Entropy production and nonlinear Fokker-Planck equations

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The entropy time rate of systems described by nonlinear Fokker-Planck equations—which are directly related to generalized entropic forms—is analyzed. Both entropy production, associated with irreversible processes, and entropy flux from the system to its surroundings are studied. Some examples of known generalized entropic forms are considered, and particularly, the flux and production of the Boltzmann-Gibbs entropy, obtained from the linear Fokker-Planck equation, are recovered as particular cases. Since nonlinear Fokker-Planck equations are appropriate for the dynamical behavior of several physical phenomena in nature, like many within the realm of complex systems, the present analysis should be applicable to irreversible processes in a large class of nonlinear systems, such as those described by Tsallis and Kaniadakis entropies.

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I. INTRODUCTION

Entropy represents one of most outstanding concepts of physics; in particular, its statistical definition in terms of probabilities allows a direct connection between the microscopic (described by statistical mechanics) and macroscopic (described by thermodynamics) worlds. The H theorem, and consequently the second law of thermodynamics, which states that the entropy of an isolated system always increases for irreversible processes, leads to the interesting phenomenon of entropy production [1–4]. Within the statistical definition of entropy, the entropy production depends directly on the time derivative of the corresponding probability; for this purpose one may use, e.g., the Boltzmann or Fokker-Planck equation in the case of continuous probabilities [4], or the master equation, when dealing with discrete probabilities [5]. Most investigations in the literature are concerned with the production of Boltzmann-Gibbs entropy (see, e.g., Refs. [4–7]) and so one makes use either of the standard master equation or of the linear Fokker-Planck equation.

In many physical situations the linear differential equations are applicable to idealized systems, characterized by specific properties, such as homogeneity, isotropy, and translational invariance, with particles interacting through short-range forces and with a dynamical behavior described by short-time memories. However, it is very common, particularly within the realm of complex systems, to find physical systems that do not fulfill these requirements, e.g., those presenting one (or more) of the following properties: spatial disorder, competing interactions, long-range interactions, long-time memories. Usually, in these cases, the associated equations have to be modified, and very frequently, nonlinearities are considered in order to take into account such effects. Among these equations one should mention the nonlinear Fokker-Planck equations (NLFPEs) [8] that are intimately related to anomalous diffusion [9]. These types of phenomena may be found in the motion of particles in porous media [10–13], the dynamics of surface growth [13], the diffusion of polymer-like breakable micelles [14], and the dynamics of interacting vortices in disordered superconductors [15–17], among other physical systems.

A general NLFPE may be written as [8,18,19]

$$\eta \frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{A(x)\Psi[P(x,t)]\}}{\partial x} + D\frac{\partial}{\partial x}\left\{\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}, \quad (1)$$

where η represents an effective friction coefficient, D is a constant with dimensions of energy, and the external force $A(x)$ is associated with a potential $\phi(x)$ $[A(x) = -d\phi(x)/dx]$. The functionals $\Psi[P(x,t)]$ and $\Omega[P(x,t)]$ should satisfy certain mathematical requirements, e.g., positiveness [18,19]; moreover, to ensure normalizability of $P(x,t)$ for all times one must impose the conditions

$$P(x,t)|_{x\to\pm\infty} = 0; \quad \left.\frac{\partial P(x,t)}{\partial x}\right|_{x\to\pm\infty} = 0; \\ A(x)\Psi[P(x,t)]|_{x\to\pm\infty} = 0 \quad (\forall t). \quad (2)$$

Although the present analysis may be carried in higher dimensions, e.g., considering N-dimensional NLFPEs, like those of Refs. [20–22], herein for simplicity we will restrict ourselves to the one-dimensional form of Eq. (1).

