ISSN 1063-780X, Plasma Physics Reports, 2007, Vol. 33, No. 5, pp. 407–419. © Pleiades Publishing, Ltd., 2007.
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NONLINEAR PHENOMENA

Effect of the Magnetic Field Curvature on the Generation of Zonal Flows by Drift-Alfvén Waves

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Received June 19, 2006; in final form, September 7, 2006

Abstract—The generation of zonal flows by drift-Alfvén waves is studied with allowance for magnetic curvature effects. The basic plasmadynamic equations relating the electrostatic potential, vector potential, and perturbed plasma density are the vorticity equation, longitudinal Ohm's law, and continuity equation. The basic equations are analyzed by applying a parametric formalism similar to that used in the theory of the generation of convective cells. In contrast to most previous investigations on the subject, consideration is given to primary modes having an arbitrary spectrum rather than to an individual monochromatic wave packet. The parametric approach so modified makes it possible to reveal a new class of instabilities of zonal flows that are analogous to two-stream instabilities in linear theory. It is shown that, in the standard theory of zonal flows, the zonal components of the vector potential and perturbed density are not excited. It is pointed out that zonal flows can be generated both in the case of a magnetic hill and in the case of a magnetic well. In the first case, the instabilities of zonal flows are analogous to negative-mass instabilities in linear theory, and, in the second case, they are analogous to two-stream instabilities.

PACS numbers: 52.35.Mw

DOI: 10.1134/S1063780X07050066

1. INTRODUCTION

Study of the generation of zonal flows by turbulence is one of the main lines of research in the present-day nonlinear theory of magnetized plasma, because zonal flows can reduce turbulence-driven anomalous transport [1]. In this context, it would be of interest to investigate the possibility of zonal flow generation by various types of fundamental turbulent modes. The best studied case is the generation of zonal flows by electrostatic electron drift modes [2] and some other electrostatic modes (see [1] and the literature cited therein, as well as more recent papers [3, 4]).

According to [5], drift-Alfvén (DA) modes [6] are among the fundamental modes in an inhomogeneous magnetized plasma. They are described by the dispersion relation

$$\omega(\omega - \omega_{*i}) - k_z^2 v_A^2 = 0. \qquad (1.1)$$

Here, ω is the mode frequency; ω_{*i} is the ion drift frequency; v_A is the Alfvén velocity; and k_z is the projection of the wave vector $\mathbf{k}$ onto the direction of the equilibrium magnetic field pointing along the z axis of a Cartesian coordinate system (x, y, z), where x is the direction in which the plasma density varies (the radial coordinate) and y is the ion diamagnetic drift direction (the poloidal coordinate). In accordance with [5, 6], perturbations described by dispersion relation (1.1) are long-wavelength ones in the sense that $k_\perp \rho_i \ll 1$, where $k_\perp$ is the transverse wavenumber and ρ_i is the ion Larmor radius.

Dispersion relation (1.1) does not take into account radial dispersion effects. This is why the modes described by this relation can be called pure or nondispersive DA modes. The question then naturally arises of whether pure DA modes can generate zonal flows. We investigate this question and show that, in the steady-state approximation (see below for details), such generation is impossible. Hence, in order to describe the generation of zonal flows by DA modes, it is necessary to account for the radial mode dispersion. From [7], we can see that DA modes are subject to the so-called curvature dispersion (see also [5]) caused by the magnetic well/hill effects. According to [8, 9], magnetic curvature effects can be described by a model gravity force g, with which dispersion relation (1.1) becomes

$$\omega(\omega - \omega_{*i}) - k_z^2 v_A^2 - g\kappa_n k_y^2/k_x^2 = 0, \qquad (1.2)$$

where κ_n is the reciprocal of the plasma density scale length (see below for details). Our main purpose here is to study whether zonal flows can be generated by the modes satisfying dispersion relation (1.2). We restrict ourselves to considering the modes such that $k_y/k_x \ll 1$, where k_x and k_y are the x and y components of the wave vector. This restriction is motivated by the fact that these modes are the most dangerous in a plasma in a sheared magnetic field [5, 7].

An important particular case of the modes described by dispersion relation (1.2) is represented by the modes with $k_z = 0$, i.e., those that obey the dispersion relation

$$\omega(\omega - \omega_{*i}) - g\kappa_n k_y^2/k_x^2 = 0. \quad (1.3)$$

For $g\kappa_n < 0$ (the case of a magnetic hill) [7] and for moderate ion drift frequencies ω_{*i}, these are the modes of a flute (or in other words, interchange) instability that is stabilized by finite ion Larmor radius effects, which come into play as the frequency ω_{*i} increases [9]. The generation of zonal flows by such modes stabilized by finite ion Larmor radius effects was studied in [10–14].

In Section 2, we present the basic equations for our problem. In this section, we use the nonlinear equations of paper [5], devoted to electrostatic modes, and of paper [15], devoted to DA vortices. We explain how the basic equations have been derived, perform their preliminary transformations, and introduce the perturbed quantities characterizing the primary modes, secondary small-scale modes (sidebands), and zonal flows. We also formulate the basic equations for zonal flows containing the amplitudes of the secondary modes and derive the basic equations for these amplitudes. In Section 3, we obtain expressions for the sideband amplitudes and give their transformations. In Section 4, we derive the dispersion relation for zonal flows, and, in Section 5, we analyze it. Section 6 is a discussion of the results of our work.

The approach of Sections 2–4 is often called parametric. It takes into account the methodological results of paper [16], which is devoted to the generation of zonal flows by the so-called small-scale DA waves (whose transverse wavelength is less than the ion Larmor radius, $k_\perp\rho_i \gg 1$), and of paper [17], which studies the generation of zonal flows by kinetic Alfvén waves. The parametric approach has its origin in the approach used in the theory of the generation of convective cells [18, 19] and deals with primary modes having an arbitrary spectrum rather than with a monochromatic packet of primary modes. As a result, the dispersion relation obtained here for zonal flows makes it possible to investigate their generation by a continuous spectrum of primary modes—the main issue in the traditional theory of the generation of zonal flows, which is based on the wave kinetic equation [20] and was summarized in [1] (see also [10]).

To avoid confusion, note that we are working here under the assumption $k_\perp\rho_i \ll 1$ (see above), in contrast to [16], where it was assumed that $k_\perp\rho_i \gg 1$. Waves with $k_\perp\rho_i \gg 1$ and $k_\perp\rho_i \ll 1$ have distinctly different properties. From the analysis to follow it will be clear that they are described by essentially different nonlinear equations. Accordingly, the regular features of the generation of zonal flows in the limiting cases $k_\perp\rho_i \gg 1$ and $k_\perp\rho_i \ll 1$ are radically different.

Since we are going to derive a dispersion relation for zonal flows generated by primary modes having an arbitrary spectrum, the question arises of whether or not there is an additional restriction on this spectrum. The question is reasonable in light of the results of [21], where it was asserted that such a restriction does indeed exist, and will be discussed in the Appendix.

2. BASIC PLASMADYNAMIC EQUATIONS AND THEIR PRELIMINARY TRANSFORMATIONS

2.1. Basic Plasmadynamic Equations

We represent the electric and magnetic fields of the perturbations, **E** and **B**, in terms of the electrostatic potential ϕ and the z component of the vector potential A through the formulas

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial A}{\partial t}\mathbf{e}_z, \quad (2.1)$$

$$\mathbf{B}_\perp = \nabla A \times \mathbf{e}_z, \quad (2.2)$$

where $\mathbf{e}_z$ is a unit vector along the equilibrium magnetic field $\mathbf{B}_0$ and c is the speed of light. The relationship between the functions A and ϕ follows from the perfect longitudinal plasma conductivity condition

$$\mathbf{E} \cdot (\mathbf{B}_0 + \mathbf{B}_\perp) = 0. \quad (2.3)$$

Substituting formulas (2.1) and (2.2) into this condition yields the equation

$$\frac{\partial A}{\partial t} + c\frac{\partial \phi}{\partial z} - \frac{c}{B_0}(\nabla A \times \nabla\phi)_z = 0, \quad (2.4)$$

which is the simplest nonlinear version of the so-called longitudinal Ohm's law or, in other terms, of the equation of longitudinal electron motion. More general versions of the longitudinal Ohm's law can be found in [16, 17].

