-iH_A\tau_j}, \tag{7.1.11}$$

where

$$m^{(j)}(0) = |e^{(j)}\rangle\langle g^{(j)}| + |g^{(j)}\rangle\langle e^{(j)}|$$

is the monopole matrix of the j-th atom [56, 57]. The transition probability to all possible atomic and field states in first-order approximation is given by

$$\Gamma_{|\omega'\rangle\to|\omega\rangle}(\Delta\omega,\tau_0,\tau_f) = g^2\sum_{\omega,i,j}\left[m^{(i)*}_{\omega\omega'} m^{(j)}_{\omega\omega'} F_{ij}(\Delta\omega,\tau_0,\tau_f)\right] \tag{7.1.12}$$

where $\Delta\omega = \omega - \omega'$, $i = 1, 2$, $j = 1, 2$ and the matrix elements are given by

$$m^{(1)}_{\omega\omega'} = \langle\omega|\, m^{(1)} \otimes 1\!\!1_2\, |\omega'\rangle$$

$$m^{(2)}_{\omega\omega'} = \langle\omega|\, 1\!\!1_1 \otimes m^{(2)}\, |\omega'\rangle\,.$$

Note that ω can be any of the energies given in Eq. (7.1.4) and also $|\omega\rangle$ can be any of the states $\{|gg\rangle, |\Psi^{\pm}\rangle, |ee\rangle\}$.

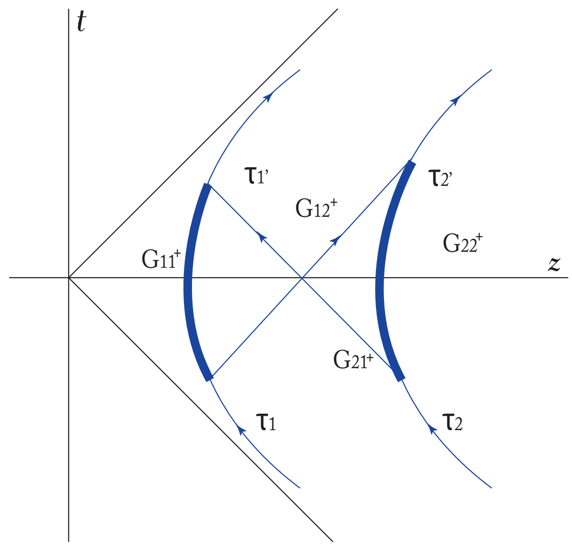

Figure 7.2: Individual and crossed correlation functions. Crossed correlation functions give the information of the mutual influences of atoms via the quantum field.

The corresponding response functions are defined by

$$F_{ij}(\Delta\omega, \tau_0, \tau_f) = \int_{\tau_0}^{\tau_f} d\tau_1 \int_{\tau_0}^{\tau_f} d\tau_1' e^{-i\Delta\omega(\tau_i(\tau_1)-\tau_j(\tau_1'))} G^+_{ij}(\tau_1, \tau_1') \frac{d\tau_j(\tau_1')}{d\tau_1'} \frac{d\tau_i(\tau_1)}{d\tau_1}, \tag{7.1.13}$$

where $G^+_{ij}(\tau_1, \tau_1') = \langle 0_M | \, \varphi\left(x_i(\tau_i(\tau_1))\right) \varphi\left(x_j(\tau_j(\tau_1'))\right) |0_M\rangle$ is the positive-frequency Wightman function in Minkowski space-time for a massless scalar field, which is given by

$$G^+_{ij}(\tau, \tau') = \frac{1}{8\pi^2} \frac{1}{\sigma(\tau, \tau')}, \tag{7.1.14}$$

where $\sigma(\tau, \tau')$ is given by

$$\begin{aligned} 2\sigma(\tau, \tau') &= (x_i(\tau) - x_j(\tau'))^2 \\ &= -(t_i(\tau) - t_j(\tau') - i\epsilon)^2 + |\mathbf{x}_i(\tau) - \mathbf{x}_j(\tau')|^2. \end{aligned}$$

We are interested in the entanglement generation of a pair of atoms travelling in different hyperbolic world lines. Hence we study the transition $|gg\rangle \to |\Psi^\pm\rangle$, with $\Delta\omega = \omega_0 > 0$. The appearance of cross terms in the transition probability has its origin in the fact of working with two atoms, both interacting with a common scalar quantum field.

We may define the total transition rate as follows

$$\mathcal{R}_{|\omega'\rangle\to|\omega\rangle}(\Delta\omega, \Delta t) = \frac{d\Gamma_{|\omega'\rangle\to|\omega\rangle}(\Delta\omega, \Delta t)}{d(\Delta t)}, \tag{7.1.15}$$

where $\Gamma_{|\omega'\rangle\to|\omega\rangle}(\Delta\omega, \Delta t)$ is given by Eq. (7.1.12). In order to present an explicit expression for the total transition rate, one needs to properly evaluate in detail each of the response functions $F_{ij}(\Delta\omega, \tau_0, \tau_f)$. This is the topic of the next Section.

7.2 Individual response functions, $F_{11}(\Delta\omega, \Delta t)$ and $F_{22}(\Delta\omega, \Delta t)$

Let us first evaluate the contribution $F_{11}(\Delta\omega)$ to the total response function. The associated Wightman function G^+_{11} is given by

$$G^+_{11}\left(\tau_1 - \tau_1'\right) = -\frac{1}{16\pi^2\alpha_1^2 \sinh^2\left(\frac{\tau_1-\tau_1'}{2\alpha_1} - \frac{i\epsilon}{\alpha_1}\right)}. \tag{7.2.1}$$

Using known series identities [47] we can rewrite (7.2.1) as

$$G^+_{11}(\tau_1 - \tau_1') = -\frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\left((\tau_1 - \tau_1') - 2i\epsilon + 2\pi i\alpha_1 n\right)^{-2}. \tag{7.2.2}$$

Changing variables to

$$\psi = \tau_1 - \tau_1' \qquad \eta = \tau_1 + \tau_1', \tag{7.2.3}$$

we have,

$$F_{11}(\Delta\omega, \Delta t) = \frac{1}{2}\int_{-\Delta t}^{\Delta t} d\psi\, (2|\psi| - 2\Delta t)\, e^{-i\Delta\omega\psi} G^+_{11}(\psi), \tag{7.2.4}$$

where $\Delta t = \tau_f - \tau_0$. The evaluation of the integral leads us to [46]

$$\begin{aligned} F_{11}(\Delta\omega, \Delta t) &= \frac{\Delta t}{2\pi^2}\left\{\pi|\Delta\omega|\Theta(-\Delta\omega) + |\Delta\omega|\left(\mathrm{Si}\Delta\omega\Delta t - \frac{\pi}{2}\right) + \frac{\pi|\Delta\omega|}{e^{2\pi\alpha_1|\Delta\omega|} - 1}\right. \\ &+ \left.\int_{\Delta t}^{\infty} d\psi \cos(\Delta\omega\psi)\left(\frac{1/(2\alpha_1)^2}{\sinh^2\psi/(2\alpha_1)} - \frac{1}{\psi^2}\right)\right\} + \frac{1}{2\pi^2}\left\{\cos\left(\Delta\omega\Delta t\right) + \log\left(\frac{\Delta t}{2\pi\epsilon}\right) - 1\right. \\ &+ \left.\int_0^{\Delta t} d\psi \frac{\cos(\Delta\omega\psi) - 1}{\psi} + \int_0^{\Delta t} d\psi\psi\cos\Delta\omega\psi\left(\frac{1/(2\alpha_1)^2}{\sinh^2\psi/(2\alpha_1)} - \frac{1}{\psi^2}\right)\right\}. \end{aligned} \tag{7.2.5}$$

For details concerning such a calculation we refer the reader the Appendix B.2. We are interested in the rate

$$R_{ij}(\Delta\omega, \Delta t) = \frac{dF_{ij}(\Delta\omega, \Delta t)}{d(\Delta t)} \tag{7.2.6}$$

which is related to the mean life of states. From the expression (B.2.3) found in Appendix B.2 we obtain that for large time intervals we have the following expression

$$\lim_{\Delta t\to\infty} R_{11}(\Delta\omega, \Delta t) = \frac{|\Delta\omega|}{2\pi}\left\{\Theta(-\Delta\omega)\left[1 + \frac{1}{e^{2\pi\alpha_1|\Delta\omega|} - 1}\right] + \Theta(\Delta\omega)\frac{1}{e^{2\pi\alpha_1\Delta\omega} - 1}\right\}. \tag{7.2.7}$$

The above equation shows us that the equilibrium between the uniformly accelerated atom and scalar field in the Minkowski vacuum state $|0_M\rangle$ is the same as that which would have been achieved had this atom followed an inertial trajectory but immersed in a bath of thermal radiation at the temperature $\beta_1^{-1} = 1/2\pi\alpha_1$.