The proof of the H theorem by using NLFPEs has been carried out by many workers in recent years [8,18,19,23–26]. In the case of a system under an external potential $\phi(x)$, the H theorem corresponds to a well-defined sign for the time derivative of the free-energy functional,

$$F = U - \theta S; \quad U = \int_{-\infty}^{\infty} dx\, \phi(x)P(x,t), \quad (3)$$

with θ representing a positive parameter with dimensions of temperature. Herein, the entropy will be considered in a very general form [18],

$$S[P] = k\Lambda[Q[P]]; \quad Q[P] = \int_{-\infty}^{\infty} dx\, g[P(x,t)]; \\ g(0) = g(1) = 0; \quad \frac{d^2 g}{dP^2} \leqslant 0, \quad (4)$$

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where k denotes a positive constant with dimensions of entropy, $\Lambda[Q[P]]$ represents a monotonically increasing functional at least once differentiable, whereas the inner functional $g[P(x,t)]$ should be at least twice differentiable. Considering the NLFPE of Eq. (1), for a well-defined sign of the time derivative of the free energy (which was considered as $(dF/dt) \leqslant 0$ in Refs. [18,19,26]), one gets that the functionals of Eq. (1) should be directly related to the entropic form,

$$-\frac{d\Lambda[Q]}{dQ}\frac{d^2g[P]}{dP^2} = \frac{\Omega[P]}{\Psi[P]}, \tag{5}$$

and we are assuming that $D = k\theta$.

As far as we know, an analysis of the entropy production within the framework of generalized entropies, i.e., considering general NLFPEs, has not been carried out in the literature; as an exception, one could mention the anomalous-diffusion analysis of Ref. [27], where the porous-medium equation [10] was used. One should be reminded that the same problem has been also investigated by means of a linear fractional-diffusion equation [28,29]. In the present work we study the entropy production related to general entropic forms, as defined in Eq. (4), associated with the general NLFPE of Eq. (1), through the relation of Eq. (5). In the next section, we derive general expressions for the entropy production, entropy flux, and dissipated energy per unit time (i.e., dissipated power). In Sec. III, we introduce a Langevin-like equation associated with the NLFPE considered herein and reobtain the expression for the dissipated power, which is valid in this case for a general external force (conservative or not). In Sec. IV, we analyze the entropy-flux and entropy-production contributions, by considering some well-known entropic forms as particular cases. Finally, in Sec. V, we present our main conclusions.

II. ENTROPY-PRODUCTION RATE

For the calculations that follow, it appears to be convenient to write Eq. (1) in the form of a continuity equation,

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x};$$
$$J(x,t) = \frac{1}{\eta}\left\{A(x)\Psi[P] - D\Omega[P]\left[\frac{\partial P(x,t)}{\partial x}\right]\right\}, \tag{6}$$

in such a way that the time derivative of the entropy defined in Eq. (4) becomes

$$\begin{aligned}\frac{d}{dt}S[P] &= k\,\frac{d\Lambda[Q[P]]}{dQ}\frac{d}{dt}\int_{-\infty}^{+\infty} dx\, g[P(x,t)] \\ &= k\,\frac{d\Lambda[Q]}{dQ}\int_{-\infty}^{+\infty} dx\,\frac{dg}{dP}\frac{\partial P}{\partial t} \\ &= -k\,\frac{d\Lambda[Q]}{dQ}\int_{-\infty}^{+\infty} dx\,\frac{dg}{dP}\frac{\partial J(x,t)}{\partial x}.\end{aligned} \tag{7}$$

Carrying an integration by parts, the above equation may be written as

$$\begin{aligned}\frac{d}{dt}S[P] = k\int_{-\infty}^{+\infty} dx\,\frac{d\Lambda[Q]}{dQ}\frac{d^2g}{dP^2}\,J(x,t) \\ \times\left\{\frac{A(x)\Psi[P] - \eta J(x,t)}{D\Omega[P]}\right\},\end{aligned} \tag{8}$$

and using the relation of Eq. (5), one obtains

$$\begin{aligned}\frac{d}{dt}S[P] = -\frac{k}{D}\int_{-\infty}^{+\infty} dx\,A(x)J(x,t) \\ +\frac{k\eta}{D}\int_{-\infty}^{+\infty} dx\,\frac{\{J(x,t)\}^2}{\Psi[P]}.\end{aligned} \tag{9}$$

Then, one may write [1–4]

$$\frac{d}{dt}S[P] = \Pi - \Phi, \tag{10}$$

where one identifies the entropy flux, representing the exchanges of entropy between the system and its neighborhood,

$$\Phi = \frac{k}{D}\int_{-\infty}^{+\infty} dx\,A(x)J(x,t), \tag{11}$$

as well as the entropy-production contribution,

$$\Pi = \frac{k\eta}{D}\int_{-\infty}^{+\infty} dx\,\frac{\{J(x,t)\}^2}{\Psi[P]}. \tag{12}$$

One should be reminded that k, η, D, and $\Psi[P(x,t)]$ were all defined previously as positive quantities, leading to the desirable result $\Pi \geqslant 0$.

