Brownian Dynamics, Time-averaging and Colored Noise

D. O. Soares-Pinto [a,1] W. A. M. Morgado [b,2]

[a] *Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud 150, CEP 22290-180, Rio de Janeiro, Brazil*

[b] *welles@fis.puc-rio.br*
Departamento de Física, Pontifícia Universidade Católica do Rio de Janeiro, C.P. 38071, 22452-970 Rio de Janeiro, Brazil

Abstract

We propose a method to obtain the equilibrium distribution for positions and velocities of a one-dimensional particle via time-averaging and Laplace transformations. We apply it to the case of a damped harmonic oscillator in contact with a thermal bath. The present method allows us to treat, among other cases, a Gaussian noise function exponentially correlated in time, e.g., Gaussian colored noise. We obtain the exact equilibrium solution and study some of its properties.

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1 Introduction

A Brownian particle is a massive particle (M), surrounded by a bath of lighter particles (m) [1]. Its motion is very irregular and varies rapidly with time. However, its averaged behavior can be well understood. The reason for this is that the mass ratio ($m/M \ll 1$) leads to the separation of time-scales for the slow Brownian's degrees of freedom from the fast bath's degrees of freedom [2]. This allows us to construct adequate methods (e.g. Fokker-Planck equations) to describe the time evolution for the probability distribution function, or equivalently, for the momenta of the distribution [3,4]. The equilibrium distribution is thus obtained as a function of the equilibrium momenta. We do not need to point out that the behavior of Brownian particles has been the

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subject of many scientific studies since Einstein's seminal papers [5,6], already a hundred years old. Interesting extensions of this old problem have attracted the interest of many groups lately [7]. The treatment of non-standard forms of the noise function has also been successfully obtained recently [8].

In this paper, we calculate the equilibrium distribution for a damped harmonically bound Brownian particle as a direct integral, by means of Laplace transforms. However, a word is necessary to explain the use of the heavy machinery of Laplace transforms upon such a well known model.

In fact, the present model is only used as an illustration for the time-averaging method we propose. Applications for more interesting systems will follow. It basically consists in arranging the full dynamical behavior, expressed by the exact time-averaged distribution, into sums of integrals that can be proved either to vanish identically or to be easily calculable. Thus, the exact equilibrium distribution is obtained. Further extensions of the method are under way for non-Gaussian noise [9].

Technically, we define the equilibrium distribution in the spirit of Boltzmann's prescription [10] as a time-averaged distribution. The equilibrium distribution calculation is done by carrying out analytically the calculation for the time average of the distribution defined below, via the use of the stochastic functions $x(t)$ and $v(t)$, solutions of the stochastic dynamics. It reads

$$\mathrm{P}_{eq}(x,v) = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \ < \delta(x - x(t))\delta(v - v(t)) >, \tag{1}$$

where $<>$ means average over the noise $\eta(t)$. For ergodic systems fulfilling

$$f(t \to \infty) = \lim_{s\to 0} s\tilde{f}(s),$$

where $\tilde{f}$ is the Laplace transform of f, we can use an equivalent form for Eq. 1, namely

$$\mathrm{P}_{eq}(x,v) = \lim_{z\to 0} z \int_0^\infty dt\, e^{-zt} < \delta(x - x(t))\delta(v - v(t)) > . \tag{2}$$

As mentioned above, the functions $x(t)$ and $v(t)$ are solutions of the Langevin equations that follow from coupling the Brownian particle to a bath of lighter particles. We assume these to influence the Brownian dynamics via an effective non-white Gaussian noise force $\eta(t)$. The effect of time-correlations leads to the renormalization of the oscillator frequency at equilibrium (all results are exact), discussed in the next section.

Our approach is straightforward and allows us to separate the reversible from the irreversible parts of the dynamics. The equilibrium distribution (the irreversible part) is obtained by opening the delta-function on Eq. 2 by Laplace transformation into an infinite series, and by keeping only non-vanishing terms in it (in the limit $z \to 0$). As will be made clear in the following, the vanishing terms (in the limit $t \to \infty \Leftrightarrow z \to 0$) do so due to their singular structure. They have poles (the same as those for the dispersion relation of the damped harmonic oscillator described by the Langevin Equation minus the random force) expressing the dissipation due to the damping. The surviving terms are the ones that do not carry the effect of the damping to the main propagator (as shall be defined below, proportional to z). The equilibrium distribution is obtained by summing over the non-vanishing terms.

The method we propose could be summarized as follows:

- Laplace transform of the Langevin equation and obtain $\tilde{x}(z)$ and $\tilde{v}(z)$;
- Laplace transform of the Equilibrium distribution defined by Equation 1;
- To eliminate the vanishing terms (in the limit $z \to 0 \Leftrightarrow t \to \infty$);
- To integrate over the the non-vanishing terms and obtain the equilibrium distribution.