Analogously, for the response function F_{22},

$$G^+_{22}\left(\tau_2(\tau_1), \tau_2(\tau_1')\right) = -\frac{1}{16\pi^2\alpha_2^2 \sinh^2\left(\frac{\tau_1-\tau_1'}{2\alpha_1} - \frac{i\epsilon}{\alpha_1}\right)}. \tag{7.2.8}$$

Performing similar steps as before, we can rewrite (7.2.8) as

$$G^+_{22}(\psi) = -\frac{1}{4\pi^2}\frac{\alpha_1^2}{\alpha_2^2}\sum_{n=-\infty}^{\infty}\left(\psi - 2i\epsilon + 2\pi i\alpha_1 n\right)^{-2}, \tag{7.2.9}$$

such that, the expression for the response function $F_{22}(\Delta\omega,\Delta t)$ yields

$$F_{22}(\Delta\omega,\Delta t) = -\frac{1}{2}e^{2a(\xi_2-\xi_1)}\int_{-\Delta t}^{\Delta t} d\psi\, G_{22}^{+}(\psi)(-2|\psi|+2\Delta t)e^{-i\Delta\omega e^{a(\xi_2-\xi_1)}\psi}. \tag{7.2.10}$$

From the expression (B.2.6) in the Appendix B.2, we obtain an asymptotic expression $\Delta t \to \infty$ for $R_{22}(\Delta\omega,\Delta t)$ which is just Eq. (7.2.7) with the replacement $\alpha_1 \to \alpha_2$. We observe that, for large time intervals, the equilibrium between the atom 2 and scalar field in the Minkowski vacuum state is the same as the equilibrium of this atom in an inertial trajectory and a bath of thermal radiation at the temperature $\beta_2^{-1} = 1/2\pi\alpha_2$.

7.3 Crossed response functions, $F_{12}(\Delta\omega,\Delta t)$ and $F_{21}(\Delta\omega,\Delta t)$

The first two response functions discussed above correspond to individual atomic transitions. Therefore one expects that the response functions $F_{12}(\Delta\omega,\Delta t)$ and $F_{21}(\Delta\omega,\Delta t)$ exhibit the existence of cross-correlations between the atoms mediated by the field. In order to unveil such a behavior, we now proceed to evaluate such contributions. It is easy to show that the positive frequency Wightman functions for both cases are equal and are given by

$$\begin{aligned} G_{21}^{+}(\tau_2(\tau_1),\tau_1') &= G_{12}^{+}(\tau_1,\tau_2(\tau_1')) \\ &= G_{c}^{+}(\tau_1-\tau_1') \end{aligned} \tag{7.3.1}$$

where

$$G_{c}^{+}(\tau_1-\tau_1') = -\frac{1}{16\pi^2\alpha_1\alpha_2}G_{c_0}^{+}(\tau_1-\tau_1') \tag{7.3.2}$$

and

$$G_{c_0}^{+}(\tau_1-\tau_1') = \left[\sinh\left(\frac{\tau_1-\tau_1'}{2\alpha_1} - \frac{4i\epsilon}{(\alpha_1+\alpha_2)} + \frac{\phi}{2}\right)\sinh\left(\frac{\tau_1-\tau_1'}{2\alpha_1} - \frac{4i\epsilon}{(\alpha_1+\alpha_2)} - \frac{\phi}{2}\right)\right]^{-1} \tag{7.3.3}$$

and

$$\cosh\phi = 1 + \frac{(\alpha_1-\alpha_2)^2 + |\Delta\mathbf{x}|^2}{2\alpha_1\alpha_2} \tag{7.3.4}$$

with $|\Delta\mathbf{x}|^2 = (x_2-x_1)^2 - (y_2-y_1)^2$. Hence one has that:

$$F_{21}(\Delta\omega,\Delta t) = \frac{1}{2}e^{a(\xi_2-\xi_1)}\int_{-\Delta t}^{\Delta t} d\psi \int_{|\psi|+2\tau_0}^{-|\psi|+2\tau_f} d\eta\, e^{-i\Delta\omega(a_-/2\alpha_1)\eta}e^{-i\Delta\omega(a_+/2\alpha_1)\psi}\, G_c^{+}(\psi), \tag{7.3.5}$$

where we used Eq. (7.2.3) and where $a_- = \alpha_2 - \alpha_1$ and $a_+ = \alpha_2 + \alpha_1$. After some algebraic manipulations, one gets

$$\begin{aligned} F_{21}(\Delta\omega,\Delta t) &= \frac{-i}{16\pi^2\alpha_1\Delta\omega a_-}\Big\{e^{-i\Delta\omega(a_-/\alpha_1)(\Delta t+\tau_0)}\left[I(\Delta\omega,\Delta t,1) + I(\Delta\omega,\Delta t,-\alpha_2/\alpha_1)\right] \\ &- e^{i\Delta\omega(a_-/\alpha_1)(\Delta t-\tau_f)}\left[I(\Delta\omega,\Delta t,-1) + I(\Delta\omega,\Delta t,\alpha_2/\alpha_1)\right]\Big\}, \end{aligned} \tag{7.3.6}$$

where

$$I(\Delta\omega,\Delta t,\sigma) \equiv \int_0^{\Delta t} d\psi e^{-i\sigma\Delta\omega\psi}G_{c_0}^{+}(\psi). \tag{7.3.7}$$

The contribution for asymptotic time interval is given by

$$\begin{aligned} \int_0^{\infty} d\psi e^{-i\sigma\Delta\omega\psi}G_{c_0}^{+}(\psi) &= \frac{4\alpha_1}{\sinh\phi}\sin(|\Delta\omega|\sigma\alpha_1\phi)\left\{\left[\nu_0 + \frac{\pi}{e^{2\pi\alpha_1\sigma|\Delta\omega|}-1} + \zeta(\Delta\omega,\sigma)\right]\Theta(-\Delta\omega)\right. \\ &+ \left.\left[\frac{\pi}{e^{2\pi\alpha_1\sigma\Delta\omega}-1} + \zeta(\Delta\omega,\sigma)\right]\Theta(\Delta\omega)\right\}, \end{aligned} \tag{7.3.8}$$

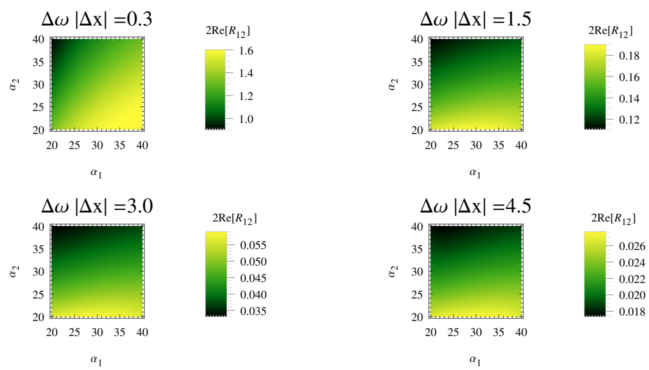

Figure 7.3: The quantity $\mathrm{Re}[R_{12}]_{|gg\rangle\to|\Psi^+\rangle}$ as a function of the inverse accelerations α_1, α_2 for different values of $\Delta\omega|\Delta\mathbf{x}|$. The quantity $\Delta\omega|\Delta\mathbf{x}|$ serves as control parameter in the study of entanglement generation. All physical quantities are given in terms of the natural units associated with the specific transition $|gg\rangle \to |\Psi^+\rangle$. Therefore, in this case ξ_1, ξ_2 and α_1, α_2 are measured in units of λ, and ω_0 is given in units of $2\pi\lambda^{-1}$, where $\lambda = 2\pi/\omega_0$. Moreover, $\mathrm{Re}[R_{12}]$ is measured in units of λ^{-1}.

where we have defined $\nu_0 = 2i\log(\alpha_1\phi) + \pi$ and the $\zeta(\Delta\omega, \sigma)$ is a combination of Hurwitz-Lerch zeta functions that is defined in the Appendix B.3. The expressions (7.3.6) and (7.3.8) clearly display a thermal Planck factor with a gray-body term, where the contributions with $\sigma = \pm 1$ contain information about the temperature β_1 and contributions with $\sigma = \pm\alpha_2/\alpha_1$ comprise the knowledge on the temperature β_2. In a similar fashion, one has that

$$
\begin{aligned}
F_{12}(\Delta\omega, \Delta t) \;&=\; \frac{i}{16\pi^2\alpha_1\Delta\omega a_-}\Big\{e^{i\Delta\omega(a_-/\alpha_1)(\Delta t+\tau_0)} \\
&\times\;\; [I(\Delta\omega, \Delta t, -1) + I(\Delta\omega, \Delta t, \alpha_2/\alpha_1)] \\
&-\;\; e^{-i\Delta\omega(a_-/\alpha_1)(\Delta t-\tau_f)}\,[I(\Delta\omega, \Delta t, 1) \\
&+\;\; I(\Delta\omega, \Delta t, -\alpha_2/\alpha_1)]\Big\}\,.
\end{aligned}
\tag{7.3.9}
$$

Observe that $F_{21}(\Delta\omega, \Delta t) = F_{12}^*(\Delta\omega, \Delta t)$. Hence the object of interest is $\mathrm{Re}[F_{12}(\Delta\omega, \Delta t)]$.

In order to study the entanglement generation, we focus attention on the particular transition $|gg\rangle \to |\Psi^+\rangle$. The corresponding matrix elements of this transition are given by

$$
m_{gs}^{(1)} = m_{gs}^{(2)} = 1/\sqrt{2}, \tag{7.3.10}
$$

and the gap energy is $E_{gs} = \omega_0$. We define the cross contribution for the total transition rate as $\mathrm{Re}[R_{12}]_{|gg\rangle\to|\Psi^+\rangle}$ which is properly evaluated in Appendix B.3, see the expression (B.3.7). The behavior of such a quantity as a function of the inverse accelerations α_1, α_2 is depicted in the Fig. 7.3, for a fixed small time interval and different spatial separations $|\Delta\mathbf{x}|$. One plainly observes the occurrence of maximum values for $\mathrm{Re}[R_{12}]$ for specific values of the accelerations. This result primarily demonstrates how $|\Delta\mathbf{x}|$ can be employed as a control parameter for entanglement generation from the vacuum state. As expected, a large value of $|\Delta\mathbf{x}|$ corresponds to a significant reduction on the magnitude of the cross contribution.

