MINI-CURSO INTENSIVO - Centro Brasileiro de Pesquisas Físicas
Coordenação de Formação Científica
Rio de Janeiro, 2 a 6 de abril de 2007

MECÂNICA ESTATÍSTICA NÃO EXTENSIVA:
Aspectos Teóricos, Experimentais, Observacionais e Computacionais

Coordenação: Prof. Constantino Tsallis

Destinado a estudantes de pós-graduação e pesquisadores de Física, Matemática, Química, Economia, Biologia, Ciências Computacionais e áreas correlatas.
Na definição da CAPES, vale 2 créditos de Pós-Graduação.
Ajuda financeira para alguns dos participantes é possível.
Inscrição: contatar a Secretaria do mini-curso cema@cbpf.br
Informações gerais: http://www.cbpf.br/NextCurso2007
Bibliografia: http://tsallis.cat.cbpf.br/biblio.htm

$$S_q = k \frac{1-\sum_i p_i^q}{q-1}$$

1ª. parte (16 horas, em Português)
Prof. Constantino Tsallis (CBPF, Brasil)

2ª. parte (8 horas, em Inglês)
Prof. Alberto Robledo (UNAM, México)

3ª. parte (8 horas, em Inglês)
Prof. Andrea Rapisarda (Universidade de Catânia, Itália)

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MECANICA ESTATISTICA NAO EXTENSIVA

ASPECTOS TEORICOS, EXPERIMENTAIS, OBSERVACIONAIS E COMPUTACIONAIS

Constantino Tsallis

Centro Brasileiro de Pesquisas Fisicas

Rio de Janeiro - Brasil

Rio de Janeiro, CBPF, Abril 2007

EXTENSIVITY OF THE NONADDITIVE ENTROPY Sq

and q - GENERALIZED CENTRAL LIMIT THEOREM:

C. T., M. Gell-Mann and Y. Sato, Proc Natl Acad Sci (USA) **102**, 15377 (2005)

C. T., M. Gell-Mann and Y. Sato, Europhysics News **36**, 186 (2005)

L.G. Moyano, C. T. and M. Gell-Mann, Europhys Lett **73**, 813 (2006)

S. Umarov, C. T., M. Gell-Mann and S. Steinberg,
cond-mat/0603593, 0606038, 0606040 and 0703533 (2006, 2007)

J.A. Marsh, M.A. Fuentes, L.G. Moyano and C. T, Physica A **372**, 183 (2006)

U. Tirnakli, C. Beck and C. T., Phys Rev E /Rapid Comm (2007), in press

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EXPERIMENTAL VERIFICATION IN COLD ATOMS:

P. Douglas, S. Bergamini and F. Renzoni, Phys Rev Lett **96**, 110601 (2006)

Sabir Umarov

Stanly Steinberg

Miguel Fuentes

Yuzuru Sato

Ugur Tirnakli

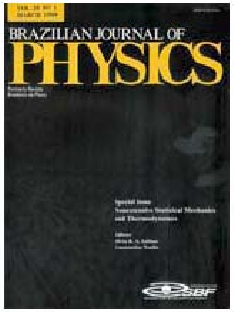
Silvio M.D. Queiros

Luis Moyano John Marsh Paul Rivkin Kenric Nelson William Thistleton

Christian Beck

E.G.D. Cohen

NONEXTENSIVE STATISTICAL MECHANICS AND THERMODYNAMICS



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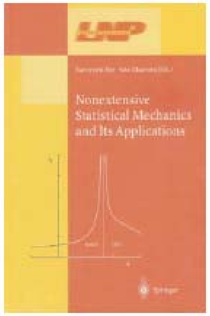

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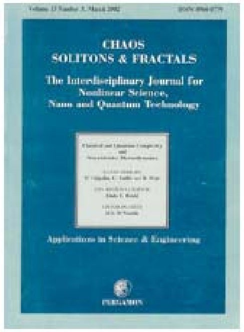

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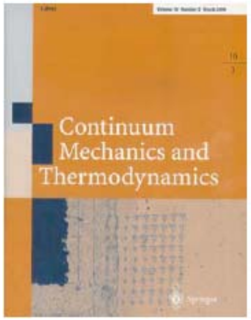

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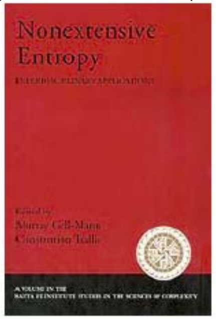

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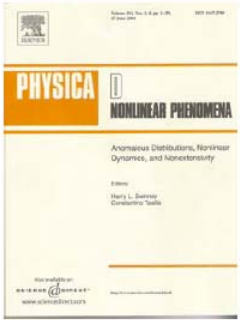

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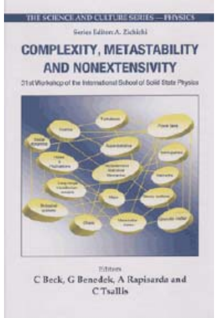
News and Expectations in Thermostatistics G Kaniadakis and M Lissia, eds Physica A **340**, Issue 1/3 (**2004**)

Trends and Perspectives in Extensive and Non-Extensive Statistical Mechanics H Herrmann, M Barbosa and E Curado, eds, Physica A **344**, Issue 3/4 (**2004**)



Complexity, Metastability and Nonextensivity, C Beck, G Benedek, A Rapisarda and C Tsallis, eds, (World Scientific, Singapore, **2005**)

Nonextensive Statistical Mechanics: New Trends, New Perspectives, JP Boon and C Tsallis, eds, Europhysics News (European Physical Society, **2005**)

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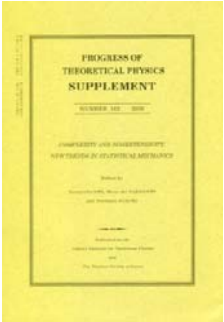

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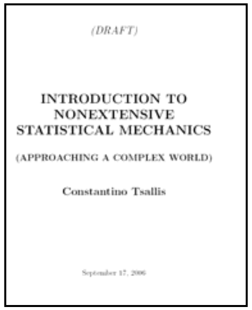


Introduction to Nonextensive Statistical Mechanics - Approaching a Complex World, C. Tsallis (in preparation)

Full bibliography (regularly updated):

http://tsallis.cat.cbpf.br/biblio.htm

2,107 articles (done by 1,550 scientists from 60 countries) which led to

> 7,520 citations of papers

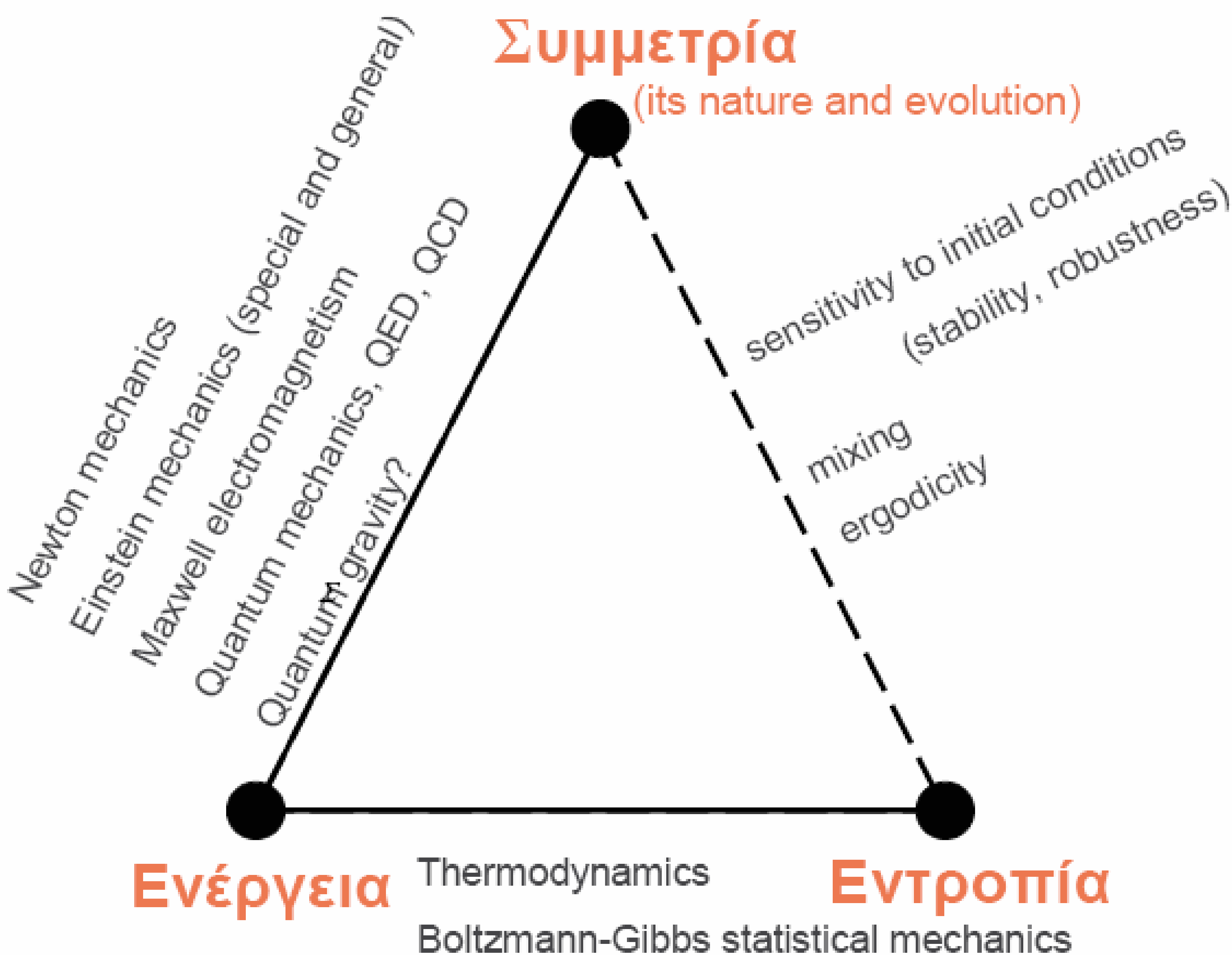
(including > 1,407 citations of the 1988 paper)

> 957 nominal citations

[31 March 2007]

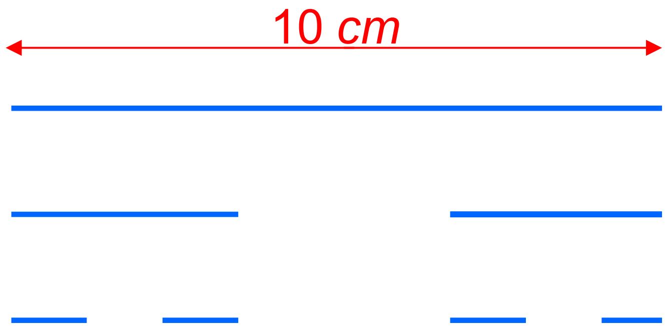
CONTRIBUTORS **(2107 MANUSCRIPTS)** [Updated 31 March 2007]

BRAZIL	303	NETHERLANDS	11	SLOVAK	3
USA	217	BELGIUM	11	CROATIA	3
ITALY	116	GREECE	11	IRELAND	3
JAPAN	105	AUSTRALIA	8	BOLIVIA	2
FRANCE	92	PORTUGAL	8	CZECK	2
CHINA	88	SOUTH AFRICA	7	FINLAND	2
ARGENTINA	79	CUBA	7	KAZAKSTAN	2
SPAIN	57	HUNGARY	7	MOLDOVA	2
GERMANY	55	IRAN	7	PHILIPINES	2
ENGLAND	40	VENEZUELA	6	PUERTO RICO	2
POLAND	37	CHILE	6	ARMENIA	1
RUSSIA	31	NORWAY	5	INDONESIA	1
TURKEY	27	SINGAPORE	4	JORDAN	1
INDIA	23	SWEDEN	4	MALAYSIA	1
CANADA	23	TAIWAN	4	SAUDI ARABIA	1
AUSTRIA	21	URUGUAY	4	SERBIA	1
MEXICO	19	ROMENIA	4	SRI LANKA	1
UKRAINE	18	EGYPT	4	UZBEKISTAN	1
SWITZERLAND	17	DENMARK	4		
ISRAEL	13	SLOVENIA	4	60 COUNTRIES	1550 SCIENTISTS
KOREA	12	BULGARIA	3		



Συμμετρία
(its nature and evolution)
Newton mechanics
Einstein mechanics (special and general)
Maxwell electromagnetism
Quantum mechanics, QED, QCD
Quantum gravity?
sensitivity to initial conditions
(stability, robustness)
mixing
ergodicity
Ενέργεια
Thermodynamics
Boltzmann-Gibbs statistical mechanics
Εντροπία

TRIADIC CANTOR SET:



$$d_F = \frac{\ln 2}{\ln 3} = 0.6309...$$

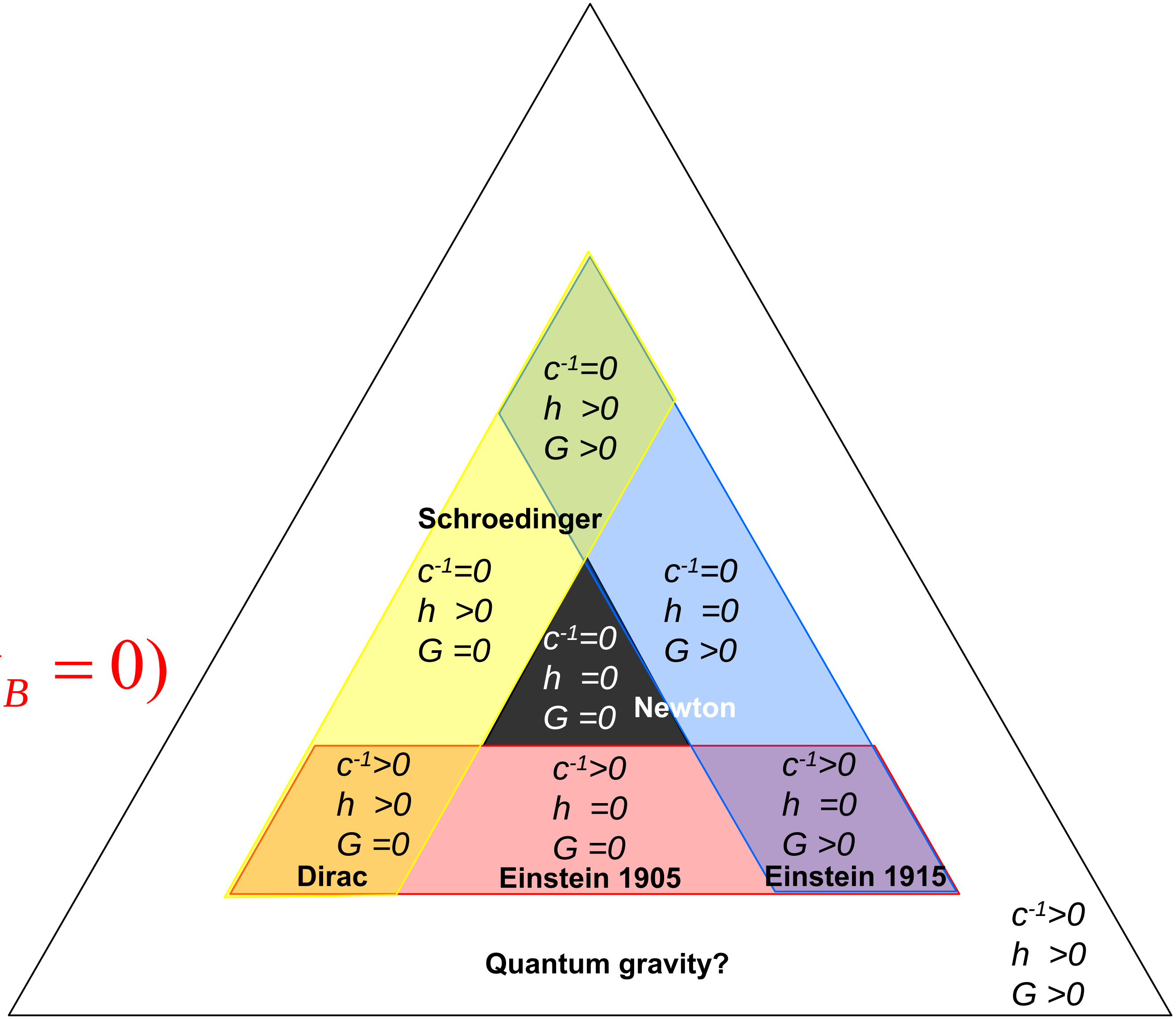
Hence the interesting measure is

$$(10\ cm)^{0.6309...} \cong 4.275\ cm^{0.6309}$$

It is the natural (or artificial or social) system itself which, through its geometrical-dynamical properties, indicates the specific informational tool --- entropy --- to be meaningfully used for the study of its thermostatistical and thermodynamical properties.

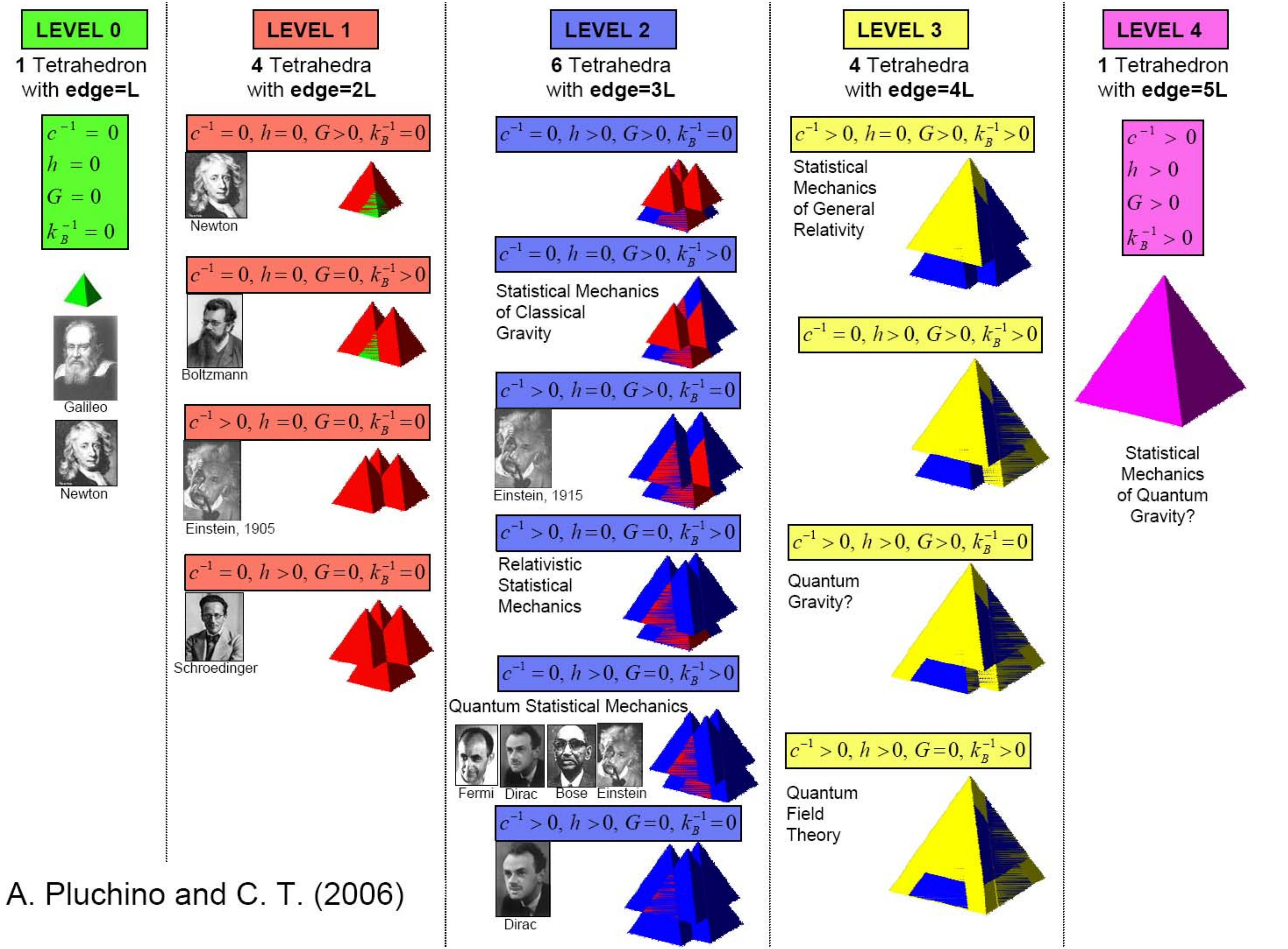
The four independent universal constants of contemporary physics: $G,\ c^{-1},\ h,\ k_B^{-1}$

$(1/k_B = 0)$

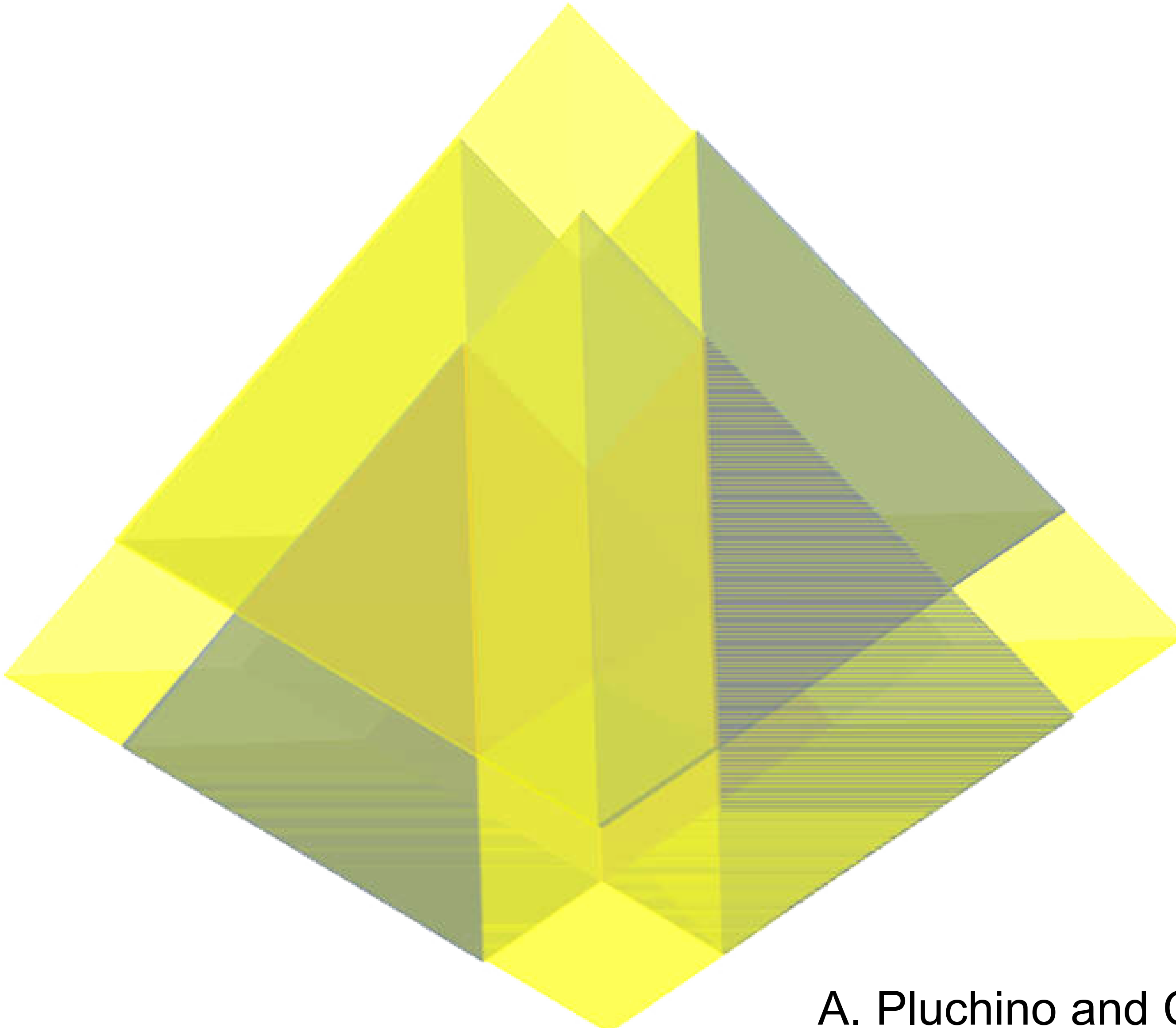


The full tetrahedron $c^{-1} > 0;\ h > 0;\ G > 0;\ k_B^{-1} > 0$ corresponds to the statistical mechanics of quantum gravity (at its center: $c^{-1} = h = G = k_B^{-1} = 0$)