This method for obtaining the equilibrium distribution via time-averages, can be extended to different systems [3].

As stressed above, an important aspect of the method is that it allows us to study time-dependent processes (e.g., equilibration), since the effect of time comes from a single propagator. The long-time behavior of any dynamical functions for the system are easily singled out in the limit $z \to 0 \Leftrightarrow t \to \infty$.

This paper is organized as follows. In Section 2 we define the model. In Sections 3 we define, and Laplace transform, the time-averaged equilibrium distribution. In Section 4 we discuss the irreversibility from the point of view of surviving and vanishing integrals. In Section 5 we select the non-vanishing contributions and obtain the equilibrium distribution in an analytical form. In section 6 we make some concluding remarks.

[3] Work is under way in order to extend the present results to more general forms of noise, in special for the case of non-Gaussian white noise.

2 Model

2.1 Langevin Equations and Laplace transforms

We model a Brownian particle subject to a damped harmonic potential. The damping is proportional to the velocity. A Langevin force $\eta(t)$, Gaussian distributed, effectively acts on the Brownian particle due to the interaction with the bath particles, not included in the dynamics. We assume that $\eta(t)$ can be successfully Laplace transformed into $\tilde{\eta}(z)$.

The system obeys the following coupled Langevin-like Equations [4]:

$$m\dot{v}(t) = -\gamma v(t) - kx(t) + \eta(t), \tag{3}$$

$$\dot{x}(t) = v(t), \tag{4}$$

where $\eta(t)$ is a colored noise (time-correlated) distributed stochastic force.

The initial conditions are given by

$$x(t=0) = x_0, \;\; v(t=0) = v_0.$$

Integrating Eqs. 3 and 4, we obtain that $x(t)$ and $v(t)$ can be expressed respectively as:

$$x(t) = x_0 + \int_0^t v(t')dt', \tag{5}$$

and

$$mv(t) = mv_0 - \gamma \int_0^t v(t')dt' - kx_0 t - k \int_0^t \int_0^{t'} v(t'')dt''dt' + \int_0^t \eta(t')dt'. \tag{6}$$

The Laplace transform of $f(t)$ is defined as

$$\tilde{f}(z) = \int_0^\infty dt\, e^{-zt} f(t), \tag{7}$$

[4] Eq. 3 is strictly of Langevin type when the noise $\eta(t)$ is white. A discussion on this can be found in reference [4], chapter IX.

with $\mathrm{Re}(z) > 0$.

By taking the Laplace transform of Equations 5 and 6, we obtain:

$$\tilde{x}(z) = A(z) + B(z)\tilde{\eta}(z), \tag{8}$$

and

$$\tilde{v}(z) = C(z) + D(z)\tilde{\eta}(z), \tag{9}$$

where

$$A(z) = \frac{(z+\beta)x_0 + v_0}{z(z+\beta) + w_0^2}, \tag{10}$$

$$B(z) = \frac{m^{-1}}{z(z+\beta) + w_0^2}, \tag{11}$$

$$C(z) = \frac{zv_0 - w_0^2 x_0}{z(z+\beta) + w_0^2}, \tag{12}$$

$$D(z) = \frac{m^{-1} z}{z(z+\beta) + w_0^2}, \tag{13}$$

where $w_0^2 = \frac{k}{m}$ and $\beta = \frac{\gamma}{m}$.

Lets observe that only the coefficients A and C carry the contributions from the initial conditions. They are the memory terms: the vanishing of their contribution will make the process irreversible.

The common denominator of the coefficients A, B, C and D is the damped harmonic oscillator's dispersion relation. Its poles are

$$\kappa_{\pm} = -\frac{\beta}{2} \pm i\frac{1}{2}\sqrt{4\omega_0^2 - \beta^2}, \tag{14}$$

where we assume that $4\omega_0^2 - \beta^2 > 0$. This will be justified in the sequence, by the end of Section 5.

We observe that the coefficients A and C carry information about the initial conditions, but not B and D, which are related to the stochastic part of the dynamics. The system will retain memory of its initial condition as far as terms depending on the coefficients A and C contribute to the final results.

We assume that the stochastic process $\eta(t)$ is of the form of Gaussian colored noise [4], in which $\eta(t)$ is uncorrelated for $t < 0$. It is more general than white noise and reduces to it in the limit $\tau \to 0$.