Next, we will show that the entropy flux of Eq. (11) is directly related to the dissipated power $\mathcal{P}$ associated with the conservative force $A(x)$,

$$\mathcal{P} = \left\langle A(x)\frac{dx}{dt}\right\rangle, \tag{13}$$

with the brackets $\langle\,\rangle$ denoting an average over the probability distribution $P(x,t)$. In this case, one may rewrite the dissipated power as

$$\begin{aligned}\mathcal{P} &= -\frac{d}{dt}U(t) = -\frac{d}{dt}\int_{-\infty}^{+\infty} dx\,\phi(x)P(x,t) \\ &= -\int_{-\infty}^{+\infty} dx\,\phi(x)\frac{\partial P(x,t)}{\partial t} = \int_{-\infty}^{+\infty} dx\,\phi(x)\frac{\partial J(x,t)}{\partial x},\end{aligned} \tag{14}$$

where we have used the continuity equation in the last equality. Integrating by parts and using the conditions of Eq. (2), one obtains

$$\mathcal{P} = -\int_{-\infty}^{+\infty} dx\,\frac{d\phi(x)}{dx}J(x,t) = \int_{-\infty}^{+\infty} dx\,A(x)J(x,t), \tag{15}$$

which may be compared with Eq. (11) to yield

$$\mathcal{P} = \frac{D}{k}\,\Phi. \tag{16}$$

This dissipated power may be expressed also in terms of the functionals of the NLFPE of Eq. (1); for that, one substitutes the current density of Eq. (6) in Eq. (15),

$$\begin{aligned}\mathcal{P} &= \frac{1}{\eta}\int_{-\infty}^{+\infty} dx\,A^2(x)\Psi[P] - \frac{D}{\eta}\int_{-\infty}^{+\infty} dx\,A(x)\Omega[P]\frac{\partial P(x,t)}{\partial x} \\ &= \frac{1}{\eta}\int_{-\infty}^{+\infty} dx\,A^2(x)\Psi[P] + \frac{D}{\eta}\int_{-\infty}^{+\infty} dx\,P(x,t)\,\frac{\partial}{\partial x}(A(x)\Omega[P]),\end{aligned} \tag{17}$$

where an integration by parts was carried in the second integral. This later equation may be written as

$$\mathcal{P} = \frac{1}{\eta}\left\langle \frac{A^2(x)\Psi[P]}{P(x,t)} + D\,\frac{\partial}{\partial x}(A(x)\Omega[P])\right\rangle. \tag{18}$$

In the next section, we will show that the above relation is very general, and it may be obtained also from a Langevin equation; as a consequence, it holds for a more general force, characterized by two contributions, a conservative and a nonconservative one.

III. THE ASSOCIATED LANGEVIN EQUATION

In this section, we will be concerned with a Langevin-like equation, defined in terms of a multiplicative noise $\zeta(t)$,

$$\eta\,\frac{dx}{dt} = f(x,t) + h(x,t)\zeta(t), \tag{19}$$

where $f(x,t)$ and $h(x,t)$ are arbitrary functions, whereas the stochastic variable $\zeta(t)$ is characterized by

$$\langle\zeta(t)\rangle = 0; \quad \langle\zeta(t)\zeta(t')\rangle = 2\eta D\delta(t-t'). \tag{20}$$