C.T., *Introduction to Nonextensive Statistical Mechanics-Approaching a Complex World* (in progress)


LEVEL 0
1 Tetrahedron with **edge=L**
$c^{-1} = 0$
$h = 0$
$G = 0$
$k_B^{-1} = 0$
Galileo
Newton
LEVEL 1
4 Tetrahedra with **edge=2L**
$c^{-1}=0,\ h=0,\ G>0,\ k_B^{-1}=0$
Newton
$c^{-1}=0,\ h=0,\ G=0,\ k_B^{-1}>0$
Boltzmann
$c^{-1}>0,\ h=0,\ G=0,\ k_B^{-1}=0$
Einstein, 1905
$c^{-1}=0,\ h>0,\ G=0,\ k_B^{-1}=0$
Schroedinger
LEVEL 2
6 Tetrahedra with **edge=3L**
$c^{-1}=0,\ h>0,\ G>0,\ k_B^{-1}=0$
$c^{-1}=0,\ h=0,\ G>0,\ k_B^{-1}>0$
Statistical Mechanics of Classical Gravity
$c^{-1}>0,\ h=0,\ G>0,\ k_B^{-1}=0$
Einstein, 1915
$c^{-1}>0,\ h=0,\ G=0,\ k_B^{-1}>0$
Relativistic Statistical Mechanics
$c^{-1}=0,\ h>0,\ G=0,\ k_B^{-1}>0$
Quantum Statistical Mechanics
Fermi
Dirac
Bose
Einstein
$c^{-1}>0,\ h>0,\ G=0,\ k_B^{-1}=0$
Dirac
LEVEL 3
4 Tetrahedra with **edge=4L**
$c^{-1}>0,\ h=0,\ G>0,\ k_B^{-1}>0$
Statistical Mechanics of General Relativity
$c^{-1}=0,\ h>0,\ G>0,\ k_B^{-1}>0$
$c^{-1}>0,\ h>0,\ G>0,\ k_B^{-1}=0$
Quantum Gravity?
$c^{-1}>0,\ h>0,\ G=0,\ k_B^{-1}>0$
Quantum Field Theory
LEVEL 4
1 Tetrahedron with **edge=5L**
$c^{-1} > 0$
$h > 0$
$G > 0$
$k_B^{-1} > 0$
Statistical Mechanics of Quantum Gravity?
A. Pluchino and C. T. (2006)

Full Tetrahedron



A. Pluchino and C. T. (2006)

HISTORICAL BACKGROUND AND PHYSICAL MOTIVATIONS FOR ATTEMPTING TO GENERALIZE BOLTZMANN-GIBBS STATISTICAL MECHANICS

ALONG THE LAST 135 YEARS…

Ludwig BOLTZMANN

Vorlesungen uber Gastheorie (Leipzig, 1896)
Lectures on Gas Theory, transl. S. Brush
(Univ. California Press, Berkeley, 1964), page 13

The forces that two molecules impose one onto the other during an interaction can be completely arbitrary, only assuming that their sphere of action is very small compared to their mean free path.

J.W. GIBBS

Elementary Principles in Statistical Mechanics - Developed with Especial Reference to the Rational Foundation of Thermodynamics

C. Scribner's Sons, New York, 1902; Yale University Press, New Haven, 1981), page 35

> *In treating of the canonical distribution, we shall always suppose the multiple integral in equation (92)* [the partition function, as we call it nowadays] *to have a finite valued, as otherwise the coefficient of probability vanishes, and the law of distribution becomes illusory. This will exclude certain cases, but not such apparently, as will affect the value of our results with respect to their bearing on thermodynamics. It will exclude, for instance, cases in which the system or parts of it can be distributed in unlimited space […]. It also excludes many cases in which the energy can decrease without limit, as when the system contains material points which attract one another inversely as the squares of their distances. […]. For the purposes of a general discussion, it is sufficient to call attention to the assumption implicitly involved in the formula (92).*

Enrico FERMI *Thermodynamics* (Dover, 1936)

The entropy of a system composed of several parts is very often equal to the sum of the entropies of all the parts. This is true if the energy of the system is the sum of the energies of all the parts and if the work performed by the system during a transformation is equal to the sum of the amounts of work performed by all the parts. Notice that these conditions are not quite obvious and that in some cases they may not be fulfilled. Thus, for example, in the case of a system composed of two homogeneous substances, it will be possible to express the energy as the sum of the energies of the two substances only if we can neglect the surface energy of the two substances where they are in contact. The surface energy can generally be neglected only if the two substances are not very finely subdivided; otherwise, it can play a considerable role.

Ettore MAJORANA

The value of statistical laws in physics and social sciences.
Original manuscript in Italian published by G. Gentile Jr. in *Scientia* **36**, 58 (1942); translated into English by R. Mantegna (2005).

This is mainly because entropy is an additive quantity as the other ones. In other words, the entropy of a system composed of several independent parts is equal to the sum of entropy of each single part. [...]
Therefore one considers all possible internal determinations as equally probable. This is indeed a new hypothesis because the universe, which is far from being in the same state indefinitively, is subjected to continuous transformations. We will therefore admit as an extremely plausible working hypothesis, whose far consequences could sometime not be verified, that all the internal states of a system are a priori equally probable in specific physical conditions. Under this hypothesis, the statistical ensemble associated to each macroscopic state A turns out to be completely defined.

Claude Elwood SHANNON

The Mathematical Theory of Communication
(University of Illinois Press, Urbana, 1949)

It is practically more useful. [...]
It is nearer to our intuitive feeling as to the proper measure. [...]
It is mathematically more suitable. [...].}

This theorem and the assumptions required for its proof, are in no way necessary for the present theory. It is given chiefly to lend a certain plausibility to some of our later definitions. The real justification of these definitions, however, will reside in their implications.

Laszlo TISZA

Generalized Thermodynamics
(MIT Press, Cambridge, Massachusetts, 1961)

The situation is different for the additivity postulate Pa2, the validity of which cannot be inferred from general principles. We have to require that the interaction energy between thermodynamic systems be negligible. This assumption is closely related to the homogeneity postulate Pd1. From the molecular point of view, additivity and homogeneity can be expected to be reasonable approximations for systems containing many particles, provided that the intramolecular forces have a short range character.

Radu BALESCU

Equilibrium and Nonequilibrium Statistical Mechanics
(John Wiley and Sons, 1975, New York)

It therefore appears from the present discussion that the mixing property of a mechanical system is much more important for the understanding of statistical mechanics than the mere ergodicity. [...] A detailed rigorous study of the way in which the concepts of mixing and the concept of large numbers of degrees of freedom influence the macroscopic laws of motion is still lacking.

Peter LANDSBERG

Thermodynamics and Statistical Mechanics (1978)

The presence of long-range forces causes important amendments to thermodynamics, some of which are not fully investigated as yet.

Is equilibrium always an entropy maximum?
J. Stat. Phys. **35**, 159 (1984)

[...] in the case of systems with long-range forces and which are therefore nonextensive (in some sense) some thermodynamic results do not hold. [...] The failure of some thermodynamic results, normally taken to be standard for black hole and other nonextensive systems has recently been discussed. [...] If two identical black holes are merged, the presence of long-range forces in the form of gravity leads to a more complicated situation, and the entropy is nonextensive.

David RUELLE

Thermodynamical Formalism -
The Mathematical Structures of Classical Equilibrium Statistical Mechanics
(page 1 of both 1978 and 2004 editions)

The formalism of equilibrium statistical mechanics -- which we shall call thermodynamic formalism -- has been developed since J.W. Gibbs to describe the properties of certain physical systems. [...] While the physical justification of the thermodynamic formalism remains quite insufficient, this formalism has proved remarkably successful at explaining facts.
The mathematical investigation of the thermodynamic formalism is in fact not completed: the theory is a young one, with emphasis still more on imagination than on technical difficulties. This situation is reminiscent of pre-classic art forms, where inspiration has not been castrated by the necessity to conform to standard technical patterns.

(page 3) *The problem of why the Gibbs ensemble describes thermal equilibrium (at least for "large systems") when the above physical identifications have been made is deep and incompletely clarified.*

[The first equation is dedicated to define the *BG* entropy form. It is introduced after the words "we define its *entropy*" without any kind of justification or physical motivation.]

Nico van KAMPEN

Stochastic Processes in Physics and Chemistry
(North-Holland, Amsterdam, 1981)

Actually an additional stability criterion is needed, see M.E. Fisher, Archives Rat. Mech. Anal. ***17****, 377 (1964); D. Ruelle, Statistical Mechanics: Rigorous Results (Benjamin, New York 1969). A collection of point particles with mutual gravitation is an example where this criterion is not satisfied, and for which therefore no statistical mechanics exists.*

Roger BALIAN

From Microphysics to Macrophysics

(Springer-Verlag, Berlin, 1991), p. 205 and 206; French edition (1982).

These various quantities are connected with one another through thermodynamic relations which make their extensive or intensive nature obvious, as soon as one postulates, for instance, for a fluid, that the entropy, considered as a function of the volume Omega and of the constants of motion such as U and N, is homogeneous of degree 1: S(x Omega, x U, x N)=x S(Omega, U, N) (Eq. 5.43). [...] Two counter-examples will help us to feel why extensivity is less trivial than it looks. [...] A complete justification of the Laws of thermodynamics, starting from statistical physics, requires a proof of the extensivity (5.43), a property which was postulated in macroscopic physics. This proof is difficult and appeals to special conditions which must be satisfied by the interactions between the particles.

L.G. TAFF

Celestial Mechanics
(John Wiley, New York, 1985)

This means that the total energy of any finite collection of self-gravitating mass points does not have a finite, extensive (e.g., proportional to the number of particles) lower bound. Without such a property there can be no rigorous basis for the statistical mechanics of such a system (Fisher and Ruelle 1966). Basically it is that simple. One can ignore the fact that one knows that there is no rigorous basis for one's computer manipulations; one can try to improve the situation, or one can look for another job.

W.C. SASLAW

Gravitation Physics of Stellar and Galactic Systems
(Cambridge University Press, Cambridge, 1985)

When interactions are important the thermodynamic parameters may lose their simple intensive and extensive properties for subregions of a given system. [...] Gravitational systems, as often mentioned earlier, do not saturate and so do not have an ultimate equilibrium state.

John MADDOX

When entropy does not seem extensive
Nature **365**, 103 (1993)

Everybody who knows about entropy knows that it is an extensive property, like mass or enthalpy. [...] Of course, there is more than that to entropy, which is also a measure of disorder. Everybody also agrees on that. But how is disorder measured? [...] So why is the entropy of a black hole proportional to the square of its radius, and not to the cube of it? To its surface area rather than to its volume?

A.C.D. van ENTER, R. FERNANDEZ and A.D. SOKAL, *Regularity Properties and Pathologies of Position-Space Renormalization-Group Transformations: Scope and Limitations of Gibbsian Theory* [J. Stat. Phys. **72**, 879-1167 (1993)]

We provide a careful, and, we hope, pedagogical, overview of the theory of Gibssian measures as well as (the less familiar) non-Gibbsian measures, emphasizing the distinction between these two objects and the possible occurrence of the latter in different physical situations.

Toward a Non-Gibbsian Point of View: Let us close with some general remarks on the significance of (non-)Gibbsianness and (non)quasilocality in statistical physics. Our first observation is that Gibbsianness has heretofore been ubiquitous in equilibrium statistical mechanics because it has been put in *by hand*: nearly all measures that physicists encounter are Gibbsian because physicists have *decided* to study Gibbsian measures! However, we now know that natural operations on Gibbs measures can sometimes lead out of this class. [...] It is thus of great interest to study which types of operations preserve, or fail to preserve, the Gibbsianness (or quasilocality) of a measure. This study is currently in its infancy. [...] More generally, in areas of physics where Gibbsianness is not put in by hand, one should expect non-Gibbsianness to be ubiquitous. This is probably the case in nonequilibrium statistical mechanics. Since one cannot expect all measures of interest to be Gibbsian, the question then arises whether there are *weaker* conditions that capture some or most of the “good” physical properties characteristic of Gibbs measures. For example, the stationary measure of the voter model appears to have the critical exponents predicted (under the hypothesis of Gibbsianness) by the Monte Carlo renormaliztion group, even though this measure is provably non-Gibbsian. One may also inquire whether there is a classification of non-Gibbsian measures according to their “degree of non-Gibbsianness”.

Floris TAKENS

Structures in Dynamics –
Finite Dimensional Deterministic Studies

Eds. H.W. Broer, F. Dumortier, S.J. van Strien and F. Takens, p. 253
(North-Holland, Amsterdam, 1991)

The values of p_i *are determined by the following dogma: if the energy of the system in the i-th state is* E_i *and if the temperature of the system is* T *then:*

$$p_i = \frac{e^{-E_i/kT}}{Z(T)}, \text{ where } Z(T) = \sum_i e^{-E_i/kT} \text{ (this last constant is taken so that } \sum_i p_i = 1).$$

This choice of p_i *is called the Gibbs distribution. We shall give no justification for this dogma; even a physicist like Ruelle disposes of this question as " deep and incompletely clarified".*

The advantages of the method of postulation are great; they are the same as the advantages of theft over honest toil.

(Bertrand Russell)

ENTROPY

ENTROPIC FORMS

	$p_i = \frac{1}{W} \quad (\forall i)$ equiprobability	$\forall p_i \; (0 \le p_i \le 1)$ $\left(\sum_{i=1}^{W} p_i = 1 \right)$
***BG* entropy (*q* =*1*)**	$k \ln W$	$-k \sum_{i=1}^{W} p_i \ln p_i$
Entropy *Sq* (*q real*)	$k \frac{W^{1-q} - 1}{1-q}$	$k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}$

Concave

Extensive

Lesche-stable

Finite entropy production per unit time

Pesin-like identity (with largest entropy production)

Composable

Topsoe-factorizable

Possible generalization of Boltzmann-Gibbs statistical mechanics

[C.T., J. Stat. Phys. **52**, 479 (1988)]

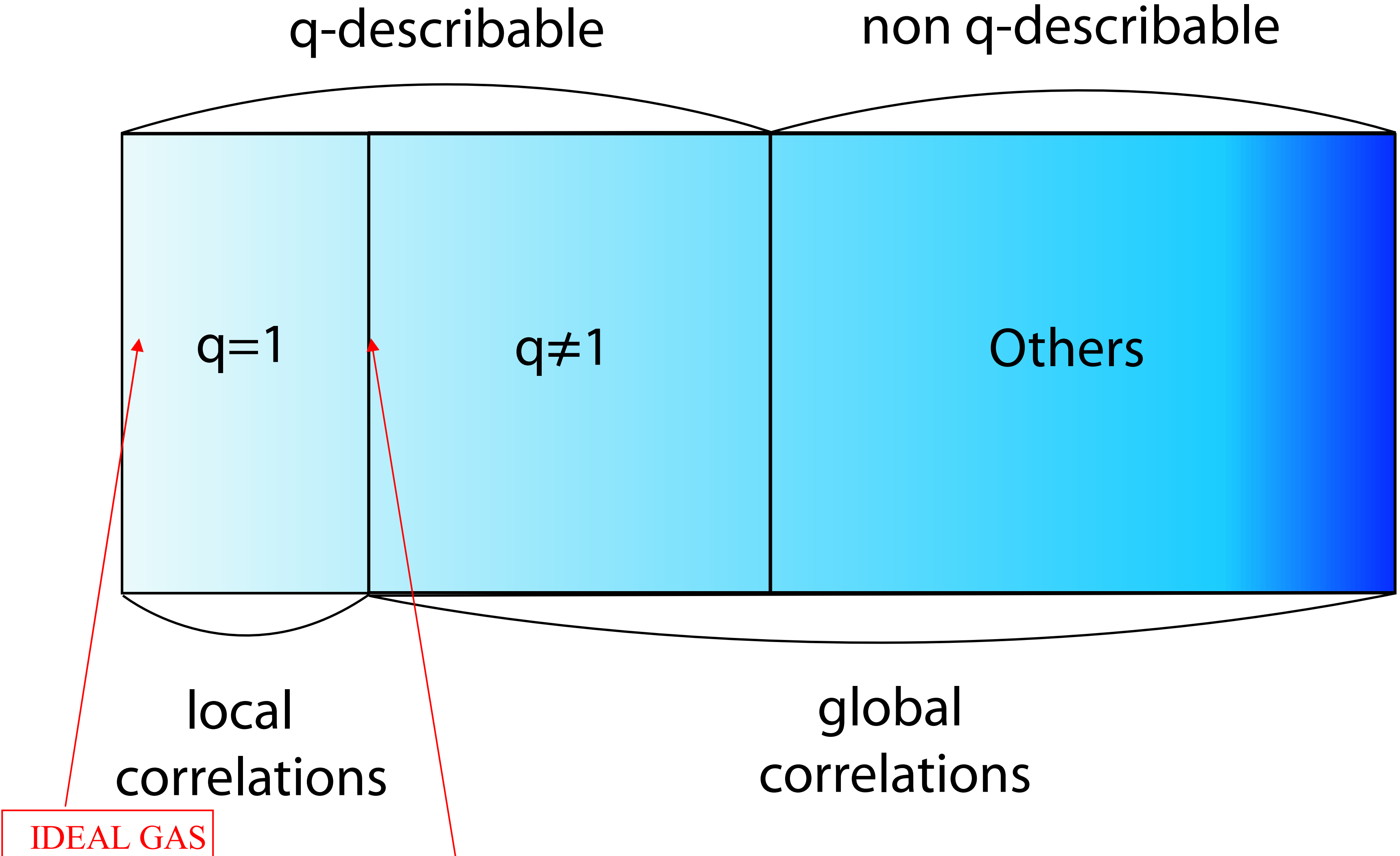
$DEFINITION : q-logarithm :$

$$\ln_q x \equiv \frac{x^{1-q}-1}{1-q} \quad (x>0)$$

$$\ln_1 x = \ln x$$

$Hence,\ the\ entropies\ can\ be\ rewritten:$

	$equal\ probabilities$	$general\ probabilities$
$BG\ entropy$ $(q=1)$	$k\ \ln W$	$k\sum_{i=1}^{W} p_i \ln\ (1/p_i)$
$entropy\ S_q$ $(q\in R)$	$k\ \ln_q W$	$k\sum_{i=1}^{W} p_i \ln_q\ (1/p_i)$

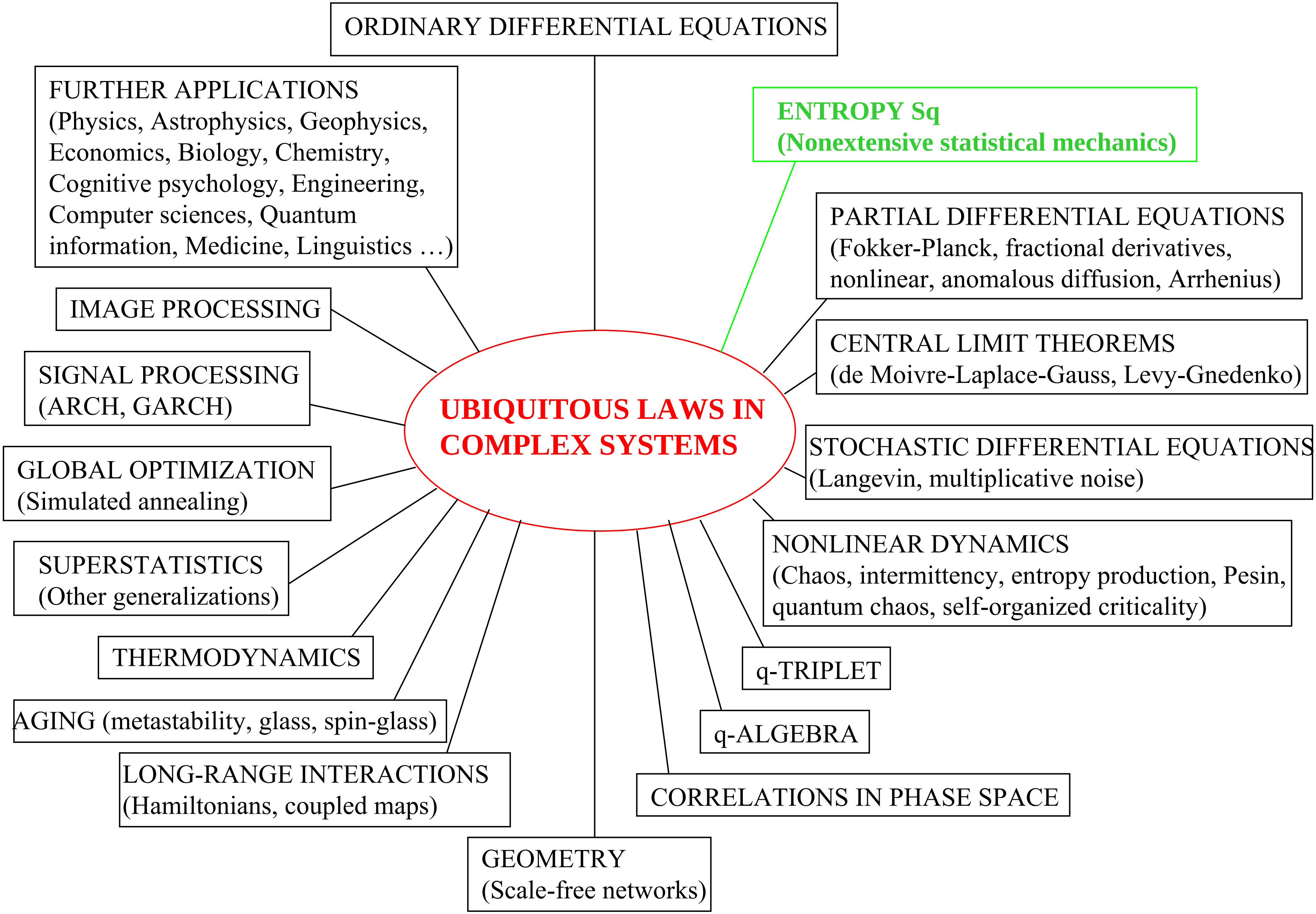


CRITICAL PHENOMENA

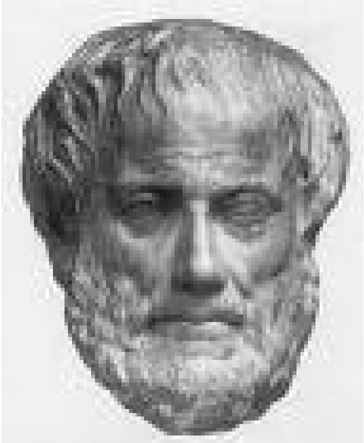
$$q = \frac{1+\delta}{2}$$ (A. Robledo, Mol Phys 103 (2005) 3025)

$$q = \frac{\sqrt{9+2c^2}-3}{c}$$ (F. Caruso and C. T., 2006)

C.T., M. Gell-Mann and Y. Sato
Europhysics News **36** (6), 186
(European Physical Society, 2005)


ORDINARY DIFFERENTIAL EQUATIONS
FURTHER APPLICATIONS (Physics, Astrophysics, Geophysics, Economics, Biology, Chemistry, Cognitive psychology, Engineering, Computer sciences, Quantum information, Medicine, Linguistics …)
ENTROPY Sq (Nonextensive statistical mechanics)
PARTIAL DIFFERENTIAL EQUATIONS (Fokker-Planck, fractional derivatives, nonlinear, anomalous diffusion, Arrhenius)
IMAGE PROCESSING
CENTRAL LIMIT THEOREMS (de Moivre-Laplace-Gauss, Levy-Gnedenko)
SIGNAL PROCESSING (ARCH, GARCH)
UBIQUITOUS LAWS IN COMPLEX SYSTEMS
STOCHASTIC DIFFERENTIAL EQUATIONS (Langevin, multiplicative noise)
GLOBAL OPTIMIZATION (Simulated annealing)
NONLINEAR DYNAMICS (Chaos, intermittency, entropy production, Pesin, quantum chaos, self-organized criticality)
SUPERSTATISTICS (Other generalizations)
THERMODYNAMICS
q-TRIPLET
AGING (metastability, glass, spin-glass)
q-ALGEBRA
LONG-RANGE INTERACTIONS (Hamiltonians, coupled maps)
CORRELATIONS IN PHASE SPACE
GEOMETRY (Scale-free networks)

ARISTOTLE (384 – 322 BC)



[Ἀριστοτέλη, Περί Ποιητικής, 1459α]

Ἔστιν δὲ μέγα μὲν τὸ ἑκάστωι τῶν εἰρημένων πρεπόντως χρῆσθαι, καὶ διπλοῖς ὀνόμασι καὶ γλώτταις, πολὺ δὲ μέγιστον τὸ μεταφορικὸν εἶναι. Μόνον γὰρ τοῦτο οὔτε παρ᾽ ἄλλου ἔστι λαβεῖν εὐφυίας τε σημεῖόν ἐστι· τὸ γὰρ εὖ μεταφέρειν τὸ τὸ ὅμοιον θεωρεῖν ἐστιν.