We also introduce the perturbed electron density n through the electron continuity equation

$$\frac{d_0 n}{dt} + V_{Ex}\frac{\partial n_0}{\partial x} + \frac{c}{4\pi e}\nabla_\parallel\nabla_\perp^2 A = 0. \quad (2.5)$$

Here, the differential operators have the form

$$\frac{d_0}{dt} = \frac{\partial}{\partial t} + \frac{c}{B_0}(\nabla\phi \times \nabla)_z, \quad (2.6)$$

$$\nabla_{\parallel} = \frac{\partial}{\partial z} - \frac{1}{B_0}(\nabla A \times \nabla)_z; \tag{2.7}$$

V_{Ex} is the x component of the drift velocity in crossed electric and magnetic fields, defined by the relationship

$$\mathbf{V}_E = c(\mathbf{e}_z \times \nabla\phi)/B_0; \tag{2.8}$$

and e is the charge of an ion. In deriving Eq. (2.5), we took into account the longitudinal Ampére's law, which in our terms reads

$$j_z = -c\nabla_{\perp}^2 A/(4\pi), \tag{2.9}$$

where j_z is the longitudinal current density. Since we ignore longitudinal ion motion, we can express the longitudinal electron velocity in the continuity equation through j_z, $v_{ze} = -j_z/(en_0)$, just as was done in deriving Eq. (2.5).

Since longitudinal ion motion is ignored, the ion continuity equation has the form

$$\partial n/\partial t + \nabla_{\perp} \cdot [(n_0 + n)\mathbf{V}_{\perp i}] = 0, \tag{2.10}$$

where $\mathbf{V}_{\perp i}$ is the transverse ion velocity. This velocity, $\mathbf{V}_{\perp i}$, can be found from the equation of transverse ion motion:

$$\begin{aligned} M_i n_0 \frac{d\mathbf{V}_{\perp i}}{dt} + \nabla \cdot \hat{\boldsymbol{\pi}}_i &= -\nabla_{\perp} p_i + en_0 \mathbf{E}_{\perp} \\ &+ \frac{e}{c}(n_0 + n)(\mathbf{V}_{\perp i} \times \mathbf{B}_0) + M_i(n_0 + n)\mathbf{g}. \end{aligned} \tag{2.11}$$

Here, $\hat{\boldsymbol{\pi}}_i$ is the so-called magnetic (oblique) viscosity tensor or equivalently the gyroviscosity tensor [5], the total time derivative has the form

$$d/dt = \partial/\partial t + \mathbf{V}_{\perp i} \cdot \nabla, \tag{2.12}$$

p_i is the ion pressure, and M_i is the mass of an ion. Taking the vector product of formula (2.12) with $\mathbf{B}_0$, we obtain

$$\begin{aligned} (n_0 + n)\mathbf{V}_{\perp i} &= n_0 \mathbf{V}_E + \frac{c}{eB_0}\mathbf{e}_z \times \nabla p_i + \frac{n_0 + n}{\omega_{Bi}}\mathbf{g} \times \mathbf{e}_z \\ &+ \frac{n_0}{\omega_{Bi}}\mathbf{e}_z \times \left(\frac{d\mathbf{V}_{\perp i}}{dt} + \frac{\nabla \cdot \boldsymbol{\pi}_i}{M_i n_0}\right), \end{aligned} \tag{2.13}$$

where $\omega_{Bi} = eB_0/(M_i c)$ is the ion cyclotron frequency. Substituting relationship (2.13) into ion continuity equation (2.10) yields

$$\begin{aligned} &\frac{d_0 n}{dt} + V_{Ex}\frac{\partial n_0}{\partial x} - \frac{g}{\omega_{Bi}}\frac{\partial n}{\partial y} \\ &- \frac{n_0}{\omega_{Bi}}\left[\nabla \times \left(\frac{d\mathbf{V}_{\perp i}}{dt} + \frac{\nabla \cdot \hat{\boldsymbol{\pi}}_i}{M_i n_0}\right)\right]_z = 0. \end{aligned} \tag{2.14}$$

Subtracting Eq. (2.14) from Eq. (2.5), we arrive at the vorticity equation (or the current closure equation)

$$\begin{aligned} &\frac{B_0}{c}\left[\nabla \times \left(\frac{d\mathbf{V}_{\perp i}}{dt} + \frac{\nabla \cdot \hat{\boldsymbol{\pi}}_i}{M_i n_0}\right)\right]_z \\ &+ \frac{v_A^2}{c}\nabla_{\parallel}\nabla_{\perp}^2 A + \frac{gB_0}{cn_0}\frac{\partial n}{\partial y} = 0, \end{aligned} \tag{2.15}$$

where $v_A = B_0/(4\pi M_i n_0)^{1/2}$ is the Alfvén velocity. The total time derivative of the velocity in Eq. (2.15) can be calculated by using formula (2.12) under the assumption that the velocity $\mathbf{V}_{\perp i}$ in it is determined by the main terms on right-hand side of relationship (2.13), $\mathbf{V}_{\perp i} = \mathbf{V}_{\perp i}^{(0)}$, i.e., by the terms that do not vanish in the limit $\omega_{Bi} \longrightarrow \infty$. In this case, the contribution of the terms on the order of n/n_0 is negligibly small, so we have

$$\mathbf{V}_{\perp i}^{(0)} = \mathbf{V}_E + \mathbf{V}_L, \tag{2.16}$$

where $\mathbf{V}_L$ is the so-called ion Larmor (diamagnetic) drift velocity, defined by the relationship

$$\mathbf{V}_L = \frac{c}{eB_0 n_0}\mathbf{e}_z \times \nabla p_i. \tag{2.17}$$

According to [5], in the approximation in which the transverse ion velocity is given by expression (2.16), we have

$$\begin{aligned} &\left[\nabla \times \left(\frac{d\mathbf{V}_{\perp i}}{dt} + \frac{\nabla \cdot \hat{\boldsymbol{\pi}}_i}{M_i n_0}\right)\right]_z \\ &= \left[\nabla \times \left(\frac{\partial}{\partial t} + \mathbf{V}_E \cdot \nabla\right)(\mathbf{V}_E + \mathbf{V}_L)\right]_z. \end{aligned} \tag{2.18}$$

Physically, this indicates that the Larmor transport effect, $(\mathbf{V}_L \cdot \nabla)\mathbf{V}_{\perp i}$, is canceled by the magnetic (oblique) viscosity effect. Inserting relationship (2.18) into Eq. (2.15) gives

$$\begin{aligned} &\frac{B_0}{c}\left[\nabla \times \frac{d_0}{dt}(\mathbf{V}_E + \mathbf{V}_L)\right]_z + \frac{v_A^2}{c}\nabla_{\parallel}\nabla_{\perp}^2 A \\ &+ \frac{gB_0}{cn_0}\frac{\partial n}{\partial y} = 0. \end{aligned} \tag{2.19}$$

As was done in dispersion relation (1.2), we now take the limit $\partial/\partial x \gg \partial/\partial y$ in Eq. (2.19). For simplicity, we assume that the equilibrium ion temperature T_{0i} is constant and ignore the ion temperature perturbation. In expression (2.17) for $\mathbf{V}_L$, we can then set $p_i = nT_{0i}$. As a result, Eq. (2.19) is reduced to

$$\begin{aligned} &\frac{\partial}{\partial x}\left\{\left(\frac{\partial}{\partial t} + \mathbf{V}_E \cdot \nabla\right)\frac{\partial}{\partial x}\left(\phi + \frac{T_{0i}}{en_0}n\right)\right. \\ &\left. + \frac{v_A^2}{c}\left(\frac{\partial}{\partial z} - \frac{1}{B_0}(\nabla A \times \nabla)_z\right)\frac{\partial A}{\partial x}\right\} + \frac{gB_0}{cn_0}\frac{\partial n}{\partial y} = 0. \end{aligned} \tag{2.20}$$

This equation can also be obtained by using formula (4.44) from [5] and formula (19) from [15].

Note that, when the correction terms are ignored, electron and ion continuity equations (2.5) and (2.14) are reduced to the equation

$$\frac{\partial n}{\partial t} + V_{Ex}\frac{\partial n_0}{\partial x} + \frac{c}{B_0}(\nabla\phi \times \nabla n)_z = 0, \qquad (2.21)$$

which can be used to express the perturbed electron density n in terms of ϕ.

2.2. Separation of Variables

We represent each of the perturbed quantities $X = (\phi, A, n)$ as

$$X = \tilde{X} + \hat{X} + \bar{X}, \qquad (2.22)$$

where $\tilde{X}$, $\hat{X}$, and $\bar{X}$ describe, respectively, the primary modes, the secondary small-scale modes (sidebands), and the secondary large-scale modes (zonal flows).