In the equations above the brackets $\langle\,\rangle$ denote time averages; as usual, we will assume that these averages coincide with those over the probability distribution $P(x,t)$ (i.e., ensemble averages), as defined in Eq. (13). Within the Stratonovich prescription, one may show that Eq. (19) is associated to the following Fokker-Planck equation [30],

$$\eta\,\frac{\partial P(x,t)}{\partial t} = -\frac{\partial[f(x,t)P(x,t)]}{\partial x} + D\frac{\partial}{\partial x}\left\{h(x,t)\frac{\partial}{\partial x}[h(x,t)P(x,t)]\right\}. \tag{21}$$

One should notice that the linear Fokker-Planck equation may be recovered from Eq. (21) in the particular case of the standard Langevin equation [31], by considering an additive noise [i.e., $h(x,t) =$ constant]; moreover, the function $f(x,t)$ defined in Eq. (19) can be associated with a general force, conservative or not. In order to recover the the NLFPE of Eq. (1), we will restrict ourselves to $h(x,t)$ written as a functional of $P(x,t)$, i.e., $h(x,t) \equiv h[P(x,t)]$; a similar procedure has already been applied in Ref. [32], through the power-dependence form, $h(x,t) \propto [P(x,t)]^{\nu}$, to derive the NLFPE of Refs. [33,34], associated with Tsallis entropy [35,36]. Moreover, $f(x,t)$ does certainly depend on the external force and may also present a functional dependence on the probability $P(x,t)$. Therefore, we write

$$h(x,t)\,\frac{\partial}{\partial x}[h(x,t)P(x,t)] = h[P(x,t)]\,\frac{\partial}{\partial P}\,\{h[P(x,t)]P(x,t)\}\,\frac{\partial P(x,t)}{\partial x} \tag{22}$$

in such a way that Eq. (21) becomes

$$\eta\,\frac{\partial P(x,t)}{\partial t} = -\frac{\partial[f(x,t)P(x,t)]}{\partial x} + D\frac{\partial}{\partial x}\left\{h[P(x,t)]\frac{\partial}{\partial P} \times \{h[P(x,t)]P(x,t)\}\,\frac{\partial P(x,t)}{\partial x}\right\}. \tag{23}$$

Comparing the equation above with Eq. (A2) of the Appendix, one has the following relations,

$$f(x,t)P(x,t) = \tilde{A}(x)\Psi[P(x,t)], \tag{24}$$

$$h[P(x,t)]\frac{\partial}{\partial P}\,\{h[P(x,t)]P(x,t)\} = \Omega[P(x,t)]. \tag{25}$$

In the present case $\tilde{A}(x)$ represents a general force, which may be written as

$$\tilde{A}(x) = A(x) + A^{*}(x), \tag{26}$$

i.e., composed by a conservative part, $A(x) = -d\phi/dx$, as well as a nonconservative one, $A^{*}(x)$.

In what follows we will derive an expression for the dissipated power associated with the general force $\tilde{A}(x)$, making use of the Langevin equation; one has that

$$\mathcal{P} = \left\langle \tilde{A}(x)\frac{dx}{dt}\right\rangle = \frac{1}{\eta}\,\langle\tilde{A}(x)f(x,t)\rangle + \frac{1}{\eta}\,\langle h(x,t)\tilde{A}(x)\zeta(t)\rangle, \tag{27}$$

where we have substituted Eq. (19). Applying Novikov's theorem [37], the second term becomes

$$\begin{aligned}\langle h(x,t)\tilde{A}(x)\zeta(t)\rangle &= D\left\langle h(x,t)\frac{\partial}{\partial x}[\tilde{A}(x)h(x,t)]\right\rangle \\ &= D\int_{-\infty}^{+\infty} dx\,h(x,t)\left\{\frac{\partial}{\partial x}[\tilde{A}(x)h(x,t)]\right\}P(x,t).\end{aligned} \tag{28}$$

Carrying integrations by parts and substituting Eq. (25), one may write this term as

$$\begin{aligned}\langle h(x,t)\tilde{A}(x)\zeta(t)\rangle &= D\int_{-\infty}^{+\infty} dx\,\frac{\partial}{\partial x}\{\tilde{A}(x)\Omega[P]\}P(x,t) \\ &= \left\langle D\frac{\partial}{\partial x}\{\tilde{A}(x)\Omega[P]\}\right\rangle.\end{aligned} \tag{29}$$

Now, using the result above together with Eq. (24), the dissipated power of Eq. (27) becomes

$$\mathcal{P} = \frac{1}{\eta}\left\langle \frac{\tilde{A}^2(x)\Psi[P]}{P(x,t)} + D\frac{\partial}{\partial x}\{\tilde{A}(x)\Omega[P]\}\right\rangle, \tag{30}$$

showing the same form of Eq. (18), derived in the case of a conservative force.