Το πιο σημαντικό από όλα τα παραπάνω είναι η δεξιοτεχνική χρήση της μεταφοράς. Διότι μόνο αυτό δεν μπορεί να διδαχθεί, ενώ είναι δείγμα ευφυΐας καθώς μια σωστή μεταφορά υποδηλώνει την ικανότητα να διακρίνει κανείς ομοιότητες ανάμεσα σε ανόμοια πράγματα.

"By far the greatest thing is to be a master of metaphor. It is the one thing that cannot be learned from others. It is a sign of genius, for a good metaphor implies an intuitive perception of similarity among dissimilars."—Aristotle

ABOUT ORDINARY DIFFERENTIAL EQUATIONS

$(i)\quad \frac{dy}{dx} = 0 \quad \text{with} \quad y(0) = 1$

$\Rightarrow y = 1$

Inverse function: $x = 1$ ← $(x \rightleftarrows y \;\; symmetry)$

$(ii)\quad \frac{dy}{dx} = 1 \quad \text{with} \quad y(0) = 1$

$\Rightarrow y = 1 + x$

Inverse function: $x = y - 1$

$(iii)\quad \frac{dy}{dx} = y \quad \text{with} \quad y(0) = 1$

$\Rightarrow y = e^x$ (EXPONENTIAL)

Inverse function: $x = \ln y$

Property: $\ln(y_A\, y_B) = \ln y_A + \ln y_B$

(iv) Unification:

$$\frac{dy}{dx} = y^q \quad (q \in \mathcal{R}) \quad \text{with} \quad y(0) = 1$$

$\Rightarrow y = [1 + (1-q)x]^{\frac{1}{1-q}} \equiv e_q^x$ (POWER-LAW)

Inverse function: $x = \frac{y^{1-q}-1}{1-q} \equiv \ln_q y$

Property: $\ln_q(y_A\, y_B) = \ln_q y_A + \ln_q y_B + (1-q)(\ln_q y_A)(\ln_q y_B)$

[$q = 1$, $q = 0$ and $q \to -\infty$ recover the three previous cases]

ABOUT MEAN VALUES

(i) Equiprobability: $p_i = \frac{1}{W} \quad (\forall i)$

$S_{BG} = k \ln W \quad (k = \text{positive constant})$

Naturally generalized into:

$S_q = k \ln_q W$

(ii) General: (not necessarily equiprobability)

$$S_{BG}(\{p_i\}) = -k \sum_{i=1}^{W} p_i \ln p_i = k \sum_{i=1}^{W} p_i \ln \frac{1}{p_i} \equiv \langle k \ln \frac{1}{p_i} \rangle$$

"surprise" or "unexpectedness"

[We verify that $p_i = \frac{1}{W} \quad (\forall i)$ recovers $S_{BG} = k \ln W$]

Naturally generalized into:

$$S_q(\{p_i\}) \equiv \langle k \ln_q \frac{1}{p_i} \rangle = k \sum_{i=1}^{W} p_i \ln_q \frac{1}{p_i}$$

"q-surprise" or "q-unexpectedness"

hence: $S_q(\{p_i\}) = k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}$

[We verify that $p_i = \frac{1}{W} \quad (\forall i)$ recovers $S_q = k \ln_q W$]

Property: $p_{ij}^{A+B} = p_i^A p_j^B \Rightarrow$

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q) S_q(A) S_q(B)$$

$[0 < p_i < 1 \quad (\forall i)]$

$$p_i^q \begin{cases} > p_i & if \quad q < 1 \\ < p_i & if \quad q > 1 \\ = p_i & if \quad q = 1 \quad (BG) \end{cases}$$

ABOUT BIAS

(i) $S_q = (\{p_i\})$ *should be invariant under permutation.*
The simplest manner is to be

$$S_q(\{p_i\}) = f(\sum_{i=1}^{W} p_i^q)$$

(ii) *The simplest function* $f(x)$ *is*

$$S_q(\{p_i\}) = a + b \sum_{i=1}^{W} p_i^q$$

(iii) *Certainty must correspond to* $S_q = 0$
In other words $p_i = 1 \quad for\ i = i_0$

$= 0 \quad otherwise$

hence $a + b = 0$
hence $S_q(\{p_i\}) = a(1 - \sum_{i=1}^{W} p_i^q)$

(iv) *For* $q \to 1$ *we must recover* $S_{BG}(\{p_i\})$.
Using $p_i^{q-1} \sim 1 + (1-q)\ln p_i$ we obtain

$$S_q(\{p_i\}) \sim -a(q-1) \sum_{i=1}^{W} p_i \ln p_i$$

consequently, by identifying $a(q-1) = k$ we obtain

$$S_q(\{p_i\}) = k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}$$

ABOUT REACTION UNDER BIAS

S. Abe
Phys Lett A **224**, 326 (1997)

(*i*) *Testing the function under* **translation** *of the bias x:*

$$S_{BG}(\{p_i\}) = -k \sum_{i=1}^{W} p_i \ln p_i = -k \left[\frac{d}{dx} \sum_{i=1}^{W} p_i^x\right]_{x=1}$$

(*ii*) *Testing the function under* **dilatation** *of the bias x:*

We replace $\frac{d}{dx}$ by Jackson's 1909 generalized derivative

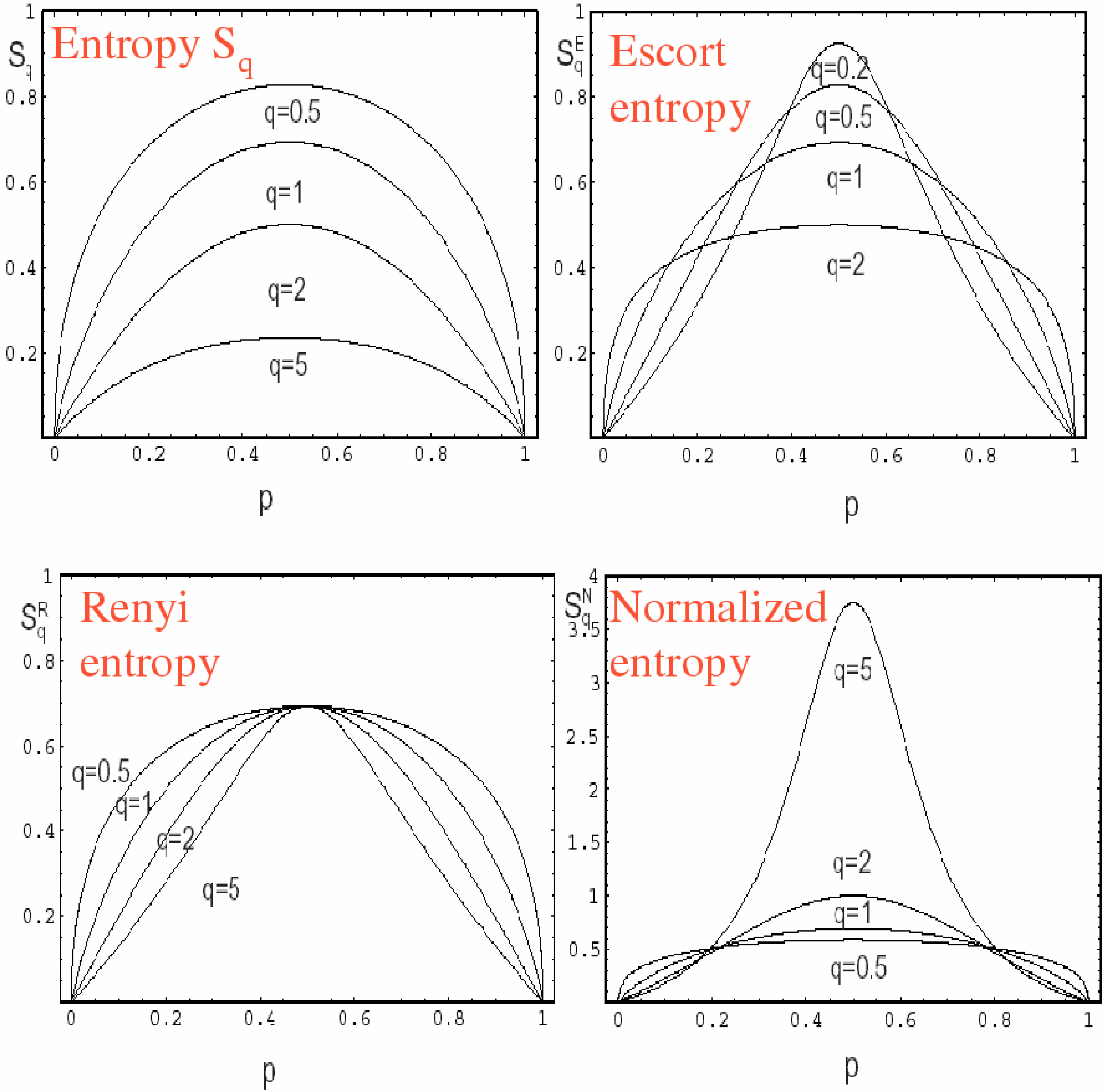
$$D_q\, h(x) \equiv \frac{h(qx)-h(x)}{qx-x} \quad \left[D_1 h(x) = \frac{dh(x)}{dx}\right]$$

and obtain

$$S_q(\{p_i\}) = -k \left[D_q \sum_{i=1}^{W} p_i^x\right]_{x=1}$$

hence

$$S_q(\{p_i\}) = k\, \frac{1-\sum_{i=1}^{W} p_i^q}{q-1}$$



Entropy S_q
S_q
q=0.5
q=1
q=2
q=5
p
Escort entropy
S_q^E
q=0.2
q=0.5
q=1
q=2
p
Renyi entropy
S_q^R
q=0.5
q=1
q=2
q=5
p
Normalized entropy
S_q^N
q=5
q=2
q=1
q=0.5
p

SANTOS THEOREM: RJV Santos, J Math Phys **38**, 4104 (1997)

(q -generalization of Shannon 1948 theorem)

IF $S(\{p_i\})$ *continuous function of* $\{p_i\}$

AND $S(p_i = 1/W,\ \forall i)$ *monotonically increases with* W

AND $\frac{S(A+B)}{k} = \frac{S(A)}{k} + \frac{S(B)}{k} + (1-q)\frac{S(A)}{k}\frac{S(B)}{k}$ (*with* $p_{ij}^{A+B} = p_i^A\, p_j^B$)

AND $S(\{p_i\}) = S(p_L, p_M) + p_L^{\,q}\ S(\{p_l / p_L\}) + p_M^{\,q}\ S(\{p_m / p_M\})$ (*with* $p_L + p_M = 1$)

THEN AND ONLY THEN

$$S(\{p_i\}) = k\frac{1-\sum_{i=1}^{W} p_i^{\,q}}{q-1} \qquad \left(q = 1 \ \Rightarrow\ S(\{p_i\}) = -k\sum_{i=1}^{W} p_i \ln p_i\right)$$

CE SHANNON (The Mathematical Theory of Communication):

"This theorem, and the assumptions required for its proof, are in no way necessary for the present theory. It is given chiefly to lend a certain plausibility to some of our later definitions. The real justification of these definitions, however, will reside in their implications.

ABE THEOREM: S Abe, Phys Lett A **271**, 74 (2000)

(q -generalization of Khinchin 1953 theorem)

IF $S(\{p_i\})$ *continuous function of* $\{p_i\}$

AND $S(p_i = 1/W,\ \forall i)$ *monotonically increases with* W

AND $S(p_1, p_2, ..., p_W, 0) = S(p_1, p_2, ..., p_W)$

AND $\frac{S(A+B)}{k} = \frac{S(A)}{k} + \frac{S(B|A)}{k} + (1-q)\frac{S(A)}{k}\frac{S(B|A)}{k}$

THEN AND ONLY THEN

$$S(\{p_i\}) = k\frac{1-\sum_{i=1}^{W} p_i^{\,q}}{q-1} \quad \left(q=1 \Rightarrow S(\{p_i\}) = -k\sum_{i=1}^{W} p_i \ln p_i\right)$$

The possibility of such theorem was conjectured by AR Plastino and A Plastino (1996, 1999).

STABILITY

or CONTINUITY

or EXPERIMENTAL ROBUSTNESS

B. Lesche
J Stat Phys 27, 419 (1982)

The entropy S is said stable iff, for any given $\varepsilon > 0$, a $\delta_\varepsilon > 0$ exists such that, independently from W,

$$\sum_{i=1}^{W} | p_i - p'_i | \leq \delta_\varepsilon \quad \Rightarrow \quad \left| \frac{S(\{p_i\}) - S(\{p_i'\})}{S_{\max}} \right| < \varepsilon$$

Hence $$\lim_{\delta \to 0} \lim_{W \to \infty} \frac{S(\{p_i\}) - S(\{p_i'\})}{S_{\max}} = 0$$

S_{BG} and S_q $(\forall q > 0)$ are stable

S. Abe, Phys Rev E **66**, 046134 (2002)

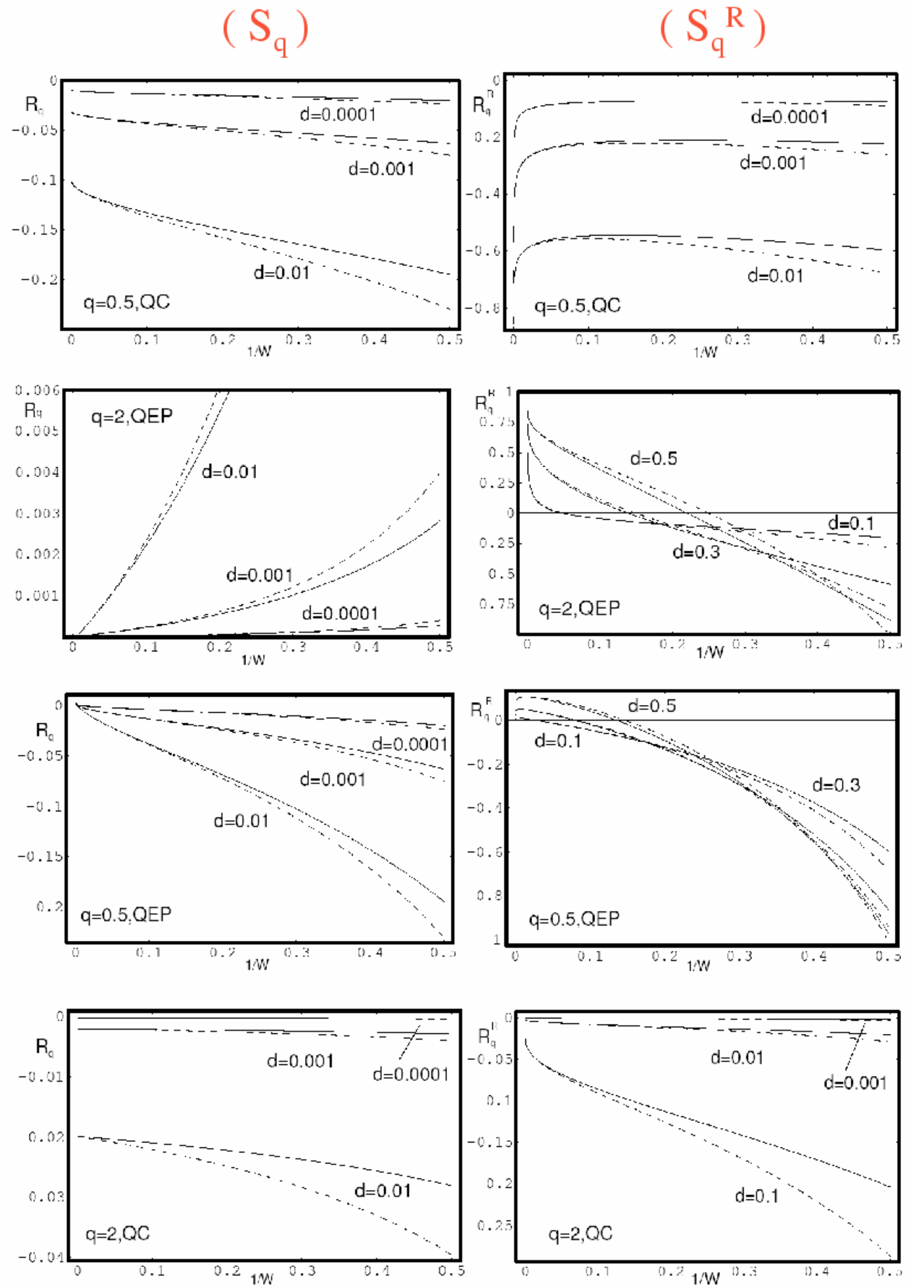
$$S_q^R(\{p_i\}) \equiv \frac{\ln \sum_{i=1}^{W} p_i^q}{q-1} \qquad \text{(Renyi entropy)}$$

$$S_q^N(\{p_i\}) \equiv \frac{S_q(\{p_i\})}{\sum_{i=1}^{W} p_i^q} \qquad \text{(Normalized entropy)}$$

$$S_q^E(\{p_i\}) \equiv \frac{1 - \left(\sum_{i=1}^{W} p_i^{1/q} \right)^{-q}}{q-1} \qquad \text{(Escort entropy)}$$

are unstable

B. Lesche (1982); S. Abe (2002); C.T. and E. Brigatti (2003)



QEP = quasi equal probabilities

QC = quasi certainty

C.T. and E. Brigatti (2003)

DEFINITIONS :

q – logarithm :

