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Nonlinear Ehrenfest's urn model

G. A. Casas,* F. D. Nobre,† and E. M. F. Curado‡

Centro Brasileiro de Pesquisas Físicas and National Institute of Science and Technology for Complex Systems, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro, Rio de Janeiro, Brazil

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Ehrenfest's urn model is modified by introducing nonlinear terms in the associated transition probabilities. It is shown that these modifications lead, in the continuous limit, to a Fokker-Planck equation characterized by two competing diffusion terms, namely, the usual linear one and a nonlinear diffusion term typical of anomalous diffusion. By considering a generalized H theorem, the associated entropy is calculated, resulting in a sum of Boltzmann-Gibbs and Tsallis entropic forms. It is shown that the stationary state of the associated Fokker-Planck equation satisfies precisely the same equation obtained by extremization of the entropy. Moreover, the effects of the nonlinear contributions on the entropy production phenomenon are also analyzed.

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I. INTRODUCTION

Ehrenfest's urn model (sometimes also called Ehrenfest's flea model) [1] has played an important role in clarifying the foundations of statistical mechanics, providing an interpretation of irreversibility in a statistical manner. The model is defined by N balls distributed in two urns (or boxes), 1 and 2, such that at each discrete instant of time s, a ball is chosen at random and moved from the box in which it is found to the other box. At the beginning of the 20th century, such a simple model was useful in explaining the heat exchange between two bodies at unequal temperatures, where the temperatures are mimicked by the number of balls in each box, and the heat exchange becomes a random process (see, e.g., Ref. [2]).

Let us then define $N_1(s)$ and $N_2(s)$ $[N_1(s) + N_2(s) = N]$ as the number of balls in each box at time s and $P(l,s)$ as the probability for finding $N_1(s) = l$ balls inside urn 1 and $N_2(s) = N - l$ balls in urn 2 at time s. Hence, at time $s + 1$ the probability $P(l,s + 1)$ for finding l balls in urn 1 follows the master equation

$$P(l,s+1) = \frac{l+1}{N} P(l+1,s) + \frac{N-l+1}{N} P(l-1,s). \tag{1}$$

Due to the randomness of the process, the move of a ball from box 1 to box 2 (from box 2 to box 1), leading to $l + 1 \to l$ $(l - 1 \to l)$, occurs with the transition probabilities appearing in the first and second terms in the right-hand side of the master equation given by the fractions of balls in boxes 1 and 2 at time s, $(l + 1)/N$ and $[N - (l - 1)]/N$, respectively. In this case, the equilibrium distribution is given by a binomial distribution of the number l, approaching a Gaussian form in the limit $N \to \infty$ [2–4]. This later result may be seen if one considers the continuum limit of Eq. (1); let us then rewrite this equation as

$$\begin{aligned} NP(l,s+1) &= l[P(l+1,s) - P(l-1,s)] \\ &\quad + [P(l+1,s)+P(l-1,s)]+NP(l-1,s) \end{aligned} \tag{2}$$

to introduce the variables

$$x = \sqrt{\frac{2D}{N}}\left(l - \frac{N}{2}\right); \quad t = \frac{s}{N};$$
$$\Rightarrow \quad \Delta x = \sqrt{\frac{2D}{N}}; \quad \Delta t = \frac{1}{N}, \tag{3}$$

where D represents a constant (to be identified later on with the diffusion constant), as well as to define a continuous one-dimensional probability for finding a ball at a given "position" x at time t, $P(x,t) = NP(l,s)$. It is important to notice that due to the symmetrization introduced in the variable x, when $N \to \infty$, one has $x \in [-\infty,\infty]$, so that in the equilibrium state one expects $\langle x \rangle = 0$. Hence, expanding the probabilities of Eq. (2) and keeping terms up to $O(1/N)$, one obtains the following linear Fokker-Planck equation [2,5]:

$$\frac{\partial P(x,t)}{\partial t} = 2\frac{\partial}{\partial x}[xP(x,t)] + D\frac{\partial^2 P(x,t)}{\partial x^2}, \tag{4}$$

which may be associated with a random walker in the presence of a parabolic confining potential $\phi(x) = x^2$, i.e., subjected to a restoring force $A(x) = -[d\phi(x)/dx] = -2x$ [6].

The Ehrenfest model of Eq. (1), which defines a procedure of uncorrelated and random actions for moving balls between the two boxes, was shown to be directly related to the Boltzmann-Gibbs (BG) entropy [7]. Consistently, considering the continuous limit above, the linear Fokker-Planck equation (4) is associated with the BG entropy by means of an H theorem [6,8,9]. It is also well known that Eq. (4) presents a Gaussian distribution as its time-dependent solution, which, for a time much smaller than the one required for the approach to the stationary state, leads to a variance $\langle x^2 \rangle \sim t$, which is typical of a linear-diffusion process [6]. Moreover, the long-time limit of this Gaussian distribution coincides with the distribution that appears from the extremization of BG entropy. Consequently, the linear Fokker-Planck equation, Gaussian distribution, linear diffusion, and BG entropy are all intimately related to uncorrelated events within the present problem.

Many attempts were made in the literature to modify Ehrenfest's urn model (see, e.g., Refs. [3,4,10,11]), although none of them has been associated with nonlinear Fokker-Planck equations (NLFPEs) [12], to our knowledge. One should

*Corresponding author: gabrielaa@cbpf.br
†fdnobre@cbpf.br
‡evaldo@cbpf.br

call attention to the modification carried out in Ref. [11], where the transition probabilities were changed to yield, in the continuous limit, a linear nonhomogeneous Fokker-Planck equation, which is usually considered for studying anomalous transport in an optical lattice. As proposed in Ref. [13], this Fokker-Planck equation presents a q-Gaussian distribution, which is typical of nonextensive statistical mechanics [14], as its stationary state; this solution, which comes from the nonhomogeneous character of the equation, has recently been considered successfully in fitting data from experiments on optical lattices [11,15].