Some considerations have to be made about our choice of colored noise. For a closed system, noise has only internal causes. Using projection operator techniques [11] one obtains a generalized Langevin Equation [12] showing that the fluctuation-dissipation theorem holds near equilibrium and has to be consistent with a retarded dissipation of the form:

$$\dot{v}(t) = -\int_0^t dt'\,\phi(t-t')v(t') + \eta(t),$$

where

$$\phi(t) = \frac{\langle \eta(t)\eta(t+\tau)\rangle}{k_B T}.$$

However, we are assuming that the much simpler form of dissipation holds, namely Eq. 3. This can be justified by assuming that in a general situation (far from equilibrium), the fluctuation-dissipation theorem does not apply and the dissipation's memory kernel and the fluctuation properties of the noise need not to be related [13]. A method for solving the fully retarded dissipation for arbitrary noise can be found in reference [13].

Another interesting situation is the case of purely external noise. In this case, noise is not described by a closed Hamiltonian and dissipation and noise fluctuations are not related. [5]

Thus, we start from Eqs. 3 and 4 and study the long-time behavior of a non-Markovian coupled system (x, v). As for the noise properties, due to its Gaussian nature, only the second order cumulant survives (zero average distributed):

$$\langle \eta(t)\rangle = 0; \tag{15}$$

$$\langle \eta(t)\eta(t')\rangle = \frac{D}{2\tau} e^{-|t-t'|/\tau}. \tag{16}$$

[5] We could imagine a simple system consisting of a charged particle immersed in a viscous fluid, driven by a strong external fluctuating electrical field.

As usual, for even l we have:

$$\langle\prod_{i=1}^{l}\eta(t_i)\rangle = \sum_{\text{arrangements}}^{\text{all pairwise}} \langle\eta(t_{i_1})\eta(t_{i_2})\rangle\ldots\langle\eta(t_{i_{l-1}})\eta(t_{i_l})\rangle,$$

for all possible combinations for $l = 2p$ even (it will have $\frac{(2p)!}{2^p p!}$ combinations of pairs of η functions).

By Laplace transforming, the η-average second cumulant reads:

$$\langle\tilde{\eta}(iq_i+\epsilon)\tilde{\eta}(iq_j+\epsilon)\rangle = \left\{\frac{D}{[i(q_i+q_j)+2\epsilon]\,[1-\tau(iq_i+\epsilon)]\,[1-\tau(iq_j+\epsilon)]}\right\} - \\ -\frac{D\tau}{2}\left\{\frac{3+\tau\,[i(q_i+q_j)+2\epsilon]-\tau^2(iq_i+\epsilon)(iq_j+\epsilon)}{[1-\tau(iq_i+\epsilon)]\,[1+\tau(iq_i+\epsilon)]\,[1-\tau(iq_j+\epsilon)]\,[1+\tau(iq_j+\epsilon)]}\right\}. \quad (17)$$

Eq. 17 reduces to the white noise case

$$\frac{D}{i(q_i+q_j)+2\epsilon},$$

as $\tau \to 0$.

The presence of time-correlated noise will affect the equilibrium distribution. As shall be seen, it has the effect of renormalizing the oscillator frequency. The white-noise equilibrium form is recovered when $\tau \to 0$.

3 Equilibrium Distribution

A convenient way of defining the long-time limit ($t \to \infty$) for a convergent equilibrium distribution is to use

$$f_{eq}^{z} = \lim_{z\to 0}\frac{\int_0^\infty dt\, e^{-zt} f(t)}{\int_0^\infty e^{-zt}dt} \quad (18)$$

for $z > 0$. This is equivalent to

$$f_{eq} = \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\, f(t),$$

whenever this expression converges. In this case we write

$$f(t \to \infty) = f_{eq} + \Delta(t),$$

with $\lim_{t\to\infty} \Delta(t) = 0$ [6]. For $z = \frac{1}{T}$, the average f_{eq}^z then reads

$$\begin{aligned} f_{eq}^z &= \lim_{z\to 0} \frac{\int_0^\infty dt\, e^{-zt} f(t)}{\int_0^\infty e^{-zt} dt} = \lim_{T\to\infty} \frac{\int_0^\infty dt\, e^{-t/T} f(t)}{T} \\ &= f_{eq} + \lim_{T\to\infty} \frac{\int_0^M dt\, e^{-t/T} \Delta(t)}{T} \\ \Rightarrow \lim_{\delta\to 0} \left| f_{eq}^z - f_{eq} \right| &\leq \lim_{\delta\to 0} \delta \to 0. \end{aligned}$$

The two forms of time averaging are equivalent. In the case the distribution defined on Equation 1 does not converge to an analytical form (as $t \to \infty$), both definitions above will fail to give convergent values.

