The transitions to chaos
Introductory Lecture

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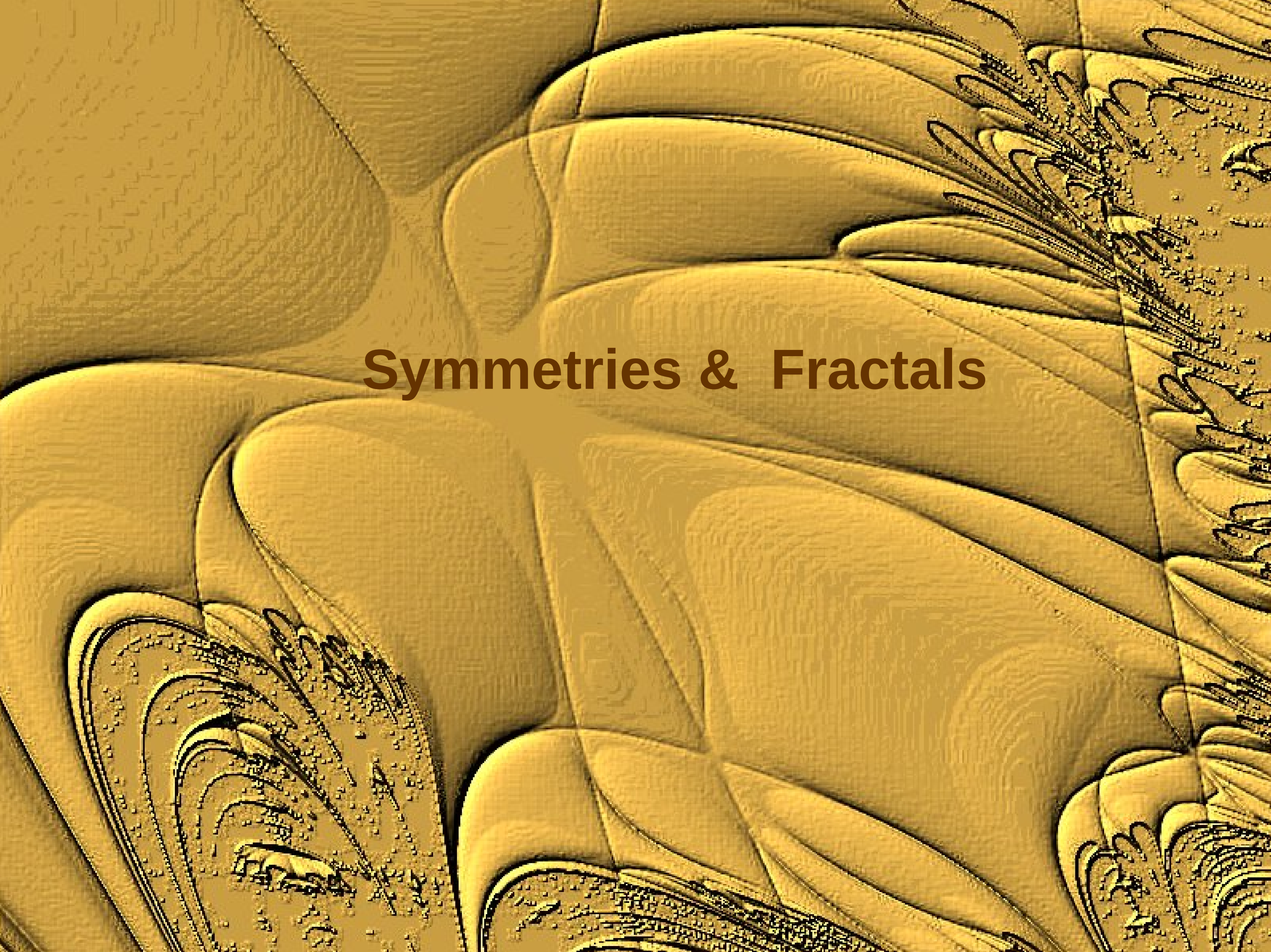
Nonextensive Statistical Mechanics
CBPF, Rio de Janeiro, April 2nd 2007

- **Symmetries**
- **Fractals**
- **Chaos**
- **Between Order & Chaos**

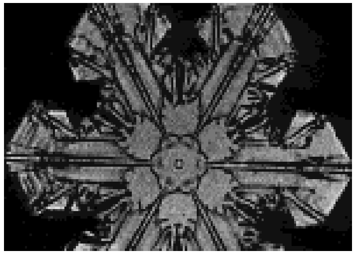
Symmetries & Fractals



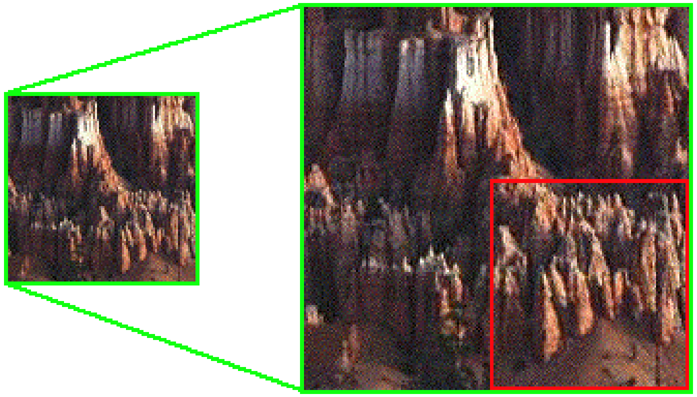
Symmetries

Translation

Reflection

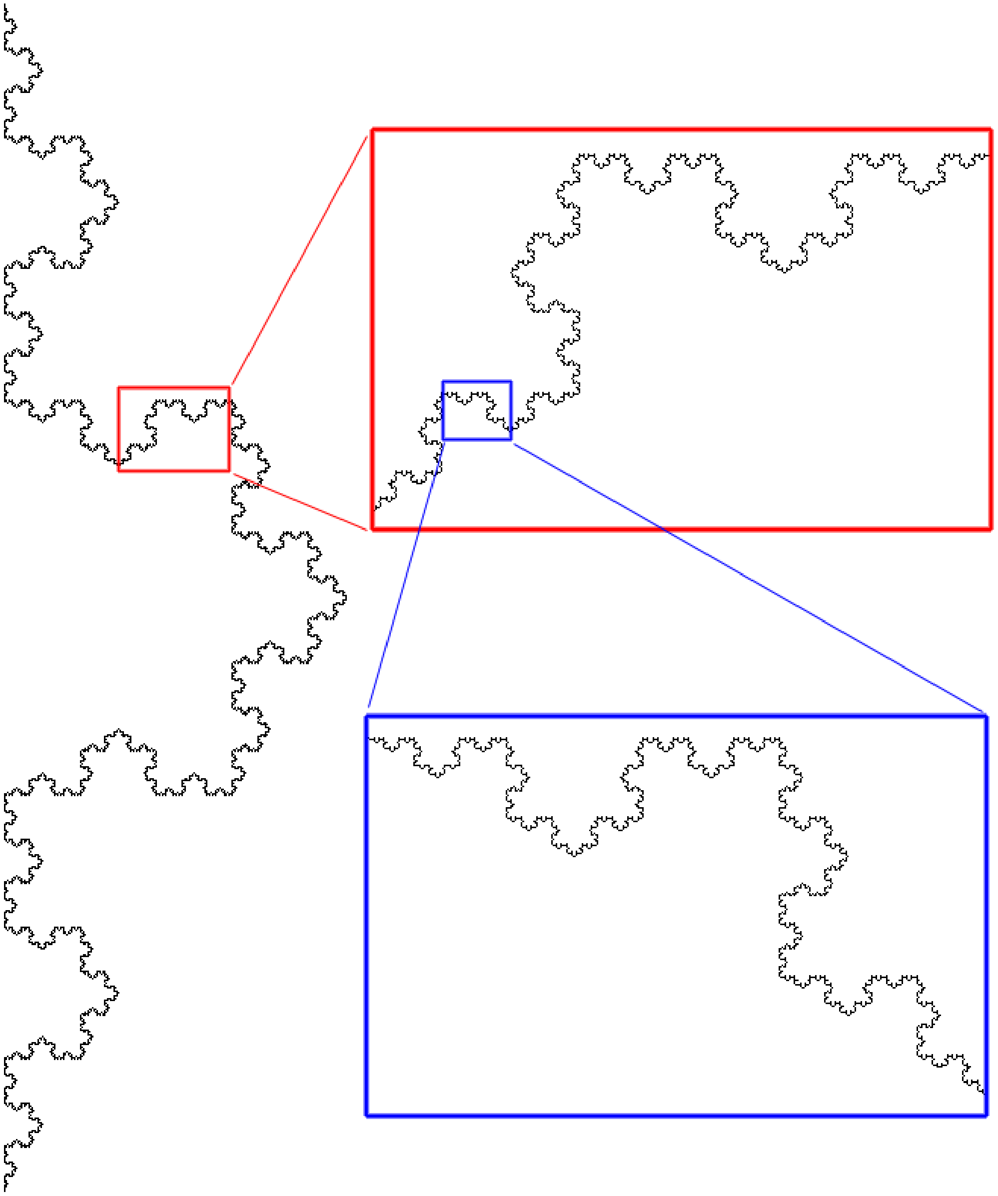


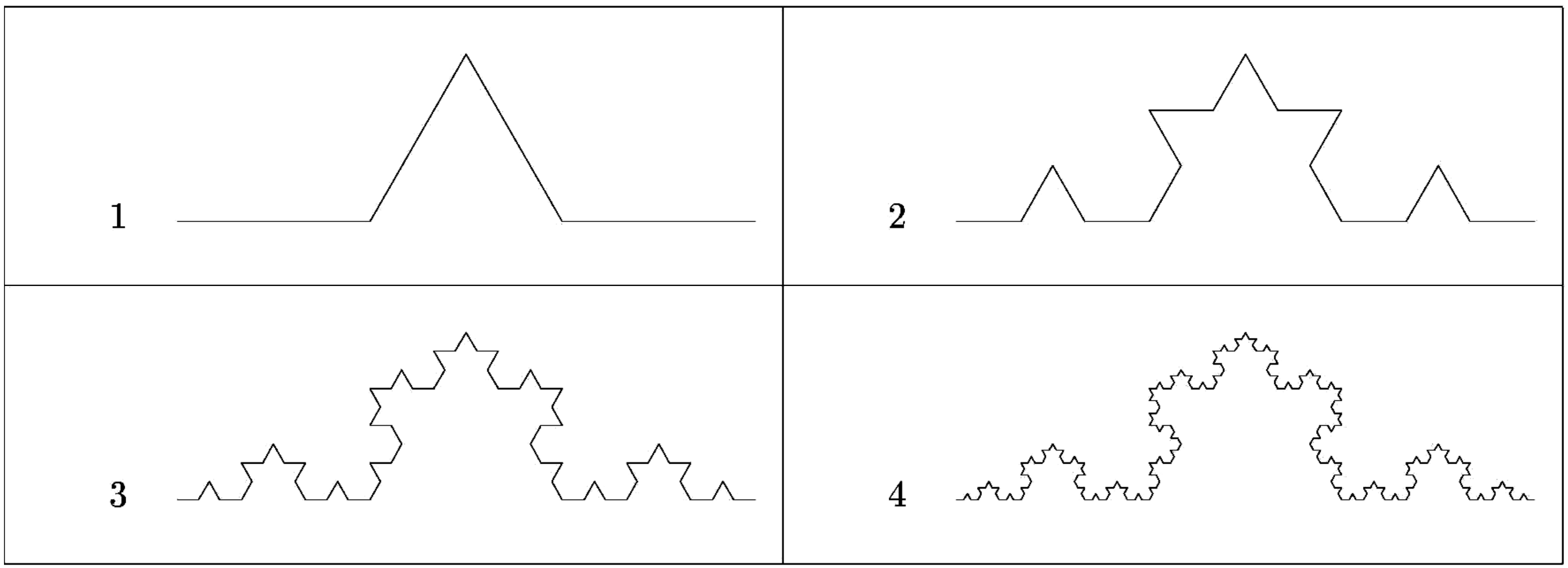
Rotation

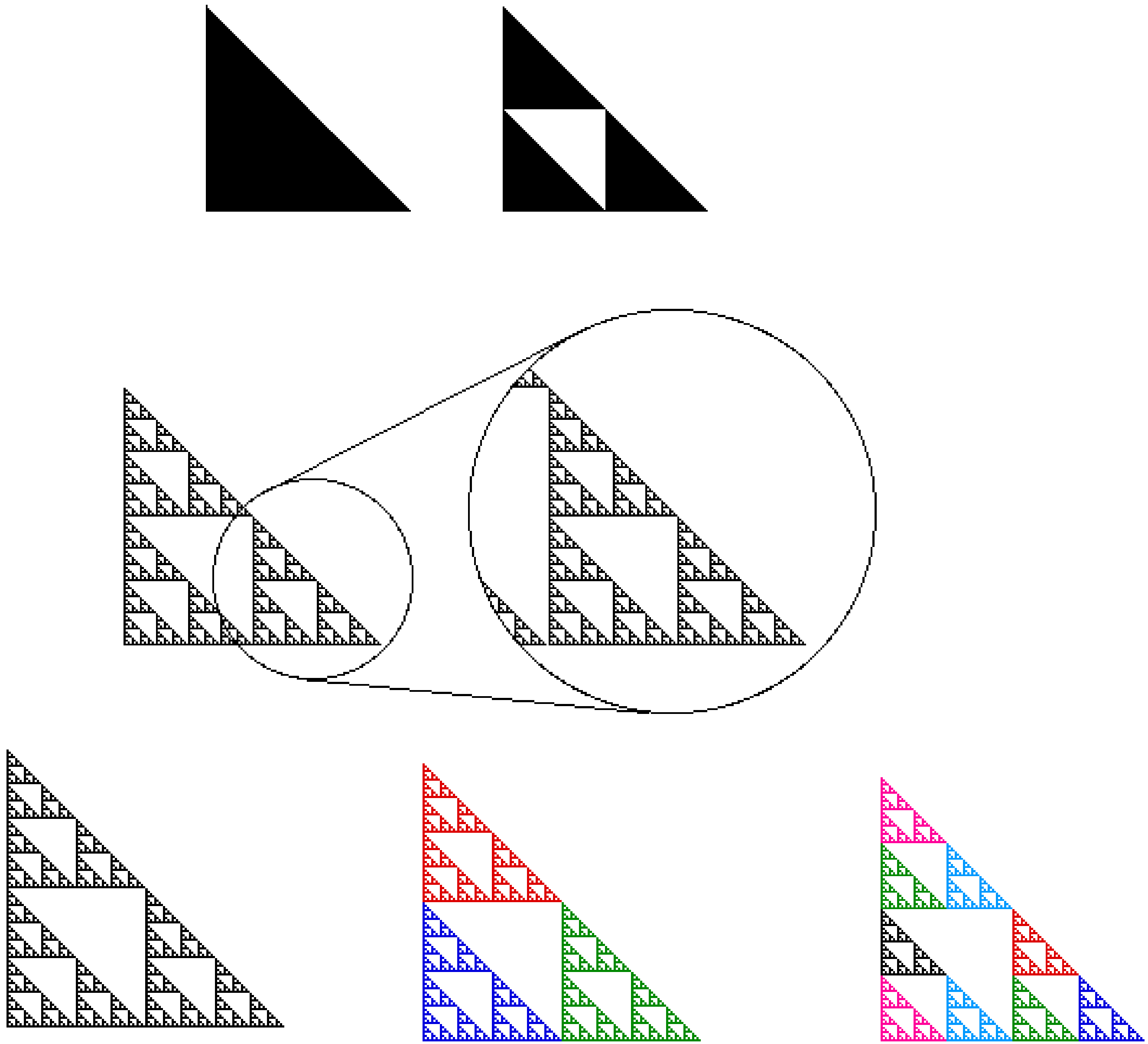


Scaling

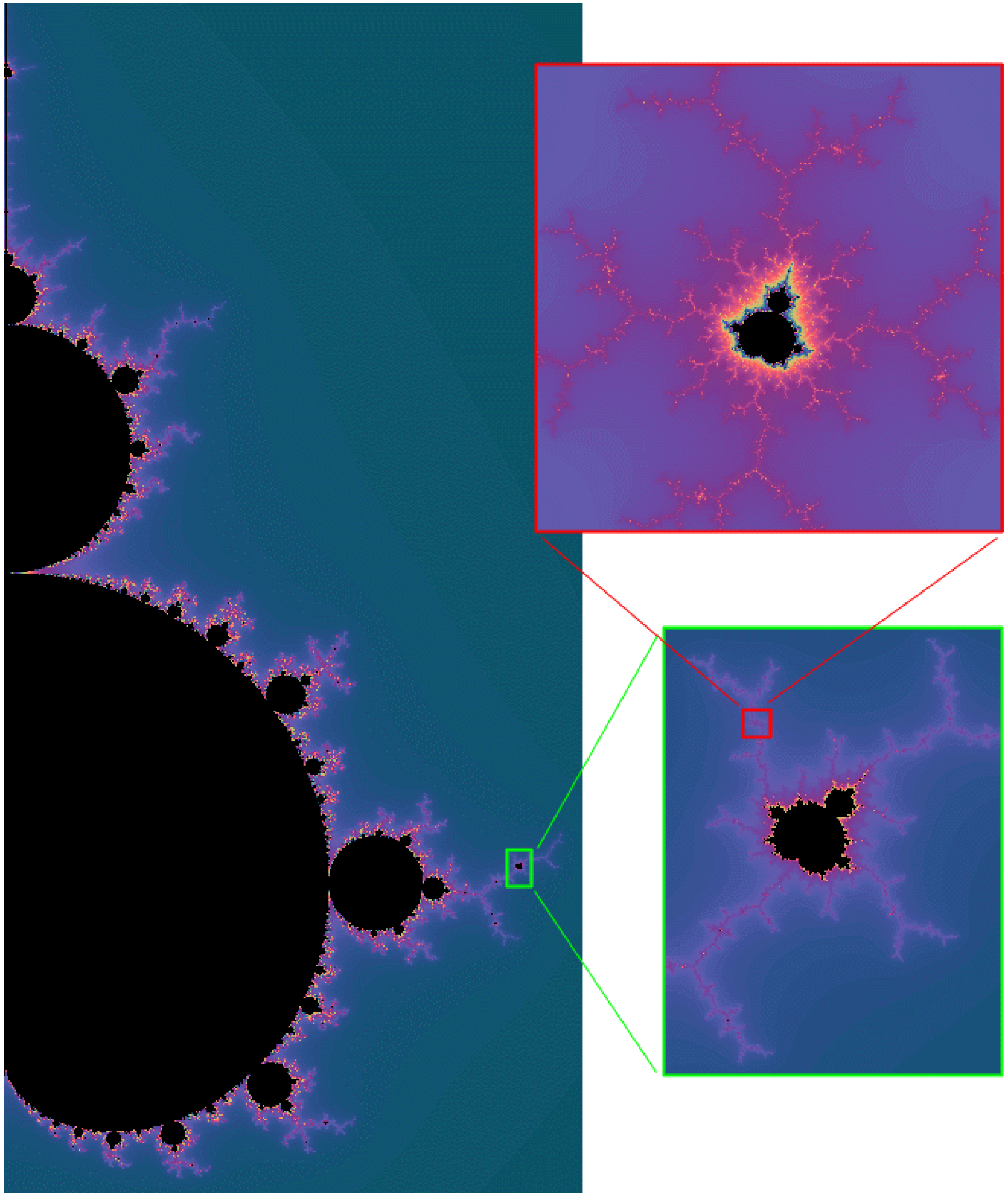
Exact self similarity



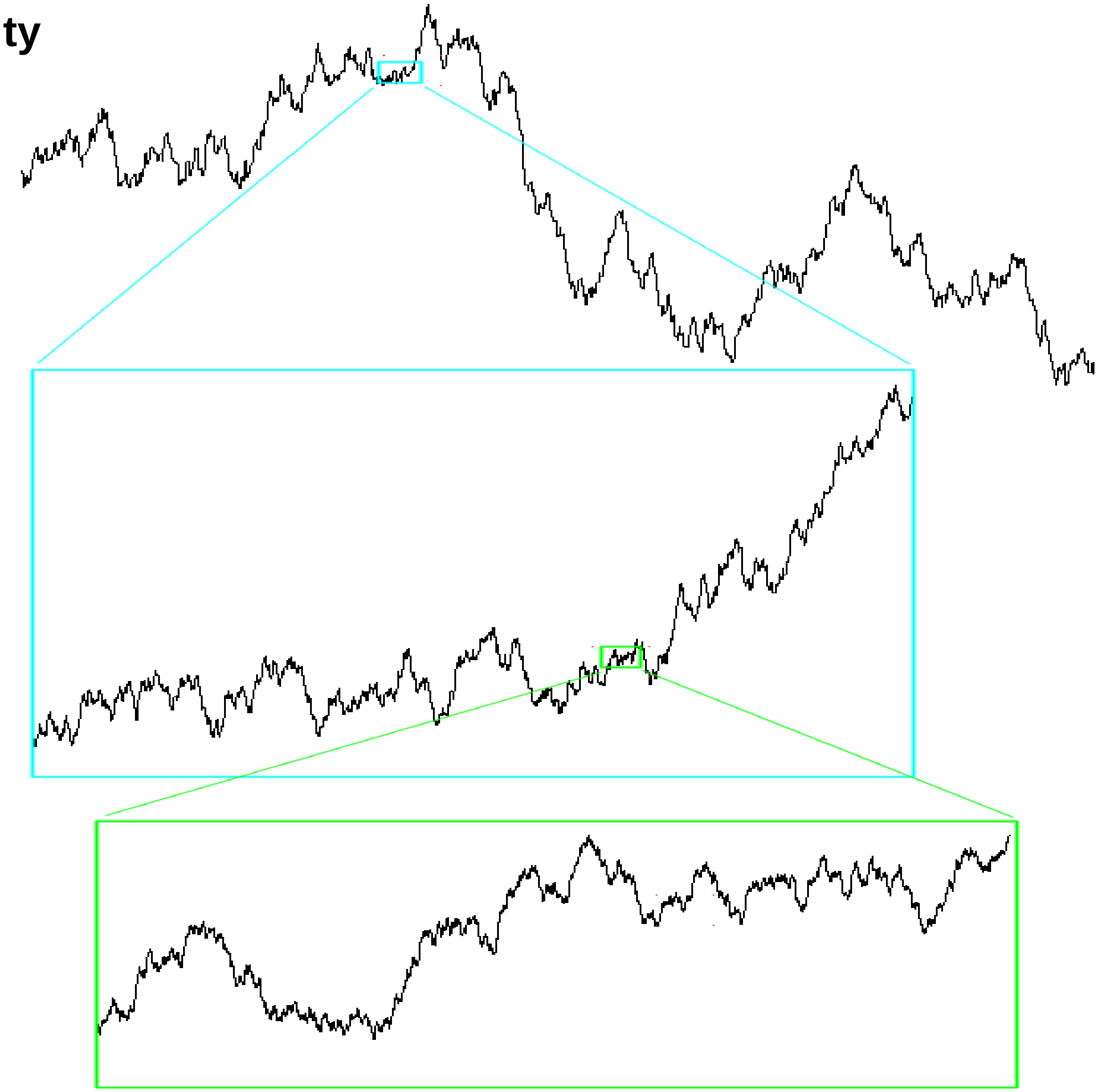

1
2
3
4

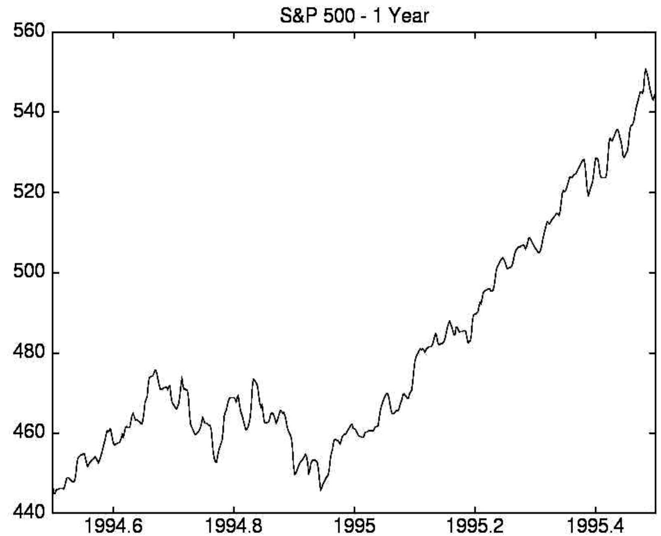


Approximate self similarity

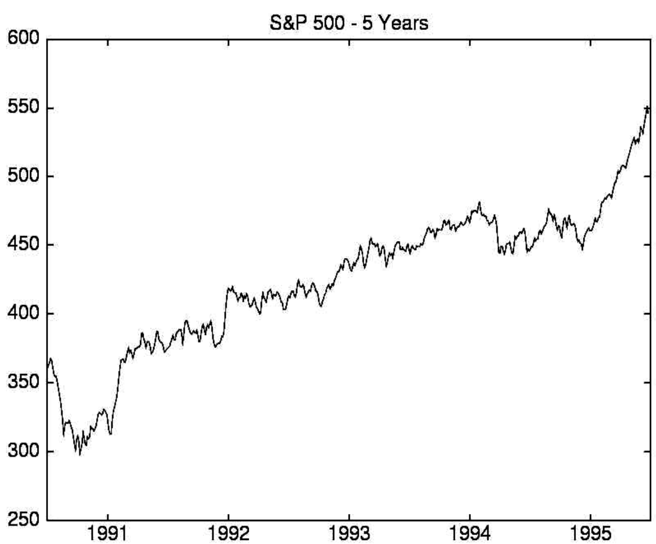


Statistical self similarity

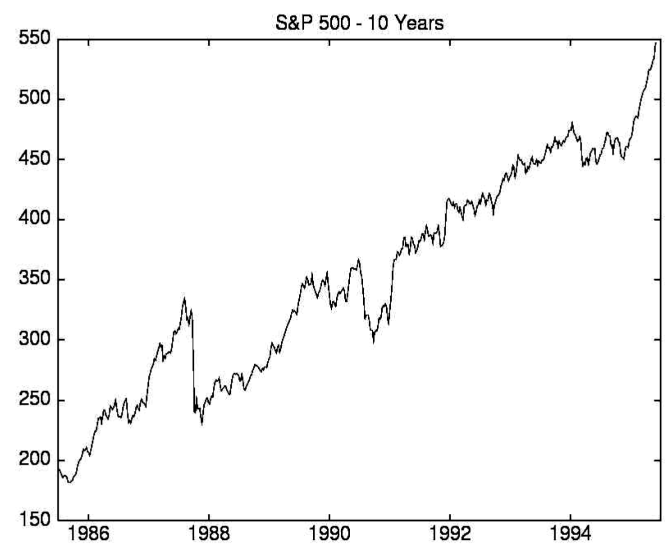



S&P 500 - 1 Year
560
540
520
500
480
460
440
1994.6
1994.8
1995
1995.2
1995.4

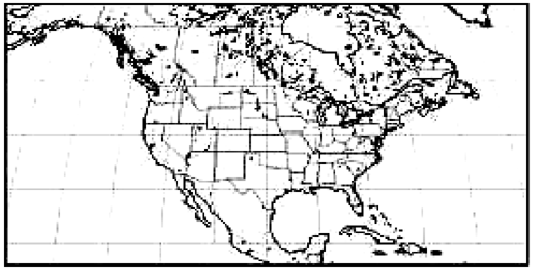
(a)


S&P 500 - 5 Years
600
550
500
450
400
350
300
250
1991
1992
1993
1994
1995

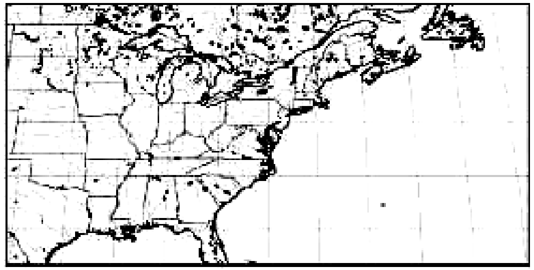
(b)


S&P 500 - 10 Years
550
500
450
400
350
300
250
200
150
1986
1988
1990
1992
1994

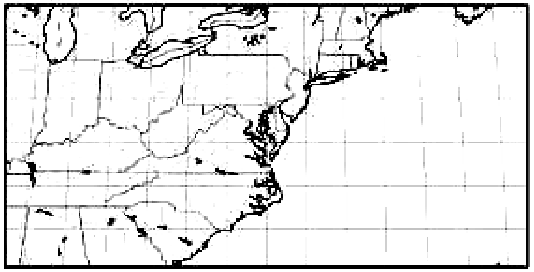
(c)



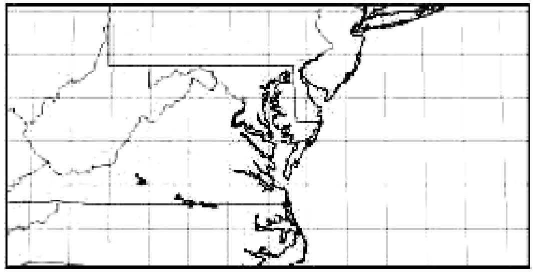
(a)



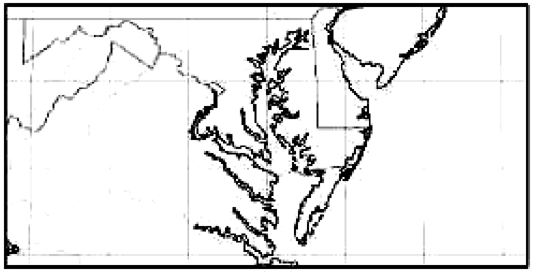
(b)



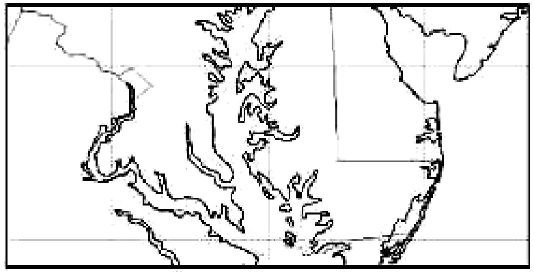
(c)



(d)

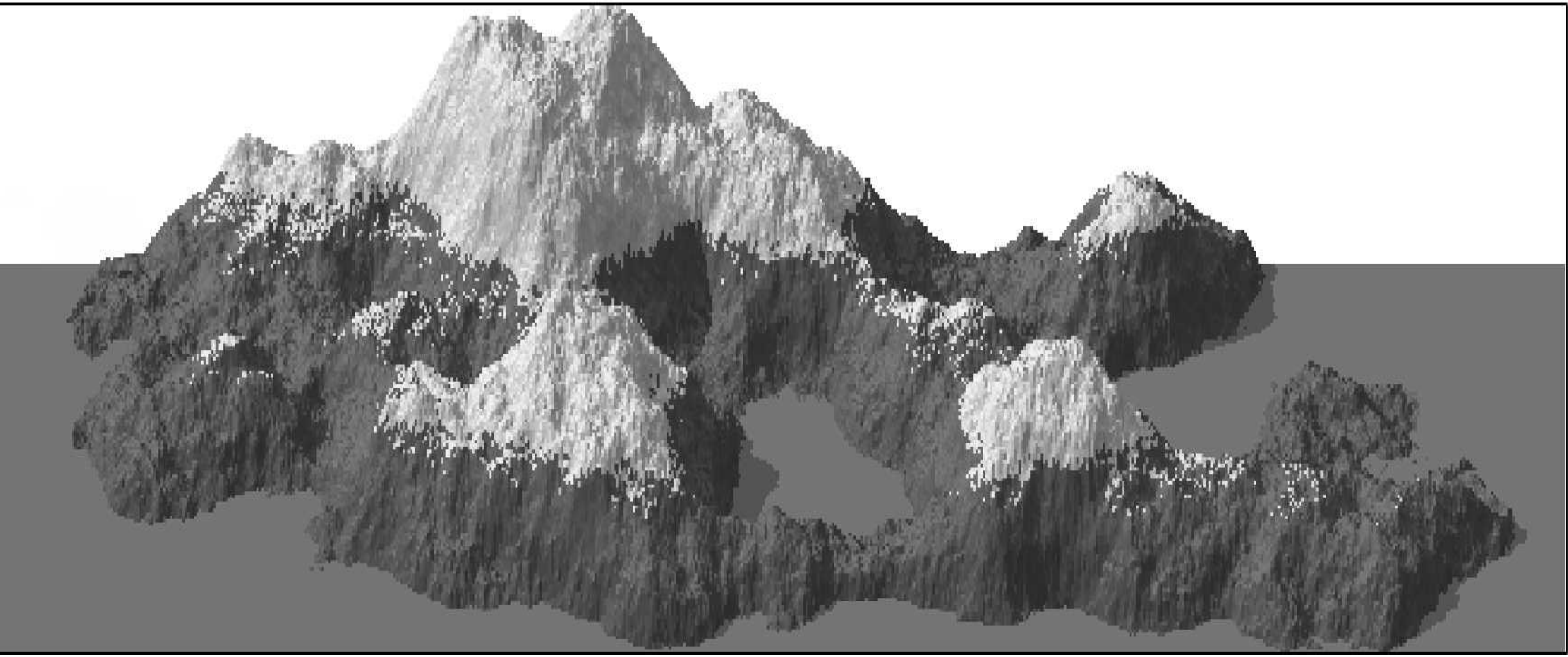


(e)

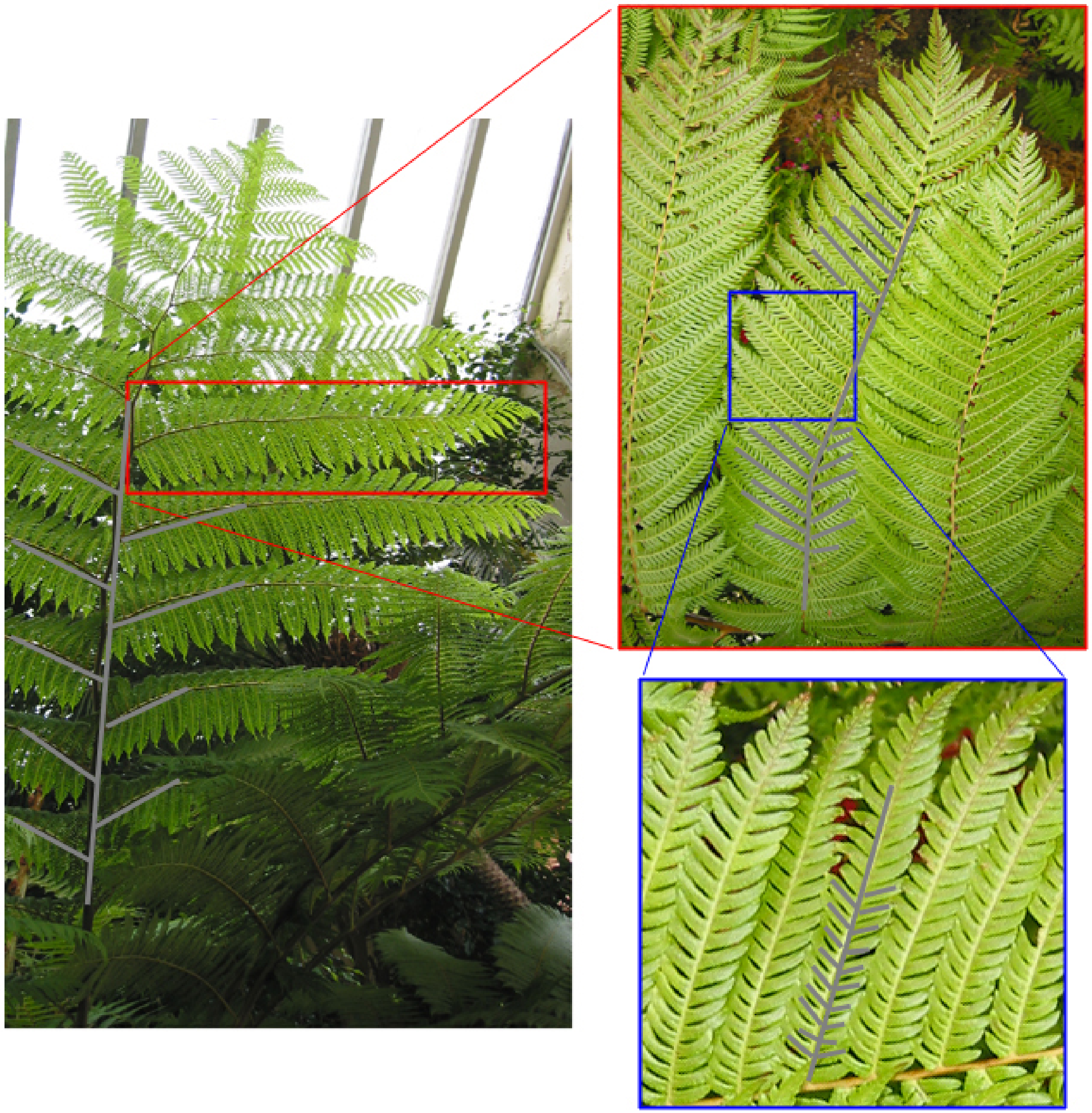


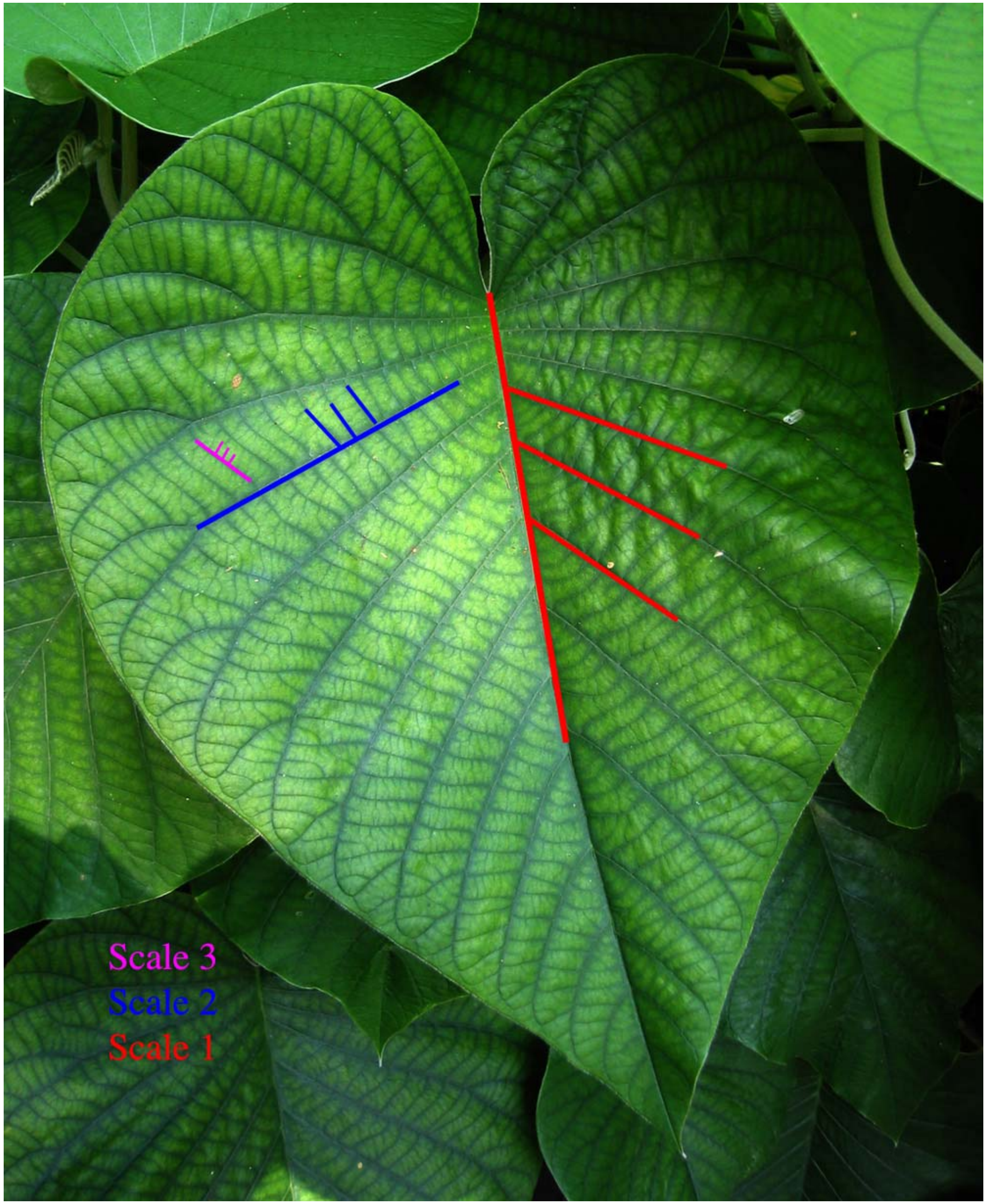
(f)

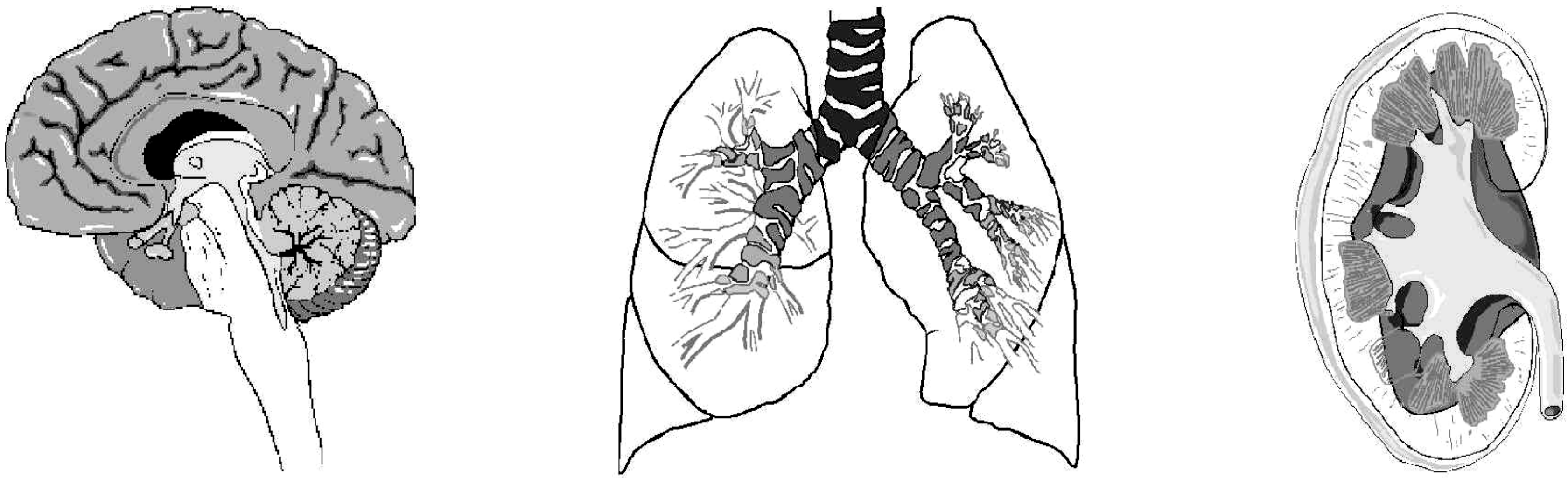
(a) (b) (c)



