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Optimal navigation for characterizing the role of the nodes in complex networks

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ABSTRACT

In this paper, we explore how the approach of optimal navigation (Cajueiro (2009) [33]) can be used to evaluate the centrality of a node and to characterize its role in a network. Using the subway network of Boston and the London rapid transit rail as proxies for complex networks, we show that the centrality measures inherited from the approach of optimal navigation may be considered if one desires to evaluate the centrality of the nodes using other pieces of information beyond the geometric properties of the network. Furthermore, evaluating the correlations between these inherited measures and classical measures of centralities such as the degree of a node and the characteristic path length of a node, we have found two classes of results. While for the London rapid transit rail, these inherited measures can be easily explained by these classical measures of centrality, for the Boston underground transportation system we have found nontrivial results.

1. Introduction

In the last decade, one of the main concerns of the statistical physics literature has been to understand the geometric and the dynamical character of complex networks. In this context, some researchers have found interesting similarities among topologies of networks that belong to different fields such as biological networks [1,2], chaotic interacting functions [3], financial institution networks [4–8], social networks [9–13], spatial graphs [14–17] and transportation networks [18,19]. They have also tried to model and understand the nature of the dynamical processes that take place in these weblike structures such as flow of information [20–22], learning [23], navigation [24–33], searchability [34–41] and synchronization [42–44]. Reviews on this subject may be found in Refs. [45–47].

In order to understand the topology and dynamics of the above-mentioned dynamical processes, centrality is an essential concept, since it is used to identify the relevance of a node in a network. There are several different measures of centrality such as *degree* of a node (the number of neighbors of a node), *closeness centrality* [48] (the inverse of the average distance from one node to all other nodes), *graph centrality* [49] (the inverse of the maximal distance from one node to all other nodes), *efficiency* [50] (the average of the inverse of the distance from one node to all other nodes), *betweenness centrality* [51] (a measure based on the counting of the number of times that a node lies in the path between the others) and *dominance power* [52] (a weighted measure of the influence of a node on all other nodes of the network relative to the influence of all other nodes).

In this paper, we explore how optimal navigation [33] can be used to evaluate the centrality of a node and to characterize its role in a network. Furthermore, we show that besides the fact that the approach of optimal navigation generalizes the way that the centrality of a node is understood in the context of random and direct navigation, there is a special motivation

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in using this approach to deal with this task. While the measures of centrality presented above consider only the geometric properties of a given network, using the measures inherited from the approach of optimal navigation, one can also consider the social context behind a network or a node. Therefore, we use this approach to measure the centrality and to characterize the role of the nodes in complex networks. As proxies for complex networks, two transportation networks are analyzed here, namely the Boston underground transportation system [18] and the London rapid transit rail [19].

The remainder of this paper is structured in the following way. Section 2 revises the problem of optimal navigation in complex networks and discusses how this approach may be used to evaluate centrality of nodes in a complex network. Section 3 describes the data of the networks of the Boston underground transportation system [18] and the London rapid transit rail [19]. In Section 4, we present the main results of this paper exploring the data of these networks. Finally, Section 5 presents the main conclusions of this work.

2. Optimal navigation in complex networks

In Ref. [33], we studied the issue of optimal navigation in complex networks as an overlap of the random walk [24,25,31,32] and the directed walk behaviors [26–30]. A walker navigates optimally if he follows the minimal cost path from a *source* node s to a *target* node t within a complex network G with nodes belonging to the set $V(G) = \{1, 2, \dots, n\}$. We have assumed that there are two types of constant costs associated to the trajectory from a *source* node s to a *target* node t, namely the *navigation cost* C_N and the *information cost* C_I. While C_N is the cost that a walker has to pay for walking an additional edge of the network, C_I is the cost that the walker has to pay in order to go in the right direction. Depending on the value of these two costs, the walker may decide to walk the next stretch randomly at a cost C_N or not randomly at a cost $C_N + C_I$. Therefore, the walker navigates optimally if in each source node i he makes his decision in order to find the expected minimal cost of the walking from the source node i to the target node t given by

$$J^\star(i) = \min_{\pi\in\Pi} E_\pi\left[\sum_{k\in\mathcal{P}(i,t)} g(k, u(k))/i\right] \tag{1}$$

where $\mathcal{P}(i, t)$ is the path from node i to the target t, the expectation $E^\pi[\cdot/i]$ is conditional on the policy π and node i, $\pi = \{u(1), u(2), \cdots, u(t), \dots, u(n)\}$ is an admissible policy belonging to the set of admissible policies Π and $u(k)$ is an admissible control belonging to the set of admissible controls $U(k) = \{0, 1\}, \forall k$. While $u(k) = 0$ means that in node k the walker decides to randomly choose the next step of his walk, $u(k) = 1$ means that in node k he intends to ask other people what the correct path to follow is. The cost per stage $g(k, u(k))$ is the cost of using $u(k)$ at node k given by

$$g(k, u(k)) = \begin{cases} C_N, & \text{if } u(k) = 0,\ \forall k \neq t \\ C_N + C_I, & \text{if } u(k) = 1,\ \forall k \neq t. \end{cases} \tag{2}$$

It is easy to show that, if $g(t, u) = 0$ and $p(t, j) = 0, \forall j \neq t$, the solution of problem (1) is given by the Bellman equation [33]

$$J^\star(i) = \min_{u\in\{0,1\}}\left[g(i, u) + \sum_{j=1}^{n} p_{ij} J(j)\right]. \tag{3}$$

Furthermore, let $u^\star$ be given by

$$u^\star(i) = \arg\min_{u\in\{0,1\}}\left[g(i, u) + \sum_{j=1}^{n} p_{ij} J(j)\right]. \tag{4}$$

$u^\star(i)$ is the optimal control that the walker has to make in node $i \in V(G)$ in order to minimize the cost of the walk.