Thus, we write the time-averaged equilibrium distribution as (see Appendix A):

$$\begin{aligned} P_{eq}(x,v) &= \lim_{z\to 0} z \int_0^\infty dt\, e^{-zt} \langle \delta(x - x(t)) \delta(v - v(t)) \rangle \\ &= \lim_{z\to 0} \lim_{\epsilon\to 0} \sum_{l,m=0}^{\infty} \int_{-\infty}^{+\infty} \frac{dQ}{2\pi} \frac{dP}{2\pi} e^{iQx+iPv} \frac{(-iQ)^l}{l!} \frac{(-iP)^m}{m!} \\ &\times \int_{-\infty}^{+\infty} \prod_{f=1}^{l} \frac{dq_f}{2\pi} \prod_{h=1}^{m} \frac{dp_h}{2\pi} \frac{z}{z - \left[\sum_{f=1}^{l} iq_f + \sum_{h=1}^{m} ip_h + (l+m)\epsilon \right]} \\ &\times \langle \prod_{f=1}^{l} \tilde{x}(iq_f + \epsilon) \prod_{h=1}^{m} \tilde{v}(ip_h + \epsilon) \rangle. \end{aligned} \tag{19}$$

The integration path for the q- and p-variables is also given in Figure B.1. Both upper and lower semi-hemispherical integration paths will contribute nothing to the total integral. We choose the upper path for our calculations.

4 Origin of irreversibility at the limit $z \to 0$

The equilibrium distribution we obtain corresponds to a sum of (infinite) products of polynomial fractions with poles given by the dispersion relation, Equa-

[6] We assume that: $\forall \delta > 0, \exists M : \forall t > M \Rightarrow |\Delta(t)| < \delta$.

tion 14. One of those propagators, in Equation 19, stands out for being the only one that contains the effect of time: the main propagator $G(z)$, given by

$$G(z) = \frac{z}{z - [i(q_1 + \ldots + q_l + p_1 + \ldots + p_m) + (l + m)\epsilon]}. \tag{20}$$

In order to evaluate the equilibrium distribution (the analysis can also be extended to the non-equilibrium case $z \neq 0$), we need to study the poles and the limiting properties of the main propagator given by Equation 20.

When integrating a given q or p–variable, by choosing the upper path on the complex plane shown in figure B.1, the pole associated with the propagator falls outside the integration path, contributing with no residues. However, after effecting all $(m + l)$ variable calculations, values associated with the poles of the the rest of the integrand in Eq. 19 might be carried through into the denominator of $G(z)$. It is essential for us to analyze $G(z)$ at the limit when $z \to 0$.

When the denominator in the main propagator is non-zero for $z \to 0$, it is clear that $\lim_{z\to 0} G(z) = 0$. A non-zero result can only be obtained, in the limit $z \to 0$, when the sum of q and p–variable is completely eliminated from $G(z)$. When all variables are eliminated, we have instead $\lim_{z\to 0} G(z) = \lim_{z\to 0} \frac{z}{z} = 1$, and the integral yields a non-zero contribution.

Since the factors A and C are not coupled with the stochastic force $\tilde{\eta}$, the q and p–variables associated with them will lead to residues only around the poles of the dispersion relation $(-i\kappa_\pm)$. The pole's values will in the end be transported all the way to the denominator of the main propagator. So, by integrating over the poles of the functions A and C, a non-zero final value to the summation $i\,[q_1 + \ldots + q_l + p_1 + \ldots + p_m] + (m + l)\epsilon$, will give a zero contribution in the limit $z \to 0$.

This is the simple way by which irreversibility manifests itself on the path to equilibrium: the memory of the initial conditions is lost as the inverse Laplace transform of terms such as

$$\sim \frac{z}{z - i(m\ (-i\kappa_\pm))} = \frac{z}{z - m\beta/2 \pm i \ldots} \to \quad e^{-\frac{m\beta t}{2}} \to 0,$$

where m is an integer.

5 Important integrals

The effect of poles from the dispersion relation in functions B and D can be eliminated since they are coupled with $\tilde{\eta}$-functions, which are associated with a factor $\delta(iq + iq^{'} + 2\epsilon)$ dependency, coming from the stochastic properties of the $\tilde{\eta}$-functions. When we integrate over the q and $q^{'}$ variables, the $\delta(iq + iq^{'} + 2\epsilon)$ dependency eliminates the effect of these variables from the main propagator in the equilibrium limit $z \rightarrow 0$. Thus, only three possible couplings may contribute to the equilibrium distribution: BB, DD, and BD.

In the following we list the non-vanishing contributions to the equilibrium distribution $\mathrm{P}_{eq}(x, v)$.

