JOURNAL OF MATHEMATICAL PHYSICS **57**, 082106 (2016)

Travelling-wave and separated variable solutions of a nonlinear Schroedinger equation

Tassos Bountis[1,a)] and Fernando D. Nobre[2,b)]
[1]*Department of Mathematics and Center for Research in Nonlinear Systems, University of Patras, Patras GR-26500, Greece*
[2]*Centro Brasileiro de Pesquisas Físicas and National Institute of Science and Technology for Complex Systems, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro–RJ, Brazil*

(Received 22 March 2016; accepted 29 July 2016; published online 17 August 2016)

Some interesting nonlinear generalizations have been proposed recently for the linear Schroedinger, Klein-Gordon, and Dirac equations of quantum and relativistic physics. These novel equations involve a real parameter q and reduce to the corresponding standard linear equations in the limit $q \to 1$. Their main virtue is that they possess plane-wave solutions expressed in terms of a q-exponential function that can vanish at infinity, while preserving the Einstein energy-momentum relation for all q. In this paper, we first present new travelling wave and separated variable solutions for the main field variable $\Psi(\vec{x},t)$, of the nonlinear Schroedinger equation (NLSE), within the q-exponential framework, and examine their behavior at infinity for different values of q. We also solve the associated equation for the second field variable $\Phi(\vec{x},t)$, derived recently within the context of a classical field theory, which corresponds to $\Psi^*(\vec{x},t)$ for the linear Schroedinger equation in the limit $q \to 1$. For $x \in \Re$, we show that certain perturbations of these q-exponential solutions $\Psi(x,t)$ and $\Phi(x,t)$ are unbounded and hence would lead to divergent probability densities over the full domain $-\infty < x < \infty$. However, we also identify ranges of q values for which these solutions vanish at infinity, and may therefore be physically important. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4960723]

I. INTRODUCTION

The birth of quantum mechanics in the beginning of the 20th century revolutionized the understanding of physics at small length scales. Nowadays, it is considered as one of the most successful theories in physics, leading to an appropriate description of an incredibly wide range of physical phenomena, although many of its fundamental concepts are quite remote from our daily experience, such as the single plane-wave solution, used for representing a free particle.[1] This solution presents a nonzero amplitude over all space, leading to a divergence of its norm, so that it is not appropriate for describing wave pulses or wave trains. However, due to its linearity, the Schroedinger equation can exhibit meaningful localized solutions constructed by defining a superposition of plane waves in the form of Fourier series, leading to the so-called wave-packet concept.

Motivated by the need for understanding a number of physical phenomena related to complex systems, alternative proposals for localized solutions have been proposed in the last years, based on modifications of the linear Schroedinger equation, one of which is to turn it to a nonlinear equation. In this context, two schemes have been pursued mostly in the literature: (i) introduction of an extra cubic term in the wave equation that makes it responsible for the modulation of some particular solutions in problems of nonlinear optics (e.g., Refs. 2 and 3); (ii) modification of the exponents of the linear terms, as considered in Refs. 4–8, a procedure that is currently employed within the framework of non-extensive statistical mechanics.[9]

[a)]Author to whom correspondence should be addressed. Electronic mail: tassosbountis@gmail.com
[b)]E-mail address: fdnobre@cbpf.br

In fact, in Ref. 4 three basic equations of quantum mechanics were modified following this procedure, resulting in a nonlinear Schroedinger equation (NLSE), as well as nonlinear versions of the Klein-Gordon and Dirac equations. As an immediate consequence, simple properties of the linear case, like conservation of probability by means of a continuity equation, become rather nontrivial objectives (see, e.g., Refs. 5, 6, and 8). Consequently, the use of latest advances in computer technology has stimulated the study of generalized equations, leading to considerable progress. Moreover, due to the aspect of nonlinearity, several distinct solutions are possible for a single equation, whose stability under small perturbations is no longer guaranteed, opening the possibility of a wide range of physical applications.[8] As is well-known, many areas of physics have already benefited from such nonlinear generalizations, like nonlinear optics, superconductivity, plasma physics, and non-equilibrium statistical mechanics.

The NLSE introduced in Ref. 4 is given by

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{x},t) = -\frac{1}{2-q}\frac{\hbar^2}{2m}\nabla^2[\Psi(\vec{x},t)]^{2-q}, \tag{1}$$

while, later on, using a classical field theory, it was verified that one must introduce in the Lagrangian an additional field $\Phi(\vec{x},t)$, which obeys the associated equation,[5]

$$i\hbar\frac{\partial}{\partial t}\Phi(\vec{x},t) = \frac{\hbar^2}{2m}[\Psi(\vec{x},t)]^{1-q}\nabla^2\Phi(\vec{x},t). \tag{2}$$

Note that, in the limit $q \to 1$, Eq. (1) reduces to the standard linear Schroedinger equation, whereas the equation above represents the complex conjugate of Eq. (1), if we identify $\Phi(\vec{x},t) = \Psi^*(\vec{x},t)$. Hence, for $q \neq 1$, one has to deal with both Equations (1) and (2), for reasons of consistency.[17]

These equations present two important features: (i) Their solutions are expressed in terms of a q-plane wave,[4–7] which generalizes the well-known plane wave, given in terms of the q-exponential function $\exp_q(u)$, which, for a purely imaginary iu, is defined as the principal value of Ref. 10,

$$\exp_q(iu) = [1 + (1-q)iu]^{1/(1-q)}, \qquad u = \vec{k}\cdot\vec{x} - \omega t, \tag{3}$$

(ii) Considering the q-plane wave solution, the new equations preserve fundamental relations of quantum physics, like the Planck and de Broglie relations, for all q, with $\vec{x} \in \Re^3$ and $t \in \Re$.

Viewing (1) and (2) as evolution equations, the probability density for finding a particle at time t in a given position $\vec{x}$ can be defined as

$$\rho(\vec{x},t) = \frac{1}{2A(\Omega)}\left[\Psi(\vec{x},t)\Phi(\vec{x},t) + \Psi^*(\vec{x},t)\Phi^*(\vec{x},t)\right], \tag{4}$$

for any q and arbitrary finite volume Ω, where $A(\Omega)$ is introduced for normalization, i.e.,

$$\int_\Omega d\vec{x}\,\rho(\vec{x},t) = 1, \qquad (\forall t). \tag{5}$$

It is important to note that what is physically relevant about the solutions of Equations (1) and (2) is their product, as given in Eq. (4), which should be finite inside the volume Ω to fulfill the above normalization condition.

Recently, the NLSE of Eq. (1) has been the object of attention of many studies, including its derivation from a hypergeometric differential equation,[11] its behavior under Galilean transformations,[7] its form in the presence of a potential,[8,12] a q-Gaussian wave-packet solution,[13] its solution by the method of separation of variables,[11] and its connection with the Bohmian approach to quantum mechanics.[14] Although some of these investigations lead to solutions that do not fulfill the continuity equation, those carried in Ref. 8, considering both Equations (1) and (2), do satisfy continuity.

In the present paper we restrict ourselves to a one-dimensional space, described by the variable $x \in \Re$. We first derive new travelling-wave solutions of the NLSE, as perturbations of the simple q-plane wave, by considering separately series expansions in two parameters $a > 0$ and $b > 0$, and examining their boundedness properties as $|u| = |kx - \omega t| \to \infty$. We also obtain, by a similar procedure, a class of separated variable solutions of the NLSE. In both cases we solve the principal equation for the field variable $\Psi(x,t)$ and the equation for the associated function $\Phi(x,t)$. As mentioned above, the product of these functions (plus its complex-conjugate) gives the probability density of Eq. (4), which must be finite inside the volume Ω, to be physically relevant.