In Ref. [33], several interesting averages of $J^\star(i)$, $i \in V(G)$, were defined in order to quantify the difficulty of navigation in the network. The average of $J^\star(i)$ over all the targets t given by $\bar{J}^N(i)$, $\forall i \in V(G)$, measures the average difficulty to go from node $i \in V(G)$ to any another node $j \in V(G) - \{i\}$. The average of all $J^\star(j), \forall j \in V(G) - \{i\}$, when $t = i$, given by $\bar{J}^H$, measures the ability that a node has to be hidden in a network. Finally, the average of $J^\star$ over all nodes and all targets measures the overall difficulty of navigating in the network. It should be clear that while $\bar{J}^N$ and $\bar{J}^H$ introduce different ways to characterize nodes in different contexts, $\bar{\bar{J}}$ is a measure related to the whole network. Since the analysis of $\bar{\bar{J}}$ was fully considered in Ref. [33], the focus of this paper is to understand how $\bar{J}^N$ and $\bar{J}^H$ can be used to characterize the nodes of a given network. In fact, depending on how C_N and C_I are defined, the variables $\bar{J}^N$ and $\bar{J}^H$ may be influenced by the geometric properties of the network or the social context behind the network. For instance, C_N may be constant over the nodes or it can be the Euclidian distance (on some metric space) from one node to the other. On the other hand, C_I may be proportional to the number of nodes or conversely be inversely proportional to the degree of the node (meaning that it is easy to get information on the most connected nodes). This variable may also include the social context: C_I in a network where roads connect cities may be proportional to the population of the city. In this paper, we consider that C_N and C_I are constant

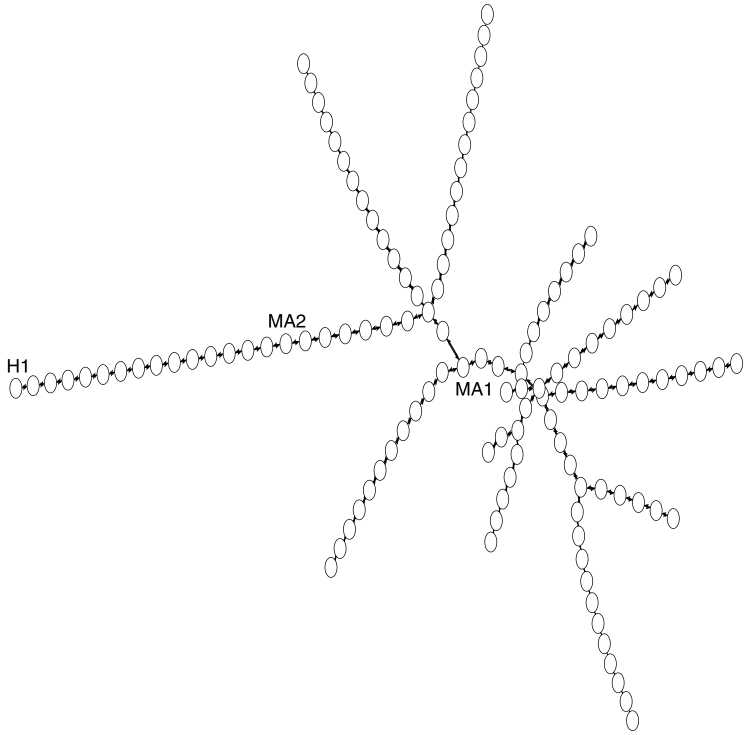

Fig. 1. This figure shows the network of the Boston underground transportation system.