5.1 Contribution from BB

The typical contributing term for the x distribution is given by the BB contributing terms below (for details see Appendix B):

$$\int_{-\infty}^{+\infty} \frac{dq_i}{2\pi}\frac{dq_j}{2\pi}\frac{z}{z - i(q_i + q_j + \diamond)}B(iq_i + \epsilon)B(iq_j + \epsilon)\langle\tilde{\eta}(iq_i + \epsilon)\tilde{\eta}(iq_j + \epsilon)\rangle =$$

$$= \frac{z}{z - i\diamond}\frac{D}{2[m + \tau^2 k - \tau\gamma][m + \tau^2 k + \tau\gamma]}\{\frac{m^2}{k\gamma} - \frac{\tau^2\gamma}{k} + \frac{\tau^2 m}{\gamma} + \tau^3\} \tag{21}$$

where $\diamond$ represents all the non integrated variables.

Consequently, the only contributing terms will be even powers $m = 2\theta$:

$$\frac{m!}{2^{\frac{m}{2}}(\frac{m}{2})!}\{\frac{1}{[(m + \tau^2 k)^2 - \tau^2\gamma^2]}[\frac{Dm^2}{2k\gamma} - \frac{\tau^2 D\gamma}{2k} + \frac{\tau^2 Dm}{2\gamma} + \frac{\tau^3 D}{2}]\}^{\frac{m}{2}} =$$

$$= \frac{(2\theta)!}{2^\theta\theta!}\{\frac{1}{[(m + \tau^2 k)^2 - \tau^2\gamma^2]}[\frac{Dm^2}{2k\gamma} - \frac{\tau^2 D\gamma}{2k} + \frac{\tau^2 Dm}{2\gamma} + \frac{\tau^3 D}{2}]\}^\theta$$

5.2 Contribution from DD

Similarly with the BB, the DD contributing terms are:

$$\int_{-\infty}^{+\infty} \frac{dp_i}{2\pi}\frac{dp_j}{2\pi}\frac{z}{z - i(p_i + p_j + \diamond)}D(ip_i + \epsilon)D(ip_j + \epsilon)\langle\tilde{\eta}(ip_i + \epsilon)\tilde{\eta}(ip_j + \epsilon)\rangle =$$

$$= \frac{z}{z - i\diamond} \frac{-D}{2[m+\tau^2 k - \tau\gamma][m+\tau^2 k + \tau\gamma]} [\frac{-m}{\gamma} - \frac{\tau^2 k}{\gamma} + \tau] \tag{22}$$

In the limit $z \to 0$, the contribution from the velocity momenta averages for the probability distribution is given by, for $n = 2\alpha$:

$$\frac{n!}{2^{\frac{n}{2}}(\frac{n}{2})!} \{ \frac{1}{[(m+\tau^2 k)^2 - \tau^2\gamma^2]} [\frac{Dm}{2\gamma} + \frac{\tau^2 Dk}{2\gamma} - \frac{\tau D}{2}] \}^{\frac{n}{2}} =$$

$$= \frac{(2\alpha)!}{2^\alpha \alpha!} \{ \frac{1}{[(m+\tau^2 k)^2 - \tau^2\gamma^2]} [\frac{Dm}{2\gamma} + \frac{\tau^2 Dk}{2\gamma} - \frac{\tau D}{2}] \}^{\alpha}$$

5.3 Contribution from BD

Similarly with the above, we can calculate the cross contribution BD except that it has an odd power of the integrated variables ($O(p)$) in the numerator while the previous ones had even powers of it ($O(1)$ and $O(p^2)$ respectively). Thus:

$$\int_{-\infty}^{+\infty} \frac{dq_i}{2\pi} \frac{dp_j}{2\pi} \frac{z}{z - i(q_i + p_j + \diamond)} B(iq_i + \epsilon) D(ip_j + \epsilon) \langle \tilde{\eta}(iq_i + \epsilon) \tilde{\eta}(ip_j + \epsilon) \rangle, \tag{23}$$

the only important term

$$\int_{-\infty}^{+\infty} \frac{dp_j}{2\pi} \frac{iz}{z - i(0 + \diamond)} \frac{m^{-2} D p_j}{(p_j - i\kappa_+)(p_j - i\kappa_-)(p_j + i\kappa_+)(p_j + i\kappa_-)} \tag{24}$$

is identically null because of the odd power of p_j. So the equilibrium distribution will be separated into a position term and a velocity term:

$$P_{eq}(x, v) = P_{eq}(x) P_{eq}(v).$$

5.4 Equilibrium distribution

Collecting all the non-vanishing terms in the $z \to 0$ limit, we write Eq. 19 as:

$$P_{eq}(x, v) = \sum_{\theta=0}^{\infty} \sum_{\alpha=0}^{\infty} \int_{-\infty}^{+\infty} \frac{dQ}{2\pi} \frac{dP}{2\pi} e^{iQx + iPv} \times$$

$$\times\frac{(-iQ)^{2\theta}}{2\theta!}\frac{(2\theta)!}{2^{\theta}\theta!}\{\frac{1}{[(m+\tau^2k)^2-\tau^2\gamma^2]}[\frac{Dm^2}{2k\gamma}-\frac{\tau^2D\gamma}{2k}+\frac{\tau^2Dm}{2\gamma}+\frac{\tau^3D}{2}]\}^{\theta}\times$$

$$\times\frac{(-iP)^{2\alpha}}{2\alpha!}\frac{(2\alpha)!}{2^{\alpha}\alpha!}\{\frac{1}{[(m+\tau^2k)^2-\tau^2\gamma^2]}[\frac{Dm}{2\gamma}+\frac{\tau^2Dk}{2\gamma}-\frac{\tau D}{2}]\}^{\alpha}=$$

$$=\int_{-\infty}^{+\infty}\frac{dQ}{2\pi}\exp\{iQx-\frac{Q^2}{2[(m+\tau^2k)^2-\tau^2\gamma^2]}[\frac{Dm^2}{2k\gamma}-\frac{\tau^2D\gamma}{2k}+\frac{\tau^2Dm}{2\gamma}+\frac{\tau^3D}{2}]\}\times$$

$$\times\int_{-\infty}^{+\infty}\frac{dP}{2\pi}\exp\{iPv-\frac{P^2}{2[(m+\tau^2k)^2-\tau^2\gamma^2]}[\frac{Dm}{2\gamma}+\frac{\tau^2Dk}{2\gamma}-\frac{\tau D}{2}]\}$$

Integrating the equation above, we finally obtain the properly normalized equilibrium distribution:

$$P_{eq}(x,v)=\sqrt{\frac{[(m+\tau^2k)^2-\tau^2\gamma^2]}{\pi[\frac{Dm^2}{k\gamma}-\frac{\tau^2D\gamma}{k}+\frac{\tau^2Dm}{\gamma}+\tau^3D]}}\exp\{-\frac{m^2[(1+\tau^2\omega_0^2)^2-\tau^2\beta^2]x^2}{[\frac{Dm^2}{k\gamma}-\frac{\tau^2D\gamma}{k}+\frac{\tau^2Dm}{\gamma}+\tau^3D]}\}\times$$
$$\times\sqrt{\frac{[(m+\tau^2k)^2-\tau^2\gamma^2]}{\pi[\frac{Dm}{\gamma}+\frac{\tau^2Dk}{\gamma}-\tau D]}}\exp\{-\frac{m^2[(1+\tau^2\omega_0^2)^2-\tau^2\beta^2]v^2}{[\frac{Dm}{\gamma}-\tau D+\frac{\tau^2Dk}{\gamma}]}\}\qquad(25)$$

On the limit $\tau\to0$ we obtain the distribution of displacement and velocity of a Brownian particle under a Gaussian white noise as expected:

$$P_{eq}(x,v)=\sqrt{\frac{k\gamma}{\pi D}}e^{-\frac{2\gamma}{D}\frac{kx^2}{2}}\sqrt{\frac{m\gamma}{\pi D}}e^{-\frac{2\gamma}{D}\frac{mv^2}{2}}\qquad(26)$$

Eq. 26 is the analytical time-averaged equilibrium distribution for the damped harmonically bound Brownian particle.

The result from Eq. 25 has to be compared with exact results in the literature. The exact solution for a Langevin equation with time-correlated noise has been obtained by means of functional methods [14,15] or by path-integral approaches [16]. In order to compare our results with these references, we allow the system to be unconfined (we assume Eq. 25 holds for $k\to0$ and $m=1$) and we integrate over the coordinate variable x. The reduced distribution obtained from Eq. 25 then reads

$$P_{eq}(v)=\sqrt{\frac{\gamma[1+\tau\gamma]}{\pi D}}\exp\{-\frac{\gamma[1+\tau\beta]v^2}{D}\}$$

$$= \left(\sqrt{\frac{\gamma}{\pi D}}\exp\{-\frac{\gamma v^2}{D}\}\right)\left(1+\tau\left[\frac{\gamma}{2}-\frac{\gamma^2 v^2}{D}\right]+O(\tau^2)\right) \tag{27}$$

This corresponds exactly (up to linear order on τ) to the results in references [15,16] when we make the change $D \to 2D$ due to slightly different definitions of the noise time-correlation. As for the result in reference [14], they are identical to all orders of τ.