One may see easily that the relation between dissipated power and entropy flux [cf. Eq. (16)] applies also for the case of the general external force of Eq. (26), described by the NLFPE of Eq. (A2). Therefore,

$$\begin{aligned}\Phi &= \frac{k}{\eta D}\left\langle \frac{\tilde{A}^2(x)\Psi[P(x,t)]}{P(x,t)} + D\frac{\partial}{\partial x}\{\tilde{A}(x)\Omega[P(x,t)]\}\right\rangle \\ &= \frac{k}{\eta D}\left\langle \frac{A^2(x)\Psi[P(x,t)]}{P(x,t)} + D\frac{\partial}{\partial x}\{A(x)\Omega[P(x,t)]\}\right\rangle \\ &\quad + \frac{k}{\eta D}\left\langle \frac{[A^{*}(x)]^2\Psi[P(x,t)]}{P(x,t)} + D\frac{\partial}{\partial x}\{A^{*}(x)\Omega[P(x,t)]\}\right\rangle \\ &\quad + \frac{2k}{\eta D}\left\langle \frac{[A(x)A^{*}(x)]\Psi[P(x,t)]}{P(x,t)}\right\rangle,\end{aligned} \tag{31}$$

where one notices three contributions for the entropy flux, coming respectively from the conservative and nonconservative forces, in addition to a mixed contribution. Splitting this later term into the two first ones, one may write $\Phi = \Phi_1 + \Phi_2$,

$$\Phi_1 = \frac{k}{\eta D}\left\langle \frac{A(x)\tilde{A}(x)\Psi[P(x,t)]}{P(x,t)} + D\frac{\partial}{\partial x}\{A(x)\Omega[P(x,t)]\}\right\rangle,$$
$$\Phi_2 = \frac{k}{\eta D}\left\langle \frac{A^*(x)\tilde{A}(x)\Psi[P(x,t)]}{P(x,t)} + D\frac{\partial}{\partial x}\{A^*(x)\Omega[P(x,t)]\}\right\rangle. \qquad (32)$$

One should stress the similarity of the entropy fluxes above with the time derivative of the ensemble average of a given function $B(x,t)$ in the Appendix [cf. Eq. (A8)]. In the particular case of the internal energy in Eq. (3), one has $B(x,t) \equiv \phi(x)$, and so comparing the above expression for Φ_1 with Eq. (A9), one notices that

$$\Phi_1 = -\frac{k}{D}\frac{dU(t)}{dt}. \qquad (33)$$

IV. SOME PARTICULAR CASES

Since the NLFPE of Eq. (1) is appropriate for describing the dynamical behavior of several nonlinear physical phenomena in nature, in this section we will work out the entropy production, as well as entropy flux, associated with well-known entropic forms. Other physical situations will be left for future investigations, such as quantum systems modeled through nonlinear classical equations of evolution, e.g., those proposed for bosons and fermions [38–41]. As pointed out in Ref. [18], a given entropic form is associated with a family of Fokker-Planck equations of the type presented in Eq. (1), with their functionals satisfying

$$\Omega[P] = a[P]b[P]; \quad \Psi[P] = a[P]P. \qquad (34)$$

All such equations are related to the same entropy, through the relation of Eq. (5), which becomes

$$-\frac{d\Lambda[Q]}{dQ}\frac{d^2g}{dP^2} = \frac{b[P]}{P}, \qquad (35)$$

whereas the current density of Eq. (6) is given by

$$J(x,t) = \frac{a[P]}{\eta}\left\{A(x)P(x,t) - D\,b[P]\left[\frac{\partial P(x,t)}{\partial x}\right]\right\}. \qquad (36)$$