Increasing interest in the study of NLFPEs has occurred recently [8,9,12,14,16–23], motivated by the fact that they appeared to be candidates for explaining a wide range of processes associated with anomalous-diffusion phenomena [24]. Based on this, generalizations of Ehrenfest's urn model by incorporating nonlinear contributions to yield NLFPEs in the continuous limit become very relevant. For this purpose, in the present work we perform modifications of the Ehrenfest model in Eq. (1) by introducing nonlinear transition probability rates, following a procedure similar to the one considered in Refs. [25,26], where NLFPEs were obtained by carrying out approximations on a master equation. Such modifications are motivated by the behavior of many complex systems, in which the transition probabilities may be directly affected by regions with different concentrations of constituents, excluded-volume effects, long-range repulsive (or attractive) interactions, and/or strong correlations; later, this procedure was justified by scaling arguments (see, e.g., Refs. [27–29]). In the next section we define the modified Ehrenfest's urn model and obtain its corresponding NLFPE, and considering a generalized H theorem, we calculate the associated entropic form. In Sec. III we investigate stationary-state solutions of the NLFPE and show that such distributions follow an equation that coincides with the one obtained by extremization of the entropy; some particular cases are analyzed. Moreover, considering the NLFPE, in Sec. IV we discuss the phenomenon of entropy production, emphasizing the effects of the nonlinear contributions. Finally, in Sec. V we present our conclusions.

II. NONLINEAR EHRENFEST'S URN MODEL AND ASSOCIATED ENTROPY

Following the model in Eq. (1), let us introduce a modified Ehrenfest's urn model,

$$P(l,s+1) = W_{l+1,l}[P]\,P(l+1,s) + W_{l-1,l}[P]\,P(l-1,s), \tag{5}$$

where the transition probabilities $W_{l+1,l}[P]$ and $W_{l-1,l}[P]$ for removing one ball from box 1 ($l+1 \to l$) and for adding one ball in box 1 ($l-1 \to l$), respectively, do not correspond to simple uncorrelated events, but rather depend on the occupation probabilities P. Inspired by Refs. [25,26], herein we consider the following ansatz for the general transition rates to recover several particular cases of interest, defined as

$$W_{l+1,l}[P] = \frac{(l+1) + w_{l+1,l}[P]}{N},$$
$$W_{l-1,l}[P] = \frac{N-(l-1) + w_{l-1,l}[P]}{N}, \tag{6}$$

where

$$\begin{aligned} w_{l\pm1,l}[P] = {} & c_1 P^{\mu-1}(l\pm1,s) + c_2 P^{\nu-1}(l,s) \\ & - c_2 P^{\nu-2}(l\pm1,s)P(l,s) - c_1 \\ & \times P^{-1}(l\pm1,s)P^{\mu}(l,s). \end{aligned} \tag{7}$$

In the equation above the dimensionless constants c_1 and c_2, as well as the exponents μ and ν, assume real values, such that the transition probabilities satisfy $0 \leqslant W_{l+1,l}[P] \leqslant 1$ and $0 \leqslant W_{l-1,l}[P] \leqslant 1$. One notices that the uncorrelated moves of Eq. (1) are recovered in the following cases: (i) $c_1 = c_2 = 0$ and (ii) nonzero c_1 and c_2, with the exponents $\mu = 0$ and $\nu = 2$. The above dependence on the occupancy probabilities is closely related to the one introduced previously in the master equation of Refs. [25,26], which has led to a very general type of NLFPE. Several terms of the equation above contribute to increasing (or decreasing) the transition probabilities $W_{l\pm1,l}[P]$, depending on the occupancy probabilities of each box. For example, if c_1 and c_2 are both positive, the term $c_1 P^{\mu-1}(l+1,s)$ increases (decreases) the transition probability $W_{l+1,l}[P]$ for $0 < \mu < 1$ ($\mu > 1$). On the other hand, the term $c_2 P^{\nu-1}(l,s)$ contributes to keeping the same number of balls (to changing the number of balls) if $0 < \nu < 1$ ($\nu > 1$) in both transition probabilities $W_{l\pm1,l}[P]$. Such effects appear to be combined in the other two terms in Eq. (7), e.g., in $c_2 P^{\nu-2}(l\pm1,s)P(l,s)$. The types of contributions in Eq. (7) should be relevant in several complex systems, in which one takes into account excluded-volume effects, long-range repulsive (or attractive) interactions, and/or strong correlations, like those mentioned in Ref. [25], where one may find phenomena such as particle diffusion in porous media [30,31], surface growth in fractals [31], and financial transactions [32].

As described in the previous section for the uncorrelated case, let us now consider the continuous limit; for that, we rewrite Eq. (5) as

$$\begin{aligned} NP(l,s+1) = {} & l[P(l+1,s) - P(l-1,s)] + [P(l+1,s) + P(l-1,s)] \\ & + NP(l-1,s) + [w_{l+1,l}P(l+1,s) + w_{l-1,l}P(l-1,s)], \end{aligned} \tag{8}$$

and substituting the transition probabilities from Eq. (7) one obtains

$$\begin{aligned} NP(l,s+1) = {} & l[P(l+1,s) - P(l-1,s)] + [P(l+1,s) + P(l-1,s)] + NP(l-1,s) \\ & + c_1[P^{\mu}(l+1,s) + P^{\mu}(l-1,s)] + c_2 P^{\nu-1}(l,s)[P(l+1,s) + P(l-1,s)] \\ & - c_2 P(l,s)[P^{\nu-1}(l+1,s) + P^{\nu-1}(l-1,s)] - 2c_1 P^{\mu}(l,s). \end{aligned} \tag{9}$$