In the case of travelling waves, we find that the above perturbations are unstable in the $q > 1$ regime, as they yield solutions $\Psi(x,t)$ and $\Phi(x,t)$ that are unbounded as x and/or t tend to infinity. However, we also find ranges of the q value for which the travelling waves have vanishing amplitude as $|u| \to \infty$ and hence can possess finite probability density over all space. Following an entirely similar approach, we discover in addition a novel class of separated variable solutions of the NLSE.

In Section II we describe how the field variables $\Psi(x,t)$ and $\Phi(x,t)$ of the NLSE are obtained as perturbations of the q-plane-wave solution. Their product is shown to vanish as $|kx - \omega t| \to \infty$, for specific ranges of q values, and lead to finite probability densities. In some cases, however, these densities are normalizable and therefore have physical meaning only over a finite domain. Analogous results are obtained in Section III for separated variable solutions of the NLSE, where densities may be integrable over the full domain $(-\infty,\infty)$. In Section IV we discuss the solutions found within the context of a continuity equation. Finally, in Section V we present our main conclusions.

II. TRAVELLING-WAVE SOLUTIONS

A. Wave solutions of the NLSE

From now on, for simplicity, we restrict ourselves to the one-dimensional case and seek travelling-wave solutions of Eqs. (1) and (2), expressed as

$$\Psi(x,t) = f(v) = f(i(kx - \omega t)), \tag{6}$$

$$\Phi(x,t) = g(v) = g(i(kx - \omega t)), \tag{7}$$

so that $v = iu = i(kx - \omega t)$, $x \in \Re$, and $t \in \Re$. Substituting these solutions in Eqs. (1) and (2) we get two second order ordinary differential equations, respectively,

$$a\,\frac{df}{dv} = \frac{d}{dv}\left[f^{1-q}\frac{df}{dv}\right], \tag{8}$$

$$a\,\frac{dg}{dv} = -f^{1-q}\frac{d^2g}{dv^2}\,, \tag{9}$$

where

$$a = \frac{2m\omega}{\hbar k^2} = \frac{\hbar\omega}{(\hbar^2k^2)/(2m)}\,. \tag{10}$$

Integrating Eq. (8), one obtains the first order ordinary differential equation,

$$\frac{df}{dv} = af^q + bf^{q-1}, \tag{11}$$

where the integration quantity b is given by

$$b = \left[f^{1-q}(v)\frac{df(v)}{dv} - af(v)\right]_{v=0}. \tag{12}$$

It is important to recall that the dimensionless positive quantity a is given as the ratio of two important energies of the problem, namely, the energy of a quantum of radiation $\hbar\omega$ (or, equivalently, the energy quantum of a one-dimensional harmonic oscillator) and the kinetic energy of the particle under investigation, $(\hbar^2k^2)/(2m)$. Moreover, the dimensionless quantity b depends on a, as well as the function $f(v)$ and its derivative at $v = 0$.

Now, in Eq. (11) one notices that the contribution bf^{q-1} is irrelevant in the limit $q = 1$, leading to the well-known plane-wave solution.[1] However, for $q \neq 1$ this term comes out naturally from the integration of the NLSE, written in the form of Eq. (8). The solution of Eq. (11), considered in Refs. 4 and 5, represents the q-plane wave,

$$f(v) = \exp_q(av) = \exp_q(iau) = [1 + (1-q)iau]^{1/(1-q)}, \tag{13}$$

corresponding to the particular case $b = 0$ in Eq. (11). This is easily verified, since $(df(v)/dv) = a[\exp_q(av)]^q$, so that one has $f(0) = 1$ and $(df(v)/dv)_{v=0} = a$, yielding $b = 0$ in Eq. (12). For $q > 1$ the complex function of Eq. (13) has bounded amplitude, which vanishes as $|u| \to \infty$, since it can be written as[10]

$$f(v) = \exp_q(av) = r_q(\cos\theta + i\sin\theta), \tag{14}$$
$$r_q = \left[1 + (1-q)^2 a^2 u^2\right]^{1/2(1-q)}, \quad \theta = \frac{1}{q-1}\tan^{-1}[(q-1)au].$$

The analysis of the additional contribution bf^{q-1} in the solution of the NLSE represents one of the main motivations of the present work, in view of the changes it incurs to the solutions considered in Refs. 4 and 5. Alternative solutions resulting from other choices of $f(0)$ and $(df(v)/dv)_{v=0}$ are now possible and will be analyzed herein. Hence, different limits need to be considered for the quantity b of Eq. (12), namely, $a \gg |b|$ (representing a physical situation of low kinetic energies) and $a \ll |b|$ (representing a physical situation of high kinetic energies).

Formally, the solution of Eq. (11) can be implicitly expressed in terms of hypergeometric functions, as shown in Ref. 15,

$$v = \frac{1}{b}\left\{\frac{f^{2-q}-1}{2-q} + \frac{a/b}{3-q}[H(1;2-q,1,a/b) - H(f;2-q,1,a/b)]\right\}, \tag{15}$$

where $H(f;\alpha,\beta,\gamma) = f^{1+\alpha}F(\frac{1+\alpha}{\beta},1,\frac{1+\alpha+\beta}{\gamma},-f^{\beta}\gamma)$ and F being the hypergeometric function. However, this is hardly a useful approach if we want to gain a deeper understanding of these solutions.

For non-zero a and b, we may also try to solve Equation (11) analytically by writing it as an integral,

$$\int \frac{f^{1-q}\,df}{af+b} = v + C. \tag{16}$$

Next, we analyze several possible solutions of Eq. (11), either by expanding the integrand in the left-hand side of the equation above as an infinite series, or directly from Eq. (11), by considering perturbations around conveniently chosen leading contributions.

Case 1: $|a/b| < 1$ in the integral of Eq. (16)
In this case, one can expand the integrand of Eq. (16), leading to

$$\int \frac{f^{1-q}\,df}{af+b} = \int \frac{f^{1-q}}{b}\sum_{k=0}^{\infty}(-1)^k(\frac{af}{b})^k df = \frac{1}{b}\sum_{k=0}^{\infty}(-1)^k(\frac{a}{b})^k\frac{f^{2-q+k}}{2-q+k} = v + C, \tag{17}$$

through which $f(v)$ is given implicitly as a function of v, provided the above series converges. For example, if we assume $|af| < |b|$, the series converges absolutely and we may rewrite Eq. (17) after some simple algebra as

$$f\left[\frac{1}{2-q} - \frac{af}{b(3-q)} + \frac{a^2f^2}{b^2(4-q)} + \cdots\right]^{1/(2-q)} = \left(bv + \frac{A}{2-q}\right)^{1/(2-q)}, \tag{18}$$

with A an arbitrary constant. We now write the above expression as

$$f\left[1 - \frac{af(2-q)}{b(3-q)} + \frac{a^2f^2(2-q)}{b^2(4-q)} + \cdots\right]^{1/(2-q)} = (A + (2-q)bv)^{1/(2-q)}. \tag{19}$$

Observe that we could expand the quantity in the brackets in Eq. (19) as a Taylor series and solve for $f(v)$ recursively, starting with the lowest-order expression $U(v) = (A + (2-q)bv)^{1/(2-q)}$, which is reminiscent of the q-exponential function with q replaced by $q-1$. Thus, if we wished to obtain bounded solutions for all v which vanish at infinity, we should also impose $q > 2$. Indeed, expanding the left hand side of Eq. (19) to second order in (a/b), we obtain

$$f(v) = U + \frac{a}{b}\frac{U^2}{(3-q)} - \left(\frac{a}{b}\right)^2\frac{U^3(q^2-3q-2)}{2(3-q)^2(4-q)} + \cdots, \tag{20}$$

$$U = A^{1/(2-q)}\left[1 + (2-q)\frac{bv}{A}\right]^{1/(2-q)} = A^{1/(2-q)}\exp_{q-1}(bv/A)\,. \tag{21}$$

This result demonstrates that, as a power series in U, $f(v) \to 0$ as $|v| \to \infty$ for $q > 2$, provided the above series converges. The solution of Eq. (20) is different from the q-plane wave, since besides $b \neq 0$, one also has $f(0) \neq 1$ and $(df(v)/dv)_{v=0} \neq a$.