The large-scale perturbations (zonal flows) are expressed as

$$\bar{X} = \bar{X}_0 \exp(-i\Omega t + iq_x x) + \text{c.c.} \qquad (2.23)$$

Here, Ω and q_x are the frequency and radial wavenumber of the zonal flow; $\bar{X}_0 \equiv (\bar{\phi}_0, \bar{A}_0, \bar{n}_0)$ are the amplitudes of the electrostatic and vector potentials of the flow and the plasma density, respectively; and the symbol c.c. stands for the complex conjugate.

In analogy with [16, 17], the primary modes are characterized by the functions $\tilde{X} = (\tilde{\phi}, \tilde{A}, \tilde{n})$ represented as

$$\tilde{X} = \sum_{\mathbf{k}} [\tilde{X}_+ \exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t) + \tilde{X}_- \exp(-i\mathbf{k}\cdot\mathbf{r} + i\omega t)]. \qquad (2.24)$$

Here, $\mathbf{k} = (k_x, k_y, k_z)$ and ω are the wave vector and frequency of the primary modes and $\tilde{X}_\pm$ are their amplitudes, satisfying the condition $\tilde{X}_- = \tilde{X}_+^*$, where the asterisk denotes the complex conjugate. From Eqs. (2.20), (2.4), and (2.21), we see that the quantities $\tilde{\phi}_+$, $\tilde{A}_+$, and $\tilde{n}_+$ are related by the formulas

$$\omega\left(\tilde{\phi}_+ + \frac{T_{0i}}{en_0}\tilde{n}_+\right) - \frac{v_A^2}{c}k_z\tilde{A}_+ + \frac{B_0}{c}\frac{gk_y}{k_x^2}\frac{\tilde{n}_+}{n_0} = 0, \qquad (2.25)$$

$$\tilde{A}_+ = \frac{ck_z}{\omega}\tilde{\phi}_+, \qquad (2.26)$$

$$\tilde{n}_+ = -\frac{ck_y n_0 \kappa_n}{B_0 \omega}\tilde{\phi}_+, \qquad (2.27)$$

where $\kappa_n \equiv \partial \ln n_0/\partial x$. From these formulas we obtain the following dispersion relation for the primary modes (cf. dispersion relation (1.2)):

$$D(\omega, \mathbf{k}) \equiv \omega - \omega_{*i} - k_z^2 v_A^2/\omega - g\kappa_n k_y^2/(\omega k_x^2) = 0, \qquad (2.28)$$

where $\omega_{*i} = ck_y\kappa_n T_{0i}/(eB_0)$ is the ion diamagnetic drift frequency.

According to representations (2.23) and (2.24), the variables describing the secondary small-scale modes are represented as a superposition of the sidebands,

$$\hat{X} = \sum_{\mathbf{k}} [\hat{X}_+ \exp(i\mathbf{k}_+\cdot\mathbf{r} - i\omega_+ t) + \hat{X}_- \exp(i\mathbf{k}_-\cdot\mathbf{r} - i\omega_- t) + \text{c.c.}], \qquad (2.29)$$

where $\omega_\pm = \Omega \pm \omega$ and $\mathbf{k}_\pm = (q_x \pm k_x, \pm k_y, \pm k_z)$. The analysis to follow will be carried out in the standard approximation such that $q_x/k_x \ll 1$ and $\Omega/\omega \ll 1$.

2.3. Basic Equations for Zonal Flows

2.3.1. Vorticity equation. Averaging Eq. (2.20) over small-scale oscillations, we obtain

$$-i\Omega\left(\bar{\phi}_0 + \frac{T_{0i}}{en_0}\bar{n}_0\right) = R_\perp, \qquad (2.30)$$

where

$$R_\perp = \frac{c}{B_0}\left\langle \frac{\partial\phi}{\partial y}\frac{\partial}{\partial x}\left(\phi + \frac{T_{0i}}{en_0}n\right) - \frac{v_A^2}{c^2}\frac{\partial A}{\partial y}\frac{\partial A}{\partial x}\right\rangle. \qquad (2.31)$$

Note that Eq. (2.30) has been derived by dividing the averaged equation (2.20) by q_x^2, i.e., by ignoring the second derivative $\partial^2/\partial x^2$. This is why the functions ϕ, A, and n, which characterize only the primary modes, cannot be substituted into right-hand side of Eq. (2.30) (cf. the discussion in the Appendix).

We express the right-hand side of expression (2.31) in terms of $\tilde{X}_\pm$ and $\hat{X}_\pm$. Using formulas (2.26) and (2.27) for $\tilde{A}_+$ and $\tilde{n}_+$ and the analogous formulas for $\tilde{A}_-$ and $\tilde{n}_-$, we then express the functions $\tilde{A}_\pm$ and $\tilde{n}_\pm$ in terms of $\tilde{\phi}_\pm$. As a result, we obtain the expression

$$R_\perp = \sum_{\mathbf{k}} \frac{ck_y}{B_0}[q_x(\tilde{\phi}_- r_+^q - \tilde{\phi}_+ r_-^q) + k_x(\tilde{\phi}_- r_+^k + \tilde{\phi}_+ r_-^k)], \qquad (2.32)$$

where

$$r_\pm^q = \hat{\phi}_\pm + \frac{T_{0i}}{en_0}\hat{n}_\pm - \frac{k_z v_A^2}{c\omega}\hat{A}_\pm, \qquad (2.33)$$

$$r_{\pm}^{k} = \left(2 - \frac{\omega_{*i}}{\omega}\right)\hat{\phi}_{\pm} + \frac{T_{0i}}{en_0}\hat{n}_{\pm} - 2\frac{k_z v_A^2}{c\omega}\hat{A}_{\pm}. \quad (2.34)$$

2.3.2. Longitudinal Ohm's law. From Eq. (2.4) we find

$$-i\Omega\bar{A}_0 = R_{\|}, \quad (2.35)$$

where $R_{\|}$ is the Reynolds stress force, which enters the longitudinal Ohm's law and is defined by the equality

$$R_{\|} = \frac{icq_x}{B_0}\left\langle A\frac{\partial\phi}{\partial y}\right\rangle. \quad (2.36)$$

In analogy with expression (2.32), relationship (2.36) leads to

$$R_{\|} = \sum_{\mathbf{k}}\frac{cq_x k_y}{B_0}\left[\tilde{\phi}_-\left(\hat{A}_+ - \frac{ck_z}{\omega}\hat{\phi}_+\right) - \tilde{\phi}_+\left(\hat{A}_- - \frac{ck_z}{\omega}\hat{\phi}_-\right)\right]. \quad (2.37)$$

2.3.3. Continuity equation. Equation (2.21) yields

$$-i\Omega\bar{n}_0 = R_n, \quad (2.38)$$

where R_n is the Reynolds stress force, which governs the generation of the zonal component of the plasma density and is equal to (cf. equality (2.36))

$$R_n = \frac{icq_x}{B_0}\left\langle n\frac{\partial\phi}{\partial y}\right\rangle. \quad (2.39)$$

This equality leads to the expression (cf. formula (2.37))

$$R_n = \sum_{\mathbf{k}}\frac{ck_y q_x}{B_0}\left[\tilde{\phi}_-\left(\hat{n}_+ + \frac{ck_y n_0\kappa_n}{B_0\omega}\hat{\phi}_+\right) - \tilde{\phi}_+\left(\hat{n}_- + \frac{ck_y n_0\kappa_n}{B_0\omega}\hat{\phi}_-\right)\right]. \quad (2.40)$$

2.4. Basic Equations for the Sidebands

Using Eqs. (2.20), (2.4), and (2.21), we arrive at the following set of equations for the sideband amplitudes:

$$\omega_{\pm}\left(\hat{\phi}_{\pm} + \frac{T_{0i}}{en_0}\hat{n}_{\pm}\right) \mp \frac{v_A^2 k_z}{c}\hat{A}_{\pm} \pm \frac{gB_0}{cn_0}\frac{k_y\hat{n}_{\pm}}{k_{x\pm}^2} = Y_{\pm}^{\phi}, \quad (2.41)$$

$$\mp ck_z\hat{\phi}_{\pm} + \omega_{\pm}\hat{A}_{\pm} = Y_{\pm}^{A}, \quad (2.42)$$

$$\pm\frac{ck_y n_0\kappa_n}{B_0}\hat{\phi}_{\pm} + \omega_{\pm}\hat{n}_{\pm} = Y_{\pm}^{n}, \quad (2.43)$$

where

$$Y_{\pm}^{\phi} = \frac{ick_y q_x}{B_0}\tilde{\phi}_{\pm}\left[\frac{k_x}{k_{x\pm}}\left(1 - \frac{\omega_{*i}}{\omega}\right)\bar{\phi}_0 \mp \frac{q_x}{k_{x\pm}}\left(\bar{\phi}_0 + \frac{T_{0i}}{en_0}\bar{n}_0\right) \pm \frac{v_A^2 k_z}{c\omega}\frac{k_{x\mp}}{k_{x\pm}}\bar{A}_0\right], \quad (2.44)$$

$$Y_{\pm}^{A} = \pm\frac{ick_y q_x}{B_0}\tilde{\phi}_{\pm}\left(\frac{ck_z}{\omega}\bar{\phi}_0 - \bar{A}_0\right), \quad (2.45)$$

$$Y_{\pm}^{n} = \mp\frac{ick_y q_x}{B_0}\tilde{\phi}_{\pm}\left(\bar{n}_0 + \frac{ck_y n_0\kappa_n}{B_0\omega}\bar{\phi}_0\right). \quad (2.46)$$

The functions $Y_{\pm}^{\phi}$, $Y_{\pm}^{A}$, and $Y_{\pm}^{n}$ can be called the sideband oscillating forces.