Now, considering the same change of variables as in Eq. (3) defining $P(x,t) = NP(l,s)$, and expanding the probabilities up to $O(1/N)$, one gets the following NLFPE:

$$\frac{\partial P(x,t)}{\partial t} = 2\frac{\partial[xP(x,t)]}{\partial x} + D\frac{\partial^2 P(x,t)}{\partial x^2} + aD\frac{\partial^2 P^{\mu}(x,t)}{\partial x^2} + bDP^{\nu-1}(x,t)\frac{\partial^2 P(x,)}{\partial x^2} - bDP(x,t)\frac{\partial^2 P^{\nu-1}(x,t)}{\partial x^2}, \tag{10}$$

where we have defined $a = (2c_1/N^{\mu})$ and $b = (2c_2/N^{\nu})$. The equation above can still be written in the general form [8,9,26]

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{A(x)\Psi[P(x,t)]\}}{\partial x} + D\frac{\partial}{\partial x}\left\{\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}, \tag{11}$$

where

$$A(x) = -2x, \quad \Psi[P(x,t)] = P(x,t),$$
$$\Omega[P(x,t)] = 1 + a\mu P^{\mu-1}(x,t) + b(2-\nu)P^{\nu-1}(x,t). \tag{12}$$

In the functional $\Omega[P(x,t)]$ (which should be a positive quantity [8,9]) one notices the presence of both linear and nonlinear contributions, with the first one being associated with random moves. On the other hand, the latter terms appear due to correlations between moves, introduced according to Eqs. (6) and (7), and are usually associated with anomalous diffusion in the present NLFPE. Hence, the correlated-move limit, or, equivalently, the anomalous-diffusion regime, corresponds to a region for x such that $a\mu P^{\mu-1}(x,t) + b(2-\nu)P^{\nu-1}(x,t) \gg 1$, whereas for $a\mu P^{\mu-1}(x,t) + b(2-\nu)P^{\nu-1}(x,t) \ll 1$, the linear diffusion (associated with random moves) dominates.

Consistent with the uncorrelated Ehrenfest model, the linear Fokker-Planck equation is recovered from Eqs. (11) and (12) by setting $a\mu = b(2-\nu) = 0$, from which one has several particular cases: (i) $a = b = 0$ and (ii) nonzero a and b, with the choices $\mu = 0$ and $\nu = 2$ for the exponents. Moreover, one should notice that apart from the constant factor D, the nonlinear diffusion terms of Eq. (10) correspond precisely to those of the NLFPE obtained in Ref. [25] by means of approximations on a master equation. In this previous work, in order to obtain these terms we also considered transition probabilities depending on $P(x,t)$. In fact, apart from being restricted to an external harmonic potential $\phi(x) = x^2$, i.e., to a restoring force $A(x) = -[d\phi(x)/dx] = -2x$, Eq. (10) differs from Eq. (2.4) of Ref. [25] by the linear diffusion contribution.

In general, the functionals $\Psi[P(x,t)]$ and $\Omega[P(x,t)]$ of Eq. (11) should satisfy certain mathematical requirements, such as differentiability and positiveness [8,9]; moreover, to ensure normalizability of $P(x,t)$ for all times one must impose the conditions

$$P(x,t)|_{x\to\pm\infty} = 0, \quad \left.\frac{\partial P(x,t)}{\partial x}\right|_{x\to\pm\infty} = 0,$$
$$A(x)\Psi[P(x,t)]|_{x\to\pm\infty} = 0 \qquad (\forall t). \tag{13}$$

An important result that follows from Eq. (11) represents the H theorem, which ensures that after a sufficiently long time, the associated system will reach a stationary state. Hence, the proof of the H theorem by using NLFPEs has been carried out by many authors in recent years [8,9,12,18–21]. In the case of a system under a given confining external potential $\phi(x)$, the H theorem corresponds to a well-defined sign for the time derivative of the free-energy functional,

$$F[P] = U[P] - \theta S[P], \quad U[P] = \int_{-\infty}^{\infty} dx\, \phi(x)P(x,t), \tag{14}$$

with θ representing a positive parameter with dimensions of temperature. Herein, the entropy may be considered in the general form [8,9,21],

$$S[P] = k\int_{-\infty}^{+\infty} dx g[P(x,t)],$$
$$g(0) = g(1) = 0, \quad \frac{d^2 g}{dx^2} \leqslant 0, \tag{15}$$

where k denotes a positive constant with dimensions of entropy, whereas the functional $g[P(x,t)]$ should be differentiable twice, at least. Considering the NLFPE in Eq. (11), for a well-defined sign of the time derivative of the free energy [which was considered as $(dF/dt) \leqslant 0$ in Refs. [8,9,21]], one finds that the functionals of Eq. (11) should be directly related to the entropic form,

$$-\frac{d^2 g[P]}{P^2} = \frac{\Omega[P]}{\Psi[P]}, \tag{16}$$

where we assumed, as usual, $D = k\theta$. Hence, substituting the functionals from Eq. (12) into Eq. (16), integrating twice, and imposing the conditions in Eq. (15), one obtains the following entropic form:

$$S[P] = -k\int_{-\infty}^{+\infty} dx\, P\ln P + ka\int_{-\infty}^{+\infty} dx\frac{P - P^{\mu}}{\mu - 1} + k\frac{b(2-\nu)}{\nu}\int_{-\infty}^{+\infty} dx\frac{P - P^{\nu}}{\nu - 1}. \tag{17}$$

One should recall that the second term on the right-hand side of this equation exists only for $\mu \neq 0$; in the particular case $\mu = 0$ this term vanishes already in the functional $\Omega[P]$ in Eq. (12). One notices well-known contributions in the above entropic form, namely, the BG, as well as two contributions of Tsallis type, characterized by the exponents μ and ν, respectively. These two latter contributions may also fall into the class of two-index entropic forms introduced in the literature [33–37] by appropriately defining the parameters a, b, μ, and ν (see, e.g., Ref. [9]). The prevalence of one particular type of entropy or a competing effect between different types of entropy will depend on the particular values of these parameters. In Ref. [22] a system of interacting vortices, typical of type-II superconductors, was shown to be described by a NLFPE of the type shown in Eqs. (11) and (12), characterized by an associated entropy in the form of Eq. (17) with $\mu = \nu = 2$, i.e., given by a sum of the BG and Tsallis contributions. Moreover, most recent studies of this system have shown that the term resulting from $\mu = \nu = 2$ prevails in the superconducting phase [38,39].