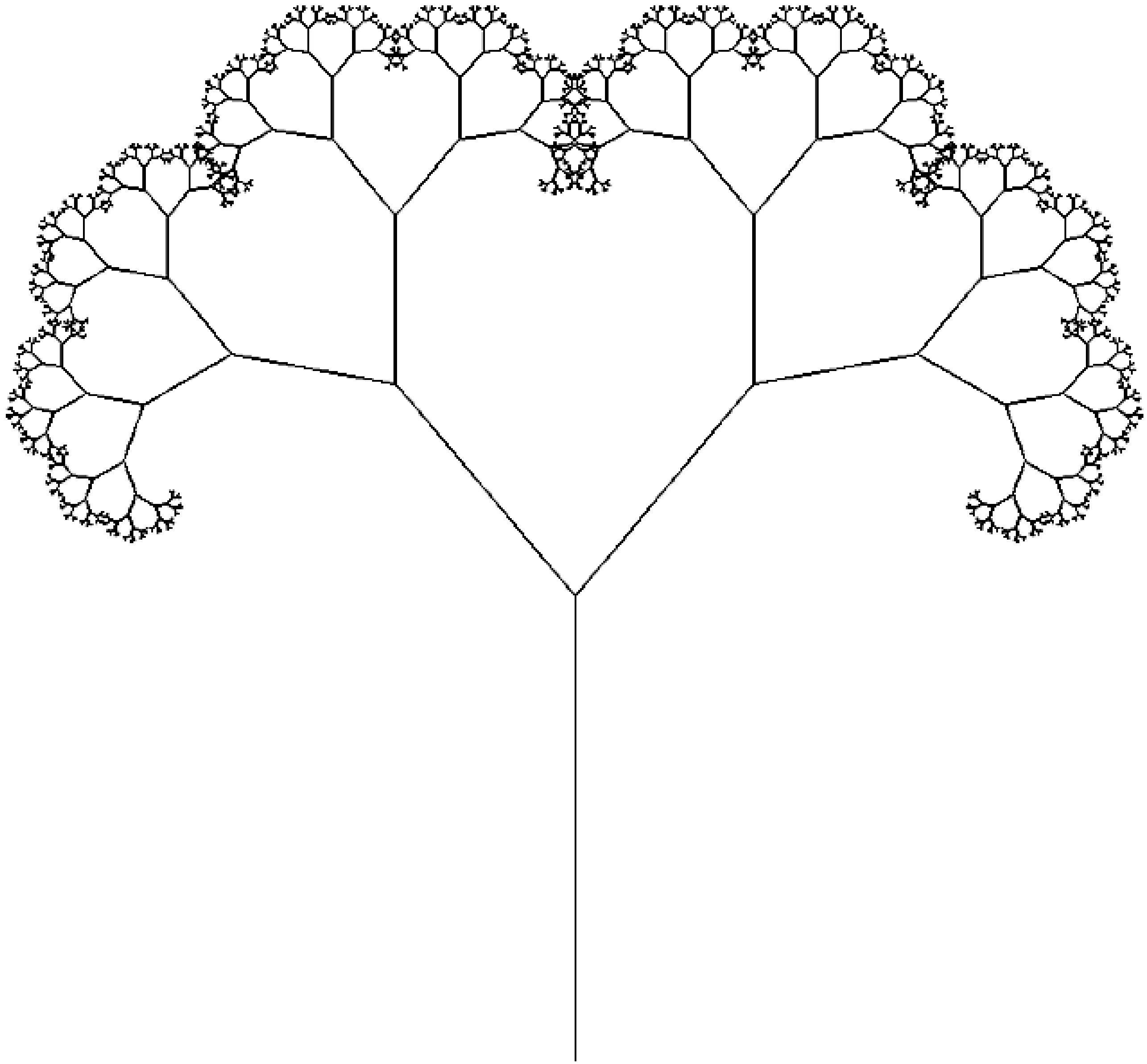
Fern leaf



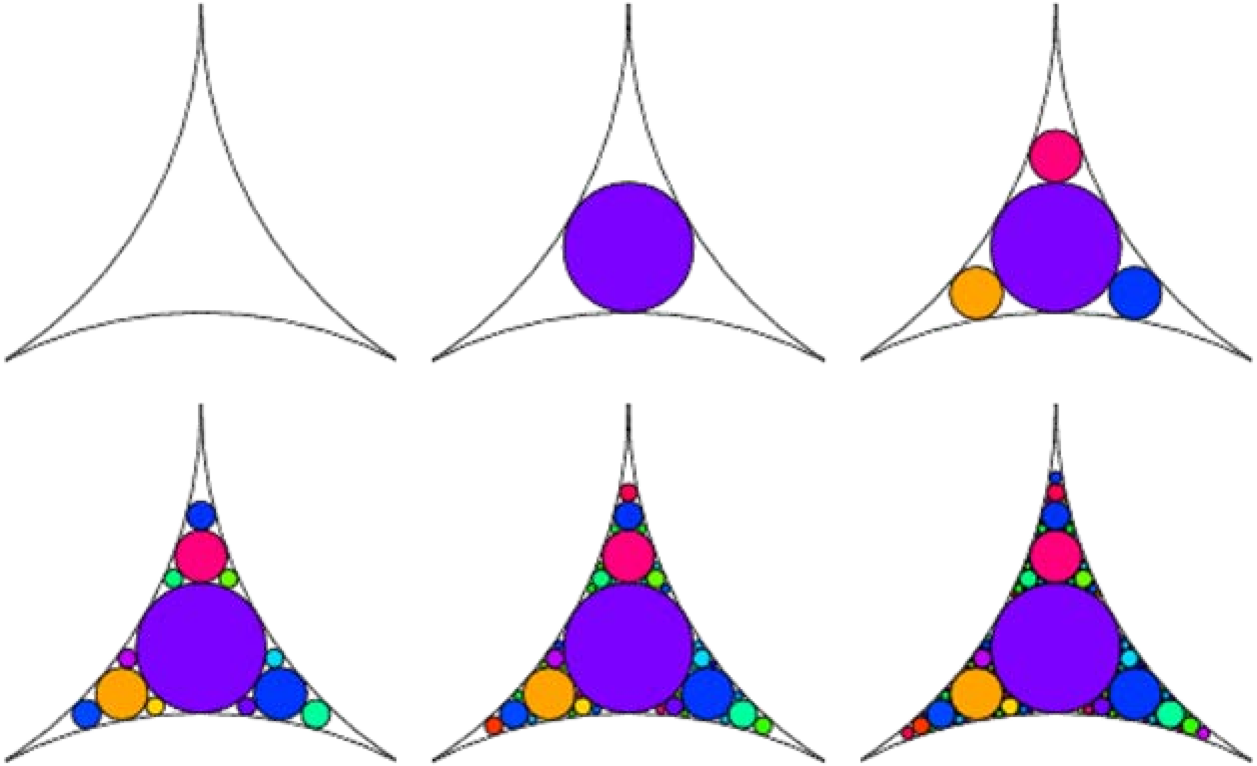

Scale 3
Scale 2
Scale 1

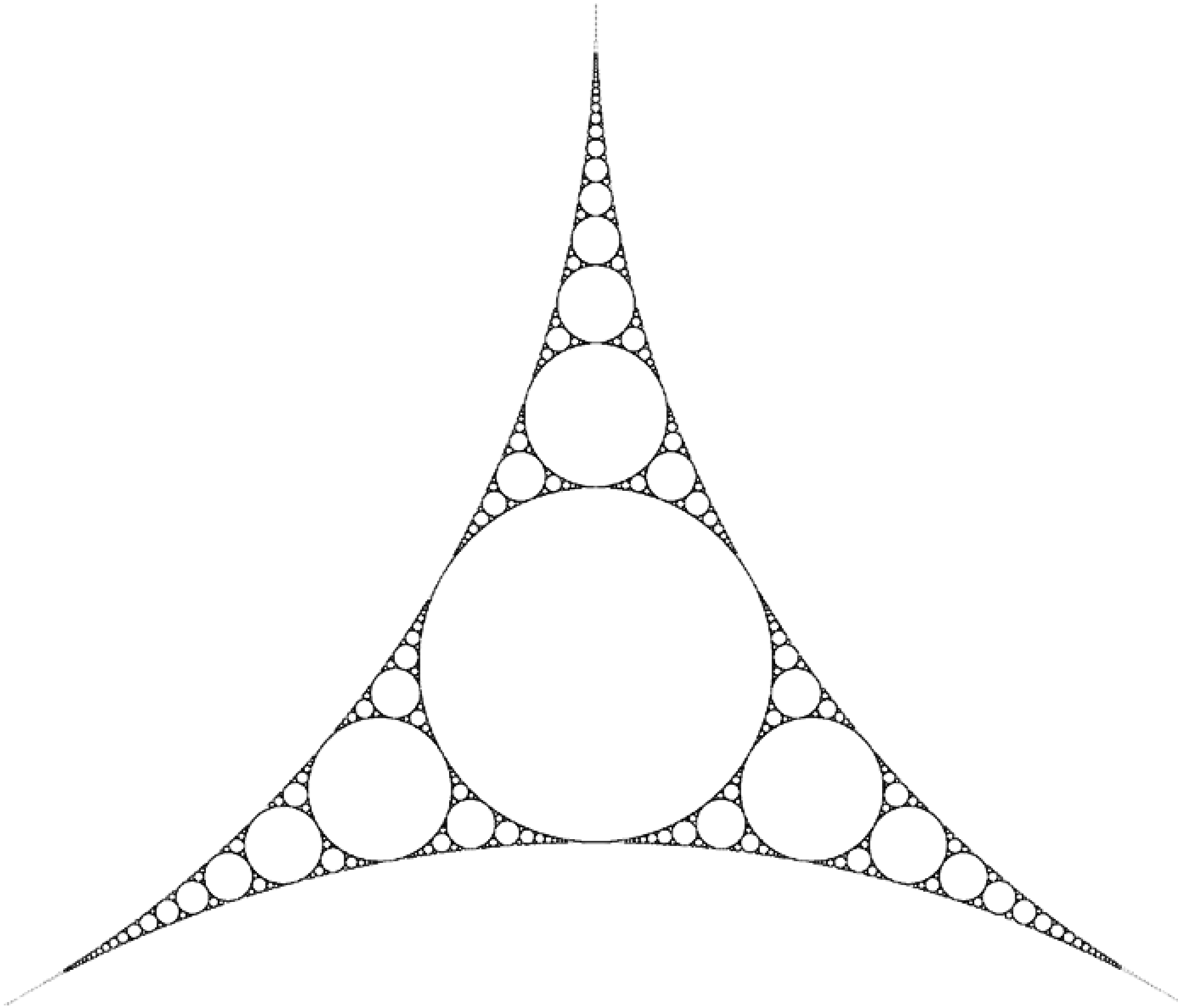


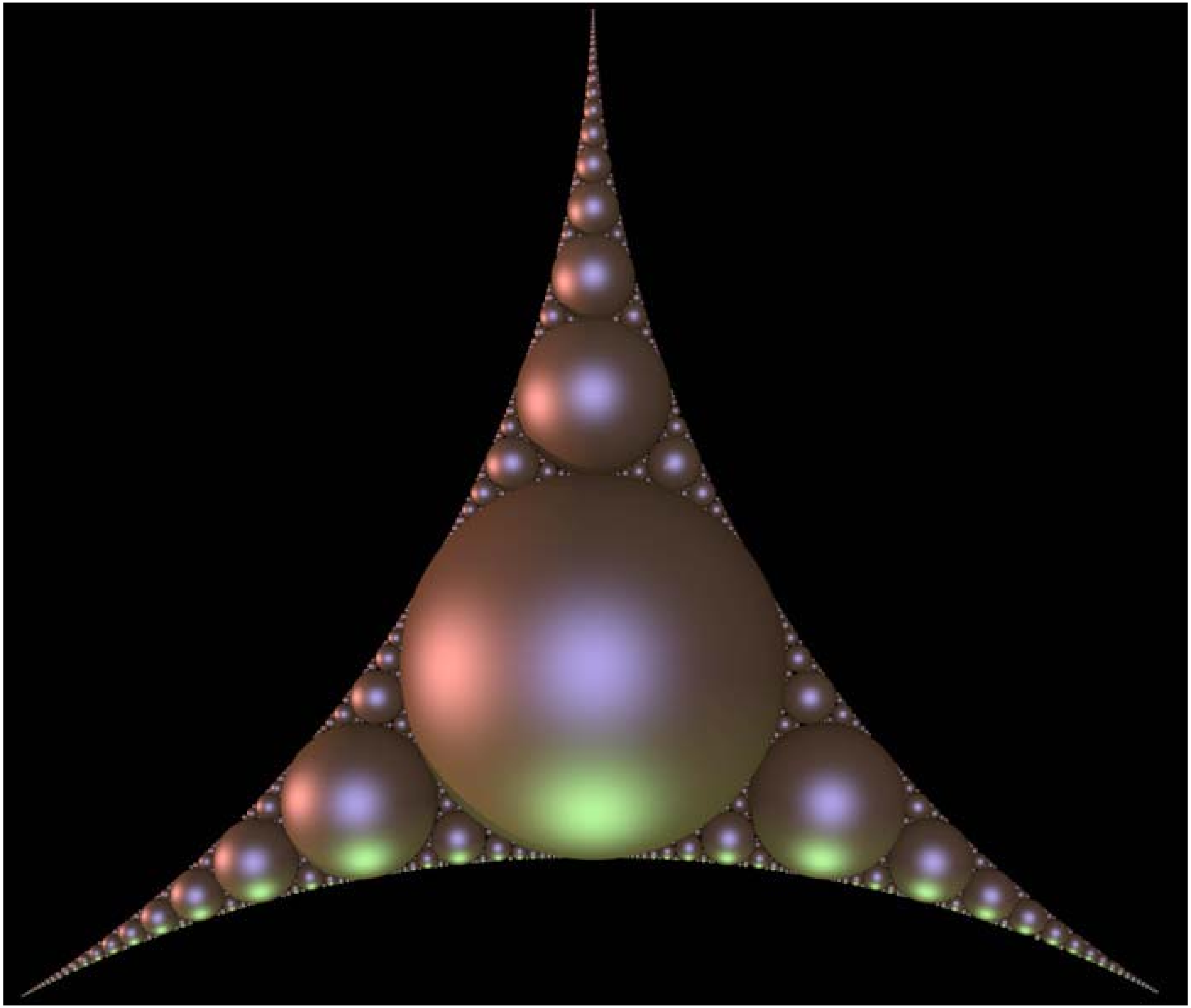
(a) (b) (c)

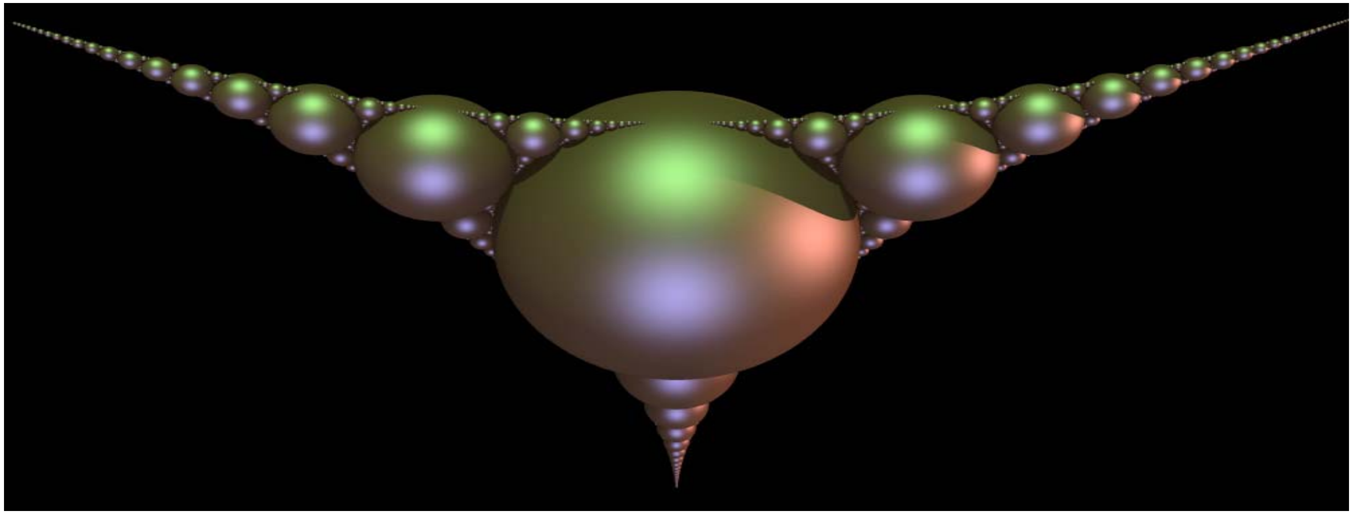


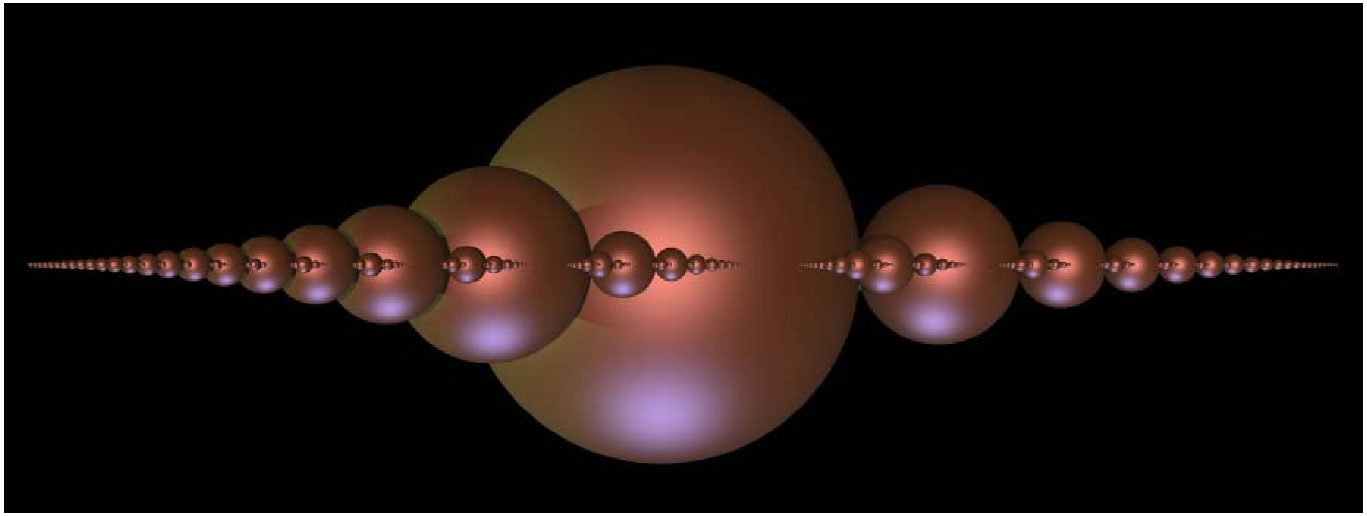
Apollonian Gasket

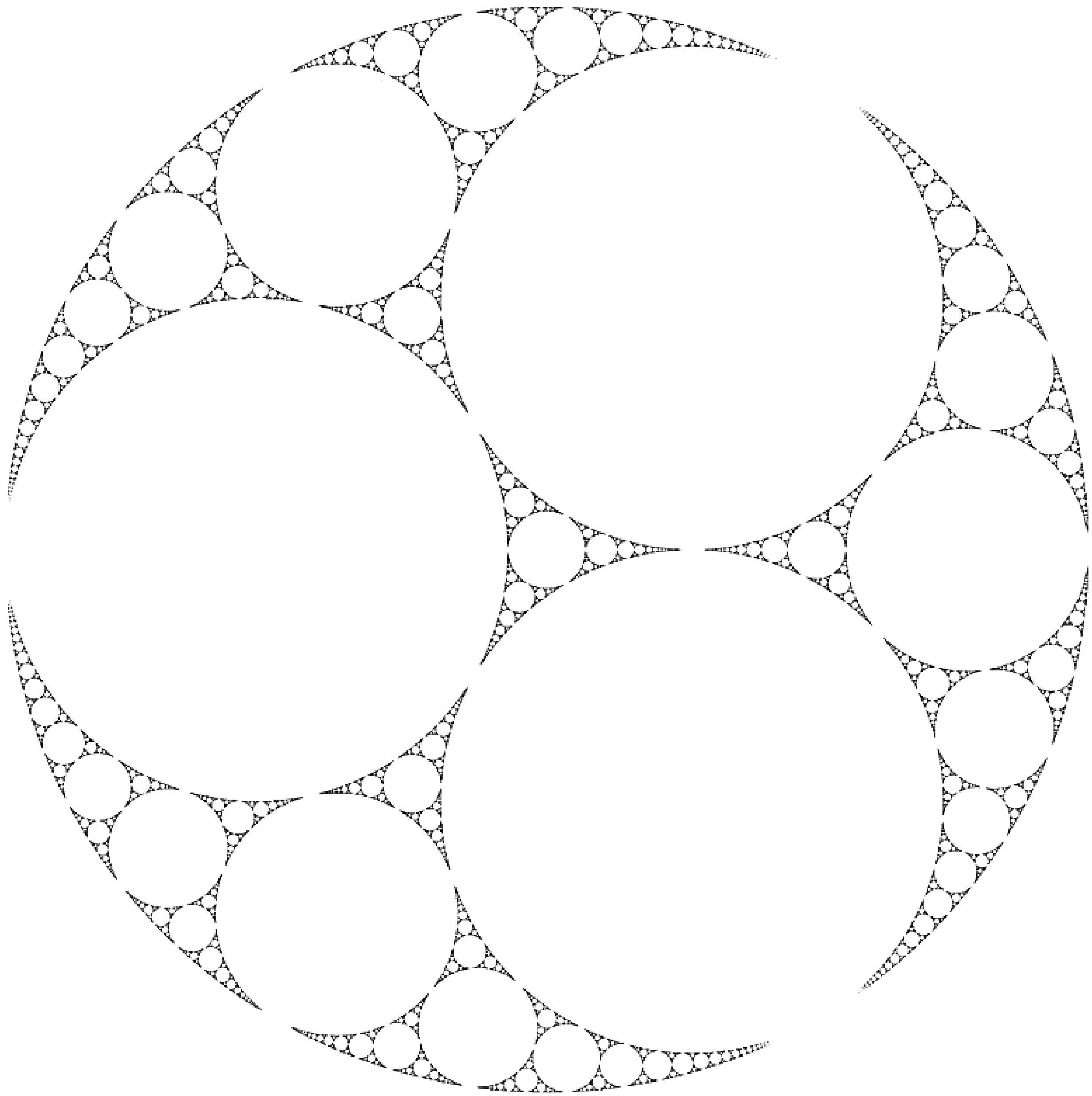


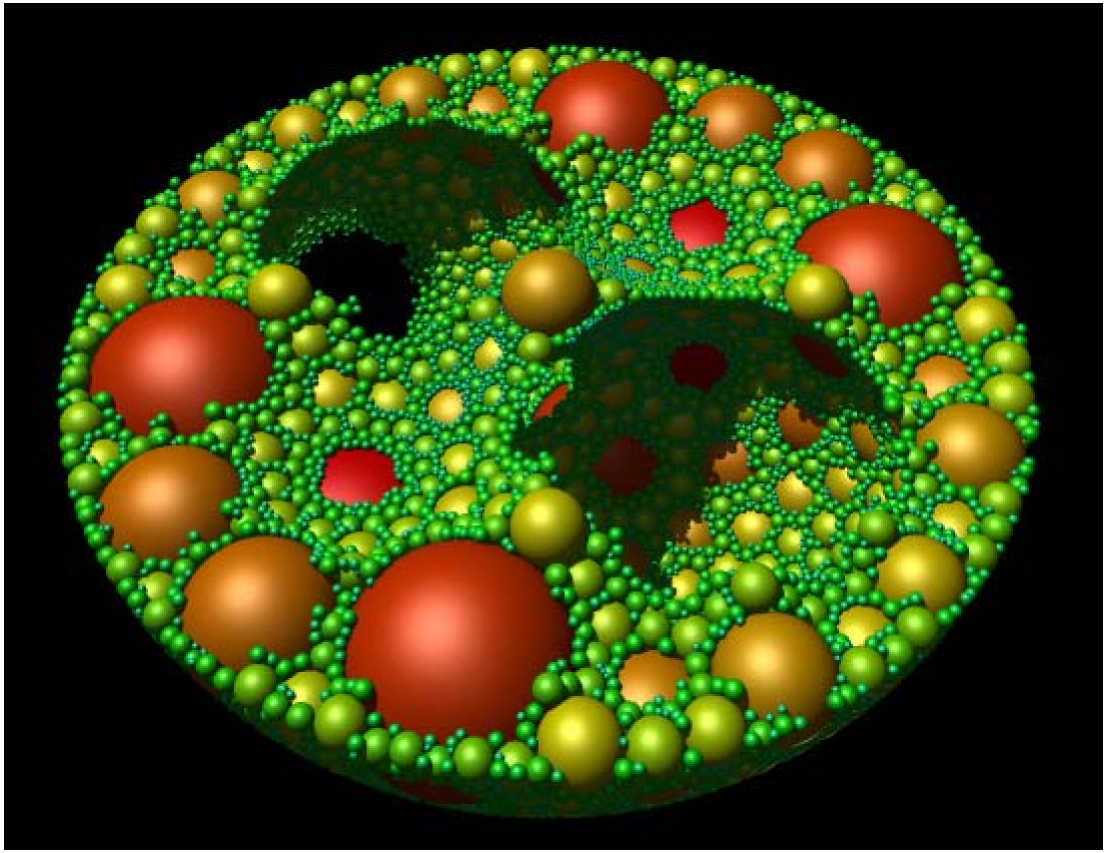


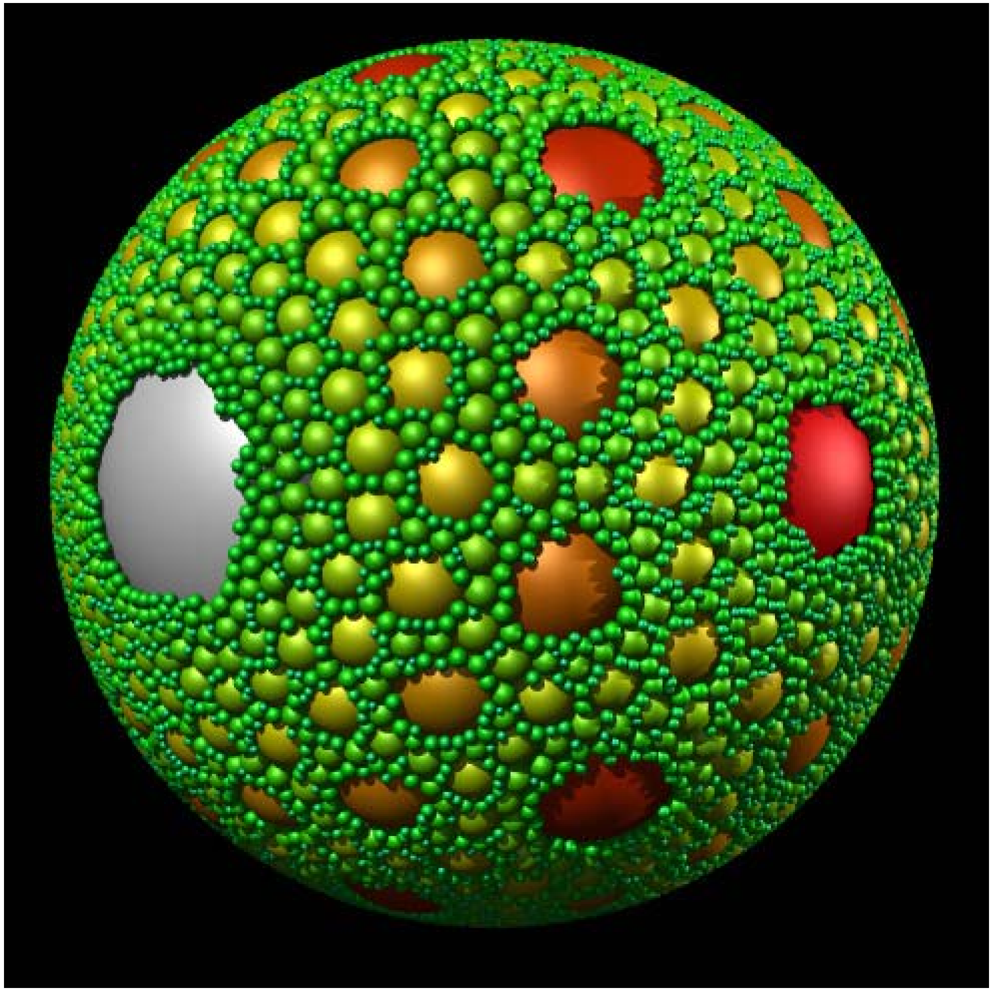


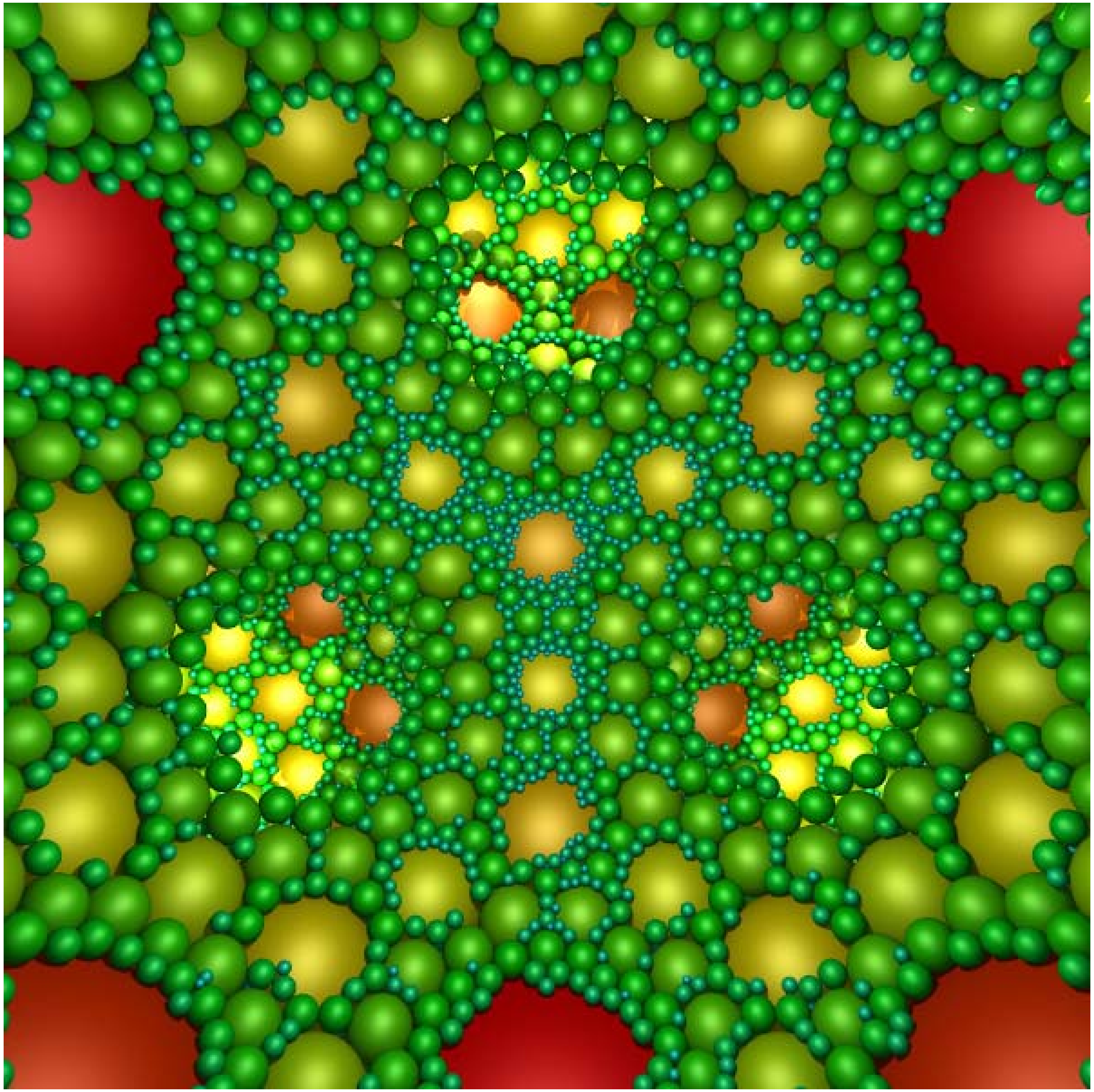


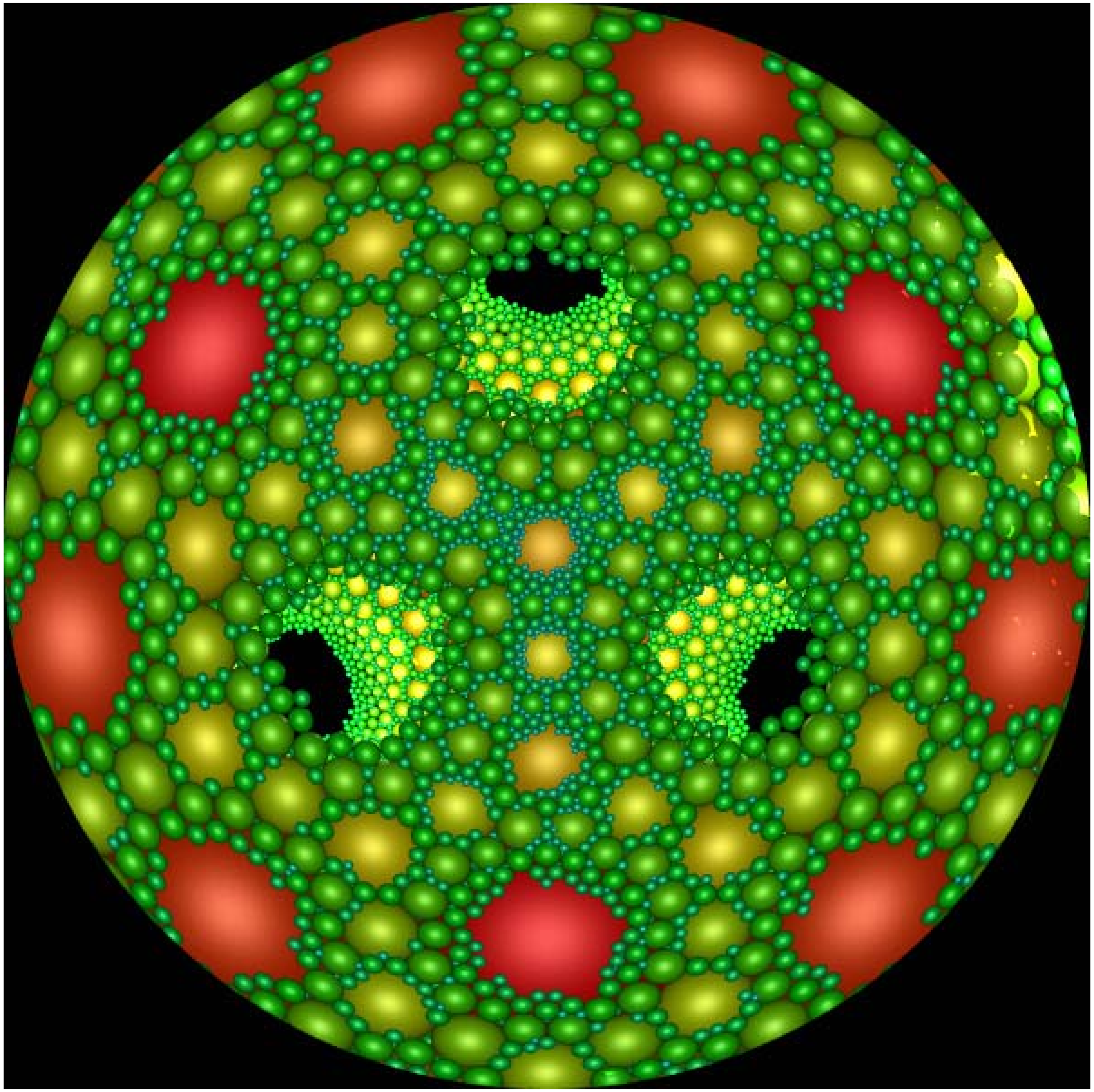




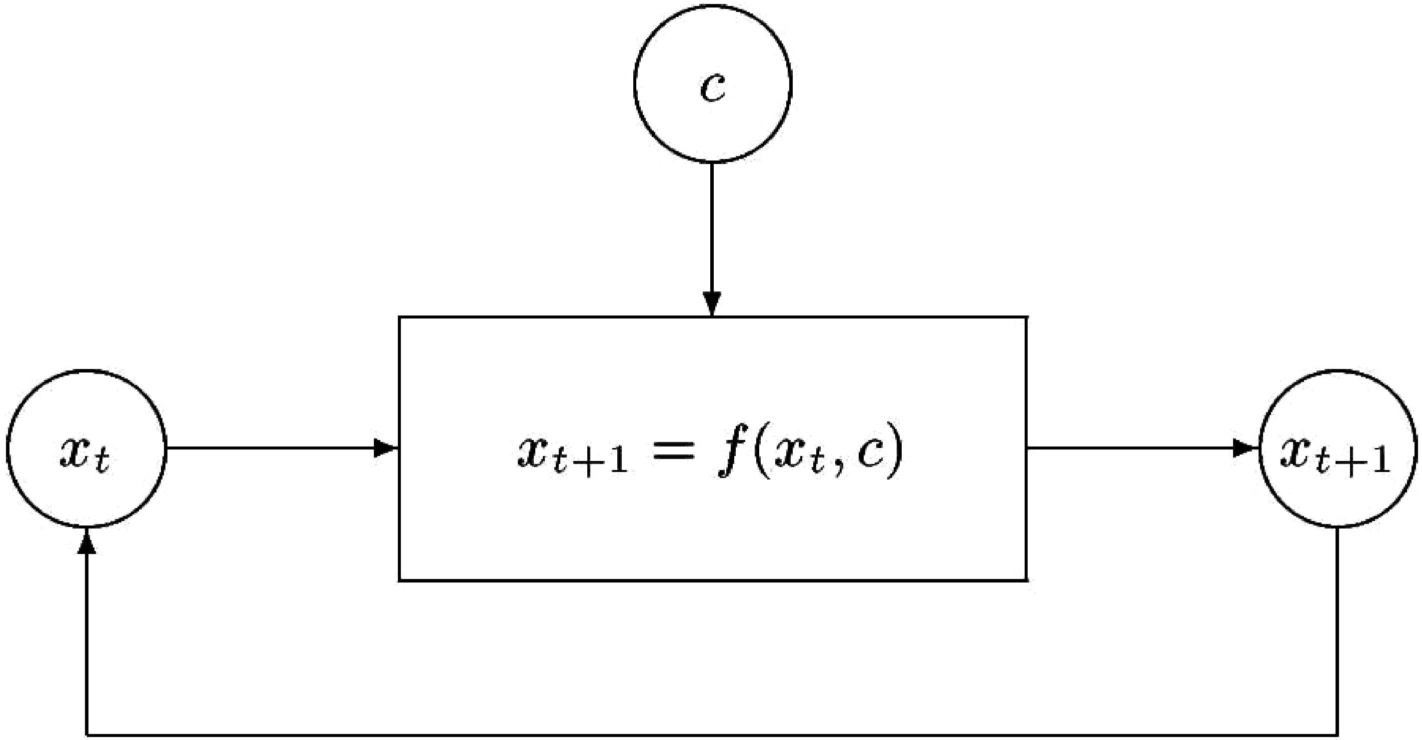






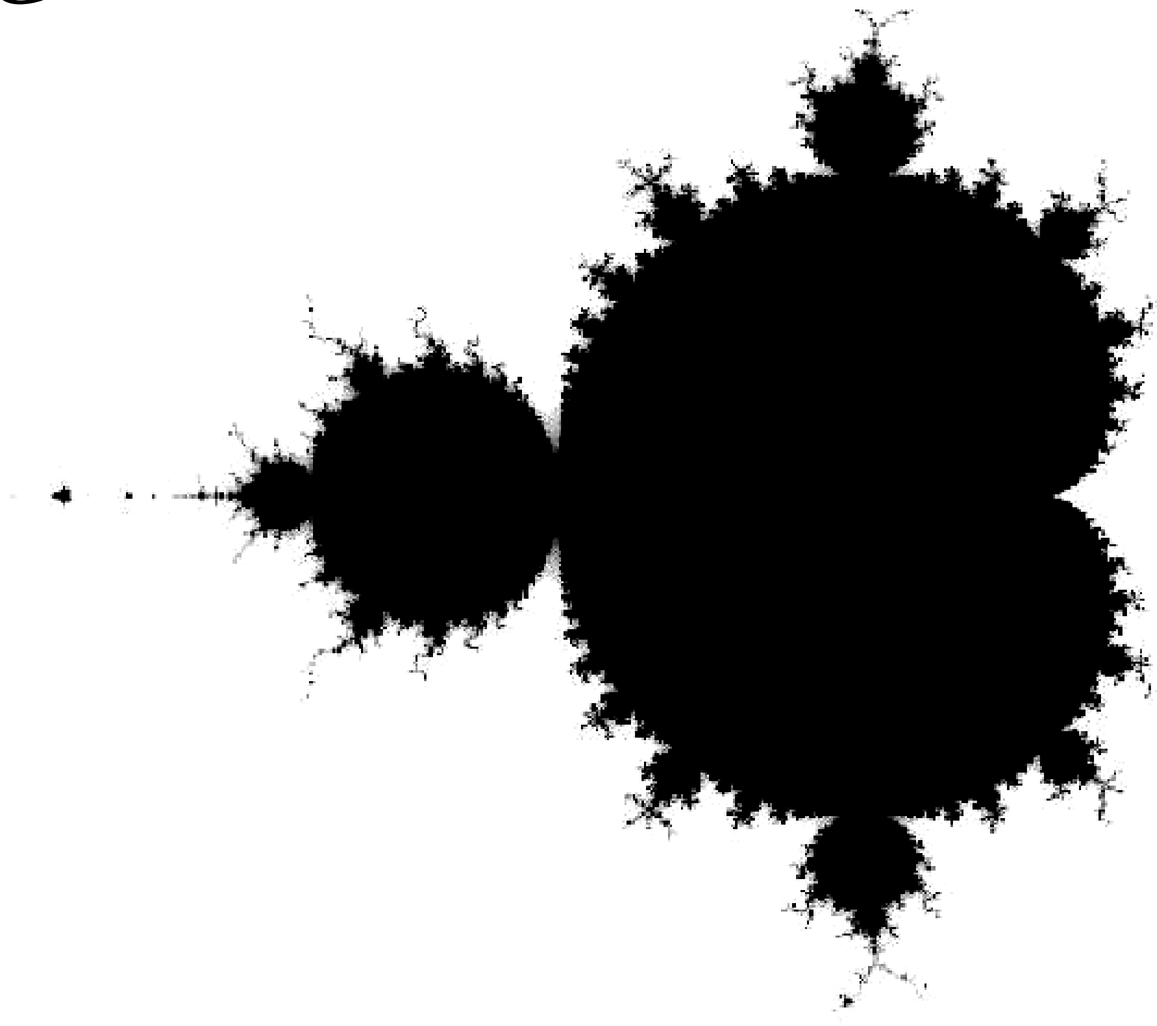


Fractals & Chaos

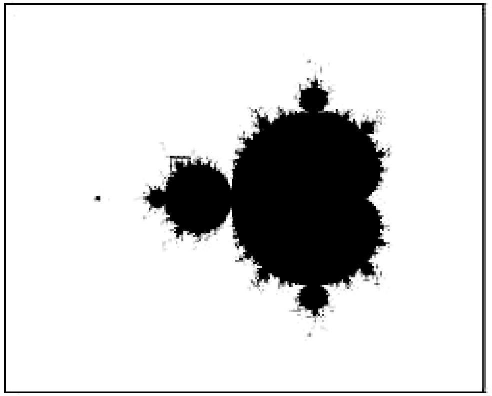


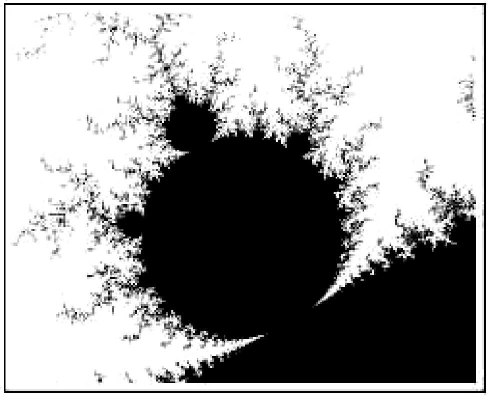
c
x_t
$x_{t+1} = f(x_t, c)$
x_{t+1}

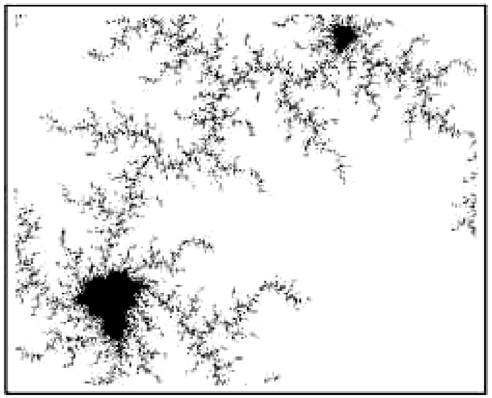
$$x_{t+1} = x_t^2 + C$$

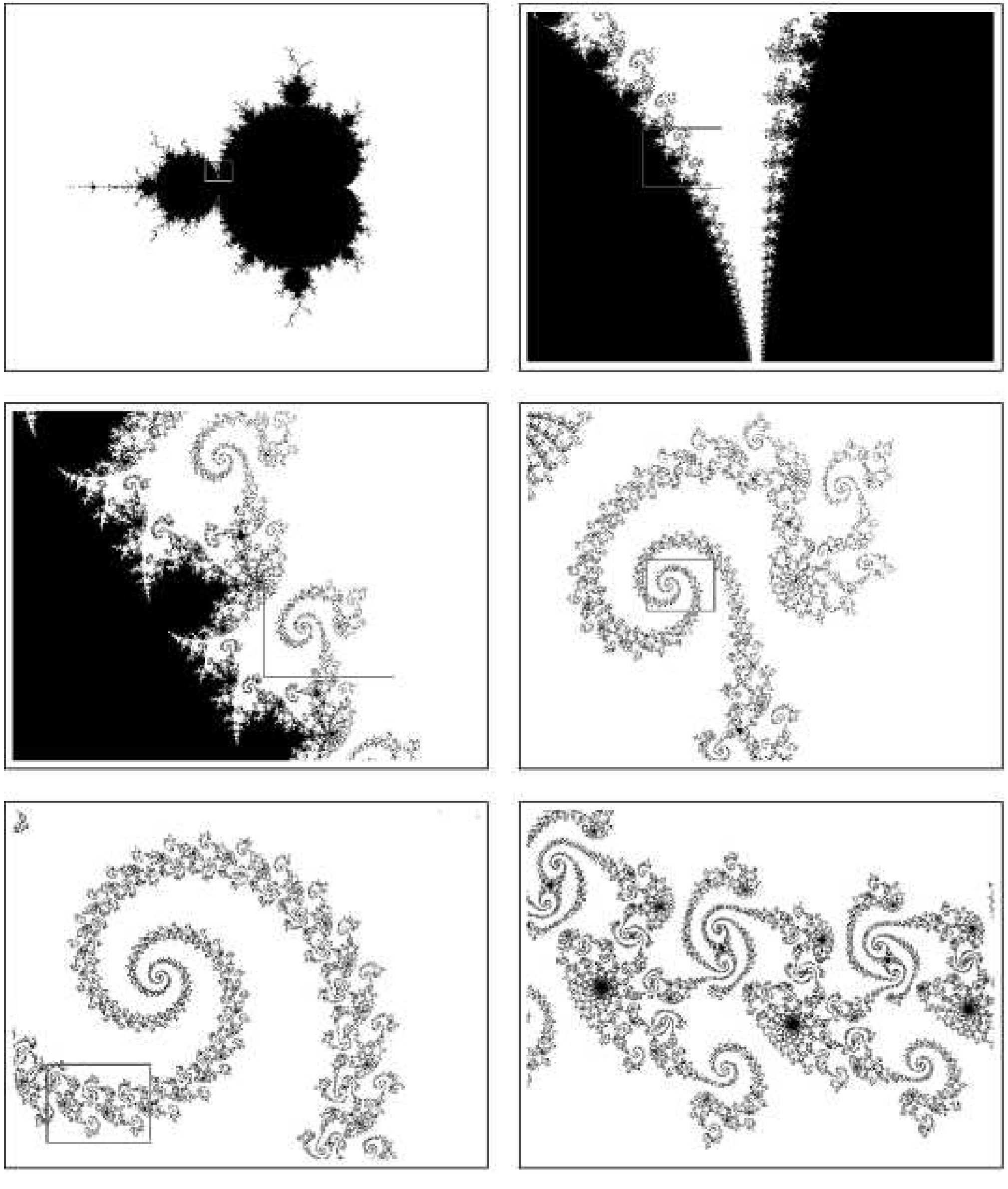


With $x_0 = 0$, for which values of C does the orbit not escape to ∞?

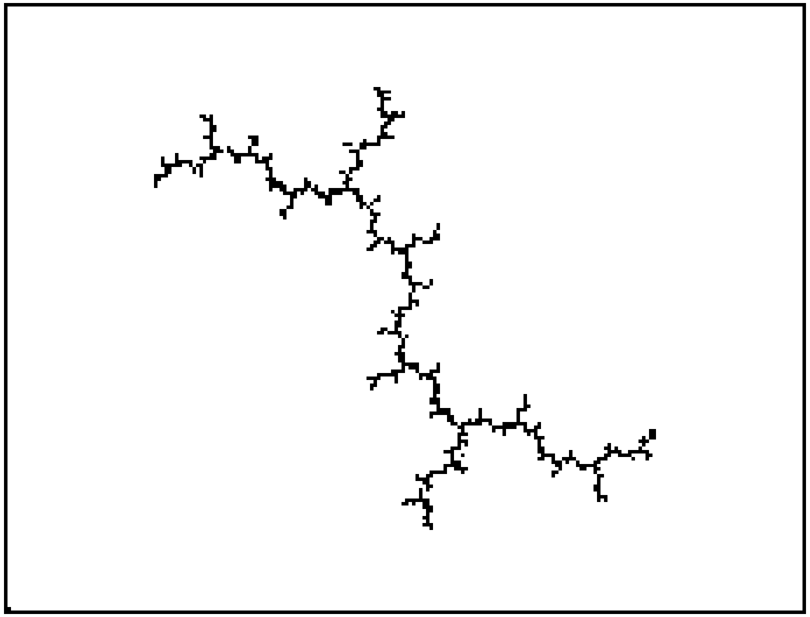




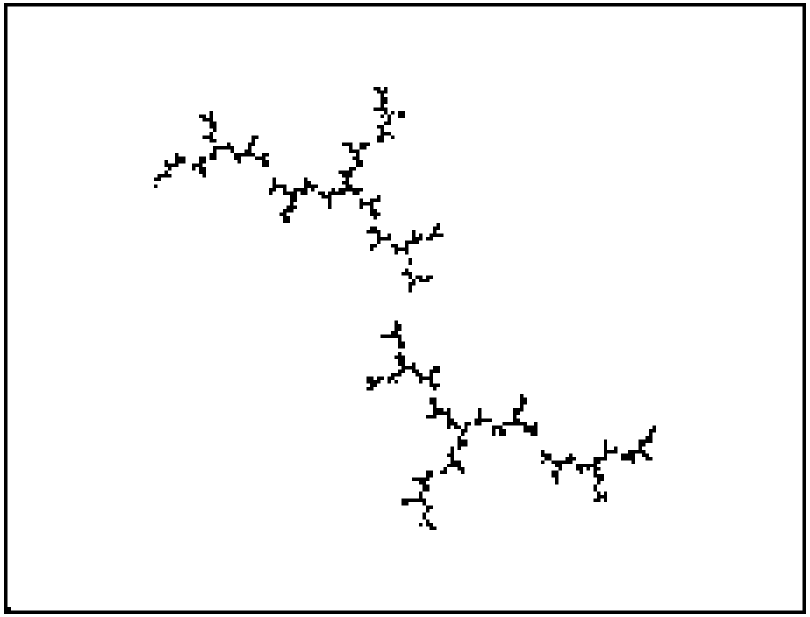




$$y_{t+1} = \pm(y_t - C)^{1/2}, \quad y_0 = 0$$

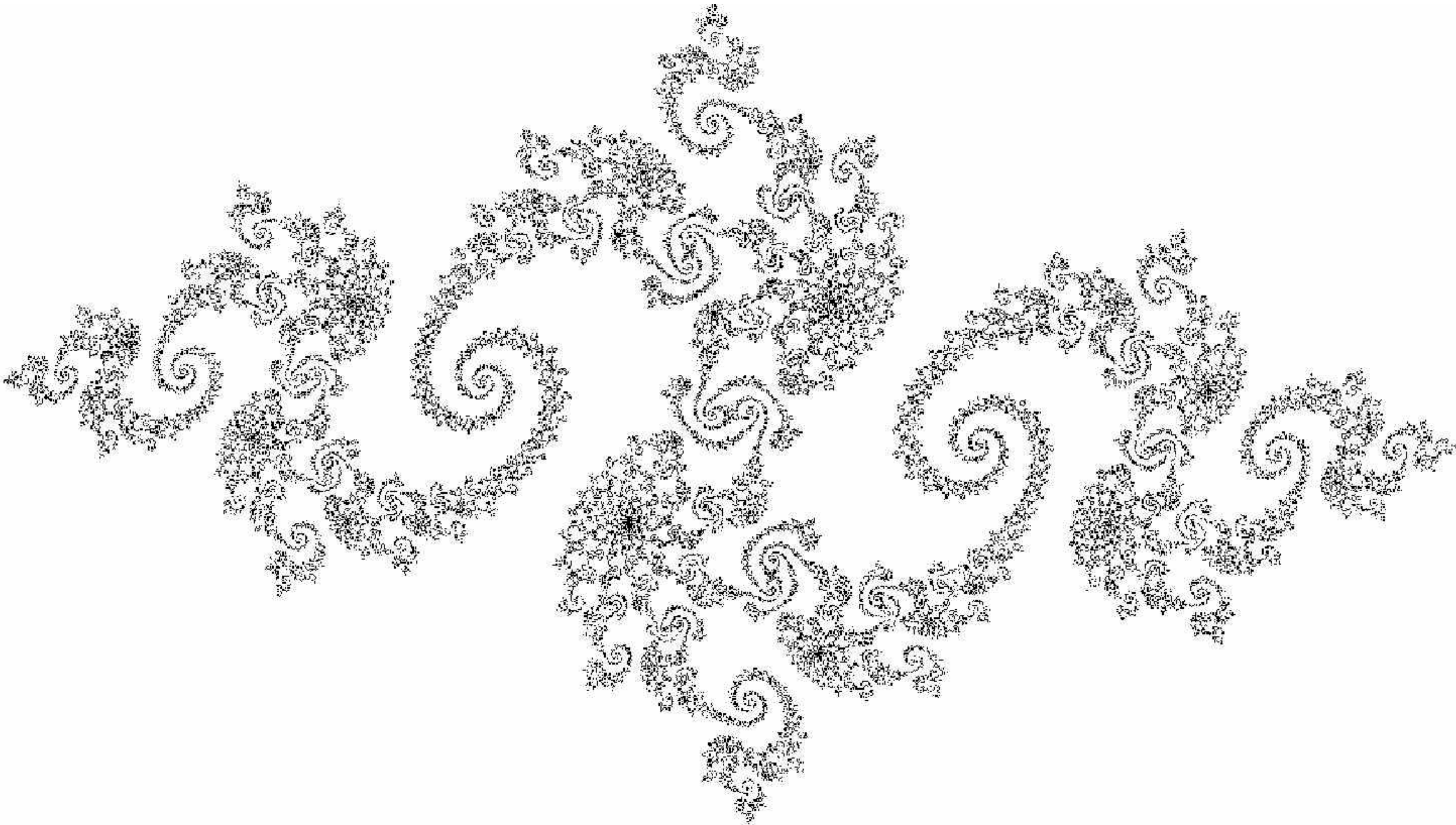


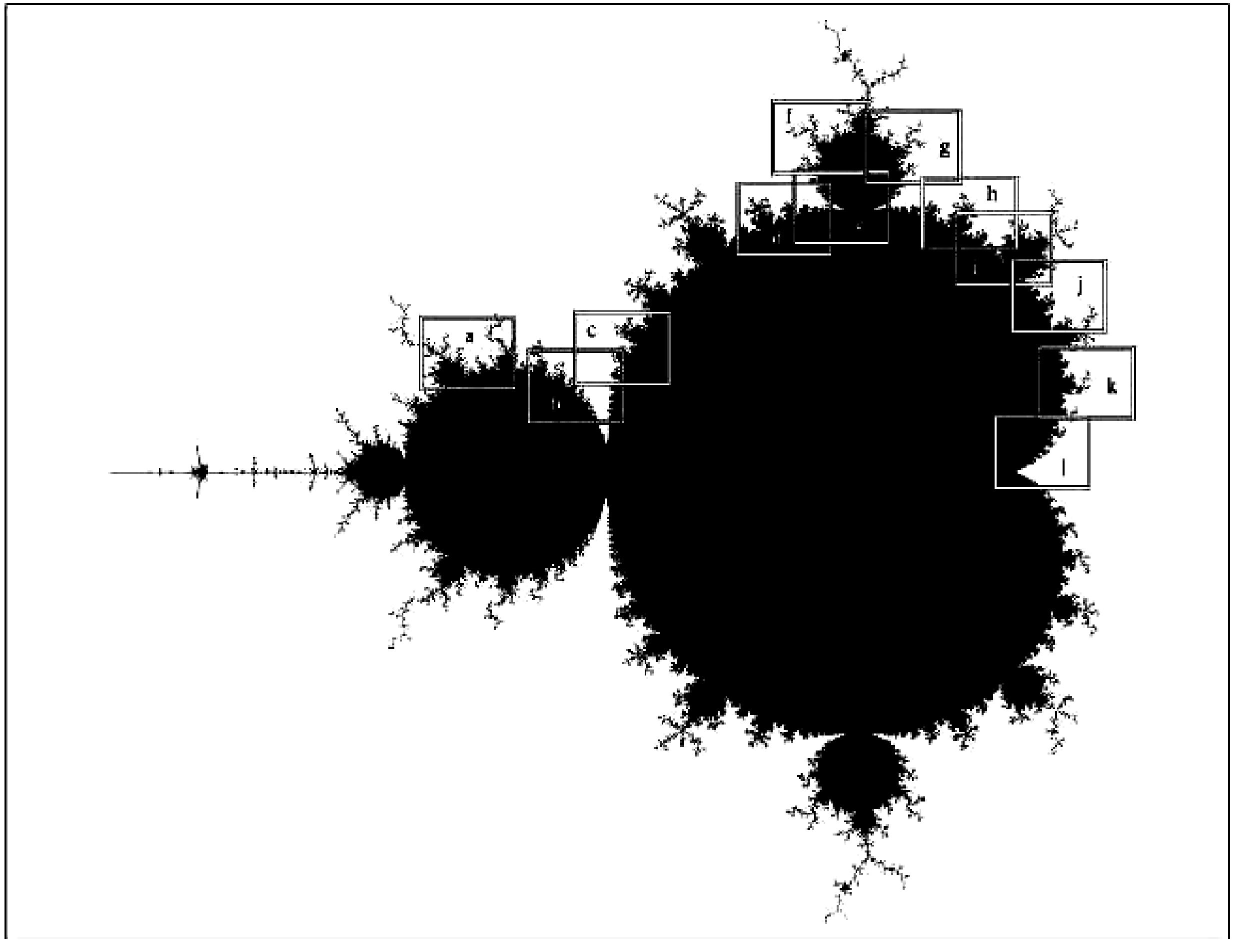
(a)