A difference between our approach and these other authors' [14–16] comes from allowing another set of coupled variables (position $\{x\}$) to interact with our variable (velocity $\{v\}$). This leads to the appearance of terms proportional to $k\tau^2$ on the equilibrium distribution. As somewhat advanced in the Appendix A of reference [16], the equilibrium distribution only depends on simple combinations of small powers of τ (on the exponent's argument).

The fluctuation-dissipation ratio $\frac{D}{2\gamma}$ plays the role of a thermal-bath temperature in the limit $\gamma, D \to 0$ with $D/2\gamma =$ constant:

$$k_B T_{bath} = \frac{D}{2\gamma}.$$

This is the reason we choose $4\omega_0^2 > \beta^2$ in Section 2.1.

However, for non-zero τ the expression for the temperature grows a little more complicated. It will be given by

$$k_B T_{bath}^{col-noise} = \frac{D[1-\tau\beta+\tau^2\omega_0^2]}{2\gamma[(1+\tau^2\omega_0^2)^2-\tau^2\beta^2]}. \tag{28}$$

From the form for the temperature and the equilibrium distribution, Eqs. 26 and 28, we observe that the equilibrium frequency is renormalized to ω_1, given exactly by

$$\omega_1^2 = \omega_0^2 \frac{[1-\tau\beta+\tau^2\omega_0^2]}{[1-\tau^2\beta^2+\tau^2\omega_0^2+\tau^3\omega_0^2\beta]}. \tag{29}$$

As can be seen, in first order in τ this corresponds to

$$\omega_1^2 \approx \omega_0^2(1-\tau\beta),$$

which reflects a decrease in the oscillating frequency due to damping. The time-correlations in $\eta(t)$ tend to make the velocities more persistent when they are highest (near the equilibrium position) thus making the system getting further away from the origin and taking longer to come back.

We are presently generalizing this method in order to treat other forms of potential gradients affecting the Brownian particle.

6 Conclusions

We present an approach for obtaining the equilibrium distribution of a simple system (in this case a Brownian particle under time correlated colored noise (with Gaussian white noise as a special case) via time-averaging (at the limit $t \to \infty$) since taking the time-average is the most correct way of describing the long time equilibrium behavior of any physical system.

Our model separates clearly the the irreversible contributions from the terms that keep track of the initial conditions (memory terms). It allows us to study each term contributing to the probability distribution for the position and velocity of the Brownian particle and to follow it as $t \to \infty$. It is shown, by Laplace transforming the distributions, that the non-contributing terms (memory terms) disappear due to the vanishing of the main propagator associated with them in the limit $z \to 0 \Leftrightarrow t \to \infty$. The non-vanishing terms can then be grouped together and integrated, yielding the exact equilibrium probability distribution, as has been confirmed by comparing with previous exact results in the literature [14–16].

In summary, the main advantage of the method is that it allows us to obtain the exact derivation of the equilibrium (and also the transient behavior if necessary, but not done in this paper) in a systematic way. It "opens" the problem into treatable parts (however ugly they look) that reach easily calculable values at infinitely long times.

Work is under way to extend the present results to other forms of noise.

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A

The detailed Laplace transformation of Equation 19 is done below:

$$
\begin{aligned}
P_{eq}(x,v) &= \lim_{z\to 0} z \int_0^{\infty} e^{-zt} \langle \delta(x - x(t))\delta(v - v(t))\rangle dt \\
&= \lim_{z\to 0} z \int_0^{\infty} dt e^{-zt} \int_{-\infty}^{+\infty} \frac{dQ}{2\pi} e^{iQx} \int_{-\infty}^{+\infty} \frac{dP}{2\pi} e^{iPv} \sum_{l=0}^{\infty} \frac{(-iQ)^l}{l!} \sum_{m=0}^{\infty} \frac{(-iP)^m}{m!} \langle x^l(t) v^m(t)\rangle \\
&= \lim_{z\to 0} z \int_0^{\infty} dt e^{-zt} \int_{-\infty}^{+\infty} \frac{dQ}{2\pi} e^{iQx} \int_{-\infty}^{+\infty} \frac{dP}{2\pi} e^{iPv} \sum_{l=0}^{\infty} \frac{(-iQ)^l}{l!} \sum_{m=0}^{\infty} \frac{(-iP)^m}{m!} \\
&\times \int_0^{\infty} \prod_{f=1}^{l} dt_{lf} \int_0^{\infty} \prod_{h=1}^{m} dt_{mh} \delta(t - t_{la})\delta(t - t_{mb}) \langle \prod_{f=1}^{l} x(t_{lf}) \prod_{h=1}^{m} v(t_{mh})\rangle \\
&= \lim_{z\to 0} z \int_0^{\infty} dt e^{-zt} \int_{-\infty}^{+\infty} \frac{dQ}{2\pi} e^{iQx} \int_{-\infty}^{+\infty} \frac{dP}{2\pi} e^{iPv} \sum_{l=0}^{\infty} \frac{(-iQ)^l}{l!} \sum_{m=0}^{\infty} \frac{(-iP)^m}{m!} \\
&\times \int_{-\infty}^{+\infty} \prod_{f=1}^{l} \frac{dq_f}{2\pi} \prod_{h=1}^{m} \frac{dp_h}{2\pi} \int_0^{\infty} \prod_{f=1}^{l} dt_{lf} \int_0^{\infty} \prod_{h=1}^{m} dt_{mh} \\
&\times e^{\sum_{a=1}^{l}(t - t_{la})(iq_a + \epsilon) + \sum_{b=1}^{m}(t - t_{mb})(ip_b + \epsilon)} \langle \prod_{f=1}^{l} x(t_{lf}) \prod_{h=1}^{m} v(t_{mh})\rangle \\
&= \lim_{z,\epsilon\to 0} \sum_{l,m=0}^{\infty} \int_{-\infty}^{+\infty} \frac{dQ}{2\pi}\frac{dP}{2\pi} e^{iQx + iPv} \frac{(-iQ)^l}{l!}\frac{(-iP)^m}{m!} \int_{-\infty}^{+\infty} \prod_{f=1}^{l} \frac{dq_f}{2\pi} \prod_{h=1}^{m} \frac{dp_h}{2\pi} \\
&\times \frac{z}{z - \left[\sum_{f=1}^{l} iq_f + \sum_{h=1}^{m} ip_h + (l+m)\epsilon\right]} \langle \prod_{f=1}^{l} \tilde{x}(iq_f + \epsilon) \prod_{h=1}^{m} \tilde{v}(ip_h + \epsilon)\rangle \quad \text{(A.1)}
\end{aligned}
$$

B

The typical contribution for the equilibrium distribution for x is (integrations follow the path described in Figure B.1):

$$
\int_{-\infty}^{+\infty} \frac{dq_i}{2\pi}\frac{dq_j}{2\pi} \frac{z}{z - i(q_i + q_j + \diamond)} B(iq_i + \epsilon) B(iq_j + \epsilon) \langle \tilde{\eta}(iq_i + \epsilon)\tilde{\eta}(iq_j + \epsilon)\rangle =
$$

$$
= \int_{-\infty}^{+\infty} \frac{dq_i}{2\pi}\frac{dq_j}{2\pi} \frac{z}{z - i(q_i + q_j + \diamond)} \frac{m^{-2}}{(q_i + i\kappa_+)(q_i + i\kappa_-)(q_j + i\kappa_+)(q_j + i\kappa_-)} \frac{D}{[i(q_i + q_j) + 2\epsilon]}
$$

$$
= \frac{z}{z - i(0+\diamond)} \int_{-\infty}^{+\infty} \frac{dq_j}{2\pi} \frac{m^{-2}D}{[q_j - i\kappa_+][q_j - i\kappa_-](q_j + i\kappa_+)(q_j + i\kappa_-)}
$$

$$
= \frac{D}{2\gamma k} \frac{z}{z - i(0+\diamond)} \qquad \text{(B.1)}
$$

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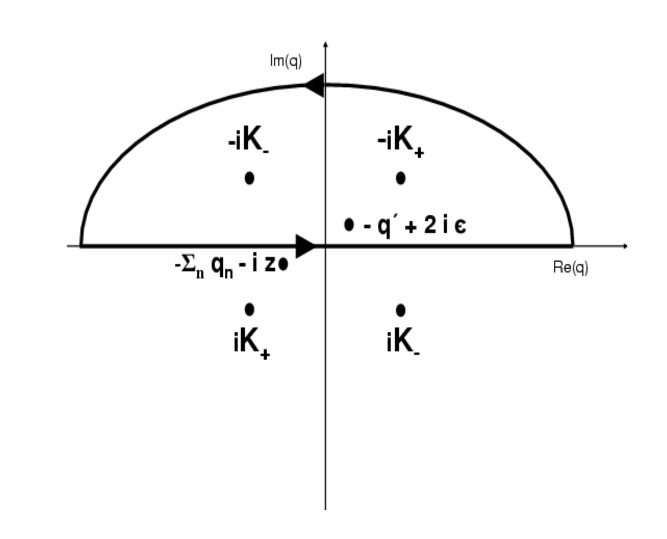

Fig. B.1. Integration path for the q or p–variables.