Let us specifically consider the condition $\Delta\omega|\Delta\mathbf{x}| \ll 1$. In this case, the behaviour of the cross contribution to the transition rate as a function of the accelerations is illustrated in Fig. 7.4. We consider not-so-great values for $\Delta\omega\Delta t$. Notice that, for atoms with different relative proper accelerations, cross correlations are negligible. Hence in this situation the dominant terms in the transition probability are those related to the individual atoms.

On the other hand, the results for greater time intervals are summarized in Fig. 7.5. Observe that the mutual influences of the atoms in this situation implies in a rather distinct interference pattern in comparison with the previous case. In addition, not only the region $\alpha_1 = \alpha_2$ in the plot produces sensible contributions to the transition rate, but we note the appearance of other regions in the plot that also provide important contributions.

In turn, for large $\Delta\omega|\Delta\mathbf{x}|$ we have a reduction in the value of the cross contributions as emphasized above. This is illustrated in Fig. 7.6). Furthermore, as in the previous figure, for increasing time intervals the dominant terms are not determined solely by the region $\alpha_1 = \alpha_2$: clearly other regions in the plot are also important to the transition rate.

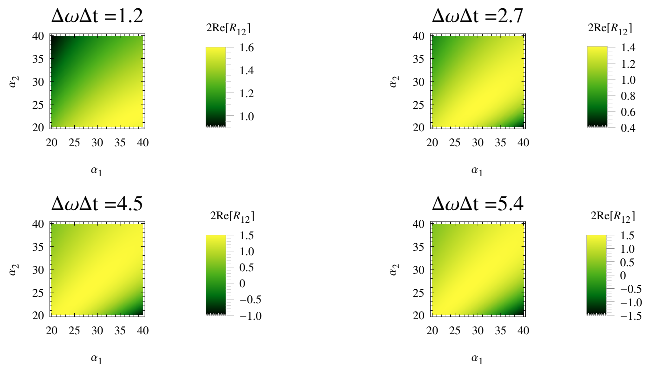

Figure 7.4: The quantity $\mathrm{Re}[R_{12}]_{|gg\rangle\to|\Psi^+\rangle}$ as a function of the inverse accelerations α_1, α_2 for different values of $\Delta\omega\Delta t$. We consider a fixed value $\Delta\omega|\Delta\mathbf{x}|= 0.3$. For increasing time intervals the dominant terms are given by the region $\alpha_1 = \alpha_2$. The inverse accelerations α_1, α_2 are measured in units of λ and $\mathrm{Re}[R_{12}]$ is measured in units of λ^{-1}.

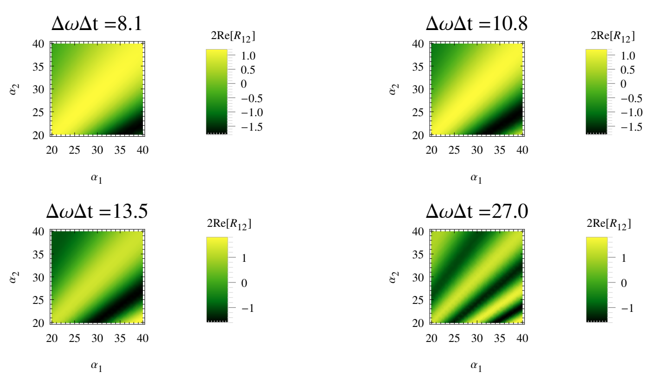

Figure 7.5: The quantity $\mathrm{Re}[R_{12}]_{|gg\rangle\to|\Psi^+\rangle}$ as a function of the inverse accelerations α_1, α_2 for different values of $\Delta\omega\Delta t$. We consider a fixed value $\Delta\omega|\Delta\mathbf{x}|= 0.3$. Here the values of $\Delta\omega\Delta t$ are significantly higher than those of the previous figure. Here it is clear that maximum values show up in other regions besides the region in which $\alpha_1 = \alpha_2$. The inverse accelerations α_1, α_2 are measured in units of λ and $\mathrm{Re}[R_{12}]$ is measured in units of λ^{-1}.

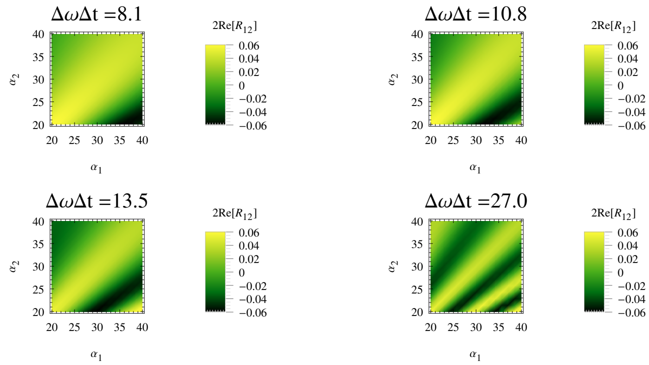

Figure 7.6: The quantity $\mathrm{Re}[R_{12}]_{|gg\rangle\to|\Psi^+\rangle}$ as a function of the inverse accelerations α_1, α_2 for different values of $\Delta\omega\Delta t$. We consider a fixed value $\Delta\omega|\Delta\mathbf{x}|= 3.0$. Again maximum values are located in other regions besides the region $\alpha_1 = \alpha_2$. The inverse accelerations α_1, α_2 are measured in units of λ and $\mathrm{Re}[R_{12}]$ is measured in units of λ^{-1}.

In virtue of the discussion just exposed, let us examine more closely the case in which $\alpha_1 = \alpha_2 = \alpha$. In this case, $G_c^+(\psi)$ becomes $G_\alpha^+(\psi)$ which is defined as

$$G_\alpha^+(\psi) = \frac{-(4\sqrt{2}\pi\alpha)^{-2}}{\sinh\left(\frac{\psi}{2\alpha} - \frac{2i\epsilon}{\alpha} + \frac{\phi}{2}\right)\sinh\left(\frac{\psi}{2\alpha} - \frac{2i\epsilon}{\alpha} - \frac{\phi}{2}\right)}. \tag{7.3.11}$$

The expression (7.3.5) then reads,

$$F_{21}(\Delta\omega, \Delta t) = \int_{-\Delta t}^{\Delta t} d\psi \int_{|\psi|+2\tau_0}^{-|\psi|+2\tau_f} d\eta e^{-i\Delta\omega\psi} G_\alpha^+(\psi). \tag{7.3.12}$$

One can resort to contour integration methods in order to evaluate the integral in ψ. Observe that the integrand have simple poles given by

$$\psi = 2\pi\alpha i n + 4i\epsilon \pm \alpha\phi, \tag{7.3.13}$$

where n is an integer. For $\Delta\omega < 0$ we make use of an infinite semicircle that we close on the upper-half $\mathrm{Im}[\psi] > 0$ plane; for $n \neq 0$ one may take the limit $\epsilon \to 0$ before solving the integral. This contour encloses the poles for $n \geq 0$ and runs in an anticlockwise direction. For $\Delta\omega > 0$ we close the contour in an infinite semicircle in the lower-half $\mathrm{Im}[\psi] < 0$ plane. Now, this contour encloses the poles for $n < 0$ and runs in the clockwise direction (see Fig. B.1). In the asymptotic limit, we have

$$\lim_{\Delta t\to\infty} R_{21}(\Delta\omega, \Delta t) = \frac{\sin(|\Delta\omega|\alpha\phi)}{2\pi\alpha\sinh\phi}\left\{\Theta(-\Delta\omega)\left[1 + \frac{1}{e^{2\pi\alpha|\Delta\omega|} - 1}\right] + \Theta(\Delta\omega)\frac{1}{e^{2\pi\alpha\Delta\omega} - 1}\right\}. \tag{7.3.14}$$

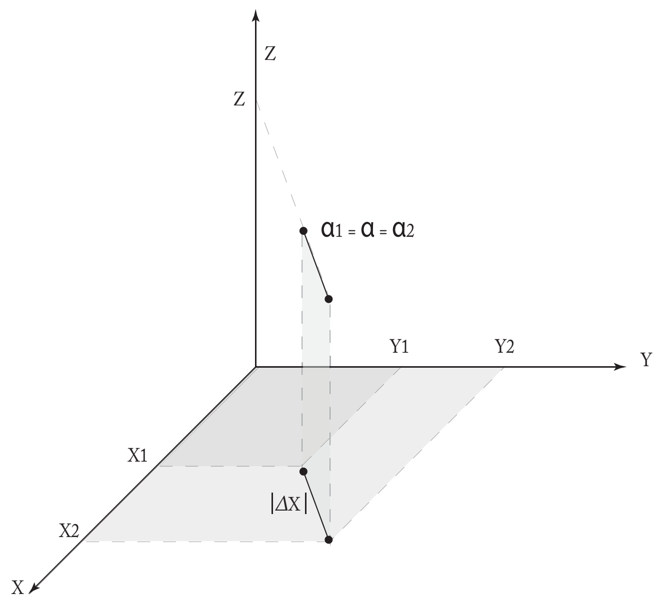

Figure 7.7: Equal-acceleration configuration. When the atoms have equal accelerations the parameter $|\Delta x|$ is used for improve the entanglement extraction.