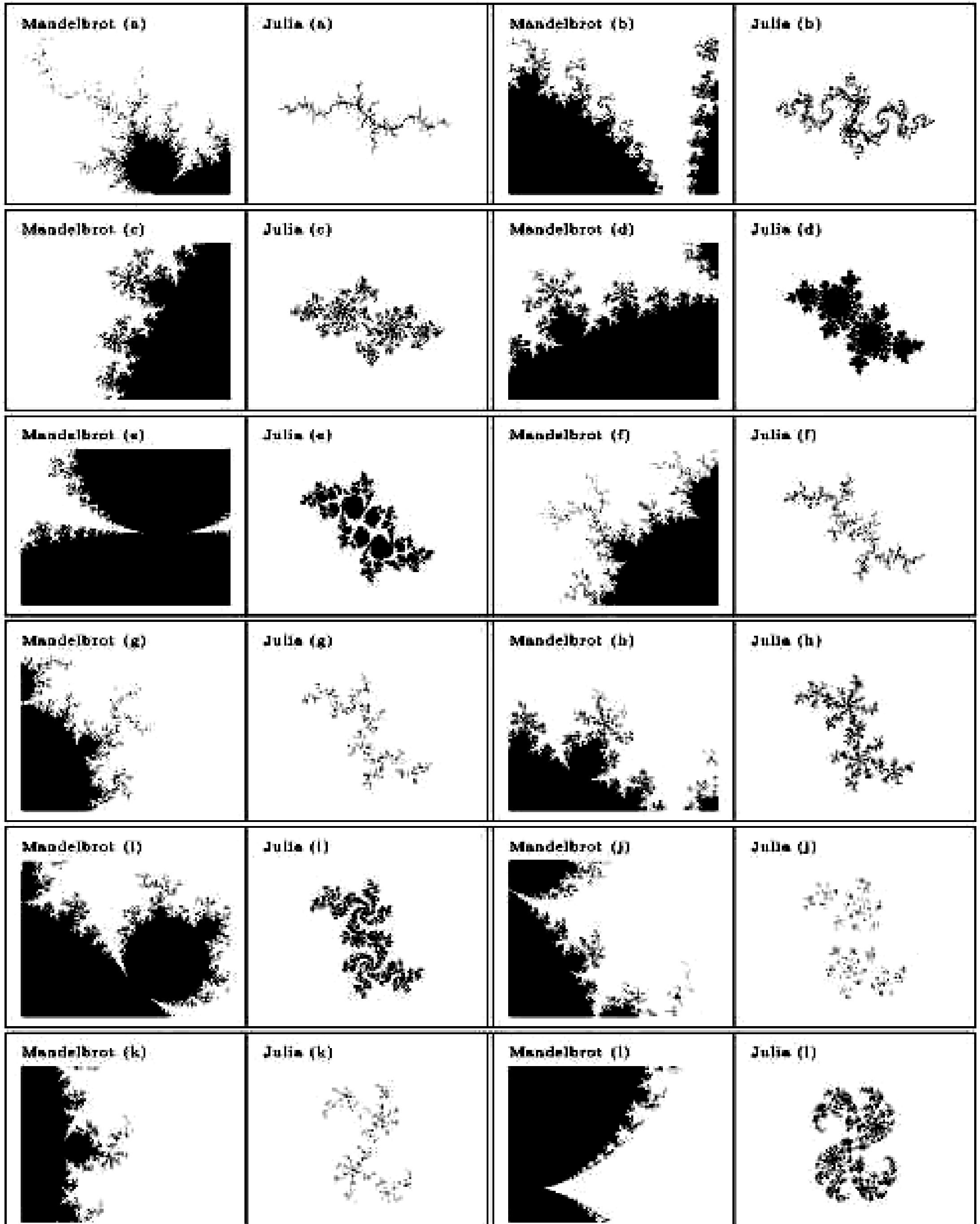


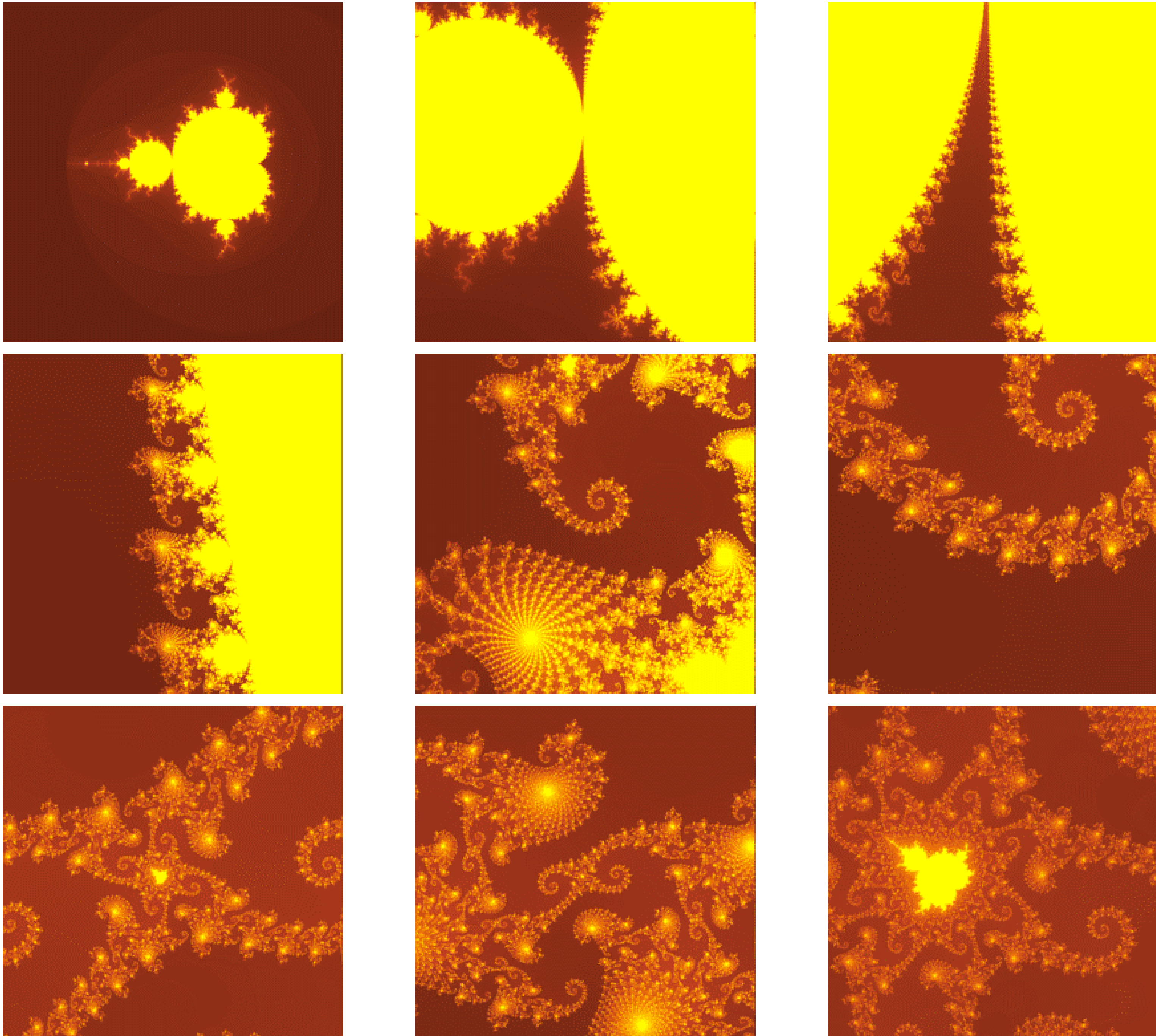
(b)

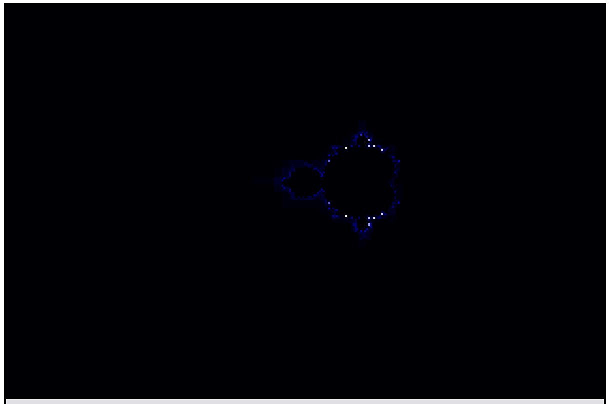
Find the limit set $\{y_t\}$ for $t \to \infty$



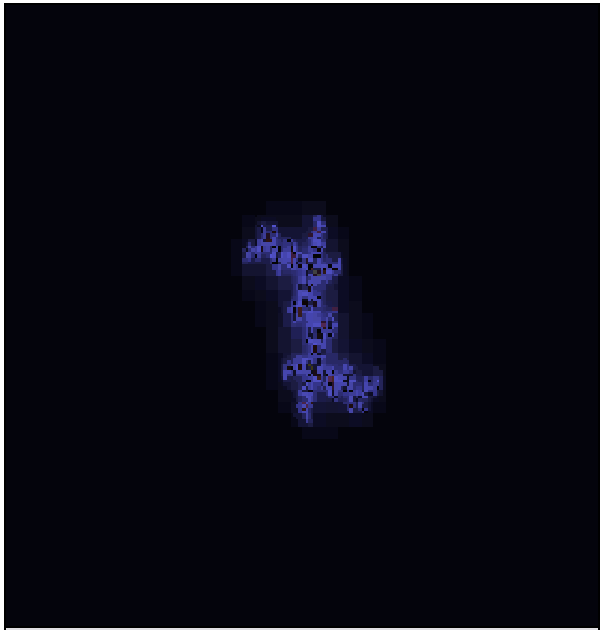

a
c
f
g
h
j
k
l


Mandelbrot (a)
Julia (a)
Mandelbrot (b)
Julia (b)
Mandelbrot (c)
Julia (c)
Mandelbrot (d)
Julia (d)
Mandelbrot (e)
Julia (e)
Mandelbrot (f)
Julia (f)
Mandelbrot (g)
Julia (g)
Mandelbrot (h)
Julia (h)
Mandelbrot (i)
Julia (i)
Mandelbrot (j)
Julia (j)
Mandelbrot (k)
Julia (k)
Mandelbrot (l)
Julia (l)

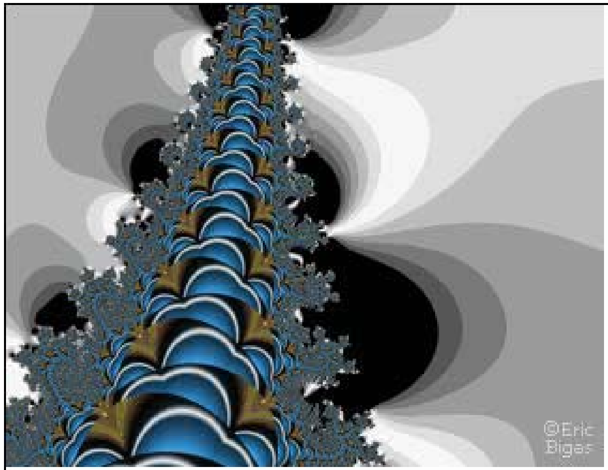




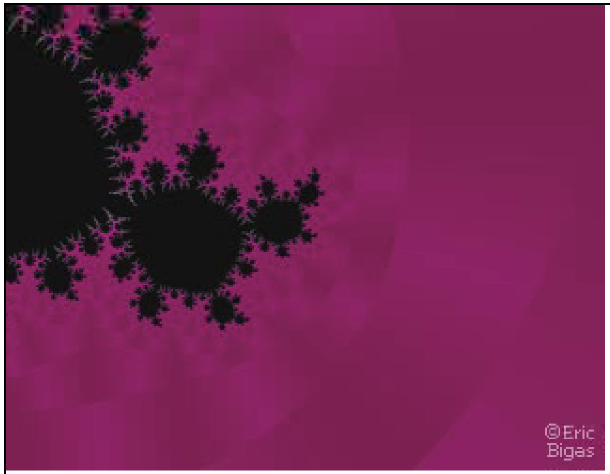
mandelz



juliaz

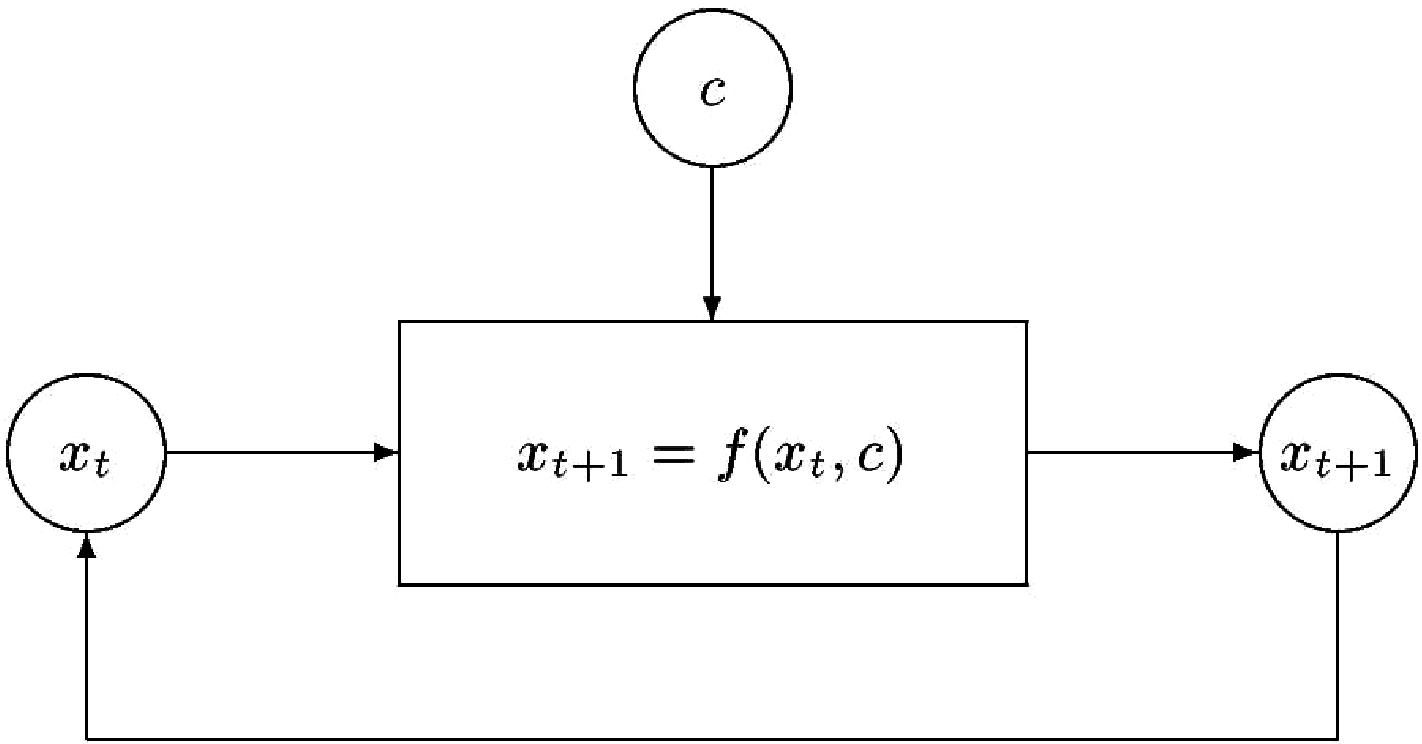


Striped_Blue_Tightrope_Tendril_pr



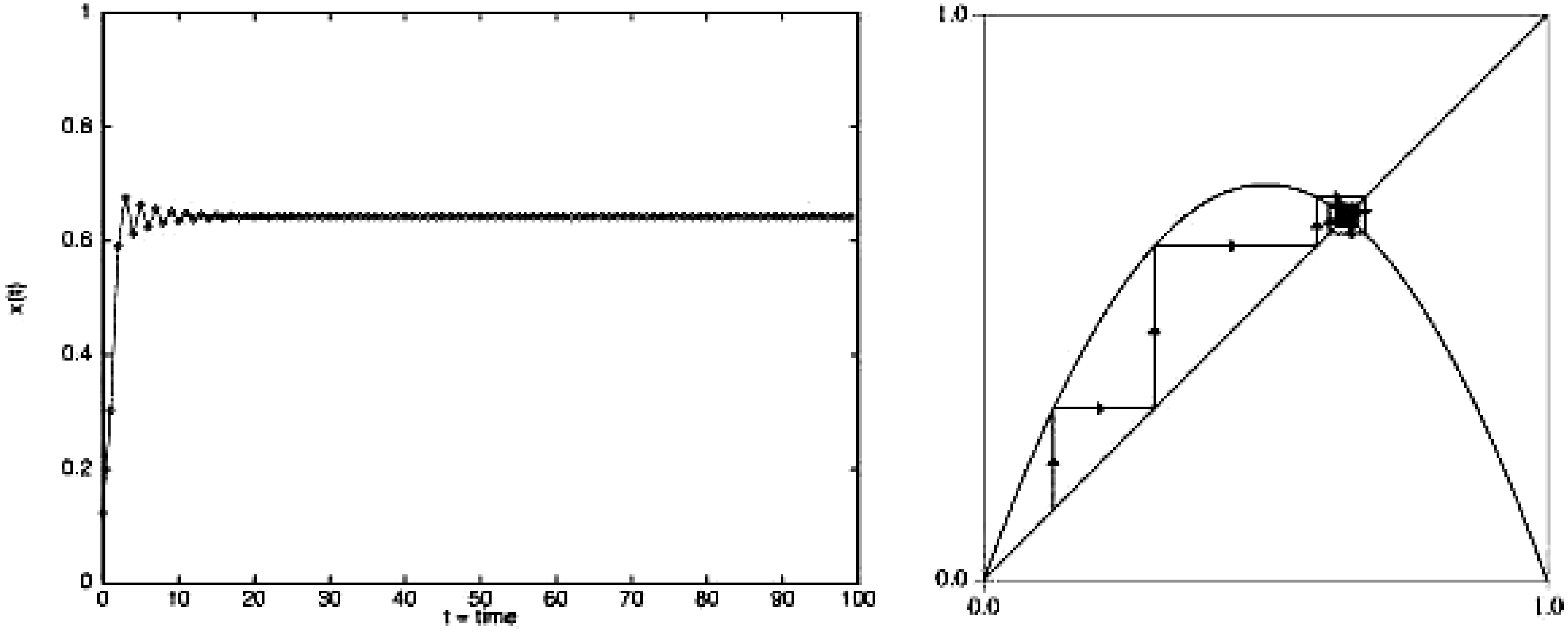
Growing_Grape_Gopherholes_pr


Between Order & Chaos



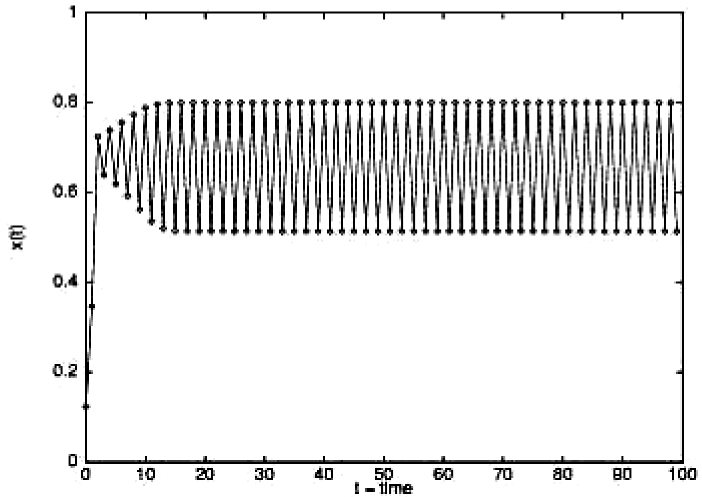
c
x_t
$x_{t+1} = f(x_t, c)$
x_{t+1}

$$x_{t+1} = r x_t (1 - x_t), \quad 0 \le x \le 1, \quad 0 \le r \le 4$$

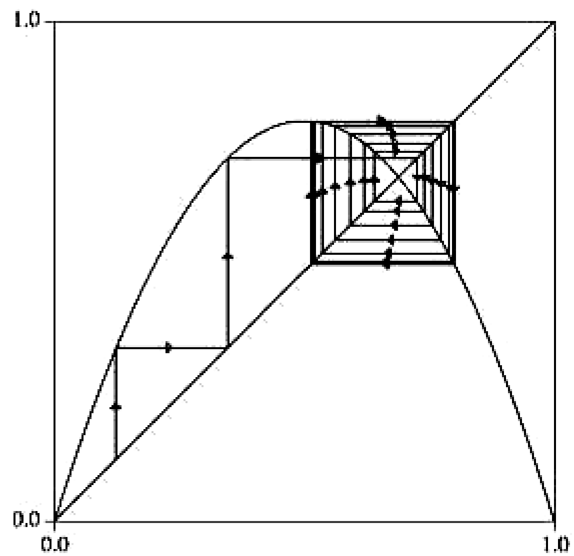


(a) (b)

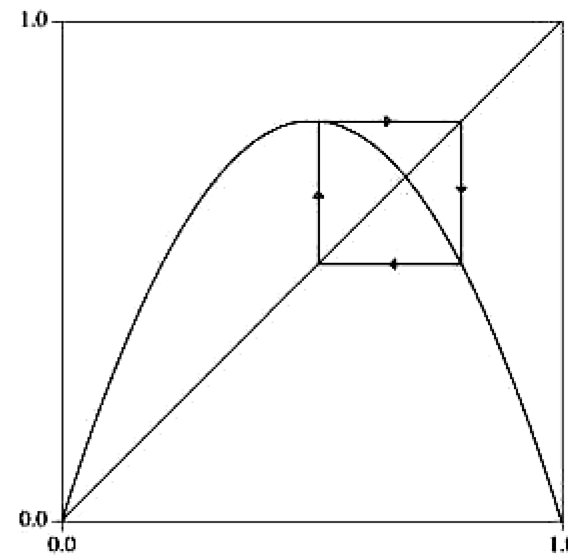
$$x_{t+1} = 1 - \mu x_t^2, \quad -1 \le x \le 1, \quad 0 \le \mu \le 2$$


1
0.8
0.6
0.4
0.2
0
x(t)
0
10
20
30
40
50
60
70
80
90
100
t = time

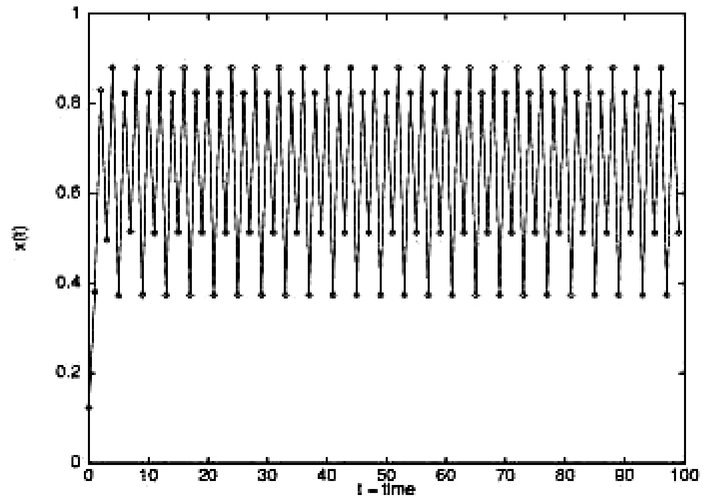
(a)


1.0
0.0
0.0
1.0

(b)

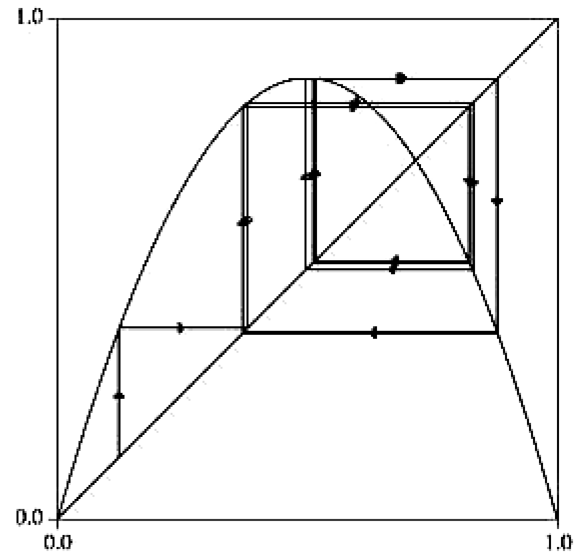

1.0
0.0
0.0
1.0

(c)



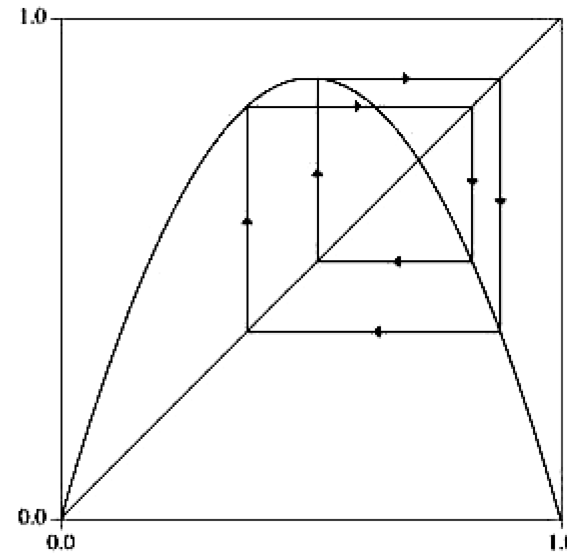
x(t)
t = time
0
0.2
0.4
0.6
0.8
1
0
10
20
30
40
50
60
70
80
90
100

(a)



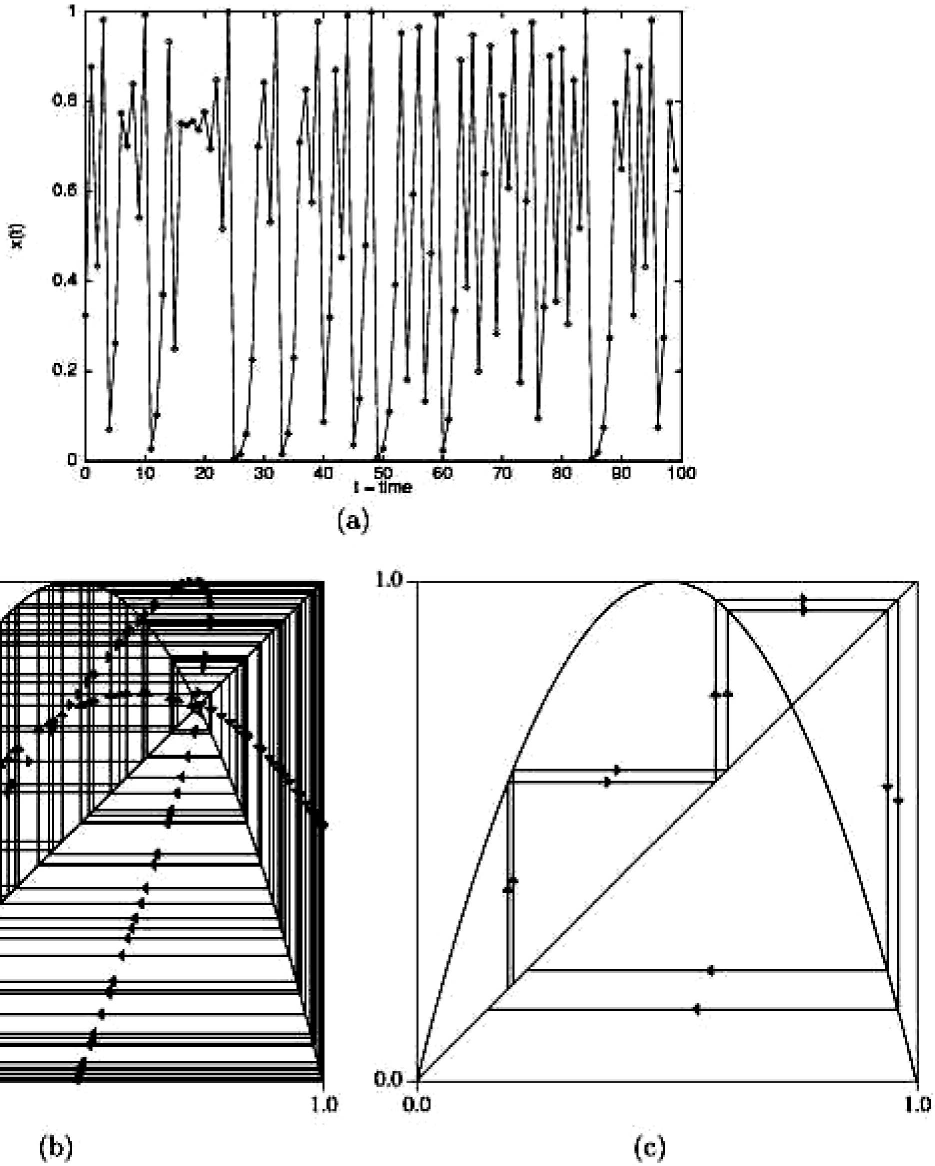
1.0
0.0
0.0
1.0

(b)

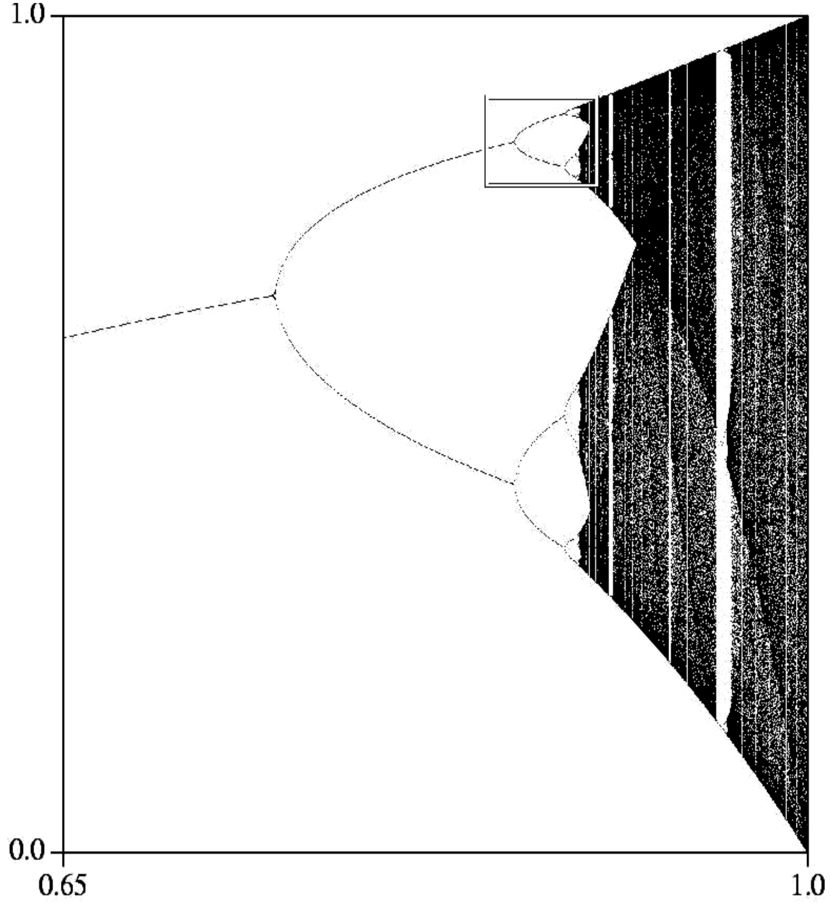


1.0
0.0
0.0
1.0

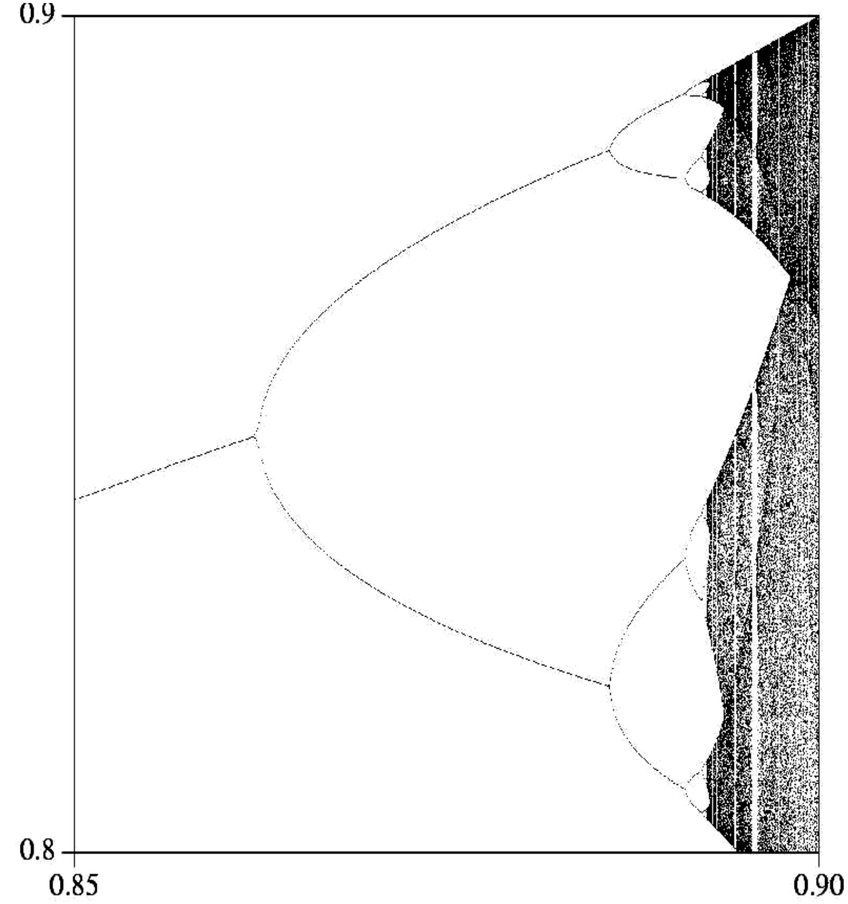
(c)


x(t)
t - time
0
0.2
0.4
0.6
0.8
1
0
10
20
30
40
50
60
70
80
90
100
(a)
1.0
0.0
0.0
1.0
(b)
1.0
0.0
0.0
1.0
(c)

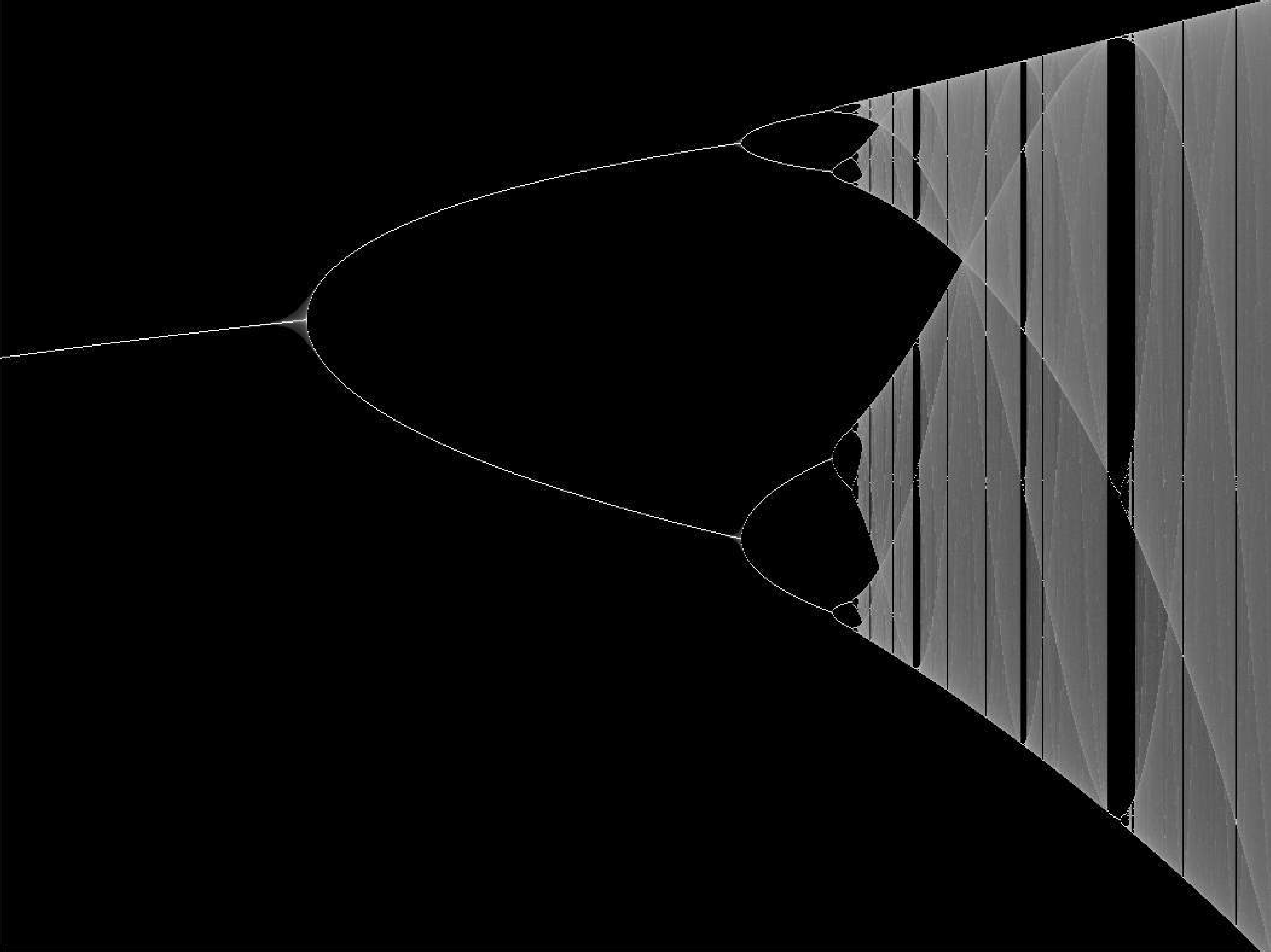
Web

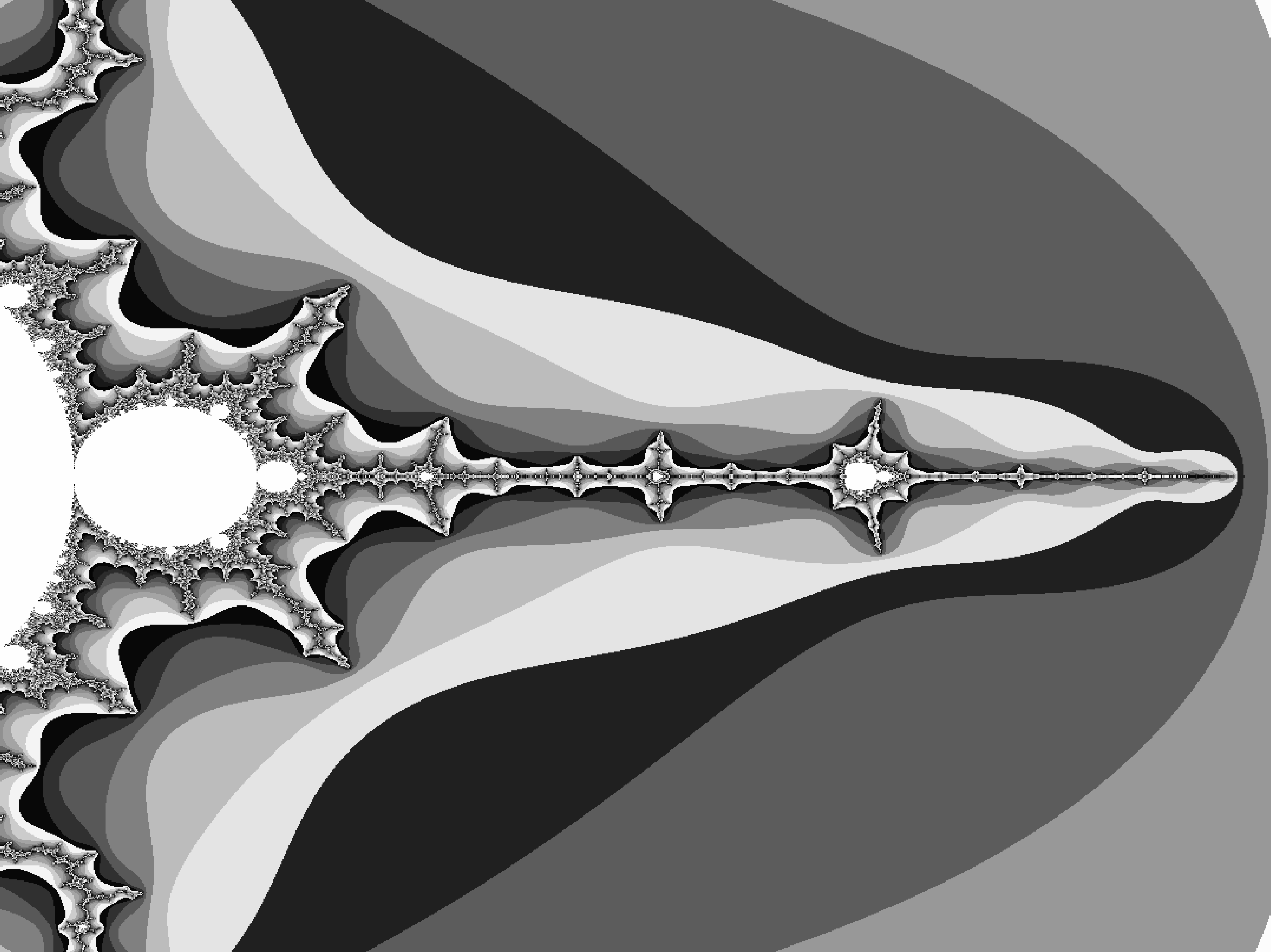


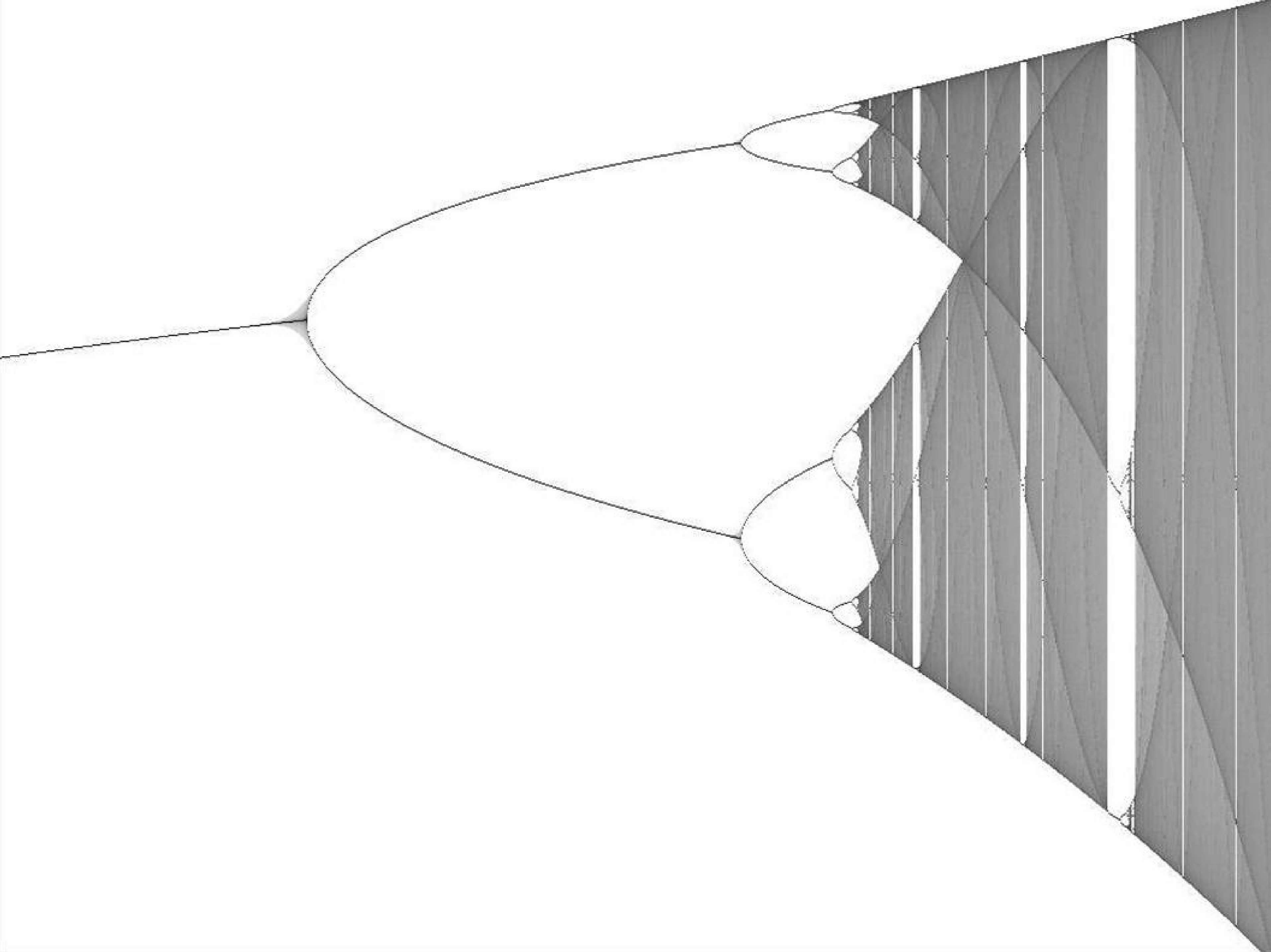
(a)



(b)