3. DERIVATION AND TRANSFORMATION OF EXPRESSIONS FOR THE SIDEBAND AMPLITUDES

3.1. Solutions to the Equations for the Sideband Amplitudes

Equations (2.41)–(2.43) have the solutions

$$\hat{\phi}_{\pm} = \frac{1}{D_{\pm}}\left[Y_{\pm}^{\phi} \pm \frac{k_z v_A^2}{c\omega_{\pm}}Y_{\pm}^{A} - \left(\frac{T_{0i}}{en_0} \pm \frac{gB_0k_y}{cn_0k_{x\pm}^2\omega_{\pm}}\right)Y_{\pm}^{n}\right], \quad (3.1)$$

$$\hat{A}_{\pm} = \frac{1}{D_{\pm}}\left[\pm\frac{ck_z}{\omega_{\pm}}Y_{\pm}^{\phi} + \left(1 \mp \frac{\omega_{*i}}{\omega} - \frac{g\kappa_n k_y^2}{\omega_{\pm}^2 k_{x\pm}^2}\right)Y_{\pm}^{A} \mp \frac{ck_z}{\omega_{\pm}}\left(\frac{T_{0i}}{en_0} \pm \frac{gB_0k_y}{cn_0k_{x\pm}^2\omega_{\pm}}\right)Y_{\pm}^{n}\right], \quad (3.2)$$

$$\hat{n}_{\pm} = \frac{1}{D_{\pm}}\left[\mp\frac{ck_y n_0\kappa_n}{B_0\omega_{\pm}}Y_{\pm}^{\phi} - \frac{k_z v_A^2 k_y n_0\kappa_n}{B_0\omega_{\pm}^2}Y_{\pm}^{A} + \left(1 - \frac{k_z^2 v_A^2}{\omega_{\pm}^2}\right)Y_{\pm}^{n}\right]. \quad (3.3)$$

Here

$$D_{\pm} = D(\Omega \pm \omega, \mathbf{q} \pm \mathbf{k}), \quad (3.4)$$

where the function $D(\omega, \mathbf{k})$ is given by the first of expressions (2.28).

3.2. Expressions for $r_{\pm}^{q}$ in Terms of the Sideband Oscillating Forces

Substituting Eqs. (3.1)–(3.3) into expression (2.33), we find

$$r_{\pm}^{q} = \frac{1}{D_{\pm}}(\alpha_{\pm}^{\phi}Y_{\pm}^{\phi} + \alpha_{\pm}^{A}Y_{\pm}^{A} + \alpha_{\pm}^{n}Y_{\pm}^{n}), \quad (3.5)$$

where

$$\alpha_\pm^\phi = 1 \mp \frac{\omega_{*i}}{\omega_\pm} \mp \frac{k_z^2 v_A^2}{\omega\omega_\pm}, \tag{3.6}$$

$$\alpha_\pm^A = \frac{k_z v_A^2}{c\omega}\left[-\left(1 \mp \frac{\omega}{\omega_\pm}\right)\left(1 \mp \frac{\omega_{*i}}{\omega_\pm}\right) + \frac{g\kappa_n k_y^2}{\omega_\pm^2 k_{x\pm}^2}\right], \tag{3.7}$$

$$\alpha_\pm^n = \mp \frac{gB_0 k_y}{cn_0 k_{x\pm}^2 \omega_\pm}\left(1 \mp \frac{k_z^2 v_A^2}{\omega\omega_\pm}\right) \pm \frac{T_{0i}}{en_0}\frac{k_z^2 v_A^2}{\omega\omega_\pm}\left(1 \mp \frac{\omega}{\omega_\pm}\right). \tag{3.8}$$

3.3. Expressions for $r_\pm^k$ in Terms of the Sideband Oscillating Forces

Using Eqs. (3.1)–(3.3), we transform expression (2.34) to

$$r_\pm^k = \frac{1}{D_\pm}(\beta_\pm^\phi Y_\pm^\phi + \beta_\pm^A Y_\pm^A + \beta_\pm^n Y_\pm^n), \tag{3.9}$$

where

$$\beta_\pm^\phi = 2 - \frac{\omega_{*i}}{\omega} \mp \frac{\omega_{*i}}{\omega_\pm} \mp \frac{2k_z^2 v_A^2}{\omega\omega_\pm}, \tag{3.10}$$

$$\beta_\pm^A = -\frac{k_z v_A^2}{c\omega}\left[\left(2 \mp \frac{\omega_{*i}}{\omega_\pm}\right)\left(1 \mp \frac{\omega}{\omega_\pm}\right) - 2\frac{g\kappa_n k_y^2}{\omega_\pm^2 k_{x\pm}^2}\right], \tag{3.11}$$

$$\beta_\pm^n = -\left[\frac{T_{0i}}{en_0}\left(1 - \frac{\omega_{*i}}{\omega} + \frac{k_z^2 v_A^2}{\omega_\pm^2} \mp \frac{2k_z^2 v_A^2}{\omega\omega_\pm}\right) \pm \frac{gB_0 k_y}{cn_0 k_{x\pm}^2 \omega_\pm}\left(2 - \frac{\omega_{*i}}{\omega} \mp \frac{2k_z^2 v_A^2}{\omega\omega_\pm}\right)\right]. \tag{3.12}$$

4. DERIVATION OF A DISPERSION RELATION FOR ZONAL FLOWS

4.1. Zero Zonal Components of the Magnetic Field and Plasma Density

We use Eqs. (3.1)–(3.3) to calculate the right-hand sides of relationships (2.37) and (2.40) for $R_\parallel$ and R_n by accounting for the terms that are formally as small as q_x^2/k_x^2 and by ignoring higher order terms. As a result, we obtain

$$R_\parallel = 0, \tag{4.1}$$

$$R_n = 0. \tag{4.2}$$

Substituting expressions (4.1) and (4.2) into Eqs. (2.35) and (2.38), we arrive at the equalities

$$\bar{A}_0 = 0, \tag{4.3}$$

$$\bar{n}_0 = 0, \tag{4.4}$$

which show that, in the approximation at hand, the zonal components of the magnetic field and plasma density are not generated.