In a similar way, we obtain the same asymptotic limit for $R_{12}(\Delta\omega, \Delta t)$. For the specific transition $|gg\rangle \to |\Psi^+\rangle$, with matrix elements given by Eq. (7.3.10), one has that

$$\mathrm{Re}[R_{12}(E_{gs})]_{|gg\rangle\to|\Psi^+\rangle} = \mathrm{Re}\left[\frac{\sin(E_{gs}\alpha\phi)}{2\pi\alpha\sinh\phi}\frac{1}{e^{2\pi\alpha\Delta\omega}-1}\right]. \tag{7.3.15}$$

Let us study in more detail the behaviour of such cross contributions as a function of α for different spatial separation $|\Delta\mathbf{x}|$. This is illustrated in Fig. 7.8. One can easily observe that for each $|\Delta\mathbf{x}|$ the function $\mathrm{Re}[R_{12}(E_{gs})]$ has a maximum value at a given acceleration $\alpha = \alpha_{max}$. As expected on the grounds of the above discussions, such maximum values decrease as $|\Delta\mathbf{x}|$ increases.

Consider the quantity α_{max} as a function of $|\Delta\mathbf{x}|$. Then we can express the acceleration as a power series in $|\Delta\mathbf{x}|$ as follows

$$\alpha_{max}(|\Delta\mathbf{x}|) = \frac{1}{|\Delta\mathbf{x}|}\left[a + b|\Delta\mathbf{x}| + c|\Delta\mathbf{x}|^2 + d|\Delta\mathbf{x}|^3 + \cdots\right]. \tag{7.3.16}$$

The graphic of this function is depicted in Fig. 7.9).

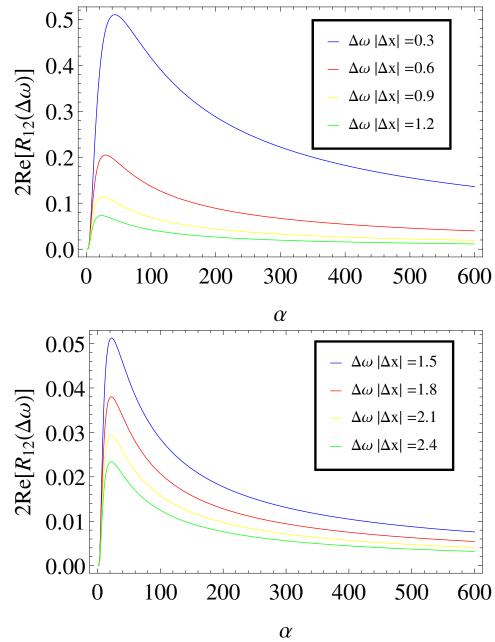

Figure 7.8: The quantity $\mathrm{Re}[R_{12}]_{|gg\rangle\to|\Psi^+\rangle}$ as a function of the inverse acceleration $\alpha_1 = \alpha_2 = \alpha$ of the atoms for different values of $\Delta\omega|\Delta\mathbf{x}|$. The inverse acceleration α is measured in units of λ and $\mathrm{Re}[R_{12}]$ is measured in units of λ^{-1}.

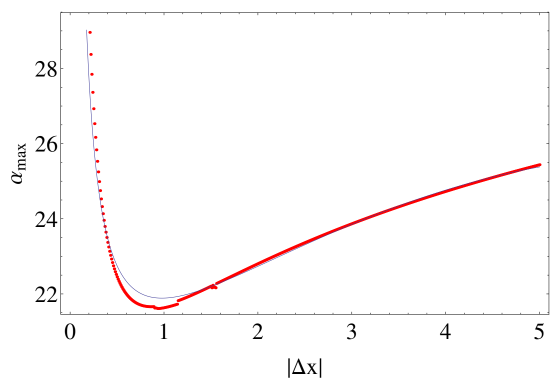

Figure 7.9: Plot of α_{max} as a function $|\Delta\mathbf{x}|$, as given by Eq. (7.3.16), for values of $a \approx 1.8928$, $b \approx 17.8431$, $c \approx 2.3334$, $d \approx -0.18037$ in natural units associated with the transition in study. Since the inverse acceleration α and $|\Delta\mathbf{x}|$ are in units of λ. The blue continuum line represents the fit and the red points represent the theoretical data obtained from the plot given in Fig. 7.8.

To conclude this Section, let us discuss the total transition rate (7.1.15) within the asymptotic time interval regime and for small distances between the atoms. We also consider the case of equal accelerations, in which the cross contributions are given by the Eq. (7.3.15). In this situation one can express the total transition rate as follows

$$\mathcal{R}_{|gg\rangle \to |\Psi^+\rangle} = R_{11} f(E_{gs}\alpha\phi), \tag{7.3.17}$$

where we have defined the function

$$f(x) = 2\left(1 + \frac{\sin x}{x}\right). \tag{7.3.18}$$

This function quantifies the influence of the crossed response functions on the entanglement between atoms, for asymptotic time intervals. Some special values are given by (n is a positive integer)

$$f[(2n+1)\pi/2] = 2\left(1 + \frac{2(-1)^n}{(2n+1)\pi}\right). \tag{7.3.19}$$

The behaviour of this function is depicted in the Fig. 7.10. There is a great oscillatory regime for large accelerations and small distances between the atoms. Since the atoms have the same z coordinate, the plot shows that for $\phi \ll 1/\Delta\omega\alpha$ cross correlations are more important for the rate (7.3.17) in comparison with cases in which the $\Delta\omega|\Delta\mathbf{x}|$ becomes larger. Therefore, the crossed response functions generate a constructive interference when the atoms are near each other in space. In turn, these interference terms vanish for large spatial separations between the atoms. Similar conclusions were reported in Refs. [52, 68, 69, 70].

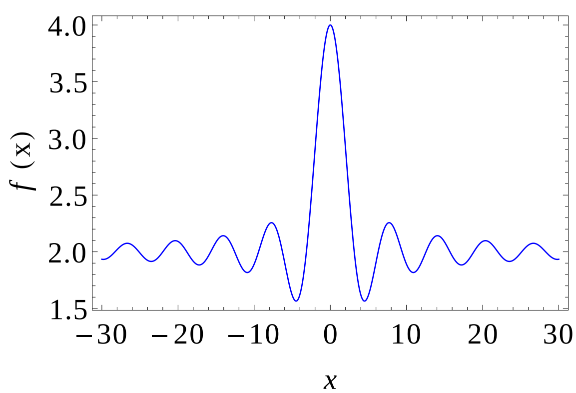

Figure 7.10: The quantity $f(x)$ as a function of $x = \Delta\omega\alpha\phi$.

7.4 Mean life of entangled states

So far we have studied the formation of entangled states through the excitation of the collective ground state $|gg\rangle$. We have demonstrated that, for uniformly accelerated atoms, the interaction with a common quantum field can act as a source of entanglement. On the other hand, such an interaction can also induce decoherence effects. Hence a natural question that emerges is whether such entangled states persist for long time intervals. A possible measurement of the decay of entangled states is given by the mean life of such states. This is defined as

$$\tau^{\ell}_{|\omega'\rangle\to|\omega\rangle}(\Delta\omega,\Delta t) = [\mathcal{R}_{|\omega'\rangle\to|\omega\rangle}(\Delta\omega,\Delta t)]^{-1}. \tag{7.4.1}$$

In order to study the stability of the entangled states under spontaneous emission processes one can study the related transition $|\Psi^{+}\rangle \to |gg\rangle$. The corresponding matrix elements of this transition are again given by Eq. (7.3.10), and the gap energy is $E_{sg} = -\omega_0$. Hence the expression (7.4.1) becomes

$$\tau^{\ell}_{|\Psi^{+}\rangle\to|gg\rangle} = \frac{2}{g^2}\Big\{R_{11}(-E_{gs},\Delta t) + R_{22}(-E_{gs},\Delta t) + 2\mathrm{Re}[R_{12}(-E_{gs},\Delta t)]\Big\}^{-1}. \tag{7.4.2}$$

The behavior of the mean life as a function of the accelerations for a relatively small time interval and with the condition $|\Delta\omega||\Delta\mathbf{x}|\ll 1$ is depicted in the Fig. 7.11. Note that such a function falls off quickly with the acceleration. This result has a clear-cut meaning: quantum entanglement disappears for sufficiently large accelerations.

Fig. 7.12 presents a similar situation as the previous figure but with $|\Delta\omega|\Delta t \gg 1$. We note the emergence of an oscillatory regime. This implies that the mean life of the entangled states displays maximum values at given accelerations of the atoms. In such a scenario one concludes that the mutual influence of atoms will contribute to the entanglement stability only for long times intervals. On the other hand, Fig. 7.13 shows that, for larger values of $|\Delta\omega||\Delta\mathbf{x}|$, the oscillations are less severe than the previous case.