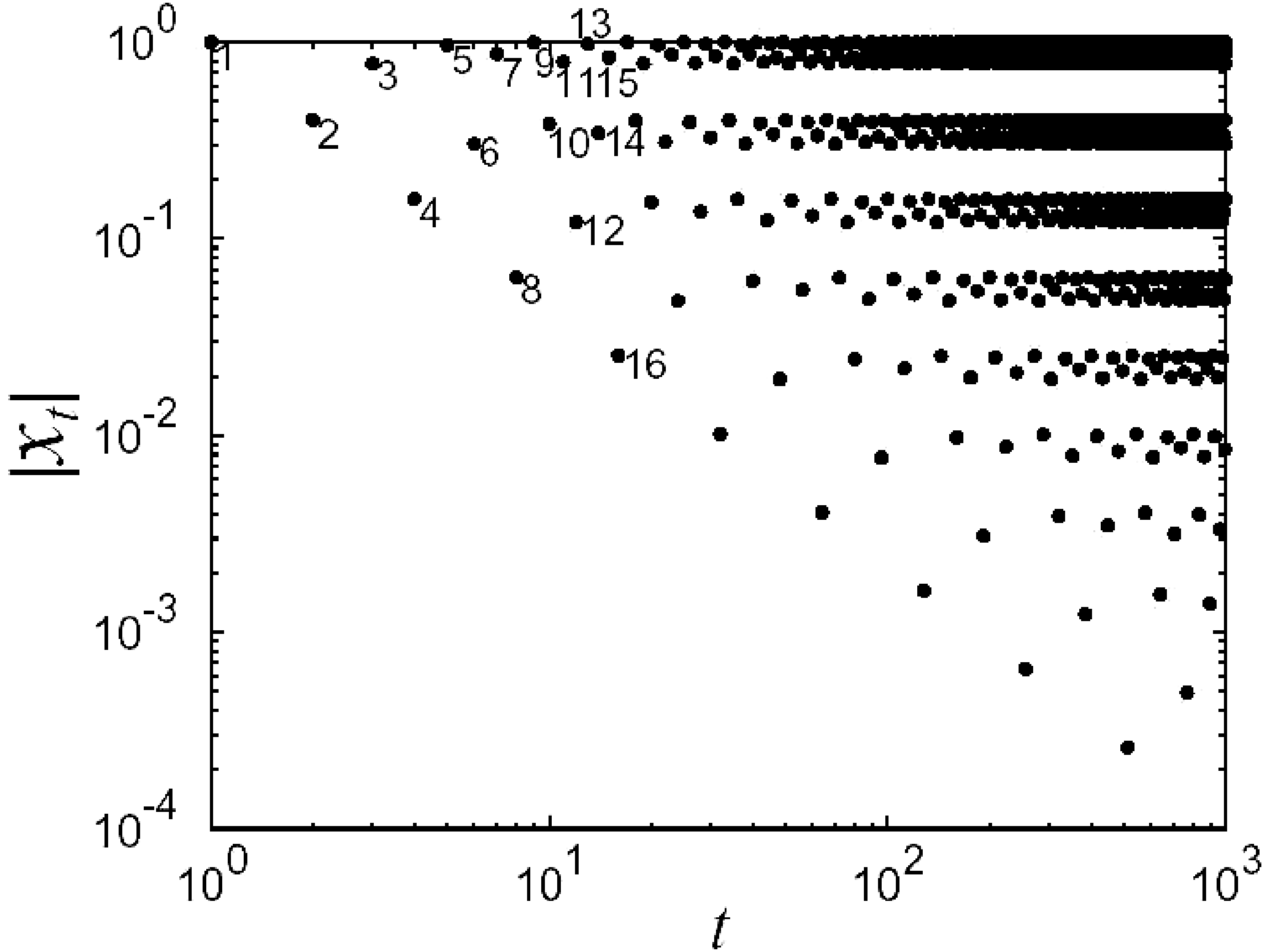




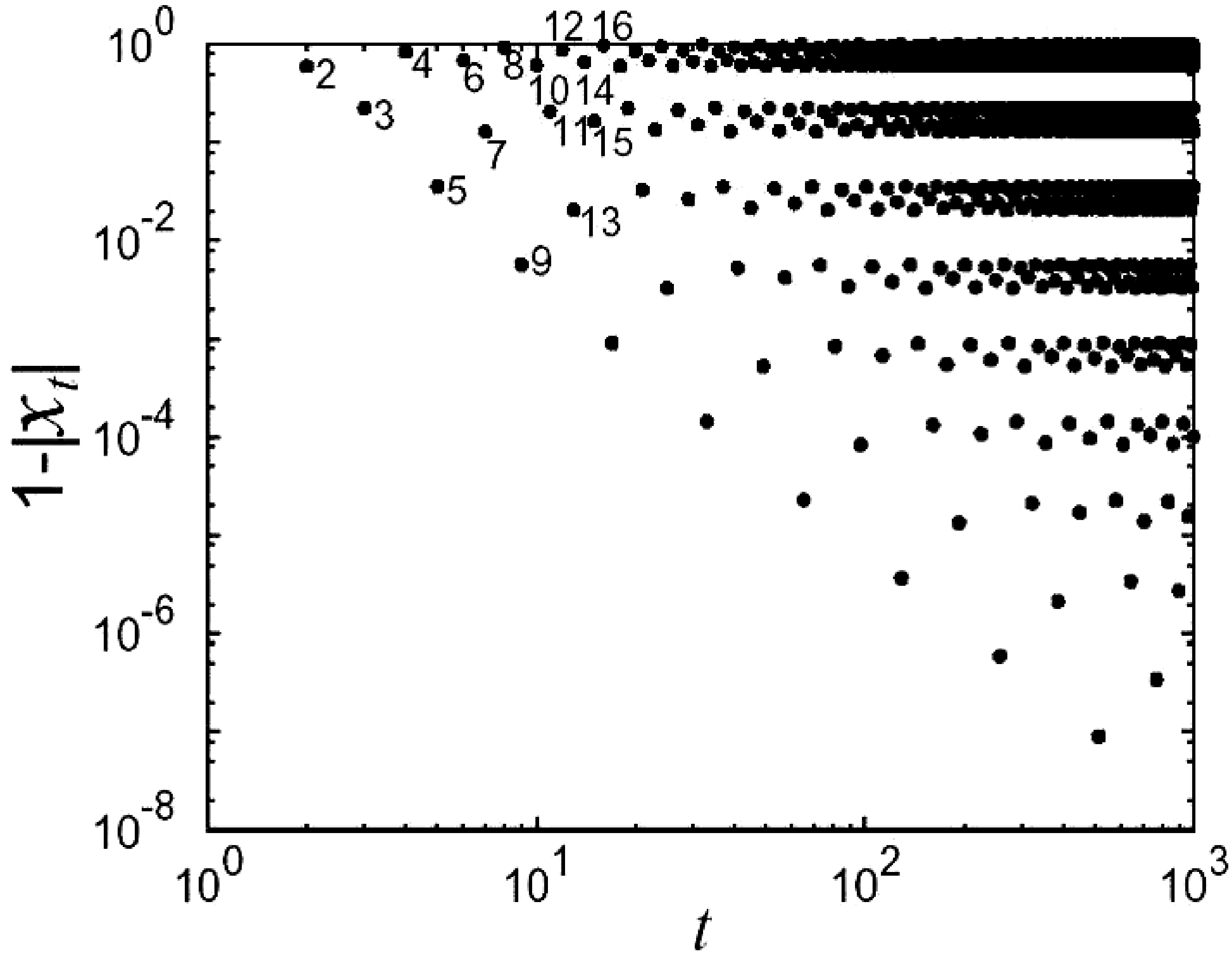


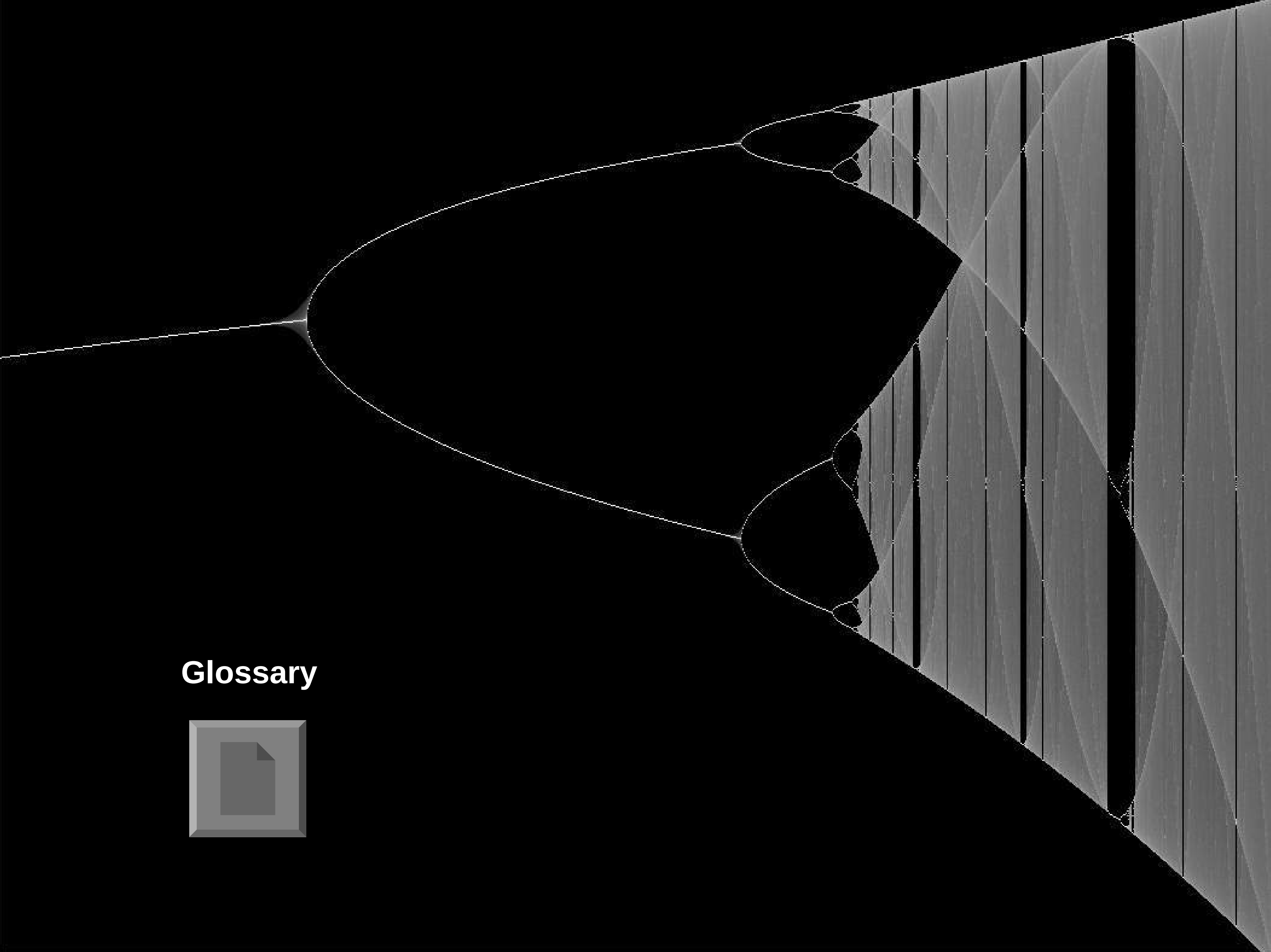
*As a Chinese proverb goes: *"A sparrow may be small but it has all the vital organs --- small but complete."* The one-dimensional logistic map is just the sparrow of nonlinear sciences. The Logistic map possesses bifurcations, stable and unstable periodic orbits, periodic windows, ergodic and mixing behaviors, homoclinic connections, chaotic orbits and some kinds of universality.*

Supercycle 2^{∞}



Supercycle 2^{∞}





Glossary

Questions?

Transitions to chaos: the realm of *q*-statistics

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Lecture Course
Nonextensive Statistical Mechanics
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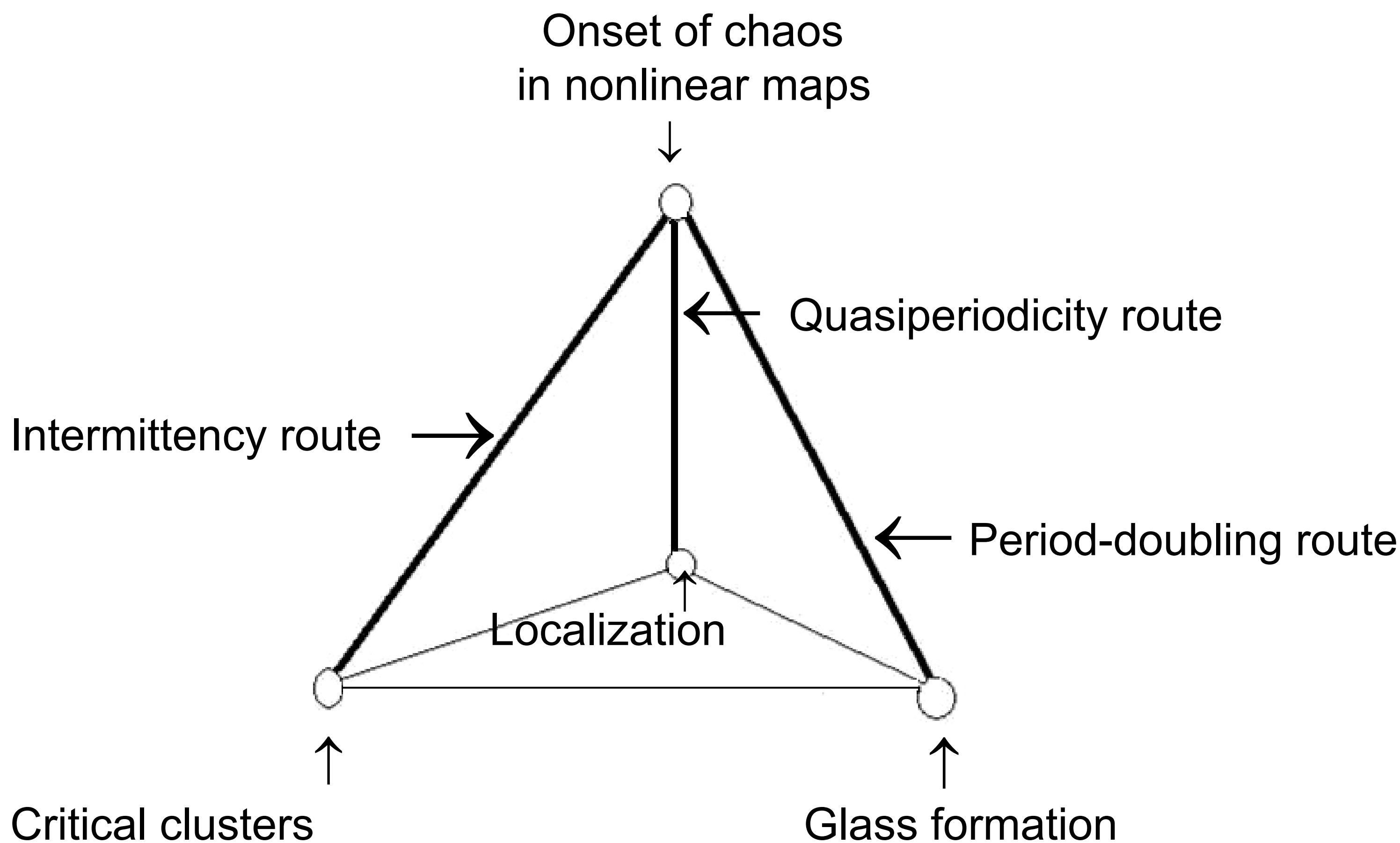
Part I.

Manifestations of *q*-statistics in the three routes to chaos

Part II.

***q*-statistics in criticality, glass formation & localization**


Onset of chaos
in nonlinear maps
Quasiperiodicity route
Intermittency route
Period-doubling route
Localization
Critical clusters
Glass formation

Incipient chaos in *d* =1 nonlinear maps

Route to chaos	Intermittency	Period doubling	Quasiperiodicity
Common properties	Vanishing ordinary Lyapunov coefficient, dynamical phase transitions (Mori's *q*-phases) power-law dynamics, *q*-sensitivity, *q*-Pesin identity		
Distinctive properties	(Also) faster than exponen-tial dynamics	Foam-like phase space	Dense phase space
Applications in condensed matter physics	Critical clusters	Glass formation	Localization
Applications in other disciplines	Information & other flows in networks, …	Protein folding, vegetation patterns, …	Mode locking, cardiac cells, Internet TCP, ...

Outline of Part I

- Critical attractors
- Pitchfork and tangent bifurcations
- The Feigenbaum attractor
- Quasiperiodic ‘golden mean’ attractor
- Sensitivity to initial conditions
- Mori’s *q*-phase transitions
- Linear growth of *q*-entropy
- Usefulness of the new theory

Outline of Part II

- Large critical fluctuations
- Multifractal geometry of critical clusters
- Cluster intermittency and equivalent map
- *q*-statistics at criticality
- Noise-perturbed onset of chaos
- Two-step glassy dynamics
- Aging at the transition to chaos
- Localization transition and quasi-periodic states

Glossary

Three routes to chaos in low-dimensional maps

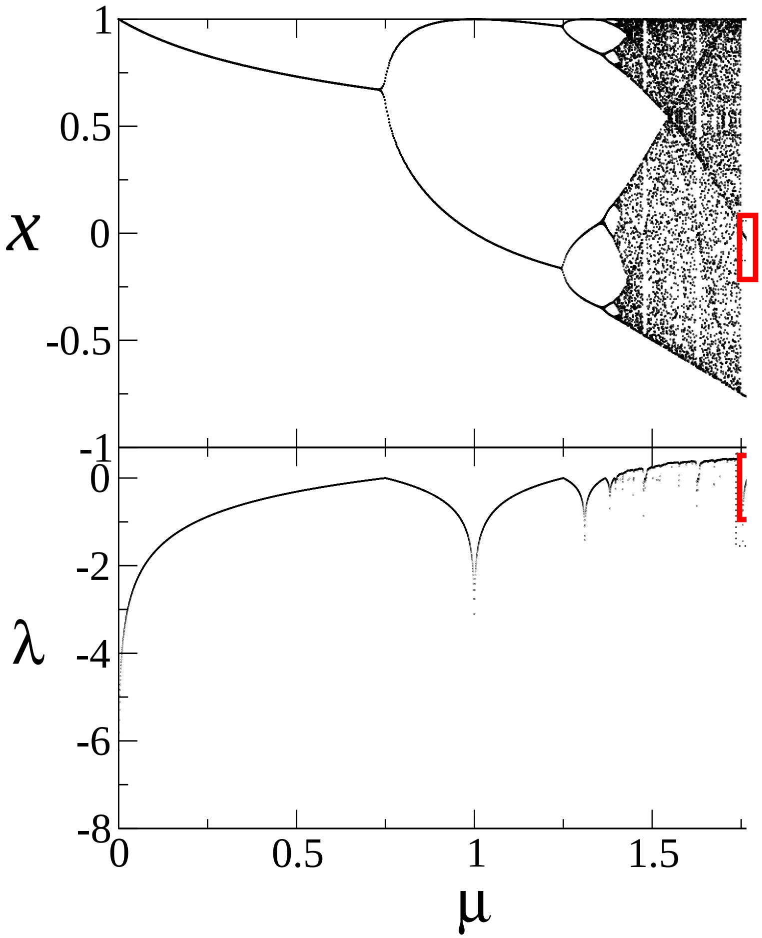
- Tangent bifurcation & intermittency
- Period doubling accumulation point
- Quasiperiodicity via the golden mean

Common features at transition between order and chaos:

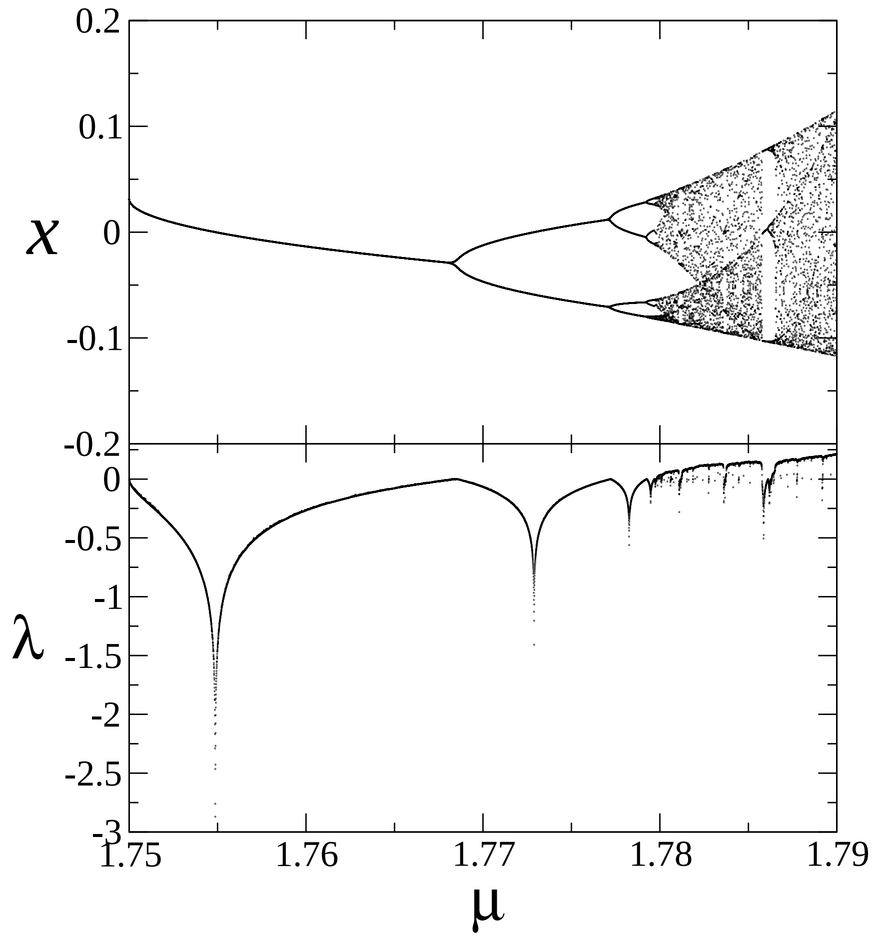
- Vanishing Lyapunov coefficient
- Fluctuations that grow indefinitely

$$x_{t+1} = f_\mu(x_t) = 1 - \mu x_t^2, \quad -1 \le x_t \le 1$$

All attractors



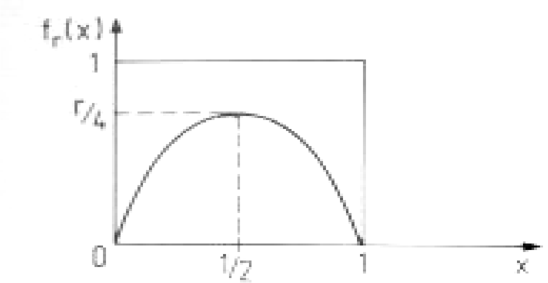
Period three window

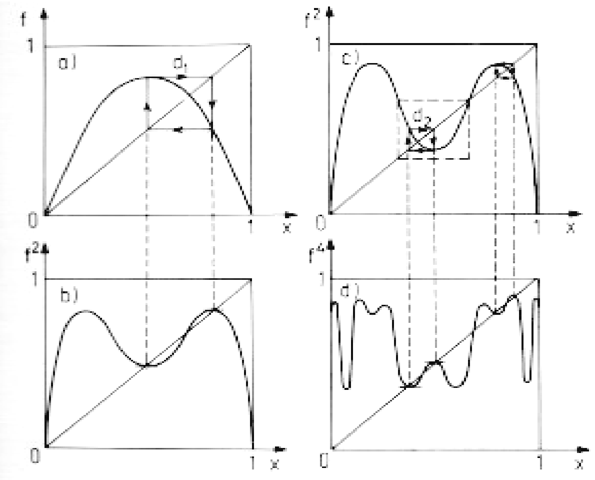


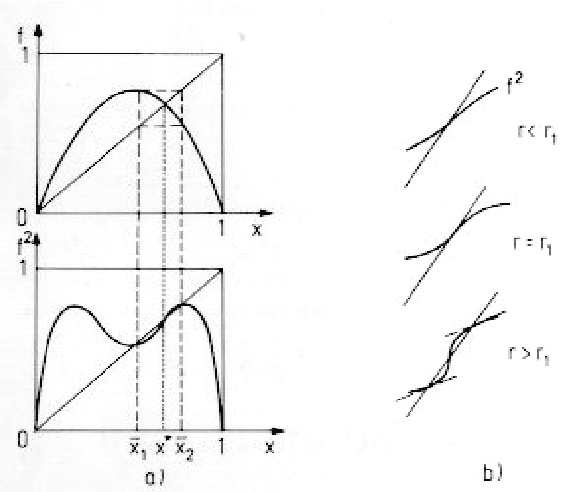
Logistic map

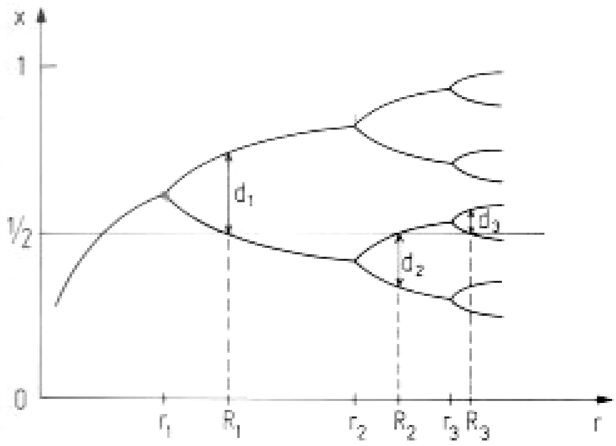
$$x_{t+1} = f_r(x_t) = r\,x_t\,(1 - x_t), \quad 0 \le x \le 1$$

$$x_{t+1} = f_\mu(x_t) = 1 - \mu\,x_t^2, \quad -1 \le x_t \le 1$$

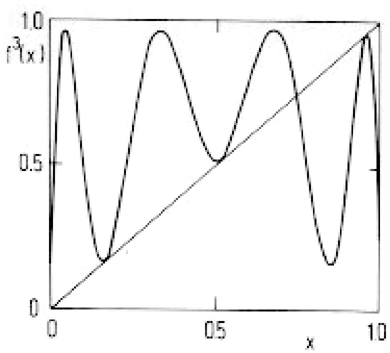


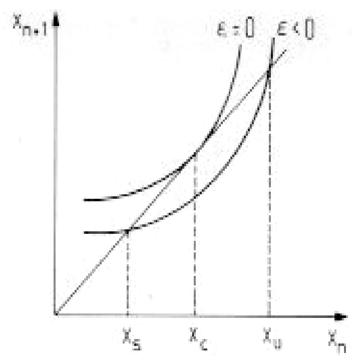


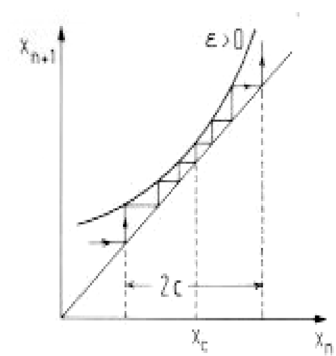


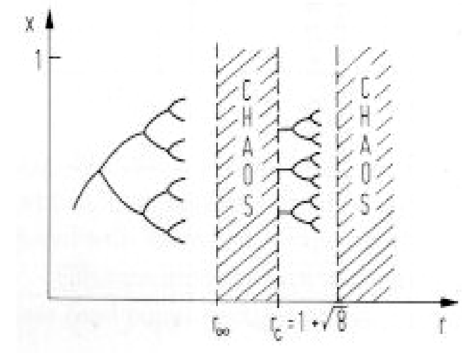


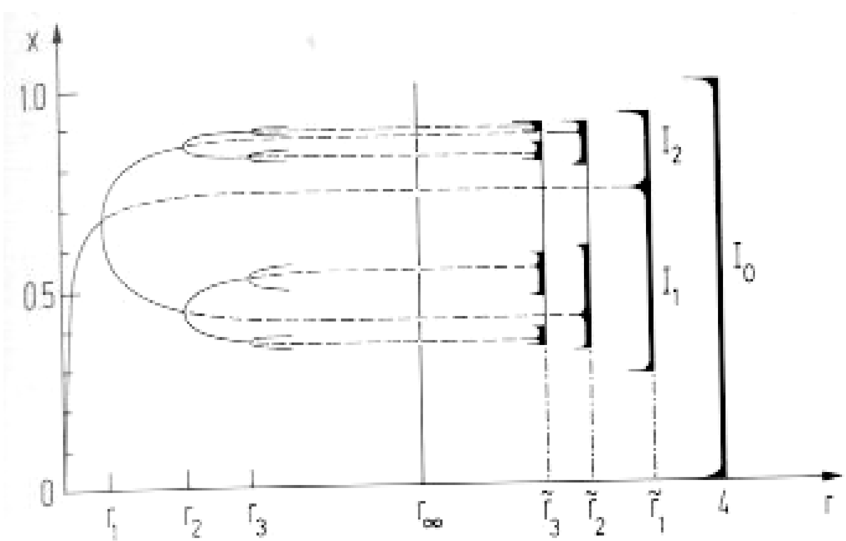
Intermittency and band splitting

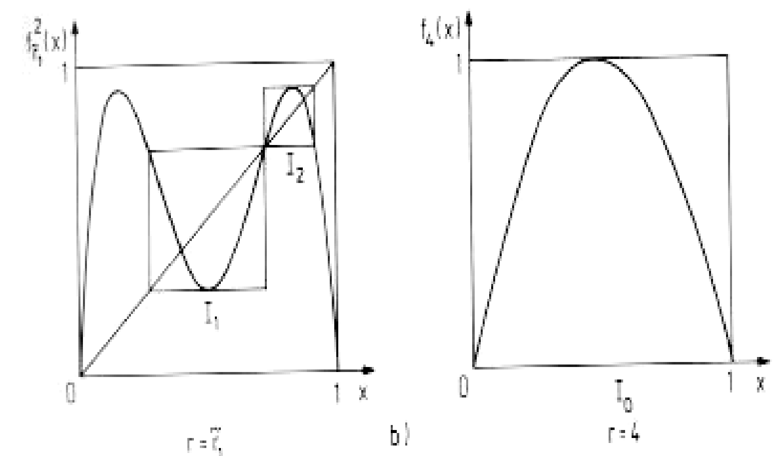




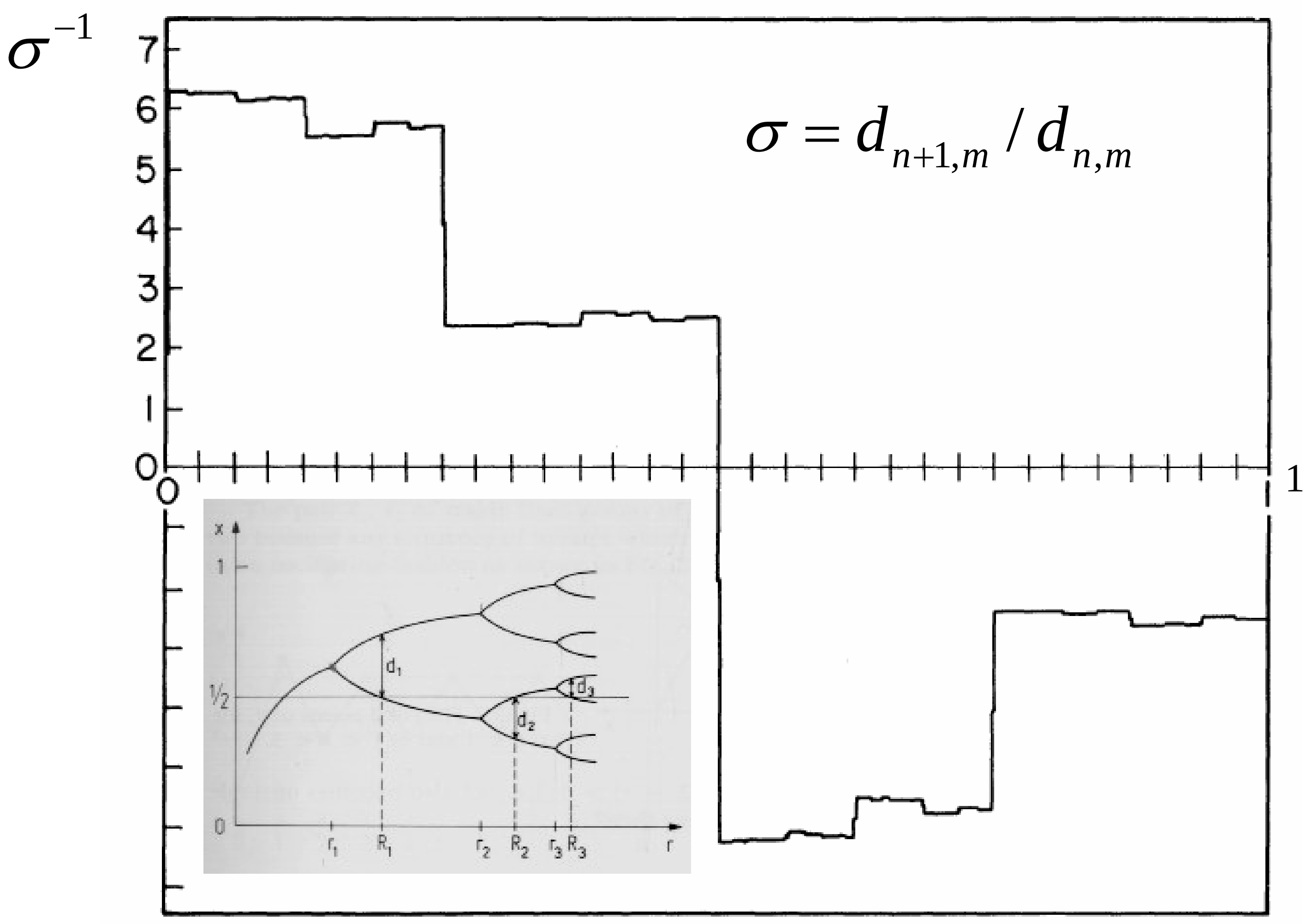








Feigenbaum's trajectory scaling function σ(*y*)



$$y = m / 2^{n+1}$$

Incidence of *q*-statistics at the transitions to chaos

Subject:

- Fluctuating dynamics at the onset of chaos (all routes)

Questions addressed:

- How much was known, say, ten years ago?
- Which are the relevant recent advances?
  Is the dynamics *fully* understood now?
- What is *q*-statistics for critical attractors?
- Is there rigorous, sensible, proof of incidence of
  q-statistics at the transitions to chaos?
- What is the relationship between *q*-statistics and the
  thermodynamic formalism?
- What is the usefulness of *q*-statistics for this problem?

Brief answers are given in the following slides

Ten years ago

- Numerical evidence of fluctuating dynamics (Grassberger & Scheunert)

- Adaptation of thermodymamic formalism to onset of chaos (Anania & Politi, Mori *et al*)

- But… implied anomalous statistics overlooked

Fluctuating dynamics at the Feigenbaum attractor

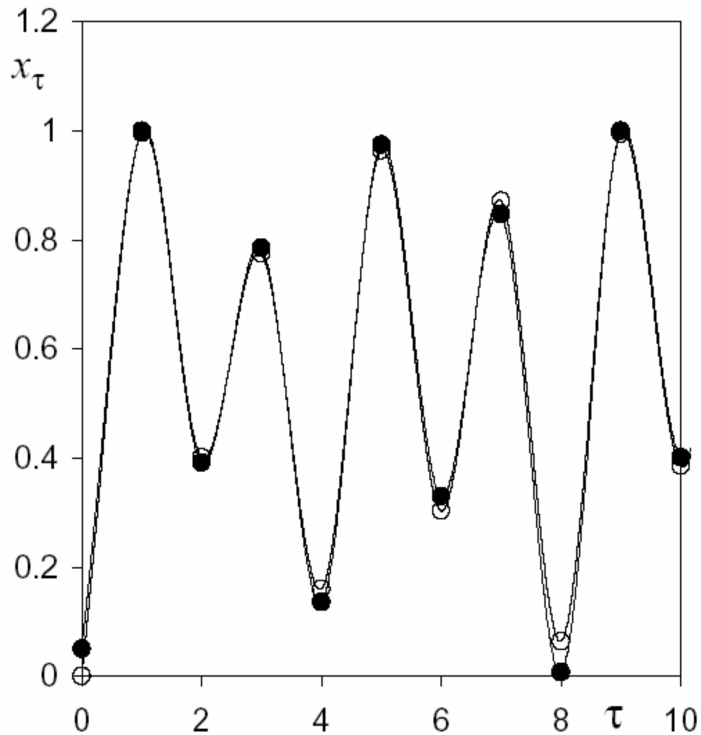
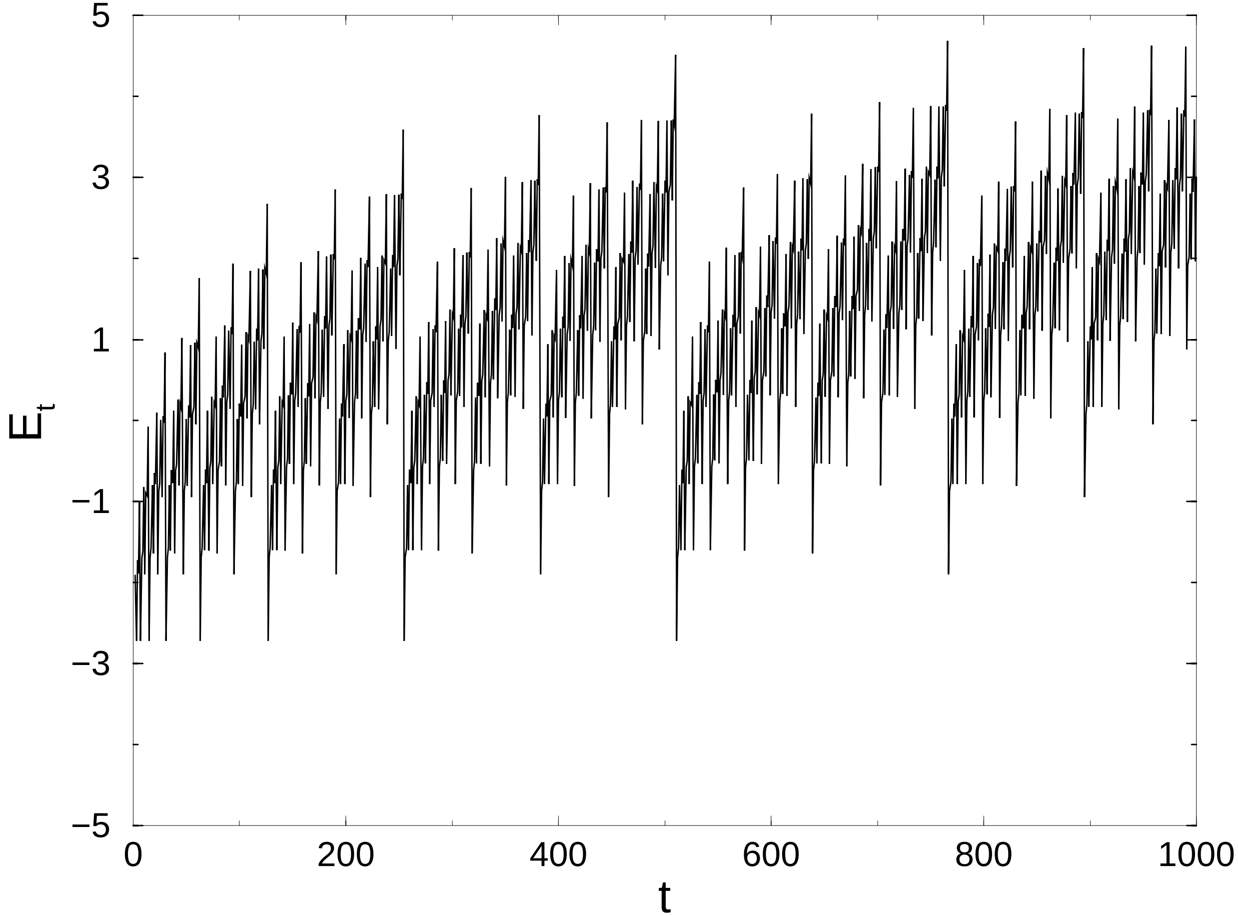


FIG. 1. Absolute values of positions of the first 10 iterations τ for two trajectories of the logistic map with initial conditions $x_0 = 0$ (empty circles) and $x_0 = \delta \simeq 10^{-1}$ (full circles). The lines are guides to the eye.

$$E_t = \sum_{i=1}^{t} \ln \left| \frac{dx_i}{dx_{i-1}} \right| \sim \ln t \qquad \lambda_t \equiv \frac{E_t}{t} = \frac{1}{t} \sum_{i=1}^{t} \ln \left| \frac{dx_i}{dx_{i-1}} \right| \sim 0 \qquad \frac{1}{\ln t} \sum_{i=1}^{t} \ln \left| \frac{dx_i}{dx_{i-1}} \right| \sim ?$$

$$E_t = \sum_{i=1}^{t} \ln\left|\frac{dx_i}{dx_{i-1}}\right|$$



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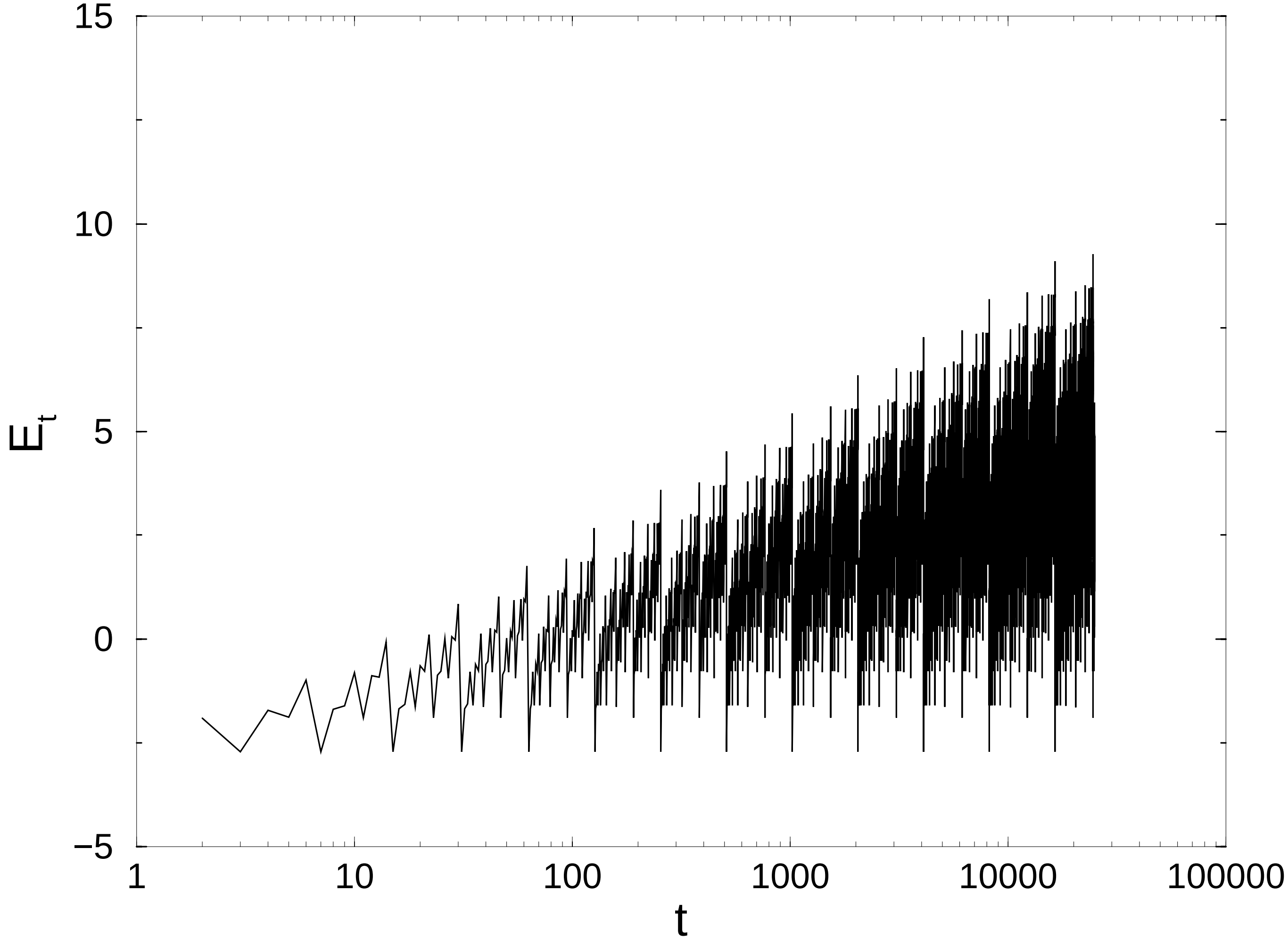

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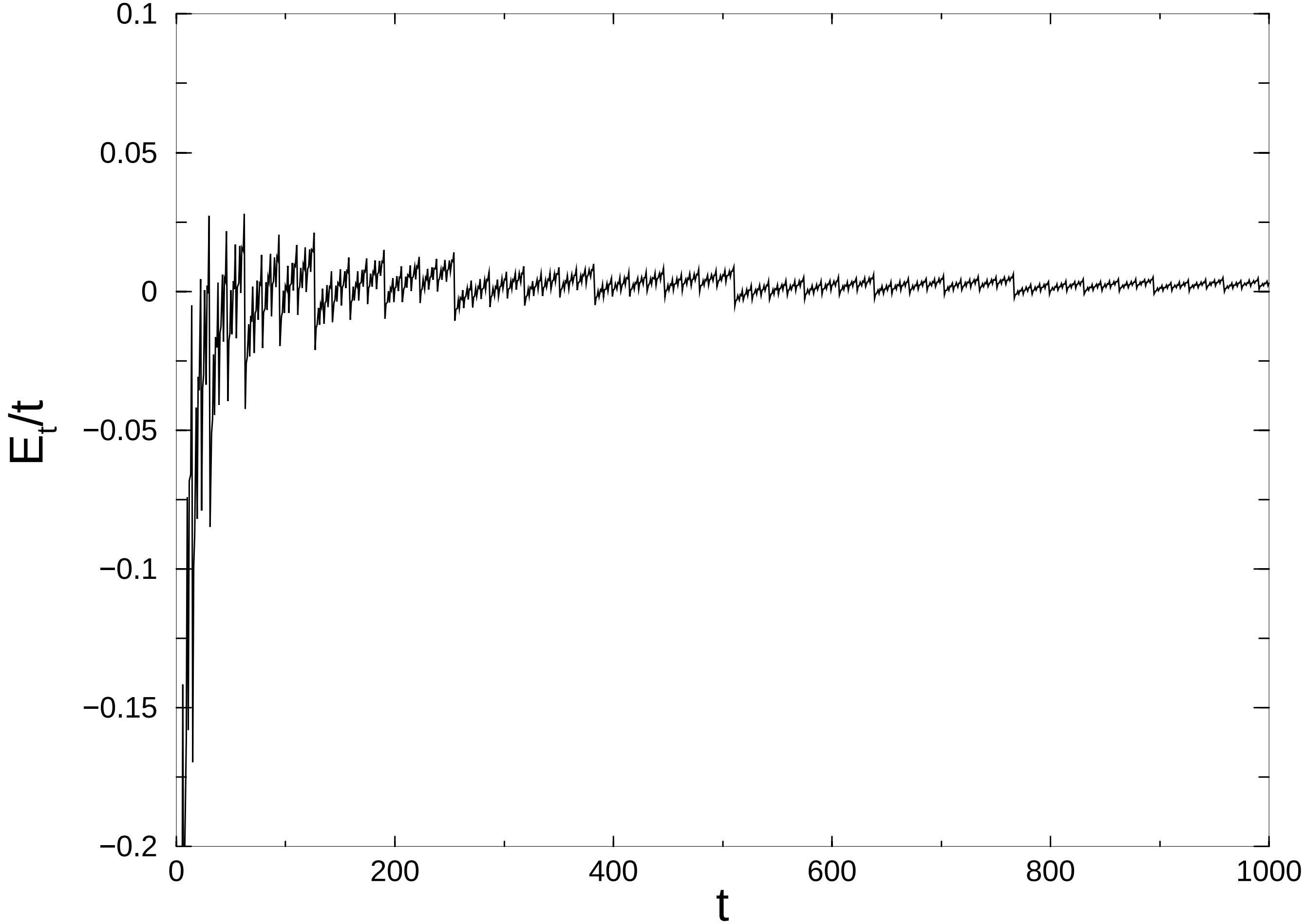

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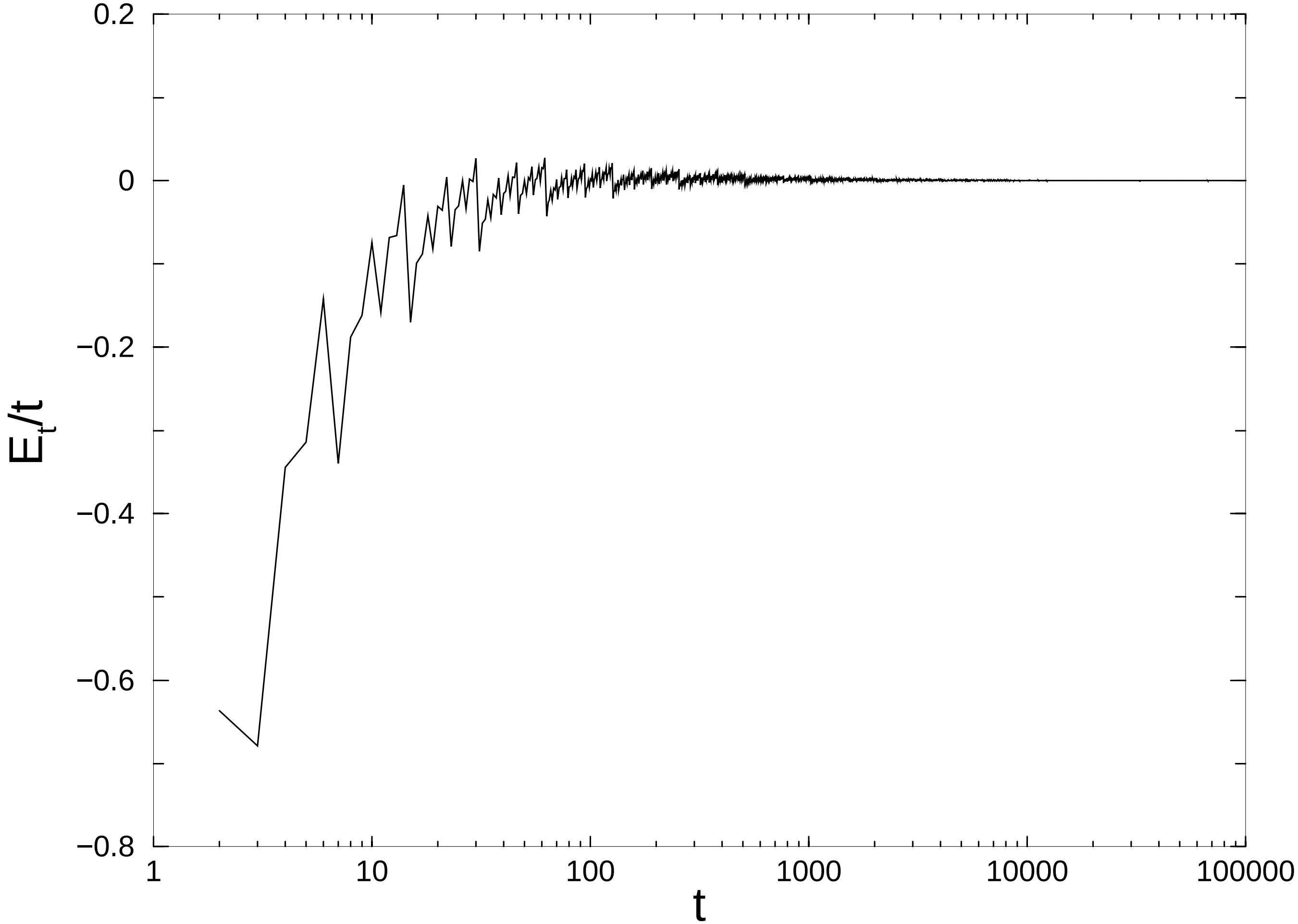