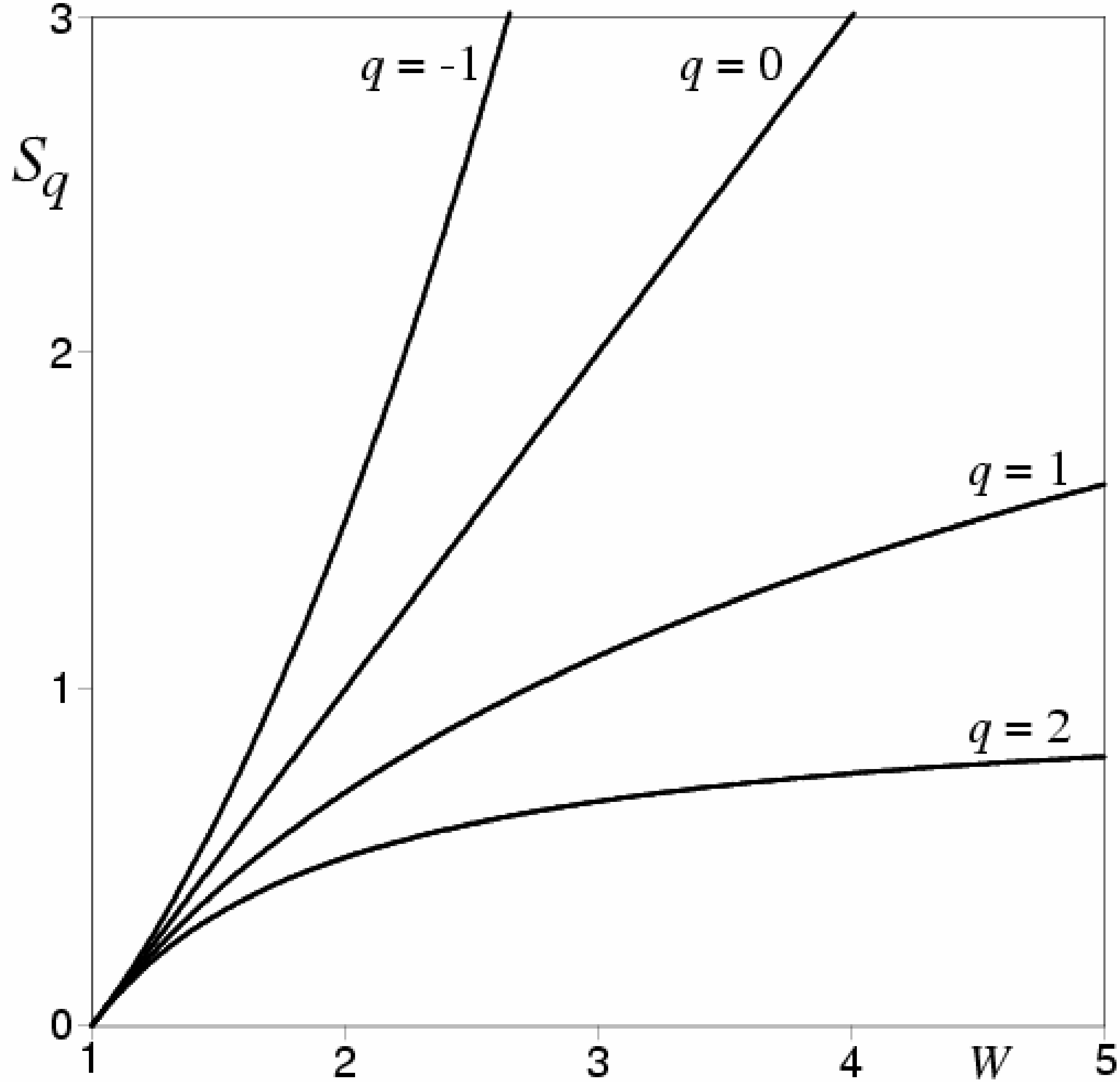
$$\ln_q x \equiv \frac{x^{1-q}-1}{1-q} \qquad (x > 0)$$

$$\ln_1 x = \ln x$$

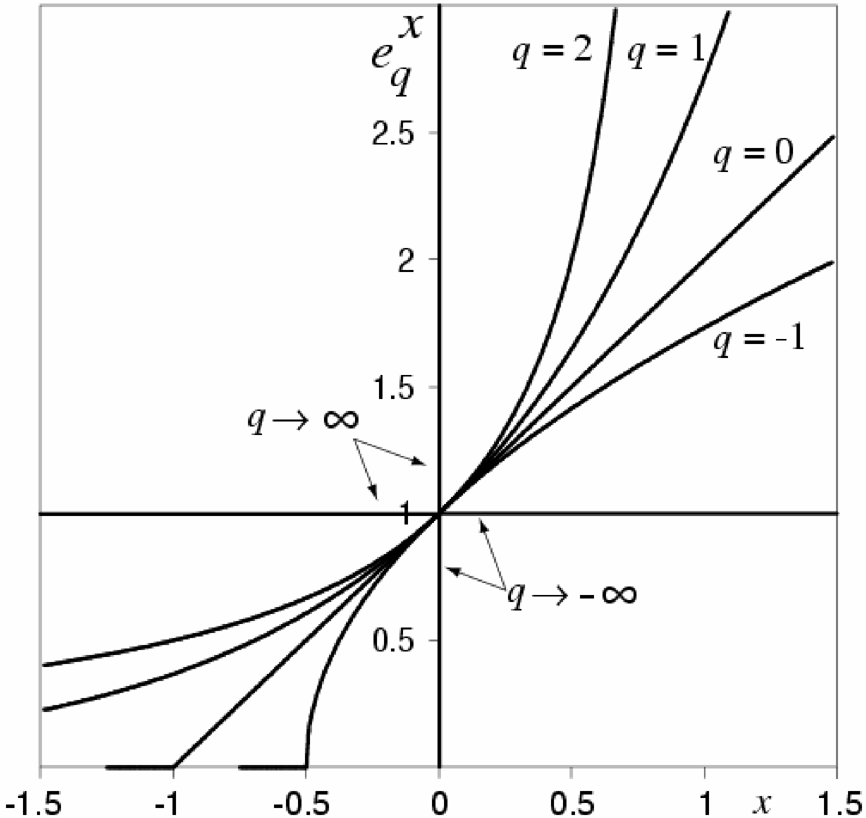
q - exponential :

$$e_q^x \equiv \begin{cases} [1+(1-q)x]^{\frac{1}{1-q}} & if \;\; 1+(1-q)x \geq 0 \\ 0 & otherwise \end{cases}$$

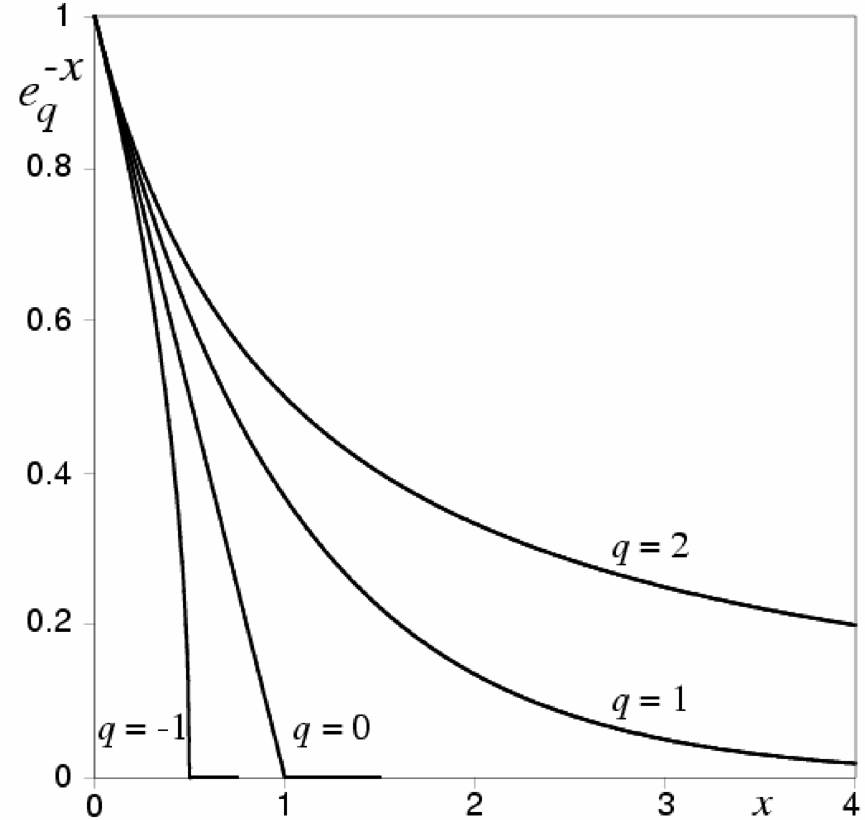
$$e_1^x = e^x$$



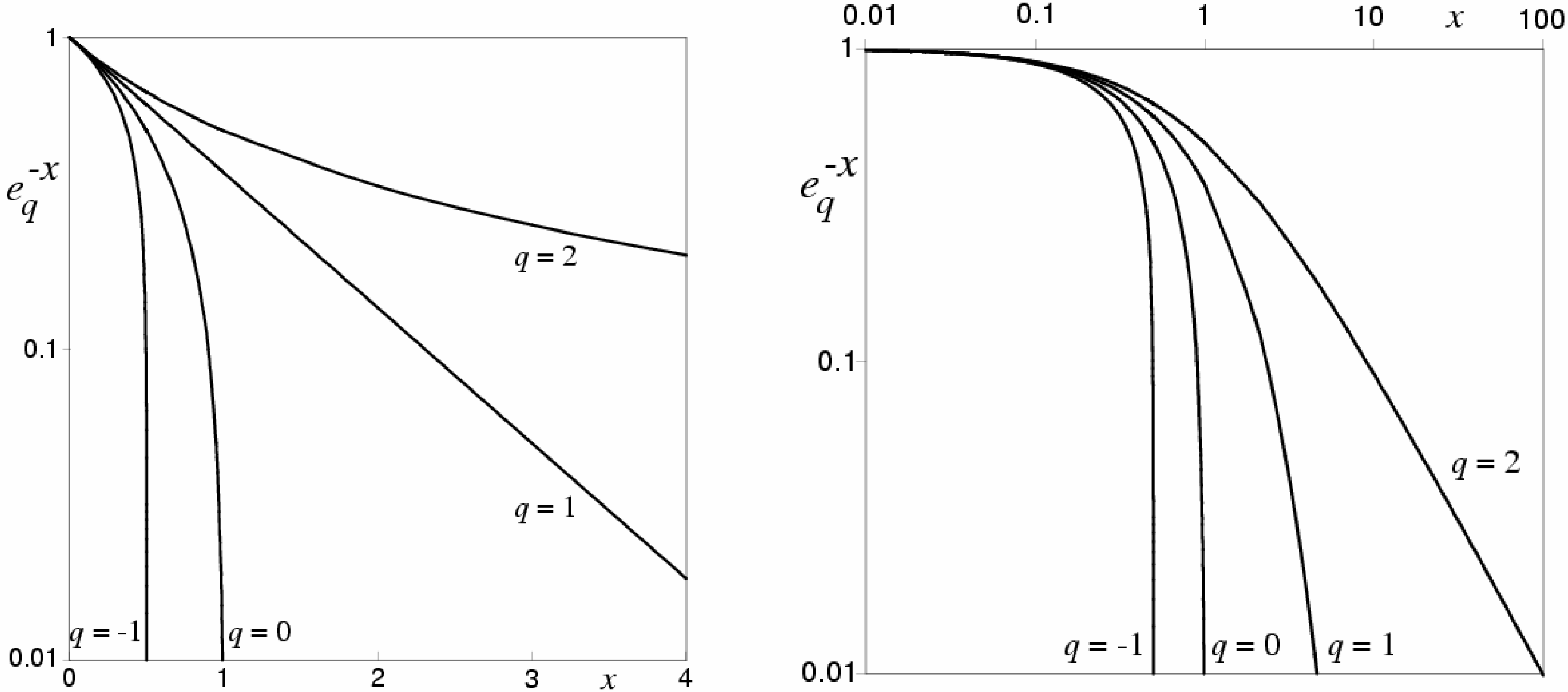
q = -1
q = 0
q = 1
q = 2
S_q
W
3
2
1
0
1
2
3
4
5



e_q^x
q = 2
q = 1
q = 0
q = -1
2.5
2
1.5
1
0.5
q → ∞
q → -∞
-1.5
-1
-0.5
0
0.5
1
x
1.5



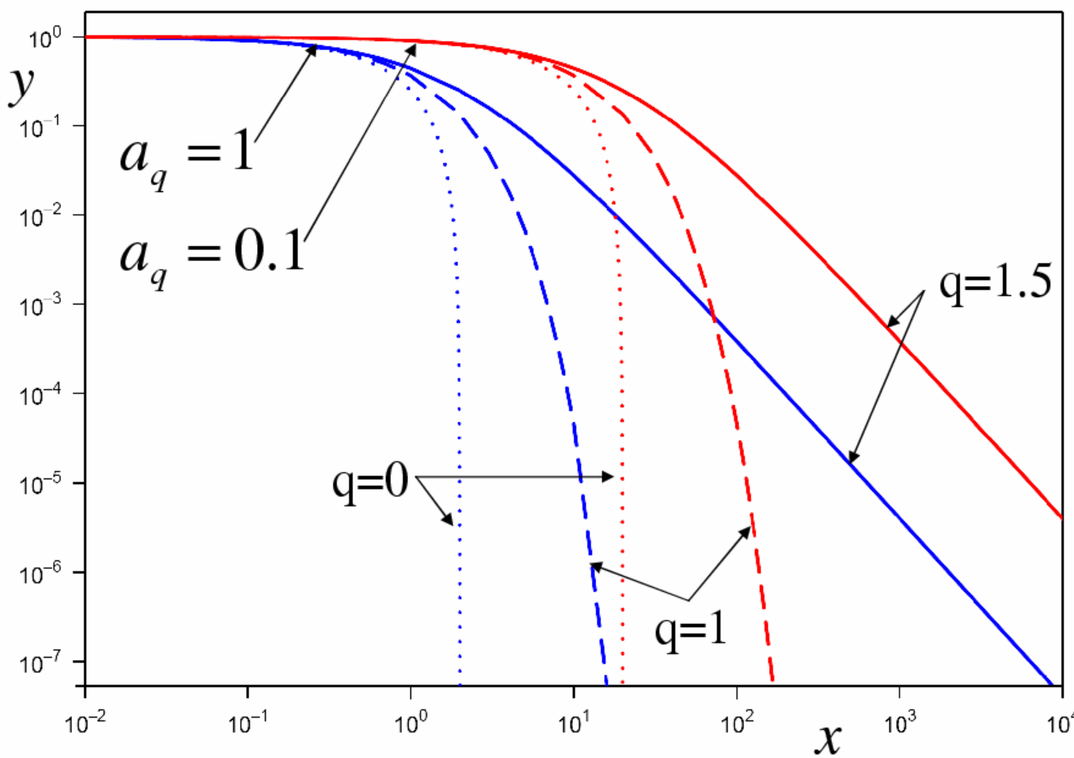
e_q^{-x}
1
0.8
0.6
0.4
0.2
0
q = 2
q = 1
q = -1
q = 0
0
1
2
3
x
4

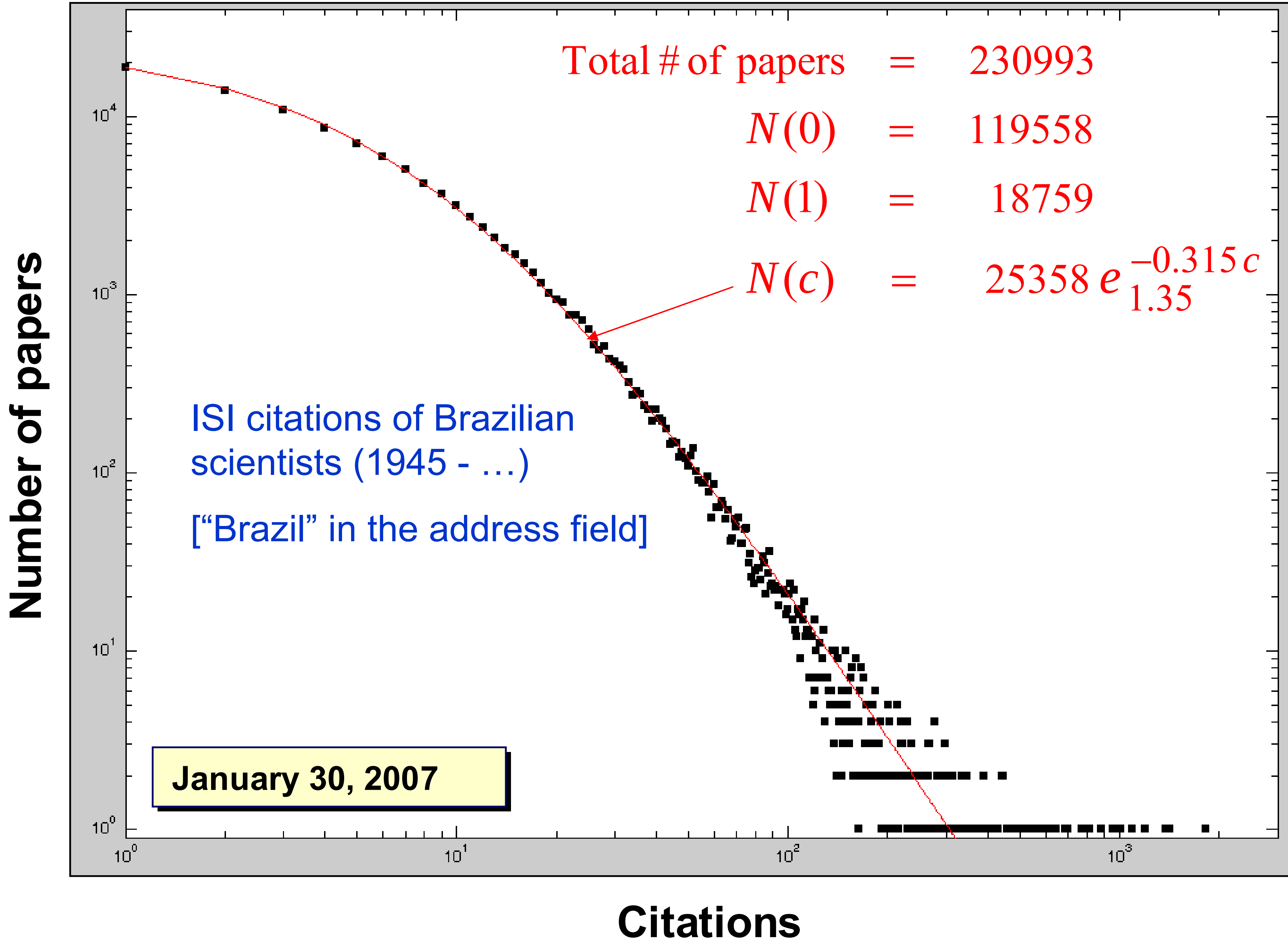


e_q^{-x}
1
0.1
0.01
0
1
2
3
4
x
$q = 2$
$q = 1$
$q = -1$
$q = 0$
0.01
0.1
1
10
x
100
e_q^{-x}
1
0.1
0.01
$q = 2$
$q = -1$
$q = 0$
$q = 1$

$$\frac{dy}{dx} = -a_q y^q \quad with \quad y(0) = 1$$

$$\Rightarrow y = \frac{1}{[1+(q-1)a_q x]^{\frac{1}{q-1}}} \equiv e_q^{-a_q x}$$





M.P de Albuquerque and D.B. Mussi (2007)

PREDECESSORS

RENYI ENTROPY $\propto \ln \sum_i p_i^q$:

M.P. Schutzenberger, Publ. Inst. Statist. Univ. Paris (1954) [*according to I. Csiszar* (1974,1978)]

A. Renyi, Proc. 4*th* *Berkeley Symposium* (1969)

ENTROPY $\propto 1-\sum_i p_i^q$:

J. Harvda and F. Charvat, Kybernetica 3, 30 (1967)

I. Vajda, Kybernetica 4, 105 (1968)

Z. Daroczy, Inf. Control 16, 36 (1970)

J. Lindhard and V. Nielsen, Det Kongelige Danske Videnskabernes Selskab
Matematisk - fysiske Meddelelser (*Denmark*) 38 (9), 1 (1971)

B.D. Sharma and D.P. Mittal, J. Math. Sci. 10, 28 (1975) [*unification of both previous entropic forms*]

A. Wehrl, Rev. Mod. Phys. 50, 221 (1978)

q − GAUSSIANS :

Gaussian distribution	*Abraham de Moivre* (1733)
	Pierre Simon de Laplace (1774)
	Robert Adrain (1808)
	Carl Friedrich Gauss (1809)
Cauchy – Lorentz – Breit	*Agustin Louis Cauchy* (~1821)
– Wigner distribution	*Hendric Antoon Lorentz* (~1880)
Student's t – distribution	*William Sealy Gosset* (1908)

THE VARIOUS FORMS OF THE NONADDITIVE q-ENTROPY Sq:

DISCRETE CLASSICAL STATES (Shannon for $q = 1$):

$$\frac{S_q}{k} = \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1} = \sum_{i=1}^{W} p_i \ln_q \frac{1}{p_i} = -\sum_{i=1}^{W} p_i^q \ln_q p_i \quad \left(with \sum_{i=1}^{W} p_i = 1\right)$$

CONTINUOUS CLASSICAL STATES (Boltzmann, Gibbs for $q = 1$):

$$\frac{S_q}{k} = \frac{1 - \int dx\, [p(x)]^q}{q-1} = \int dx\, p(x) \ \ln_q \frac{1}{p(x)} = -\int dx\, [p(x)]^q \ \ln_q p(x) \ \left(with \int dx\, p(x) = 1\right)$$

Particular case: $p(x) = \sum_{i=1}^{W} p_i \ \delta(x - x_i) \ \Rightarrow \ \frac{S_q}{k} = \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}$

QUANTUM STATES (von Neumann for $q = 1$):

$$\frac{S_q}{k} = \frac{1 - Tr\,\rho^q}{q-1} = Tr\ (\rho \ \ln_q \rho^{-1}) = -Tr\ (\rho^q \ \ln_q \rho) \quad (with\ Tr\,\rho = 1)$$

Particular case: $\rho_{ij} = p_i \ \delta_{ij} \ \Rightarrow \ \frac{S_q}{k} = \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}$

$$S_q(N,t) \ versus \ \ t$$

DISSIPATIVE MAPS:

Strongly chaotic (i.e., maximal Lyapunov exponent > 0)

Weakly chaotic (i.e., maximal Lyapunov exponent = 0)

CONSERVATIVE MAPS:

Strongly chaotic (i.e., maximal Lyapunov exponent > 0)

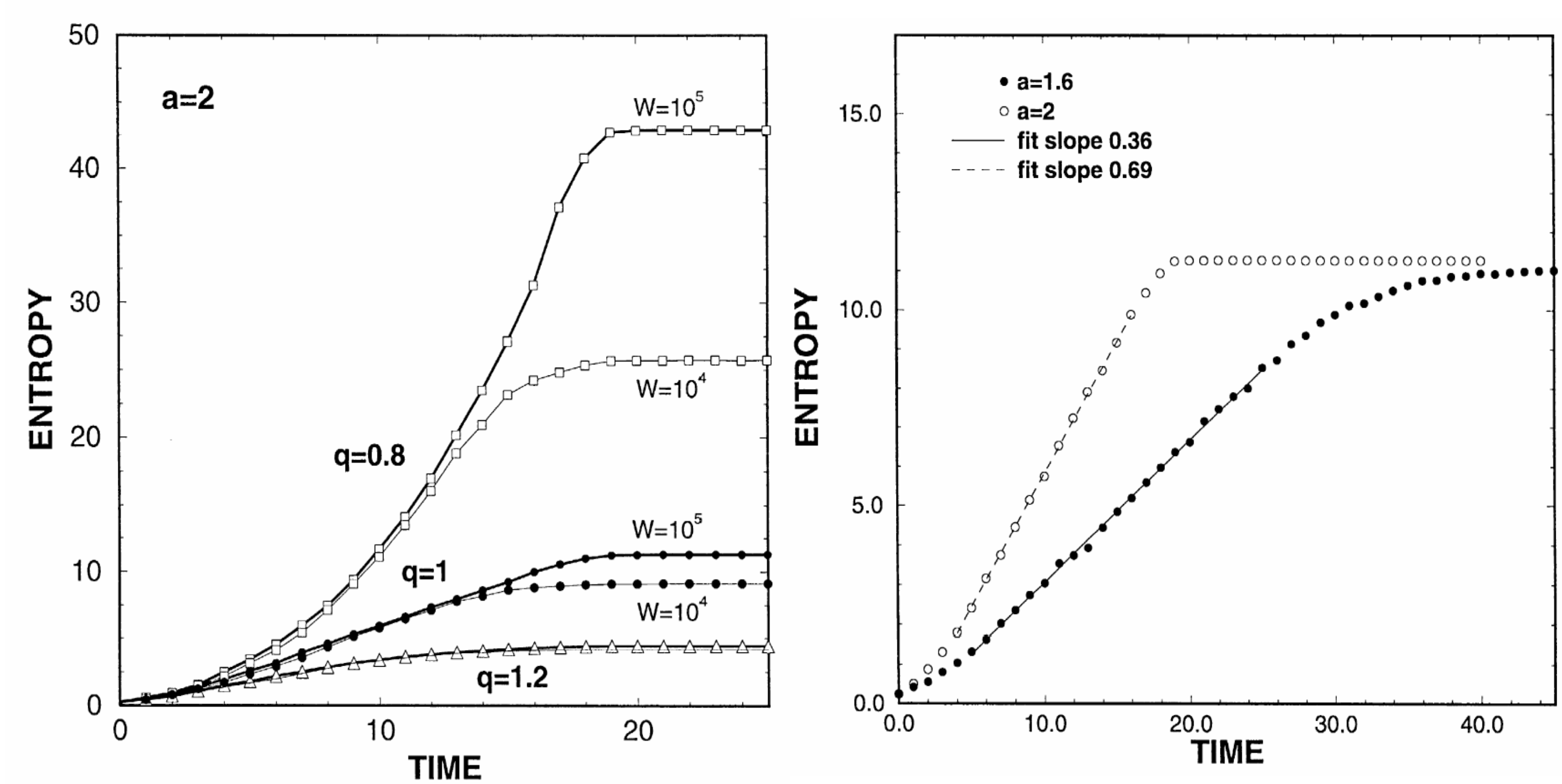
Weakly chaotic (i.e., maximal Lyapunov exponent = 0)