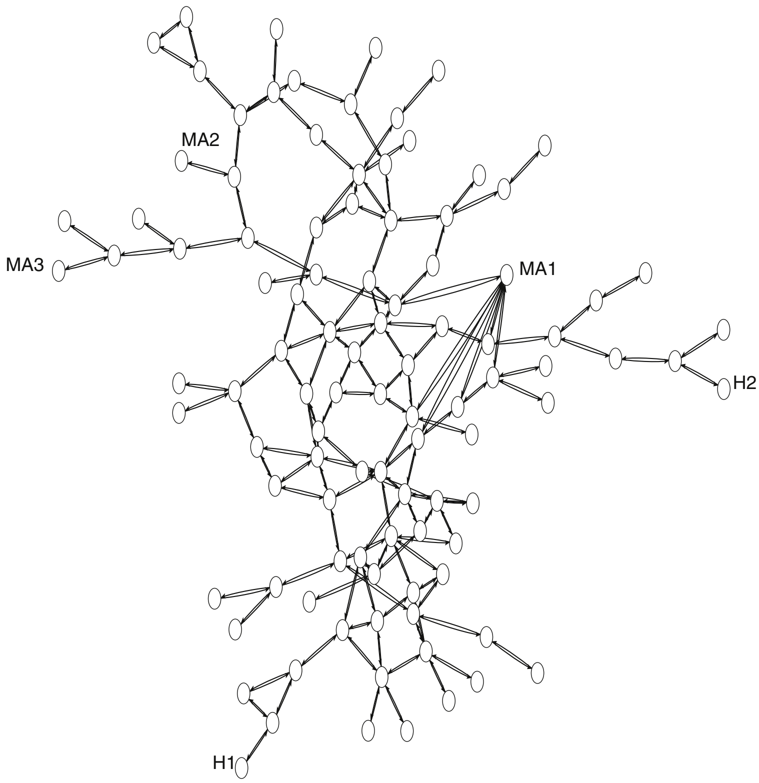

Fig. 2. This figure shows the network of the London rapid transit rail.

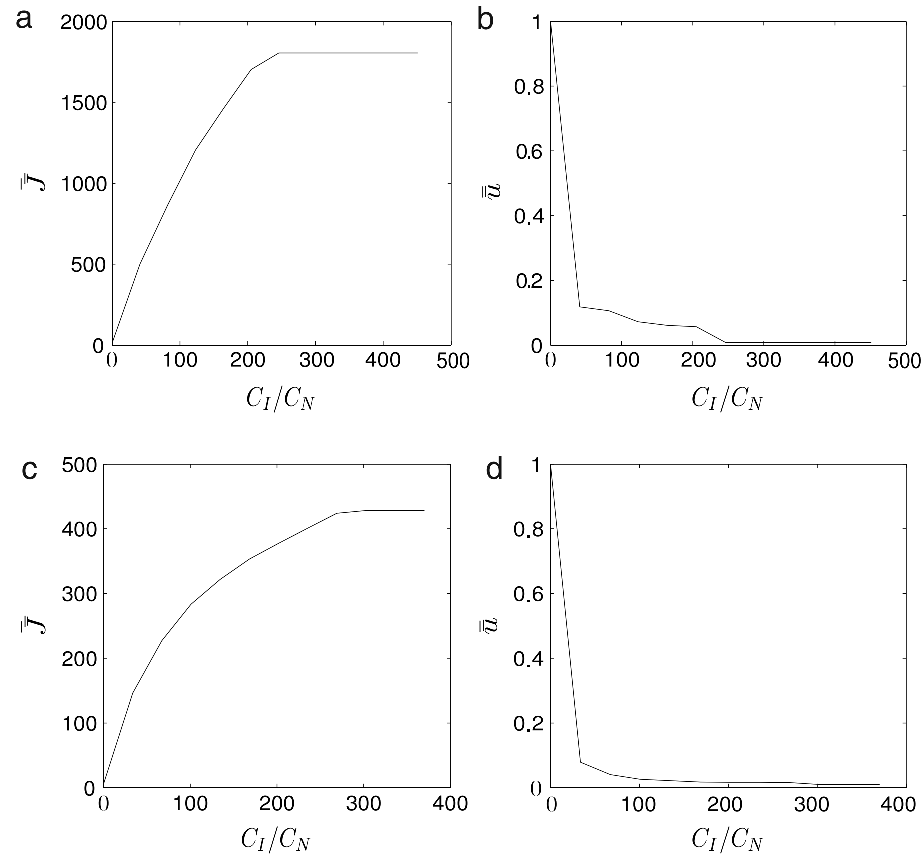

Fig. 3. (a) and (c) show $\bar{\bar{J}}$, respectively, for the Boston underground transportation system and for the London rapid transit rail for several values of C_I/C_N. (b) and (d) show $\bar{\bar{u}}$, respectively, for the Boston underground transportation system and for the London rapid transit rail for several values of C_I/C_N.

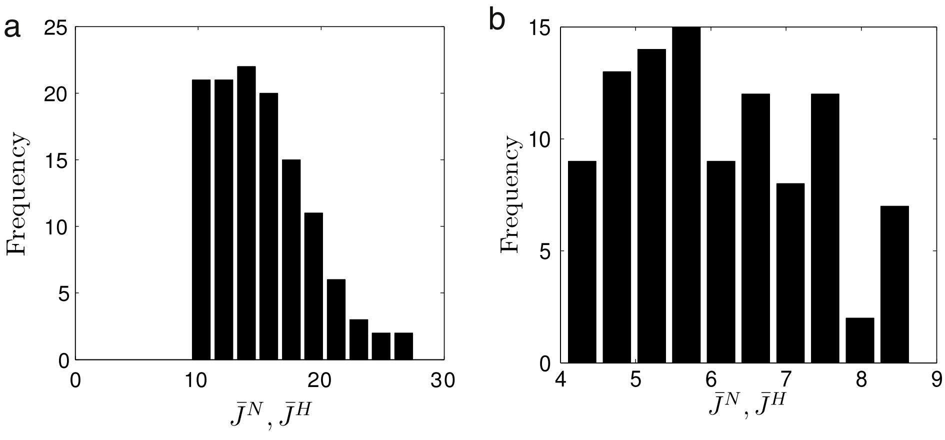

Fig. 4. These figures show the empirical distribution of the values of $\bar{J}^N(i)$ $(= \bar{J}^H(i))$ for each node $i \in V(G)$ for the case of direct navigation: (a) for the Boston underground transportation system; (b) for the London rapid transit rail.

throughout the network and we only vary the ratio C_I/C_N to understand the effect of this ratio on the characteristics of the nodes. Therefore, for each value of the ratio C_I/C_N, one may expect different roles of the nodes in a given network.