4.2. Simplification of the Expressions for the Sideband Oscillating Forces and Calculation of the Contributions of the Sideband Amplitudes to the Averaged Vorticity Equation

With equalities (4.3) and (4.4), relationships (2.44)–(2.46) are reduced to

$$Y_\pm^\phi = \frac{ick_y q_x}{B_0 k_{x\pm}}\left[\left(1 - \frac{\omega_{*i}}{\omega}\right)k_x \mp q_x\right]\tilde{\phi}_\pm\bar{\phi}_0, \tag{4.5}$$

$$Y_\pm^A = \pm\frac{ic^2 k_y q_x k_z}{B_0\omega}\tilde{\phi}_\pm\bar{\phi}_0, \tag{4.6}$$

$$Y_\pm^n = \mp\frac{ic^2 k_y^2 q_x n_0 \kappa_n}{B_0^2\omega}\tilde{\phi}_\pm\bar{\phi}_0. \tag{4.7}$$

4.2.1. Calculation of $r_\pm^q$. In order to calculate the quantity $r_\pm^q$, it is sufficient to take into account only the leading-order terms in $D_\pm$, i.e., to set

$$D_\pm = D^{(0)}, \tag{4.8}$$

where

$$D^{(0)} = \left(2 - \frac{\omega_{*i}}{\omega}\right)\Omega + 2\frac{k_y^2 g\kappa_n}{\omega k_x^3}q_x. \tag{4.9}$$

Using expressions (3.6)–(3.8) and (4.5)–(4.7), we then reduce expression (3.5) to

$$r_\pm^q = \pm\frac{ick_y q_x}{B_0 D^{(0)}}\frac{g\kappa_n k_y^2}{\omega^2 k_x^2}\left(2 - \frac{\omega_{*i}}{\omega}\right)\tilde{\phi}_\pm\bar{\phi}_0. \tag{4.10}$$

4.2.2. Calculation of $r_\pm^k$. We seek the quantity $r_\pm^k$ in the form of a power series in q_x and Ω by representing it as

$$r_\pm^k = r_\pm^{k(0)} + r_\pm^{k(1)}. \tag{4.11}$$

From expressions (3.9) and (4.5)–(4.7) we see that the leading-order term of the series is given by the equality

$$r_\pm^{k(0)} = \pm\frac{ick_yq_x}{B_0D^{(0)}}\tilde{\phi}_\pm\bar{\phi}_0 \times\left[\left(1-\frac{\omega_{*i}}{\omega}\right)\beta_\pm^{\phi(0)}+\frac{ck_z}{\omega}\beta_\pm^{A(0)}-\frac{ck_yn_0\kappa_n}{B_0\omega}\beta_\pm^{n(0)}\right]. \quad (4.12)$$

Expressions (3.10)–(3.12) give

$$\beta_\pm^{\phi(0)} = 2\frac{g\kappa_nk_y^2}{\omega^2k_x^2}, \quad (4.13)$$

$$\beta_\pm^{A(0)} = 2\frac{k_zv_A^2}{c\omega}\frac{g\kappa_nk_y^2}{\omega^2k_x^2}, \quad (4.14)$$

$$\beta_\pm^{n(0)} = -2\frac{gB_0k_y}{cn_0k_x^2\omega}\left(1-\frac{k_z^2v_A^2}{\omega^2}\right). \quad (4.15)$$

With expressions (4.13)–(4.15), equality (4.12) transforms to

$$r_\pm^{k(0)} = \pm2\frac{ick_yq_x}{B_0D^{(0)}}\frac{g\kappa_nk_y^2}{\omega^2k_x^2}\left(2-\frac{\omega_{*i}}{\omega}\right)\tilde{\phi}_\pm\bar{\phi}_0. \quad (4.16)$$

In accordance with relationships (3.4) and (2.23), we calculated the quantity $r_\pm^{k(1)}$ by using, instead of formula (4.8), the expression

$$D_\pm = D^{(0)}\pm D^{(1)}, \quad (4.17)$$

where

$$D^{(1)} = -\left(1-\frac{\omega_{*i}}{\omega}\right)\frac{\Omega^2}{\omega}-\frac{g\kappa_nk_y^2q_x}{\omega k_x^3}\left(2\frac{\Omega}{\omega}+3\frac{q_x}{k_x}\right). \quad (4.18)$$

Equality (3.5) then becomes

$$r_\pm^{k(1)} = \mp\frac{D^{(1)}}{D^{(0)}}r_\pm^{k(0)}+\frac{ick_yq_x}{B_0D^{(0)}}\tilde{\phi}_\pm\bar{\phi}_0\rho_\pm^{(1)}, \quad (4.19)$$

where

$$\rho_\pm^{(1)} = \pm\left(1-\frac{\omega_{*i}}{\omega}\right)\beta_\pm^{\phi(1)}-\left(2-\frac{\omega_{*i}}{\omega}\right)\frac{q_x}{k_x}\beta_\pm^{\phi(0)} \pm\frac{ck_z}{\omega}\beta_\pm^{A(1)}\mp\frac{ck_yn_0\kappa_n}{B_0\omega}\beta_\pm^{n(1)}. \quad (4.20)$$

Relationships (3.10)–(3.12) give

$$\beta_\pm^{\phi(1)} = \pm\frac{\Omega}{\omega}\left(\frac{\omega_{*i}}{\omega}+2\frac{k_z^2v_A^2}{\omega^2}\right), \quad (4.21)$$

$$\beta_\pm^{A(1)} = \mp\frac{k_zv_A^2}{c\omega}\left[\left(2-\frac{\omega_{*i}}{\omega}\right)\frac{\Omega}{\omega}+4\frac{g\kappa_nk_y^2}{\omega^2k_x^2}\left(\frac{\Omega}{\omega}+\frac{q_x}{k_x}\right)\right], \quad (4.22)$$

$$\beta_\pm^{n(1)} = \pm\frac{gB_0k_y}{cn_0k_x^2\omega}\left[\left(2-\frac{\omega_{*i}}{\omega}\right)\left(\frac{\Omega}{\omega}+2\frac{q_x}{k_x}\right) -4\frac{k_z^2v_A^2}{\omega^2}\left(\frac{\Omega}{\omega}+\frac{q_x}{k_x}\right)\right]. \quad (4.23)$$

From relationships (4.21)–(4.23) and (4.13) we obtain

$$\rho_\pm^{(1)} = -2\frac{g\kappa_nk_y^2}{k_x^2\omega^2}\left[\left(1-\frac{\omega_{*i}}{\omega}\right)\frac{\Omega}{\omega}+2\left(2-\frac{\omega_{*i}}{\omega}\right)\frac{q_x}{k_x}\right]. \quad (4.24)$$

4.3. Dispersion Relation for Zonal Flows

With formulas (4.10), (4.11), (4.16), (4.19), and (4.24), expression (2.32) is reduced to

$$R_\perp = i\Omega\bar{\phi}_0\sum_{\mathbf{k}}\frac{F(\mathbf{k})}{(\Omega-q_xV_g)^2}. \quad (4.25)$$

Here, $V_g = V_g(\mathbf{k})$ is the zonal radial group velocity, defined by the equality

$$V_g = -2\frac{k_y^2g\kappa_n}{\omega k_x^3(2-\omega_{*i}/\omega)}, \quad (4.26)$$

and the function $F(\mathbf{k})$ has the form

$$F(\mathbf{k}) = -\frac{c^2k_y^2q_x^2I_{\mathbf{k}}g\kappa_nk_y^2}{B_0^2k_x^2(2\omega-\omega_{*i})^2} \times\left[2\left(2-\frac{\omega_{*i}}{\omega}\right)^2+\frac{\omega_{*i}^2}{\omega^2}+4\frac{k_z^2v_A^2}{\omega^2}\right], \quad (4.27)$$

where

$$I_{\mathbf{k}} = 2\tilde{\phi}_+\tilde{\phi}_-. \quad (4.28)$$

Substituting expression (4.25) into Eq. (2.30) and taking into account equalities (4.4) and (4.9), we switch from summation over $\mathbf{k}$ to integration over this variable to obtain the sought-for dispersion relation for zonal flows:

$$1+\int\frac{F(\mathbf{k})d\mathbf{k}}{(\Omega-q_xV_g)^2} = 0. \quad (4.29)$$

Dispersion relation (4.29) differs substantially from dispersion relation (58) for zonal flows from [16]. The difference stems from the fact that, in our problem, we assume that $k_\perp\rho_i \ll 1$ (see the Introduction), whereas in [16], it was assumed that $k_\perp\rho_i \gg 1$.

5. ANALYSIS OF THE DISPERSION RELATION FOR ZONAL FLOWS

5.1. General Considerations

5.1.1. Impossibility of generating zonal flows by nondispersive DA modes. Setting

$$g = 0 \tag{5.1}$$

in expression (4.27), we find

$$F(\mathbf{k}) = 0. \tag{5.2}$$

In this case, dispersion relation (4.29) cannot be satisfied. This indicates that nondispersive (pure) DA modes do not generate zonal flows. Let us discuss the essence of this remarkable effect.

A particular case of DA modes is represented by Alfvén modes in a homogeneous plasma that are described by the dispersion relation

$$\omega^2 = k_z^2 v_A^2. \tag{5.3}$$

In [22] (see also [23]), it was noted that nondispersive Alfvén modes cannot generate zonal flows. The physical reason for this was explained in [22]: in contrast to the electrostatic electron drift modes, the nonlinear dynamics of Alfvén modes is affected not only by the Reynolds stress but also by the Maxwell stress, which completely counterbalances the Reynolds stress. This counterbalancing effect was discussed in quite a number of papers (see [17] and the literature cited therein). Switching from dispersion relation (5.3) to dispersion relation (1.1), we can see that the ion drift effect shifts the eigenfrequency of Alfvén modes. It might then be expected that, due to this shift, the balance between the Reynolds and Maxwell stresses would be incomplete. However, from Eq. (2.20) we can see that, along with the Reynolds and Maxwell stresses (described by the nonlinear terms with ϕ and A, respectively), there is also the ion drift stress, described by the nonlinear terms with n. Because of the interplay between these three stresses, the resulting nonlinear force generating the zonal flows vanishes.