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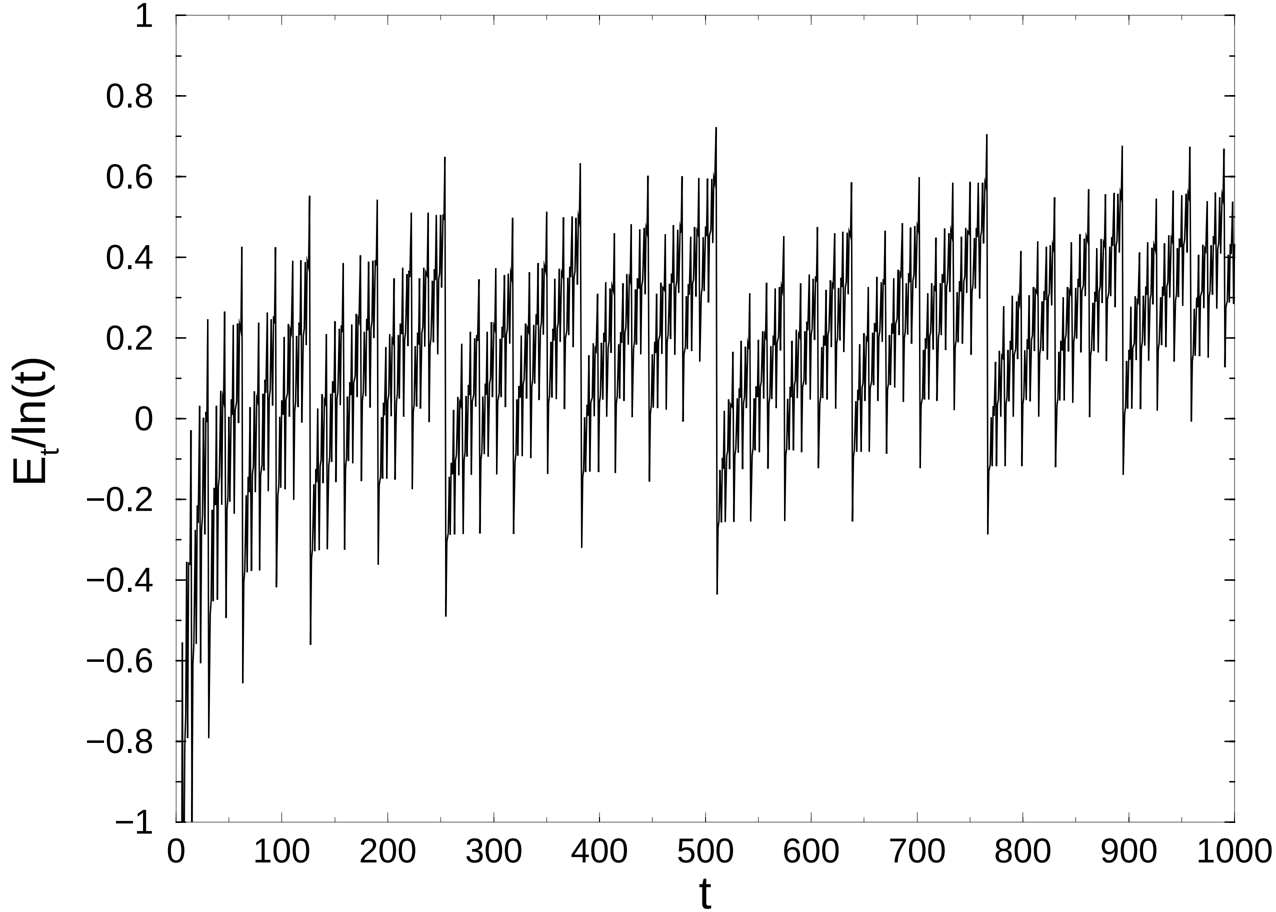

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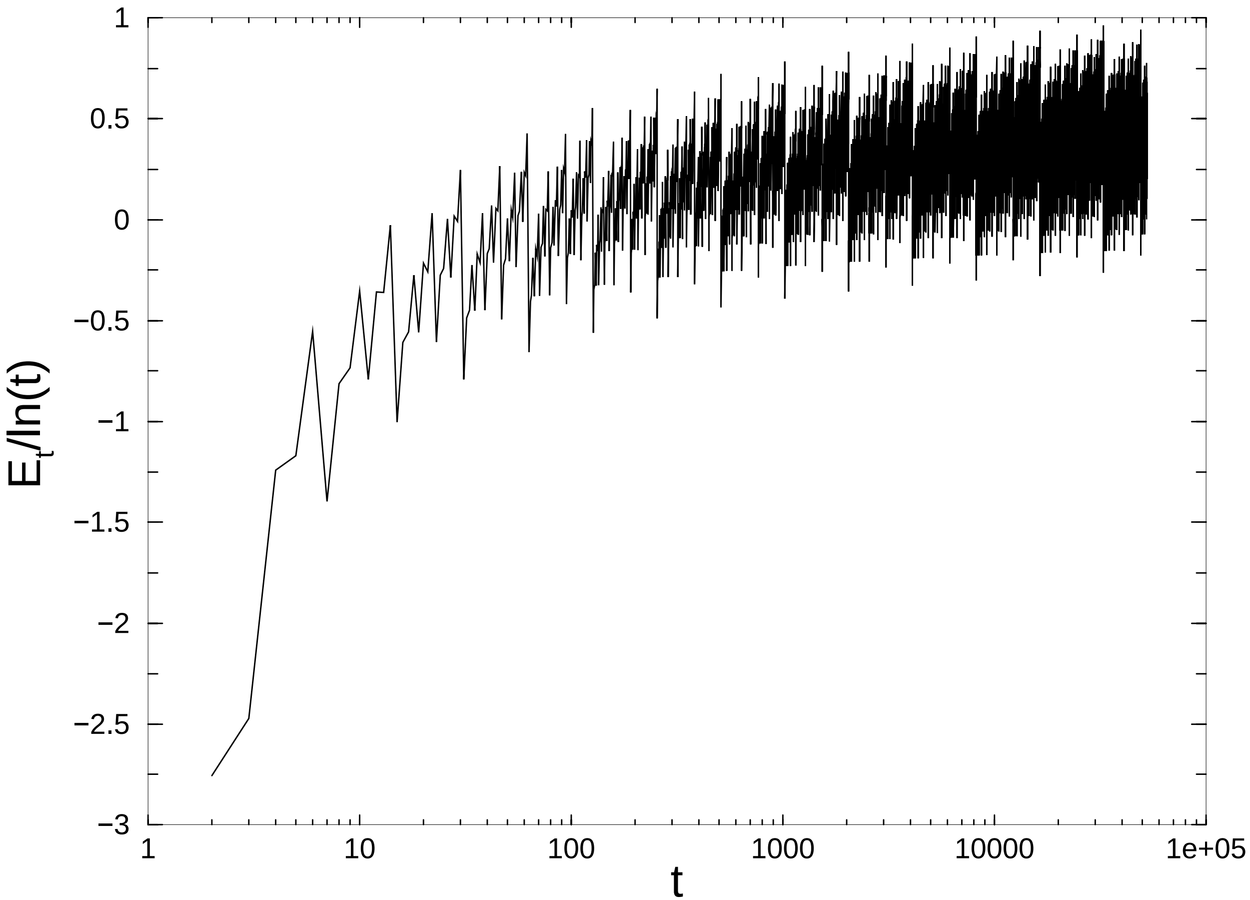


Thermodynamic approach for attractor dynamics
(Mori and colleagues ~1989)

- 'Special' Lyapunov coefficients

$$\lambda(t,x_0)=\frac{1}{\ln t}\sum_{i=0}^{t-1}\ln\left|\frac{df(x_i)}{dx_i}\right|,\quad t>>1,\qquad P(\lambda;t)=t^{-\psi(\lambda)}P(0;t)$$

- Partition function

$$Z(t,\mathrm{q})\equiv\int d\lambda\, P(\lambda;t)W(\lambda,t)^{1-\mathrm{q}},\quad W(\lambda,t)=t^{\lambda}$$

- Free energies

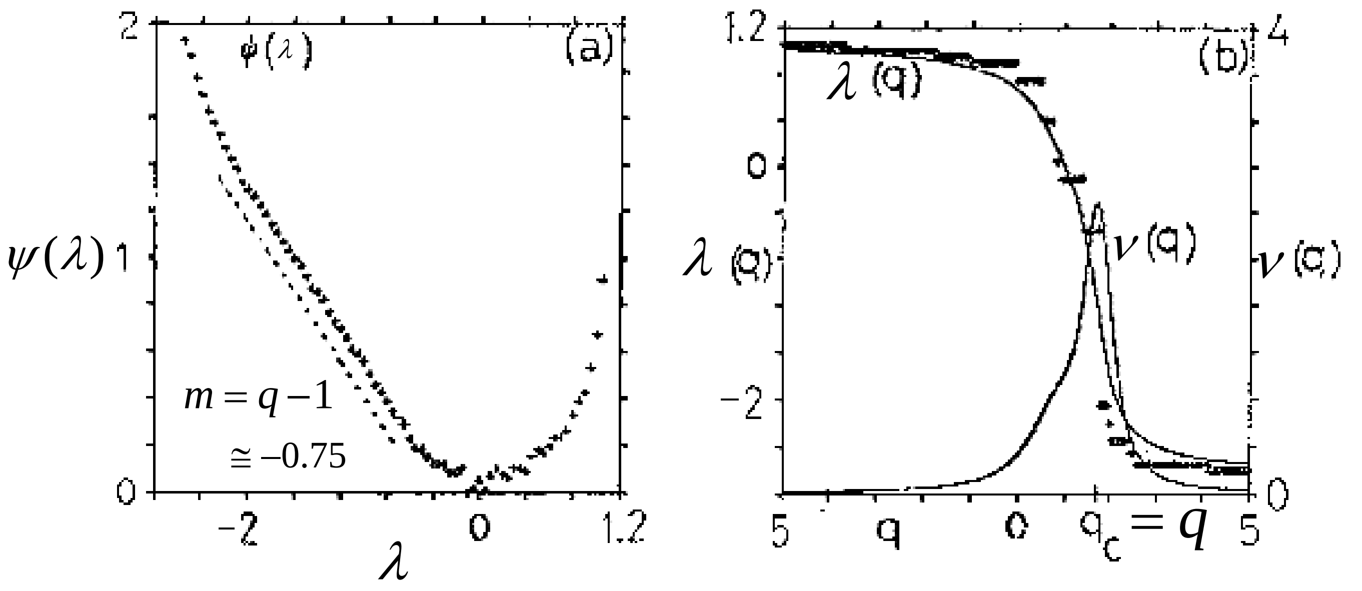
$$\phi(\mathrm{q})\equiv\lim_{t\to\infty}\frac{\ln Z(t,\mathrm{q})}{\ln t},\qquad \psi(\lambda)=\phi(\mathrm{q})-\lambda(\mathrm{q}-1)$$

- Equation of state and susceptibility

$$\lambda(\mathrm{q})=\frac{d\phi(\mathrm{q})}{d\mathrm{q}},\quad \upsilon(\mathrm{q})=\frac{d\lambda(\mathrm{q})}{d\mathrm{q}}$$

q ~ 'magnetic field' λ ~ 'magnetization'

Mori's *q*-phase transition at the period doubling onset of chaos



$$P(\lambda;t) = t^{-\psi(\lambda)} P(0;t),$$

Static spectrum

$$\tau(\mathrm{q}) = f(\alpha) + \alpha \mathrm{q}\,,$$

$$\tau(\mathrm{q}) = (\mathrm{q}-1) D_{\mathrm{q}}$$

Dynamic spectrum

$$\phi(\mathrm{q}) = \psi(\lambda) + \lambda(\mathrm{q}-1)$$

- Is theTsallis index *q* the value of **q** at the Mori transition?

Today

- Rigorous, analytical, results for the three routes to chaos (e.g. sensitivity to initial conditions)

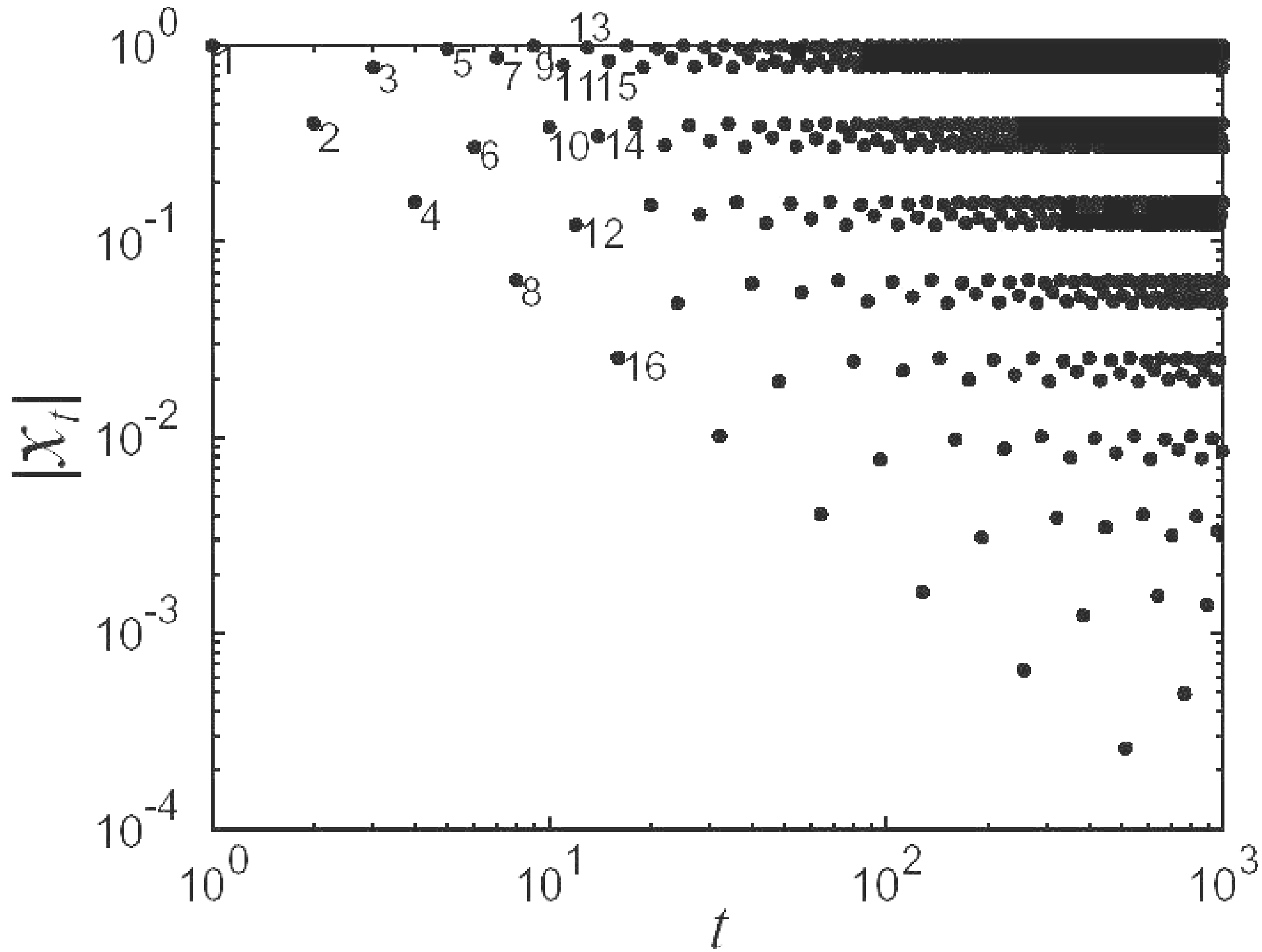
- Hierarchy of dynamical *q*-phase transitions

- Link between thermodymamics and *q*-statistics

- Temporal extensivity of *q*-entropy

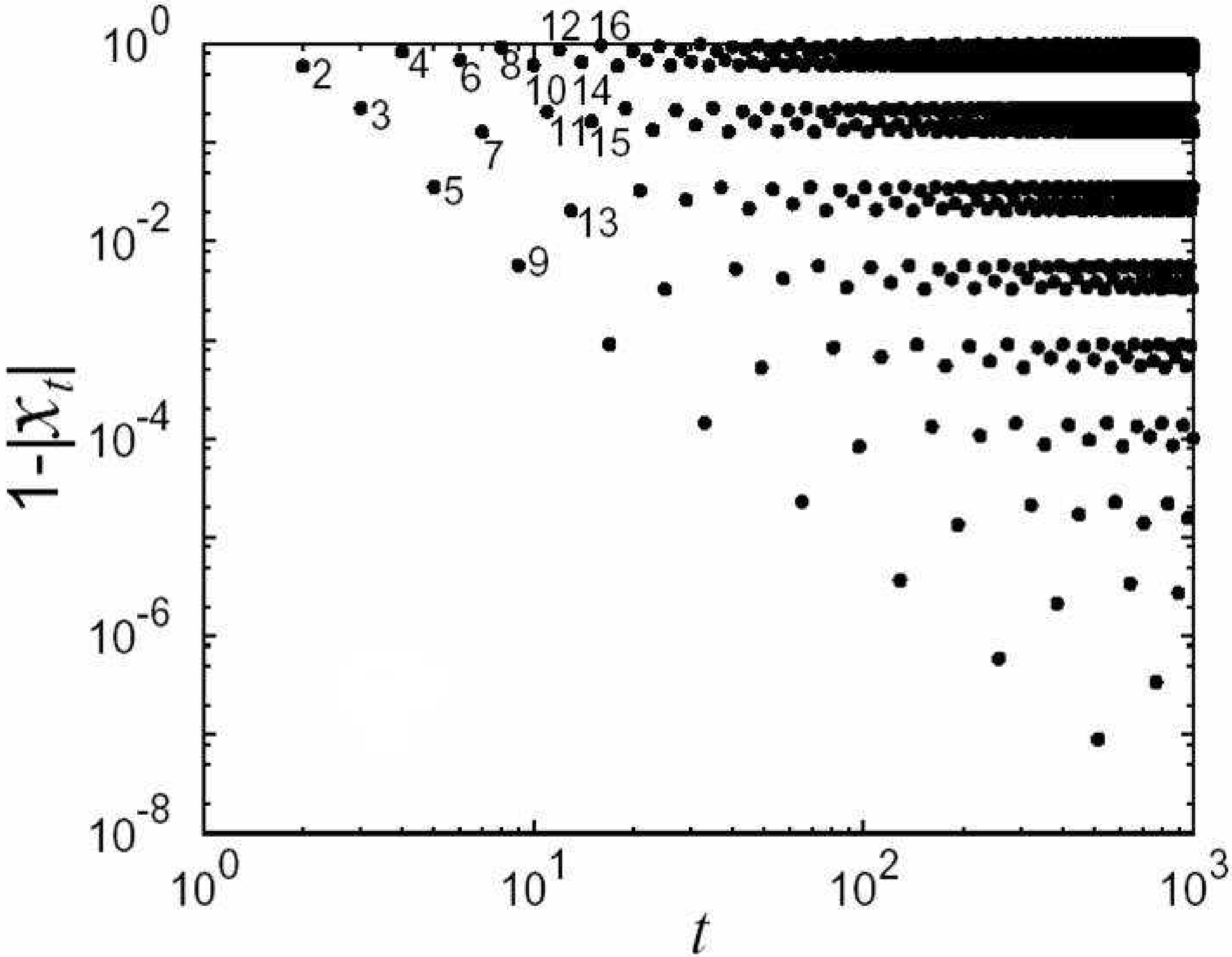
- *q* values determined from theoretical arguments

Trajectory on the Feigenbaum attractor



$$x_{t+1} = f_{\mu_\infty}(x_t) = 1 - \mu_\infty\, x_t^2, \quad -1 \le x_t \le 1$$

2^{∞} - supercycle



q-statistics for critical attractors

Sensitivity to initial conditions

$$\xi(x_0, t) \equiv \lim_{\Delta x_0 \to 0} \frac{\Delta x_t}{\Delta x_0}$$

- **Ordinary statistics:**

$$\xi(x_0, t) = \exp[\lambda_1(x_0) t]$$

(independent of x_0 for $t \to \infty$)

- ***q* statistics:**

$$\xi(x_0, t) = \exp_q[\lambda_q(x_0) t]$$

(dependent on x_0 for all t)

- ***q*-exponential function:**

$$\exp_q(x) \equiv [1 + (1-q)\, x]^{\frac{1}{1-q}}$$

- **Basic properties:**

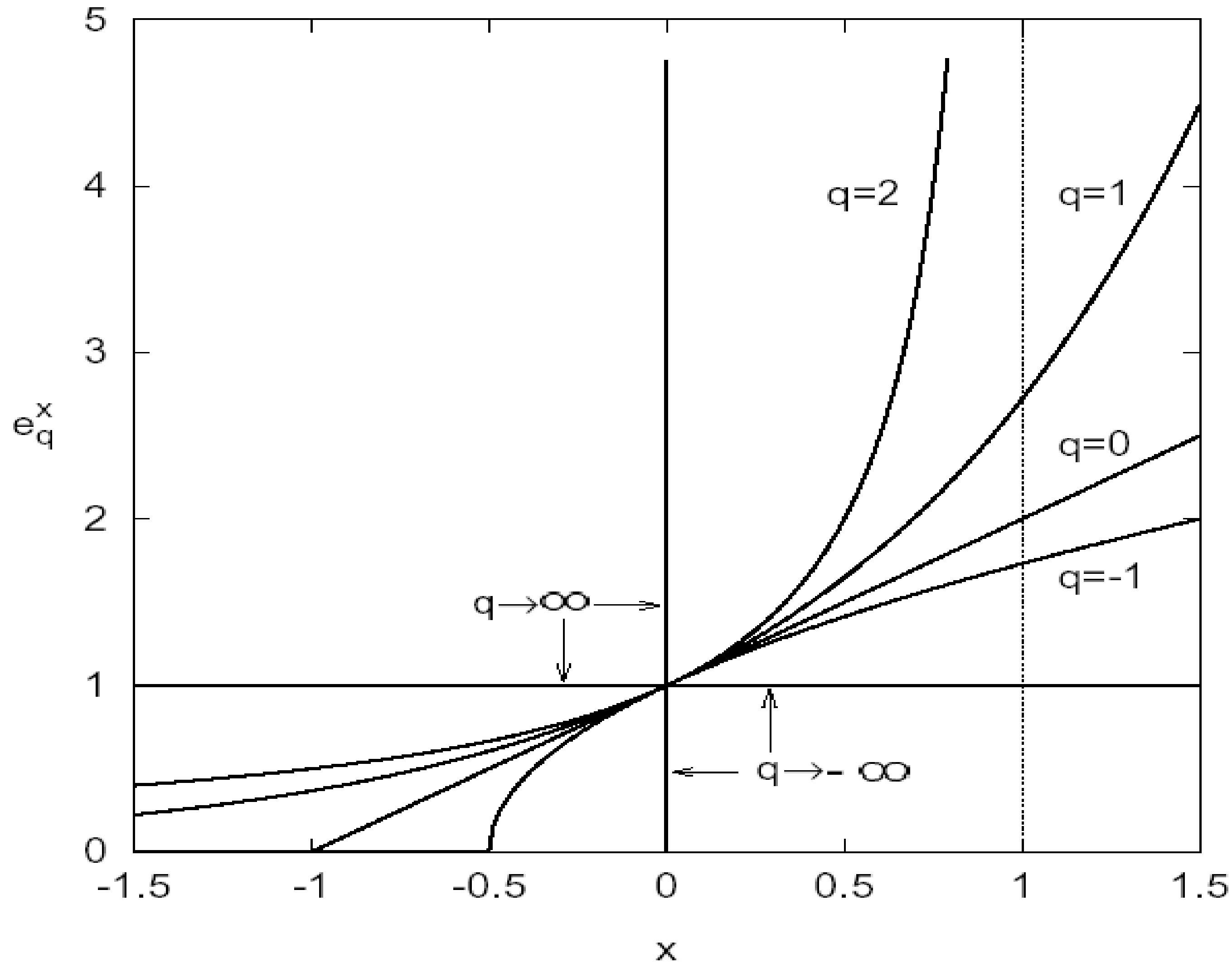
$$\exp(x) = \lim_{q \to 1} \exp_q(x)$$

$$\exp_q(-x) = [\exp_Q(x)]^{-1}, \quad Q = 2 - q$$

$$d \exp_q(x) / dx = [\exp_q(x)]^q$$

$$x_{t+1} = f_\mu(x_t), \quad a \le x_t \le b, \quad A \le \mu \le B$$

q-exponential function



Entropic expression for Lyapunov coefficient

$$\lambda_q \equiv \lim_{t\to\infty} \frac{1}{t} \left[S_q(t) - S_q(0) \right]$$

• **Ordinary statistics:**

$$S_1 = -\sum_i p_i \ln p_i$$

• ***q* statistics:**

$$S_q = -\sum_i p_i^q \ln_q p_i \quad \text{or} \quad -\sum_i p_i \ln_Q p_i$$

• ***q*-logarithmic function:** $\ln_q(y) \equiv \dfrac{y^{1-q} - 1}{1-q} \quad (y \in R^+; q \in R)$

• **Basic properties:**

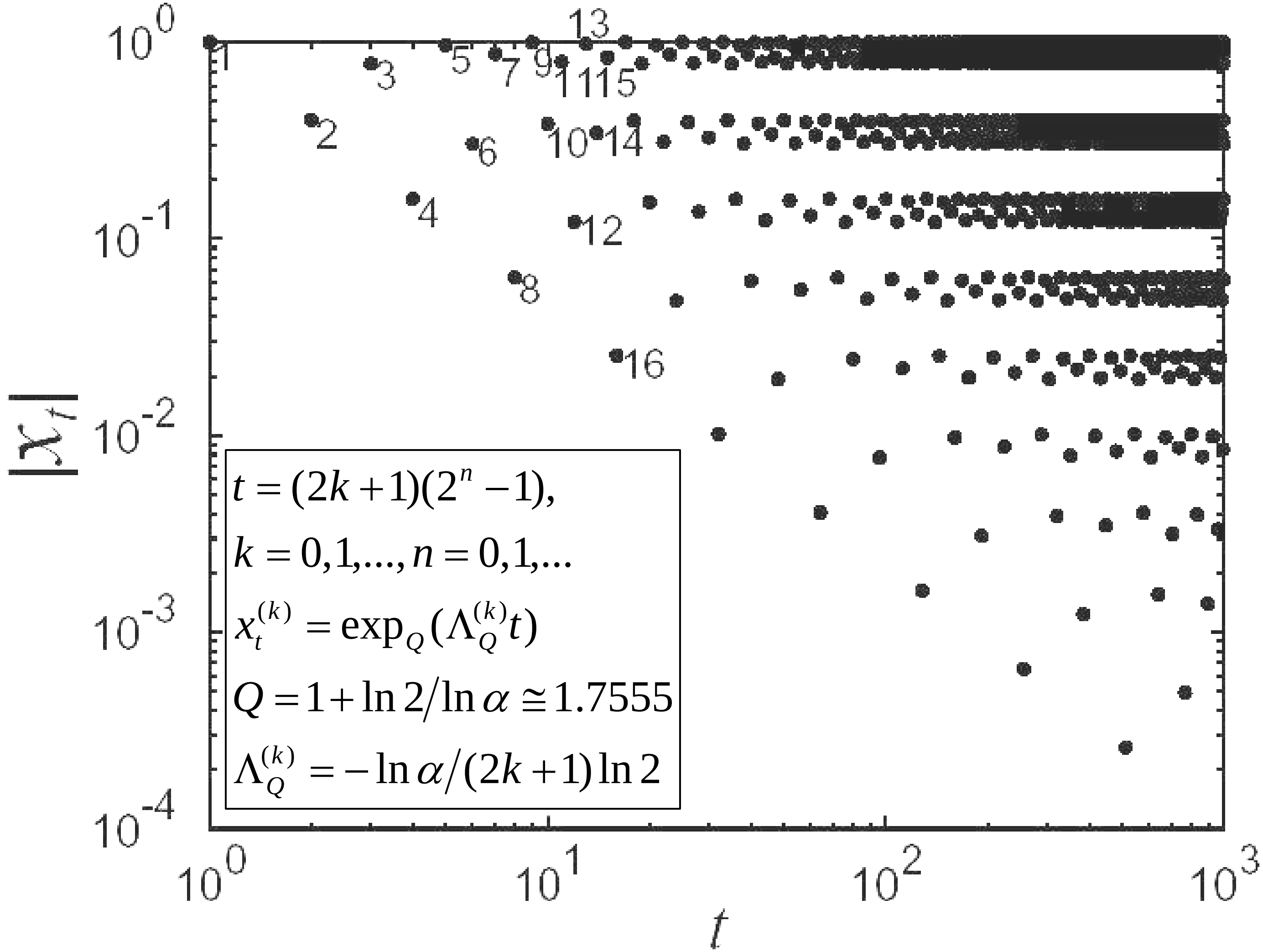
$$\ln(y) = \lim_{q\to 1} \ln_q(y)$$

$$\ln_q(y) = -\ln_Q(1/y), \quad Q = 2-q$$

$$\ln_q(\exp_q(x)) = \exp_q(\ln_q(x)) = x$$

Analytical results for the sensitivity

2^{∞} - supercycle



Power laws, *q*-exponentials and two-time scaling

$$\xi_t(x_{in}) = \alpha^k, \quad x_{in} = 1$$

$$t = (2l+1)(2^k - 1), \quad k = 1,2,\ldots, \quad l = 0,1,\ldots$$

$$\alpha^k \equiv \left(1 + \frac{t}{2l+1}\right)^{\ln\alpha/\ln 2} \Rightarrow$$

$$\xi_t(x_0) = \exp_q\left[\lambda_q^{(l)} t\right], \quad q = 1 - \frac{\ln 2}{\ln\alpha}, \quad \lambda_q^{(l)} = \frac{\ln\alpha}{(2l+1)\ln 2},$$

$$\xi_t(x_0) = \exp_q\left[\lambda_q^{(0)} t / t_w\right], \quad t = (2l+1)2^k - 1, \quad t_w = 2l+1$$

Sensitivity to initial conditions within the Feigenbaum attractor

- Starting at the most crowded (*x*=1) and finishing at the most sparse (*x*=0) region of the attractor

$$\xi_t(x_0) = \exp_q\left[\lambda_q^{(l)} t\right], \quad q = 1 - \frac{\ln 2}{(z-1)\ln\alpha}, \quad \lambda_q^{(l)} = \frac{(z-1)\ln\alpha}{(2l+1)\ln 2},$$

$$t = (2l+1)2^k - 1, \quad l = 0,1,\ldots, k = 0,1,\ldots$$

- Starting at the most sparse (*x*=0) and finishing at the most crowded (*x*=1) region of the attractor

$$\xi_t(x_0) = \exp_{2-q}\left[\lambda_{2-q}^{(l)} t\right], \quad 2-q = 1 + \frac{\ln 2}{(z-1)\ln\alpha}, \quad \lambda_{2-q}^{(l)} = -\frac{2(z-1)\ln\alpha}{(2l+1)\ln 2},$$

$$t = (2l+1)2^k + 1, \quad l = 0,1,\ldots, k = 0,1,\ldots$$

Thermodynamic approach and *q*-statistics

Mori's definition for Lyapunov coefficient at onset of chaos

$$\lambda(t, x_{in}) \equiv \frac{1}{\ln t} \ln \left| \frac{dg^{(t)}(x_{in})}{dx_{in}} \right|,$$

is equivalent to that of same quantity in *q*-statistics

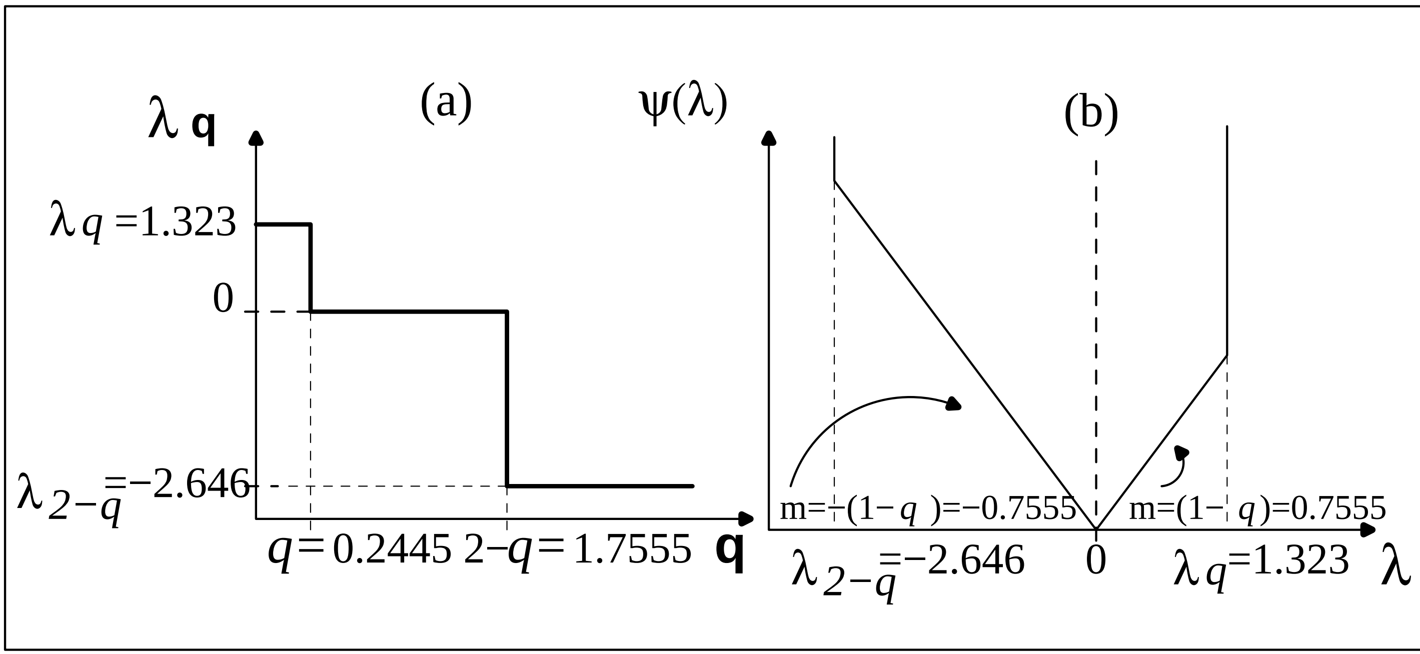
$$\lambda(t, x_{in} = 1) = \frac{1}{t} \ln_q \left| \frac{dg^{(t)}(x_{in})}{dx_{in}} \right|_{x_{in}=1} = \frac{\ln \alpha}{(2l+1)\ln 2} = \lambda_q^{(l)},$$

$$t = (2l+1)2^k - 1, \quad k = 0,1,\ldots, l = 0,1,\ldots$$

Two-scale Mori's λ(q) and ψ(λ) for period-doubling threshold

Dynamic spectrum

$$P(\lambda;t)=t^{-\psi(\lambda)}P(0;t),$$



$$q=1-\frac{\ln 2}{\ln\alpha}\cong 0.2445 \qquad \lambda_q=\frac{\ln\alpha}{\ln 2}\cong 1.3236 \qquad m=1-q=\frac{\ln 2}{\ln\alpha}\cong 0.7555$$

Hierarchical family of *q*-phase transitions

Trajectory scaling function σ(*y*) → sensitivity ξ(t)

- $$\sigma_n(m) = \frac{d_{n+1,m}}{d_{n,m}}, \quad d_{n,m} = \left| x_m - x_{m+2^{n-1}} \right|$$
- $$\sigma(y) = \lim_{n\to\infty} \sigma_n(m), \quad y = \frac{m}{2^n}$$
- $$\xi_t(m) = \left| \frac{d_{n,m+t}}{d_{n,m}} \right| \cong \left| \frac{\sigma_n(m-1)}{\sigma_n(m)} \right|^{\pm n}, \quad t = 2^n \pm 1$$

Spectrum of *q*-Lyapunov coefficients with common index *q*

- Successive approximations to σ(*y*),

$$\sigma(y) = \alpha_j^{-1}, \quad a_j \le y < a_{j+1}, \quad j = 0,1,\ldots,J, \quad J = 1,2,\ldots,$$

lead to:

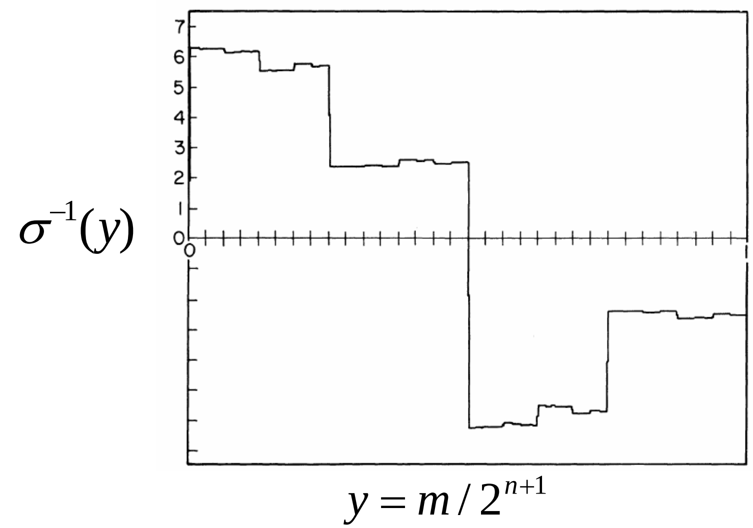
$$\xi_t(x_0) = \left|\frac{\alpha_j}{\alpha_{j+1}}\right|^{\pm n} = \exp_q\left[\lambda_q^{(k)} t\right], \quad \text{where}$$

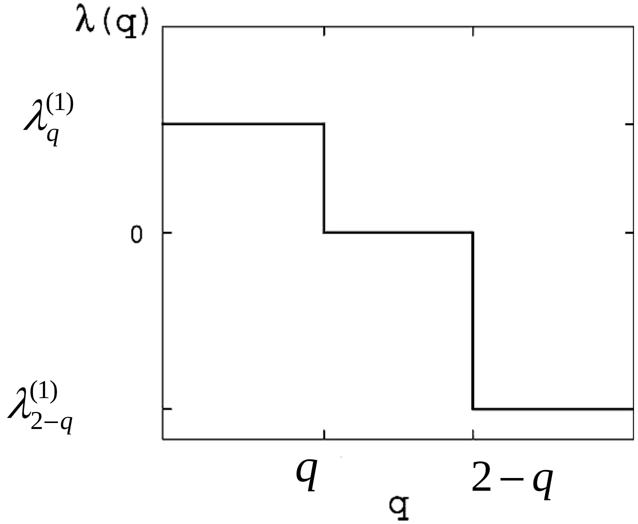
$$q = 1 + \frac{\ln 2}{\ln(\alpha_j / \alpha_{j+1})}, \quad \lambda_q^{(k)} = \frac{\ln(\alpha_j / \alpha_{j+1})}{(2k+1)\ln 2},$$

and similarly with *Q=2-q*

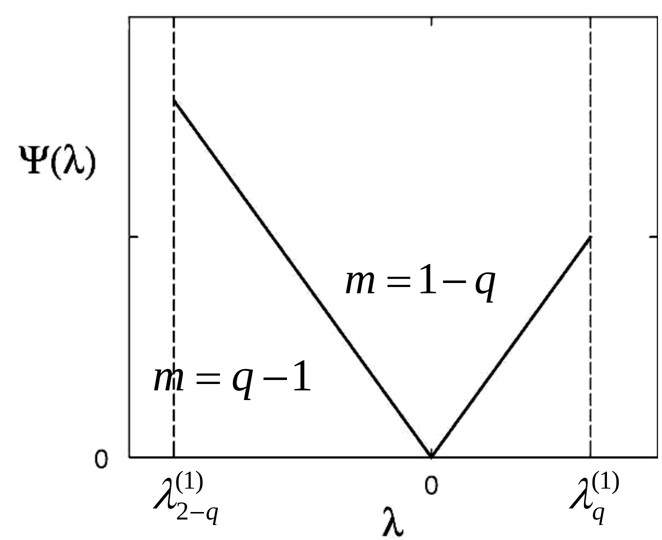
Infinite family of *q*-phase transitions

- Each discontinuity in σ(y) leads to a couple of *q*-phase transitions





$$\lambda(\mathrm{q}) = \frac{d\phi(\mathrm{q})}{d\mathrm{q}}$$



$$\psi(\lambda) = \phi(\mathrm{q}) - \lambda(\mathrm{q}-1)$$

Temporal extensivity of the *q*-entropy

Precise knowledge of dynamics implies that

$$P(\lambda,t)W(\lambda,t) = \delta(\lambda-\lambda_q^{(l)})\exp_q(\lambda_q^{(l)}t)$$

therefore

$$Z(t,\mathrm{q}) \equiv W(\lambda_q^{(l)},t)^{1-\mathrm{q}} = \left[1+(1-q)\lambda_q^{(l)}t\right]^{(1-\mathrm{q})/(1-q)}$$

and

$$Z(t,\mathrm{q}) \equiv \sum_{i=1}^{W}\left[p_i(t)\right]^{\mathrm{q}} = 1+(1-\mathrm{q})S_{\mathrm{q}}(t),$$

with

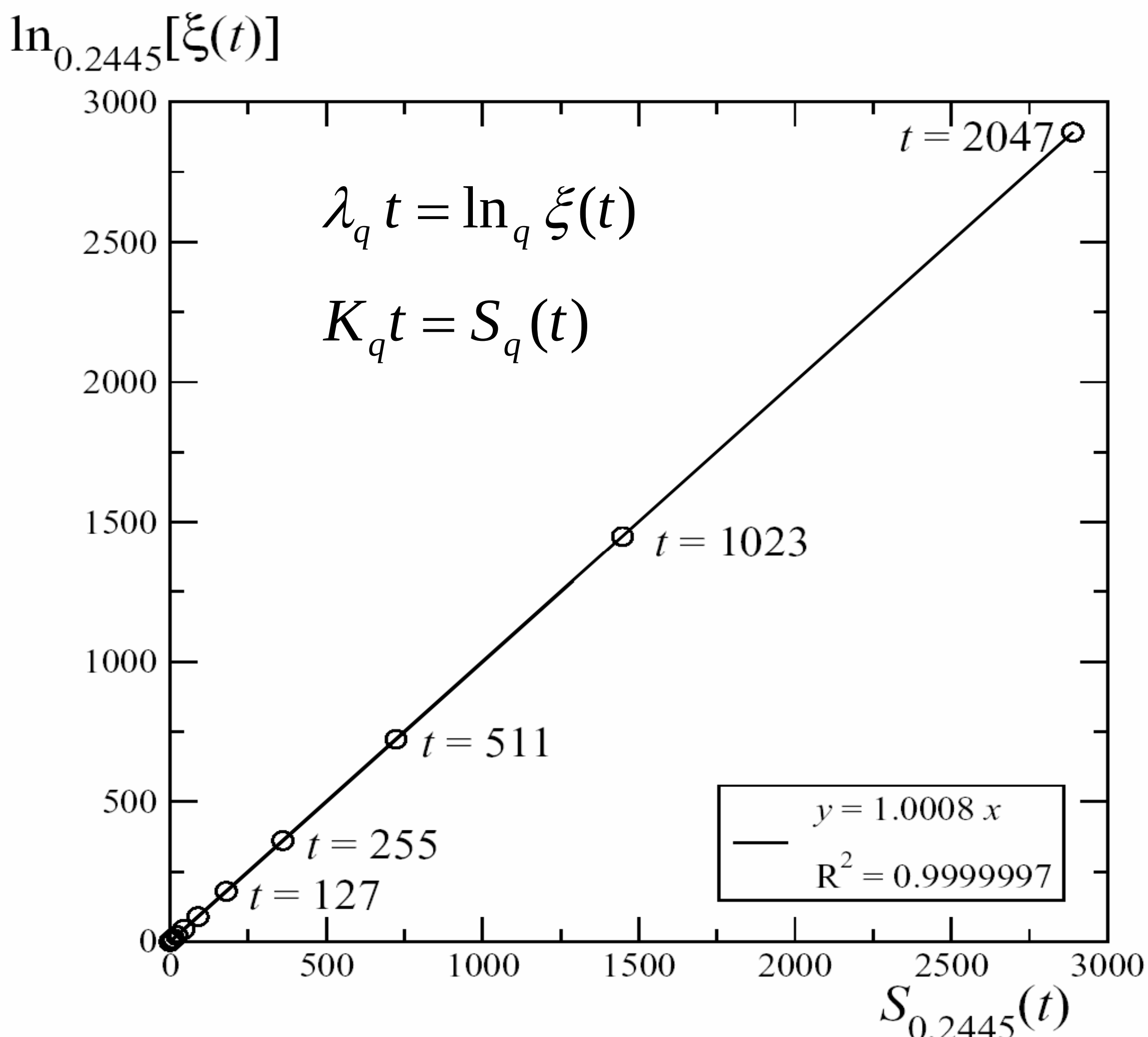
$$p_i(t) = \left[W(t)\right]^{-1} \quad \text{for all } i \quad \text{and} \quad S_{\mathrm{q}} = \ln_{\mathrm{q}} W.$$

When q=*q*

$$S_q(t) = \lambda_q^{(l)}t,$$

$$t = (2l+1)(2^k-1), \quad k = 1,2,..., \quad \text{with fixed} \quad l = 0,1,...$$

Linear growth of S_q



F. Baldovin and A. Robledo (2003).

q-statistics at the pitchfork and tangent bifurcations

Trajectories at the bifurcation points

- **Logistic map of order *ζ***

$$f_\mu(x) = 1 - \mu|x|^\zeta,\ \zeta > 1,\ -1 \le x \le 1,\ 0 \le \mu \le 2$$

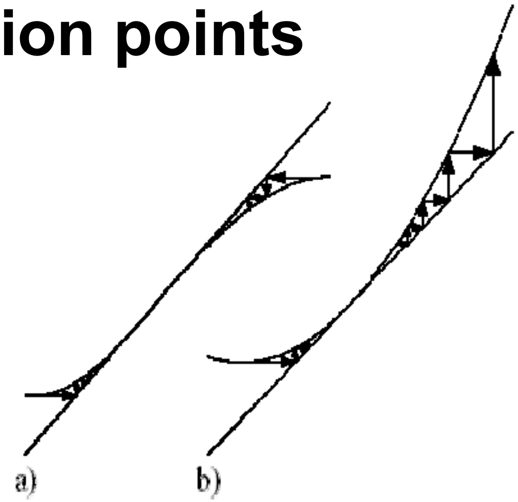


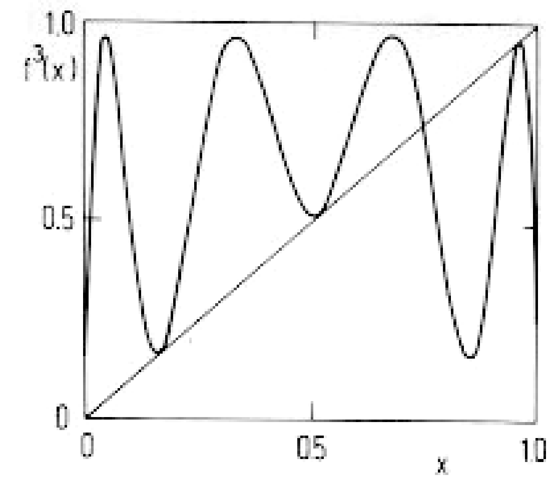
Figure 1: Schematic form of $f^{(n)}$ at the a) pitchfork and b) tangent bifurcations.