Finally, here we can define $\bar{u}^N(i)$ and $\bar{\bar{u}}$ analogously to $\bar{J}^N(i)$ and $\bar{\bar{J}}$. Since $u^\star$ belongs to the set $\{0, 1\}$, $\bar{u}^N(i)$ and $\bar{\bar{u}}$ belong to the interval $[0, 1]$. While $\bar{u}^N(i)$ closes to 1 implies that given the values of C_N and C_I it is more costly to walk randomly in node i than asking people the correct direction, $\bar{u}^N(i)$ closes to 0 means the opposite. Furthermore, $\bar{\bar{u}}$ measures the aggregated effect of the costs C_N and C_I on the choice of asking the correct direction or not.

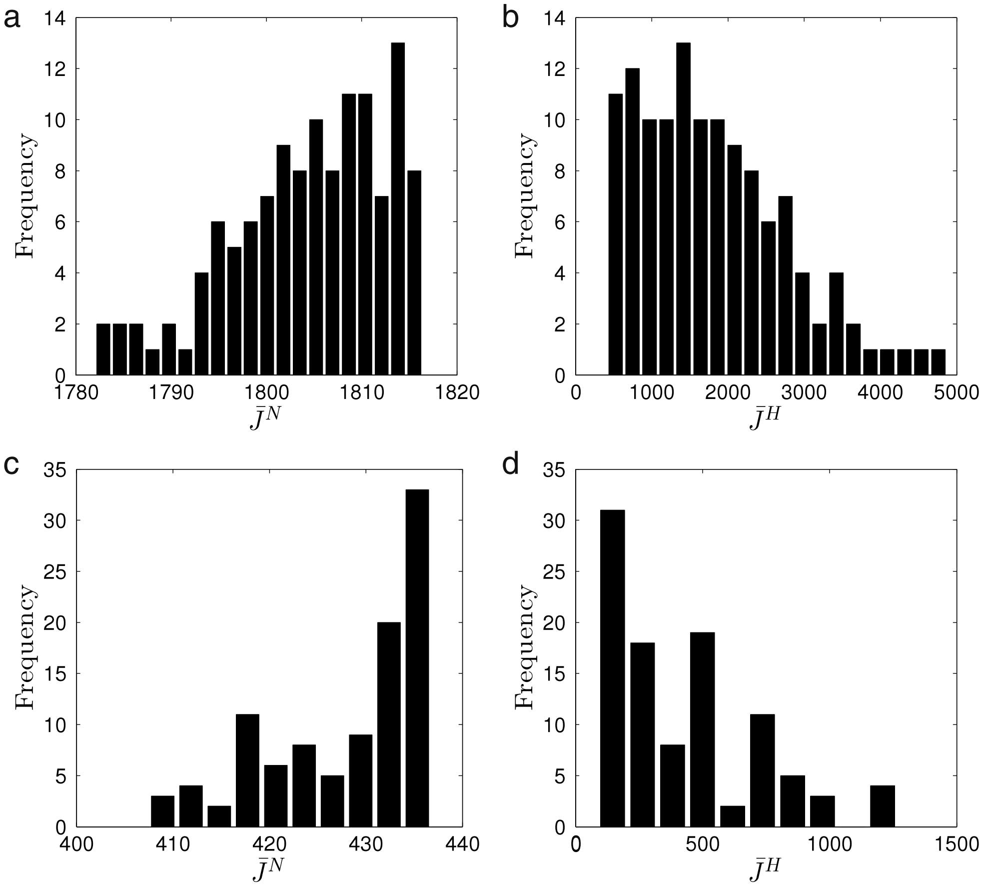

Fig. 5. These figures show the empirical distribution of the values of $\bar{J}^N(i)$ and $\bar{J}^H(i)$ for each node $i \in V(G)$ for the case of random navigation: (a) and (b) for the Boston underground transportation system; (c) and (d) for the London rapid transit rail.

As in Ref. [33], in order to find the averages of $J^\star(i)$, $i \in V(G)$, we have to find the solutions of the Bellman equation by means of the value iteration algorithm [53,54].

3. Data

The data considered in this work is the data of the network of the Boston underground transportation system [18] presented in Fig. 1 and the data of the network of the London rapid transit rail [19] presented in Fig. 2. Both networks are unweighted and undirected. While the former has 123 nodes, the later has 101 nodes.

4. Main results

Fig. 3 presents the variables $\bar{\bar{J}}$ and $\bar{\bar{u}}$ associated to the solutions of the Bellman equation (3) for the Boston underground transportation system and London rapid transit rail. This figure is very elucidative. Note that in Fig. 3(a) and (c) $\bar{\bar{J}}$ increases with the values of C_I/C_N until the regime transition (characterized by the point where the existence of directed walkers ceases). Fig. 3(b) and (d) show that when the value of C_I/C_N increases, it is no longer optimal to choose $u(i) = 1$ for most nodes i of the network.