In the next section we will study stationary-state solutions of the NLFPE defined by Eqs. (11) and (12); we will show that such distributions follow an equation that coincides with the one obtained by extremization of the entropy in Eq. (17) and we will analyze some particular cases.

III. EQUILIBRIUM DISTRIBUTIONS

Let us first consider the stationary-state distributions of the NLFPE defined by Eqs. (11) and (12); for that, we rewrite this NLFPE in the form of a continuity equation,

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x}, \tag{18}$$

where the probability current density is given by

$$J(x,t) = -2xP(x,t) - D[1 + a\mu P^{\mu-1}(x,t) + b(2-\nu)P^{\nu-1}(x,t)]\frac{\partial P(x,t)}{\partial x}. \tag{19}$$

As required by an appropriate conservation of probability, the stationary-state solution $P_{\rm st}(x)$ is obtained by setting $J_{\rm st}(x) = 0$,

$$J_{\rm st}(x) = -2xP_{\rm st} - D\big[1 + a\mu P_{\rm st}^{\mu-1} + b(2-\nu)P_{\rm st}^{\nu-1}\big]\frac{\partial P_{\rm st}}{\partial x} = 0, \tag{20}$$

which, after an integration over x, yields

$$\ln(P_{\rm st}) + \frac{a\mu}{\mu-1}P_{\rm st}^{\mu-1} + \frac{b(2-\nu)}{\nu-1}P_{\rm st}^{\nu-1} = \frac{1}{D}(C - x^2), \tag{21}$$

where C represents an integration constant.

Now, we will extremize the entropy of Eq. (17) under the constraints of probability normalization and internal energy definition according to Eq. (14). Supposing a unique extremizing state (usually referred to as equilibrium), a direct consequence of the H theorem is that the system should reach this equilibrium state after a sufficiently long time. The maximum-entropy principle is established herein by extremizing the functional [40]

$$\Im[P(x,t)] = \frac{S[P(x,t)]}{k} + \alpha\left(1 - \int dx\, P(x,t)\right) + \beta\left(U - \int dx\phi(x)P(x,t)\right), \tag{22}$$

with α and β representing Lagrange multipliers. The extremization of this functional yields

$$\left.\left(\frac{dg[P]}{dP} - \alpha - \beta\phi(x)\right)\right|_{P=P_{\rm eq}(x)} = 0, \tag{23}$$

where $P_{\rm eq}(x)$ represents the probability distribution at equilibrium. Now, considering the entropy of Eq. (17), one finds

$$-(\ln P_{\rm eq} + 1) + \frac{a}{\mu-1}\left(1 - \mu P_{\rm eq}^{\mu-1}\right) + \frac{b(2-\nu)}{\nu(\nu-1)}\left(1 - \nu P_{\rm eq}^{\nu-1}\right) = \alpha + \beta\phi(x). \tag{24}$$

Using $\phi(x) = x^2$, the equation above can be rewritten as

$$\ln P_{\rm eq} + \frac{a\mu}{\mu-1}P_{\rm eq}^{\mu-1} + \frac{b(2-\nu)}{\nu-1}P_{\rm eq}^{\nu-1} = \beta(C' - x^2). \tag{25}$$

Apart from an additive constant, the above equation, which holds for the state that extremizes the entropy, *coincides* with Eq. (21) for the stationary state of the corresponding NLFPE through the identifications $P_{\rm eq}(x) \leftrightarrow P_{\rm st}(x)$ and by considering the Lagrange multiplier $\beta = D^{-1} = (k\theta)^{-1}$. This remarkable result shows the consistency of the connection between the entropic functional in Eq. (17) and the NLFPE defined by Eqs. (11) and (12), already stated by Eq. (16) through the H theorem. Herein, this connection is reinforced by showing that the stationary distribution of the NLFPE coincides with the equilibrium distribution obtained independently through the maximum-entropy principle.

Next, we solve Eq. (25) [or, equivalently, Eq. (21)] in some particular cases. Since the power terms in Eq. (17) are readily associated with Tsallis entropy [14], herein, for simplicity we will be restricted to those situations where $\mu = \nu = 2 - q$, where q denotes Tsallis's entropic index. With this choice, Eq. (25) becomes

$$\ln P_{\rm eq}(x) + \frac{a(2-q) + bq}{1-q}\, P_{\rm eq}^{1-q}(x) = \beta(C' - x^2), \tag{26}$$

which may be written also in terms of the q logarithm, $\ln_q(u) \equiv (u^{1-q} - 1)/(1-q)$ $[\ln_1(u) = \ln(u)]$, as

$$\ln[P_{\rm eq}(x)] + [a(2-q) + bq]\ln_q[P_{\rm eq}(x)] = \beta(C'' - x^2). \tag{27}$$

From the equation above one sees that the equilibrium distribution $P_{\rm eq}(x)$ will result from a competition between these logarithms, such that its final form will depend on the choices of a, b, and q; below we consider some typical values of q.

(a) *Case with* $q = 1$. This particular case corresponds to BG entropy, as well as to the linear Fokker-Planck equation, for which Eq. (27) leads to

$$(1 + a + b)\ln[P_{\rm eq}(x)] = \beta(C'' - x^2). \tag{28}$$

The solution of the equation above is given by the standard Gaussian distribution,

$$P_{\rm eq}(x) = \left(\frac{1}{\pi A}\right)^{1/2} \exp\left(-\frac{x^2}{A}\right), \tag{29}$$

where the constant A is found by imposing normalization and depends on the constants in Eq. (28).

(b) *Case with* $q = 0$. Substituting $q = 0$ into Eq. (26), one obtains

$$\ln P_{\rm eq}(x) + 2aP_{\rm eq}(x) = \beta(C' - x^2), \tag{30}$$

which is similar to the equilibrium equation found in Ref. [22] for a system of interacting vortices, which is typical of type-II superconductors. This system was shown to be described by a NLFPE like the one in Eqs. (11) and (12), characterized by an associated entropy in the form of Eq. (17) with $\mu = \nu = 2$.