DISSIPATIVE MAPS

LOGISTIC MAP:

$$x_{t+1} = 1 - a\,x_t^2 \quad (0 \le a \le 2;\ -1 \le x_t \le 1;\ t = 0, 1, 2, \ldots)$$

(strong chaos, i.e., **positive** Lyapunov exponent)



V. Latora, M. Baranger, A. Rapisarda and C. T., Phys. Lett. A **273**, 97 (2000)

We verify

$$K_1 = \lambda_1 \quad (Pesin-like\ identity)$$

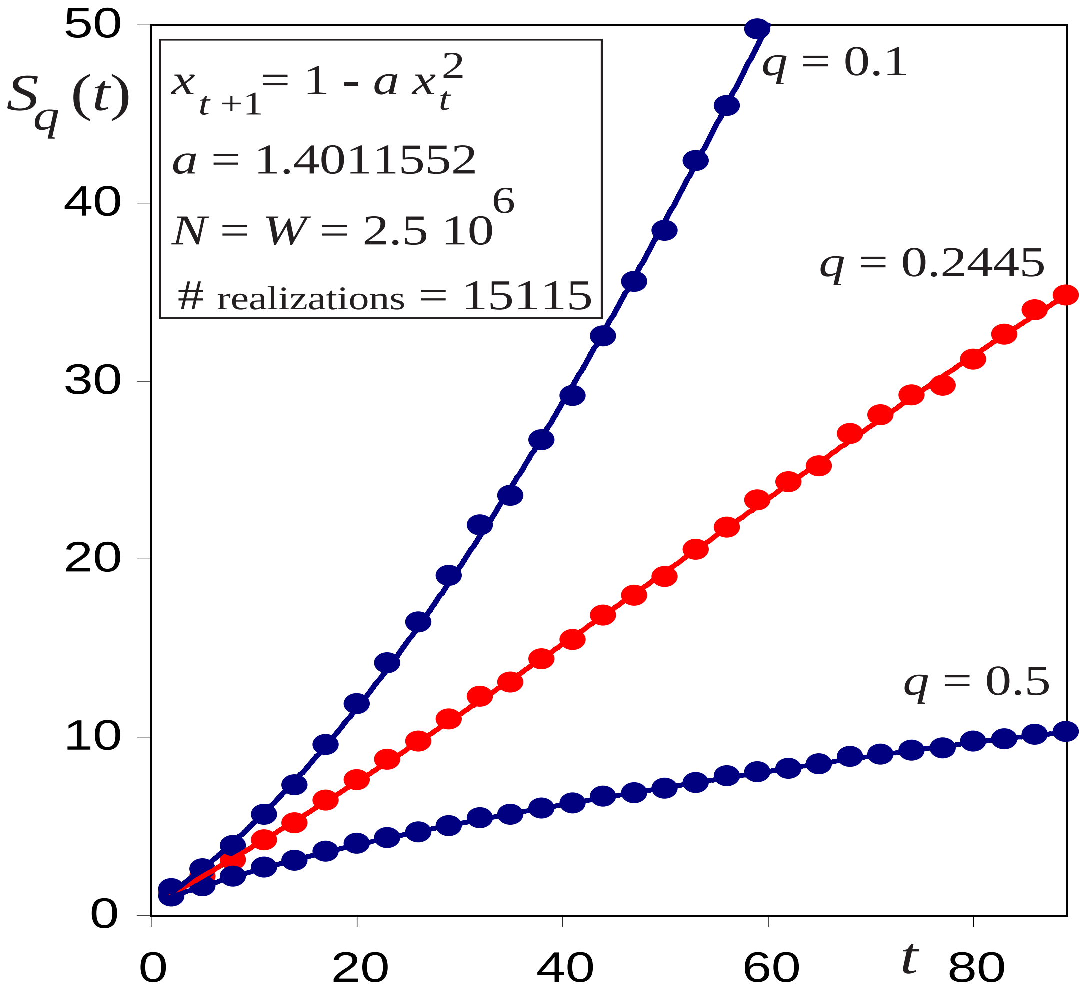
where

$$K_1 \equiv \lim_{t\to\infty} \frac{S_1(t)}{t}$$

and

$$\xi(t) \equiv \lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)} = e^{\lambda_1 t}$$

(weak chaos, i.e., **zero** Lyapunov exponent)



C. T. , A.R. Plastino and W.-M. Zheng, Chaos, Solitons & Fractals **8**, 885 (1997)
M.L. Lyra and C. T. , Phys. Rev. Lett. **80**, 53 (1998)
V. Latora, M. Baranger, A. Rapisarda and C. T. , Phys. Lett. A **273**, 97 (2000)
E.P. Borges, C. T. , G.F.J. Ananos and P.M.C. Oliveira, Phys. Rev. Lett. **89**, 254103 (2002)
F. Baldovin and A. Robledo, Phys. Rev. E **66**, R045104 (2002) and **69**, R045202 (2004)
G.F.J. Ananos and C. T. , Phys. Rev. Lett. **93**, 020601 (2004)
E. Mayoral and A. Robledo, Phys. Rev. E **72**, 026209 (2005), and references therein

It can be proved that

$$K_q = \lambda_q \quad (q-generalized \;\; Pesin-like\ identity)$$

where

$$K_q \equiv \lim_{t\to\infty} \sup \left\{ \frac{S_q(t)}{t} \right\}$$

and

$$\xi(t) \equiv \sup\left\{ \lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)} \right\} = e_q^{\lambda_q\, t}$$

with

$$\frac{1}{1-q} = \frac{1}{\alpha_{\min}} - \frac{1}{\alpha_{\max}} = \frac{\ln \alpha_F}{\ln 2} \quad and \quad \lambda_q = \frac{1}{1-q}$$

$$\left[x_{t+1} = 1 - a\,|\,x_t\,|^z \;\;\Rightarrow\;\; \frac{1}{1-q(z)} = \frac{1}{\alpha_{\min}(z)} - \frac{1}{\alpha_{\max}(z)} = (z-1)\frac{\ln \alpha_F(z)}{\ln 2} \right]$$

EDGE OF CHAOS OF THE LOGISTIC MAP:

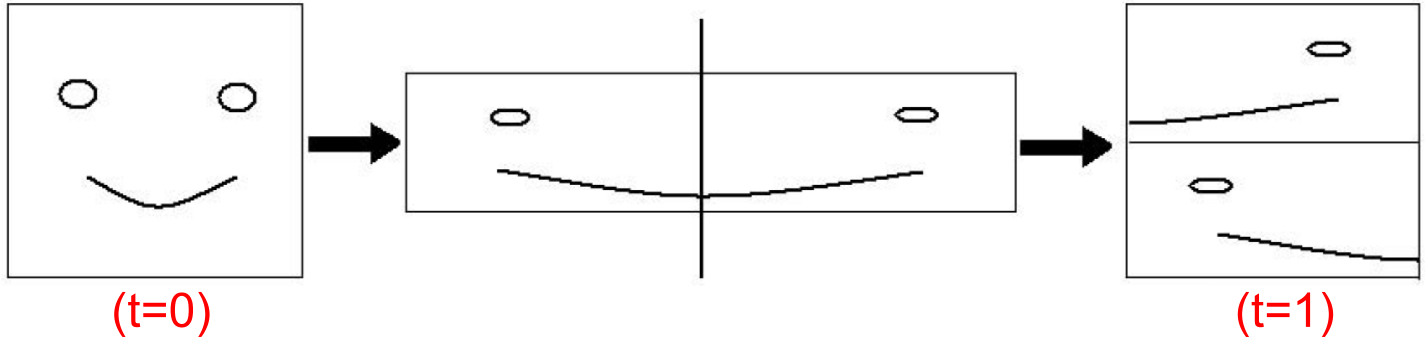
(Using result in http://pi.lacim.uqam.ca/piDATA/feigenbaum.txt)

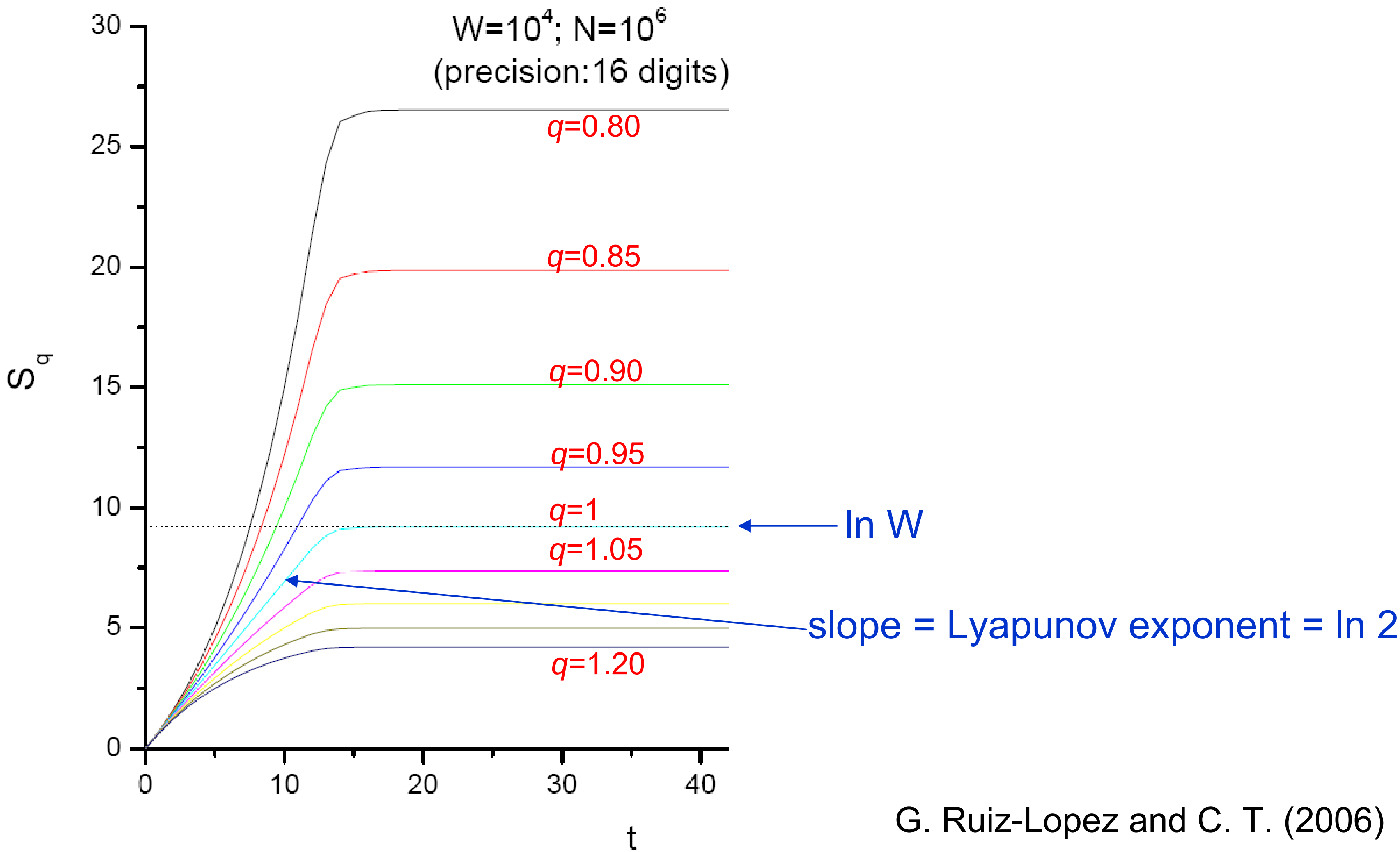
q =

0.244487701341282066198770423404680405234446935490057673670365O
98632774967276655866575515622685754070628834964038272830606360O
1937303318189645513410812778097921943860270831944900524658135 21
5031745349520749404481654609490874483340567236224664880833 3072
1423189871458729926815484967746078648218345690633702059468 20461
8990216753214575461174383050084968604088469694917043674789 91506
0166464910602178348278899938183825225545823380381131180318 05448
2367579449903970743954661463408155531687885350301138214914 11266
2463289401303701523549365714712699179210216226888330296754 05780
6307068223688104320157903521237407354446029700060552504231 42028
0891935788112397319779748442351524560409264467095795703046 58614
1295664796666877436832404920227573930047508953118551795587 20483
9926968968275558524450244365268256094237801280330948779544 03542
5248590433797618027118300045735855507389411367587844006291 35630
4216745416940921356986032078590881998593590073193368010699 67496
7079044560924186321120541305473939857955444103476122225921 36846
219346009360… (1018 meaningful digits)

CONSERVATIVE MAPS

BAKER MAP:





G. Ruiz-Lopez and C. T. (2006)

THE CASATI-PROSEN TRIANGLE MAP:

G. Casati and T. Prosen,
Phys. Rev. Lett. **83**, 4729 (1999) and **85**, 4261 (2000)

"While exponential instability is sufficient for a meaningful statistical description, it is not known whether or not it is also necessary."

$$y_{t+1} = y_t + \alpha \ \mathrm{sgn}(x_t) + \beta \quad (\mathrm{mod}\, 2)$$

$$x_{t+1} = x_t + y_{t+1} \quad (\mathrm{mod}\, 2)$$

(α and β independent irrationals)

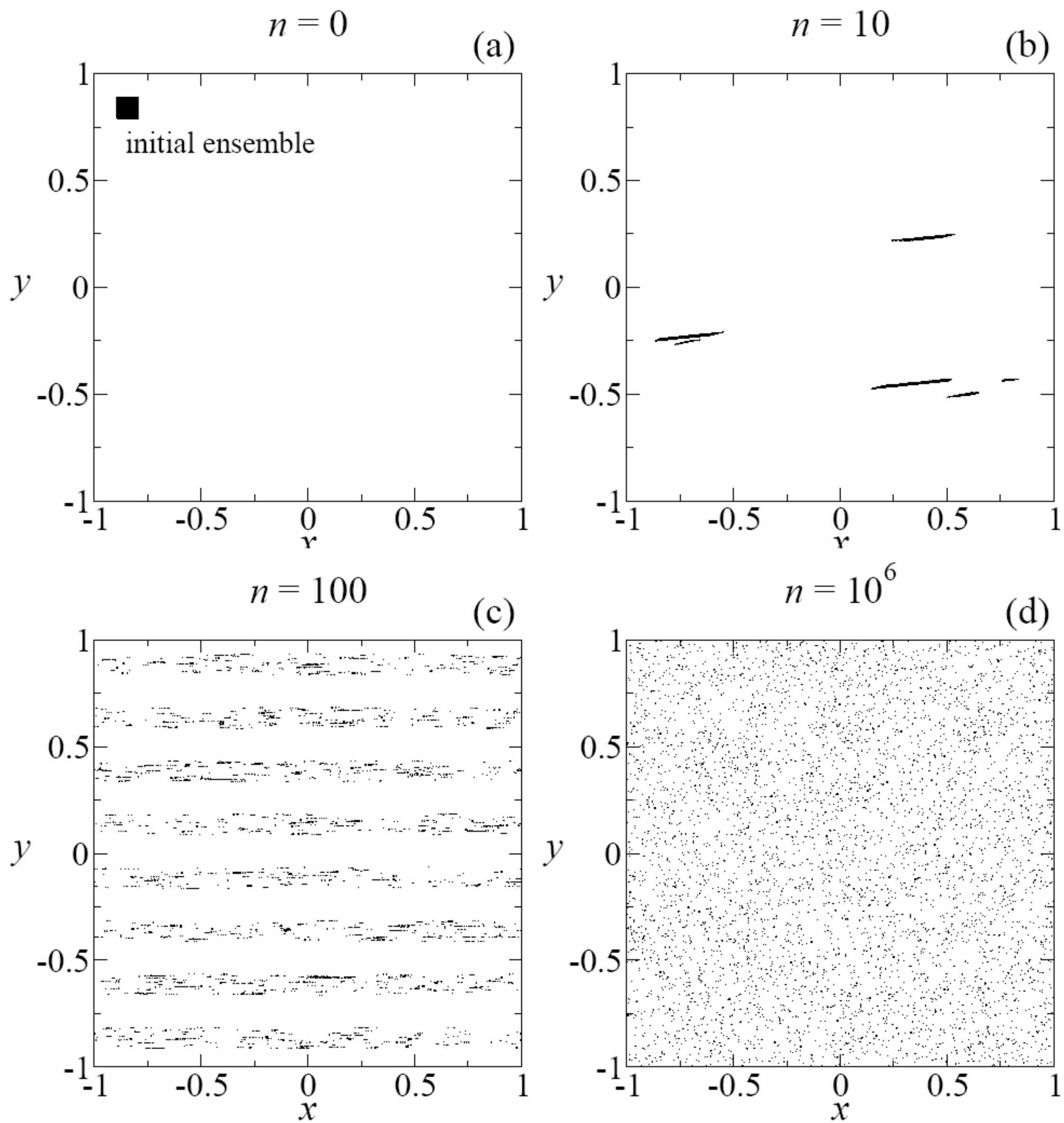
e.g., $(\alpha,\beta)=\left((1/2)(\sqrt{5}-1)-(1/e),\ (1/2)(\sqrt{5}-1)+(1/e)\right)$

This map is conservative, mixing, ergodic and nevertheless with zero Lyapunov exponent!

Furthermore $\xi \equiv \lim_{\Delta X(0)\to 0} \dfrac{\Delta X(t)}{\Delta X(0)} \propto t$

CASATI-PROSEN TRIANGLE MAP [Casati and Prosen, Phys Rev Lett **83**, 4729 (1999) and **85**, 4261 (2000)]

(two-dimensional, conservative, mixing, ergodic, vanishing maximal Lyapunov exponent)



G. Casati, C. T. and F. Baldovin, Europhys. Lett. **72**, 355 (2005)

NONEXTENSIVITY OF THE CASATI-PROSEN MAP:

[G. Casati, C.T. and F. Baldovin, Europhys Lett **72**, 355 (2005)]

Answer to the above equation:

It is not necessary: a meaningful statistical description is possible with zero Lyapunov exponent!

[Essentially because an integrable system has zero Lyapunov exponent but the opposite is not true]

In general, $\xi = [1 + (1\text{-q})\lambda_q t]^{1/(1-q)}$

hence, $\xi \propto t \Rightarrow q = 0$

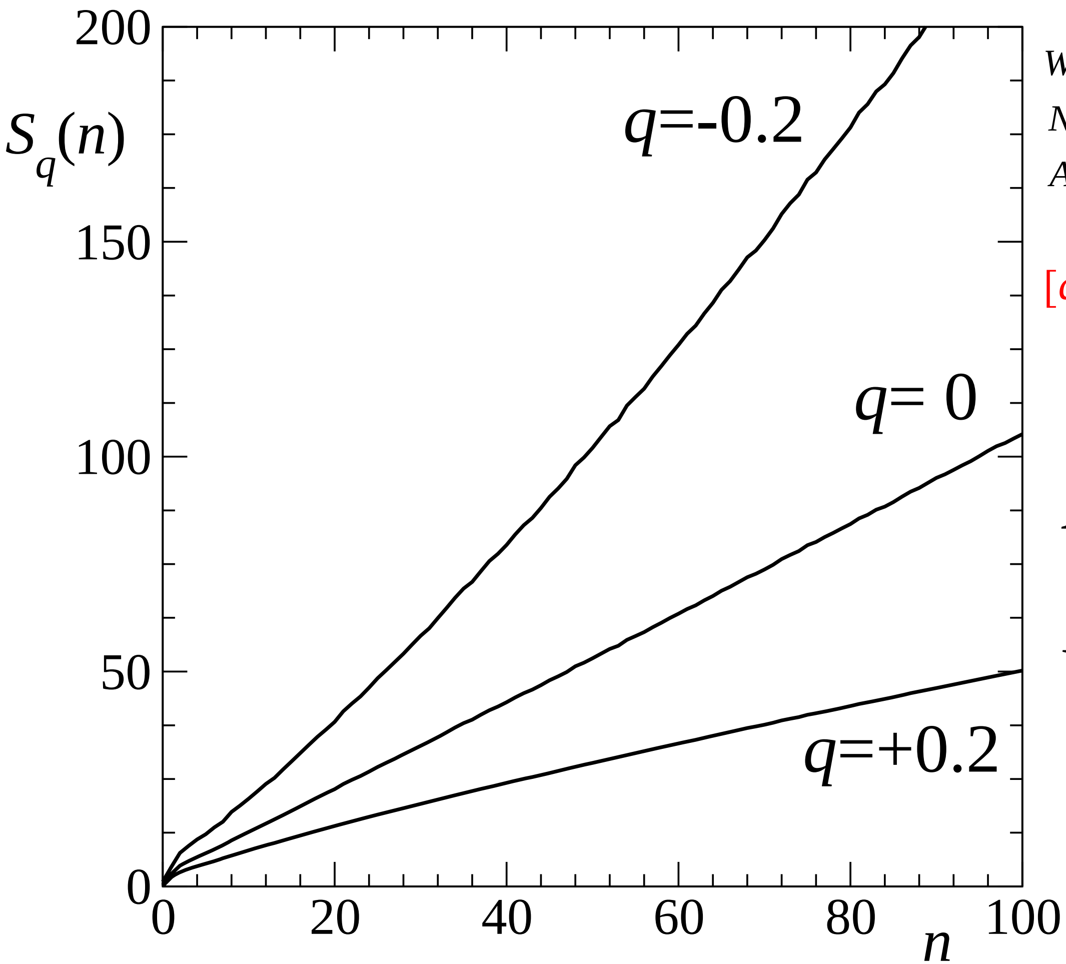
Consistently, we expect

$$(i)\ S_q(t) \equiv \frac{1 - \sum_{i=1}^{W} [p_i(t)]^q}{q-1} \propto t \text{ only for } q = 0$$

$$(ii) K_q \equiv \lim \frac{S_q(t)}{} = \lambda \qquad \text{for } a = 0$$

CASATI-PROSEN TRIANGLE MAP [Casati and Prosen, Phys Rev Lett **83**, 4729 (1999) and **85**, 4261 (2000)]

(two-dimensional, conservative, mixing, ergodic, vanishing maximal Lyapunov exponent)