Next, we try to understand the role of each node on the networks that represent the Boston underground transportation system and the London rapid transit rail. Our analysis here is mainly based on the measures $\bar{J}^N$ and $\bar{J}^H$ introduced in Section 2. This analysis is divided into three cases. The first case considers the situation where in these networks there are only directed walkers, i.e., $C_I/C_N = 0$. The second case considers the other extreme where there are only random walkers, i.e., a high enough value of C_I/C_N. The third case is an intermediate situation, where in one part of the nodes there are random walkers and in the other part there are directed walkers. It is clear that the most common situation for the specific case of a subway network is the first case where the cost of information is very low. However, since this is not true for all complex networks, we also consider the other situations here. In fact, it is likely that in most complex networks the cost of information is not too

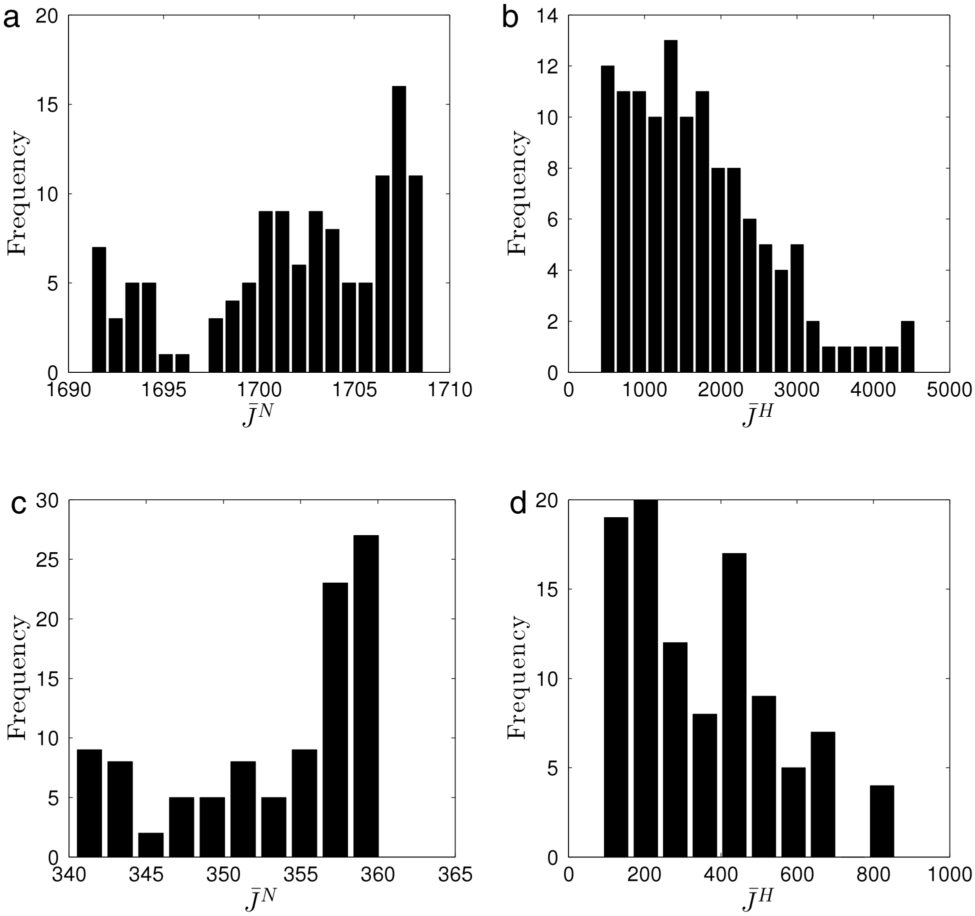

Fig. 6. These figures show the empirical distribution of the values of $\bar{J}^N(i)$ and $\bar{J}^H(i)$ for each node $i \in V(G)$ for the case of intermediate values of C_I/C_N: (a) and (b) for the Boston underground transportation system; (c) and (d) for the London rapid transit rail.

high, but it is not equal to zero either. As mentioned in Section 2, the costs C_N and C_I may reflect the social context behind the network. Consider for instance two networks where places are connected by roads. The first network represents nodes belonging to a metropolis and the second network represents nodes belonging to a desert. It is likely that while in the first network it is very easy to obtain information, in the second network, the same does not happen. To get information in the second case, one may have to deviate from the main road to ask the correct direction or one may try to make a long distance call (if it is possible!). Finally, although the high cost of information and only the presence of random walkers are very strong assumptions, they are possible hypotheses in some situations.

Fig. 4 shows the histograms of the values of $\bar{J}^N(i)$ for each node $i \in V(G)$ for the case of direct navigation, i.e., $C_I/C_N = 0$. Note that in the case of direct navigation $\bar{J}^N(i) = \bar{J}^H(i)$ for each node $i \in V(G)$. This happens because since the traveler always walks using the shortest path between two nodes, the walking is symmetrical. This is not true for the case $C_I/C_N \neq 0$. Fig. 5 shows the histograms of the values of $\bar{J}^N(i)$ and $\bar{J}^H(i)$ for each node $i \in V(G)$ for the case of random navigation. Fig. 6 shows the histograms of the values of $\bar{J}^N(i)$ and $\bar{J}^H(i)$ for each node $i \in V(G)$ for intermediate values of C_I/C_N. Three conclusions may be drawn from these figures: (1) Excluding the situation that $C_I/C_N = 0$, two different measures are associated to each node, namely a definition of centrality associated to the ability of easily accessing the other nodes of the network and the measure that characterizes the ability of being hidden in the network. (2) The values of $\bar{J}^N$ and $\bar{J}^H$ vary strongly among the nodes. (3) The networks that represent the Boston underground transportation system and London rapid transit rail are very different in nature. For instance, in Fig. 4, one may note that the variability of the values of $\bar{J}^N$ for the nodes of the Boston underground transportation system is much larger than for the case of the London rapid transit rail. This later assertion may also be intuitively noted, if one compares Fig. 1 with Fig. 2.