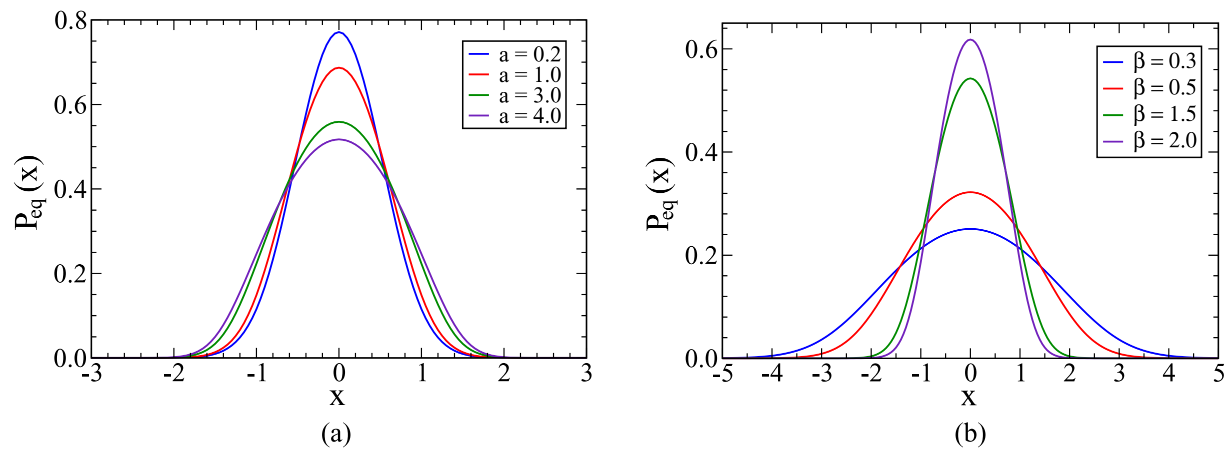

FIG. 1. (Color online) The equilibrium distribution $P_{\text{eq}}(x)$ of Eq. (32) is represented vs x in the following cases: (a) $\beta = 2$ and typical values of a (increasing values of a from top to bottom); (b) $a = 1$ and typical values of β (increasing values of β from bottom to top). The parameter C' is found in each case by imposing normalization.

Writing Eq. (30) as

$$2a P_{\text{eq}}(x) \exp[2a P_{\text{eq}}(x)] = 2a \exp[\beta(C' - x^2)], \tag{31}$$

one identifies the form $Xe^X = Y$, which defines the implicit W-Lambert function, such that $X = W(Y)$ (see, e.g., Ref. [41]). Hence,

$$P_{\text{eq}}(x) = \frac{1}{2a} W\{2a \exp[\beta(C' - x^2)]\}. \tag{32}$$

Although Eq. (30) does not present an explicit analytical solution, the equilibrium distribution above recovers two well-known limiting cases: (i) $2a \ll 1$, where the Gaussian-distribution behavior dominates, and (ii) $2a \gg 1$, where the W-Lambert function approaches a parabola, corresponding to the relevant limit for the superconducting phase of real type-II superconductors [38,39]. It should be mentioned that the solution of Eq. (32) matches precisely the one in Ref. [22] by a simple redefinition of the parameters a and β, as well as of the constant C'. Both parameters a and β affect directly the width of the distribution, as shown in Fig. 1. In Fig. 1(a) we exhibit the above equilibrium distribution for $\beta = 2$ and typical values of the parameter a; one sees that the width of the corresponding distribution increases for increasing values of a. On the other hand, one concludes from Fig. 1(b) that the Lagrange parameter β is related to the inverse of the width of the distribution in the sense that larger values of β yield smaller widths and vice versa.

(c) *Case with a general q*. One can write Eq. (26) as

$$\begin{aligned}(1-q)\ln P_{\text{eq}}(x) + [a(2-q) + bq]P_{\text{eq}}^{1-q}(x)\\ = (1-q)\beta(C' - x^2),\end{aligned} \tag{33}$$

leading to

$$\begin{aligned}[a(2-q) + bq]P_{\text{eq}}^{1-q}(x)\exp\{[a(2-q) + bq]P_{\text{eq}}^{1-q}(x)\}\\ = [a(2-q) + bq]\exp[(1-q)\beta(C' - x^2)],\end{aligned} \tag{34}$$

where one identifies a form similar to the one in Eq. (31), for which the W-Lambert function appears,

$$\begin{aligned}&[a(2-q) + bq]P_{\text{eq}}^{1-q}(x)\\ &\quad = W\{[a(2-q) + bq]\exp[(1-q)\beta(C' - x^2)]\}.\end{aligned} \tag{35}$$

Hence, one gets the following solution:

$$P_{\text{eq}}(x) = \left[\frac{W\{[a(2-q) + bq]\exp[(1-q)\beta(C' - x^2)]\}}{a(2-q) + bq}\right]^{1/(1-q)}, \tag{36}$$

where one should have $[a(2-q) + bq] > 0$ (in agreement with Ref. [25]), and for each set of q, a, and b, the parameter C' is found by imposing normalization. One should notice that the distribution above recovers that of Eq. (32) in the particular case $q = 0$.

In Fig. 2 we exhibit the above equilibrium distribution for $\beta = 1$, typical values of the parameters a and b, and $q < 1$ [Figs. 2(a) and 2(b)] and $q > 1$ [Figs. 2(c) and 2(d)]. The cases $q < 1$ yield essentially short-tailed distributions, and one sees that the parameters a and b are directly related to the width of the distribution in the sense that larger values for these parameters yield larger widths, as can be seen by comparing Figs. 2(a) and 2(b). On the other hand, the cases $q > 1$ lead to a behavior typical of long-tailed distributions, and for a better visualization, they are shown in log-linear representations. As for the cases with $q < 1$, by increasing the parameters a and b, one increases the width at midheight, as can be seen by comparing Figs. 2(c) and 2(d).