Case 2: $|b/a| < 1$ **in the integral of Eq. (16)**
Alternatively, one could have expanded the integral in Eq. (16) in powers of $b/(af)$ to arrive at an equation similar to Eq. (19) (with a q-exponential on the right side), which could also be solved iteratively for f. This time, however, for convergence we would have to assume $|f| > |b/a|$, which would not allow f to vanish at infinity. For example, the first correction to f would yield

$$f(v) = (B + (1-q)av)^{1/(1-q)} - \frac{b}{qa} + \cdots, \tag{22}$$

implying that $f(v) \to$ constant as $|v| \to \infty$ for $q > 1$, with B an arbitrary constant. Similar conclusions are reached when we calculate higher-order corrections.

Case 3: $|b/a| < 1$ **with q-plane wave as leading term in Eq. (11)**
On the other hand, one might have attempted to solve Eq. (11) perturbatively as a series in powers of (b/a), starting with the q-exponential $\exp_q(av)$ at leading order, to find that

$$f(v) = f_0(v) + \frac{b}{a}f_1(v) + \left(\frac{b}{a}\right)^2 f_2(v) + \cdots, \tag{23}$$

$$f_0(v) = \exp_q(av), \tag{24}$$

$$f_1(v) = \frac{1}{q}\left[\exp_q^q(av) - 1\right], \tag{25}$$

$$f_2(v) = \frac{-1}{(q+1)}\exp_q^q(av) + \frac{1}{2q}\exp_q^{2q-1}(av) + \frac{q-1}{2q(q+1)}\exp_q^{-1}(av), \tag{26}$$

so that $f(0) = 1$. One should notice that the main difference between this solution and the previous one [Case 2] concerns the value of $f(0)$. Unfortunately, as in the case of the above expansion (22), a constant term enters in the first order correction $f_1(v)$ above. Moreover, in the second-order terms [Eq. (26)] one sees that terms of the form $\exp_q^{-1}(av)$ have to be included, which leads to divergent solutions as $|v| \to \infty$, for $q > 1$. Indeed, the above analysis seems to imply that the q-plane waves of the NLSE of Eq. (1) possess some unstable directions in the space of all solutions. On the other hand, if we limit ourselves to $q < 1$, the first and second terms in f_2 (as well as f_1) would require $q < 0$ for convergence. Even in that case, however, the zeroth-order term f_0 would be unbounded as $|v| \to \infty$.

Case 4: $|a/b| < 1$ **and $(q-1)$-plane wave as leading term in Eq. (11)**
Let us now consider bf^{q-1} as the leading term in Eq. (11) and expand $f(v)$ in powers of (a/b) as follows:

$$f(v) = \exp_{q-1}(bv) + \frac{a}{b}f_1(v) + \left(\frac{a}{b}\right)^2 f_2(v) + \cdots, \tag{27}$$

under the condition $q > 2$ to ensure that $f(v) \to 0$ as $|v| \to \infty$. The only difference with the previous analysis that led to Eq. (20) [Case 1] is that here we can impose $f(0) = 1$, while in Eq. (20) there is an arbitrary constant A that determines the value of $f(0)$. Indeed, substituting (27) in (11) and equating terms of first order in (a/b) yields

$$\frac{df_1(v)}{dv} = (q-1)\exp_{q-1}^{q-2}(bv)f_1 + \exp_{q-1}^q(bv), \tag{28}$$

with $f_1(0) = 0$. This equation is easily solved to provide

$$f_1(v) = \frac{1}{(q-3)}\left[\exp_{q-1}^{q-1}(bv) - \exp_{q-1}^2(bv)\right]. \tag{29}$$

If we now proceed to second and higher orders, we find that new q–exponential terms enter with powers that preserve the boundedness of the solution, provided $q > 2$. For example, the second-order contribution $f_2(v)$, besides the $\exp_{q-1}^{q-1}(bv)$ term of the homogeneous equation, also contains the terms $\exp_{q-1}^{2q-3}(bv)$, $\exp_{q-1}^{q}(bv)$, $\exp_{q-1}^{3}(bv)$, all of which are clearly finite for $q > 2$ and vanish as $|v| \to \infty$. Since the above treatment is equivalent to the one that led to Eq. (20), we expect that here also the property $f(v) \to 0$ as $|v| \to \infty$ holds at all orders.

It is important to mention that some of the cases considered above [Cases 1 and 4, as well as Cases 2 and 3] correspond to physically equivalent situations and so, their corresponding solutions are expected to coincide. Indeed, these solutions differ by arbitrary multiplicative and/or additive constants. Hence, if they lead to a normalizable probability density in Eq. (4), these constants can redefined by imposing the normalization condition, yielding equivalent solutions.

Next we will evaluate the possible solutions for the second field $g(v)$, defined in Eq. (7) corresponding to the above expansions. Thus, we will be able to check whether the product $f(v)g(v)$ is finite and integrable over the full domain $-\infty < v < \infty$, so as to ensure a normalizable probability density in Eq. (4). Of course, in the case of divergences, one can still limit the applicability of these solutions to a finite domain. This is taken up in Sec. II B, both for the case of series expansions in powers of (a/b) and power series in (b/a).

B. Travelling wave solutions of the associated NLSE equation

Let us now turn to the associated equation (2) to see what type of solutions for the second field $\Phi(x,t)$ corresponds to the travelling wave solutions of Eq. (1), discussed in Subsection II A. Let us first consider the following ansatz:

$$g(v) = C_1[f(v)]^{\alpha} + C_2, \tag{30}$$

where C_1, C_2, and α are real quantities and substitute in Eq. (9) to obtain,

$$a f^{\alpha-1}\frac{df}{dv} = -f^{1-q}\frac{d}{dv}\left(f^{\alpha-1}\frac{df}{dv}\right), \tag{31}$$

which, after multiplying both sides by f^{q-1}, and integrating, yields

$$\frac{a}{\alpha+q-1} f^{\alpha+q-1} = -f^{\alpha-1}\frac{df}{dv} + C, \tag{32}$$

where C is an integration constant. Now, multiplying the whole equation by $f^{1-\alpha}$, one gets that