When $C_I/C_N = 0$ on the Boston underground transportation system, the node that requires minimal access information is referred to as acronym "MA1" and the best node to be hidden is referred to as acronym "H1" in Fig. 1. When $C_I/C_N = 205$, on the Boston underground transportation system the node that requires minimal access information is referred to as acronym "MA2" in Fig. 1 and the best node to be hidden is also referred to as "H1". Finally, for high enough values of C_I/C_N, "H1" is

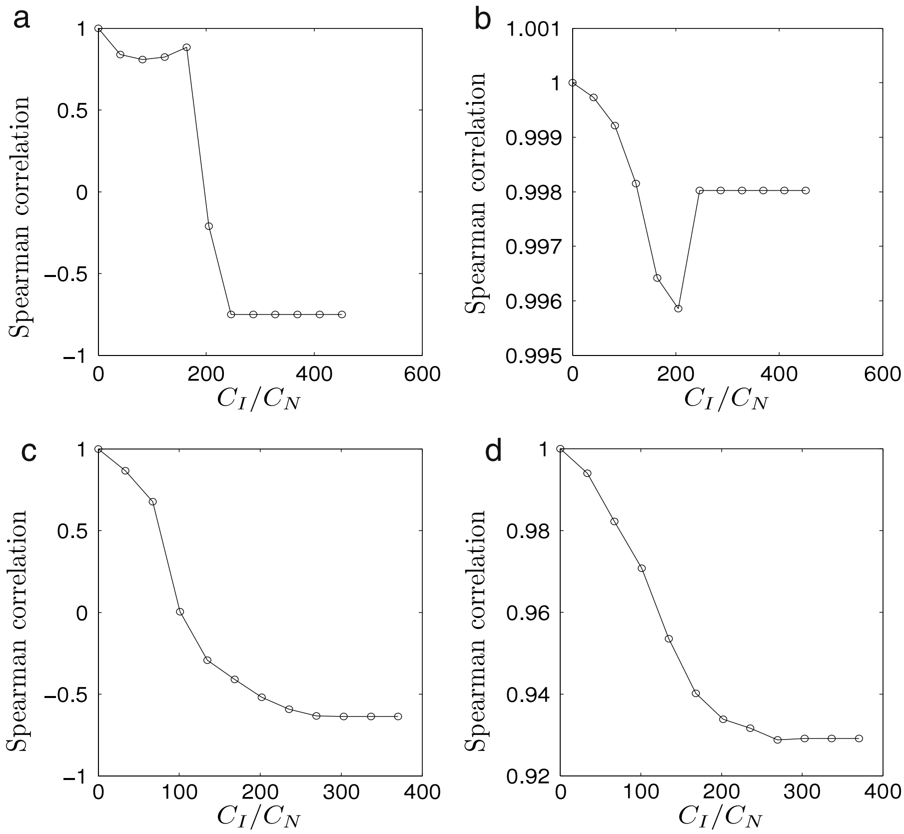

Fig. 7. The Spearman correlation coefficients between $\bar{J}^N$ evaluated in $C_I/C_N = 0$ and $C_I/C_N \neq 0$: (a) for the Boston underground transportation system; (c) for the London rapid transit rail. The Spearman correlation coefficients between $\bar{J}^H$ evaluated in $C_I/C_N = 0$ and $C_I/C_N \neq 0$: (b) for the Boston underground transportation system; (d) for the London rapid transit rail.

both the minimal access node and the best place to be hidden. This may seen quite surprising, but random walkers get lost frequently and node "H1" is located on the largest branch of the network. Therefore, in the random walker regime, a walker located in "H1" easily accesses more nodes than in any other place of the network. This node is also the best place to be hidden, since it is located at the end of the largest branch of the network. In the Boston underground transportation system "MA1" is the station known as "Copley". It is located at Copley square — the heart of the Back Bay Historic District. It is a place of historical significance and the farmer's market. "H1" is the station known as Boston college and it is at the end of the branch of the green line known as Commonwealth Avenue Branch.

When $C_I/C_N = 0$, on the London rapid transit rail the node that requires minimal access information is referred to as acronym "MA1" and the best node to be hidden is referred to as acronym "H1" in Fig. 2. When $C_I/C_N = 135$, on the London rapid transit rail the node that requires minimal access information is referred to as acronym "MA2" in Fig. 2 and the best node to be hidden is referred to "H2". Finally, for high enough values of C_I/C_N, "MA3" is the minimal access node and "H2" is also the best place to be hidden. In the London rapid transit rail "MA1" is the station known as "King's Cross St. Pancras", which is the biggest interchange station of the London Underground. "H1" is the station known as Epping, which is the north-eastern terminus of the Central Line. "MA2" is the Uxbridge station — the terminus of the Uxbridge branches of both the Metropolitan Line and the Piccadilly Line. "H2" is the Mill Hill East station, which is on the High Barnet branch of the Northern Line, and is the terminus, and only station, of a branch from Finchley Central station. "MA3" is the Amersham station, which is a terminus of the Metropolitan Line branch of the London Underground.