W = 4000 × 4000 *cells*

N = 1000 *initial conditions randomly chosen in one cell*

Average done over 100 *initial cells*

$[q = 0 \;\rightarrow\; \textit{linear correlation} = 0.99993]$

Also $\xi = e_0^{\lambda_0 \, t}$

with $\lambda_0 = \lim_{n\to\infty} \frac{S_0(n)}{n} = 1$

q - generalization of Pesin (- like) theorem

G. Casati, C. T. and F. Baldovin, Europhys. Lett. **72**, 355 (2005)

$$S_q(N,t)\ versus\ \ N$$

HYBRID PASCAL - LEIBNITZ TRIANGLE

(N = 0) $1 \times \frac{1}{1}$

(N = 1) $1 \times \frac{1}{2}$ $1 \times \frac{1}{2}$

(N = 2) $1 \times \frac{1}{3}$ $2 \times \frac{1}{6}$ $1 \times \frac{1}{3}$

(N = 3) $1 \times \frac{1}{4}$ $3 \times \frac{1}{12}$ $3 \times \frac{1}{12}$ $1 \times \frac{1}{4}$

(N = 4) $1 \times \frac{1}{5}$ $4 \times \frac{1}{20}$ $6 \times \frac{1}{30}$ $4 \times \frac{1}{20}$ $1 \times \frac{1}{5}$

(N = 5) $1 \times \frac{1}{6}$ $5 \times \frac{1}{30}$ $10 \times \frac{1}{60}$ $10 \times \frac{1}{60}$ $5 \times \frac{1}{30}$ $1 \times \frac{1}{6}$

$$\Sigma = 1 \quad (\forall \ N)$$

Blaise **Pascal** (1623-1662)
Gottfried Wilhelm **Leibnitz** (1646-1716)
Daniel **Bernoulli** (1700-1782)

$(N=2)$

A \ B	1	2	
1	$p^2+\kappa$	$p(1-p)-\kappa$	p
2	$p(1-p)-\kappa$	$(1-p)^2+\kappa$	1- p
	p	1- p	1

EQUIVALENTLY:

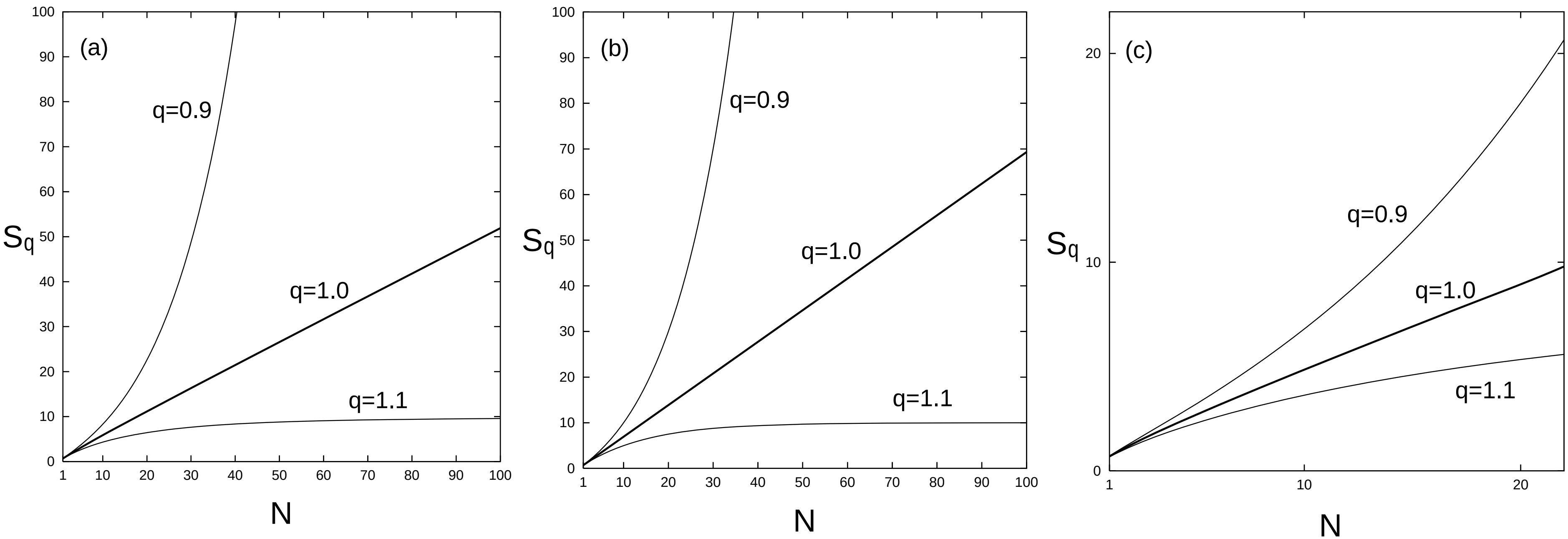
$(N=0)$ $\quad 1\times 1$

$(N=1)$ $\quad 1\times p \quad 1\times(1-p)$

$(N=2)$ $\quad 1\times[p^2+\kappa] \quad 2\times[p(1-p)-\kappa] \quad 1\times[(1-p)^2+\kappa]$

$q = 1$ SYSTEMS

i.e., such that $S_1(N) \propto N \quad (N \to \infty)$



Leibnitz triangle

$$\left(p_{N,0} = \frac{1}{N+1} \right)$$

N independent coins

$$\left(p_{N,0} = p^N \quad \text{with } p = 1/2 \right)$$

Stretched exponential

$$\left(p_{N,0} = p^{N^\alpha} \quad \text{with } p = \alpha = 1/2 \right)$$

(All three examples **strictly** satisfy the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA **102**, 15377 (2005)

Asymptotically scale-invariant (d=2)

$(N=0)$	1				
$(N=1)$	$1/2$	$1/2$			
$(N=2)$	$1/3$	$1/6$	$1/3$		
$(N=3)$	$3/8$	$5/48$	$5/48$	0	
$(N=4)$	$2/5$	$3/40$	$1/20$	0	0

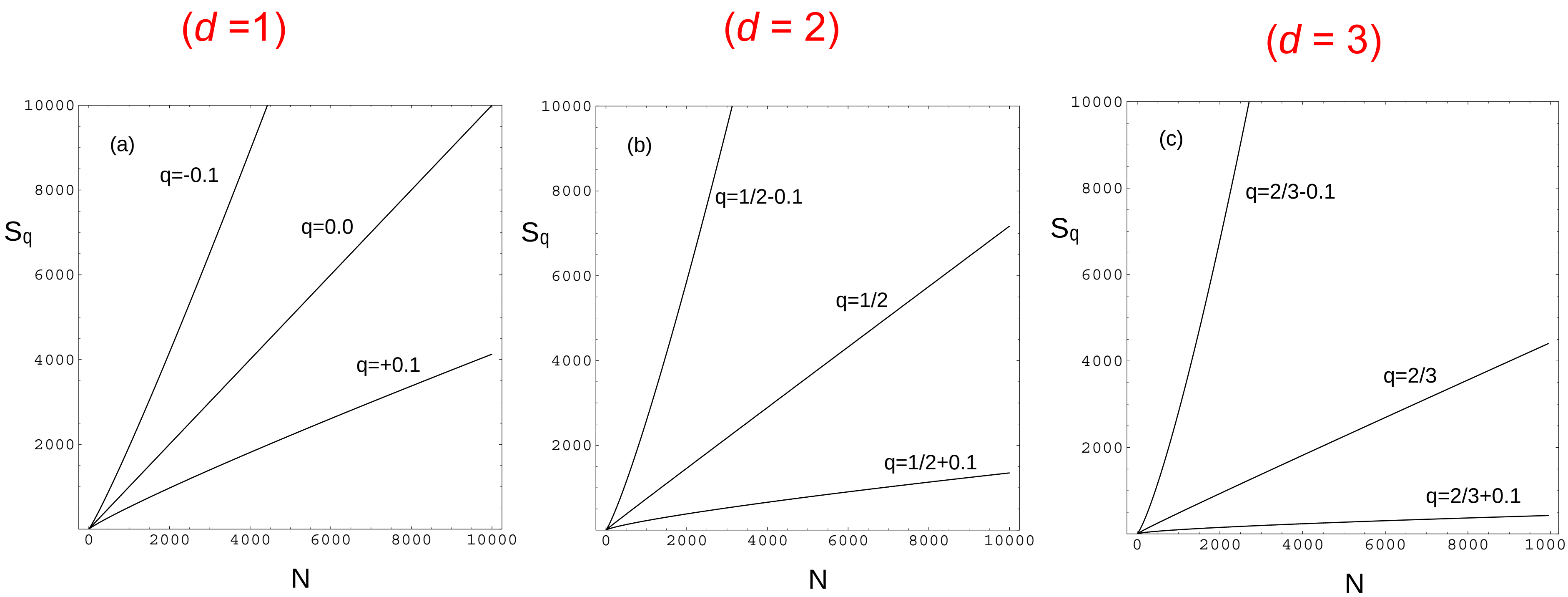
d+1

(It **asymptotically** satisfies the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA **102**, 15377 (2005)

$q \neq 1$ SYSTEMS

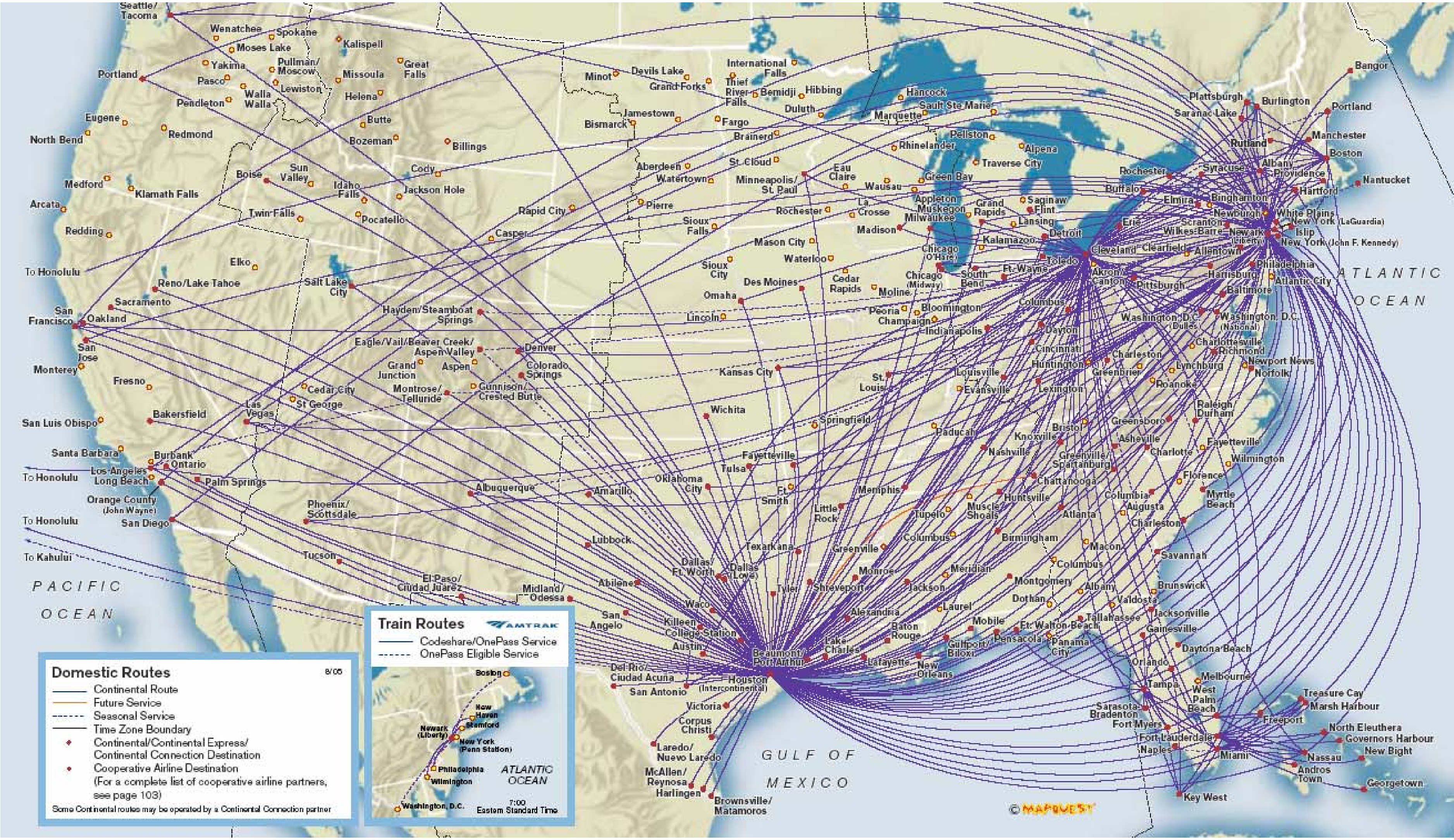
i.e., such that $S_q(N) \propto N \quad (N \to \infty)$



$$q = 1 - \frac{1}{d}$$

(All three examples **asymptotically** satisfy the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA **102**, 15377 (2005)



Continental Airlines

If A *and* B *are* *independent*,

i.e., if $p_{ij}^{A+B} = p_i^{A} \, p_j^{B}$,

then

$$S_{BG}(A+B) = S_{BG}(A) + S_{BG}(B)$$

whereas

$$S_q(A+B) = S_q(A) + S_q(B) + \frac{1-q}{k_B} S_q(A) \, S_q(B)$$

$$\neq S_q(A) + S_q(B) \quad (if \;\; q \neq 1)$$

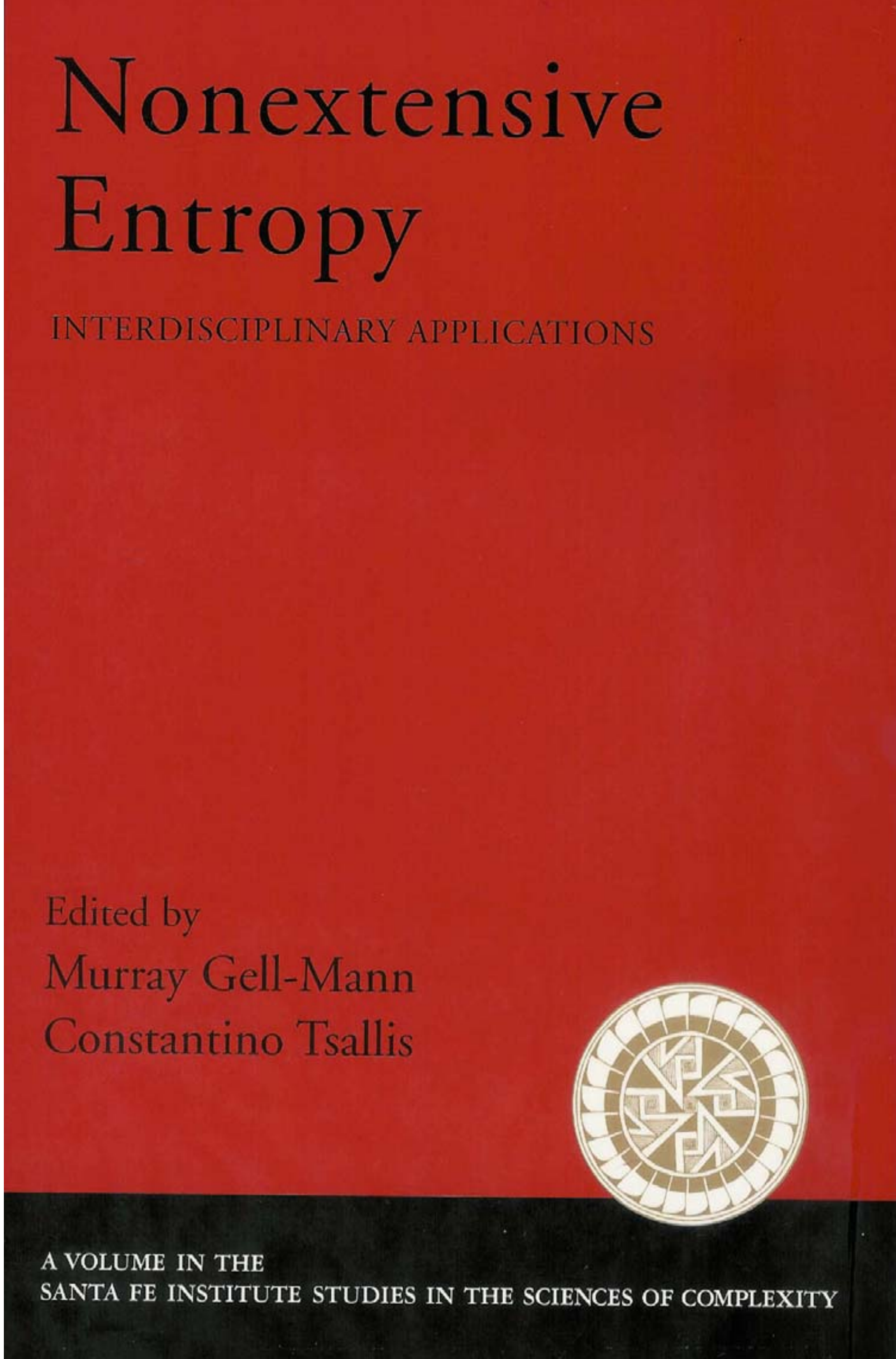
But if A *and* B *are* *especially (globally) correlated*

then

$$S_q(A+B) = S_q(A) + S_q(B)$$

whereas

$$S_{BG}(A+B) \neq S_{BG}(A) + S_{BG}(B)$$



ADDITIVITY: O. Penrose, *Foundations of Statistical Mechanics: A Deductive Treatment* (Pergamon, Oxford, 1970), page 167

An entropy is additive if, for two probabilistically independent systems A and B,

$$S(A+B)=S(A)+S(B)$$

Hence, S_{BG} and S_q^{Renyi} $(\forall q)$ are additive, and S_q $(\forall q \neq 1)$ is nonadditive .

EXTENSIVITY:

Consider a system $\Sigma \equiv A_1 + A_2 + \ldots + A_N$ made of N (not necessarily independent) identical elements or subsystems A_1 and A_2, ..., A_N. An entropy is extensive if

$$0 < \lim_{N \to \infty} \frac{S(N)}{N} < \infty \text{ , } i.e., \; S(N) \propto N \quad (N \to \infty)$$

CONSEQUENTLY:

The additive entropies S_{BG} and S_q^{Renyi} are extensive if and only if the N subsystems are (strictly or asymptotically) independent; otherwise, S_{BG} and S_q^{Renyi} are nonextensive.

The nonadditive entropy S_q $(q \neq 1)$ is extensive for special values of q if the subsystems are specially (globally) correlated.

MEPHISTOPHELES:

Denn eben wo Begriffe fehlen,

Da stellt ein Wort zur rechten Zeit sich ein.

Wolfgang von Goethe

[*Faust I, Vers 1995, Schuelerszene (1808)*]

For at the point where concepts fail,

At the right time a word is thrust in there.

King Thutmosis III
18th Dynasty
c. 1460 B. C.

TYPICAL EXAMPLES FOR THE CASE OF EQUAL PROBABILITIES:

1) *If* $W(N) \sim A\,\mu^N \quad (A>0,\ \mu>1,\ N\to\infty)$

then $\dfrac{S_{BG}(N)}{k} = \ln\,[W(N)] \sim (\ln\mu)\,N$, *i.e.*, S_{BG} *is extensive!*

whereas $\dfrac{S_q(N)}{k} = \ln_q\,[W(N)] = \dfrac{[W(N)]^{1-q}-1}{1-q} \sim \dfrac{A^{1-q}}{1-q}\,\mu^{N(1-q)} \neq constant\ N,$

i.e., S_q *is nonextensive for* $q<1$ (*and neither is for* $q>1$).

2) *If* $W(N) \sim B\,N^\rho \quad (B>0,\ \rho>0,\ N\to\infty)$

then $\dfrac{S_{BG}(N)}{k} = \ln\,[W(N)] \sim \rho\,\ln N$, *i.e.*, S_{BG} *is nonextensive*

whereas $\dfrac{S_q(N)}{k} = \ln_q\,[W(N)] = \dfrac{[W(N)]^{1-q}-1}{1-q} \sim \dfrac{B^{1-q}}{1-q}\,N^{\rho(1-q)}$, *i.e.*, $S_{1-\frac{1}{\rho}}$ *is extensive!*

3) *If* $W(N) \sim C\,\mu^{N^\gamma} \quad (C>0,\ \mu>1,\ \gamma\neq1,\ N\to\infty)$

Hence, there is no value of q (*neither* $q=1$ *nor* $q\neq1$) *for which* S_q *is extensive.*

HOW IT WORKS?

S. Abe, Phys Lett A **271**, 74 (2000) and Physica A **368**, 430 (2006)

We consider generic subsystems A and B:

$$p_{ij}(A+B) = p_i(A)\,p_j(B\,|\,A) = p_i(A\,|\,B)\,p_j(B) \quad \text{(Bayes theorem)}$$

implies

$$\frac{S_q(A+B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B\,|\,A)}{k} + (1-q)\frac{S_q(A)}{k}\frac{S_q(B\,|\,A)}{k}$$

$$= \frac{S_q(B)}{k} + \frac{S_q(A\,|\,B)}{k} + (1-q)\frac{S_q(B)}{k}\frac{S_q(A\,|\,B)}{k}$$

hence

$$S_q(A+B) = S_q(A) + S_q(B\,|\,A) + \frac{1-q}{k} S_q(A)\,S_q(B\,|\,A)$$

$$= S_q(B) + S_q(A\,|\,B) + \frac{1-q}{k} S_q(B)\,S_q(A\,|\,B)$$

A special class of correlations exists such that, for a special value of q,

$$S_q(A\,|\,B) + \frac{1-q}{k} S_q(B)\,S_q(A\,|\,B) = S_q(A)$$

and

$$S_q(B\,|\,A) + \frac{1-q}{k} S_q(A)\,S_q(B\,|\,A) = S_q(B)$$

hence $S_q(A+B) = S_q(A) + S_q(B)$ (extensivity!)