$$\frac{df}{dv} = -\frac{a}{\alpha+q-1} f^{q} + C f^{1-\alpha}. \tag{33}$$

This equation is not compatible with Eq. (11), since by comparing these two equations, one readily sees that it is not possible to match both terms on their right-hand-sides. This means that the ansatz of Eq. (30) is not valid in general, except in some particular cases, more specifically for $(a = 0, b \neq 0)$, or $(a \neq 0, b = 0)$. In these situations, one can make convenient choices for C and α to match each one of these terms separately. These choices will correspond to the leading-order contributions of the expansions of both functions $f(v)$ and $g(v)$, to be discussed next. (i) $a = 0$, $\alpha = 2 - q$, and $b = C$: the leading terms are of the type $f(v) = D_1\exp_{q-1}(bD_2 v)$ [cf. Eq. (21)], or $f(v) = \exp_{q-1}(bv)$ [cf. Eq. (24)] and $g(v) = C_1[f(v)]^{2-q} + C_2$. (ii) $b = C = 0$ and $\alpha = -q$: the leading terms in this case are $f(v) = \exp_q(av)$ and $g(v) = C_1[\exp_q(av)]^{-q} + C_2$. Choosing $C_1 = 1$ and $C_2 = 0$, these latter solutions correspond precisely to those found in Ref. 5. One should notice that the types of solutions (i) and (ii) correspond, respectively, to the leading terms found for $f(v)$ in the cases $|a/b| < 1$ and $|b/a| < 1$ of the previous analysis. Hence, below we will analyze Eq. (9) considering the solutions for $f(v)$ discussed in Cases 1 and 3 of the Subsection II A, and verify for each one of them the ansatz of Eq. (30) for the leading contribution of $g(v)$.

Case $|a/b| < 1$: In this case we write Eq. (20) in the form

$$f(v) = U + \frac{a}{b}\lambda_1(q)U^2 + \left(\frac{a}{b}\right)^2 \lambda_2(q)U^3 + \cdots, \quad U = (A + (2-q)bv)^{1/(2-q)}, \tag{34}$$

where the coefficients $\lambda_1(q), \lambda_2(q), \ldots$ are obtained from Eq. (20), e.g., $\lambda_1(q) = 1/(3-q)$. Then, we set $dg/dv = h(v)$, substitute in Eq. (9), and expand again in powers of (a/b) to obtain a differential equation for $h(v)$,

$$h'(v) = -aU^{q-1}\left[1 + \frac{a}{b}\,\mu_1(q)U + \left(\frac{a}{b}\right)^2 \mu_2(q)U^2 + \cdots\right] h(v), \tag{35}$$

where the coefficients $\mu_1(q), \mu_2(q), \ldots$ are obtained from the expansion of $[f(v)]^{(q-1)}$, e.g., $\mu_1(q) = (q-1)\lambda_1(q) = (q-1)/(3-q)$.

To solve the equation above, we insert $h(v) = h_0(v) + (a/b)h_1(v) + (a/b)^2 h_2(v) + \cdots$ in Eq. (35), perform simple integrations, and obtain one by one the terms in this expansion by equating like powers of (a/b), so that

$$h(v) = 1 - \frac{a}{b}U - \frac{1}{2}\left(\frac{a}{b}\right)^2 [\mu_1(q) - 1]U^2 + \cdots, \tag{36}$$

where we have set $h_0(v) = 1$ with no loss of generality. Finally, there is one more set of integrations to perform, in order to evaluate $g(v)$ as a power series in (a/b), using the fact that $h(v) = g_0'(v) + (a/b)g_1'(v) + (a/b)^2 g_2'(v) + \cdots$. These integrations yield

$$g_0(v) = v,\ \ g_1(v) \propto U^{3-q},\ \ g_2(v) \propto U^{4-q}, \ldots,\ \ g_n(v) \propto U^{2+n-q}, \tag{37}$$

with $g(v)$ to first order in (a/b) given by

$$g(v) = v - \frac{a}{b}\frac{U^{3-q}}{(3-q)b} - \frac{1}{2b}\left(\frac{a}{b}\right)^2 \frac{\mu_1(q) - 1}{4-q} U^{4-q} + \cdots, \tag{38}$$

provided $q \neq 1 + n,\ \ n = 2, 3, \ldots$. Then, using Eq. (20), one has

$$f(v)g(v) = \left[U + \frac{a}{b}\frac{U^2}{3-q} + \cdots\right]\left[v - \frac{a}{b}\frac{U^{3-q}}{(3-q)b} + \cdots\right]. \tag{39}$$

It is important to recall that the wavefunctions $\Phi(x,t)$ and $\Psi(x,t)$ given in Eqs. (6) and (7) should yield a finite result when they multiply each other to yield the probability density of Eq. (4). Hence, a divergent solution for one of the fields as $|v| \to \infty$ can still satisfy Eq. (4) provided the other field converges fast enough to compensate for this divergence. Otherwise, one should consider these solutions in a finite volume Ω for normalization, as in the case, e.g., with $q = 1$.[1]

Recall that we set $q > 2$ to have solutions vanishing at infinity. Requiring now that the product $f(v)g(v)$ also be finite as $v \to \pm\infty$ necessitates that the first term in this product, i.e., $vU \propto |v|^{(3-q)/(2-q)}$, also vanishes in that limit. This happens only if $2 < q < 3$. Moreover, it is easy to see that, under this condition, $f(v)g(v)$ remains finite even when higher-order terms are included in the expansion of Eq. (39). It would not be possible, however, for this quantity to be integrable over the domain $-\infty < v < \infty$, unless its first integrated term $\propto |v|^{(5-2q)/(2-q)}$ also vanishes at infinity. This implies that we must further restrict the range of q to $2 < q < 2.5$, if we wish to have physically meaningful results for all $x \in \Re$. Of course, in cases where the integral diverges calculations can always be limited to a finite interval Ω to satisfy the normalization condition of Eq. (5).

Case $|b/a| < 1$: Let us now turn to the solution of the associated equation corresponding to the field variable $f(v)$ expanded in powers of (b/a). Note that this should perhaps be the most natural perturbation approach to follow, since it develops the solution about the q-plane wave of Eq. (3), discussed in Refs. 4–8. To this end, we need to solve the associated equation (9) for $g(v) = g_0(v) + (b/a)g_1(v) + (b/a)^2 g_2(v) + \cdots$ with $f(v)$ given by Eqs. (23)–(26).

We start by introducing the auxiliary function $h(v) = dg/dv = g'(v)$ and writing its expansion in the form

$$h(v) = h_0(v) + \frac{b}{a}\,h_1(v) + \left(\frac{b}{a}\right)^2 h_2(v) + \cdots = g_0'(v) + \frac{b}{a}\,g_1'(v) + \left(\frac{b}{a}\right)^2 g_2'(v) + \cdots, \tag{40}$$

in terms of which we can write Equation (9) as

$$h'(v) = -a f_0^{q-1}\left[1 + \frac{b}{a}\frac{f_1}{f_0} + \left(\frac{b}{a}\right)^2 \frac{f_2}{f_0} + \cdots\right]^{q-1} h(v). \tag{41}$$