Let us also say a few words about nondispersive ion drift modes described by the dispersion relation (see Eq. (1.1) with $k_z = 0$ or Eq. (1.3) with $g = 0$)

$$\omega = \omega_{*i}. \tag{5.4}$$

From Eq. (2.20) we can find that, in this case, the Reynolds stress is counteracted by the ion drift stress in such a way that the resulting stress is zero.

5.1.2. Effect of the magnetic curvature on the linear instabilities and on the generation of zonal flows. Two classes of instabilities of zonal flows. From the standpoint of the linear theory of instabilities, the term with g in dispersion relation (1.2) is important not because it is dispersive in character but because it describes the MHD stability of the plasma in the corresponding confinement system (see [5], Section 11, and book [24] for details). The reason is that, when $g\kappa_n > 0$, the roots of Eq. (1.2) are real for arbitrary values of ω_{*i} and k_z, so the plasma in the system is MHD stable. This is the case of a magnetic well. On the other hand, when $g\kappa_n < 0$, the roots are complex for sufficiently small values of ω_{*i} and k_z. This corresponds to MHD instability—the case of a magnetic hill. In this context, the question naturally arises of how the pattern of the generation of zonal flows by the modes described by dispersion relation (1.2) in the case of a magnetic well differs from that in the case of a magnetic hill.

From definition (4.27), we can see that the sign of the quantity $g\kappa_n$ exactly corresponds to the sign of the function $F(\mathbf{k})$. For a magnetic hill,

$$g\kappa_n < 0, \tag{5.5}$$

this function is positive definite,

$$F(\mathbf{k}) > 0. \tag{5.6}$$

In [17], it was noted that, in this case, dispersion relation (4.29) for zonal flows is analogous to the dispersion relation describing linear negative-mass instabilities that were considered in Section 4.1 of [25]. In the theory of the generation of zonal flows by electrostatic electron drift waves (cf. [26]), the dispersion relations have the same structure. A remarkable property of this class of the instabilities of zonal flows is that the flows can be generated by an individual monochromatic wave packet. For this case, the function $F(\mathbf{k})$ can be represented as

$$F(\mathbf{k}) = \Omega_0^2\delta(\mathbf{k} - \mathbf{k}_0), \tag{5.7}$$

where Ω_0^2 is a positive constant. Dispersion relation (4.29) then implies that zonal flows are unstable with the growth rate

$$\operatorname{Im}\Omega = \Omega_0. \tag{5.8}$$

For a magnetic well, i.e., when

$$g\kappa_n > 0, \tag{5.9}$$

the function $F(\mathbf{k})$ defined by expression (4.27) is negative,

$$F(\mathbf{k}) < 0. \tag{5.10}$$

If we consider an individual monochromatic packet of DA waves, then, instead of representation (5.7), we must write

$$F(\mathbf{k}) = -\Omega_0^2\delta(\mathbf{k} - \mathbf{k}_0). \tag{5.11}$$

In this case, dispersion relation (4.29) for zonal flows is reduced to

$$1 - \frac{\Omega_0^2}{(\Omega - q_x V_{g0})^2} = 0, \tag{5.12}$$

where $V_{g0} = V_g(\mathbf{k}_0)$. From Sections 1–3 of book [27], devoted to the linear theory of two-stream instabilities, we can see that dispersion relation (5.12) is analogous to that for an individual cold beam. Such a dispersion relation has no roots with $\mathrm{Im}\,\Omega > 0$, which corresponds to instability. Nevertheless, for two monochromatic packets of DA waves, in place of representation (5.11), we have

$$F(\mathbf{k}) = -\Omega_1^2\delta(\mathbf{k}-\mathbf{k}_1) - \Omega_2^2\delta(\mathbf{k}-\mathbf{k}_2), \quad (5.13)$$

where Ω_1^2 and Ω_2^2 are positive. Instead of dispersion relation (5.12), we then arrive at the following dispersion relation for zonal flows:

$$1 - \frac{\Omega_1^2}{(\Omega - q_x V_{g1})^2} - \frac{\Omega_2^2}{(\Omega - q_x V_{g2})^2} = 0. \quad (5.14)$$

This dispersion relation is similar to that for two cold beams. It is well known that the latter dispersion relation describes two-stream hydrodynamic instability. This analogy suggests the existence a class of instabilities of zonal flows that are similar to two-stream instabilities and are triggered by DA modes in the case of a magnetic well. Two-stream-like instabilities driven by kinetic Alfvén waves were thoroughly analyzed in [17]. It is also possible to reveal a fairly broad class of two-stream-like instabilities driven by DA modes (see [17, 27]). The simplest example of such instabilities will be considered below.

5.1.3. Criterion for the class of instabilities of zonal flows in terms of the sign of the quantity $\omega^{-1}\partial^2\omega/\partial k_x^2$. In [17], it was noted that the criterion for negative-mass instabilities or two-stream instabilities is formulated in terms of the sign of the quantity $\omega^{-1}\partial^2\omega/\partial k_x^2$ (the Lighthill stability criterion; see [28] for details). According to [17], the reason for this is that the condition

$$\omega^{-1}\partial^2\omega/\partial k_x^2 < 0 \quad (5.15)$$

implies the onset of negative-mass instabilities of zonal flows, whereas the condition

$$\omega^{-1}\partial^2\omega/\partial k_x^2 > 0 \quad (5.16)$$

corresponds to two-stream instabilities. Let us find out whether such a rule is applicable to the instabilities of zonal flows that are triggered by DA modes modified by the magnetic curvature effects.

From equality (4.26) we have (cf. definition (4.27))

$$\frac{1}{\omega}\frac{\partial^2\omega}{\partial k_x^2} \equiv \frac{1}{\omega}\frac{\partial V_g}{\partial k_x} = \frac{2k_y^2 g\kappa_n}{\omega^4 k_x^2(2-\omega_{*i}/\omega)^3}[2(2\omega-\omega_{*i})^2 + \omega_{*i}^2 + 4k_z^2 v_A^2]. \quad (5.17)$$

We can see that, for the modes under consideration, the difference $2 - \omega_{*i}/\omega$ is positive. We can then write

$$\mathrm{sgn}\left(\frac{1}{\omega}\frac{\partial^2\omega}{\partial k_x^2}\right) = \mathrm{sgn}(g\kappa_n). \quad (5.18)$$

In the context of the above discussion of the criterion for the class of instabilities of zonal flows in terms of $\mathrm{sgn}(g\kappa_n)$, we can conclude that the Lighthill criterion is applicable to the modes in question.

5.2. Explicit Form of the Dispersion Relation for Zonal Flows in the Case of a Magnetic Hill and of an Individual Monochromatic Wave Packet

In the case of a magnetic hill, we can introduce the hill-related instability growth rate γ_{MH} through the relationship

$$\gamma_{MH}^2 = -g\kappa_n k_y^2/k_x^2. \quad (5.19)$$

In terms of this growth rate, dispersion relation (1.2) for the primary modes is represented as

$$\omega(\omega-\omega_{*i}) - k_z^2 v_A^2 + \gamma_{MH}^2 = 0. \quad (5.20)$$

In our analysis, we assume that the frequency of the primary modes is real, $\mathrm{Im}\,\omega = 0$. Dispersion relation (5.20) implies that primary modes occur when

$$\gamma_{MH}^2 \le \gamma_{MH,\,\mathrm{crit}}^2 \equiv k_z^2 v_A^2 + \omega_{*i}^2/4. \quad (5.21)$$

This is the MHD stability condition. Having introduced the growth rate squared, γ_{MH}^2, we can rewrite dispersion relation (4.29) for zonal flows excited by an individual monochromatic wave packet as

$$\left[\left(2-\frac{\omega_{*i}}{\omega}\right)\Omega - 2\frac{q_x}{k_x\omega}\gamma_{MH}^2\right]^2 = -\Gamma_0^2\frac{\gamma_{MH}^2}{\omega^2}\left[2\left(2-\frac{\omega_{*i}}{\omega}\right)^2 + \frac{\omega_{*i}^2}{\omega^2} + 4\frac{k_z^2 v_A^2}{\omega^2}\right], \quad (5.22)$$

where

$$\Gamma_0^2 = c^2 k_y^2 q_x^2 I_{\mathbf{k}}/B_0^2. \quad (5.23)$$

Let us examine the consequences of dispersion relation (5.22).