A MANY-BODY HAMILTONIAN ILLUSTRATION OF THE EXTENSIVITY OF S_q FOR ANOMALOUS VALUES OF q

SPIN ½ XY FERROMAGNET WITH TRANSVERSE MAGNETIC FIELD:

$$\hat{\mathcal{H}} = -\sum_{j=1}^{N-1} \left[(1+\gamma)\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + (1-\gamma)\hat{\sigma}_j^y \hat{\sigma}_{j+1}^y + 2\lambda \hat{\sigma}_j^z \right]$$

$|\gamma| = 1 \quad \rightarrow$ *Ising ferromagnet*

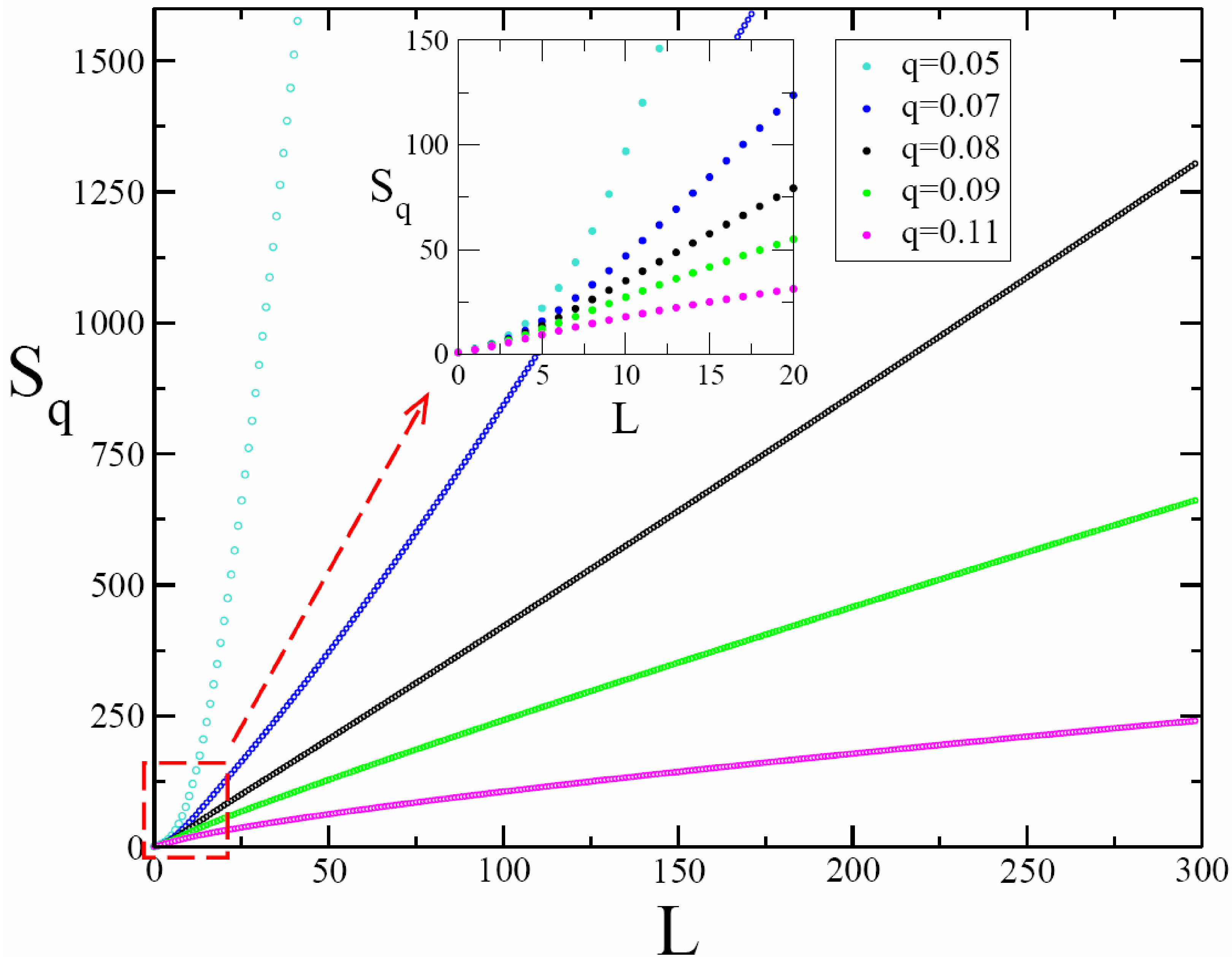
$0 < |\gamma| < 1 \quad \rightarrow$ *anisotropic XY ferromagnet*

$\gamma = 0 \quad \rightarrow$ *isotropic XY ferromagnet*

$\lambda \equiv$ *transverse magnetic field*

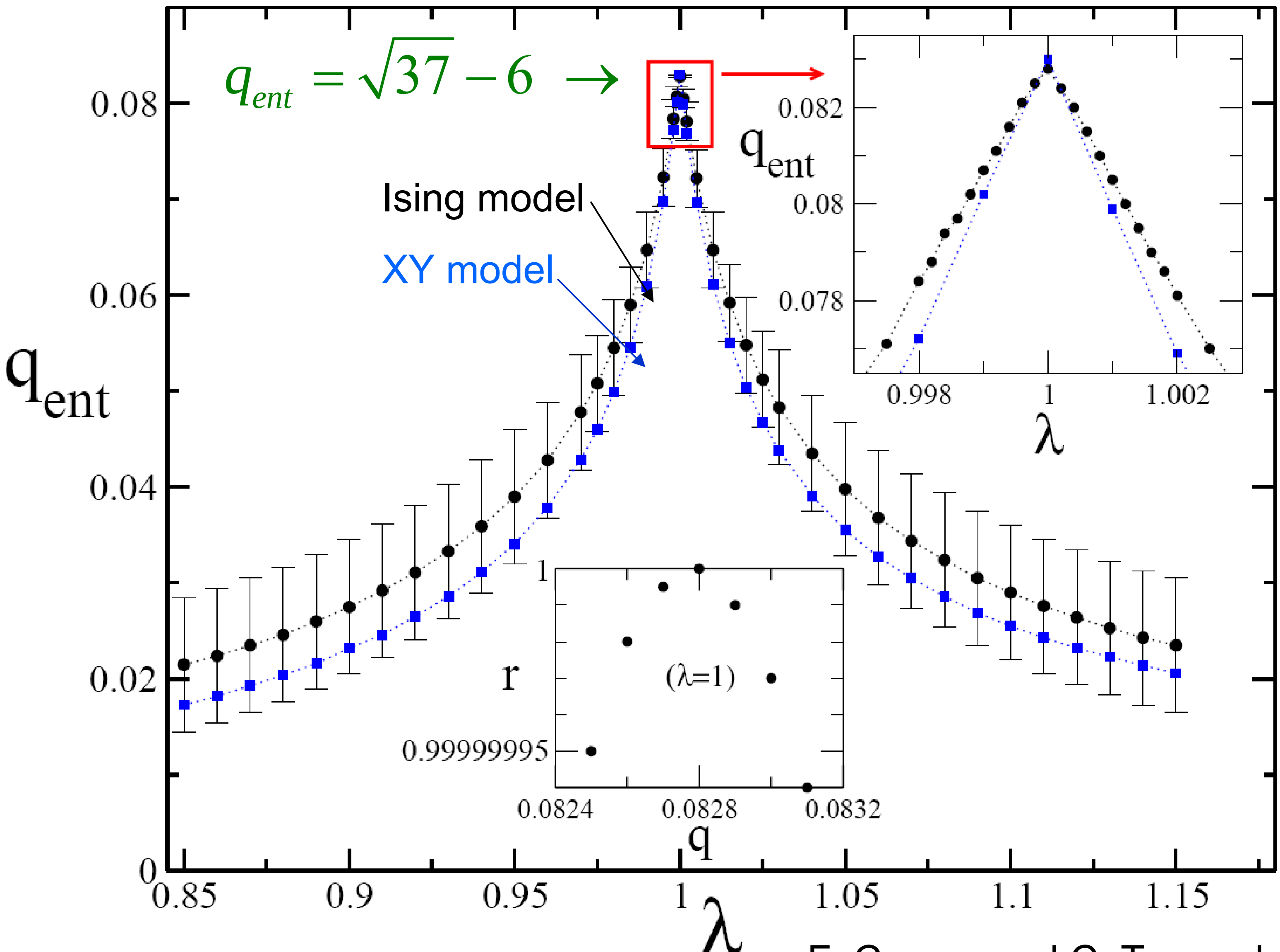
$L \equiv$ *length of a block within a* $N \rightarrow \infty$ *chain*

F. Caruso and C. T., cond-mat/0612032



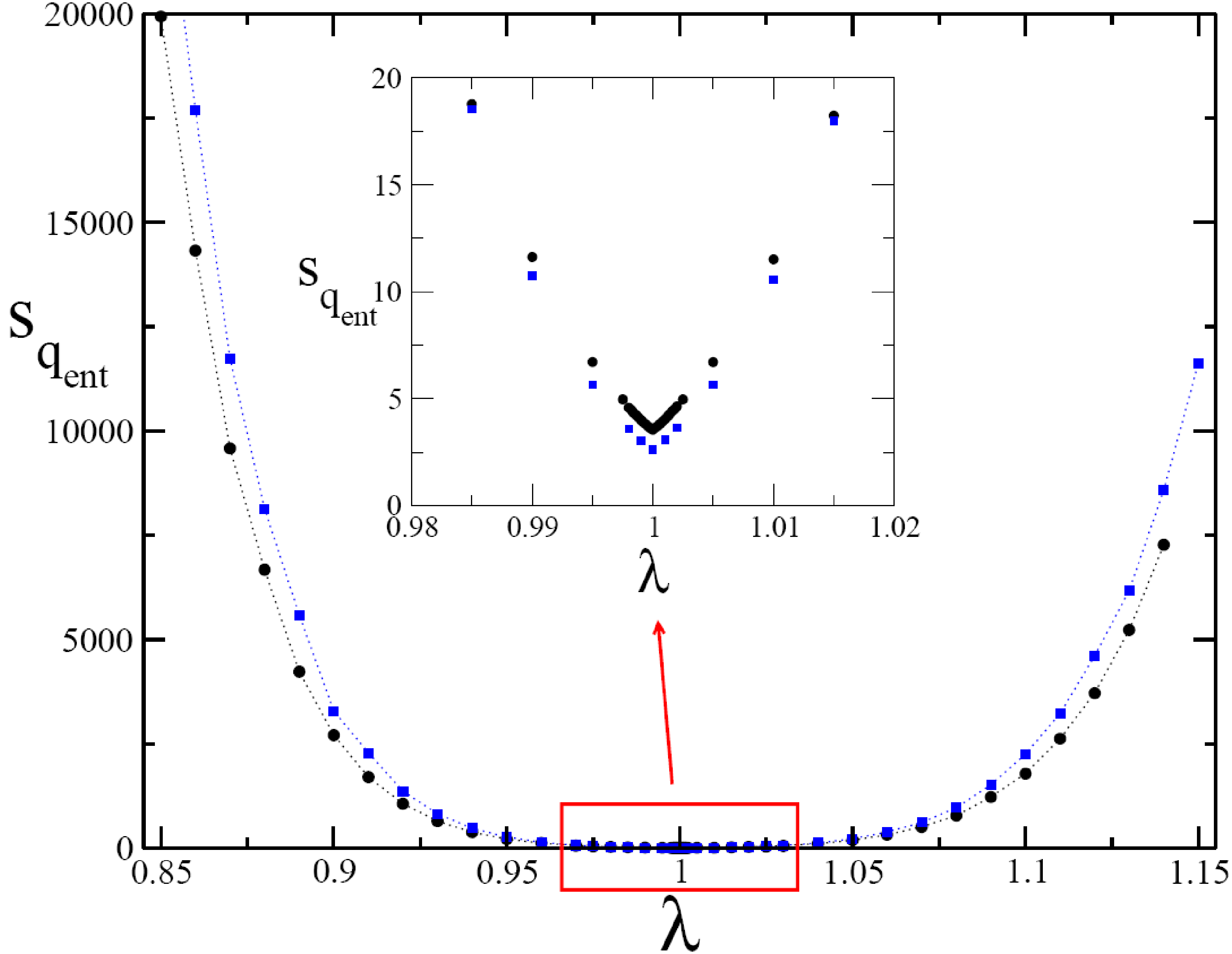
F. Caruso and C. T., cond-mat/0612032

$$S_{q_{ent}}(L) \sim s_{q_{ent}} L \qquad (L \to \infty)$$



F. Caruso and C. T., cond-mat/0612032

$$\lambda = 1 \Rightarrow S_{q_{ent}}(L) \sim s_{q_{ent}} L \qquad (L \to \infty) \qquad with \quad s_{q_{ent}} = 3.56 \pm 0.03$$



F. Caruso and C. T., cond-mat/0612032

Using a Quantum Field Theory result

in P. Calabrese and J. Cardy, JSTAT P06002 (2004)

we obtain, at the critical transverse magnetic field,

$$q_{ent} = \frac{\sqrt{9+c^2}-3}{c}$$

with $c \equiv$ *central charge in conformal field theory*

Hence

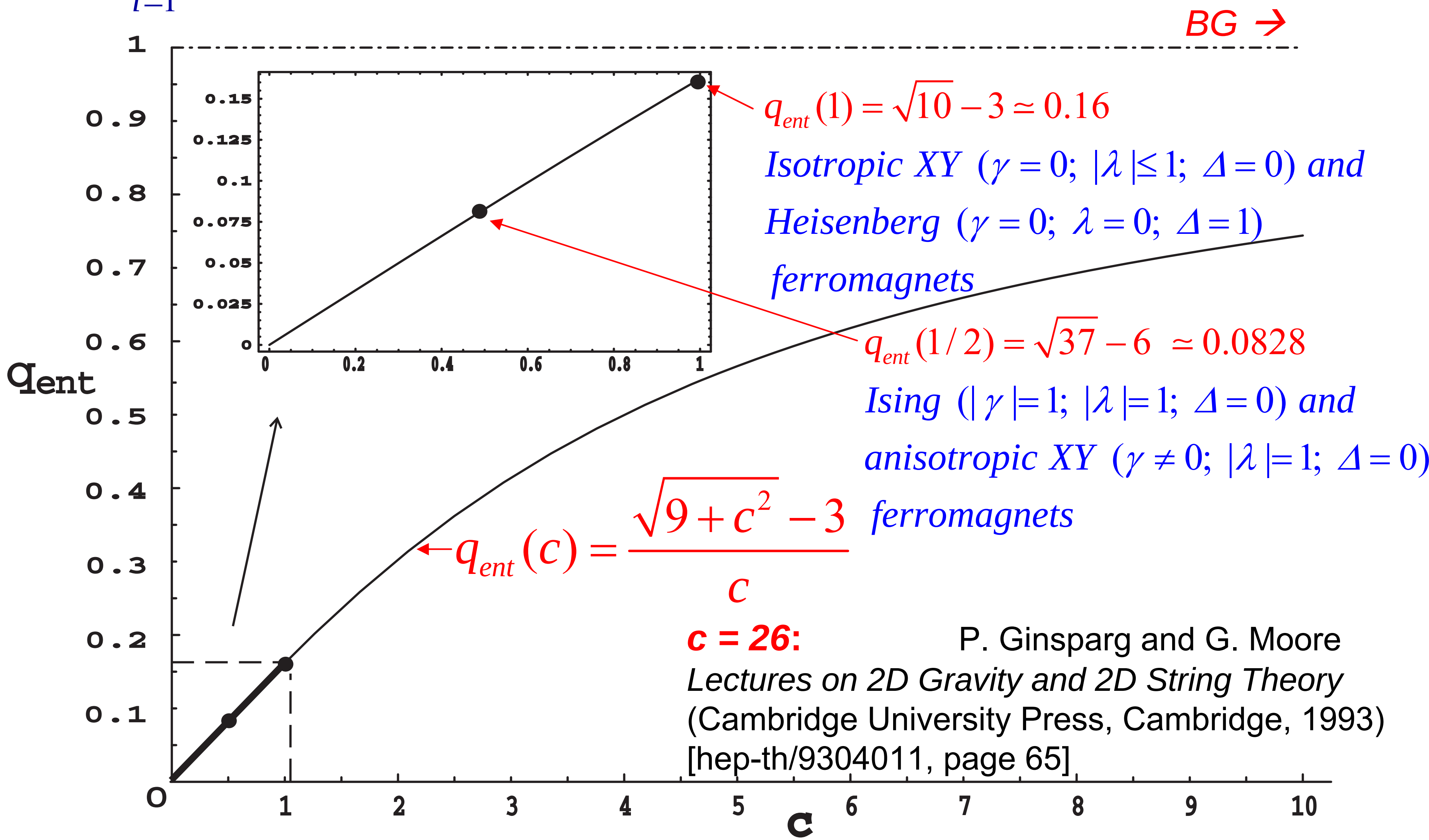
Ising and anisotropic XY ferromagnets $\Rightarrow\ c = \frac{1}{2}\ \Rightarrow\ q_{ent} = \sqrt{37}-6\ \simeq 0.0828$

and

Isotropic XY ferromagnet $\Rightarrow\ c = 1\ \Rightarrow\ q_{ent} = \sqrt{10}-3 \simeq 0.16$

F. Caruso and C. T., cond-mat/0612032

$$H = -\sum_{i=1}^{N-1}\left[(1+\gamma)\ \sigma_i^x\sigma_{i+1}^x + (1-\gamma)\ \sigma_i^y\sigma_{i+1}^y + \Delta\ \sigma_i^z\sigma_{i+1}^z + 2\lambda\ \sigma_i^z\right]$$



F. Caruso and C. T., cond-mat/0612032

In other words,

$$S_{\left[\sqrt{9+c^2}-3\right]c^{-1}}(L) \propto L \qquad \text{(extensive!)}$$

whereas

$$S_{BG}(L) \propto \ln L \qquad \text{(nonextensive!)}$$

REVISITING THE DIFFUSION OF ONE ELECTRON IN A MANY-BODY QUANTUM HAMILTONIAN

1D ANDERSON MODEL WITH LONG-RANGE CORRELATED DISORDER (METAL-INSULATOR TRANSITION):

F.A.B.F. de Moura and M.L. Lyra, Phys Rev Lett **81**, 3735 (1998)

One electron in a disordered linear chain [the disorder comes from a randomly-correlated potential characterized by a spectral density $s(k) \propto k^{-\alpha}$ $(\alpha \geq 0)$]:

$$H = \sum_{n=1}^{N} \varepsilon_n \,|n><n| + t_{\mathrm{H}} \sum_{n} \left[|n><n+1| + |n><n-1|\right] \qquad (t_{\mathrm{H}} = 1)$$

$$\textit{with} \quad \varepsilon_n = \sum_{k=1}^{N/2} \left[k^{-\alpha}\,|2\pi/N|^{1-\alpha}\right]^{1/2} \cos\left(\frac{2\pi nk}{N} + \Phi_k\right)$$

where Φ_k *are* $N/2$ *independent random phases uniformly distributed in* $[0, 2\pi]$

$\alpha = 2\mathrm{H} + 1$ (H $\equiv$ Hurst exponent)

$\alpha = 0 \;\Rightarrow\;$ *standard Anderson model [white noise spectrum, i.e., no correlation from site to site), i.e.,* $<\varepsilon_n \varepsilon_{n'}> = <\varepsilon_n{}^2> \delta_{n,n'}$]

$\alpha \to \infty \;\Rightarrow\;$ *crystal (no disorder)*

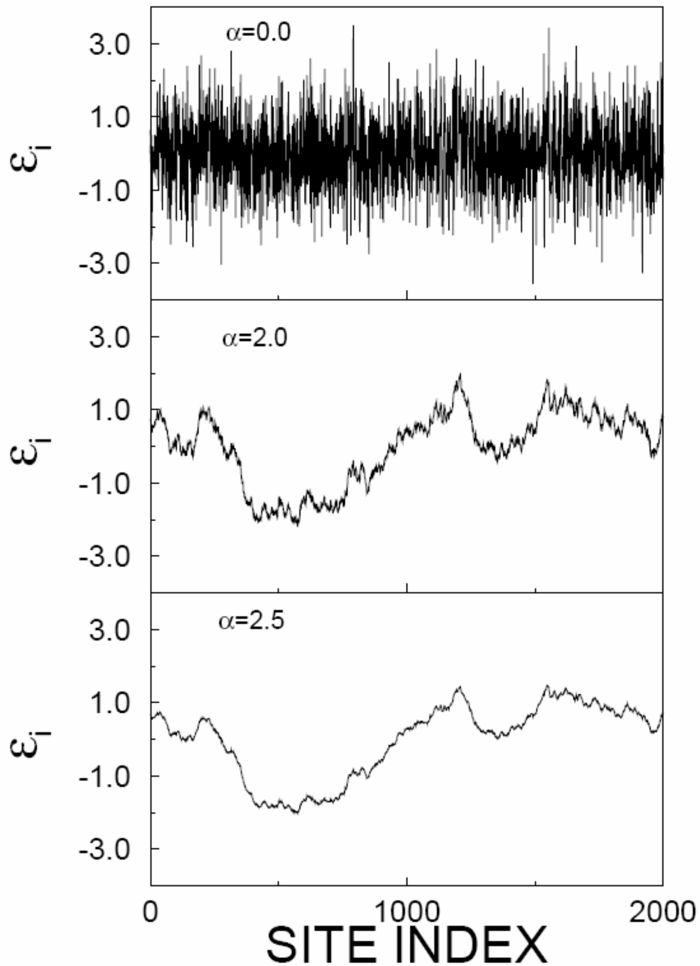


FIG. 1. Typical on-site energy landscapes generated from relation (2) with $N = 4096$: $\alpha = 0.0$: uncorrelated random sequency; $\alpha = 2.0$: trace of a usual Brownian motion; $\alpha = 2.5$: trace of a fractional Brownian motion with persistent increments. Notice the smoothening of the energy landscape for increasing values of α.

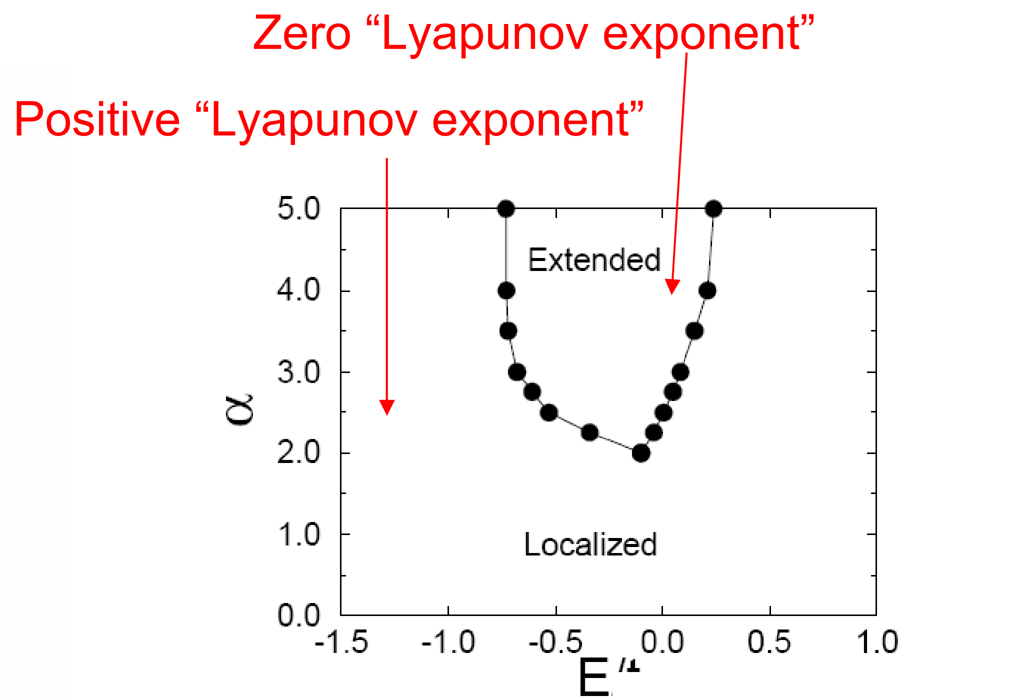
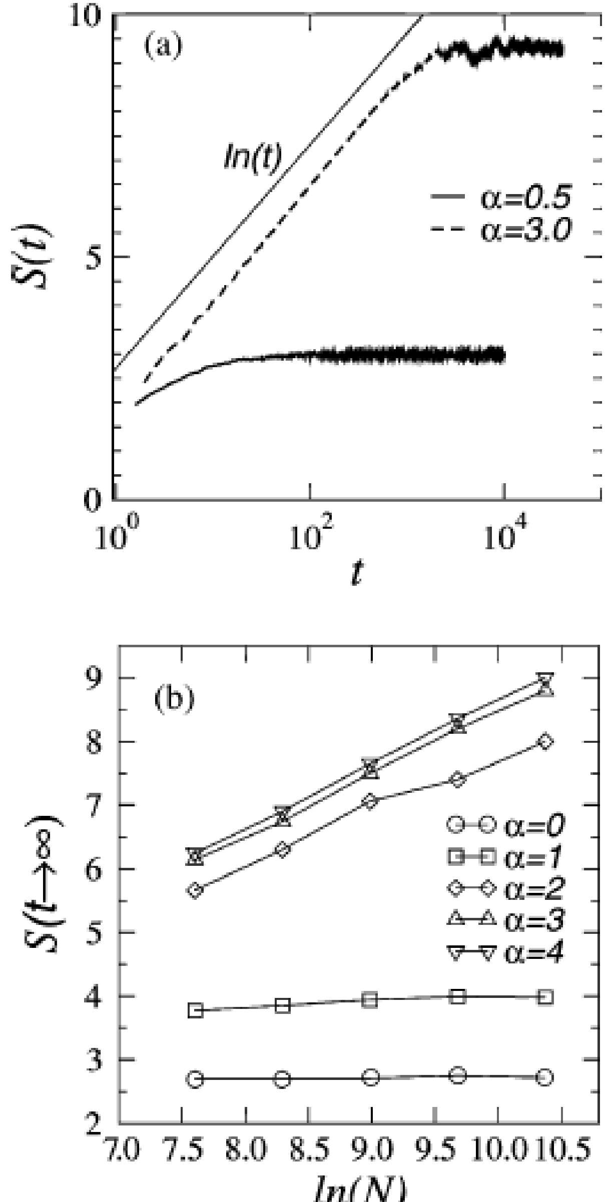


FIG. 5. Phase diagram in the $(E/t, \alpha)$ plane. Data were obtained from chains with 10^4 sites, $\Delta\epsilon/t = 1.0$, and the same random phases sequency. The phase of extended states emerges for $\alpha > 2$, and its width saturates as $\alpha \to \infty$. The band of allowed states ranges approximately from $-4.0 < E < 4.0$ and is independent of α by construction (see text).