We now expand the expression inside the brackets in powers of (b/a) and obtain a sequence of linear equations for the functions $h_n(v), n = 0,1,2,\ldots$. Solving these equations, substituting in Eq. (40), and integrating once more, we obtain the $g_n(v)$ functions as follows:

$$g_0(v) = [1 + (1-q)av]^{\frac{q}{q-1}}, \tag{42}$$

$$g_1(v) = c_1 P^{q/(q-1)} + c_2 P^{1/(q-1)} + c_3 P^{(1+q)/(q-1)}, \tag{43}$$

$$\begin{aligned} g_2(v) = {} & d_1 P^{q/(q-1)} + d_2 P^{1/(q-1)} + d_3 P^{(1+q)/(q-1)} + \\ & + d_4 P^{(2+q)/(q-1)} + d_5 P^{2/(q-1)} + d_6 P^{(2-q)/(q-1)}, \end{aligned} \tag{44}$$

where $P = [1 + (1-q)av]$ and the coefficients c_i and d_i are constants depending on q. Note that the above terms above can also be written in terms of the q-exponential of Eq. (3) as follows:

$$g_0(v) = [\exp_q(av)]^{-q}, \tag{45}$$

$$g_1(v) = c_1[\exp_q(av)]^{-q} + c_2[\exp_q(av)]^{-1} + c_3[\exp_q(av)]^{-(1+q)}, \tag{46}$$

$$\begin{aligned} g_2(v) = {} & d_1[\exp_q(av)]^{-q} + d_2[\exp_q(av)]^{-1} + d_3[\exp_q(av)]^{-(1+q)} + \\ & + d_4[\exp_q(av)]^{-(2+q)} + d_5[\exp_q(av)]^{-2} + d_6[\exp_q(av)]^{-(2-q)}. \end{aligned} \tag{47}$$

As we observe from these expressions, all terms composing $g(v)$ to second order in (b/a) are bounded as $|v| \to \infty$ for $0 < q < 1$.

Hence, considering $f(v)$ given by Eqs. (23)–(26) together with the above results for $g(v)$ we have one has

$$\begin{aligned} f(v)g(v) = {} & \left\{\exp_q(av) + \frac{b}{qa}\left([\exp_q(av)]^q - 1\right) + \cdots\right\}\left\{[\exp_q(av)]^{-q}\right. \\ & \left. + \frac{b}{a}\left(c_1[\exp_q(av)]^{-q} + c_2[\exp_q(av)]^{-1} + c_3[\exp_q(av)]^{-(1+q)}\right) + \cdots\right\}, \end{aligned} \tag{48}$$

from which one identifies the solution found in Ref. 5 as the leading-order contribution, $f(v)g(v) = [\exp_q(av)]^{(1-q)}$, leading to $f(v)g(v) + f^*(v)g^*(v) = 2$. In this case, one has a constant probability density in Eq. (4), $\rho(x,t) = 1/\Omega\ \forall(x,t)$, so that the continuity equation is satisfied trivially, i.e., $(\partial\rho)/(\partial t) = 0$. From Eq. (48) one sees that some higher order terms [of order (b/a)] also lead to constant values for $\rho(x,t)$, whereas other contributions yield terms that may approach zero as $|v| \to \infty$ (e.g., $[\exp_q(av)]^{-q}$, which converges as $|v| \to \infty$, for $0 < q < 1$); similar behavior may be found for terms of $O[(b/a)^2]$, as well as in higher-order contributions. However, due to the constant contributions for $\rho(x,t)$, we encounter again the same difficulty found in the standard plane wave (case $q = 1$), of non-integrability of the probability density in the whole domain $-\infty < x < \infty$, so that one has to restrict calculations to a finite interval Ω, in order to satisfy the normalization condition of Eq. (5).

In Sec. III, we will explore other type of solutions for Equations (1) and (2), namely, those characterized by the property of separation of variables. We will consider more general solutions than those achieved in Refs. 8 and 11, which recover the previous ones as particular limits.

III. SEPARATED VARIABLE SOLUTIONS OF THE NLSE

A. Solutions of the principal NLSE

Let us now attempt to solve the nonlinear Schroedinger equation (1) by separating variables, looking for solutions of the form

$$\Psi(x,t) = \mathcal{G}(t)\mathcal{F}(x), \tag{49}$$

where, as usual, x and t $(t > 0)$ are real variables, representing respectively, position and time. Inserting this expression in Eq. (1) and equating each side of the equation that depends on a different variable to $-\varepsilon$ ($\varepsilon > 0$), we obtain

$$\frac{-i\hbar}{\mathcal{G}^{2-q}}\frac{d\mathcal{G}}{dt} = \frac{\hbar^2}{2m\mathcal{F}}\frac{d}{dx}(\mathcal{F}^{1-q}\mathcal{F}'(x)) = -\varepsilon, \tag{50}$$

where $\mathcal{F}' = d\mathcal{F}/dx$. Solving the time-dependent equation in (50), one immediately gets

$$\mathcal{G}(t) = \exp_{2-q}(-i\varepsilon t/\hbar) = \left[1 - (q-1)\frac{i\varepsilon t}{\hbar}\right]^{1/(q-1)}, \tag{51}$$

with $\mathcal{G}(0) = 1$. Interestingly enough, this solution is bounded as $t \to \infty$ if and only if $q < 1$! Its real and imaginary parts are oscillatory functions with magnitude $\rho_q(t) = \left[1 + (q-1)^2(\varepsilon t/\hbar)^2\right]^{1/2(q-1)}$.

On the other hand, the x-equation in Eq. (50),

$$(1-q)\frac{\mathcal{F}'^2}{\mathcal{F}} + \mathcal{F}'' = -\frac{2m\varepsilon}{\hbar^2}\mathcal{F}^q, \tag{52}$$

possesses the following solution:[8,11]

$$\mathcal{F}(x) = \left[1 + \frac{(1-q)}{2}\, iKx\right]^{2/(1-q)} = \exp_{(q+1)/2}(iKx), \quad K = \sqrt{\frac{4m\varepsilon}{(3-q)\hbar^2}}. \tag{53}$$

B. Separated variable solutions of the associated equation

Let us now turn to the associated equation (2) to see what type of second field $\Phi(x,t)$ corresponds to the separated variable solutions of Eq. (1), we have just proposed in Eq. (49). Using the functions $\mathcal{F}(x)$ and $\mathcal{G}(t)$ of Subsection III A [cf. Eq. (49)], Eq. (2) becomes

$$i\hbar\frac{\partial\Phi(x,t)}{\partial t} = \frac{\hbar^2}{2m}[\mathcal{F}(x)\mathcal{G}(t)]^{1-q}\frac{\partial^2\Phi(x,t)}{\partial x^2}. \tag{54}$$

Seeking again separated variable solutions of the form $\Phi(x,t) = F(x)G(t)$, the probability density of Eq. (4) may be written as

$$\rho(x,t) = \frac{1}{2A(L)}\left\{\mathcal{G}(t)G(t)\mathcal{F}(x)F(x) + [\mathcal{G}(t)G(t)\mathcal{F}(x)F(x)]^*\right\}, \tag{55}$$

where now the normalization factor $A(L)$ depends on a typical length of the system L. In terms of these types of solutions, Eq. (54) becomes

$$\frac{i\hbar}{\mathcal{G}^{1-q}G}\frac{dG}{dt} = \frac{\hbar^2}{2mF}(\mathcal{F}^{1-q}F''(x)) = -\varepsilon', \tag{56}$$

where ε' is an arbitrary quantity with dimensions of energy. Solving the t-dependent equation yields

$$G(t) = \left[\exp_{2-q}(-i\varepsilon t/\hbar)\right]^{-\varepsilon'/\varepsilon} = [\mathcal{G}(t)]^{-\varepsilon'/\varepsilon}. \tag{57}$$

The equation for the $F(x)$ part,

$$\frac{\hbar^2}{2mF}(\mathcal{F}^{1-q}F''(x)) = -\varepsilon', \tag{58}$$

may be solved through the ansatz $\mathcal{F}(x) = [F(x)]^{2/\gamma}$, see Eq. (53), where γ is a real number. The above solution holds provided the following condition is satisfied:

$$\gamma(\gamma + q - 1)\varepsilon = 2(3-q)\varepsilon', \tag{59}$$

relating the energies ε and ε'. It is important to notice that the solution above differs from the one of Ref. 8 through the parameters ε' and γ; this previous solution is recovered by choosing $\gamma = -2$, so that $\varepsilon' = \varepsilon$ and

$$G(t) = [\mathcal{G}(t)]^{-1}, \quad F(x) = [\mathcal{F}(x)]^{-1} \;\Rightarrow\; \Psi(x,t)\Phi(x,t) = \mathcal{F}(x)\mathcal{G}(t)F(x)G(t) = 1, \tag{60}$$

leading to a constant probability density in Eq. (55), i.e., $\rho(x,t) = 1/A(L)$.