- **Conjugate *q*-indexes**

$$q = 1 - z^{-1} = 2 - Q^{-1} \quad \Leftrightarrow \quad \exp_q(x) \equiv [\exp_Q(x/Q)]^Q$$

- **Two universality classes (for all ζ)**

Pitchfork	Tangent
$q = 5/3$	$q = 3/2$



Trajectories at the bifurcation points

- **Logistic map of order ζ**

$$f_\mu(x) = 1 - \mu|x|^\zeta,\ \zeta > 1,\ -1 \le x \le 1,\ 0 \le \mu \le 2$$

- **Map at bifurcation point**

$$f^{(n)}(x) = x + u|x|^z + O(|x|^z)$$

- **RG fixed-point solution**

$$f^*(f^*(x)) = \alpha^{-1} f^*(\alpha x) \quad \Rightarrow \quad x'^{-(z-1)} = x^{-(z-1)} - (z-1)u, \quad \alpha = 2^{\frac{1}{z-1}}$$

$$x' = f^*(x) = x \exp_z(xux^{z-1}) = [1-(z-1)ux^{z-1}]^{-\frac{1}{z-1}}, \quad x^{z-1} \equiv |x|^{z-1}\operatorname{sgn}(x)$$

- **Trajectories**

$$x_t = x_0 \exp_z(ax_0^{z-1}t) = x_0[1-(z-1)ax_0^{z-1}t]^{-\frac{1}{z-1}}, \quad u = at$$

$$\Delta x_t^{-(z-1)} + \varepsilon_t \Delta x_t^{-z} = \Delta x_0^{-(z-1)} + \varepsilon_0 \Delta x_0^{-z} - (z-1)at$$

q-generalized Lyapunov coefficient

- **Sensitivity to initial conditions** $f^{*(m)}(x) = m^{-\frac{1}{z-1}} f^{*}(m^{\frac{1}{z-1}}x), \quad m = 1, 2, \ldots$

$$\xi(x_0,t) \equiv \lim_{\Delta x_0 \to 0} \frac{\Delta x_t}{\Delta x_0} = [1-(z-1)ax_0^{z-1}\, t]^{-\frac{z}{z-1}}, \quad u = at, \quad q = 2 - z^{-1}$$

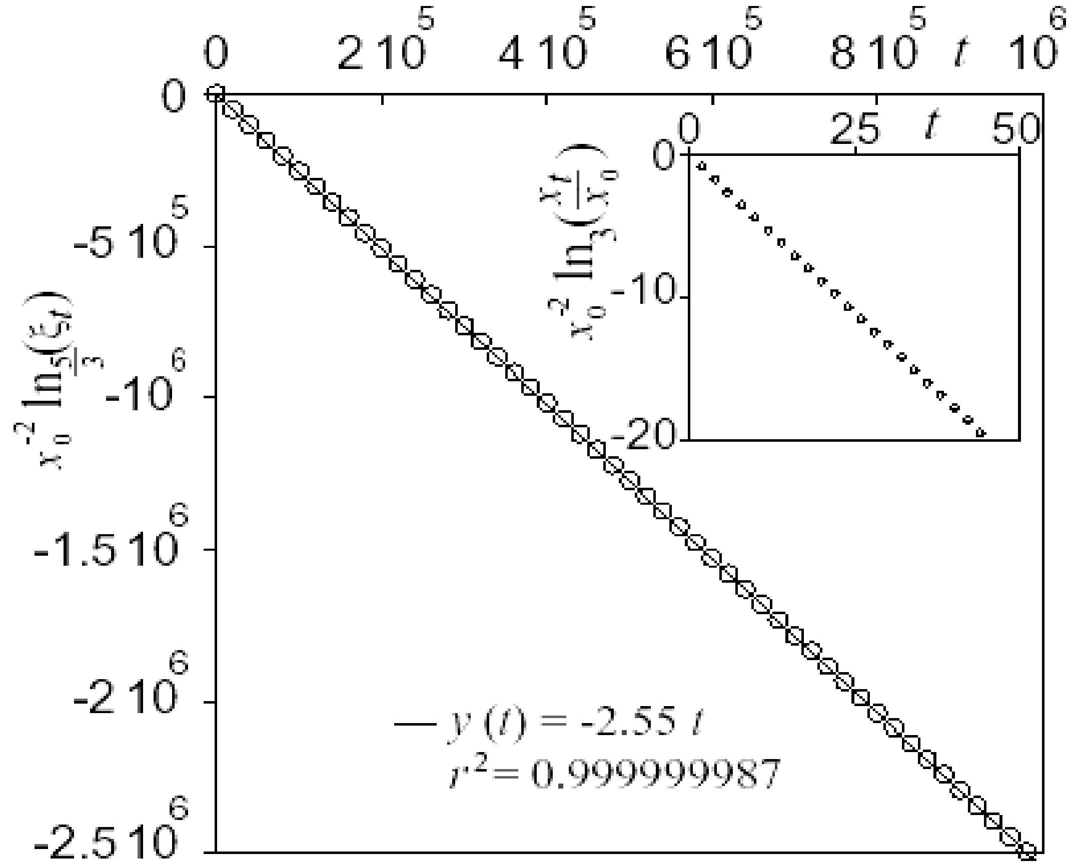
- ***q*-Lyapunov coefficient**

$$\lambda_q(x_0) = zax_0^{z-1} \qquad \overline{\lambda_q} = \int dx_0\, \lambda_q(x_0)\, \rho(x_0) = a, \quad \rho(x_0) \propto |x_0|^{-(z-1)}$$

- **Two universality classes (for all ζ)**

Pitchfork	Tangent
$f^{(n)}(x) = x + ux^3 + O(x^3), \quad u < 0$	$f^{(n)}(x) = x + ux^2 + O(x^2), \quad u > 0$
$q = 5/3$	$q = 2/3$

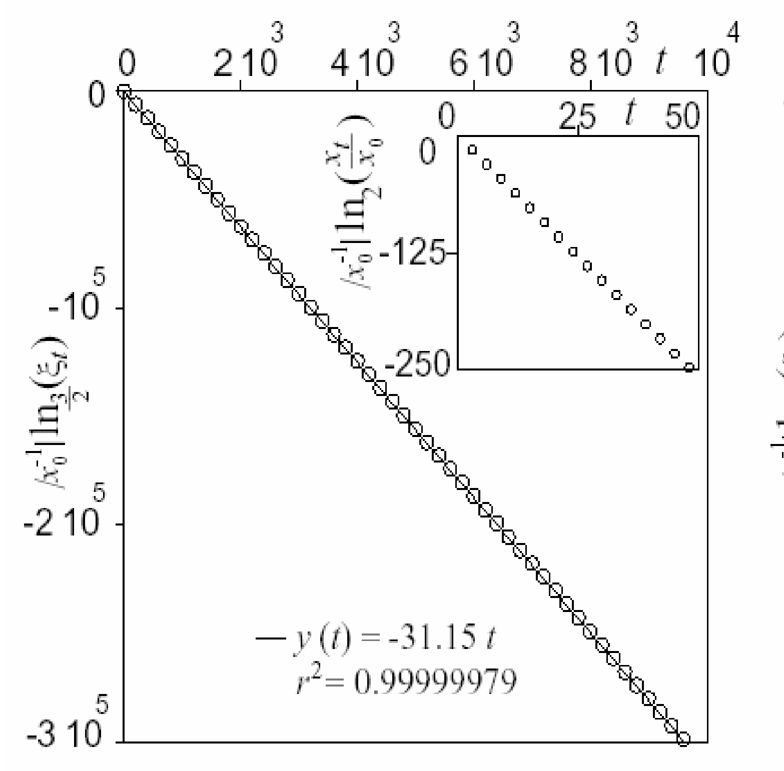
Pitchfork bifurcations

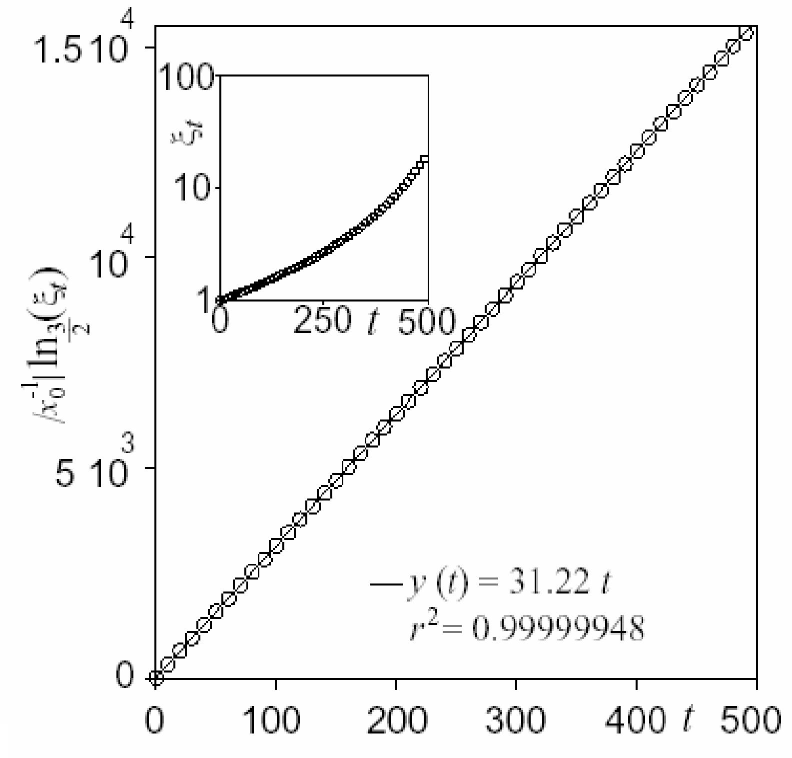


$$\lambda_q(x_0) = zax_0^{z-1}, \quad zat = x_0^{1-z} \ln_q \xi_t$$

Baldovin & Robledo (2002)

Tangent bifurcations



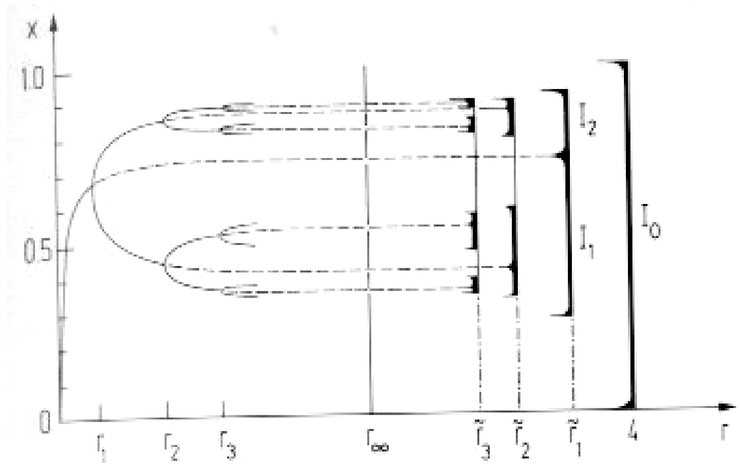


$$\lambda_q(x_0) = zax_0^{z-1}, \quad zat = x_0^{1-z}\ln_q \xi_t$$

Baldovin & Robledo (2002)

Crossover from *q* to BG statistics

Crossover from *q* to BG statistics at the onset of chaos: Shift into band chaos or noise perturbation



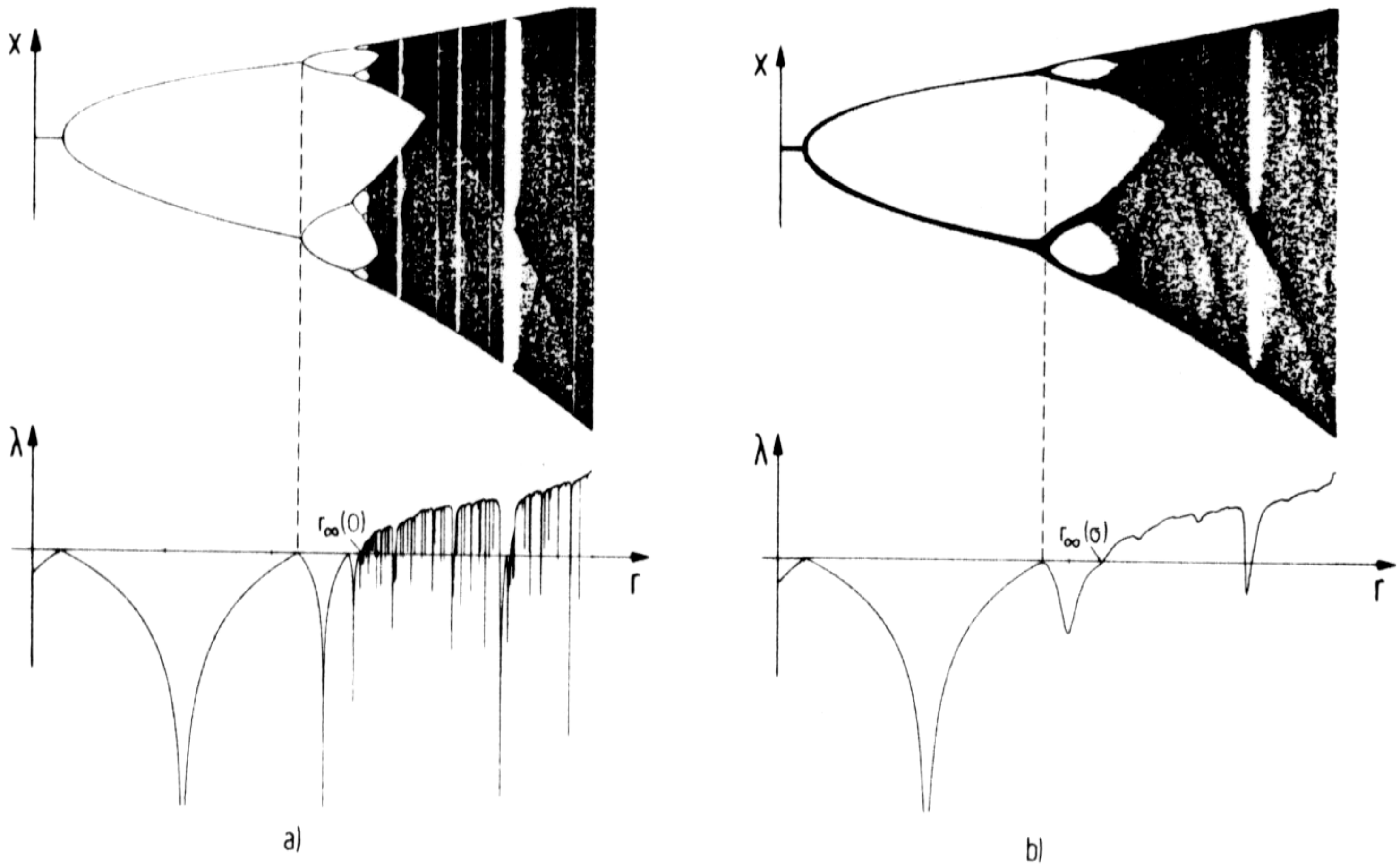
For some $\mu > \mu_\infty$ attractor consists of 2^N bands

$$\lambda_1 \propto 2^{-N} \approx \Delta\mu^{\kappa}, \quad \kappa = \ln 2 / \ln \delta, \quad \Delta\mu \equiv \mu - \mu_\infty$$

$t < 2^N$ orbital expansion rate grows as $\ln t$ and $\xi_t = \exp_q(\lambda_q t)$

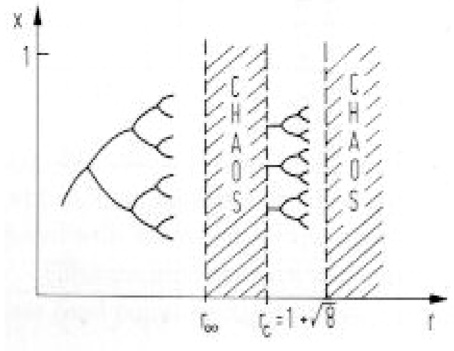
$t > 2^N$ orbital expansion rate grows as t and $\xi_t = \exp(\lambda_1 t)$

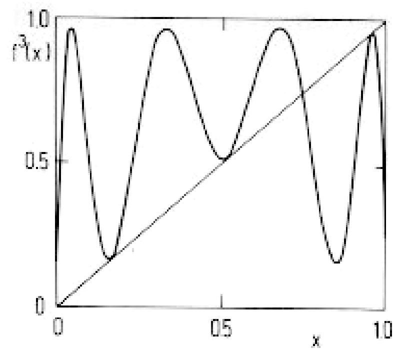
$$x_{t+1} = f_{\mu}(x_t) + \xi_t \, \sigma$$

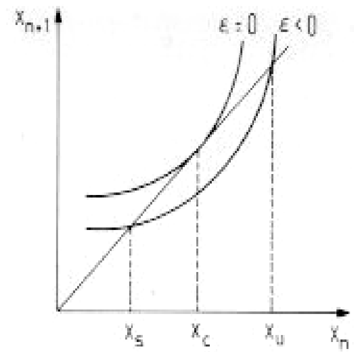


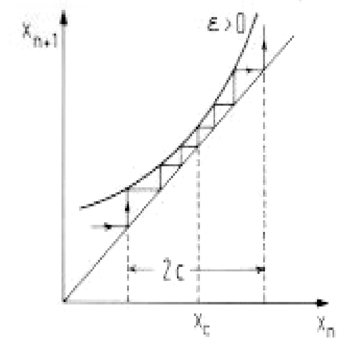
$\sigma = 0$ $\sigma \neq 0$

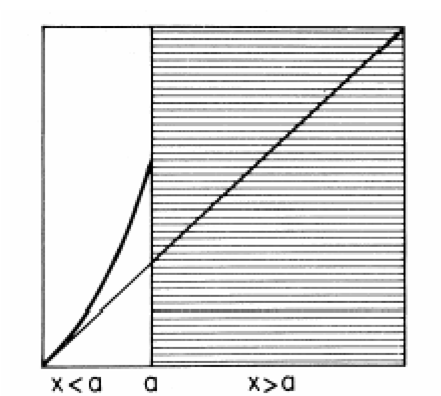
Crossover from *q* to BG statistics at the tangent bifurcations: Shift from tangency, feedback mechanism, or noise

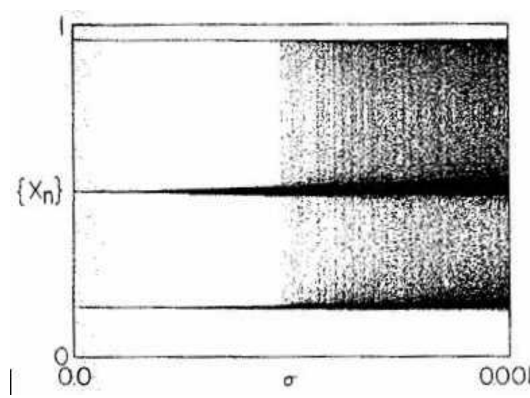


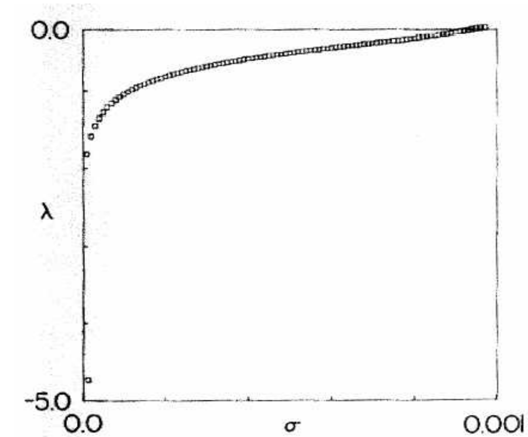












Concluding remarks

- Usefulness of *q*-statistics at the transitions to chaos

q-statistics and the transitions to chaos

- The fluctuating dynamics on a critical multifractal attractor has been determined exactly (e.g. via the universal function σ)

- *A posteriori*, comparison has been made with Mori's and Tsallis' formalisms

It was found that:

- The entire dynamics consists of a family of *q*-phase transitions

- The structure of the sensitivity is a two-time *q*-exponential

- Tsallis' *q is* the value that Mori's field **q** takes at a *q*-phase transition

- There is a discrete set of *q* values determined by universal constants

- The entropy S_q grows linearly with time when **q** = *q*

Where to find our statements and results explained

- “Critical attractors and *q*-statistics”,
  A. Robledo, Europhys. News, 36, 214 (2005)

- “Incidence of nonextensive thermodynamics in temporal scaling at Feigenbaum points”,
  A. Robledo, Physica A 370, 449 (2006)

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"The renormalization group and optimization of non-extensive entropy: criticality in non-linear one-dimensional maps"
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"Sensitivity to initial conditions at bifurcations in one-dimensional non-linear maps: rigorous non-extensive solutions"
Europhysics Letters 60, 518 (2002).

- Baldovin, F., Robledo, A.,
"RG universal dynamics at the onset of chaos in logistic maps and non-extensive statistical mechanics"
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"Criticality in non-linear one-dimensional maps: RG universal map and non-extensive entropy"
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"Multifractality and nonextensivity at the edge of chaos of unimodal maps"
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“Universal glassy dynamics at noise-perturbed onset of chaos. A route to ergodicity breakdown”,
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“The noise-perturbed onset of chaos in logistic maps and the dynamics of glass formation”
Physical Review E 72, 066213 (2005).

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Physica A 370, 286 (2006).

Fluctuating dynamics at the quasiperiodic onset of chaos

Alberto Robledo

Lecture Course
Nonextensive Statistical Mechanics
2-6 April 2006, CBPF, Rio de Janeiro, Brazil

Subject:

- Fluctuating dynamics at the onset of chaos (all routes)

Questions addressed:

- How much was known, say, ten years ago?
- Which are the relevant recent advances?
  Is the dynamics *fully* understood now?
- What is *q*-statistics for critical attractors?
- Is there rigorous, sensible, proof of incidence of
  q-statistics at the transitions to chaos?
- What is the relationship between *q*-statistics and the
  thermodynamic formalism?
- What is the usefulness of *q*-statistics for this problem?

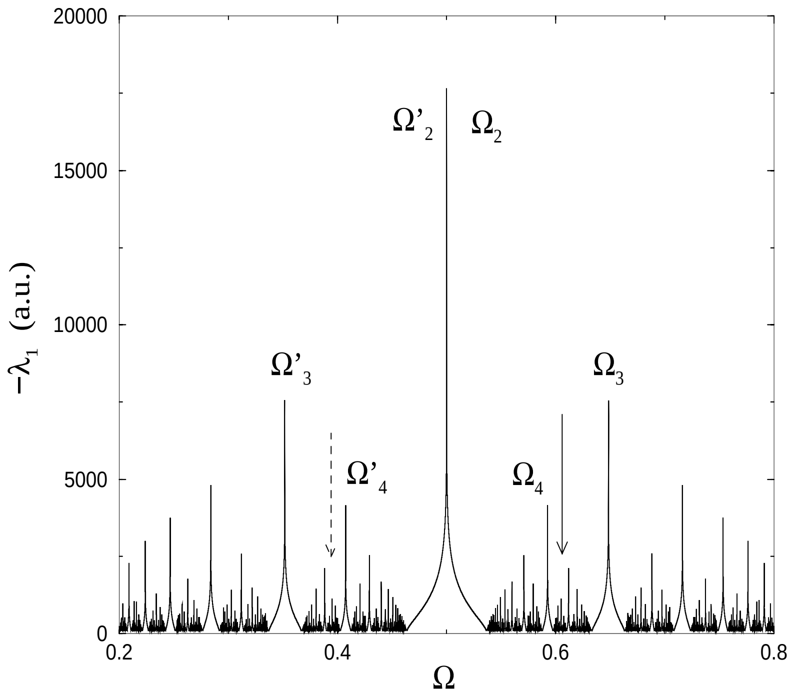
Brief answers are given in the following slides

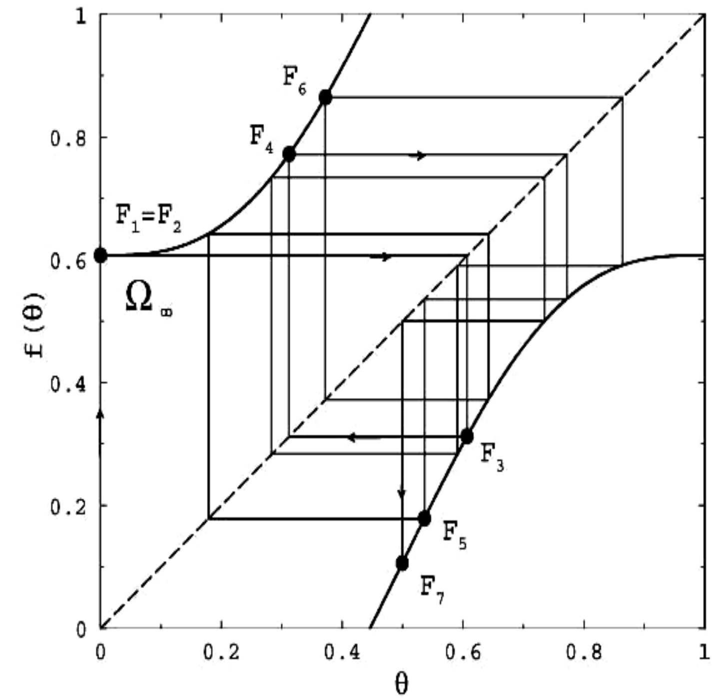
Ten years ago

- Numerical evidence of fluctuating dynamics (Grassberger & Scheunert)
- Adaptation of ‘thermodymamic’ formalism to onset of chaos (Anania & Politi, Mori *et al*)
- But… implied anomalous statistics left largely unexplored

Critical circle map

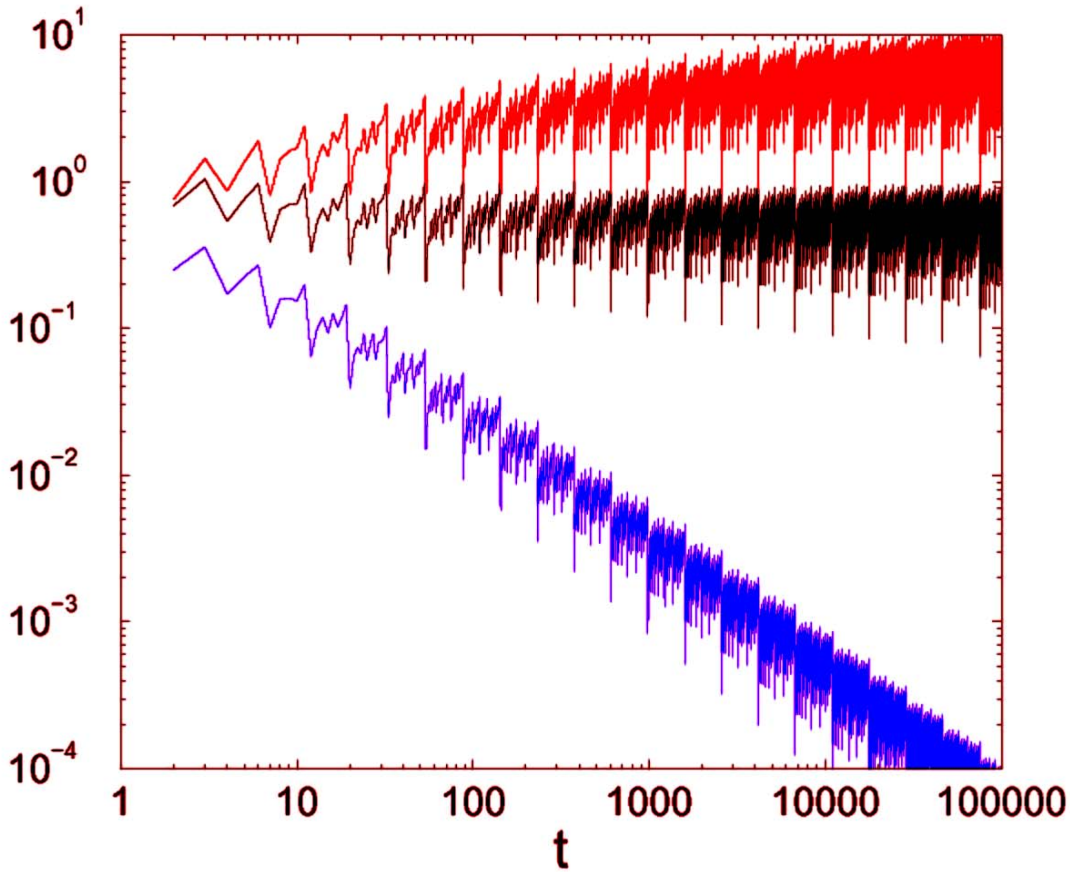
$$\theta_{t+1} = \theta_t + \Omega - \frac{1}{2\pi}\sin(2\pi\theta_t), \quad \text{mod}1$$





Dynamics at the 'golden-mean' quasiperiodic attractor

Red: $\sum_{i=1}^{t} \ln\left|\frac{dx_i}{dx_{i-1}}\right|$ Black: $\frac{1}{\ln t}\sum_{i=1}^{t} \ln\left|\frac{dx_i}{dx_{i-1}}\right|$ Blue: $\frac{1}{t}\sum_{i=1}^{t} \ln\left|\frac{dx_i}{dx_{i-1}}\right|$



Thermodynamic approach for attractor dynamics
(Mori and colleagues ~1989)

- Density of Lyapunov coefficients

$$\lambda(t, x_0) = \frac{1}{\ln t}\sum_{i=0}^{t-1} \ln\left|\frac{df(x_i)}{dx_i}\right|, \quad t >> 1, \qquad P(\lambda;t) = t^{-\psi(\lambda)}P(0;t)$$

- Partition function

$$Z(t,\mathrm{q}) \equiv \int d\lambda\, P(\lambda;t)[W(\lambda,t)]^{1-\mathrm{q}}, \quad W = t^{\lambda}$$

- Free energies

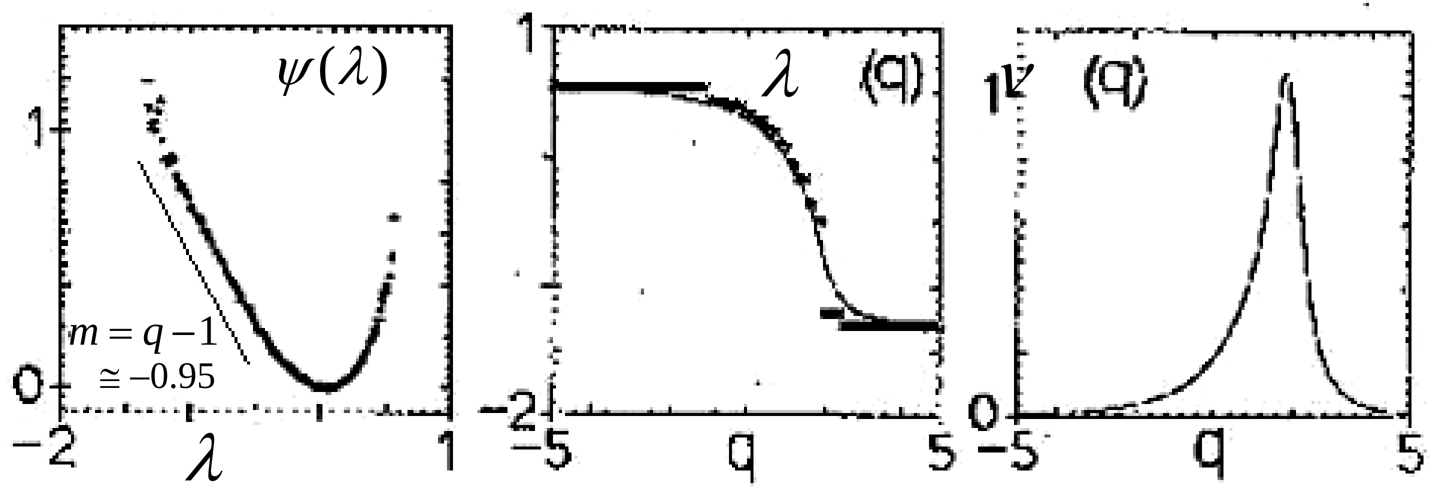
$$\phi(\mathrm{q}) \equiv \lim_{t\to\infty}\frac{\ln Z(t,\mathrm{q})}{\ln t}, \qquad \psi(\lambda) = \phi(\mathrm{q}) - \lambda(\mathrm{q}-1)$$

- Equation of state and susceptibility

$$\lambda(\mathrm{q}) = \frac{d\phi(\mathrm{q})}{d\mathrm{q}}, \quad \upsilon(\mathrm{q}) = \frac{d\lambda(\mathrm{q})}{d\mathrm{q}}$$

q ~ ‘magnetic field’ λ ~ ‘magnetization’

Mori's *q*-phase transition at the quasiperiodic onset of chaos



$$P(\lambda;t) = t^{-\psi(\lambda)} P(0;t),$$

Static spectrum

$$\tau(\mathrm{q}) = f(\alpha) + \alpha \mathrm{q}\,,$$
$$\tau(\mathrm{q}) = (\mathrm{q}-1) D_{\mathrm{q}}$$

Dynamic spectrum

$$\phi(\mathrm{q}) = \psi(\lambda) + \lambda(\mathrm{q}-1)$$

- Is theTsallis index *q* the value of q at the Mori transition?

Today

- Rigorous, exact, analytical results for the three routes to chaos (e.g. sensitivity to initial conditions)

- Hierarchy of dynamical *q*-phase transitions

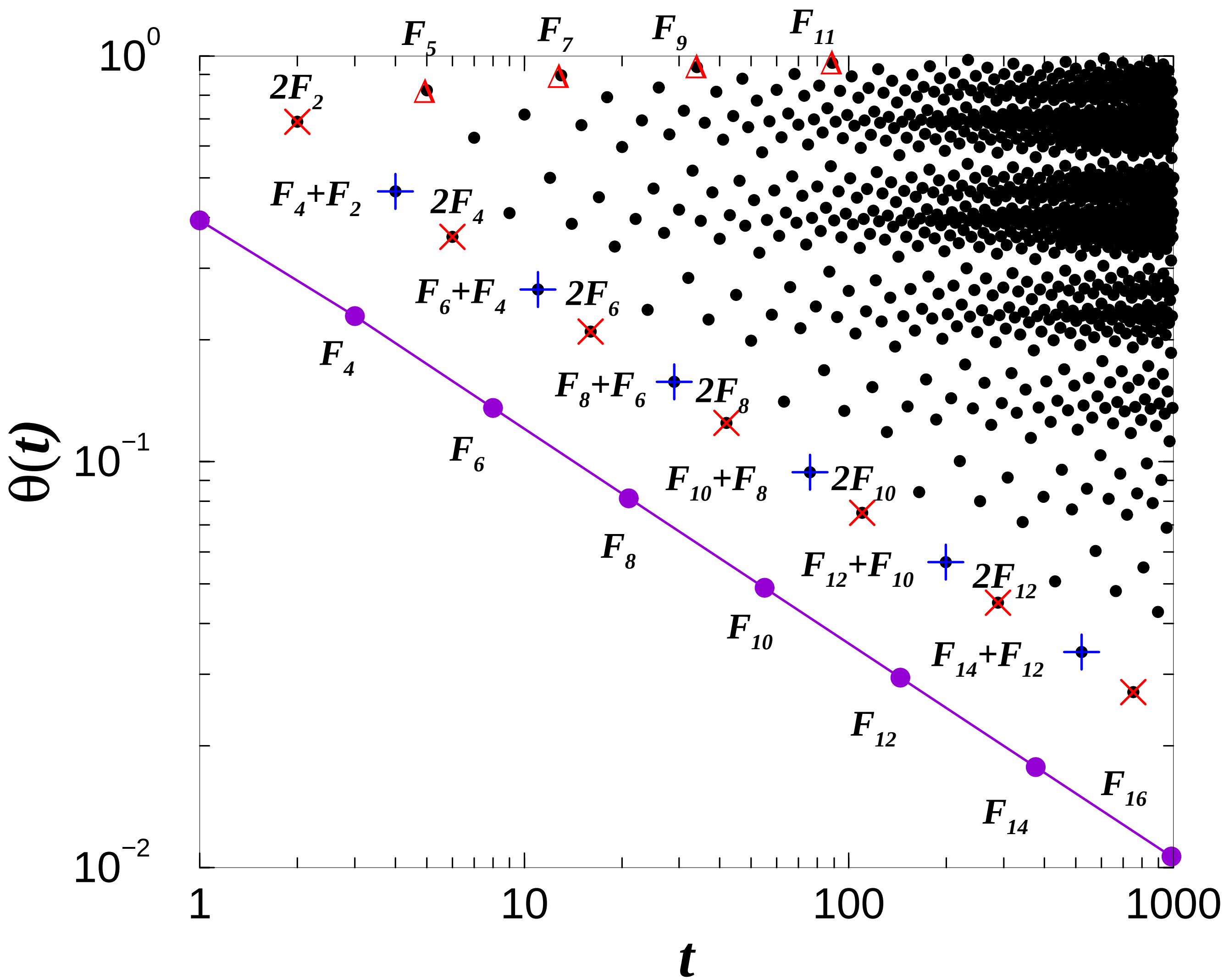
- Link between thermodymamics and *q*-statistics

- Temporal extensivity of *q*-entropy

- *q* values determined from theoretical arguments

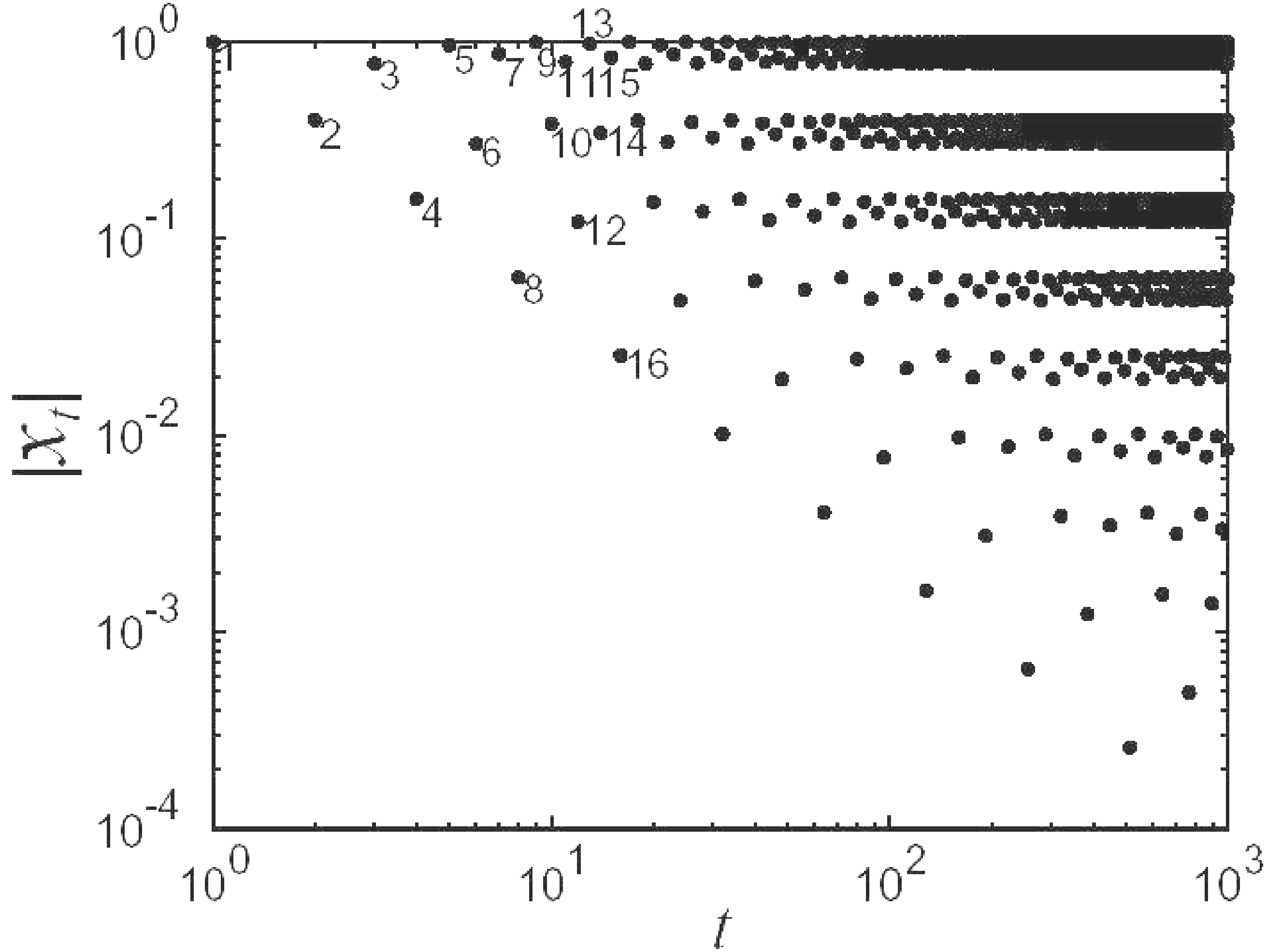
Onset of chaos in the critical circle map (golden-mean entry point)

Trajectory <u>on</u> the ‘golden-mean’ quasiperiodic attractor



$$\theta_{t+1} = \theta_t + \Omega_\infty - \frac{1}{2\pi}\sin(2\pi\theta_t), \quad \bmod 1$$

Trajectory on the Feigenbaum attractor



q-statistics for critical attractors

Sensitivity to initial conditions

$$\xi(x_0, t) \equiv \lim_{\Delta x_0 \to 0} \frac{\Delta x_t}{\Delta x_0}$$

- **BG statistics:**

$$\xi(x_0, t) = \exp[\lambda_1(x_0)\, t]$$

(independent of x_0 for $t \to \infty$)

- ***q* statistics:**

$$\xi(x_0, t_k) = \exp_q[\lambda_q(x_0)\, t_k], k = 0,1,\ldots$$

(dependent on x_0 for all t)

- ***q*-exponential function:**

$$\exp_q(x) \equiv [1 + (1-q)\, x]^{\frac{1}{1-q}}$$

- **Basic properties:**

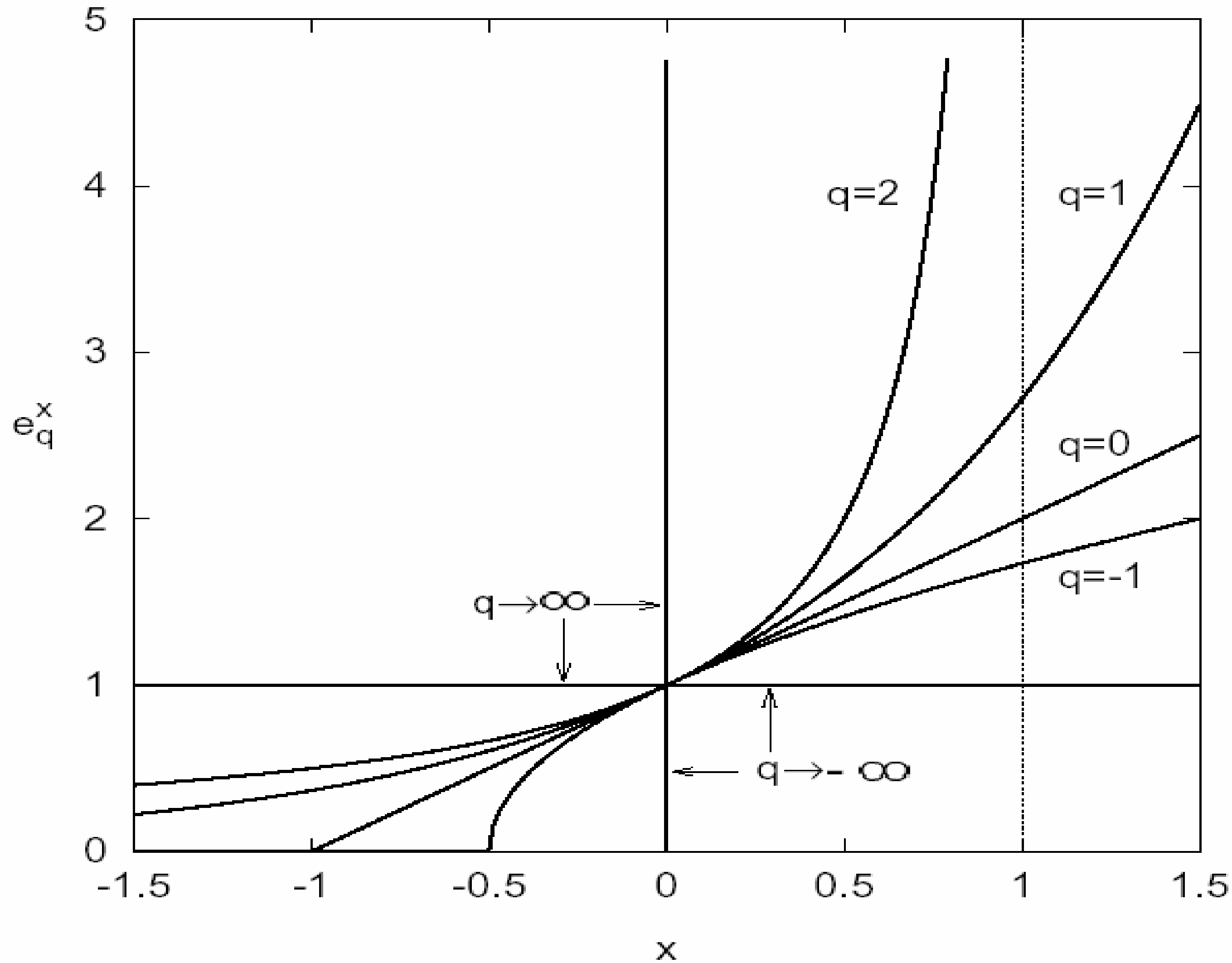
$$\exp(x) = \lim_{q \to 1} \exp_q(x)$$

$$\exp_q(-x) = [\exp_Q(x)]^{-1}, \quad Q = 2 - q$$

$$d \exp_q(x) / dx = [\exp_q(x)]^q$$

$$x_{t+1} = f_\mu(x_t), \quad a \le x_t \le b, \quad A \le \mu \le B$$

q-exponential function



Entropic expression for Lyapunov coefficient

$$\lambda_q \equiv \frac{1}{t_k}\,[\,S_q(t_k) - S_q(0)\,]$$

- **BG statistics:**

$$S_1 = -\sum_i p_i \ln p_i$$

- ***q* statistics:**

$$S_q = -\sum_i p_i^q \ln_q p_i \quad \text{or} \quad -\sum_i p_i \ln_Q p_i$$

- ***q*-logarithmic function:** $\ln_q(y) \equiv \dfrac{y^{1-q} - 1}{1-q} \quad (y \in R^+; q \in R)$

- **Basic properties:**

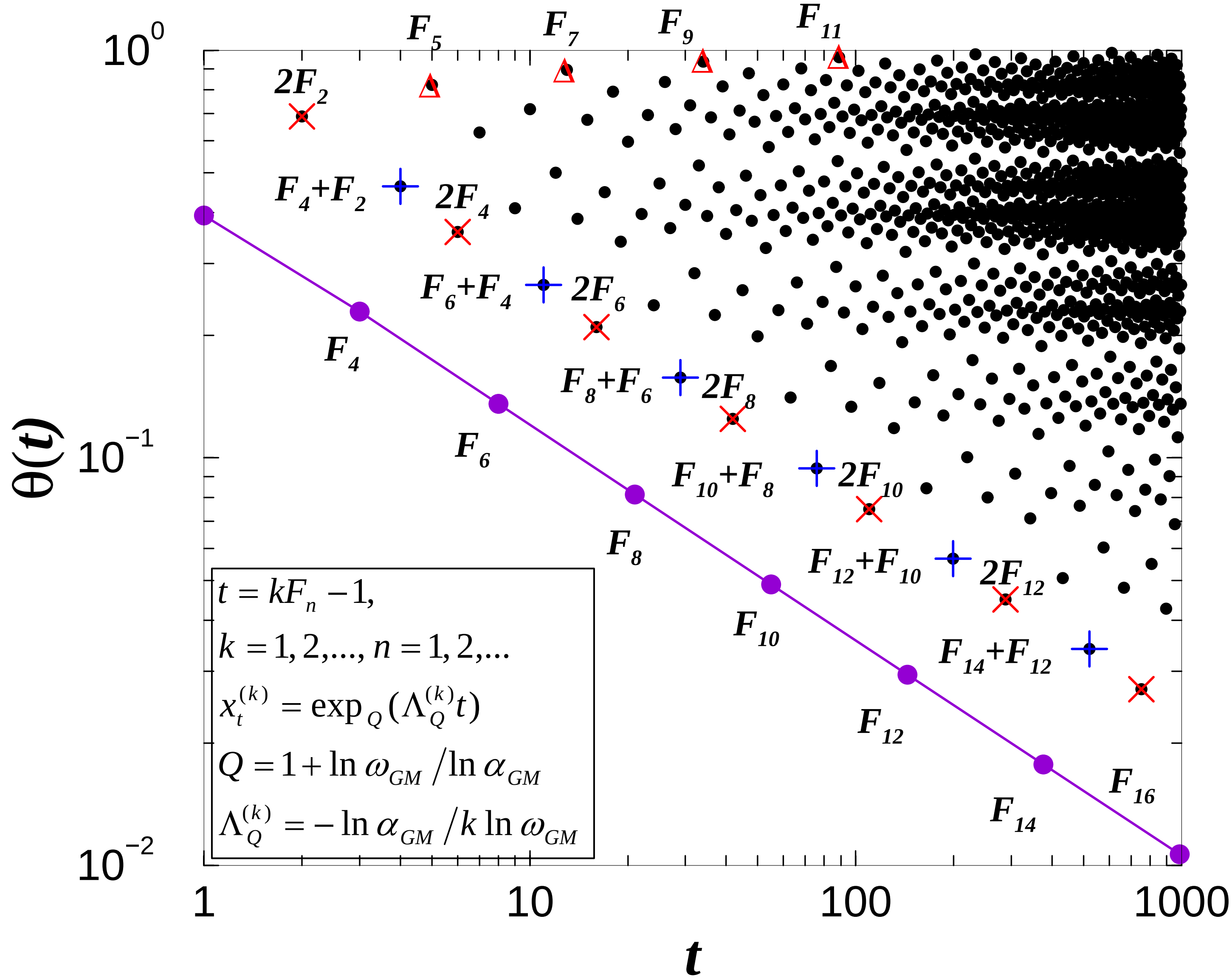
$$\ln(y) = \lim_{q \to 1} \ln_q(y)$$

$$\ln_q(y) = -\ln_Q(1/y), \quad Q = 2-q$$

$$\ln_q(\exp_q(x)) = \exp_q(\ln_q(x)) = x$$

Analytical results for the sensitivity

F_∞ - supercycle



Power laws, *q*-exponentials and two-time scaling

$$\xi_t(\theta_0) = \alpha_{gm}^{2k}, \quad \theta_0 = 0,$$

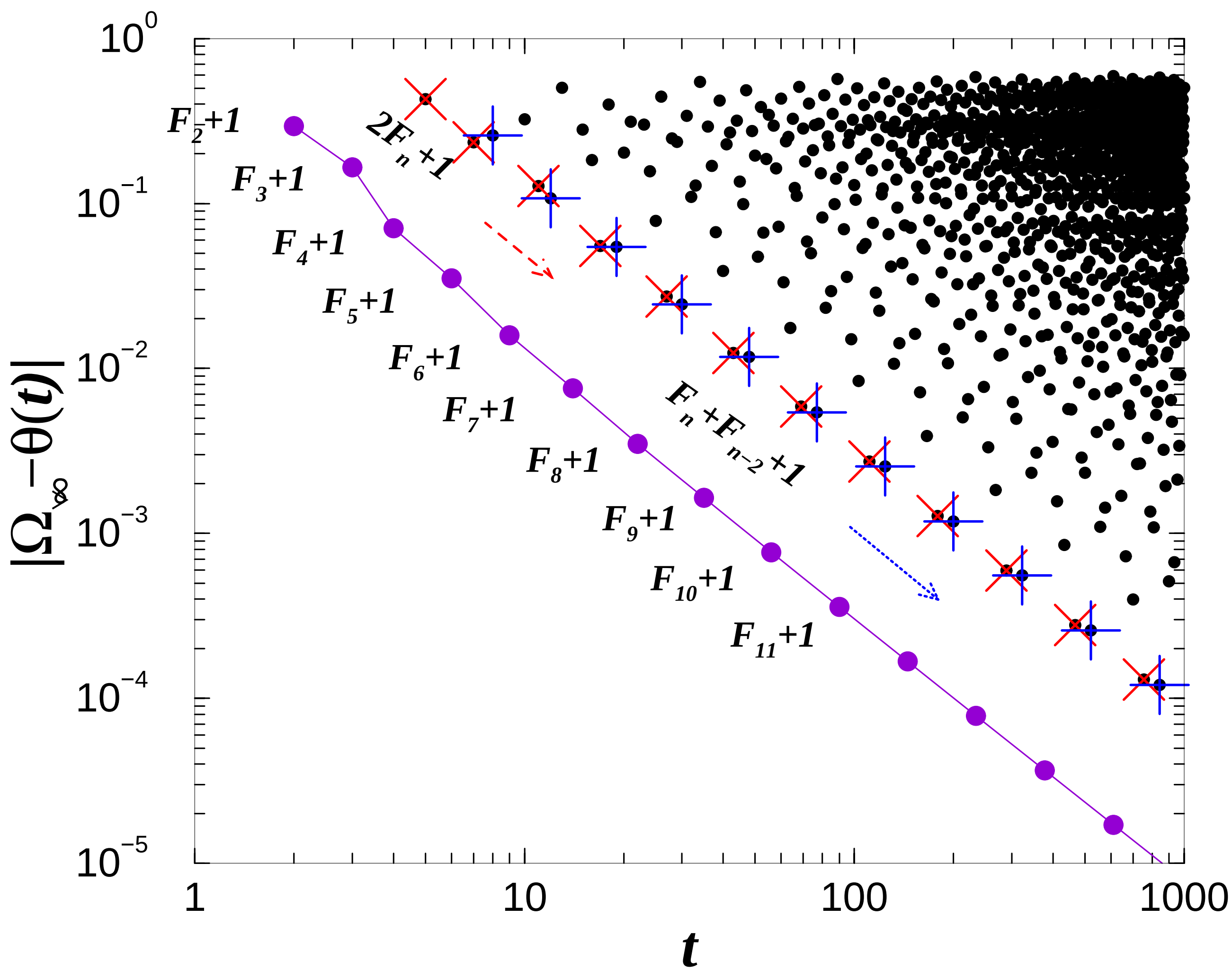
$$t = (l-m)F_k + mF_{k-2} - 1, \quad k = 1,2,\ldots, \; l = 1,2,\ldots, \; m = 0,1,\ldots$$

$$\alpha_{gm}^{k} \cong \left(1 + \frac{t}{l-m+mw_{gm}^2}\right)^{\frac{2\ln\alpha_{gm}}{\ln w_{gm}}}, \quad w_{gm}^2 = \lim_{k\to\infty}\frac{F_{k-2}}{F_k}, \quad \Rightarrow$$

$$\xi_t(\theta_0) = \exp_q\left[\lambda_q^{(l,m)} t\right], \quad q = 1 + \frac{\ln w_{gm}}{2\ln\alpha_{gm}}, \quad \lambda_q^{(l,m)} = \frac{\ln\alpha_{gm}}{(l-m+mw_{gm}^2)\ln w_{gm}},$$

$$\xi_t(\theta_0) = \exp_q\left[\lambda_q^{(0,0)} t / t_w\right], \quad t = (l-m)F_k + mF_{k-2} - 1, \quad t_w = l - m + mw_{gm}^2.$$