IV. ENTROPY-PRODUCTION ANALYSIS

Associating the present generalization of Ehrenfest's urn problem with the NLFPE defined by Eqs. (11) and (12), as done in the previous section, it is possible to analyze the entropy production resulting from the irreversible process characterized by moving balls between urns. In this case, one may write [42,43]

$$\frac{d}{dt}S[P] = \Pi - \Phi, \tag{37}$$

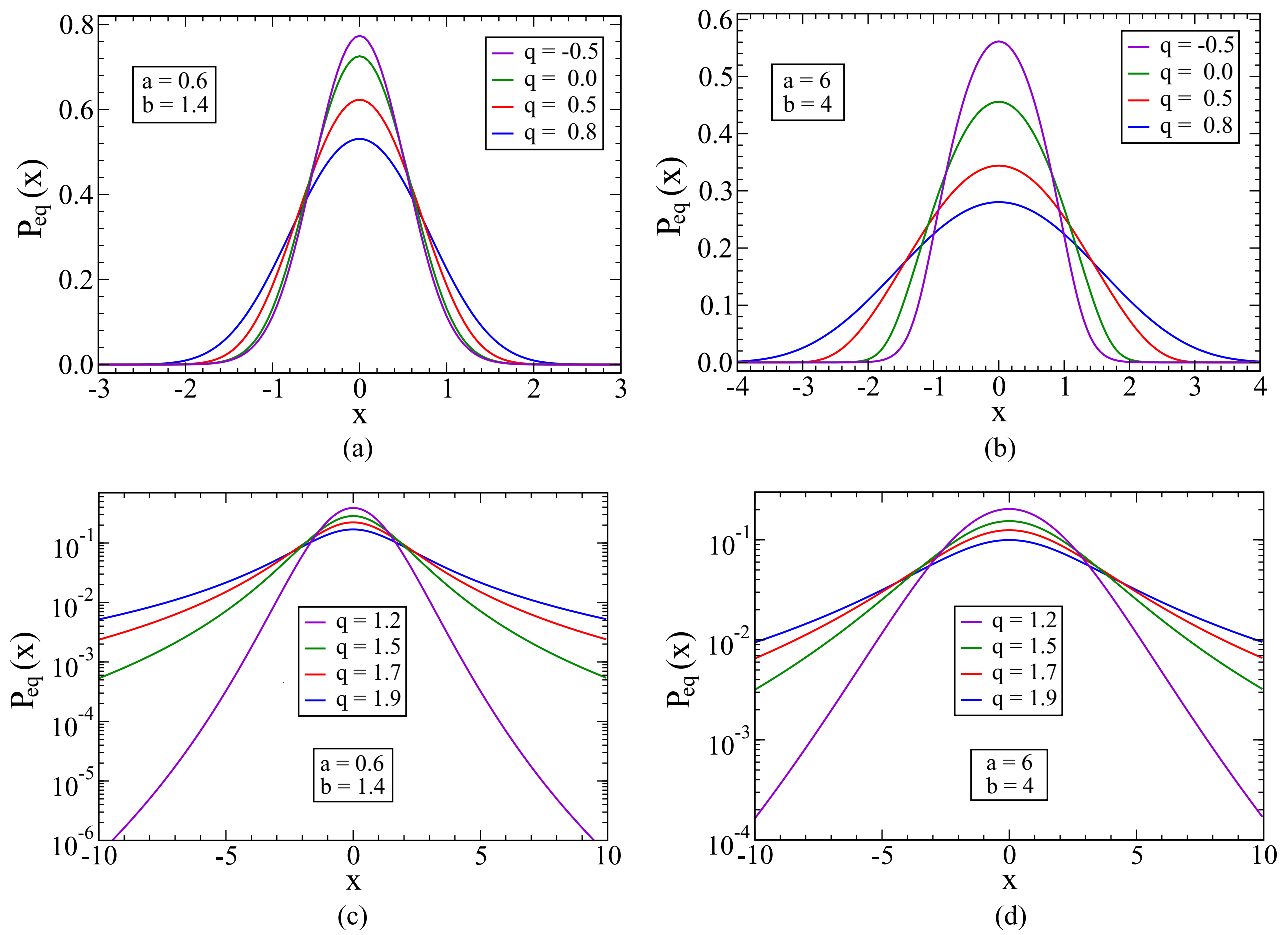

FIG. 2. (Color online) The equilibrium distribution $P_{\rm eq}(x)$ of Eq. (36) is represented vs x for $\beta = 1$, typical values of the parameters a and b, and (a) and (b) $q < 1$ (in a linear-linear representation) and (c) and (d) $q > 1$ (in a log-linear representation). In all cases, the maximum $P_{\rm eq}(x = 0)$ decreases by increasing values of q.

where one identifies the entropy flux, associated with the exchanges of entropy between the system and its neighborhood (represented herein by the confining potential $\phi(x) = x^2$, i.e., by a restoring force $A(x) = -[d\phi(x)/dx] = -2x$) [44],

$$\Phi = \frac{k}{D}\int_{-\infty}^{+\infty} dx A(x)J(x,t). \tag{38}$$

Additionally, one has the entropy-production contribution,

$$\Pi = \frac{k}{D}\int_{-\infty}^{+\infty} dx \frac{\{J(x,t)\}^2}{\Psi[P]}, \tag{39}$$

and since k, D, and $\Psi[P(x,t)]$ were all defined previously as positive quantities, one has the desirable result $\Pi \geqslant 0$.