Fig. 7 presents the Spearman correlation coefficients between $\bar{J}^N$ ($\bar{J}^H$) evaluated in $C_I/C_N = 0$ (the vector of the characteristic path lengths of the nodes) and $C_I/C_N \neq 0$ for the Boston underground transportation system and for the London rapid transit rail. This figure reinforces the fact that the role of each node is strongly dependent on the value of C_I/C_N. One should note that the Spearman correlation coefficients of $\bar{J}^N$ ($\bar{J}^H$) evaluated for both transportation systems present the expected signals. However, while the Spearman correlation coefficients of $\bar{J}^H$ for the London rapid transit rail present an expected monotonic behavior, the same does not happen for the Spearman correlation coefficients of $\bar{J}^H$ evaluated for the Boston underground transportation system. Fig. 7(c) shows that, for low values of C_I/C_N, smaller characteristic length implies easier navigation. On the other hand, for high values of C_I/C_N, the opposite happens, since the walker gets lost easily

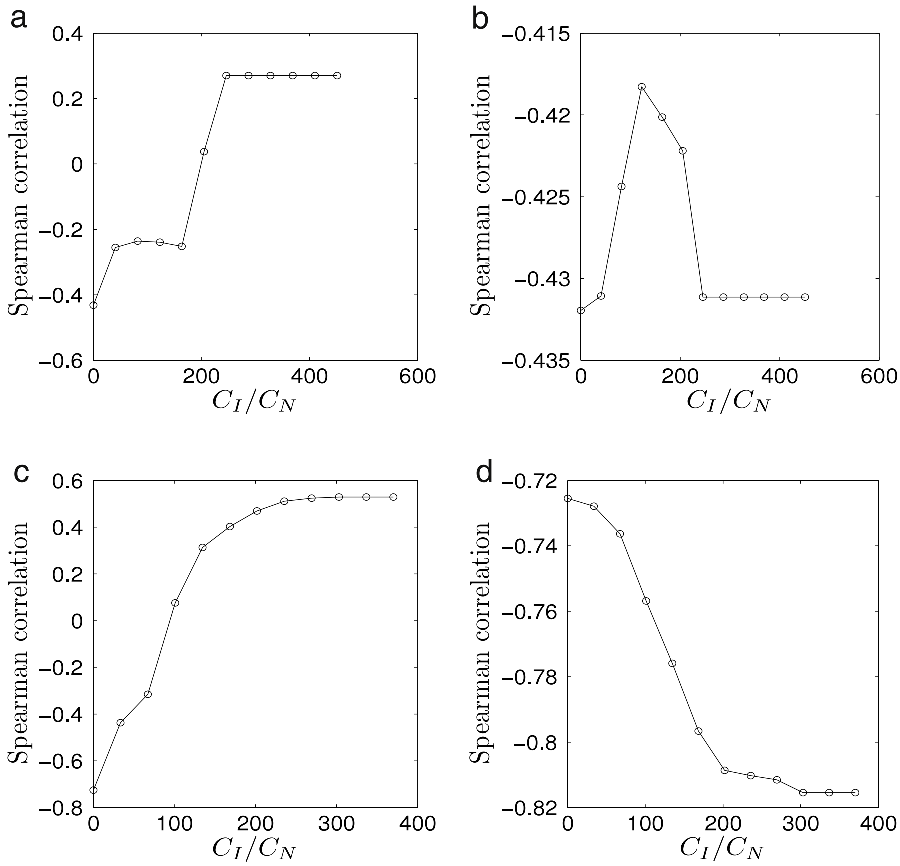

Fig. 8. The Spearman correlation coefficients between $\bar{J}^N$ and the vector of degrees of each node: (a) for the Boston underground transportation system; (c) for the London rapid transit rail. The Spearman correlation coefficients between $\bar{J}^H$ and the vector of degrees of each node: (b) for the Boston underground transportation system; (d) for the London rapid transit rail.

on the most central nodes in situations closer to the random walk regime. Furthermore, the decreasing behavior of the curve presented in Fig. 7(d) means that having a large characteristic path length is more important to be hidden for small values of C_I/C_N than for large values of C_I/C_N.

Fig. 8 presents the Spearman correlation coefficients between $\bar{J}^N$ ($\bar{J}^H$) and the vector of degrees of each node. For the case of the Spearman correlation coefficients for the London rapid transit rail, the results presented in Fig. 8(c) show that for small values of C_I/C_N, larger degree implies in easier navigation. On the other hand, for large values of C_I/C_N, larger degree implies more difficult navigation. Furthermore, the results presented in Fig. 8(d) shows that a node with larger degree in general is easier to be found. The decreasing behavior of the curve in this figure means that having a small degree is more relevant to be hidden for large values of C_I/C_N than for small values of C_I/C_N. Although the Spearman correlation coefficients of $\bar{J}^H$ for the Boston underground transportation system present the expected signals which are the same as those of the Spearman correlation coefficients for the London rapid transit rail, they do not present the monotonic behavior.