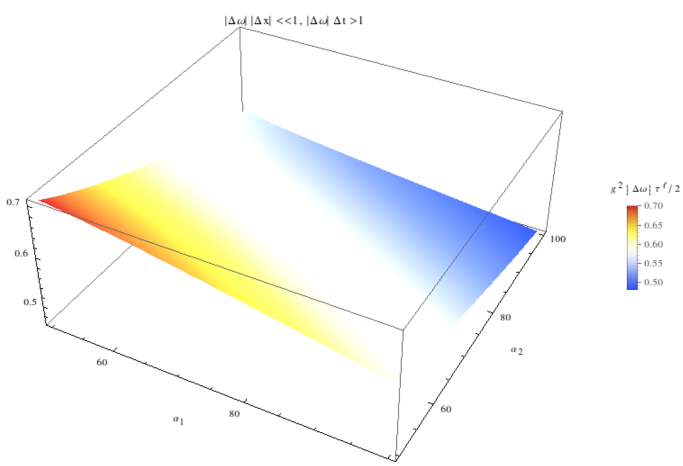

Figure 7.11: The quantity $g^2|\Delta\omega|\tau^{\ell}/2$ as a function of the accelerations of the atoms. We consider the fixed values $|\Delta\omega||\Delta\mathbf{x}|= 0.3$ and $|\Delta\omega|\Delta t = 1.2$. The inverse accelerations α_1, α_2 are measured in units of λ

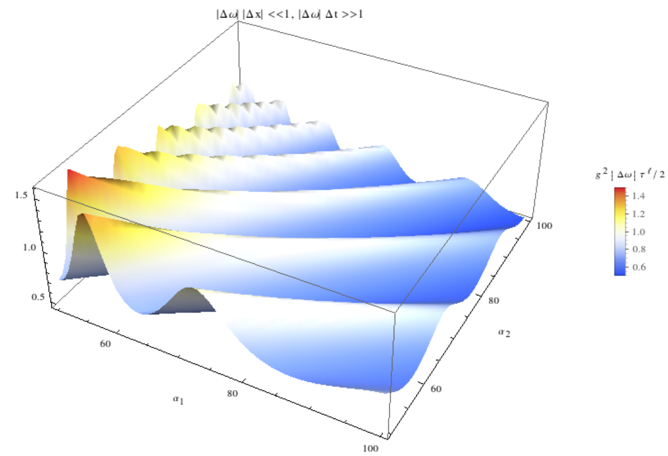

Figure 7.12: The quantity $g^2|\Delta\omega|\tau^\ell/2$ as a function of the accelerations of the atoms. We consider the fixed values $|\Delta\omega||\Delta\mathbf{x}|= 0.3$ and $|\Delta\omega|\Delta t = 60.0$. We have the presence of interference effects that provide more stability for the entangled states. The inverse accelerations α_1, α_2 are measured in units of λ

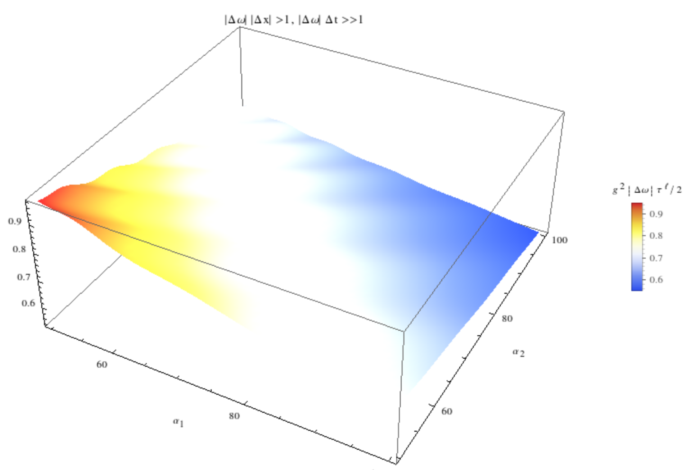

Figure 7.13: The quantity $g^2|\Delta\omega|\tau^\ell/2$ as a function of the accelerations of the atoms. We consider the fixed values $|\Delta\omega||\Delta\mathbf{x}|= 3$ and $|\Delta\omega|\Delta t = 60.0$. The inverse accelerations α_1, α_2 are measured in units of λ

Chapter 8

Concluding remarks

The thermal nature of the Unruh effect is still a mystery. However it is not the only one; we have other situations where we can perceive the vacuum as a black body radiating, such as moving mirror, universe in expansion and black holes. All of them are covering a situation where relativity, quantum mechanics, gravity and thermodynamics are involved. The unexpected and exotic properties of the quantum vacuum gave the vacuum an important role in the fundamental physics. Why? Since we have defined the vacuum as a state of minimum energy, we automatically open a wide set of extraordinary possibilities. The fundamental characteristic of the above mentioned physical situations is the existence of horizons, a surface that determine two (or more) causally disconnected regions. We could say that thermal radiation has a topological origin.

These manifestations of the vacuum lead us to think that there is something more fundamental that the structure we are using. With the asymptotic freedom, these important effects of vacuum allow us to perceive how we can surpass the obstacles to achieve the unification of the forces of Nature. These kind of results as we presented here, show that there is a thermodynamic connection in a fundamental level.

On the other hand, these results strengthen the fact that the entanglement is becoming an observer dependent phenomena. With this kind of situations we can perform experimental devices where these local properties of the entanglement will bring information about the space-time structure. Because in relativistic quantum information while a detector sees the same thermal bath in diverse scenarios, the way in which two separated detectors become entangled senses a difference [59]. Quantum correlations acquired by space-like separated detectors interacting with the same quantum field are dramatically sensitive to the space-time background and the state of motion of the detectors [59]. These systems that are currently treated in relativistic quantum information are a potential useful tool to probe the geometry of the space-time, stablish quantum communication satellite network and observe directly quantum gravity effects as quantum fluctuations of curvature.

Therefore, the main results of the work are presented in the last chapter where we have studied two identical uniformly accelerated two-level atoms weakly coupled with a massless scalar field prepared in the Minkowski vacuum state. We have shown the possibility of generation of entanglement between such two atoms initially prepared in the ground state. We also found that the associated response function contains terms related to cross correlations between the atoms mediated by the field. Since the atoms move along different world lines, such crossed terms present thermal contributions with different temperatures. The crossed terms of the response function are modulated by an oscillating function. In addition, such contributions are accompanied by a gray body factor, composed by a linear combination of derivatives of Hurwitz-Lerch zeta-functions, whose arguments have information on the accelerations and the energy gap. The appearance of the aforementioned gray body factor may be understood in the sense of Ref. [53], in which the authors demonstrate the emergence of a thermal noise produced by the fluctuations of the fields and field correlations between the two trajectories. Moreover, we also presented a general expression for the mean life of the entangled states. In general, we found that atoms with same acceleration will be less correlated for an increasing $|\Delta\mathbf{x}|$, as expected.

With the results obtained here we can conclude that there is a non zero probability for the system of two accelerated atoms to become entangled. It is clear that thermal effects can produce decoherence, but we are evidencing that there is an interference pattern produced by the presence of the interaction between the two atoms and the field where we can extract coherence quantum properties as it is the entanglement enhancement. Furthermore, decoherence effects and open quantum systems have been studied by modelling dissipative environment as a collection of harmonic oscillators, spin chains and others [71],[72],[73],[74], our environment here is a quantum field in space-time background, following this, can we talk about the space-time as a collection of harmonic oscillators or a spin set? When we are talking about gravitational waves, this could not be an ambiguous idea.

Our treatment can be extended to a complex scalar field, spinor fields or electromagnetic field and in the infinite acceleration limit, extend this for the Schwarzschild spacetime. Since the presence of boundaries will affect the vacuum fluctuations of the quantum field, a natural extension for this work is to discuss the mean life times of these accelerated entangled in presence of a reflecting plane. In order to continue the investigation of boundary effects, it can be generalized to the problem where we have more than one boundary. In the case of the presence of two parallel reflecting planes, we may compute the green function as an infinite image sum [12]. Thinking of experimental setups, these situations, would give us a technique to improve coherence in quantum information processing and new interesting results in cavity-QED. On the other hand, to clarify explicitly the contributions of vacuum fluctuations and radiation reaction, it can be used to this situation the formalism presented by Dalibard, Dupont-Roc and Cohen-Tannoudji [70, 75, 76] (known as DDC formalism). It can be interesting to obtain a general expression for different accelerations and on-time switching for the contributions of vacuum fluctuations and radiation reaction.

Furthermore, in order to clarify certain questions about the stochastic properties involved in the Unruh effect, such as quantum noise, correlation and dissipation, it can be applied the influence functional method introduced by Raval, B. L. Hu and Anglin [53]. Finally following [59] and taking into consideration the interference pattern structure where we have regions of high entanglement stability, obtained in this work, we could add one inertial detector and study the properties involved in a quantum teleportation protocol, such negativity, fidelity and other control parameters.

Let us remark the following final words to clarify the thermodynamic connection. James Clerk Maxwell, with its treatise on electricity and magnetism, introduced the electromagnetic waves, a new form of matter. In the same form, quantum field theory gives us a wide spectrum of new forms of matter, that were condensed in the standard model. In order to show the existence of these new forms of matter, a great effort was made as we can observe in the particle accelerators, where, apart from consolidating experimentally the standard model, new forms are being discovered, for example, the pentaquark. Furthermore, with the existence of gravitational waves, we are able to explore the universe not only with electromagnetic waves, but with gravitational waves.

There is a quantum theory or a fully interactive theory for three of the four fundamental interactions. As we see above, the proposal of an existence of ether in order to understand failures in the classical theory of electromagnetic waves propagation, was a proposal with a geometrical basis, in the same form, currently a geometrical question is inquiring us about the behaviour of gravitational waves in the framework of a quantum field theory, it could be exposing certain failures of the theory in order to find a quantum theory of gravity.