From now on, we restrict our analysis to the situation investigated in the previous section, which is typical of nonextensive statistical mechanics, by considering the case $\mu = \nu = 2 - q$; the probability current in Eq. (19) becomes

$$J(x,t) = -2xP(x,t) - D\{1 + [a(2-q) + bq]P^{1-q}(x,t)\} \times \frac{\partial P(x,t)}{\partial x}. \tag{40}$$

The above expression may be written as

$$J(x,t) = -2xP(x,t) + J^{(\rm l)}(x,t) + J^{(\rm nl)}(x,t), \tag{41}$$

where

$$J^{(\rm l)}(x,t) = -D\frac{\partial P(x,t)}{\partial x}, \tag{42}$$

$$J^{(\rm nl)}(x,t) = -D[a(2-q) + bq]P^{1-q}(x,t)\frac{\partial P(x,t)}{\partial x}. \tag{43}$$

We have split the above probability current into linear and nonlinear diffusion contributions, respectively. The first one, $J^{(\rm l)}(x,t)$, corresponds to the diffusion contribution to the probability current due to random moves (i.e., without correlations), whereas $J^{(\rm nl)}(x,t)$ carries the nonlinearities, associated with anomalous diffusion, and as discussed in Sec. II, it should be related to correlations between moves. It is important to recall that the equilibrium solution of Eq. (36) implies $[a(2-q) + bq] > 0$ (which represents a condition also found in Ref. [25]). Hence, for the x region such that $[a(2-q) + bq]P^{1-q}(x,t) \gg 1$ [i.e., $|J^{(\rm nl)}(x,t)| \gg |J^{(\rm l)}(x,t)|$], the anomalous diffusion, produced by correlated moves, prevails, whereas for $[a(2-q) + bq]P^{1-q}(x,t) \ll 1$ [i.e., $|J^{(\rm nl)}(x,t)| \ll |J^{(\rm l)}(x,t)|$], the linear diffusion, associated with random moves, dominates.

Let us then consider a situation such that at the initial time both urns have approximately the same number of balls, i.e., $N_1(s) \approx N/2$ and $N_2(s) \approx N/2$, so that the variable x

defined in Eq. (3) is small at the beginning of the evolution. This represents a typical situation that occurs with the time-dependent solution $P(x,t)$ of the NLFPE in Eqs. (11) and (12) (see, e.g., Refs. [16,17]). Hence, in this regime, the diffusion contributions in Eq. (41) prevail over the confining one, leading to $\Pi \gg \Phi$; particularly, the nonlinear contribution $J^{(\mathrm{nl})}(x,t)$ yields an increase in the total entropy production in Eq. (39). Therefore, the correlations between moves, introduced herein in the exchange transition rates, by means of nonlinear dependences on the occupancy probabilities result in an enlargement of the total entropy production of the present model. On the other hand, the approach to the stationary state occurs for sufficiently long times such that the confining contribution in Eq. (41) becomes the same order as the diffusion ones; although small in this regime, the nonlinear diffusion contribution still yields an increment with respect to the linear one for the total entropy production.

V. CONCLUSIONS

We have carried out modifications of Ehrenfest's urn model by introducing nonlinear terms in the associated transition probabilities. These nonlinearities, which were set by means of the dependence on the occupancy probabilities of both urns, were argued to be related to correlations between moves. We have shown that these modifications lead, in the continuous limit, to a Fokker-Planck equation characterized by two competing diffusion terms, namely, the usual linear one and a nonlinear diffusion term typical of anomalous diffusion.

Considering such a Fokker-Planck equation, an H theorem led to the associated entropy, which was shown to be given by a sum of Boltzmann-Gibbs and Tsallis entropic forms. Two independent procedures, namely, the extremization of the entropy and the stationary state of the corresponding Fokker-Planck equation, have yielded the same equation for the equilibrium distribution, which was solved in some particular cases.

Furthermore, by using the associated Fokker-Planck equation we have also analyzed the phenomenon of entropy production. It was shown that for short times of the evolution, when the entropy-production contribution prevails over the entropy flux, the correlations between moves, introduced herein by means of nonlinear dependences on the occupancy probabilities in the exchange transition rates, result in an enlargement of the total entropy production of the present model.

In the same way that the uncorrelated Ehrenfest's urn model has played an important role in the statistical interpretation of some irreversible processes, the present modified model is expected to provide further insights into irreversible processes that occur in many complex systems, described by nonlinear equations, for which generalized entropic forms are applicable. The dependence of the transition probabilities on the occupancy probabilities considered herein is closely related to the one introduced previously in the master equation in Refs. [25,26], which has led to a very general type of nonlinear Fokker-Planck equation. Particularly, this model should be relevant for physical systems in which the transition probabilities may be directly affected by some characteristics usually found in complex systems, such as (i) excluded-volume effects, which yield restrictions in position transitions, (ii) charged constituents, characterized by long-range Coulomb interactions, e.g., found in ionic solutions, where regions characterized by larger (or smaller) concentrations of ions influence the diffusion of other ions, and (iii) regions with distinct concentrations of constituents, like in social and biological systems, which affect directly the associated transition probabilities. The application of the present formalism to some of these problems is in progress.

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[1] P. Ehrenfest and T. Ehrenfest, Über zwei bekannte Einwände gegen das Boltzmannsche H-Theorem, Phys. Z. **8**, 311 (1907).

[2] M. Kac, Random walk and the theory of Brownian motion, Am. Math. Mon. **54**, 369 (1947).

[3] W. Feller, *An Introduction to Probability Theory and Its Applications*, 3rd ed. (Wiley, New York, 1968), Vol. 1.

[4] C. Godrèche and J. M. Luck, Nonequilibrium dynamics of urn models, J. Phys.: Condens. Matter **14**, 1601 (2002).

[5] J. Nagler, C. Hauert, and H. G. Schuster, Self-organized criticality in a nutshell, Phys. Rev. E **60**, 2706 (1999).

[6] L. E. Reichl, *A Modern Course in Statistical Physics*, 2nd ed. (Wiley, New York, 1998).

[7] M. J. Klein, Entropy and the Ehrenfest urn model, Physica **22**, 569 (1956).

[8] V. Schwämmle, F. D. Nobre, and E. M. F. Curado, Consequences of the H theorem from nonlinear Fokker-Planck equations, Phys. Rev. E **76**, 041123 (2007).