Another view of the F_∞- supercycle



Sensitivity to initial conditions within the golden-mean quasiperiodic attractor

- Starting at the most crowded ($\theta = \Omega_\infty$) and finishing at the most sparse (θ=0) region of the attractor

$$\xi_t(\theta_0) = \exp_q\left[\lambda_q^{(l,m)} t\right], \quad q = 1 + \frac{\ln w_{gm}}{2\ln \alpha_{gm}}, \quad \lambda_q^{(l,m)} = \frac{2\ln \alpha_{gm}}{(l - m + m w_{gm}^2)\ln w_{gm}}$$

$$t = (l-m)F_k + mF_{k-2} - 1, \quad k = 1,2,\ldots, l = 1,2,\ldots, m = 0,1,\ldots$$

- Starting at the most sparse (θ=0) and finishing at the most crowded ($\theta = \Omega_\infty$) region of the attractor

$$\xi_t(\theta_0) = \exp_{2-q}\left[\lambda_{2-q}^{(l,m)} t\right], \quad 2 - q = 1 + \frac{\ln w_{gm}}{2\ln \alpha_{gm}}, \quad \lambda_q^{(l,m)} = \frac{2\ln \alpha_{gm}}{(l - m + m w_{gm}^2) w_{gm} \ln w_{gm}}$$

$$t = (l-m)F_k + mF_{k-2} + 1, \quad k = 1,2,\ldots, l = 1,2,\ldots, m = 0,1,\ldots$$

Thermodynamic approach and *q*-statistics

Mori's definition for Lyapunov coefficient at onset of chaos

$$\lambda(t,\theta_0) \equiv \frac{1}{\ln t} \ln\left|\frac{dg^{(t)}(\theta_0)}{d\theta_0}\right|,$$

is equivalent to that of same quantity in *q*-statistics

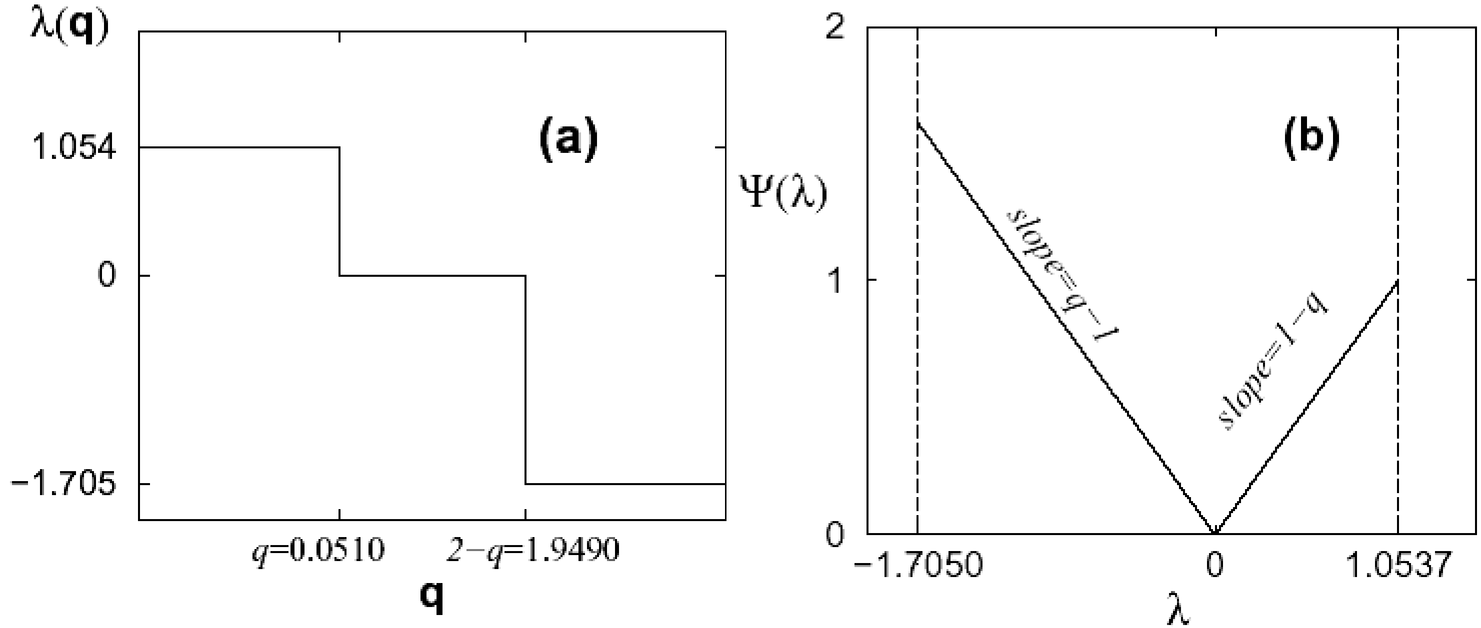
$$\lambda(t,\theta_0 = 0) = \frac{1}{t} \ln_q \left|\frac{dg^{(t)}(\theta_0)}{d\theta_0}\right|_{\theta_0 = 0} = \frac{2\ln\alpha_{gm}}{(l - m + m w_{gm}^2)\ln w_{gm}} = \lambda_q^{(l,m)},$$

$$t = (l-m)F_k + mF_{k-2} + 1, \quad k = 1, 2, \ldots, l = 1, 2, \ldots, m = 0, 1, \ldots$$

Two-scale Mori's λ(q) and ψ(λ) for golden-mean threshold

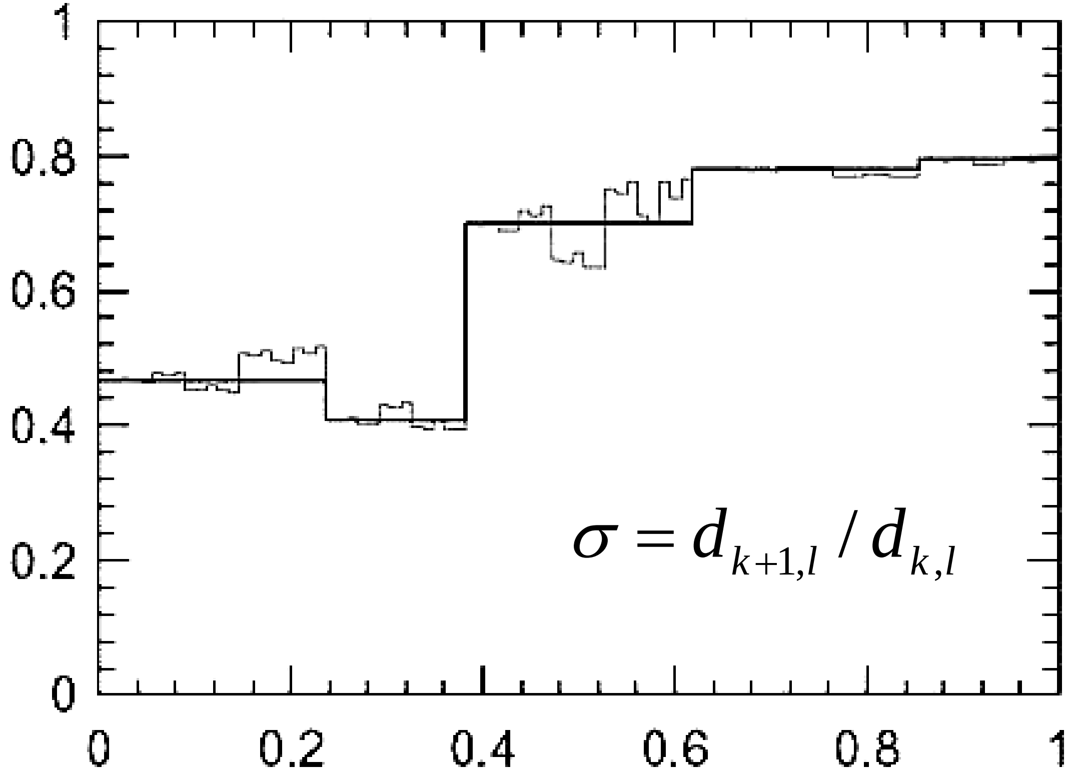
Dynamic spectrum

$$P(\lambda;t) = t^{-\psi(\lambda)} P(0;t),$$



$$q = 1 - \frac{\ln w_{gm}}{2\ln \alpha_{gm}} \cong 0.0510 \qquad \lambda_q = \frac{2\ln \alpha_{gm}}{\ln w_{gm}} \cong 1.0537 \qquad m = 1 - q = \frac{\ln w_{gm}}{\ln \alpha_{gm}} \cong 0.949$$

Trajectory scaling function σ(*y*) for golden mean threshold



$\sigma(y)$

$\sigma = d_{k+1,l} / d_{k,l}$

$y = l / F_k$

Hierarchical family of *q*-phase transitions

Trajectory scaling function σ(y) → sensitivity ξ(t)

- $$\sigma_k(l) = \frac{d_{k+1,l}}{d_{k,l}}, \quad d_{k,l} = \left|\theta_l - \theta_{l+F_{k-1}}\right|$$
- $$\sigma(y) = \lim_{k\to\infty} \sigma_k(l), \quad y = \frac{l}{F_k}$$
- $$\xi_t(l) = \left|\frac{d_{k,l+t}}{d_{k,l}}\right| \cong \left|\frac{\sigma_k(l-1)}{\sigma_k(l)}\right|^{\pm k}, \quad t = F_k \pm 1$$

Spectrum of *q*-Lyapunov coefficients with common index *q*

- Successive approximations to σ(*y*),

$$\sigma(y) = \alpha_j^{-1}, \quad a_j \le y < a_{j+1}, \quad j = 0, 1, \ldots, J, \quad J = 1, 2, \ldots,$$

lead to:

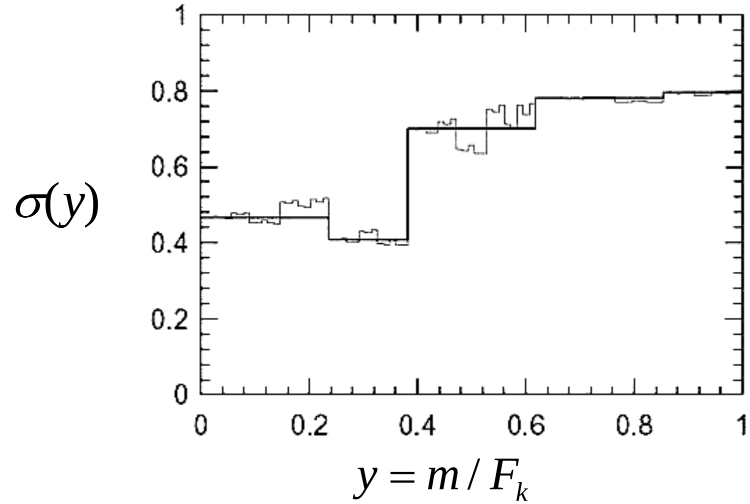
$$\xi_t(x_0) = \left| \frac{\alpha_j}{\alpha_{j+1}} \right|^{\pm k} = \exp_q \left[\lambda_q^{(l,m)} t \right], \quad \text{where}$$

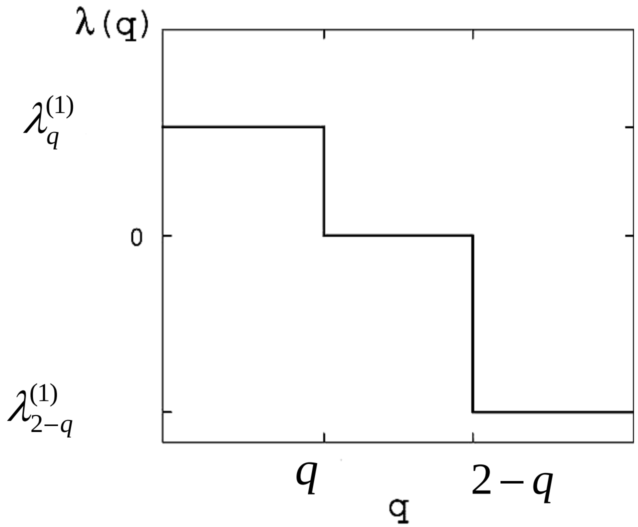
$$q = 1 + \frac{\ln w_{gm}}{\ln(\alpha_j / \alpha_{j+1})}, \quad \lambda_q^{(l,m)} = \frac{\ln(\alpha_j / \alpha_{j+1})}{(l - m + m w_{gm}^2) \ln w_{gm}};$$

and similarly with *Q=2-q*

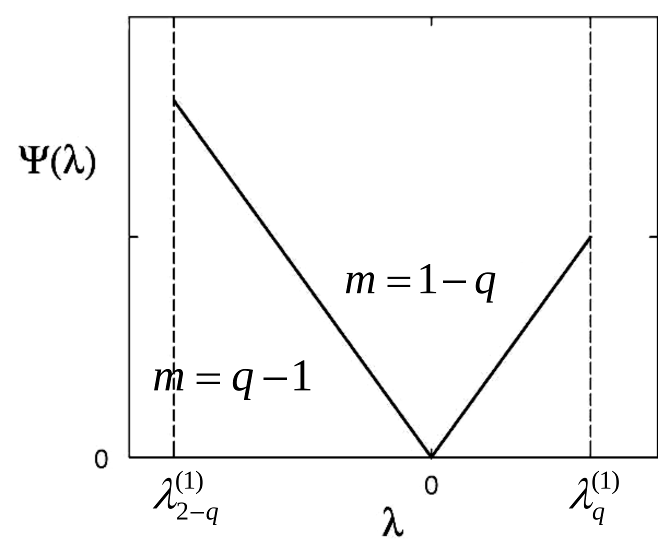
Infinite family of *q*-phase transitions

- Each discontinuity in σ(y) leads to a couple of *q*-phase transitions





$$\lambda(\mathrm{q}) = \frac{d\phi(\mathrm{q})}{d\mathrm{q}}$$



$$\psi(\lambda) = \phi(\mathrm{q}) - \lambda(\mathrm{q} - 1)$$

Temporal extensivity of the *q*-entropy

Precise knowledge of dynamics implies that

$$P(\lambda,t)W(\lambda,t)=\delta(\lambda-\lambda_q^{(l,m)})\,\exp_q(\lambda_q^{(l,m)}t)$$

therefore

$$Z(t,\mathrm{q})\equiv W(\lambda_q^{(l,m)},t)^{1-\mathrm{q}}=\left[1+(1-q)\lambda_q^{(l,m)}t\right]^{(1-\mathrm{q})/(1-q)}$$

and

$$Z(t,\mathrm{q})\equiv\sum_{i=1}^{W}\left[p_i(t)\right]^{\mathrm{q}}=1+(1-\mathrm{q})S_{\mathrm{q}}(t),$$

with

$$p_i(t)=\left[W(t)\right]^{-1}\quad\text{for all } i\quad\text{and}\quad S_{\mathrm{q}}=\ln_{\mathrm{q}}W.$$

When q=*q*

$$S_q(t)=\lambda_q^{(l,m)}t$$

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  A. Robledo, Europhys. News, 36, 214 (2005)

- "Incidence of nonextensive thermodynamics in temporal scaling at Feigenbaum points",
  A. Robledo, Physica A 370, 449 (2006)

- "Fluctuating dynamics at the quasiperiodic onset of chaos, Tsallis *q*-statistics and Mori's *q*-phase transitions", H. Hernández Saldaña, A. Robledo, Physica A 370, 286 (2006)

Concluding remarks

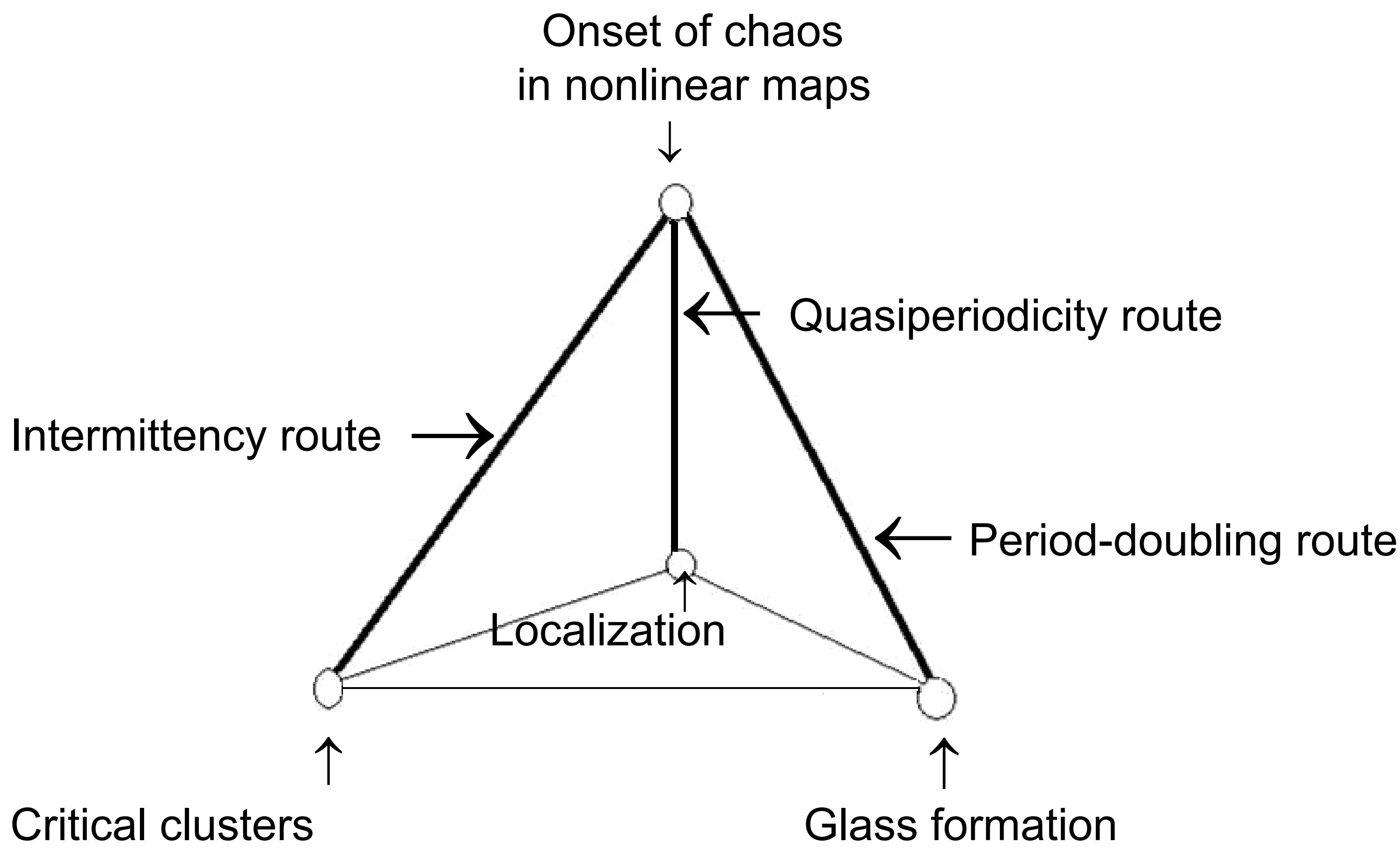
- Usefulness of *q*-statistics at the transitions to chaos

q-statistics and the transitions to chaos

- The fluctuating dynamics on a critical multifractal attractor has been determined exactly (e.g. via the universal function σ)
- *A posteriori*, comparison has been made with Mori's and Tsallis' formalisms

It was found that:

- The entire dynamics consists of a family of *q*-phase transitions
- The structure of the sensitivity is a two-time *q*-exponential
- Tsallis' *q is* the value that Mori's field q takes at a *q*-phase transition
- There is a discrete set of *q* values determined by universal constants
- The entropy S_q grows linearly with time when q= *q*


Onset of chaos
in nonlinear maps
Quasiperiodicity route
Intermittency route
Period-doubling route
Localization
Critical clusters
Glass formation

Transitions to chaos: the realm of *q*-statistics

Part II.

q-statistics in critical fluctuations, glass formation and localization

- Large critical fluctuations
- Multifractal geometry of critical clusters
- Intermittency and equivalent map
- *q*-statistics at criticality
- Noise-perturbed onset of chaos
- Two-step glassy dynamics
- Aging at the transition to chaos
- Localization transition and quasi-periodic states

Incipient chaos in *d* =1 nonlinear maps

Route to chaos	Intermittency	Period doubling	Quasiperiodicity
Common properties	Vanishing ordinary Lyapunov coefficient, dynamical phase transitions (Mori's *q*-phases) power-law dynamics, *q*-sensitivity, *q*-Pesin identity		
Distinctive properties	(Also) faster than exponen-tial dynamics	Foam-like phase space	Dense phase space
Applications in condensed matter physics	Critical clusters	Glass formation	Localization
Applications in other disciplines	Information & other flows in networks, …	Protein folding, vegetation patterns, …	Mode locking, cardiac cells, Internet TCP, ...

q-statistics in large critical clusters

Outline:

- **Introduction**

  - Why are critical clusters interesting?
  - Are the properties of critical clusters orthodox?

- **Dynamical properties**

  - Cluster instability, intermittency & *q*-statistics
  - 1/f and other noise in clusters

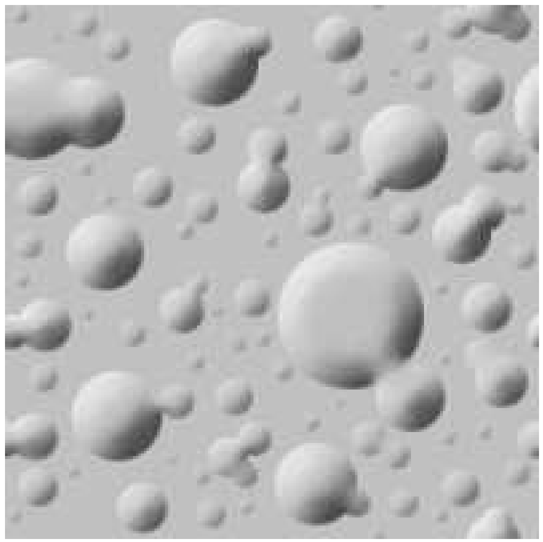
- **Static properties**

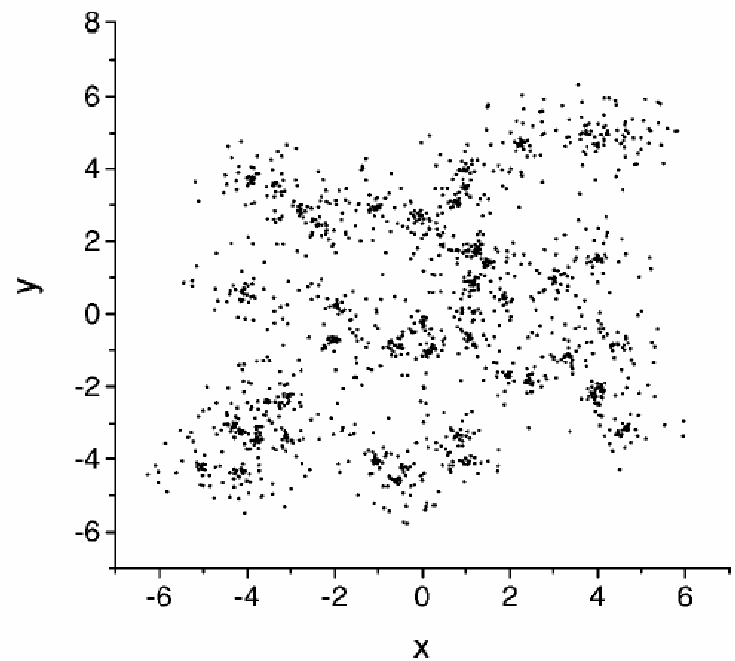
  - Cluster properties from dominant configurations
  - Extensivity of cluster entropy & *q*-statistics

- **Conclusions**

  - What are the theoretical implications of the anomalous behavior?

Critical fluctuations
(temporary clusters of order parameter ϕ)

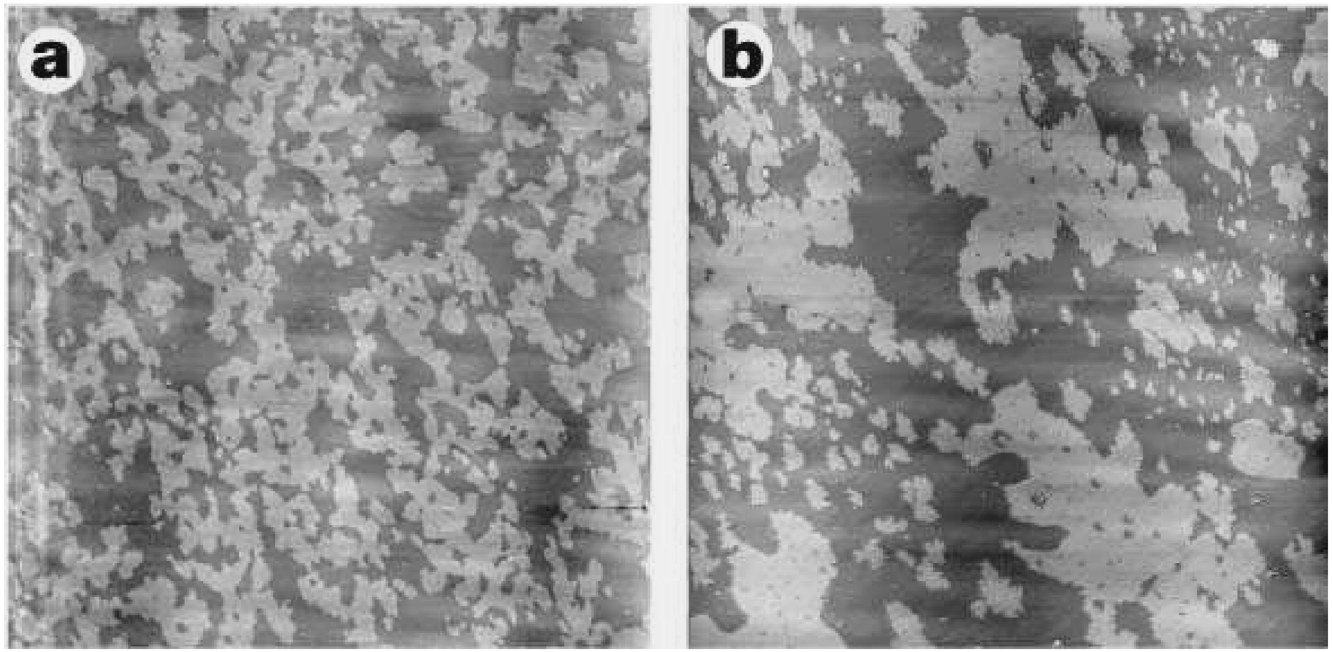




Critical isotherm exponent δ

Critical phenomena: Fluctuations caught in the act

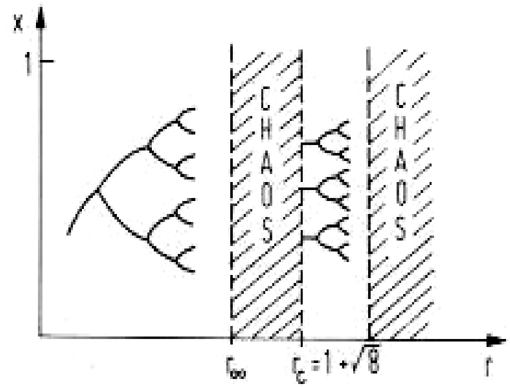
Nielsen, Bjørnholm & Mouritsen, Nature 404, 352 (23 March 2000)

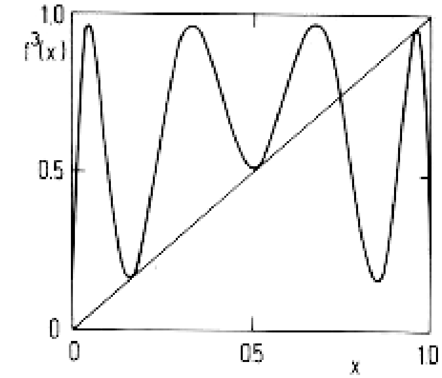


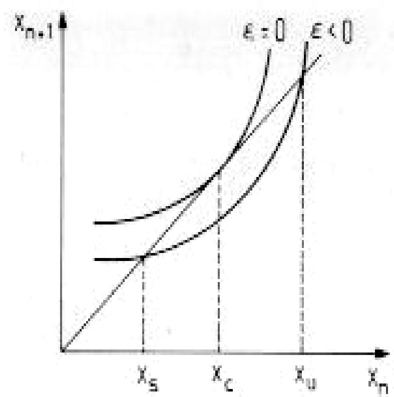
a,b, Images of dimyristoyl phosphatidylcholine (25x25 μm^2) and dipalmitoyl phosphatidylcholine (20x20 μm^2) monolayers at their respective critical points. The monolayers have been transferred from an air–water interface to solid mica supports. The patterns correspond to lipid domains of one phase immersed into the other. The height difference between the light and dark areas is about 5 Å.

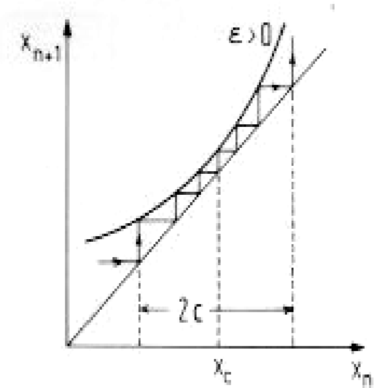
$$D_f = \frac{d\delta}{\delta + 1}, \; d = 1,2,3,\ldots \qquad d_f = d - \frac{2}{\delta - 1}, \; d = 1,2,\ldots$$

Dynamical properties









$$\phi_{t+1} = \varepsilon + \phi_t + \upsilon \phi_t^{\,z}, \quad z > 1$$

Intermittency of critical clusters

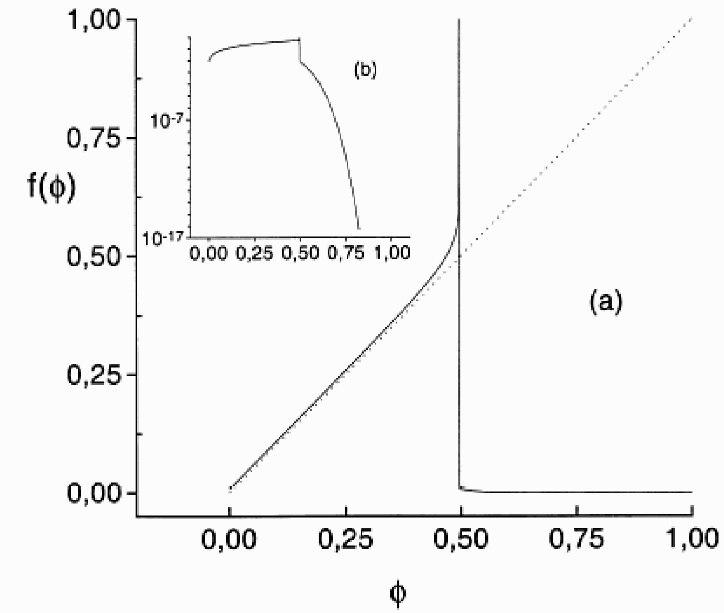


Fig. 1. (a) The critical map for $g_1 = 20$, $g_2 = 0.01$, $R = 5$ and $\delta = 5$. The value of g_1 fixes the maximum of the map at $\phi \approx 0.5$. (b) The inserted figure represents the same map in logy-scale.

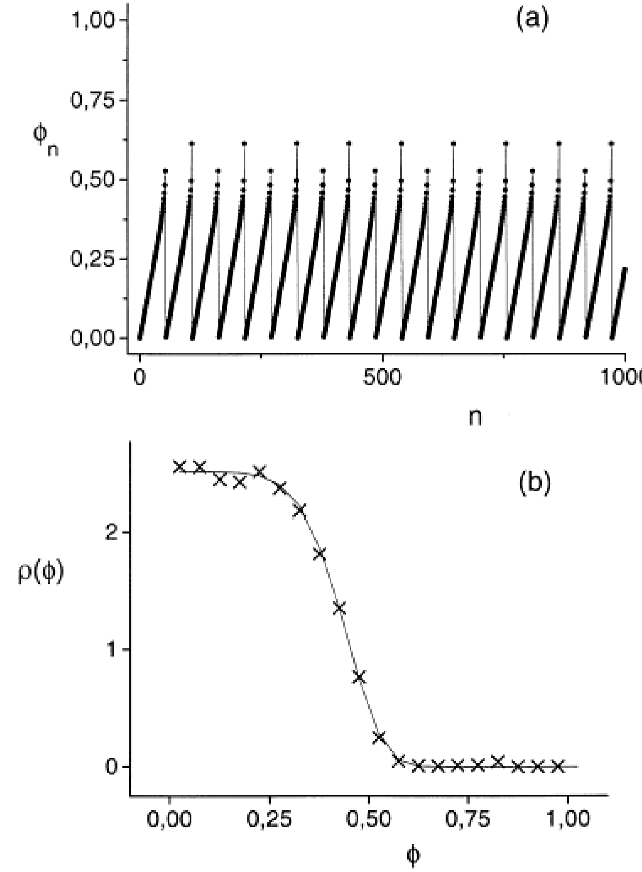


Fig. 2. (a) A typical trajectory of the map presented in Fig. 1. (b) The invariant density resulting from the chaotic trajectories of the map of Fig. 1 (crosses). In the same figure we plot the normalized statistical weight (solid line) characterizing the configurations of the critical system according to Eq. (11).

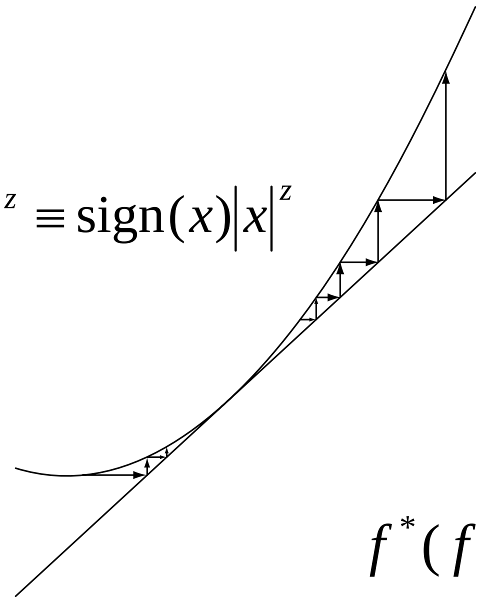
$$\rho(\overline{\phi}) = \exp(-\Psi_c)/Z, \quad \overline{\phi} = R^{-1}\int_R \phi(x)dx$$

$$\overline{\phi}_{t+1} = \varepsilon + \overline{\phi}_t + \upsilon\overline{\phi}_t^{\,\delta+1},$$

$$\varepsilon \sim R^{-1}; \quad z = \delta + 1$$

Antoniou, Contoyiannis & Diakonos 2000-2

Hu & Rudnick (1982) RG fixed-point map for tangent bifurcation



$$x^z \equiv \mathrm{sign}(x)|x|^z$$

$$x' = f^{(n)}(x) = x + u x^z + \ldots, z > 1$$

$$x'^{-(z-1)} = x^{-(z-1)} - (z-1)u$$

$$x' = f^*(x) = x \exp_z(u x^{z-1}) \equiv x\left[1-(z-1)u x^{z-1}\right]^{-\frac{1}{z-1}}$$

$$f^*(f^*(x)) = \alpha^{-1} f^*(\alpha x), \quad \alpha = 2^{1/(z-1)}$$

$$f^{*(t)}(x) = t^{-1/(z-1)} f^*(t^{1/(z-1)} x), \quad t = 2^n, \quad n = 1,2,\ldots$$

$$x_t = f^{*(t)}(x_0) = x_0[1-(z-1)a x_0{}^{z-1} t]^{-\frac{1}{z-1}}, \quad x_0 > 0, \quad at = u$$

$$x_t = x_0 \exp_z[a x_0{}^{z-1} t], \;\; \xi(x_0,t) \equiv \lim_{\Delta x_0 \to 0} \frac{\Delta x_t}{\Delta x_0} = \exp_q(z a x_0^{z-1} t), \quad q = 2 - z^{-1}$$

Sensitivity to initial conditions

$$\xi(x_0, t) \equiv \lim_{\Delta x_0 \to 0} \frac{\Delta x_t}{\Delta x_0}$$

- **BG statistics:**

$$\xi(x_0, t) = \exp[\lambda_1(x_0)\, t]$$

(independent of x_0 for $t \to \infty$)

- ***q* statistics:**

$$\xi(x_0, t) = \exp_q[\lambda_q(x_0)\, t]$$

(dependent on x_0 for all t)

- ***q*-exponential function:**

$$\exp_q(x) \equiv [1 + (1-q)\, x]^{\frac{1}{1-q}}$$

- **Basic properties:**

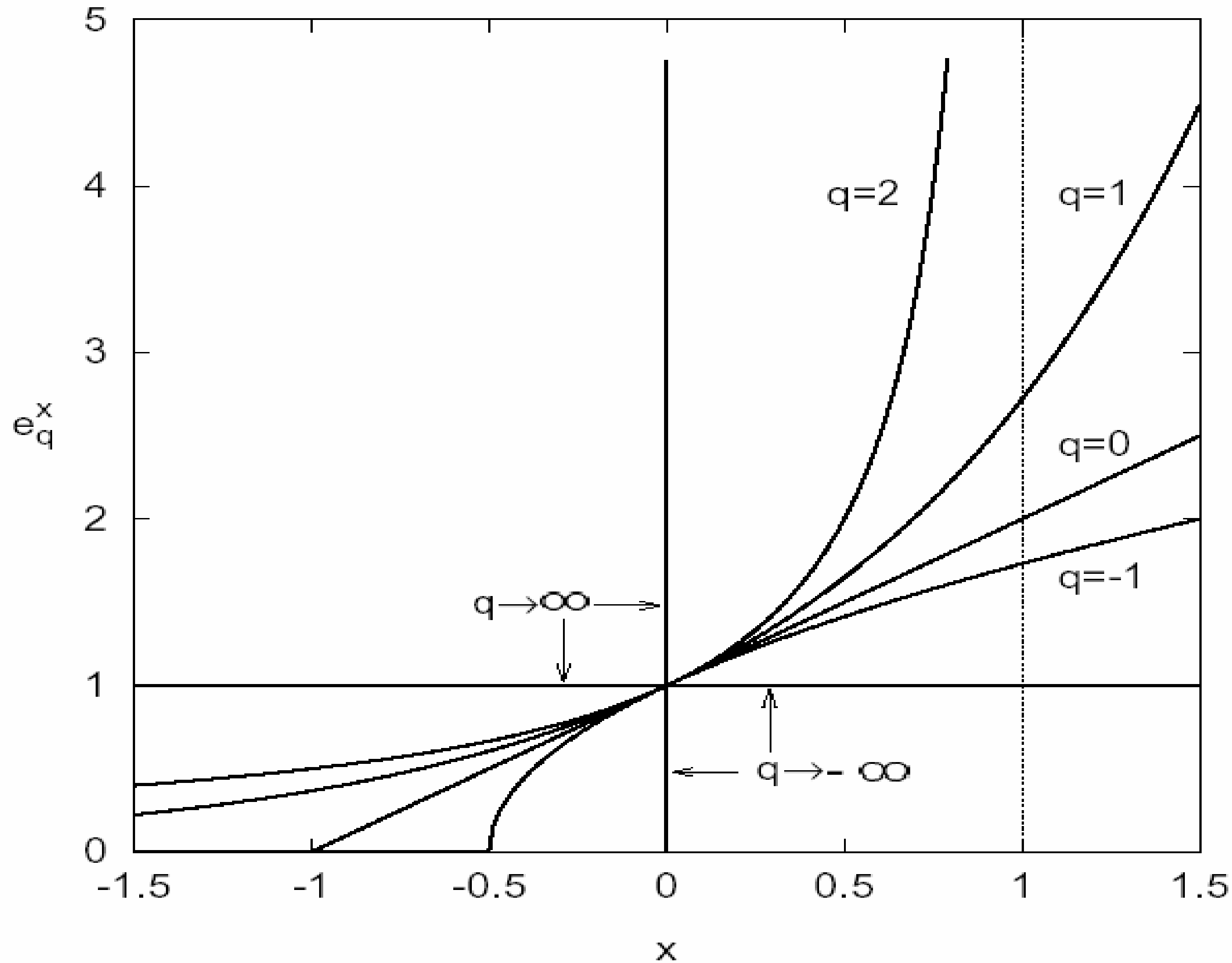
$$\exp(x) = \lim_{q \to 1} \exp_q(x)$$

$$\exp_q(-x) = [\exp_Q(x)]^{-1}, \quad Q = 2 - q$$

$$d \exp_q(x) / dx = [\exp_q(x)]^q$$

$$x_{t+1} = f_\mu(x_t), \quad a \le x_t \le b, \quad A \le \mu \le B$$

q-exponential function



Entropic expression for Lyapunov coefficient

$$\lambda_q \equiv \lim_{t\to\infty} \frac{1}{t} [\, S_q(t) - S_q(0) \,]$$

- **BG statistics:**

$$S_1 = -\sum_i p_i \ln p_i$$

- ***q* statistics:**

$$S_q = -\sum_i p_i^q \ln_q p_i \quad \text{or} \quad -\sum_i p_i \ln_Q p_i$$

- ***q*-logarithmic function:** $\ln_q(y) \equiv \dfrac{y^{1-q} - 1}{1-q} \quad (y \in R^+; q \in R)$

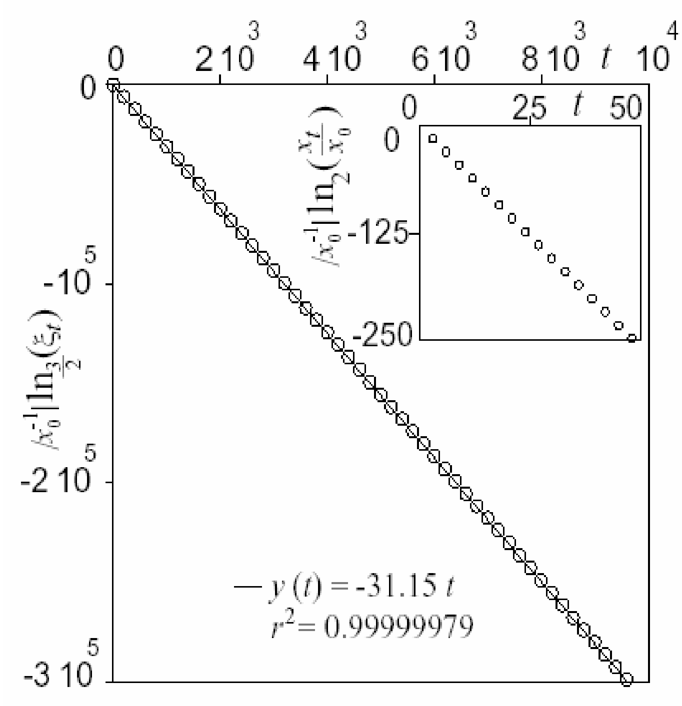
- **Basic properties:**

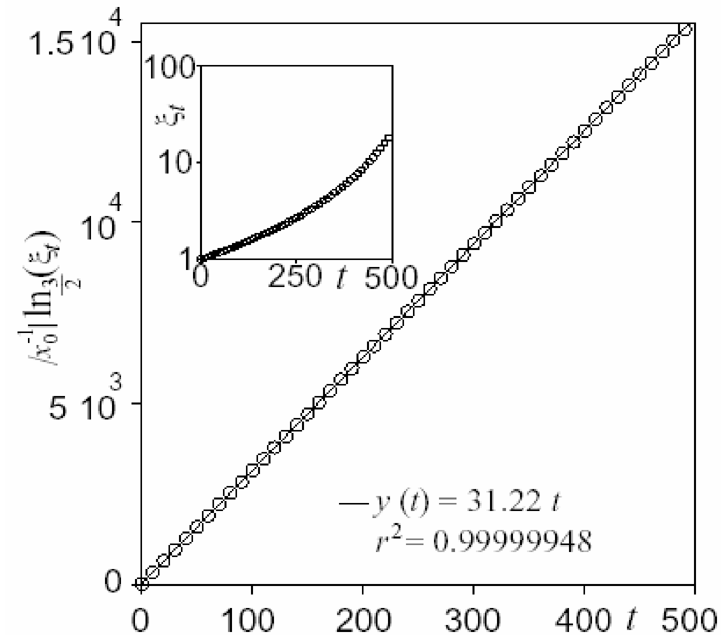
$$\ln(y) = \lim_{q\to 1} \ln_q(y)$$

$$\ln_q(y) = -\ln_Q(1/y), \quad Q = 2-q$$

$$\ln_q(\exp_q(x)) = \exp_q(\ln_q(x)) = x$$

Tangent bifurcations





$$\lambda_q(x_0) = zax_0^{z-1}, \quad zat = x_0^{1-z} \ln_q \xi_t$$

Baldovin & Robledo (2002)

$1/f^{1/\delta}$ noise in critical clusters

$P(R)$ probability for the occurrence of a laminar episode of length *R*

$$P(R) \sim R^{-z/(1-z)} \qquad \delta = z-1$$

Spectrum:

$$S_f = \int_0^{\infty} dt\, L^{-1}\left\{[1-\hat{P}(s)]^{-1}\right\}\cos(2\pi f t)$$

$$S_f \sim \frac{1}{f|\ln f|^2}, \quad z=2, \qquad S_f \sim f^{-1/(z-1)}, \quad z>2,$$

Procaccia, I., Shuster, H., 1983: Universal 1/f noise in dynamical systems. *Phys. Rev. A.* **28**,1210.

West, B.J., Shlesinger, M.F., 1989: On the ubiquity of 1/f noise. *Int'l J. Mod. Phys. B* **3**, 795.