The freedom for choosing the functional $a[P]$ generates different Fokker-Planck equations, characterized by distinct dynamical behaviors, although presenting the same stationary state and same entropic form. These characteristics are reflected in the current density above, where the entropic form is identified through the functional $b[P]$, whereas the possible dynamical behaviors, described by the NLFPE of Eq. (1), are distinguished by means of the multiplicative functional $a[P]$. These two functionals will appear in the time-dependent quantities defined in Eqs. (10)–(12), leading to the following entropy-flux and entropy-production contributions,

$$\Phi = \frac{k}{\eta D}\int_{-\infty}^{+\infty} dx\, A(x)a[P]\Big\{A(x)P(x,t) - D\,b[P]\left[\frac{\partial P(x,t)}{\partial x}\right]\Big\}, \qquad (37)$$

$$\Pi = \frac{k}{\eta D}\int_{-\infty}^{+\infty} dx\,\frac{a[P]}{P(x,t)}\Big\{A(x)P(x,t) - D\,b[P]\left[\frac{\partial P(x,t)}{\partial x}\right]\Big\}^2. \qquad (38)$$

Interestingly, although the entropic form does not depend explicitly on $a[P]$, this functional appears naturally in the time derivative dS/dt through its associated Fokker-Planck equation; however, one should notice that in each family the simplest Fokker-Planck equation associated with a given entropic form is obtained by considering $a[P] = 1$. In what follows we consider some examples of well-known entropic forms, within these simplest Fokker-Planck equations.

A. Boltzmann-Gibbs entropy

For completeness, we consider herein the production of the Boltzmann-Gibbs entropy, which has already been studied by many authors, making use of the linear Fokker-Planck equation (see, e.g., Refs. [4,6,7]). In this case one has

$$S_{\rm BG} = -k\int_{-\infty}^{+\infty} dx\, P(x,t)\ln P(x,t), \qquad (39)$$

which when compared with Eq. (4) corresponds to $\Lambda \equiv I$ (identity operator) and $g[P] = -P(x,t)\ln P(x,t)$. From Eq. (35) one gets $b[P] = 1$, leading to

$$\Phi = \frac{k}{\eta D}\int_{-\infty}^{+\infty} dx\, A(x)\Big\{A(x)P(x,t) - D\left[\frac{\partial P(x,t)}{\partial x}\right]\Big\}, \qquad (40)$$

$$\Pi = \frac{k}{\eta D}\int_{-\infty}^{+\infty} dx\,\frac{1}{P(x,t)}\Big\{A(x)P(x,t) - D\left[\frac{\partial P(x,t)}{\partial x}\right]\Big\}^2. \qquad (41)$$

One should recognize the above expressions as those already known in the literature [4,6,7].

B. Tsallis entropy

Tsallis entropy is defined in terms of a real parameter q [35,36],

$$S_q = k\int_{-\infty}^{+\infty} dx\,\frac{P(x,t) - P^q(x,t)}{q-1}, \qquad (42)$$

corresponding to $\Lambda \equiv I$ and $g[P] = [P(x,t) - P^q(x,t)]/(q-1)$. From Eq. (35) one gets $b[P] = qP^{q-1}$ and so

$$\Phi = \frac{k}{\eta D}\int_{-\infty}^{+\infty} dx\, A(x)\Big\{A(x)P(x,t) - qD\,[P(x,t)]^{q-1}\left[\frac{\partial P(x,t)}{\partial x}\right]\Big\}, \qquad (43)$$

$$\Pi = \frac{k}{\eta D}\int_{-\infty}^{+\infty} dx\,\frac{1}{P(x,t)}\Big\{A(x)P(x,t) - qD\,[P(x,t)]^{q-1}\left[\frac{\partial P(x,t)}{\partial x}\right]\Big\}^2. \qquad (44)$$

Considering $A(x) = 0$, the above entropy-production contribution recovers the one obtained previously from an anomalous-diffusion equation [27].