Since Eq. (59) is quadratic in γ, one may still have another solution in the case $\varepsilon' = \varepsilon$. In fact, these two solutions are $\gamma = -2$ and $\gamma = 3 - q$; the later one leads to

$$G(t) = [\mathcal{G}(t)]^{-1}; \quad F(x) = [\mathcal{F}(x)]^{(3-q)/2} \;\Rightarrow\; \Psi(x,t)\Phi(x,t) = \mathcal{F}(x)F(x) = [\mathcal{F}(x)]^{(5-q)/2}, \tag{61}$$

so that the probability density of Eq. (55) becomes time-independent,

$$\rho(x,t) = \frac{1}{A(L)}\,\Re[\mathcal{F}(x)F(x)] = \frac{1}{A(L)}\,\Re\left[\left(1 + \frac{(1-q)}{2}\,i\,Kx\right)^{(5-q)/(1-q)}\right], \tag{62}$$

where $\Re[..]$ stands for the real part of [..].

One should notice that Eq. (59) may also be considered in more general situations where $\varepsilon' = f(q)\varepsilon$, conditioned to $f(q=1) = 1$, in such a way to recover the expected results in the limit $q = 1$. In Subsection III C, we explore one of these cases, which presents interesting results, to be discussed next.

C. Probability density for a particular solution

Let us now explore the above solutions by considering a choice for γ different from that of Ref. 8, namely, $\gamma = q - 3$, which yields $\varepsilon' = (2-q)\varepsilon$ in Eq. (59). This particular choice has two important properties: (i) It recovers the case $\varepsilon' = \varepsilon$ for $q = 1$; (ii) It relates the two energies ε and ε' through the factor $(2-q)$, which is very common in nonextensive statistical mechanics, usually considered as a duality $q \leftrightarrow (2-q)$, under which several physical properties remain invariant. With this choice, one obtains

$$G(t) = [\mathcal{G}(t)]^{q-2} \quad\Rightarrow\quad \mathcal{G}(t)G(t) = [\mathcal{G}(t)]^{q-1} = 1 - (q-1)\frac{i\varepsilon t}{\hbar}, \tag{63}$$

as well as,

$$\mathcal{F}(x)F(x) = \left[1 + \frac{(1-q)}{2}\,i\,K\,x\right]^{-1} = \frac{1 - \frac{(1-q)}{2} i\,K\,x}{1 + \frac{(1-q)^2}{4}\,K^2\,x^2}, \tag{64}$$

see Eq. (53). The probability density of Eq. (55) becomes

$$\rho(x,t) = \frac{1}{A(L)}\,\frac{1 + (q-1)(\varepsilon/\hbar)\sqrt{\xi_q(\varepsilon)}\,xt}{1 + \xi_q(\varepsilon)x^2}, \qquad \xi_q(\varepsilon) = \frac{(q-1)^2}{4}\,K^2, \tag{65}$$

which recovers $\rho(x,t) = 1/A(L)$ for $q = 1$, whereas $\rho(0,0) = 1/A(L)$ for any q.

Let us now restrict ourselves to $1 < q < 3$. Due to its odd part in x, the probability density in Eq. (65) exhibits the important property of integrability over a symmetric interval for all t, since

$$\int_{-L/2}^{L/2} dx\,\rho(x,t) = \frac{1}{A(L)}\int_{-L/2}^{L/2} \frac{dx}{1 + \xi_q(\varepsilon)x^2} = \frac{2}{\sqrt{\xi_q(\varepsilon)}\,A(L)}\tan^{-1}\left(\frac{\sqrt{\xi_q(\varepsilon)}\,L}{2}\right) = 1 \qquad (\forall t), \tag{66}$$

using the normalization condition

$$A(L) = \frac{2}{\sqrt{\xi_q(\varepsilon)}}\tan^{-1}\left(\frac{\sqrt{\xi_q(\varepsilon)}\,L}{2}\right). \tag{67}$$

In the limit $L \to \infty$, the above normalization factor becomes

$$\lim_{L\to\infty} A(L) = \frac{\pi}{\sqrt{\xi_q(\varepsilon)}} = \frac{2\pi}{q-1}\sqrt{\frac{(3-q)\hbar^2}{4m\varepsilon}}, \tag{68}$$

which is finite for any $1 < q < 3$, but diverges as $q \to 1$. One should recall the analogy between the above result and the integrability of q-plane wave solution found in Ref. 4; although the probability density of Eq. (4) is not integrable over all space, the q-plane wave of Ref. 4 is square integrable over all space for $1 < q < 3$, since the integral of $|\Psi(x,t)|^2$ is finite.[5,16]

In Fig. 1 we exhibit three-dimensional plots of the probability density in Eq. (65) for $x \geq 0$ and some typical values of q, namely, $q = 5/4$ [Fig. 1(a)], $q = 3/2$ [Fig. 1(b)], $q = 2$ [Fig. 1(c)],