As it occurred in 1900 with the black body radiation, a thermodynamic phenomena, where the electromagnetic waves are the principal physical entity, nowadays, the semi-classical treatment of gravity is showing us thermodynamic phenomena, where we have *particles* been created by external gravitational fields, places where we have gravitational waves too. As it occurred in 1900 with the black body radiation that the classical theory fails and the great mind of Planck gave us the begin of quantum mechanics, can a great change of paradigm occur again with these thermodynamic phenomena exposed by quantum field theory in curved space-time?.

The failures of the modern theories in the search of a model beyond the standard model are showing us that our consciousness is still in a certain form in the classical paradigm. Our task must be to identify how these failures have a classical foundation and transform them. As we see above, the construction of mental images is resulting insufficient to understand the new paradigm, can this alternative perception of the world with thinking without images, topology without points trough the symphony of the universe, the melody and the thermodynamic phenomena of vacuum, bring us the pathway beyond the standard model to grand unification theory? As Planck said ”an act of despair... I was ready to sacrifice any of my previous convictions about physics”. The only safe aspect about the present theory is that it will show that it is so unsuitable like his predecessors.

Appendix A

Thermal States

The black-body radiation, that we have found in the Unruh effect and in the expressions of the finite time response functions of the entangled detectors, has been studied as a electromagnetic radiation emitted by a hot body. This radiation is also known as thermal light. The properties of this thermal states are studied by applying the laws of statistical mechanics to the radiation within an enclosed cavity at a temperature T. In this appendix we explore the origin of the Planck factor that appears in the Unruh effect, specifically in the mean particle number, and why we can say that this effect has a thermal nature.

The radiation pattern consists of a continuous spectrum of oscillating modes, with the energy density within the angular frequency range ω to $\omega + d\omega$ given by the Planck's law (recovering constants)

$$\rho(\omega, T)d\omega = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{e^{\hbar\omega/k_B T} - 1}d\omega. \tag{A.0.1}$$

The expression (A.0.1) has sense if the energy of the radiation is quantized. We can consider each individual mode as a harmonic oscillator of angular frequency ω. The energy has the well known form

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \tag{A.0.2}$$

where as before n is a positive integer.

We consider a single radiation mode at angular frequency ω inside the cavity. The probability that there will be n photons in the mode is given by the Boltzmann's Law:

$$\mathcal{P}_\omega(n) = \frac{e^{-E_n/k_B T}}{\sum_{n=0}^{\infty} e^{-E_n/k_B T}}. \tag{A.0.3}$$

Substituting the definition of energy (A.0.2), the probability reads

$$\mathcal{P}_\omega(n) = \frac{e^{-n\hbar\omega/k_B T}}{\sum_{n=0}^{\infty} e^{-n\hbar\omega/k_B T}}, \tag{A.0.4}$$

which can be written in the form

$$\mathcal{P}_\omega(n) = \frac{x^n}{\sum_{n=0}^{\infty} x^n}, \tag{A.0.5}$$

being

$$x = e^{-\hbar\omega/k_B T}. \tag{A.0.6}$$

Using the series identity

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad x < 1, \tag{A.0.7}$$

we find

$$\mathcal{P}_\omega(n) = e^{-n\hbar\omega/k_BT}\left(1 - e^{-\hbar\omega/k_BT}\right). \tag{A.0.8}$$

The mean photon number is given by

$$\begin{aligned}
\bar{n} &= \sum_{n=0}^{\infty} n\mathcal{P}_\omega(n) \\
&= \sum_{n=0}^{\infty} nx^n(1-x) \\
&= x(1-x)\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) \\
&= x(1-x)\frac{d}{dx}\left(\frac{1}{1-x}\right) \\
&= x(1-x)\frac{1}{(1-x)^2} \\
&= \frac{x}{1-x},
\end{aligned} \tag{A.0.9}$$

which, with the definition (A.0.6), gives the Planck formula

$$\bar{n} = \frac{1}{e^{\hbar\omega/k_BT} - 1}. \tag{A.0.10}$$

The probability can be expressed in terms of the Planck formula

$$\mathcal{P}_\omega(n) = \frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n. \tag{A.0.11}$$

This distribution is called the Bose-Einstein distribution.

The magnitude of the energy fluctuations from the mean value at thermal equilibrium is given by

$$\left\langle \Delta E^2 \right\rangle = k_BT^2\frac{\partial \langle E \rangle}{\partial T}. \tag{A.0.12}$$

Then the energy fluctuations of the black body radiation in the angular frequency range ω to $\omega + d\omega$ yields

$$\begin{aligned}
\left\langle \Delta E^2 \right\rangle d\omega &= k_BT^2\frac{\partial}{\partial T}(V\rho d\omega) \\
&= k_BT^2 V d\omega\frac{\partial \rho}{\partial T} \\
&= \left(\hbar\omega\rho + \frac{\pi^2c^3}{\omega^2}\rho^2\right)V d\omega,
\end{aligned} \tag{A.0.13}$$

where V is the volume of the cavity and ρ is the spectral energy density. These energy fluctuations can be written in the following form

$$\begin{aligned}
\left\langle \Delta E^2 \right\rangle d\omega &= \text{density of states} \times \text{energy fluctuations per mode} \times \text{volume} \\
&= \frac{\omega^2}{\pi^2 c^3} d\omega \times \left\langle (\Delta(n\hbar\omega))^2 \right\rangle \times V \\
&= \frac{\omega^2}{\pi^2 c^3} (\Delta n)^2 (\hbar\omega)^2 V d\omega.
\end{aligned} \tag{A.0.14}$$

Comparing (A.0.13) with (A.0.14) we find that

$$(\Delta n)^2 = \frac{\pi^2 c^3}{\hbar\omega^3}\rho + \left(\frac{\pi^2 c^3}{\hbar\omega^3}\rho\right)^2. \tag{A.0.15}$$

If we express the spectral density (A.0.1) in terms of $\bar{n}$, we have

$$\rho = \frac{\hbar\omega^3}{\pi^2 c^3}\bar{n}, \tag{A.0.16}$$

then

$$(\Delta n)^2 = \bar{n} + \bar{n}^2. \tag{A.0.17}$$

The Einstein's understanding assume that the first term in (A.0.13) is due to the particle nature of the light, while the second is given by the thermal fluctuations of the energy of the electromagnetic radiation. This last term has a classical origin and is called the wave noise. The first term has its origin in the quantization of energy of the electromagnetic radiation. It is given by the photon nature of light.

Appendix B

Further developments

B.1 Convergence values of interest

In this appendix we enlist some sums results of the text

$$\sum_{k=-\infty}^{\infty} e^{2\pi\sigma\Delta\omega k} = \Theta(-\Delta\omega)\left[1+\frac{1}{e^{2\pi\sigma|\Delta\omega|}-1}\right]+\Theta(\Delta\omega)\left[\frac{1}{e^{2\pi\sigma\Delta\omega}-1}\right] \tag{B.1.1}$$

$$\begin{aligned}
\sum_{l=1}^{\infty}\frac{\sin(\Delta\omega L l)}{l\left(2\alpha+\frac{(lL)^2}{4\alpha}\right)} &= \frac{e^{-iL\Delta\omega}}{8\sqrt{\alpha^2}\left(8\alpha^2+L^2\right)}\left\{-iL^2\,{}_2F_1\left(1,\frac{2i\sqrt{2}\alpha}{L}+1;\frac{2i\sqrt{2}\alpha}{L}+2;e^{-iL\omega}\right)\right.\\
&+ iL^2 e_2^{2iL\Delta\omega}F_1\left(1,\frac{2i\sqrt{2}\alpha}{L}+1\frac{2i\sqrt{2}\alpha}{L}+2;e^{iL\omega}\right)\\
&+ L\left(2\sqrt{2}\alpha-iL\right)\,{}_2F_1\left(1,1-\frac{2i\sqrt{2}\alpha}{L};2-\frac{2i\sqrt{2}\alpha}{L};e^{-iL\omega}\right)\\
&+ iL\left(L+2i\sqrt{2}\alpha\right)e^{2iL\omega}\,{}_2F_1\left(1,1-\frac{2i\sqrt{2}\alpha}{L};2-\frac{2i\sqrt{2}\alpha}{L};e^{iL\omega}\right)\\
&- 2\sqrt{2}\alpha L\,{}_2F_1\left(1,\frac{2i\sqrt{2}\alpha}{L}+1;\frac{2i\sqrt{2}\alpha}{L}+2;e^{-iL\omega}\right)\\
&+ 2\sqrt{2}\alpha L e^{2iL\omega}\,{}_2F_1\left(1,\frac{2i\sqrt{2}\alpha}{L}+1;\frac{2i\sqrt{2}\alpha}{L}+2;e^{iL\omega}\right)\\
&- 2iL^2e^{iL\omega}\log\left(1-e^{-iL\omega}\right)+2iL^2e^{iL\omega}\log\left(1-e^{iL\omega}\right)\\
&- 16i\alpha^2e^{iL\omega}\log\left(1-e^{-iL\omega}\right)+16i\alpha^2e^{iL\omega}\log\left(1-e^{iL\omega}\right)
\end{aligned} \tag{B.1.2}$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function defined for $|z|< 1$ by the power series