F.A.B.F. de Moura and M.L. Lyra, Phys Rev Lett **81**, 3735 (1998)

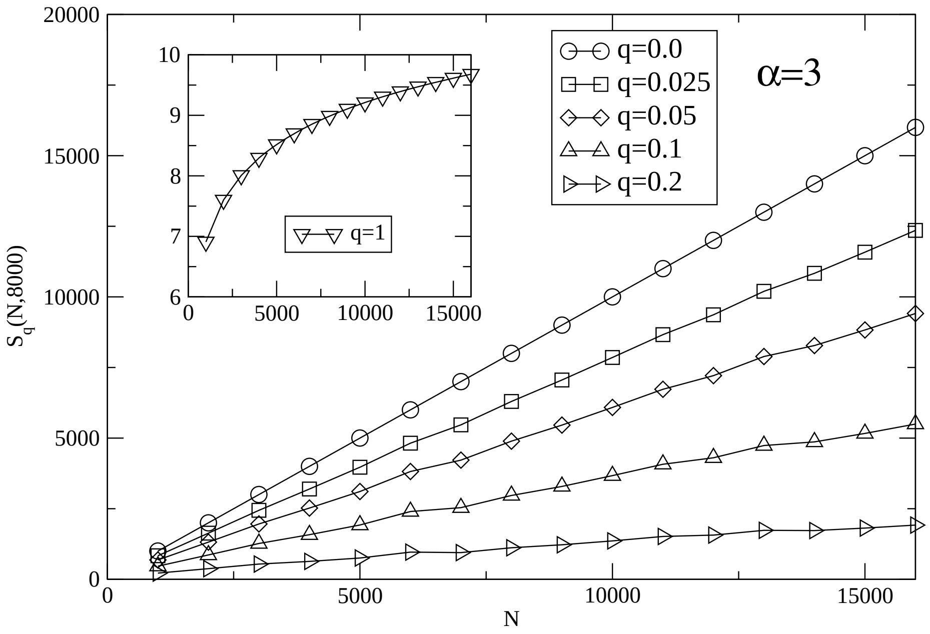
$S_{BG}(8000,t)$

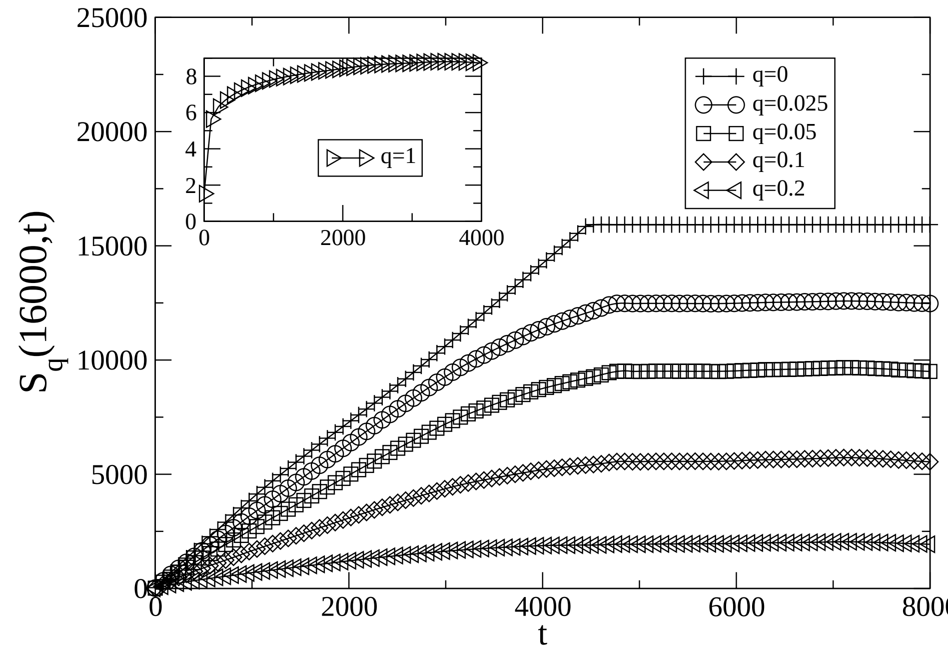
$S_{BG}(N,t>>1)$

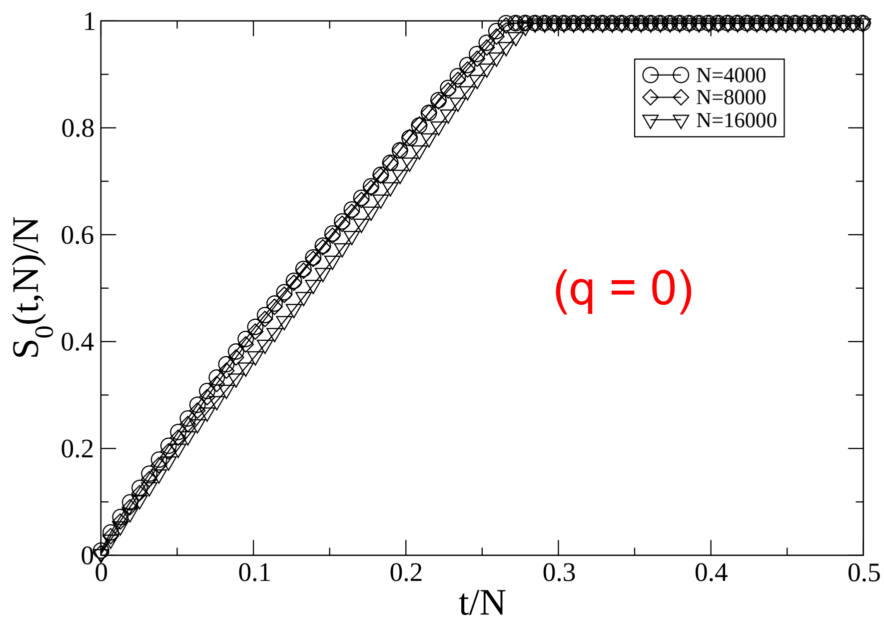


B. Santos, L.P. Viana, M.L. Lyra and F.A.B.F. de Moura, Sol State Comm **138**, 585 (2006)

REVISITING THE PREVIOUS RESULTS:



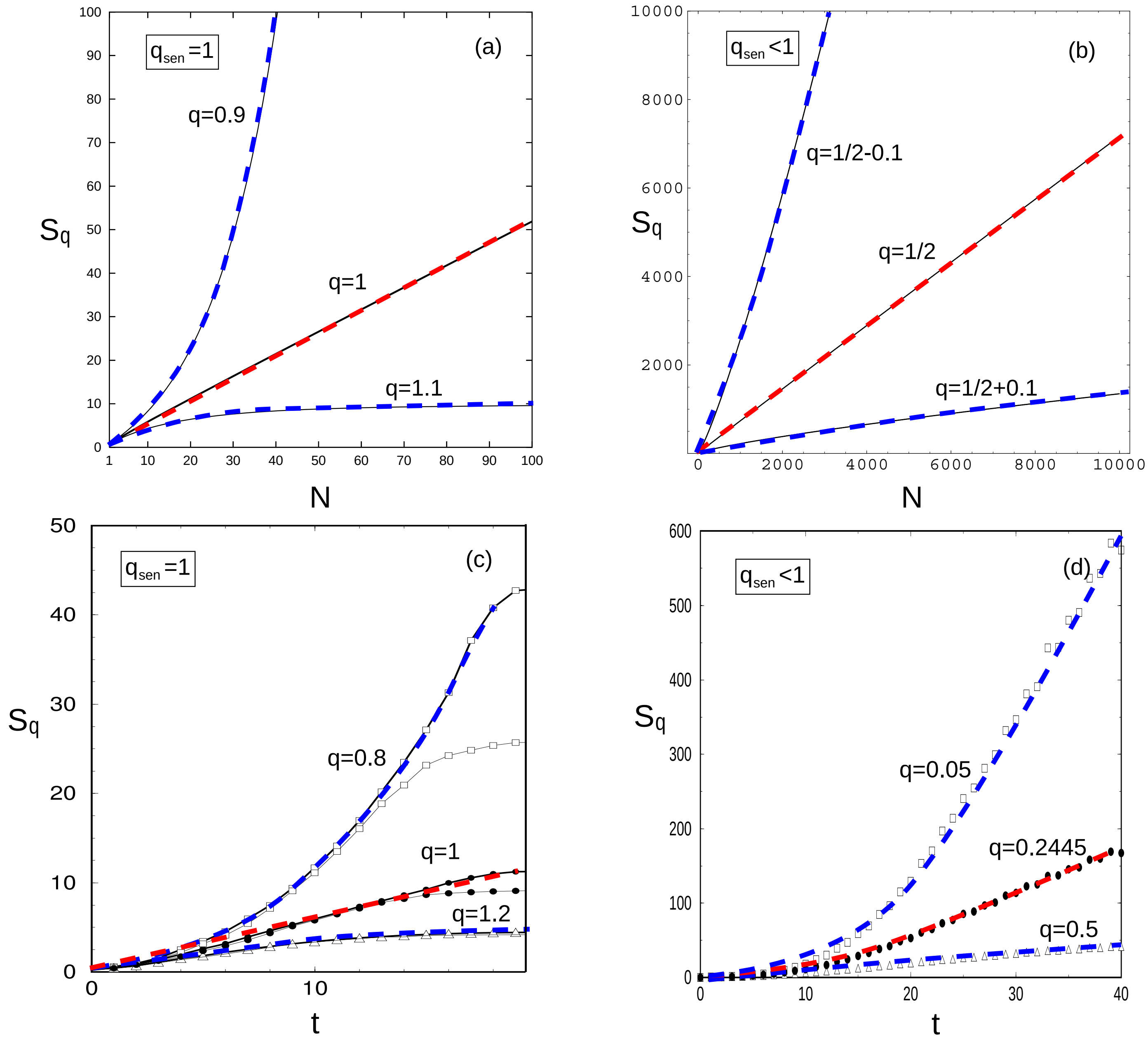




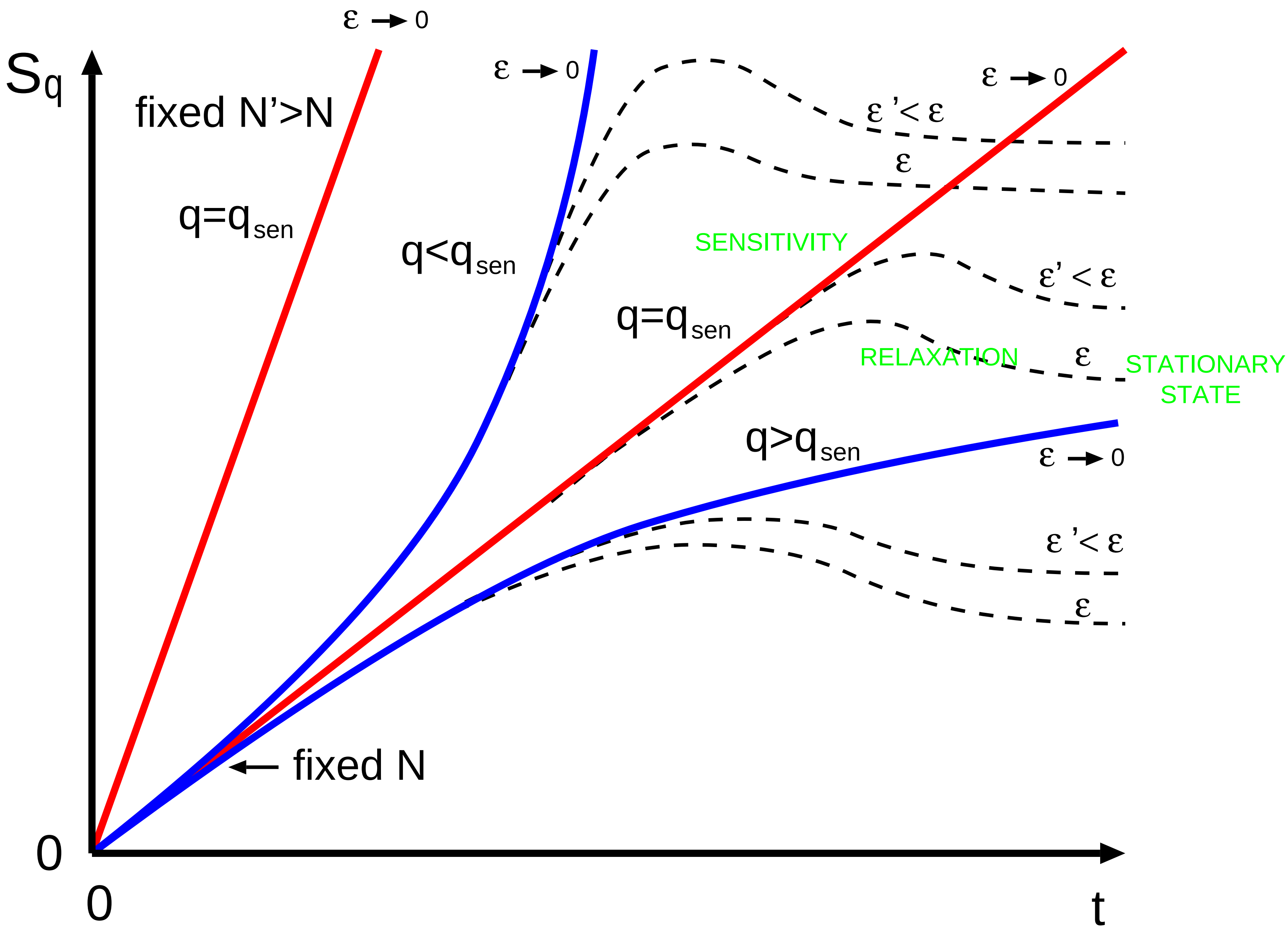
$(\alpha = 3)$

F.A.B.F. de Moura and M.L. Lyra (2007)

$$S_q(N,t) \; versus \; (t,N)$$



C.T., M. Gell-Mann and Y. Sato, Europhysics News **36** (6), 186 (2005) [European Physical Society]



C.T., M. Gell-Mann and Y. Sato, Europhysics News **36** (6) (Nov-Dec 2005)
Special Issue *Nonextensive Statistical Mechanics – New Trends, New Perspectives*
(European Physical Society)

A conjecture for $S_q(N,t)$:

For $q = q_{sen}$, $N \to \infty$ *and* $t \to \infty$ *play essentially the same role.*
In particular,

i) Under conditions of ***infinitely fine*** *graining in phase space,*

$$S_{q_{sen}}(N,t) \sim K_{q_{sen}}(N)\; t \;\propto\; N\; t$$

$$\left(q_{sen} = 1 \;\Rightarrow\; K_1 = \sum_{j/\lambda_1^{(j)}>0} \lambda_1^{(j)},\ \textit{i.e., }\text{Pesin}-\textit{like identity for finite } N\right)$$

ii) Under conditions of ***finite*** *graining in phase space,*

$$\lim_{t\to\infty} S_{q_{sen}}(N,t) \propto N$$

(Clausius)

C.T., M. Gell-Mann and Y. Sato
Europhysics News **36**, 186 (2005)

NONEXTENSIVE STATISTICAL MECHANICS
AND
THERMODYNAMICS

C. T.

Possible generalization of Boltzmann-Gibbs statistics

J Stat Phys **52**, 479 (1988)

E.M.F. Curado and C. T.

Generalized statistical mechanics: connection with thermodynamics

J Phys A **24**, L69 (1991)

[Corrigenda: **24**, 3187 (1991) and **25**, 1019 (1992)]

C. T., R.S. Mendes and A.R. Plastino

The role of constraints within generalized nonextensive statistics

Physica A **261**, 534 (1998)

NONEXTENSIVE STATISTICAL MECHANICS AND THERMODYNAMICS (CANONICAL ENSEMBLE):

Extremization of the functional

$$S_q[p_i] \equiv k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}$$

with the constraints

$$\sum_{i=1}^{W} p_i = 1 \quad \text{and} \quad \frac{\sum_{i=1}^{W} p_i^q E_i}{\sum_{i=1}^{W} p_i^q} = U_q$$

yields

$$p_i = \frac{e_q^{-\beta_q (E_i - U_q)}}{\mathbb{Z}_q}$$

with $\beta_q \equiv \frac{\beta}{\sum_{i=1}^{W} p_i^q}$, $\beta \equiv$ *energy Lagrange parameter*, *and* $\mathbb{Z}_q \equiv \sum_{i=1}^{W} e_q^{-\beta_q (E_i - U_q)}$

We can rewrite $p_i = \dfrac{e_q^{-\beta_q' E_i}}{Z_q'}$

with $\beta_q' \equiv \dfrac{\beta_q}{1+(1-q)\beta_q U_q}$, and $Z_q' \equiv \sum_{i=1}^{W} e_q^{-\beta_q' E_i}$

And we can prove

(i) $\dfrac{1}{T} = \dfrac{\partial S_q}{\partial U_q}$ with $T \equiv \dfrac{1}{k\beta}$

(ii) $F_q \equiv U_q - TS_q = -\dfrac{1}{\beta}\ln_q Z_q$ where $\ln_q Z_q = \ln_q \mathbb{Z}_q - \beta U_q$

(iii) $U_q = -\dfrac{\partial}{\partial \beta}\ln_q Z_q$

(iv) $C_q \equiv T\dfrac{\partial S_q}{\partial T} = \dfrac{\partial U_q}{\partial T} = -T\dfrac{\partial^2 F_q}{\partial T^2}$

(*i.e., the Legendre structure of Thermodynamics is q-invariant!*)

SOME FORM-INVARIANT RELATIONS (arbitrary q)

CLAUSIUS INEQUALITY AND BOLTZMANN H-THEOREM (macroscopic time irreversibility)

Mariz, Phys Lett A **165** (1992) 409; Ramshaw, Phys Lett A **175** (1993) 169; Abe and Rajagopal, Phys Rev Lett **91** (2003)

$$\beta\,\delta Q_q \le \delta S_q \quad ; \qquad q\,\frac{dS_q}{dt} \ge 0$$

EHRENFEST THEOREM (correspondence principle)

Plastino and Plastino, Phys Lett A **177** (1993) 177

$$\frac{d}{dt}\left\langle \hat{O} \right\rangle_q = \frac{i}{\hbar}\left\langle \left[\hat{H}, \hat{O} \right] \right\rangle_q$$

FACTORIZATION OF LIKELIHOOD FUNCTION (Einstein's 1910 reversal of Boltzmann's formula; thermodynamically independent systems)

Caceres and Tsallis, unpublished (1993); Chame and Mello, J Phys A **27** (1994) 3663; Tsallis, Chaos, Solitons and Fractals **6** (1995) 539

$$W_q(A+B) = W_q(A)\,W_q(B)$$

ONSAGER RECIPROCITY THEOREM (microscopic time reversibility)

Caceres, Physica A **218** (1995) 471; Rajagopal, Phys Rev Lett **76** (1996) 3469; Chame and Mello, Phys Lett A **228** (1997) 159

$$L_{jk} = L_{kj}$$

KRAMERS AND KRONIG RELATION (causality)

Rajagopal, Phys Rev Lett **76** (1996) 3469

PESIN EQUALITY (mixing; Kolmogorov-Sinai entropy and Lyapunov exponent)

Tsallis, Plastino and Zheng, Chaos, Solitons and Fractals **8** (1997) 885; Baldovin and Robledo, Phys. Rev. E **69**, 045202(R) (2004).

$$K_q = \begin{cases} \lambda_q & \text{if } \lambda_q > 0 \\ 0 & \text{otherwise} \end{cases}$$

WHY USING ESCORT DISTRIBUTIONS FOR THE CONSTRAINTS?

1) The optimizing probability distribution is automatically invariant with regard to uniform translation of the energy eigenvalues (zero-point invariance).

2) The constraints $\sum_i p_i = 1$ and $\sum_i P_i \, E_i = constant$, where $P_i \equiv \frac{p_i^q}{\sum_j p_j^q}$ and $p_i \equiv \frac{P_i^{1/q}}{\sum_j P_j^{1/q}}$, are finite up to a common upper bound for q ($e.g$, for $E(x) \propto x^2$, it must be $q < 3$).

3) $\frac{de_q^x}{dx} = (e_q^x)^q \neq e_q^x$ yields $\{P_i\}$ instead of $\{p_i\}$ in the steepest-descent-method calculation of the stationary-state distribution [*Abe and Rajagopal, J Phys A* 33, 8733 (2000)].

4) The conditional entropy naturally appears [*Abe, Phys Lett A* 271, 74 (2000)] as a q-expectation value without involving any optimization principle.

5) The principle of minimal relative entropy is consistent as a rule of statistical inference (1980 Shore-Johnson axioms) only if [*Abe and Bagci, Phys Rev E* 71, 016139 (2005)] we select the q-expectation values if we use the entropy S_q.