C. Kaniadakis entropy

Kaniadakis entropy is also defined in terms of a real parameter κ [42,43],

$$S_\kappa = -\frac{k}{2\kappa}\int_{-\infty}^{+\infty} dx\left(\frac{1}{1+\kappa}[P(x,t)]^{1+\kappa} - \frac{1}{1-\kappa}[P(x,t)]^{1-\kappa}\right), \tag{45}$$

from which one identifies $\Lambda \equiv I$ and

$$g[P] = -\frac{1}{2\kappa}\left(\frac{1}{1+\kappa}[P(x,t)]^{1+\kappa} - \frac{1}{1-\kappa}[P(x,t)]^{1-\kappa}\right). \tag{46}$$

Substituting into Eq. (35) one obtains $b[P] = (P^\kappa + P^{-\kappa})/2$ and so

$$\Phi = \frac{k}{\eta D}\int_{-\infty}^{+\infty} dx A(x)\left\{A(x)P(x,t) - D\frac{[P(x,t)]^\kappa + [P(x,t)]^{-\kappa}}{2}\left[\frac{\partial P(x,t)}{\partial x}\right]\right\}, \tag{47}$$

$$\Pi = \frac{k}{\eta D}\int_{-\infty}^{+\infty} dx \frac{1}{P(x,t)}\left\{A(x)P(x,t) - D\frac{[P(x,t)]^\kappa + [P(x,t)]^{-\kappa}}{2}\left[\frac{\partial P(x,t)}{\partial x}\right]\right\}^2. \tag{48}$$

D. Renyi entropy

Renyi entropy is defined as [44]

$$S_q^R = k\frac{\ln\left\{\int_{-\infty}^{+\infty} dx[P(x,t)]^q\right\}}{1-q}, \tag{49}$$

from which one finds [18]

$$\Lambda[Q[P]] = \frac{\ln Q[P]}{1-q};\quad \frac{d\Lambda[Q[P]]}{dQ} = \frac{1}{(1-q)Q[P]};\quad g[P] = [P(x,t)]^q, \tag{50}$$

leading to

$$b[P] = \frac{q[P(x,t)]^{q-1}}{\int_{-\infty}^{+\infty} dx[P(x,t)]^q}. \tag{51}$$

The corresponding flux and entropy production are given respectively by

$$\Phi = \frac{k}{\eta D}\int_{-\infty}^{+\infty} dx A(x)\left\{A(x)P(x,t) - \frac{qD[P(x,t)]^{q-1}}{\int_{-\infty}^{+\infty} dx[P(x,t)]^q}\left[\frac{\partial P(x,t)}{\partial x}\right]\right\}, \tag{52}$$

$$\Pi = \frac{k}{\eta D}\int_{-\infty}^{+\infty} dx \frac{1}{P(x,t)}\left\{A(x)P(x,t) - \frac{qD[P(x,t)]^{q-1}}{\int_{-\infty}^{+\infty} dx[P(x,t)]^q}\left[\frac{\partial P(x,t)}{\partial x}\right]\right\}^2. \tag{53}$$

It is important to be reminded that the quantities above are physically relevant only in the interval $0 < q < 1$, for which the entropy of Eq. (49) presents the appropriate concavity required by the H theorem [18].

V. CONCLUSIONS

We have analyzed the entropy rate of systems described by nonlinear Fokker-Planck equations. The Fokker-Planck equations considered are very general, written in terms of two functionals of the probability $P(x,t)$, appearing respectively in the drift and diffusion terms. These equations are directly related to generalized entropies by means of the H theorem and are expected to describe appropriately several nonlinear phenomena in nature. We have worked out the two contributions associated with time variations of the entropy, namely, the entropy flux and entropy production, the second being always positive for irreversible processes, as expected. Some examples for generalized entropic forms are analyzed, and particularly, the entropy production and flux associated with the Boltzmann-Gibbs entropy and the linear Fokker-Planck equation are recovered as particular cases. The present analysis is relevant for irreversible processes in many physical systems for which generalized entropic forms are applicable, like those exhibiting anomalous diffusion. Although the above study was restricted, for simplicity, to one dimension, it would be interesting to investigate in future works whether novelties could appear in higher dimensions.