5.3. Quasi-Pure DA Modes in the Case of a Weak Magnetic Hill

According to dispersion relation (5.22), the magnetic curvature effects play a decisive role in the nonlinear dynamics of DA modes. On the other hand, in describing the linear dynamics of the modes, these effects can be ignored, provided that the magnetic hill is sufficiently weak, $\gamma_{MH}^2 \ll \gamma_{MH,\,\mathrm{crit}}^2$. Consequently, in addition to pure DA modes, we can consider quasi-pure DA modes—those whose properties can be studied based on dispersion relation (1.1) in which the gravity force is ignored. In this case, the curvature effects manifest themselves only in the terms describing the coupling between the zonal flows and the primary modes (see the factor γ_{MH}^2 on the right-hand side of dispersion relation (5.22)) and the term describing the radial propagation of zonal flows (see the term with γ_{MH}^2 on the left-hand side of dispersion relation (5.22)).

5.3.1. Quasi-pure Alfvén modes. Using dispersion relation (5.3), we can reduce dispersion relation (5.22) for quasi-pure Alfvén modes to

$$\left(\Omega - \frac{q_x}{k_x\omega}\gamma_{MH}^2\right)^2 = -3\Gamma_0^2\frac{\gamma_{MH}^2}{k_z^2 v_A^2}. \qquad (5.24)$$

As a result, we have

$$\mathrm{Im}\,\Omega = 3^{1/2}\Gamma_0\frac{\gamma_{MH}}{|k_z| v_A}, \qquad (5.25)$$

$$\mathrm{Re}\,\Omega = \frac{q_x}{k_x\omega}\gamma_{MH}^2. \qquad (5.26)$$

In Section 6, it will be explained that Γ_0 is the growth rate of the zonal flow generated by a monochromatic packet of electrostatic electron drift waves. Expression (5.25) for the growth rate of the zonal flow generated by quasi-pure Alfvén modes is seen to contain an additional small parameter on the order of $\gamma_{MH}/|k_z|v_A$.

5.3.2. Quasi-pure ion drift modes. For $k_z = 0$, quasi-pure ion drift modes are described by dispersion relation (5.4). In this case, dispersion relation (5.22) is reduced to

$$\left(\Omega - \frac{2q_x}{k_x\omega_{*i}}\gamma_{MH}^2\right)^2 = -3\Gamma_0^2\frac{\gamma_{MH}^2}{\omega_{*i}^2} \qquad (5.27)$$

and, instead of expressions (5.25) and (5.26), we have

$$\mathrm{Im}\,\Omega = 3^{1/2}\Gamma_0\frac{\gamma_{MH}}{|\omega_{*i}|}, \qquad (5.28)$$

$$\mathrm{Re}\,\Omega = \frac{2q_x}{k_x\omega_{*i}}\gamma_{MH}^2. \qquad (5.29)$$

By analogy with expression (5.25), growth rate (5.28) is as small compared with Γ_0 as the ratio of γ_{MH} to the frequency of the pure ion drift modes.

5.3.3. Quasi-pure slow DA modes at $k_z v_A \ll \omega_{*i}$. For $k_z v_A \ll \omega_{*i}$, dispersion relation (1.1) describes not only quasi-pure ion drift modes (see dispersion relation (5.4)) but also the branch of slow DA modes with the dispersion

$$\omega = -k_z^2 v_A^2/\omega_{*i}. \qquad (5.30)$$

In this case, dispersion relation (5.22) transforms to

$$\left(\Omega - 2\frac{q_x\gamma_{MH}^2}{k_x\,\omega_{*i}}\right)^2 = -3\Gamma_0^2\frac{\gamma_{MH}^2\omega_{*i}^2}{k_z^4 v_A^4}. \qquad (5.31)$$

This dispersion relation for zonal flows leads to the growth rate

$$\mathrm{Im}\,\Omega = 3^{1/2}\Gamma_0\frac{\gamma_{MH}|\omega_{*i}|}{k_z^2 v_A^2}, \qquad (5.32)$$

whereas the real part of the frequency is given by formula (5.29). From dispersion relation (5.30), we can see that growth rate (5.32) is as small as the ratio of γ_{MH} to the frequency of the primary modes.

5.4. Simplest Example of the Generation of Zonal Flows in the Case of a Magnetic Well

Turning to expression (4.27), we find that the quantity Ω_1^2 in expression (5.13) is equal to

$$\Omega_1^2 = \Gamma_1^2 C_1^2. \qquad (5.33)$$

Here, the quantity Γ_1^2 is defined by formula (5.23) with the replacement $k_y^2 I_{\mathbf{k}} \longrightarrow k_{y1}^2 I_{\mathbf{k}1}$, and

$$C_1^2 = g\kappa_n\left\{\frac{k_y^2}{k_x^2(2\omega-\omega_{*i})^2}\times\left[2\left(2-\frac{\omega_{*i}}{\omega}\right)^2 + \frac{\omega_{*i}^2}{\omega^2} + 4\frac{k_z^2 v_A^2}{\omega^2}\right]\right\}_{\mathbf{k}=\mathbf{k}_1}. \qquad (5.34)$$

Let us consider standing primary modes, i.e., a combination of two modes having the same amplitude and the same absolute value of the wave vector, $|\mathbf{k}|$, and propagating in opposite directions. In this case, dispersion relation (5.14) is reduced to

$$1 - \frac{\Gamma_1^2 C_1^2}{(\Omega - q_x V_{g1})^2} - \frac{\Gamma_1^2 C_1^2}{(\Omega + q_x V_{g1})^2} = 0. \qquad (5.35)$$

Formally, dispersion relation (5.35) coincides with the dispersion relation for two cold beams of equal density [27]. Using the results of Section 1.5.1 of [27], where this dispersion relation was investigated, we can see that it describes the instability of a zonal flow with the maximum growth rate

$$(\mathrm{Im}\,\Omega)_{\max} = \Gamma_1 C_1/2, \qquad (5.36)$$

which is achieved at

$$q_x = q_{x\,\mathrm{opt}} \equiv 3^{1/2}\Gamma_1 C_1/(4V_{g1}). \tag{5.37}$$

For $q_x < q_{x\,\mathrm{opt}}$, we are dealing with an analogous instability but with a slower growth rate.

6. DISCUSSION OF THE RESULTS

We have investigated the generation of zonal flows by DA waves modified by magnetic curvature effects, which are modeled by a gravity force. Our study is based on the vorticity equation; longitudinal Ohm's law; the continuity equation; and Eqs. (2.20), (2.4), and (2.21), which relate the electrostatic potential, vector potential, and perturbed plasma density. These equations have been analyzed by applying a parametric formalism similar to that used in the theory of the generation of convective cells [18, 19] (see also [16]). In contrast to most of the previous investigations on the subject (except for [17]), we have considered primary modes with an arbitrary spectrum rather than a monochromatic wave packet. The parametric approach so modified makes it possible to reveal a new class of instabilities of zonal flows—those that are analogous to two-stream instabilities in linear theory [27].

As was explained in Section 2.1, vorticity equation (2.20), used in our analysis, is quite nontrivial: in the hydrodynamic approximation, it should be derived by taking into account magnetic viscosity in the equation of motion. Hydrodynamic equations containing magnetic viscosity were originally obtained by Braginskii [29]. That the ion drift effect should be described with allowance for magnetic viscosity was first pointed out in [30, 31]. It is only when the magnetic viscosity is taken into account that the ion drift effect is described in the same manner as in Eq. (2.20) (see the term with $T_{0i}n$ in this equation). Note also that vorticity equation (2.20) with $\nabla_\parallel = 0$ and $g = 0$ was obtained later by an alternative kinetic method described in [5]. Although the magnetic viscosity is not explicitly introduced into the formalism of [5], it plays an important role in the hydrodynamic interpretation of the results [5].