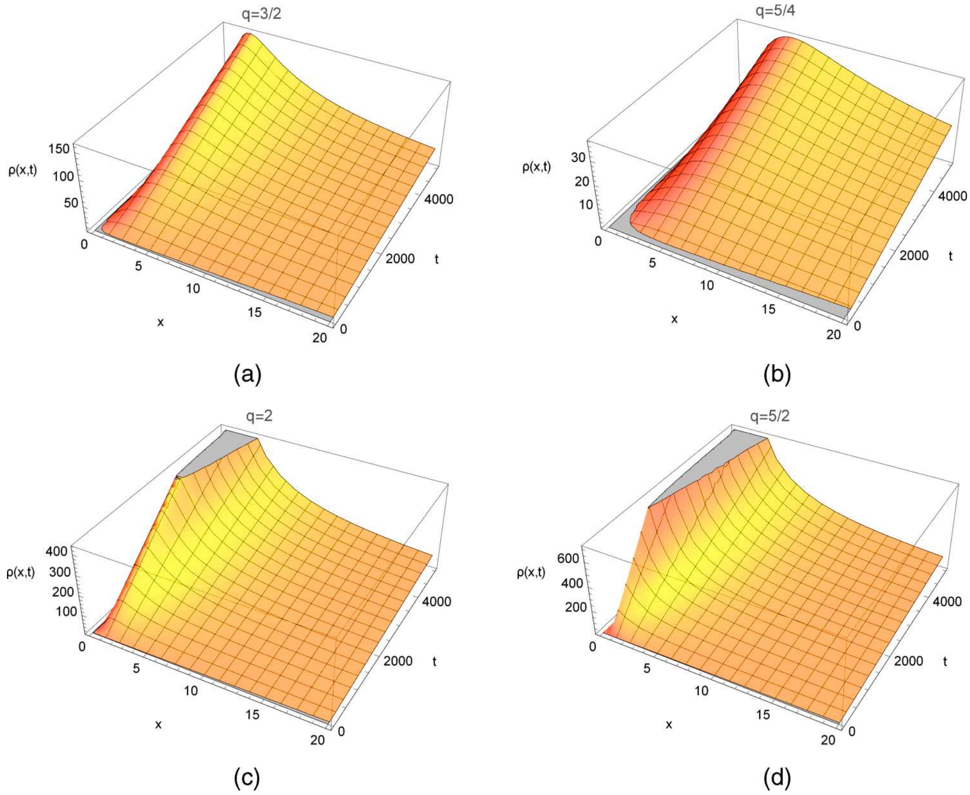

FIG. 1. The probability density $\rho(x,t)$ of Eq. (65) is represented in three-dimensional plots for increasing values of q in the range $1 < q < 3$. For convenience, we have considered $m = \hbar = \varepsilon = 1$, so that all quantities represented, namely, x, t, and $\rho(x,t)$, become dimensionless.

and $q = 5/2$ [Fig. 1(d)]. In all cases, the initial distribution $\rho(x,0)$ is a Lorentzian and it evolves in time, becoming steeper near the origin for larger times. This behavior becomes more pronounced for larger values of q, so that the probability for finding the particle near $x = 0$ approaches a delta function.

The normalization property of Eq. (66) ensures the preservation of the norm for any symmetric interval, $-L/2 \leq x \leq L/2$, and all times,

$$\left(\frac{\partial}{\partial t}\right) \int_{-L/2}^{L/2} dx\, \rho(x,t) = 0\,, \qquad (\forall t), \tag{69}$$

so that by imposing normalization for $\rho(x,0)$, we have normalized $\rho(x,t)$ for all later times.

Moreover, for finite values of L, one can calculate the average position of the particle,

$$\langle x \rangle = \int_{-L/2}^{L/2} dx\, x\rho(x,t) = \frac{q-1}{A(L)}\frac{\varepsilon}{\hbar}\sqrt{\xi_q(\varepsilon)}\, t \int_{-L/2}^{L/2} \frac{x^2 dx}{1+\xi_q(\varepsilon)x^2}, \tag{70}$$

which leads to

$$\langle x \rangle = (q-1)\frac{\varepsilon}{\hbar}\frac{1}{\sqrt{\xi_q(\varepsilon)}}\left[\frac{\sqrt{\xi_q(\varepsilon)}\,L}{2}\left(\tan^{-1}\left(\frac{\sqrt{\xi_q(\varepsilon)}\,L}{2}\right)\right)^{-1} - 1\right] t\,, \tag{71}$$

showing a linear increase with time. Finally, we may consider L large (but still finite), so that $\tan^{-1}\left(\sqrt{\xi_q(\varepsilon)}\,L/2\right)$ can be expanded in powers of $1/L$. Thus, one gets

$$\langle x \rangle = (q-1)\frac{\varepsilon}{\hbar}\frac{1}{\sqrt{\xi_q(\varepsilon)}}\left[\frac{\sqrt{\xi_q(\varepsilon)}\,L}{\pi} + \left(\frac{4}{\pi^2} - 1\right) + \frac{16}{\pi^3}\frac{1}{\sqrt{\xi_q(\varepsilon)}\,L} + \cdots\right] t\,, \tag{72}$$

which in units of the length L leads to

$$\frac{\langle x\rangle}{L} = (q-1)\frac{\varepsilon}{\hbar}\left[\frac{1}{\pi} + \left(\frac{4}{\pi^2}-1\right)\frac{1}{\sqrt{\xi_q(\varepsilon)}\,L} + \frac{16}{\pi^3}\frac{1}{\xi_q(\varepsilon)\,L^2} + \cdots\right] t\,. \tag{73}$$

Thus, for sufficiently large sizes L, one finds to leading order,

$$\frac{\langle x\rangle}{L} \approx (q-1)\frac{\varepsilon t}{\pi\hbar}\,, \tag{74}$$

showing that, for any $1 < q < 3$, the average position (in units of the length L) increases linearly with εt.

One should notice that Eqs. (71)–(74) may be rewritten in the form $\langle x\rangle = v_p t$, from which one identifies a particle "velocity," presenting the following properties: (i) $v_p > 0$ for $q > 1$, whereas $v_p = 0$ in the limit $q = 1$; (ii) v_p increases linearly with both ε and L. The increase of the velocity with ε is expected, whereas its enlargement with L appears as an intriguing property. This may be interpreted as if the particle recognizes the finite size of the system; one should call the attention to the fact that in order to obtain a finite integral in Eq. (70) the size L has to remain finite, and so the velocity v_p cannot grow indefinitely.

In Sec. IV we discuss the solutions we have obtained thus far in the context of a continuity equation.

IV. CONTINUITY EQUATION

In general, the probability density of Eq. (4), constructed from the solutions of Equations (1) and (2), satisfies the balance equation,[8]

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = R, \tag{75}$$

where

$$\vec{J} = \frac{i\hbar}{4mA(\Omega)}\left[-\Psi^{1-q}(\vec{\nabla}\Psi)\Phi + (\Psi^*)^{1-q}(\vec{\nabla}\Psi^*)\Phi^* + \Psi^{2-q}(\vec{\nabla}\Phi) - (\Psi^*)^{2-q}(\vec{\nabla}\Phi^*)\right], \tag{76}$$

and

$$R = \frac{i(1-q)\hbar}{4mA(\Omega)}\left[\Psi^{1-q}(\vec{\nabla}\Psi)\cdot(\vec{\nabla}\Phi) - (\Psi^*)^{1-q}(\vec{\nabla}\Psi^*)\cdot(\vec{\nabla}\Phi^*)\right]. \tag{77}$$

One should notice that, considering the q-plane wave as a solution of Eq. (1), i.e., $\Psi(\vec{x},t) = \exp_q(iu)$, the field $\Phi(\vec{x},t) = [\Psi(\vec{x},t)]^{-q}$ is a solution of Eq. (2), leading to the probability density $\rho(\vec{x},t) = 1/A(\Omega)$ for all values of q, and consequently, satisfying the continuity equation.[5] This represents a peculiar property of the q-plane-wave solution, since for general solutions one has $R \neq 0$, for $q \neq 1$. The fact that $R \neq 0$ implies the breakdown of the continuity equation, so that the probability density $\rho(\vec{x},t)$ is not conserved in time; in this case, the solutions for the fields $\Psi(\vec{x},t)$ and $\Phi(\vec{x},t)$ cannot be used within the standard probabilistic context of quantum mechanics.[1] However, even in such cases, one can still consider Equations (1) and (2) as describing physical phenomena, which do not require the conservation of $\rho(\vec{x},t)$.