[9] V. Schwämmle, E. M. F. Curado, and F. D. Nobre, A general nonlinear Fokker-Planck equation and its associated entropy, Eur. Phys. J. B **58**, 159 (2007).

[10] M. J. Klein, Generalization of the Ehrenfest urn model, Phys. Rev. **103**, 17 (1956).

[11] E. Lutz, Power-law tail distributions and nonergodicity, Phys. Rev. Lett. **93**, 190602 (2004).

[12] T. D. Frank, *Nonlinear Fokker-Planck Equations: Fundamentals and Applications* (Springer, Berlin, 2005).

[13] L. Borland, Ito-Langevin equations within generalized thermostatistics, Phys. Lett. A **245**, 67 (1998).

[14] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics* (Springer, New York, 2009).

[15] E. Lutz and F. Renzoni, Beyond Boltzmann-Gibbs statistical mechanics in optical lattices, Nat. Phys. **9**, 615 (2013).

[16] A. R. Plastino and A. Plastino, Non-extensive statistical mechanics and generalized Fokker-Planck equation, Phys. A (Amsterdam, Neth.) **222**, 347 (1995).

[17] C. Tsallis and D. J. Bukman, Anomalous diffusion in the presence of external forces: Exact time-dependent solutions and their thermostatistical basis, Phys. Rev. E **54**, R2197 (1996).

[18] M. Shiino, Free energies based on generalized entropies and H-theorems for nonlinear Fokker-Planck equations, J. Math. Phys. **42**, 2540 (2001).

[19] T. D. Frank and A. Daffertshofer, H-theorem for nonlinear Fokker-Planck equations related to generalized thermostatistics, Phys. A (Amsterdam, Neth.) **295**, 455 (2001).

[20] P. H. Chavanis, Generalized thermodynamics and Fokker-Planck equations: Applications to stellar dynamics and two-dimensional turbulence, Phys. Rev. E **68**, 036108 (2003).

[21] V. Schwämmle, E. M. F. Curado, and F. D. Nobre, Dynamics of normal and anomalous diffusion in nonlinear Fokker-Planck equations, Eur. Phys. J. B **70**, 107 (2009).

[22] J. S. Andrade, Jr., G. F. T. da Silva, A. A. Moreira, F. D. Nobre, and E. M. F. Curado, Thermostatistics of overdamped motion of interacting particles, Phys. Rev. Lett. **105**, 260601 (2010).

[23] M. S. Ribeiro, F. D. Nobre, and E. M. F. Curado, Time evolution of interacting vortices under overdamped motion, Phys. Rev. E **85**, 021146 (2012).

[24] J. P. Bouchaud and A. Georges, Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications, Phys. Rep **195**, 127 (1990).

[25] E. M. F. Curado and F. D. Nobre, Derivation of nonlinear Fokker-Planck equations by means of approximations to the master equation, Phys. Rev. E **67**, 021107 (2003).

[26] F. D. Nobre, E. M. F. Curado, and G. Rowlands, A procedure for obtaining general nonlinear Fokker-Planck equations, Phys. A (Amsterdam, Neth.) **334**, 109 (2004).

[27] J. P. Boon and J. F. Lutsko, Nonlinear diffusion from Einstein's master equation, Europhys. Lett. **80**, 60006 (2007).

[28] J. F. Lutsko and J. P. Boon, Generalized diffusion: A microscopic approach, Phys. Rev. E **77**, 051103 (2008).

[29] J. F. Lutsko and J. P. Boon, Microscopic theory of anomalous diffusion based on particle interactions, Phys. Rev. E **88**, 022108 (2013).

[30] M. Muskat, *The Flow of Homogeneous Fluids through Porous Media* (McGraw-Hill, New York, 1937).

[31] H. Spohn, Surface dynamics below the roughening transition, J. Phys. I **3**, 69 (1993).

[32] L. Borland, Option pricing formulas based on a non-Gaussian stock price model, Phys. Rev. Lett. **89**, 098701 (2002).

[33] G. Kaniadakis, Non-linear kinetics underlying generalized statistics, Phys. A (Amsterdam, Neth.) **296**, 405 (2001).

[34] G. Kaniadakis, Statistical mechanics in the context of special relativity, Phys. Rev. E **66**, 056125 (2002).

[35] E. P. Borges and I. Roditi, A family of nonextensive entropies, Phys. Lett. A **246**, 399 (1998).

[36] R. Hanel and S. Thurner, A comprehensive classification of complex statistical systems and an axiomatic derivation of their entropy and distribution functions, Europhys. Lett. **93**, 20006 (2011).

[37] R. Hanel and S. Thurner, Generalized (c, d)-entropy and aging random walks, Entropy **15**, 5324 (2013).

[38] F. D. Nobre, A. M. C. Souza, and E. M. F. Curado, Effective-temperature concept: A physical application for nonextensive statistical mechanics, Phys. Rev. E **86**, 061113 (2012).

[39] E. M. F. Curado, A. M. C. Souza, F. D. Nobre, and R. F. S. Andrade, Carnot cycle for interacting particles in the absence of thermal noise, Phys. Rev. E **89**, 022117 (2014).

[40] R. Balian, *From Microphysics to Macrophysics* (Springer, Berlin, 1991), Vols. 1 and 2.

[41] S. R. Valluri, M. Gil, D. J. Jeffrey, and Shantanu Basu, The Lambert *W* function and quantum statistics, J. Math. Phys. **50**, 102103 (2009).

[42] I. Prigogine, *Introduction of the Thermodynamics of Irreversible Processes* (Wiley, New York, 1967).

[43] P. Glansdorff and I. Prigogine, *Thermodynamic Theory of Structure, Stability and Fluctuations* (Wiley, New York, 1971).

[44] G. A. Casas, F. D. Nobre, and E. M. F. Curado, Entropy production and nonlinear Fokker-Planck equations, Phys. Rev. E **86**, 061136 (2012).