The non-monotonic behavior of the Spearman correlation coefficients evaluated for the Boston underground transportation system presented in Figs. 7(d) and 8(d) were already detected in Fig. 1 and in the analysis presented in Section 4. In this section we noted that "H1" in the three situations low C_I/C_N, intermediate C_I/C_N and high C_I/C_N is the best place to be hidden. Also, one should note that in this network most node degrees are equal to 2 and the only relevant characteristic is the distance from the center of the network.

Finally, in order to understand the decision making process that takes place in the network, Fig. 9 presents the Spearman correlation coefficients between $\bar{u}^N$ and the vector of characteristic lengths of each node and the Spearman correlation coefficients between $\bar{u}^N$ and the vector of degrees of each node. One should note that all the plots in this figure present zero Spearman correlation in the extremes due to the trivial behaviors in these situations. Fig. 9(a) and (c) show that directed walkers are more common on the nodes with smallest characteristic length. However, the importance of this phenomenon decreases when C_I/C_N increases, since the cost of information increases. Fig. 9(b) and (d) show that directed walkers are more common on the largest degree nodes. Again, the importance of this phenomenon also decreases when C_I/C_N increases, since the cost of information increases.

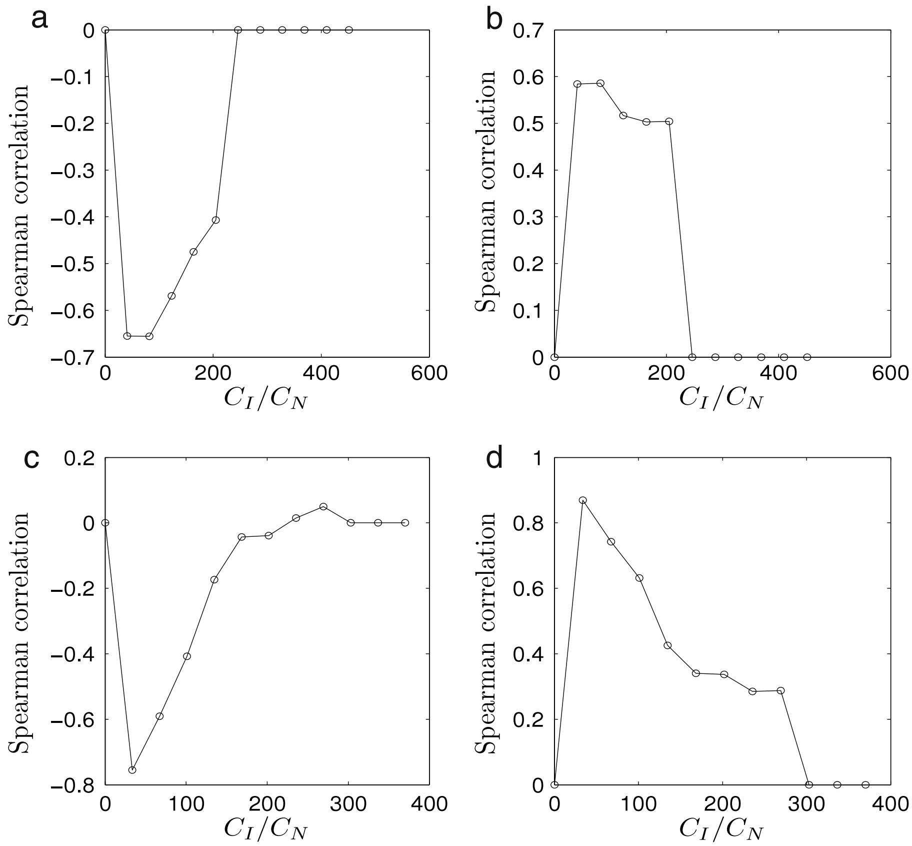

Fig. 9. The Spearman correlation coefficients between $\bar{u}^N$ and the vector of characteristic lengths of each node: (a) for the Boston underground transportation system; (c) for the London rapid transit rail. The Spearman correlation coefficients between $\bar{u}^N$ and the vector of degrees of each node: (b) for the Boston underground transportation system; (d) for the London rapid transit rail.

5. Conclusions

The measures inherited from the approach of optimal navigation [33] have been considered in order to analyze the role of individual nodes in complex networks. We have shown that these measures are strongly dependent on the ratio C_I/C_N. While in the directed walk regime there is no difference between arriving at a node or leaving a node, the same does not happen for the other regimes.

Evaluating the correlations between $\bar{J}^N$ ($\bar{J}^H$) and the vector of characteristic lengths of each node and the correlations between $\bar{J}^N$ ($\bar{J}^H$) and the vector of degrees of each node, we have found two classes of results. While for the London rapid transit rail, the measures $\bar{J}^N$ and $\bar{J}^H$ can be easily explained by other measures of centrality such as the characteristic path length (the inverse of the closeness centrality) and the degree, for the Boston underground transportation system we have found nontrivial results.

Finally, although this was not fully explored here since it requires special data, using this methodology it is also possible to include the social context behind a network in the definition of the centrality of a node. The social context is a new ingredient which is not considered by the measures of centrality that arise in random navigation [25], directed navigation [26,27] and the classical measures of centrality [48,51,52,49,50]. According to the results it may affect strongly the centrality and the other characteristics of a node.

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