Indeed, most of the solutions we have presented here do not satisfy the continuity equation in general. However, in some special cases it is possible to verify $R = 0$:

(i) The leading order terms of Eq. (39), corresponding to the expansion in the case $|a/b| < 1$, remarkably *vanish to second order in* (a/b),

$$\begin{aligned} R &= \frac{i(1-q)\hbar}{4mA(\Omega)}\left[f^{1-q}(v)f'(v)g'(v) - f^{1-q}(-v)f'(-v)g'(-v)\right] \\ &= \frac{i(1-q)\hbar}{4mA(\Omega)}\left[1 - 1 + O[(a/b)^3]\right], \end{aligned} \tag{78}$$

where, as usual, the primes denote differentiations with respect to v. It would be interesting to study further this result to show whether continuity is indeed obeyed to all orders.

(ii) Also in the expansion for $|b/a| < 1$ [cf. Eq. (48)] one has that the leading-order contribution gives $f(v)g(v) = [\exp_q(av)]^{(1-q)}$, yielding $f(v)g(v) + f^*(v)g^*(v) = 2$, in agreement with Ref. 5. Moreover, some higher-order terms give constant values, so that the continuity equation follows trivially for these terms.

(iii) In the case of separated variables one has that the solution of Ref. 8, corresponding to $\gamma = -2$, which yields $\varepsilon' = \varepsilon$ in Eq. (59), leads to a constant probability density in Eq. (55), i.e., $\rho(x,t) = 1/A(L)$. However, the interesting choice corresponding to $\varepsilon' = (2-q)\varepsilon$ yields the preservation of the norm for all times [cf. Eq. (69)].

V. CONCLUSIONS

Recently, it has been shown that it is possible to generalize the Schroedinger, Klein-Gordon, and Dirac equations of relativistic and quantum physics, by introducing a real parameter q in such a way that the new equations (a) become nonlinear and reduce to their classical linear version in the limit $q \to 1$, (b) conserve the Planck, de Broglie, and Einstein energy-momentum relations for all q, and, most importantly, (c) possess plane wave solutions expressed in terms of an oscillating q-exponential function whose amplitude vanishes at infinity by a power law.

In this paper, we have sought to obtain new solutions of the q-generalized Schroedinger equation for the main field variable $\Psi(\vec{x},t)$ in the form of travelling waves and separated variables, using the formalism of q-exponential functions and restricting our study to one spatial dimension x and time t. We also found, in every case, corresponding solutions of the associated equations for a second field variable $\Phi(\vec{x},t)$, required by the Euler-Lagrange equations of classical field theory.

We were able to construct two types of travelling wave solutions of the nonlinear Schroedinger equation which vanish at infinity. Considering the main field in the form $\Psi(x,t) = f(v) = f(i(kx - \omega t))$, we have shown that these types of solutions obey the equation

$$\frac{df}{dv} = af^q + bf^{q-1}, \tag{79}$$

where b is an integration constant, whereas $a = (\hbar\omega)/[(\hbar^2k^2)/(2m)]$ appears as a dimensionless ratio between two typical energies, namely, the quantum of radiation and the kinetic energy of the particle. These types of solutions are more general than the previous q-plane wave of Refs. 4 and 5 [recovered from Eq. (79) as the particular case $b = 0$].

Although the general solutions of Eq. (79) can be expressed in terms of hypergeometric functions, we have explored these solutions perturbatively, in both cases $|a/b| < 1$ and $|b/a| < 1$, the latter representing perturbations around the q-plane wave solution. Our first discovery was that, when we attempted to develop these solutions as perturbations of the corresponding q-plane waves, we found terms of the expansion that, as x and/or t go to infinity, either become unbounded or tend to a non-zero constant, implying that there are directions in solution space where these q-plane waves are unstable. It is important to emphasize, however, that these results were obtained for expansions with terms up to second order and, therefore, it would be very interesting to study them at higher orders to prove the convergence of the corresponding series. Even, in the case of non-convergence, the solutions may still be considered on a finite x-interval, as is usually done for the particular case $q = 1$.

Finally, a new class of separated variable solutions was obtained, which vanish at infinity and may thus yield integrable probability densities. These solutions were achieved by considering an energy ε, associated with the Schroedinger equation for the main field variable $\Psi(\vec{x},t)$, and an energy ε', associated with the equation for the second field variable $\Phi(\vec{x},t)$, where in principle, $\varepsilon' \neq \varepsilon$; the choice $\varepsilon' = (2-q)\varepsilon$ was analyzed in detail. Considering $\varepsilon' \neq \varepsilon$ seems to be a natural alternative, since these energies come from two non-symmetric equations and may possibly be associated with two strongly correlated particles. One should recall that the previous separated variable solutions were restricted to the simpler case $\varepsilon' = \varepsilon$.[8]

We, therefore, conclude that it would be very interesting to explore these new solutions in terms of their physical relevance. One way to do this would be to examine their stability properties under time evolution, by solving either perturbatively or numerically the full nonlinear q-generalized

Schroedinger equation. Although the solutions found herein do not generally satisfy the continuity equation (only in some particular cases), they may still be useful for a wide variety of phenomena that do not require conservation of probability, such as follows: (a) creation or annihilation processes that usually occur with particles while they propagate in space, typical of many-body systems; (b) decrease or growth of populations, like those that take place in biological systems.

ACKNOWLEDGMENTS

We acknowledge many useful conversations with Professor C. Tsallis. One of us (T.B.) is grateful for the support provided by the research project MACOMSYS co-financed by the European Union (European Social Fund = ESF) and Greek National Funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: *THALES. Investing in knowledge society through the European Social Fund.* F.D.N. thanks partial financial support from CNPq and FAPERJ (Brazilian funding agencies).

[1] R. L. Liboff, *Introductory Quantum Mehanics*, 4th ed. (Addison Wesley, San Francisco, 2003).
[2] C. Sulem and P.-L. Sulem, *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse* (Springer, New York, 1999).
[3] A. C. Scott, *The Nonlinear Universe* (Springer, Berlin, 2007).
[4] F. D. Nobre, M. A. Rego-Monteiro, and C. Tsallis, Phys. Rev. Lett. **106**, 140601 (2011).
[5] F. D. Nobre, M. A. Rego-Monteiro, and C. Tsallis, Europhys. Lett. **97**, 41001 (2012).
[6] M. A. Rego-Monteiro and F. D. Nobre, J. Math. Phys. **54**, 103302 (2013).
[7] A. R. Plastino and C. Tsallis, J. Math. Phys. **54**, 041505 (2013).
[8] A. R. Plastino, A. M. C. Souza, F. D. Nobre, and C. Tsallis, Phys. Rev. A **90**, 062134 (2014).
[9] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics* (Springer, New York, 2009).
[10] E. P. Borges, J. Phys. A **31**, 5281 (1998).
[11] A. Plastino and M. C. Rocca, Phys. Lett. A **379**, 2690 (2015).
[12] I. V. Toranzo, A. R. Plastino, J. S. Dehesa, and A. Plastino, Physica A **392**, 3945 (2013).
[13] S. Curilef, A. R. Plastino, and A. Plastino, Physica A **392**, 2631 (2013).
[14] F. Pennini, A. R. Plastino, and A. Plastino, Physica A **403**, 195 (2014).
[15] C. Tsallis, G. Bemski, and R. S. Mendes, Phys. Lett. A **257**, 93 (1999).
[16] M. Jauregui and C. Tsallis, J. Math. Phys. **51**, 063304 (2010).
[17] Herein we adopt the notation of Refs. 5 and 6, where the fields $\Psi(\vec{x}, t)$ and $\Phi(\vec{x}, t)$ represent dimensionless quantities, rescaled by their corresponding amplitudes.