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{B.1.3}$$

where $(q)_n$ is the Pochhammer symbol defined by

$$(q)_n = \begin{cases} 1 & \text{for } n = 0 \\ q(q+1)\cdots(q+n-1) & \text{for } n = 0 \end{cases}. \tag{B.1.4}$$

The functions $f_{s_1}^{(c)}$ and $f_{s_2}^{(c)}$ are defined as follows

$$\begin{aligned} f_{s_1}^{(c)} &= \sum_{k=1}^{\infty} \frac{\sin(\Delta\omega(2z-kL))}{(2z-kL)\left(\sqrt{\frac{z^2}{\alpha^2}+1} - \frac{zLk}{2\alpha^2\sqrt{\frac{z^2}{\alpha^2}+1}}\right)} \\ &= \frac{i}{2L\alpha} e^{-i\Delta\omega(L+2z)} \sqrt{z^2+\alpha^2} \left\{ e^{4i\Delta\omega z} \left[\Phi\left(e^{-i\Delta\omega L}, 1, \frac{L-2z}{L}\right) - \Phi\left(e^{-i\Delta\omega L}, 1, \frac{Lz-2z^2-2\alpha^2}{Lz}\right)\right]\right. \\ &+ \left. e^{2i\Delta\omega z} \left[\Phi\left(e^{i\Delta\omega L}, 1, \frac{L-2z}{L}\right) + \Phi\left(e^{i\Delta\omega L}, 1, \frac{Lz-2z^2-2\alpha^2}{Lz}\right)\right]\right\} \end{aligned} \tag{B.1.5}$$

$$\begin{aligned} f_{s_2}^{(c)} &= \sum_{k=1}^{\infty} \frac{\sin(\Delta\omega(2z-kL))}{(2z+kL)\left(\sqrt{\frac{z^2}{\alpha^2}+1} + \frac{zLk}{2\alpha^2\sqrt{\frac{z^2}{\alpha^2}+1}}\right)} \\ &= \frac{-i}{2L\alpha} e^{-i\Delta\omega(L+2z)} \sqrt{z^2+\alpha^2} \left\{ -\Phi\left(e^{-i\Delta\omega L}, 1, \frac{L+2z}{L}\right) + \Phi\left(e^{-i\Delta\omega L}, 1, \frac{Lz+2z^2+2\alpha^2}{Lz}\right)\right. \\ &+ \left. e^{2i\Delta\omega(L+2z)} \left[\Phi\left(e^{i\Delta\omega L}, 1, \frac{L+2z}{L}\right) - \Phi\left(e^{i\Delta\omega L}, 1, \frac{Lz+2z^2+2\alpha^2}{Lz}\right)\right]\right\} \end{aligned} \tag{B.1.6}$$

B.2 Explicit calculation of F_{11} and F_{22}

In this Appendix we concisely perform the evaluation of the individual contributions of the atoms to the total response function. In order to study the contribution (7.2.4), we perform the Fourier transform with the help of contour integration methods. From the expression (7.2.2) one notes the existence of second order poles of the form

$$\psi = 2i\epsilon - 2\pi i\alpha_1 n, \tag{B.2.1}$$

where n is an integer. One must treat separately the cases of $n \neq 0$ and $n = 0$. For $n \neq 0$ we may take the limit $\epsilon \to 0$ before solving the integral. For $\Delta\omega < 0$ we make use of a semicircle of radius R that we close on the upper-half $Im[\psi] > 0$ plane. This contour encloses the poles for $n \geq 0$ and runs in an anticlockwise direction. For $\Delta\omega > 0$ we close the contour in a semicircle of radius R in the lower-half $Im[\psi] < 0$ plane. Now, this contour encloses the poles for $n < 0$ and runs in the clockwise direction (see Fig. B.1). We consider the limit $R \to \infty$ such that the contribution from the arcs will vanish by the Jordan's lemma. We obtain, for the atom 1

$$\begin{aligned} F_{11}(\Delta\omega, \Delta t) &= \frac{\Delta t}{2\pi^2} \left\{ \pi|\Delta\omega|\Theta(-\Delta\omega) + |\Delta\omega|\left(\mathtt{Si}\Delta\omega\Delta t - \frac{\pi}{2}\right) + \frac{\pi|\Delta\omega|}{e^{2\pi\alpha_1|\Delta\omega|}-1}\right. \\ &+ \left. \int_{\Delta t}^{\infty} d\psi \cos(\Delta\omega\psi)\left(\frac{1/(2\alpha_1)^2}{\sinh^2\psi/(2\alpha_1)} - \frac{1}{\psi^2}\right)\right\} + \frac{1}{2\pi^2}\left\{\cos(\Delta\omega\Delta t) + \log\left(\frac{\Delta t}{2\pi\epsilon}\right) - 1\right. \\ &+ \left. \int_0^{\Delta t} d\psi \frac{\cos(\Delta\omega\psi)-1}{\psi} + \int_0^{\Delta t} d\psi\psi\cos\Delta\omega\psi\left(\frac{1/(2\alpha_1)^2}{\sinh^2\psi/(2\alpha_1)} - \frac{1}{\psi^2}\right)\right\}. \end{aligned} \tag{B.2.2}$$

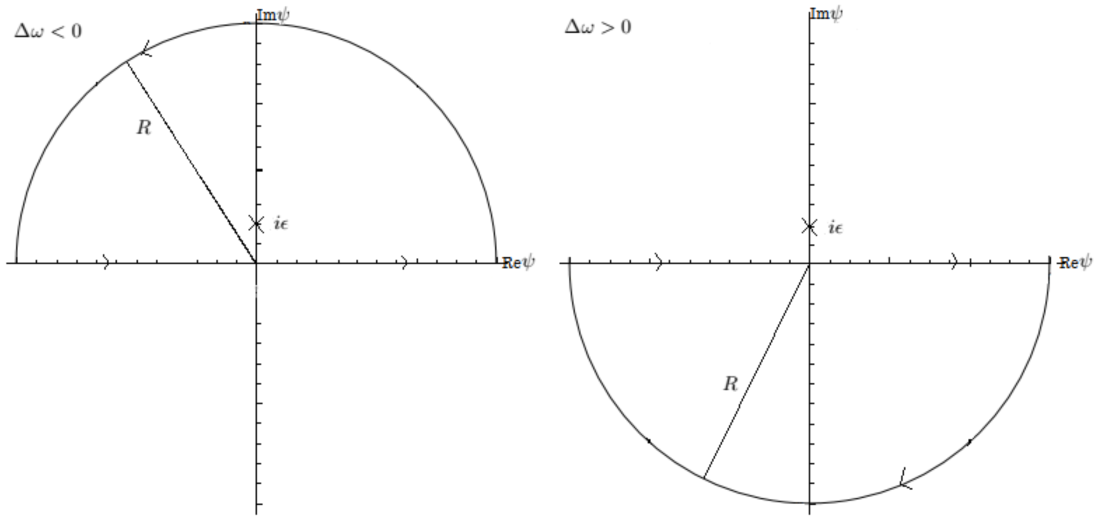

Figure B.1: Contour used to perform the integral of $R_{11}(\Delta\omega, \Delta t)$ and $R_{22}(\Delta\omega, \Delta t)$.

This is the expression (7.2.5). By the definition (7.2.6), we have

$$\begin{aligned} R_{11}(\Delta\omega, \Delta t) &= \frac{|\Delta\omega|}{2\pi}\left\{\Theta(-\Delta\omega) + \frac{1}{e^{2\pi\alpha_1|\Delta\omega|} - 1} + \frac{\cos(\Delta\omega\Delta t)}{\pi\Delta\omega\Delta t} + \frac{\mathtt{Si}\Delta\omega\Delta t}{\pi} - \frac{1}{2}\right\} \\ &+ \frac{1}{2\pi^2}\int_{\Delta t}^{\infty} d\psi\cos(\Delta\omega\psi)\left(\frac{1/(2\alpha_1)^2}{\sinh^2\psi/(2\alpha_1)} - \frac{1}{\psi^2}\right), \end{aligned} \tag{B.2.3}$$

where $\mathtt{Si}(z)$ is the sine integral function given by [82]

$$\mathtt{Si}(z) = \int_0^z \frac{\sin t}{t}dt. \tag{B.2.4}$$

In an analogous way, for the individual contribution of the atom 2 given by Eq. (7.2.10) one has that (using the same contour depicted in the Fig. (B.1))

$$\begin{aligned} F_{22}(\Delta\omega, \Delta t) &= \frac{\Delta t}{2\pi^2}\left\{\pi|\Delta\omega|\Theta(-\Delta\omega) + |\Delta\omega|\left(\mathtt{Si}\Delta\omega\Delta t - \frac{\pi}{2}\right) + \frac{\pi|\Delta\omega|}{e^{2\pi\alpha_2|\Delta\omega|} - 1}\right. \\ &+ \left.\int_{\Delta t}^{\infty} d\psi\cos(\Delta\omega\psi)\left(\frac{1/(2\alpha_2)^2}{\sinh^2\psi/(2\alpha_2)} - \frac{1}{\psi^2}\right)\right\} + \frac{1}{2\pi^2}\left\{\cos(\Delta\omega\Delta t) + \log\left(\frac{\Delta t}{2\pi\epsilon}\right) - 1\right. \\ &+ \left.\int_0^{\Delta t} d\psi\frac{\cos(\Delta\omega\psi) - 1}{\psi} + \int_0^{\Delta t} d\psi\psi\cos\Delta\omega\psi\left(\frac{1/(2\alpha_2)^2}{\sinh^2\psi/(2\alpha_2)} - \frac{1}{\psi^2}\right)\right\}, \end{aligned} \tag{B.2.5}$$

and consequently

$$\begin{aligned} R_{22}(\Delta\omega, \Delta t) &= \frac{|\Delta\omega|}{2\pi}\left\{\Theta(-\Delta\omega) + \frac{1}{e^{2\pi\alpha_2|\Delta\omega|} - 1} + \frac{\cos(\Delta\omega\Delta t)}{\pi\Delta\omega\Delta t} + \frac{\mathtt{Si}\Delta\omega\Delta t}{\pi} - \frac{1}{2}\right\} \\ &+ \frac{1}{2\pi^2}\int_{\Delta t}^{\infty} d\psi\cos(\Delta\omega\psi)\left(\frac{1/(2\alpha_2)^2}{\sinh^2\psi/(2\alpha_2)} - \frac{1}{\psi^2}\right). \end{aligned} \tag{B.2.6}$$