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APPENDIX

In this Appendix we will derive an expression for the time derivative of the ensemble average of a given function $B(x,t)$,

$$\langle B\rangle = \int_{-\infty}^{+\infty} dx B(x,t)P(x,t). \tag{A1}$$

For that, we will make use of a NLFPE similar to the one of Eq. (1),

$$\eta\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{\tilde{A}(x)\Psi[P(x,t)]\}}{\partial x} + D\frac{\partial}{\partial x}\left\{\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}, \tag{A2}$$

written also as

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x};\quad J(x,t) = \frac{1}{\eta}\left\{\tilde{A}(x)\Psi[P] - D\Omega[P]\left[\frac{\partial P(x,t)}{\partial x}\right]\right\}, \tag{A3}$$

where $\tilde{A}(x)$ represents now a general external force. One has that

$$\frac{d}{dt}\langle B\rangle = \frac{d}{dt}\int_{-\infty}^{+\infty} dx B(x,t)P(x,t) = \int_{-\infty}^{+\infty} dx B(x,t)\frac{\partial P(x,t)}{\partial t} + \int_{-\infty}^{+\infty} dx \frac{\partial B(x,t)}{\partial t}P(x,t) = \int_{-\infty}^{+\infty} dx B(x,t)\frac{\partial P(x,t)}{\partial t} + \left\langle\frac{\partial B(x,t)}{\partial t}\right\rangle. \tag{A4}$$

Using Eq. (A3) and carrying an integration by parts, the first term becomes

$$\int_{-\infty}^{+\infty} dx\, B(x,t)\frac{\partial P(x,t)}{\partial t} = \int_{-\infty}^{+\infty} dx\, \frac{\partial B(x,t)}{\partial x} J(x,t) = \frac{1}{\eta}\int_{-\infty}^{+\infty} dx\, \frac{\partial B(x,t)}{\partial x}\left\{\tilde{A}(x)\Psi[P] - D\Omega[P]\frac{\partial P(x,t)}{\partial x}\right\}, \quad \text{(A5)}$$

and a further integration on the last term of the equation above leads to

$$\int_{-\infty}^{+\infty} dx\, B(x,t)\frac{\partial P(x,t)}{\partial t} = \frac{1}{\eta}\int_{-\infty}^{+\infty} dx \left\{\frac{\partial B(x,t)}{\partial x}\frac{\tilde{A}(x)\Psi[P]}{P(x,t)} + D\frac{\partial}{\partial x}\left(\frac{\partial B(x,t)}{\partial x}\Omega[P]\right)\right\} P(x,t). \quad \text{(A6)}$$

Consequently, Eq. (A4) may be written as

$$\frac{d}{dt}\langle B\rangle = \frac{1}{\eta}\left\langle \frac{\partial B(x,t)}{\partial x}\frac{\tilde{A}(x)\Psi[P]}{P(x,t)} + D\frac{\partial}{\partial x}\left(\frac{\partial B(x,t)}{\partial x}\Omega[P]\right)\right\rangle + \left\langle\frac{\partial B(x,t)}{\partial t}\right\rangle, \quad \text{(A7)}$$

and particularly, in the case where B does not depend explicitly on time ($\partial B/\partial t = 0$),

$$\frac{d}{dt}\langle B\rangle = \frac{1}{\eta}\left\langle \frac{\partial B(x)}{\partial x}\frac{\tilde{A}(x)\Psi[P]}{P(x,t)} + D\frac{\partial}{\partial x}\left(\frac{\partial B(x)}{\partial x}\Omega[P]\right)\right\rangle. \quad \text{(A8)}$$

Equations (A7) and (A8) express the time derivative $d\langle B\rangle/dt$ for a system obeying the NLFPE of Eq. (A2) in the presence of a general external force $\tilde{A}(x)$, which may be written in terms of two contributions, like in Eq. (26). Therefore, for the particular case of the internal energy defined in Eq. (3), one has $B(x) \equiv \phi(x)$, in such a way that Eq. (A8) becomes

$$\frac{d}{dt}U(t) = -\frac{1}{\eta}\left\langle \frac{A(x)\tilde{A}(x)\Psi[P]}{P(x,t)} + D\frac{\partial}{\partial x}(A(x)\Omega[P])\right\rangle. \quad \text{(A9)}$$

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