As in most of the previous papers on the theory of the generation of zonal flows (see, e.g., [1, 3, 4, 10–14, 16–19, 21, 26]), we assume that the primary modes under consideration have real frequencies. However, it is then natural to ask whether the amplitude of the primary modes is assumed to remain at the fluctuation level or to grow far above this level due to some physical mechanisms. Following numerous earlier studies on the subject, we consider that such mechanisms do exist, but they are beyond the applicability limits of our "ideal" equations in the linear approximation. A linear mechanism whereby the primary DA modes grow can be exemplified by their dissipative growth caused by thermal conductivity and viscosity effects—a mechanism that was considered in [32] without allowance for the curvature of the equilibrium magnetic field. It is obvious that primary waves also may well be excited by nonlinear effects, e.g., those occurring during the decay of other waves. An example of such decay instabilities of Alfvén waves was considered in [33], where the ion drift effects, as well as the magnetic curvature effects, were ignored. In accordance with the general principles of nonlinear plasma theory (see, e.g., [34]), we assume that, because of the interplay between the linear and nonlinear effects, primary waves evolve to a certain steady state. On the whole, such effects can be called "traditional." By assumption, we consider the generation of zonal flows after the primary waves have relaxed to the corresponding steady state. The problem in question can also be studied in a more general formulation in which zonal flows are generated on a characteristic time scale comparable to that of the traditional effects, but this issue goes beyond the scope of the present paper.

In the case of a magnetic hill, $\kappa_n g < 0$, dispersion relation (1.2) describes both stable and unstable linear modes. The modes are expected to be unstable for sufficiently small values of ω_{*i} and $k_z v_A$. Investigation of the generation of zonal flows by unstable waves is beyond the scope of our work because the characteristic growth rates of the instabilities of zonal flows are slow in comparison with the linear growth rates of the primary waves. In order to analyze their generation, it is necessary to develop a nonlinear theory of unstable primary modes with allowance for the processes whereby the modes evolve to a steady state. According to [34], it may be supposed that the corresponding traditional nonlinear effects responsible for the evolution to the steady state will also modify the linear dispersion relation for primary modes. If this is the case, it is necessary to use a dispersion relation other than relation (1.2). The development of this theory may be the subject of future research.

We restricted ourselves to analyzing primary modes with $k_y \ll k_x$ (see the discussion in the Introduction) and used the standard approximation such that $\Omega \ll \omega$ and $q_x \ll k_x$. We have shown that, when only the effects proportional to q_x^2/k_x^2 are taken into account, the zonal components of the vector potential and perturbed plasma density are not generated, $\bar{A}_0 = 0$ and $\bar{n}_0 = 0$ (see equalities (4.3) and (4.4)).

One of the main results of our work is that we have derived a dispersion relation for zonal flows, namely, relation (4.29), where the function $F(\mathbf{k})$ is defined by expression (4.27). We have examined the main properties of this dispersion relation and have considered its physical consequences in some simplest limiting cases. A more detailed analysis of this dispersion relation can be postponed to a future study.

We have shown that zonal flows can be generated both in the case of a magnetic hill and in the case of a magnetic well. In the first case, the instabilities of zonal flows are analogous to negative-mass instabilities in linear theory, and, in the second case, they are analogous to two-stream instabilities.

It seems reasonable to compare the results of our analysis with those on the generation of zonal flows by the simplest types of electrostatic drift modes described by the dispersion relation

$$\omega = \omega_{*e}, \tag{6.1}$$

where ω_{*e} is the electron drift frequency.

Dispersion relation (6.1) does not contain the radial dispersion effects. In this context, the modes described by dispersion relation (6.1) may be called pure electron drift modes. The physical mechanism whereby pure electron drift modes generate zonal flows is the Reynolds stress, which takes part in the nonlinear dynamics of the modes. The generation of zonal flows by these modes is most pronounced in the case of an individual monochromatic wave packet, when the growth rate is equal to (see [26])

$$\mathrm{Im}\,\Omega = \Gamma_0, \tag{6.2}$$

where Γ_0 is defined by formula (5.23).

Our analysis shows that, because of a complete mutual balance between the averaged stresses, pure DA waves do not generate zonal flows. In a curved magnetic field, the generation is possible, however. If the magnetic curvature effects are weak enough so that the notion of quasi-pure DA modes can be introduced, then the growth rates of unstable zonal flows generated by these modes are as slow compared with growth rate (6.2) as the ratio of the characteristic magnetic-curvature-related frequencies (i.e., the quantity $|g\kappa_n|^{1/2}k_y/k_x$) to the frequencies of the quasi-pure modes (cf. expressions (5.25), (5.28), and (5.32), and also cf. expression (6.2) versus expression (5.36)). On the other hand, as this ratio increases, the growth rates of unstable zonal flows can become as fast as that given by expression (6.2).

We have shown that, in order to reveal the instability of zonal flows in the case of a magnetic well, it is necessary to consider primary modes with double-peaked spectra. The simplest of such spectra is represented by a superposition of two pump waves. We have also explained that, in contrast to the assertion of [21], there is no restriction on the steady-state spectrum of the primary modes.

ACKNOWLEDGMENTS

We are grateful to A.I. Smolyakov for useful discussions. This work was supported in part by the Russian Foundation for Basic Research (project no. 06-02-16767), the RF Program for State Support of Leading Scientific Schools (grant no. 2024.2003.2), the Department of Atomic Science and Technology of the RF Ministry of Atomic Industry, the National Program of Support for Excellence Groups (PRONEX) of the National Council for Scientific and Technological Development (CNPq) (Brazil), and the State of São Paulo Foundation for the Support of Research (FAPESP) (Brazil).

APPENDIX

Is There a Restriction on the Spectra of Primary Modes in the Problem of the Generation of Zonal Flows?

In [21], the generation of zonal flows by electrostatic electron drift waves was studied by using the vorticity equation

$$\left[\frac{\partial}{\partial t} + (V_0 + V_{*e})\frac{\partial}{\partial y}\right]\phi^{>} - \rho_s^2\left(\frac{\partial}{\partial t} + \mathbf{V}_E\cdot\nabla\right)\nabla_\perp^2\phi = 0. \tag{A.1}$$

Here, $\phi^{>} \equiv \tilde{\phi} + \hat{\phi}$, $V_0 = (c/B_0)\partial\bar{\phi}/\partial x$ is the zonal component of the drift velocity in crossed fields, V_{*e} is the electron drift velocity, ρ_s^2 is the square of the ion-sound Larmor radius (i.e., the ion Larmor radius calculated in terms of the electron temperature), and $\nabla_\perp^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Averaging Eq. (A.1) over small-scale oscillations, we obtain

$$\frac{\partial^2}{\partial x^2}\frac{\partial\bar{\phi}}{\partial t} = \frac{c}{B_0}\frac{\partial^2\Lambda}{\partial x^2}, \tag{A.2}$$

where

$$\Lambda = \left\langle\frac{\partial\phi^{>}}{\partial x}\frac{\partial\phi^{>}}{\partial y}\right\rangle. \tag{A.3}$$

In expression (A.3), we ignore the sidebands—the function $\hat{\phi}$—by setting $\phi^{>} = \tilde{\phi}$, where $\tilde{\phi}$ is given by representation (2.24). Equation (A.3) then becomes

$$\Lambda = \sum_{\mathbf{k}} k_x k_y I_{\mathbf{k}}, \tag{A.4}$$

where $I_{\mathbf{k}}$ is described by formula (4.28). From Eq. (A.4) we can see that the quantity Λ is independent of x and therefore does not contribute to Eq. (A.2). Hence, for arbitrary values of $I_{\mathbf{k}}$, we have

$$\partial^2\Lambda/\partial x^2 \equiv 0. \tag{A.5}$$

At the same time, in Eq. (A.2) from [21], the derivatives $\partial^2/\partial x^2$ were omitted with the result that (see [21], Eq. (2))

$$\frac{\partial\bar{\phi}}{\partial t} = \frac{c}{B_0}\Lambda. \tag{A.6}$$

From the requirement that the primary modes be steady-state, the following restriction on their spectra was obtained (see [21], relationship (6)):

$$\Lambda = 0. \qquad (A.7)$$

However, this procedure for reducing Eq. (A.2) to Eq. (A.6) is justified only when Λ depends on x and cannot be used for Λ determined by Eq. (A.4). Accordingly, restriction (A.7) obtained in [21] turns out to be incorrect.

It is Eq. (2.30) that is equivalent to Eq. (A.6). According to the explanations given in Section 2, we derived Eq. (2.30) from the equation analogous to Eq. (A.2) by ignoring the second derivative $\partial^2/\partial x^2$ in it. Consequently, by analogy with the case of electrostatic electron drift waves, there is no restriction on the spectrum of primary modes in our problem.

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Translated by I.A. Kalabalyk