B.3 Explicit calculation of F_{12} and F_{21}

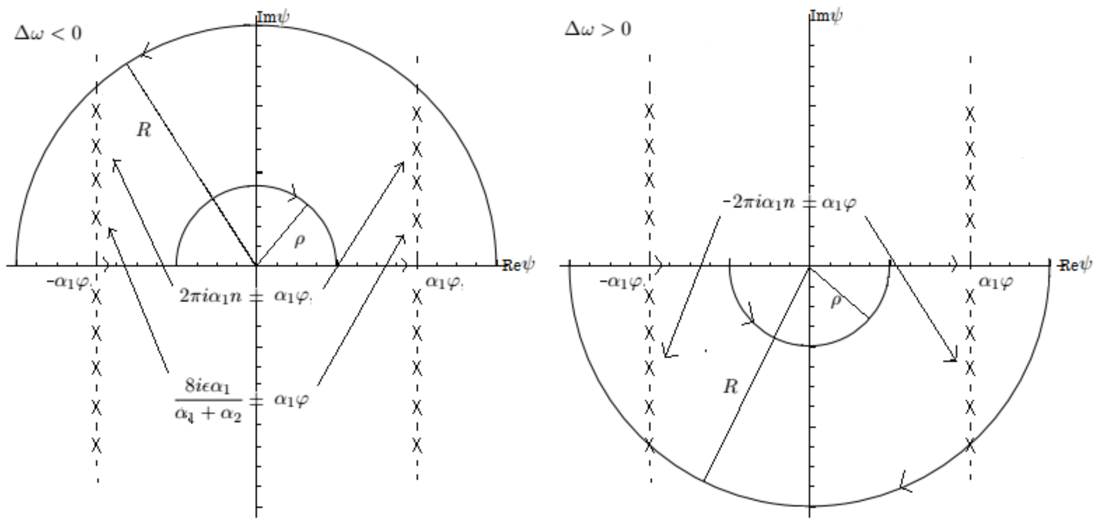

Figure B.2: Contour used for perform the integral of $R_{12}(\Delta\omega, \Delta t)$ and $R_{21}(\Delta\omega, \Delta t)$.

In this Appendix we perform the explicit evaluation of the cross contributions F_{12} and F_{21}. As above, we shall employ the method of residues. The integral (7.3.7) can be expressed as,

$$I(\Delta\omega, \Delta t, \sigma) = \int_0^\infty d\psi e^{-i\sigma\Delta\omega\psi} G^+_{c_0}(\psi) - \int_{\Delta t}^\infty d\psi e^{-i\sigma\Delta\omega\psi} G^+_{c_0}(\psi). \tag{B.3.1}$$

For the first term on the right-hand side of the expression (B.3.1), the simple poles of the integrand are given by

$$\psi_n = 2\pi i\alpha_1 n + \frac{8i\epsilon\alpha_1}{\alpha_1 + \alpha_2} \pm \alpha_1\phi, \tag{B.3.2}$$

where n is an integer. Making use of the following auxiliary integrals

$$\oint_C dz e^{-i\Delta\omega\sigma z} \log(z) G^+_{c_0}(z), \qquad \oint_C dz e^{-i\Delta\omega\sigma z} G^+_{c_0}(z),$$

we shall perform the integral for $\Delta\omega < 0$ with a contour such that it encloses the poles for $n \geq 0$ and runs in an anticlockwise direction. For processes with $\Delta\omega > 0$ we employ a contour such that it encloses the poles for $n < 0$ and runs in the clockwise direction (in this case one may perform the limit $\epsilon \to 0$ before the evaluation of integral, see Fig.B.2). In the limit $\rho \to 0$ and $R \to \infty$ and we obtain the following expression

$$\begin{aligned}\int_0^\infty d\psi e^{-i\sigma\Delta\omega\psi} G^+_{c_0}(\psi) &= \frac{4\alpha_1}{\sinh\phi} \sin(|\Delta\omega|\sigma\alpha_1\phi) \left\{ \left[\nu_0 + \frac{\pi}{e^{2\pi\alpha_1\sigma|\Delta\omega|} - 1} + \zeta(\Delta\omega, \sigma) \right] \Theta(-\Delta\omega) \right. \\ &+ \left. \left[\frac{\pi}{e^{2\pi\alpha_1\sigma\Delta\omega} - 1} + \zeta(\Delta\omega, \sigma) \right] \Theta(\Delta\omega) \right\}. \end{aligned} \tag{B.3.3}$$

This is the expression (7.3.8), where we have defined the function

$$\zeta(\Delta\omega, \sigma) = \zeta_1(\Delta\omega, \sigma) + \cot(|\Delta\omega|\alpha_1\sigma\phi)\zeta_2(\Delta\omega, \sigma), \tag{B.3.4}$$

with

$$\begin{aligned}\zeta_1(\Delta\omega,\sigma) &= -ie^{-2\pi\alpha_1\sigma\Delta\omega}\Big\{\mathrm{Re}\left(\Phi^{(0,1,0)}\left(e^{-2\pi\alpha_1\sigma\Delta\omega},0,\chi\right)\right)\\ &+ e^{2\pi\alpha_1\sigma\Delta\omega}\left[1+\log(2\alpha_1\sqrt{\pi})+e^{2\pi\alpha_1\sigma\Delta\omega}\,\mathrm{Re}\left(\Phi^{(0,1,0)}\left(e^{2\pi\alpha_1\sigma\Delta\omega},0,\chi\right)\right)\right]\Big\}\end{aligned} \tag{B.3.5}$$

and

$$\begin{aligned}\zeta_2(\Delta\omega,\sigma) &= 8ie^{-2\pi\alpha_1\sigma\Delta\omega}\Big\{\mathrm{Im}\left(\Phi^{(0,1,0)}\left(e^{-2\pi\alpha_1\sigma\Delta\omega},0,\chi\right)\right)\\ &- e^{2\pi\alpha_1\sigma\Delta\omega}\left[\frac{\pi}{2}+e^{2\pi\alpha_1\sigma\Delta\omega}\,\mathrm{Im}\left(\Phi^{(0,1,0)}\left(e^{2\pi\alpha_1\sigma\Delta\omega},0,\chi\right)\right)\right]\Big\}.\end{aligned} \tag{B.3.6}$$

In the above, $\Phi\left(z,s,\alpha\right)$ the Hurwitz–Lerch zeta-function, $\Phi^{(0,1,0)}\left(z,s,\alpha\right)$ is its first derivative with respect to its second variable and $\chi = 1+(i\phi/2\pi)$. Recalling Eq. (7.3.6), we observe that the contributions with $\sigma = 1$ are associated with $\beta_1^{-1} = 1/2\pi\alpha_1$ whereas the contributions with $\sigma = \alpha_2/\alpha_1$ are related to $\beta_2^{-1} = 1/2\pi\alpha_2$. The contribution of the cross correlations to the total transition rate is given by the real part of the following expression

$$\begin{aligned}\left[R_{12}\right]_{|g\rangle\to|s\rangle} &= \frac{i}{16\pi^2\alpha_1^2E_{gs}a_-}\Big\{iE_{gs}a_-\Big[e^{iE_{gs}(a_-/\alpha_1)(\Delta t+\tau_0)}\left[I(E_{gs},\Delta t,-1)+I(E_{gs},\Delta t,\alpha_2/\alpha_1)\right]\\ &+ e^{-iE_{gs}(a_-/\alpha_1)(\Delta t-\tau_f)}\left[I(E_{gs},\Delta t,1)+I(E_{gs},\Delta t,-\alpha_2/\alpha_1)\right]\Big]\\ &+ \alpha_1G^+_{c_0}(\Delta t)\Big[e^{iE_{gs}(a_-/\alpha_1)\tau_0}\left(e^{iE_{gs}\Delta t\alpha_2/\alpha_1}+e^{iE_{gs}\Delta t(2\alpha_2-\alpha_1)/\alpha_1}\right)\\ &- e^{iE_{gs}(a_-/\alpha_1)\tau_f}\left(e^{-iE_{gs}\Delta t\alpha_2/\alpha_1}+e^{-iE_{gs}\Delta t(2\alpha_2-\alpha_1)/\alpha_1}\right)\Big]\Big\},\end{aligned} \tag{B.3.7}$$

where $E_{gs} = \omega_0$.

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