A Landau approach for single clusters

- Two-stage calculation strategy:

$$Z = \int D[\phi]\, Z_\phi, \qquad Z_\phi = \exp(-\Psi[\phi]/kT)$$

- Landau-Ginzburg-Wilson free energy:

off criticality

$$\Psi[\phi] = \int dr^d \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} r_0 \phi^2 + u_0 \phi^4 \right], \quad r_0 = a_0 \Delta T, \quad u_0 > 0$$

at criticality

$$\Psi_c[\phi] = a \int dr^d \left[\frac{1}{2}(\nabla\phi)^2 + b|\phi|^{\delta+1} \right], \delta = 5 \text{ when } d = 3$$

(short range forces)

Cluster properties from dominant configurations

- Saddle-point configurations from Euler-Lagrange equation:

$e.g.\ d=1$

$$\frac{d^2\phi}{dx^2} = -\frac{dV}{d\phi}, \quad V=-b|\phi|^{\delta+1} \quad \Rightarrow \quad U=\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 - b|\phi|^{\delta+1}$$

(particle at position ϕ and time x under potential V)

- Most probable profile:

$$U=0 \quad \Rightarrow \quad \phi(x)=A|x-x_0|^{-2/(\delta-1)}, \; x_0 \sim (\delta-1)^{-1}\phi_0^{-(\delta-1)/2}$$

- *q*-exponential magnetization:

$$\phi(x)=\phi_0 \exp_q(kx)=\phi_0[1+(1-q)kx]^{1/1-q},$$

$$q=(1+\delta)/2, \quad k=\beta\phi_0^{-(\delta-1)/2}, \quad \beta=\sqrt{2b}$$

- Total cluster magnetization:

$$\Phi(R) = \int_0^R dx\phi(x) = \Phi_0 \left\{ \left[\exp_q(kR) \right]^{2-q} - 1 \right\}, \quad R < x_0$$

$$\frac{d\Phi(R)}{dR} = \phi_0 \exp_q(kR) = \phi_R, \quad R < x_0,$$

- Scaling of cluster magnetization profile with size *R*:

$$\phi^{(R)}(\phi_0) = R^{-1/(z-1)}\phi(R^{1/(z-1)}\phi_0), \quad R = 2^n, n = 1,2,...$$

$$\phi_R = \phi^{(R)}(\phi_0) = \phi_0[1-(z-1)\phi_0^{z-1}\beta R]^{-\frac{1}{z-1}}$$

- Cluster fractal dimension:

$$D_f = 1 - \frac{1}{2q} = \frac{\delta}{\delta+1} \quad \left(= \frac{5}{6}, \delta = 5 \right)$$

$(d = 1)$

$$d_f = \frac{2-q}{1-q} = \frac{\delta-3}{\delta-1} \quad \left(= \frac{1}{2}, \delta = 5 \right)$$

- Extensivity of cluster entropy:

at criticality

$$\delta = 5, \quad q = 3 \quad \Rightarrow \qquad S_q = S(R) = \ln_q \phi_R \equiv \frac{\phi_R^{\ 1-q} - 1}{1-q} \quad \Rightarrow S_q \sim R,$$

off criticality

$$\delta = 1, \quad q = 1 \quad \Rightarrow \qquad S_1 = \ln \phi_R \quad \Rightarrow S_1 \sim R,$$

- **Partial conclusions**

A. Robledo, Molec. Phys. 103, 3020 (2005)

A. Robledo, Europhys. News 36, 214 (2005)

Aging at the edge of chaos

Phenomenology of supercooling and glass formation

(presumed manifestations of ergodicity breaking)

- **Two-step relaxation**

  - Power-law decay towards and away from a plateau

- **Adam-Gibbs formula**

  - Connection between kinetics and thermodynamics

- **Aging**

  - History-dependent relaxation

Noisy period-doubling transitions to chaos

- **Bifurcation gap**

    - Noise fluctuations smear sharp features

- **Suppression of subharmonics**

    - Connection with noise-free parameter

- **Scaling of Lyapunov coefficient with noise amplitude**

    - Analogy with finite magnetization

Dynamics at noise-perturbed onset of chaos

- **Structure of correlations**

  - Time translation invariance

- **Duration of the plateau**

  - Running into the bifurcation gap

- **Diffusion through a cell map**

  - From normal diffusion to subdiffusion & arrest

Map counterpart of Adam-Gibbs law & aging at the onset of chaos

- **Entropy of noisy trajectories**

  - Link between 'landscape' and dynamical properties

- **Attractor position subsequences**

  - A waiting time for each subsequence

- **Relaxation through intricate trajectories**

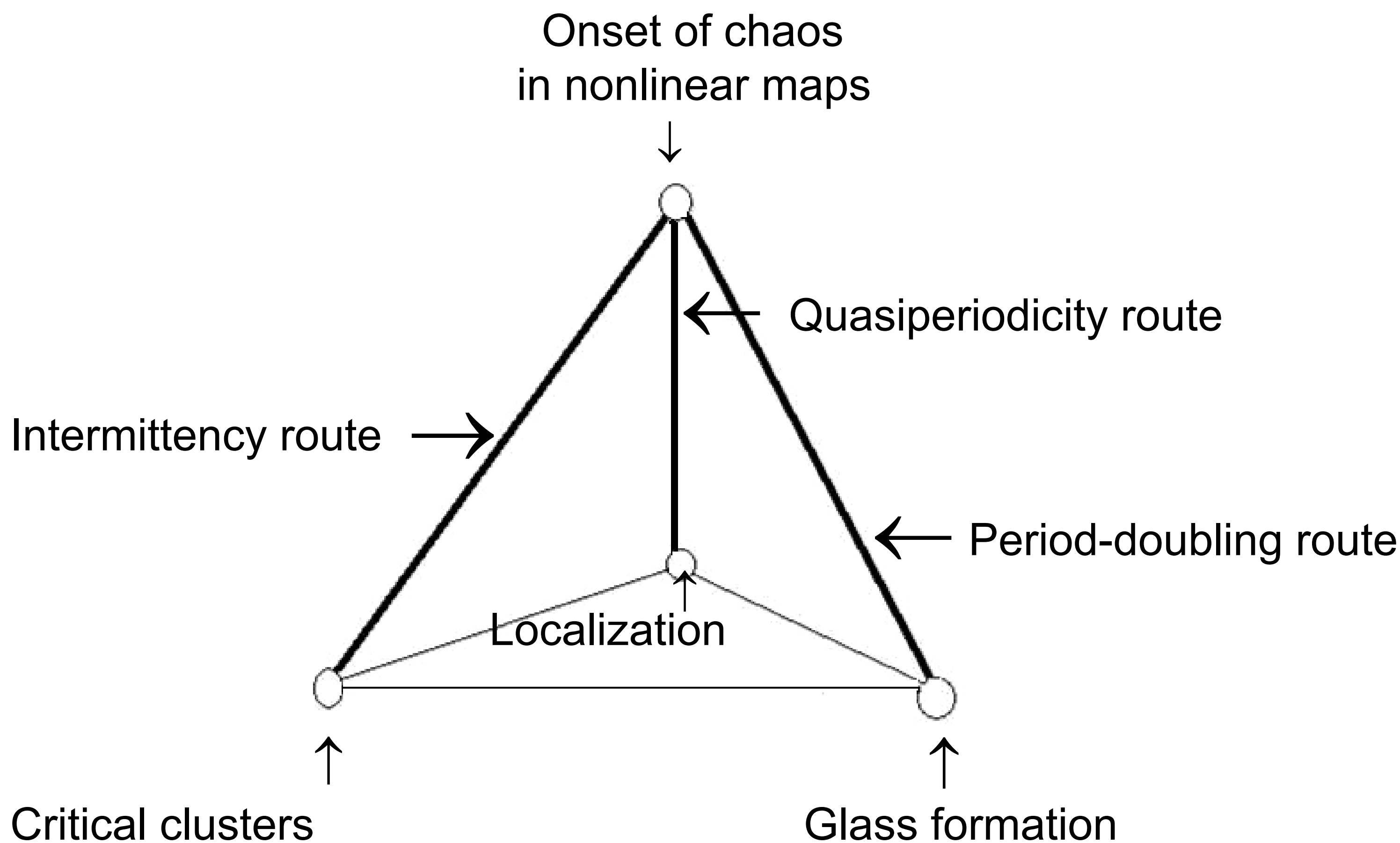
  - Waiting-time scaling of correlations

Partial conclusions

Robledo, A.,
Physics Letters A 328, 467 (2004).

Baldovin, F., Robledo, A.,
Physical Review E 72, 066213 (2005).

Summary and discussion


Onset of chaos
in nonlinear maps
Quasiperiodicity route
Intermittency route
Period-doubling route
Localization
Critical clusters
Glass formation

Incipient chaos in *d* =1 nonlinear maps

Route to chaos	Intermittency	Period doubling	Quasiperiodicity
Common properties	Vanishing ordinary Lyapunov coefficient, dynamical phase transitions (Mori's *q*-phases) power-law dynamics, *q*-sensitivity, *q*-Pesin identity		
Distinctive properties	(Also) faster than exponen-tial dynamics	Foam-like phase space	Dense phase space
Applications in condensed matter physics	Critical clusters	Glass formation	Localization
Applications in other disciplines	Information & other flows in networks, …	Protein folding, vegetation patterns, …	Mode locking, cardiac cells, Internet TCP, ...

Current questions on generalized statistics

- When does Boltzmann-Gibbs statistics stop working?
- Where, and in that case why, does *q*-statistics apply?
- Is ergodicity & mixing failure the basic playground for applicability of generalized statistics?
- Are critical states - infinite correlation length or vanishing Lyapunov coefficient - outside BG statistics?
- Is there a firm connection between critical phenomena and transitions to chaos?
- Is there a strong link between glassy dynamics and transitions to chaos?

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“The renormalization group and optimization of non-extensive entropy: criticality in non-linear one-dimensional maps”
Physica A 314, 437 (2002).

- Baldovin, F., Robledo, A.,
“Sensitivity to initial conditions at bifurcations in one-dimensional non-linear maps: rigorous non-extensive solutions”
Europhysics Letters 60, 518 (2002).

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“RG universal dynamics at the onset of chaos in logistic maps and non-extensive statistical mechanics”
Physical Review E 66, 045104 (R) (2002).

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“Criticality in non-linear one-dimensional maps: RG universal map and non-extensive entropy”
Physica D 193, 153(2004).

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“Non-extensive Pesin identity: Exact RG analytical results for the dynamics at the edge of chaos of the logistic map”
Physical Review E69, 045202 (R) (2004).

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“Multifractality and nonextensivity at the edge of chaos of unimodal maps”
Physica A 340, 219 (2004).

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"Universal glassy dynamics at noise-perturbed onset of chaos.
A route to ergodicity breakdown",
Physics Letters A 328, 467 (2004).

- Robledo, A.,
"Critical fluctuations, intermittent dynamics and Tsallis statistics",
Physica A 344, 631 (2004).

- Mayoral, E., Robledo, A.,
"Tsallis' *q* index and Mori's *q* phase transitions at edge of chaos"
Physical Review E 72, 026029 (2005).

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"Unorthodox properties of critical clusters"
Molecular Physics (Special Widom Issue) 103, 3025 (2005).

- **Baldovin, F., Robledo, A.,**
"The noise-perturbed onset of chaos in logistic maps and the dynamics of glass formation"
Physical Review E 72, 066213 (2005).

- **Hernández-Saldaňa, H., Robledo, A.,**
"Dynamics at the quasiperiodic onset of chaos, Tsallis *q*-statistics and Mori's *q*-phase thermodynamics"
Physica A 370, 286 (2006).

Supercooled liquid

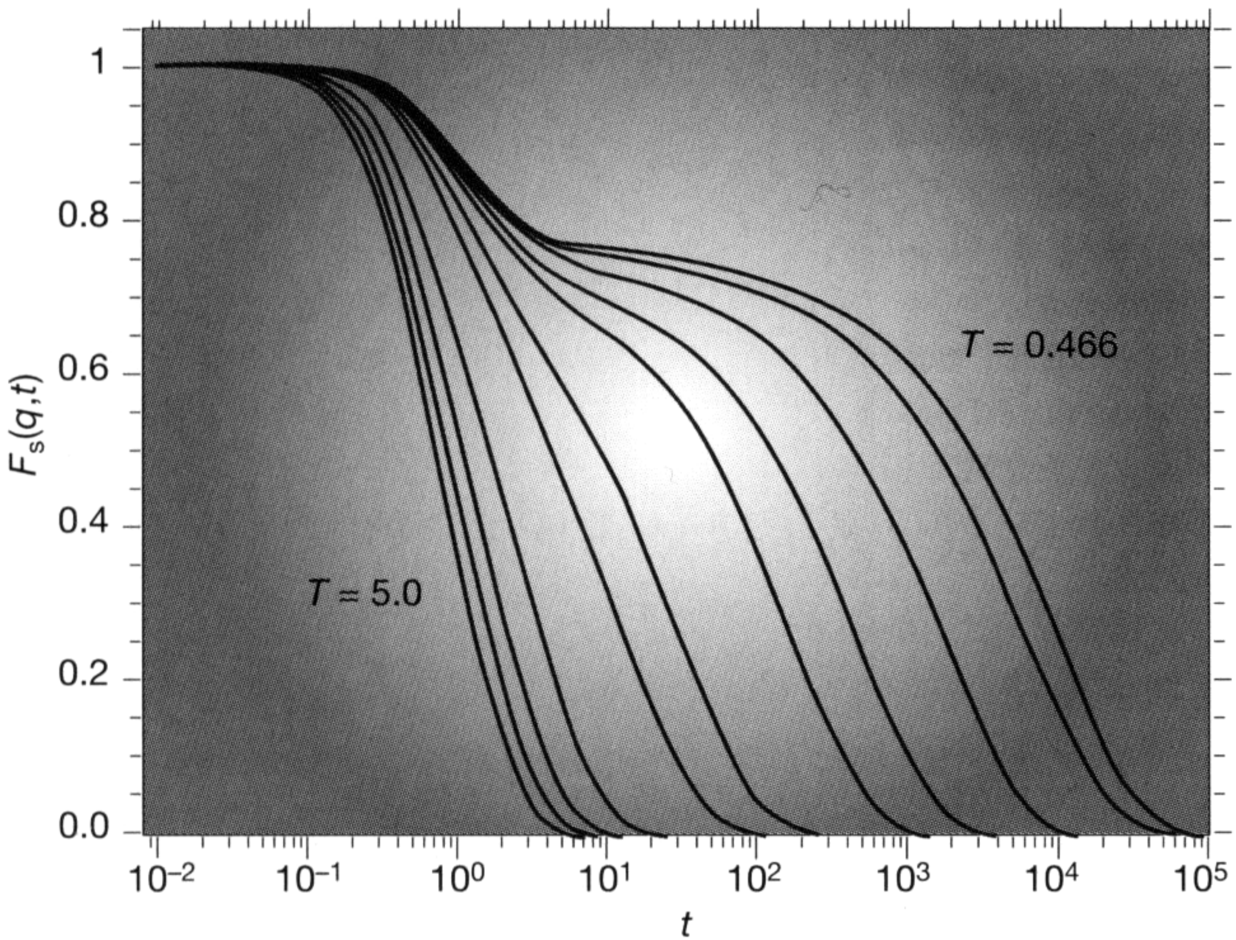


Figure 9 Evolution of the self-intermediate scattering function for A-type atoms for the same supercooled Lennard–Jones mixture as in Fig. 6, at $q\sigma_{AA} = 7.251$,

Adam-Gibbs formula

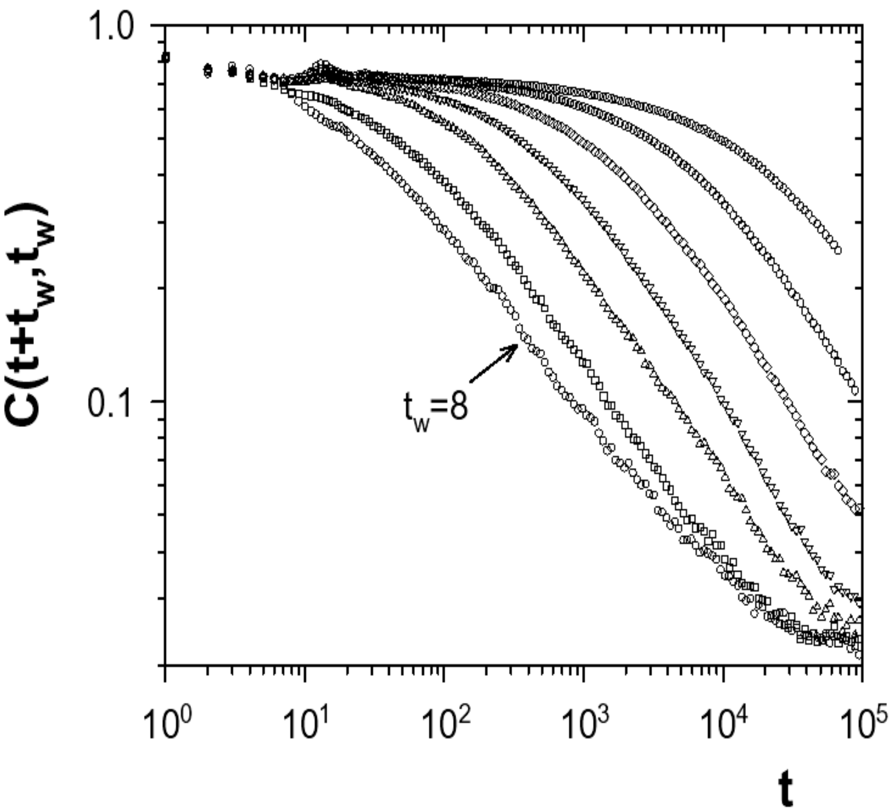
$$t_x = A \exp (B/TS_c)$$

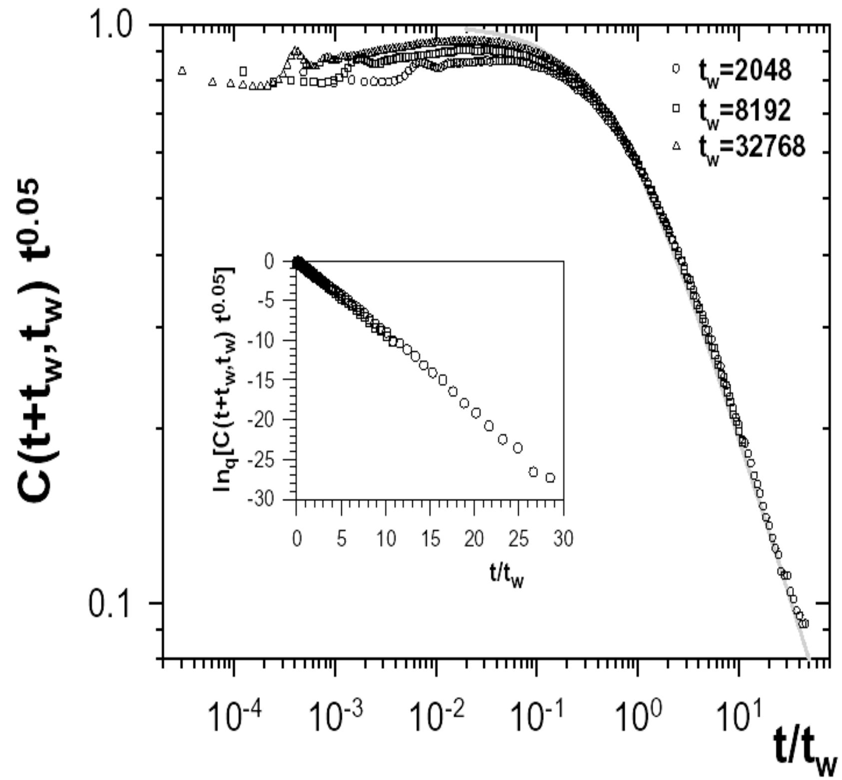
t_x is a relaxation time (or, equivalently, the viscosity) and $\boldsymbol{S_c}$, the configurational entropy, is related to the number of minima of the system's multidimensional potential energy surface – the 'energy landscape'.

Diverging relaxation time as entropy vanishes

$$\boxed{t_x \to \infty \quad \text{and} \quad S_c \to 0 \quad \text{as} \quad \sigma \to 0}$$

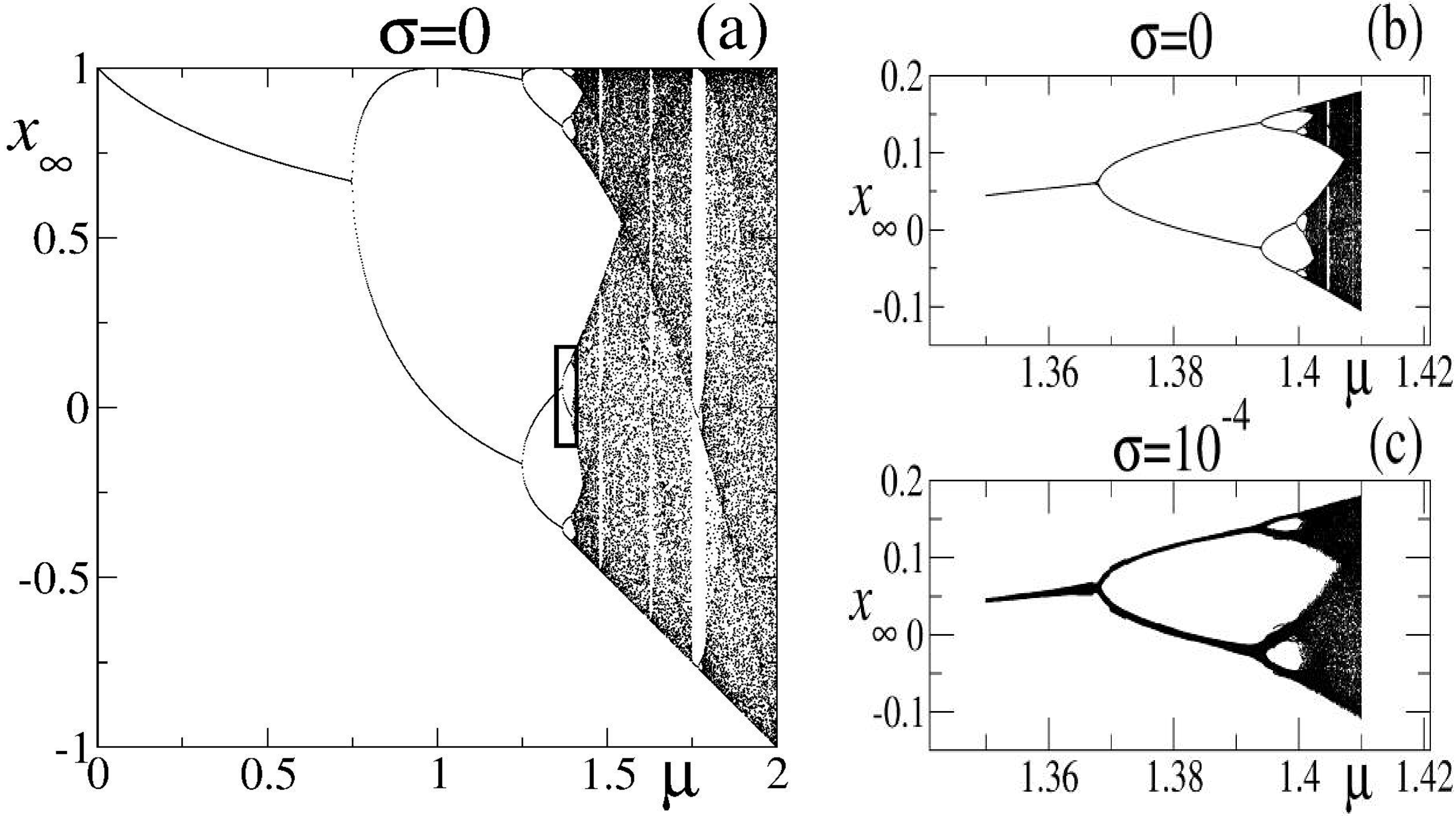
Aging in coupled rotors



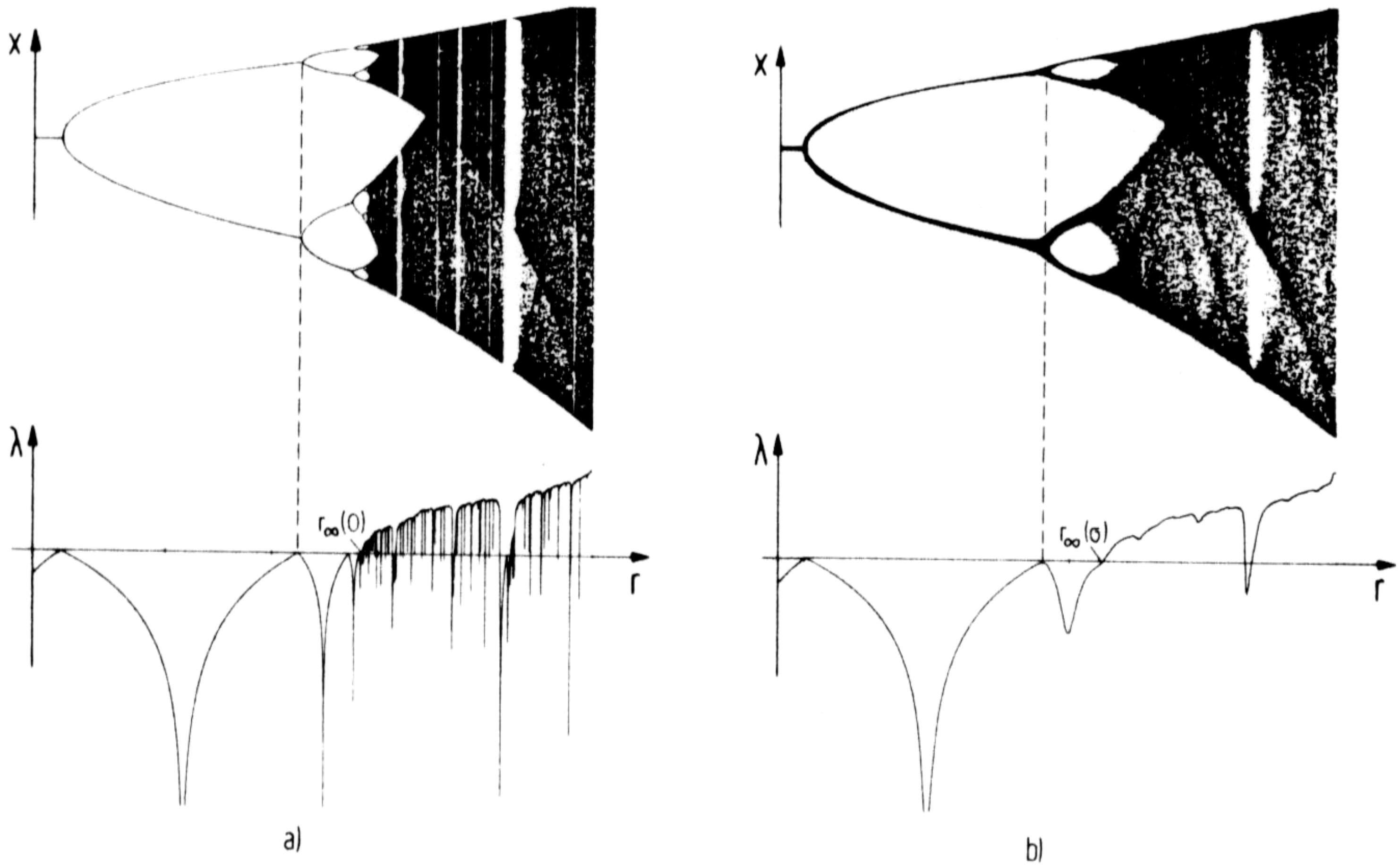


Montemurro, Tamarit and Anteneodo (2003)

$$x_{t+1} = f_\mu(x_t) + \xi_t\,\sigma$$

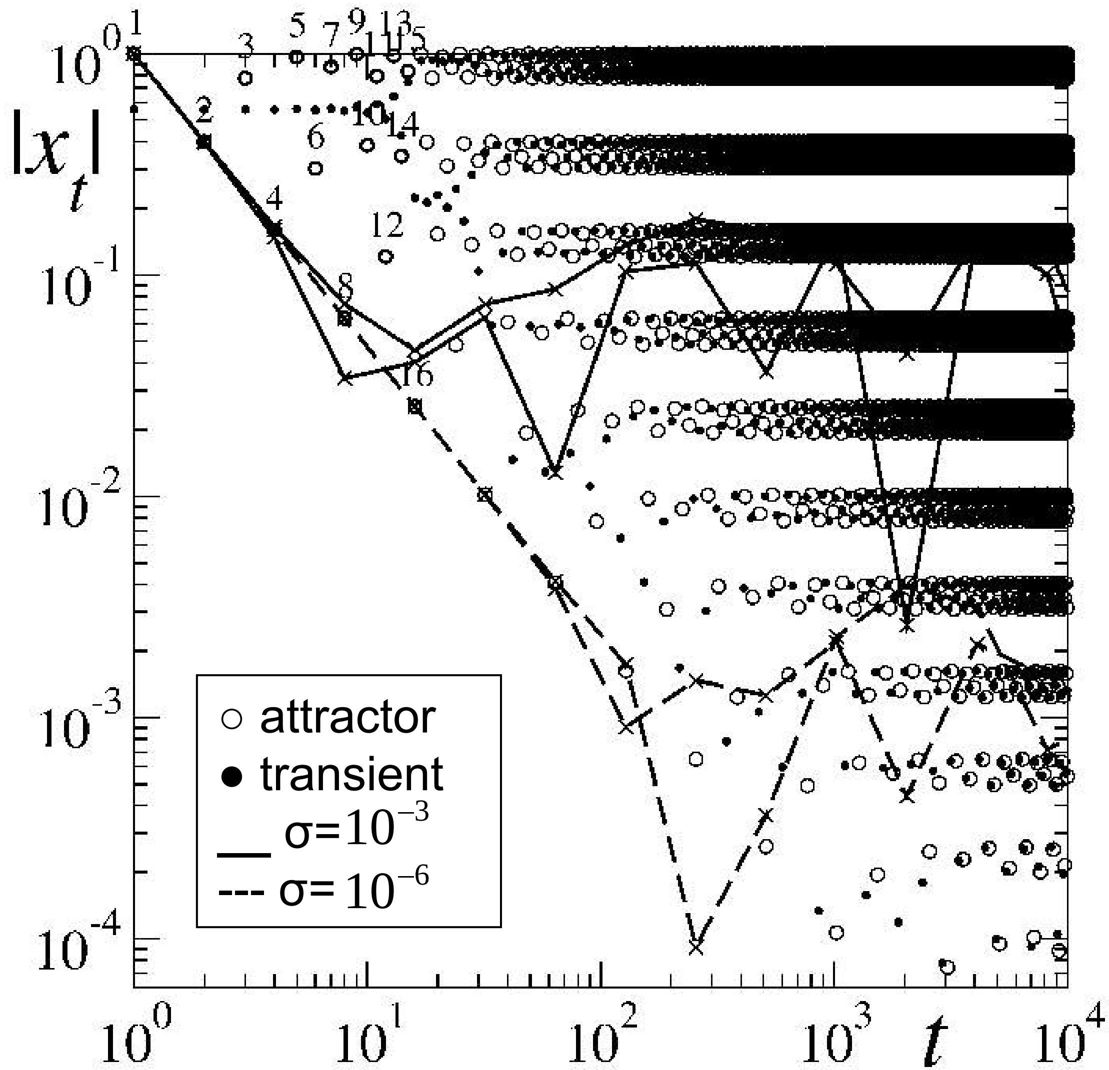


$$x_{t+1} = f_r(x_t) + \xi_t \sigma$$



$\sigma = 0$ $\sigma \neq 0$

Running into the bifurcation gap



Scaling of Lyapunov exponent

$$(R_\infty - R_n) \propto \delta^{-n}$$

$$\sigma_n \propto \mu^{-n}$$

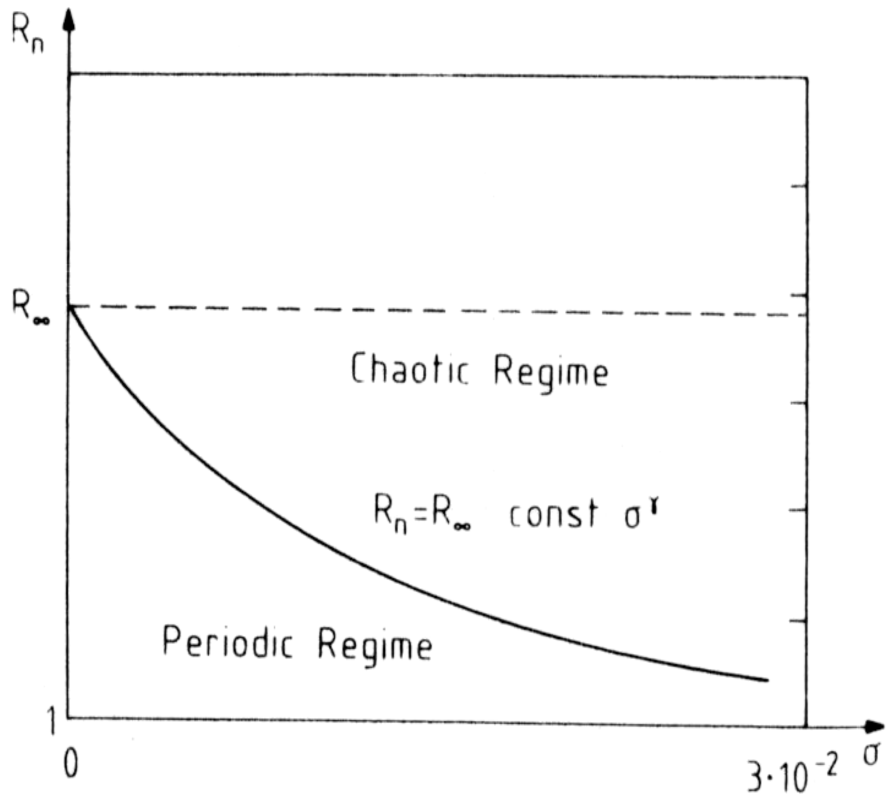
$$\Rightarrow \qquad (R_\infty - R_n) \propto \sigma_n^{\gamma}$$

$$\mu^{-1} = \frac{1}{4\alpha}\sqrt{2\left(1 + \frac{1}{\alpha^2}\right)}\,, \quad \alpha = 2.50290\ldots \qquad \gamma = \log\delta/\log\mu.$$

$$\lambda = r^{\beta}\lambda_0[r^{-1/\gamma}\sigma]\ ; \quad \beta = \log 2/\log\delta\ ; \quad r = R_\infty - R$$

$$\lambda = \sigma^{\theta}\lambda_1[r\sigma^{-\gamma}]\ ; \quad \theta = \log 2/\log\mu$$

Bifurcation gap



$$(R_\infty - R_n) \propto \sigma_h^\gamma$$

$$\gamma = \log \delta / \log \mu.$$

Fig. 37: Suppression of the periodic regime by the presence of external noise for the logistic map (after Crutchfield, Farmer and Huberman, 1982).

• Unperturbed orbit with $x_{in} = 0$

$$\lim_{n\to\infty} (-\alpha)^{-n} g(\alpha^n x) = g(x) \;\Longrightarrow\; x_\tau = \left| g^{(\tau)}(x_{in}) \right| = \tau^{-1/1-q} \left| g(\tau^{1/1-q} x_{in}) \right|$$

where $\alpha = 2^{-1/(1-q)}$ and $q = 1 - \ln 2/\ln\alpha \simeq 0.2455.$

For $x_\tau = \left| g^{(\tau)}(0) \right|$ we find $x_{2^n} = d_n = \alpha^{-n}$

or $x_t = \exp_{2-q}(-\lambda_q t)$, $t = \tau - 1$ and $\lambda_q = \ln\alpha/\ln 2$.

Similarly for all time subsequences $\tau = (2k+1)2^{n-1}, k = 0, 1, 2, n = 1, 2, 3, ...$

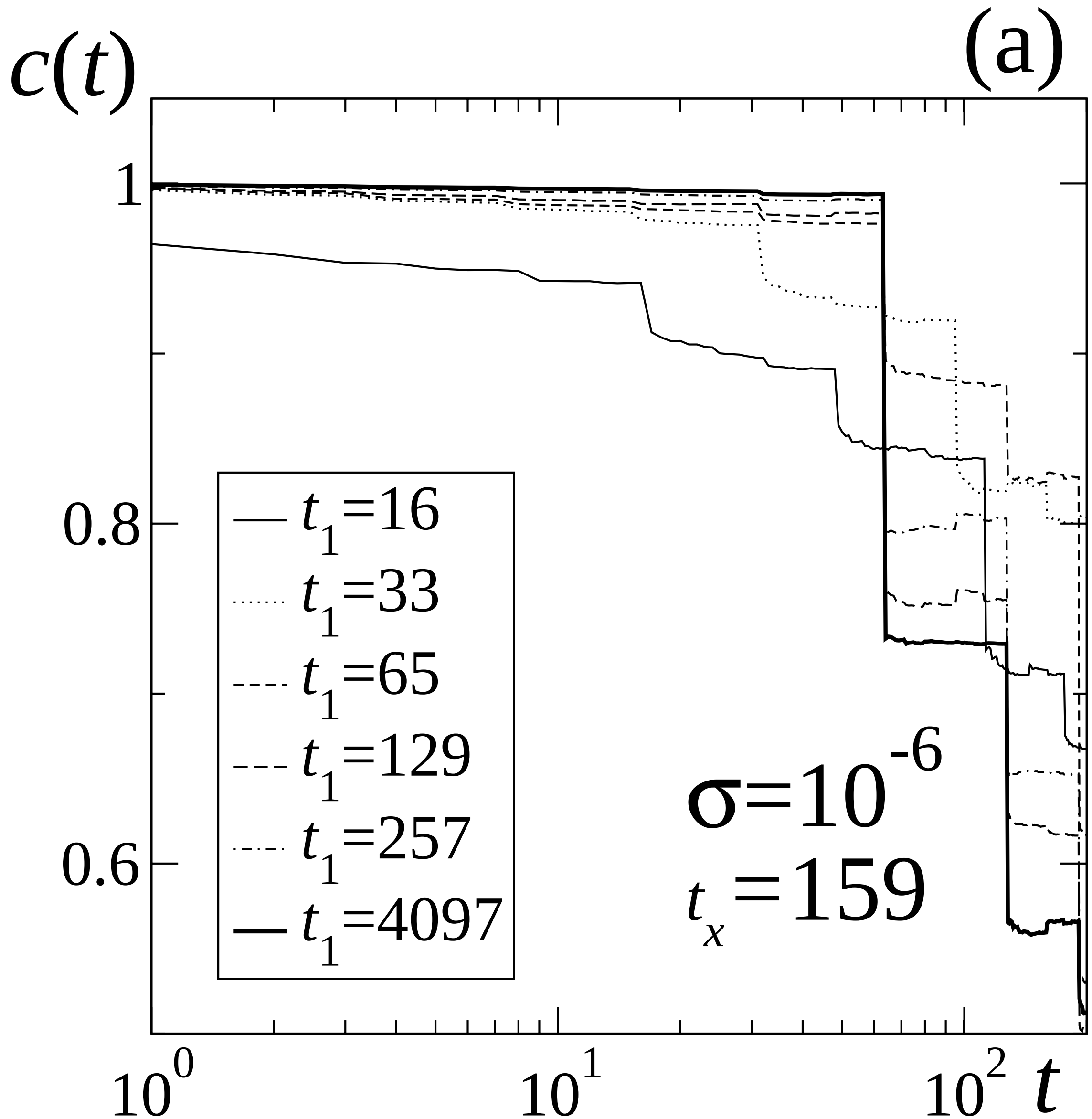
• Noise-perturbed orbit with $x_{in} = 0$

$$\lim_{n\to\infty} (-\alpha)^{-n}[g(\alpha^n x) + \xi\sigma\kappa^n G_\lambda(\alpha^n x)] \;\Longrightarrow\; x_\tau = \tau^{-1/1-q} \left| g(\tau^{1/1-q} x) + \xi\sigma\tau^{1/1-r} G_\lambda(\tau^{1/1-q} x) \right|$$

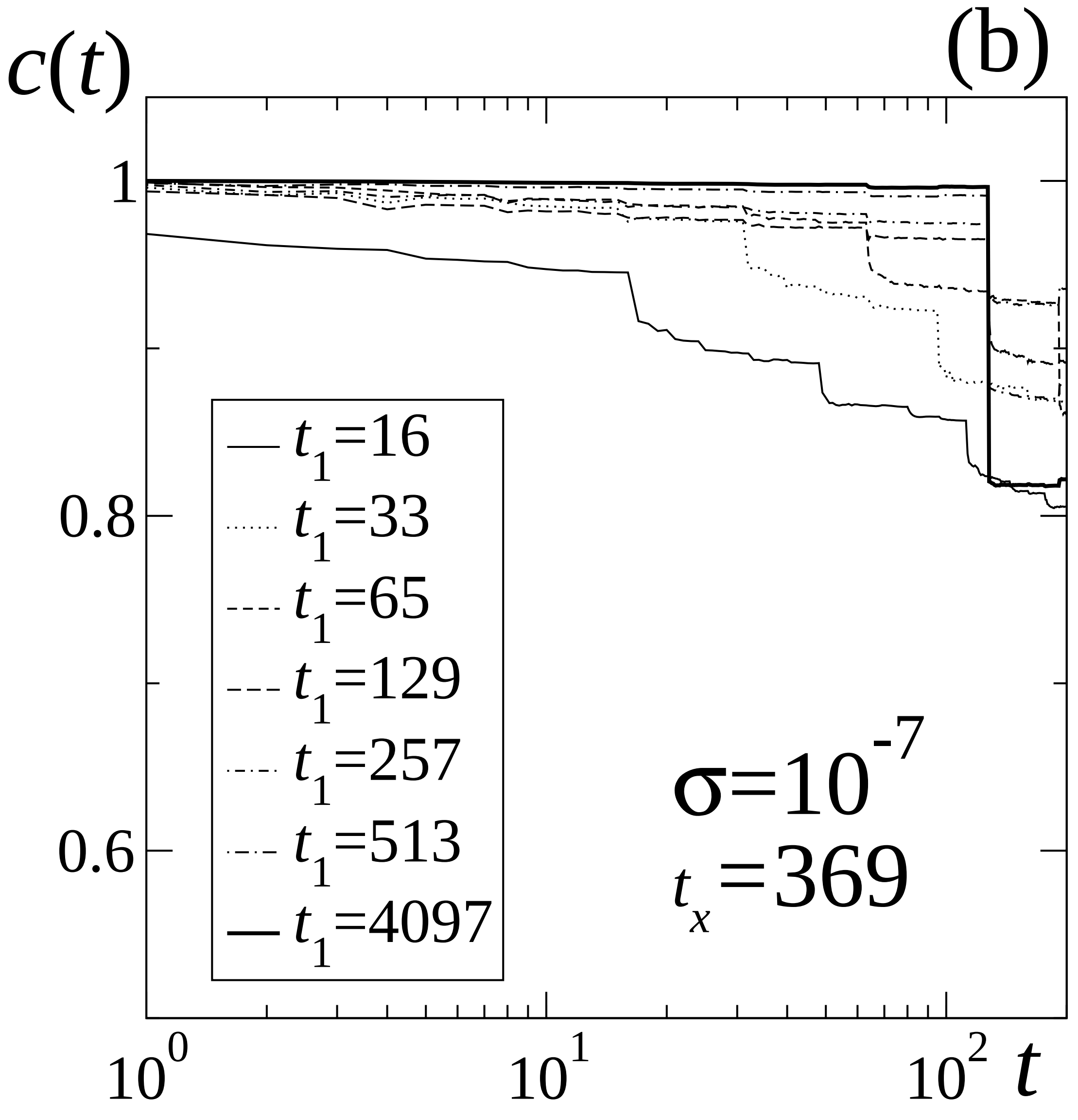
where $\kappa \simeq 6.619$ and $r = 1 + \ln 2/\ln\kappa \simeq 0.6332$

For $x_\tau = \left| g^{(\tau)}(0) \right|$ we find $x_t = \exp_{2-q}(-\lambda_q t)\,[1 + \xi\sigma \exp_r(\lambda_r t)]$, $\lambda_r = \ln\kappa/\ln 2$

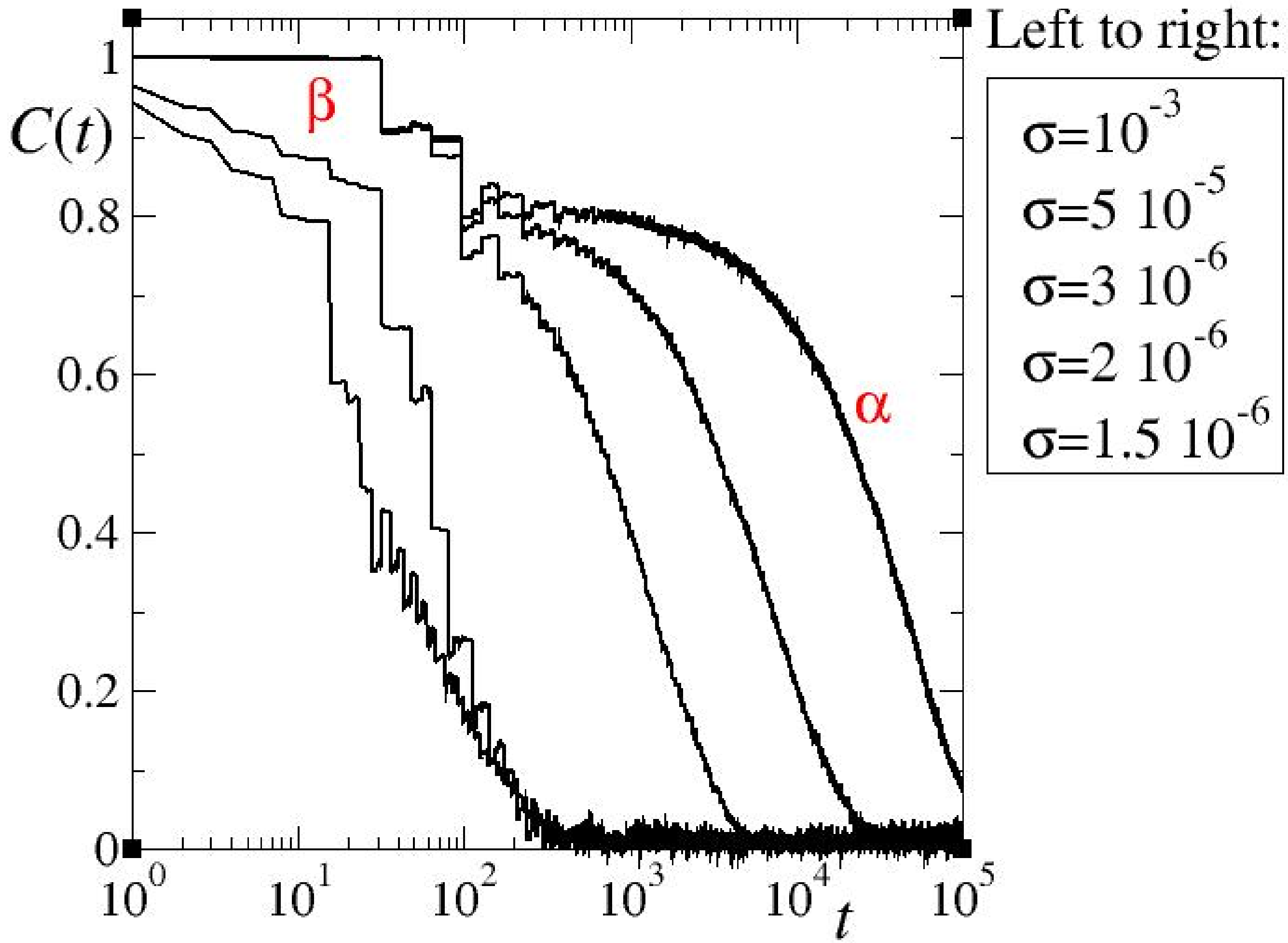
Onset of time translation invariance



Onset of time translation invariance



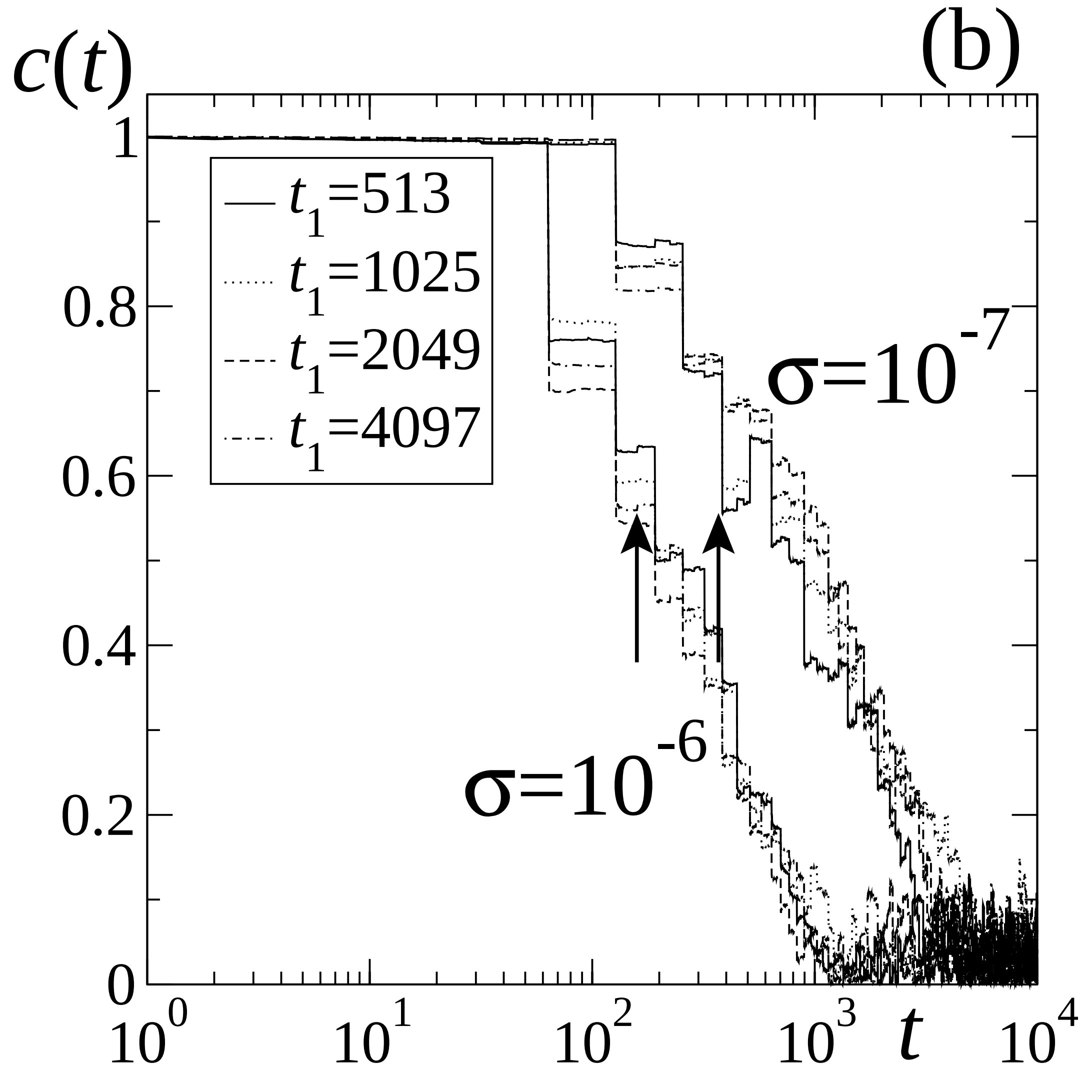
Two-step relaxation



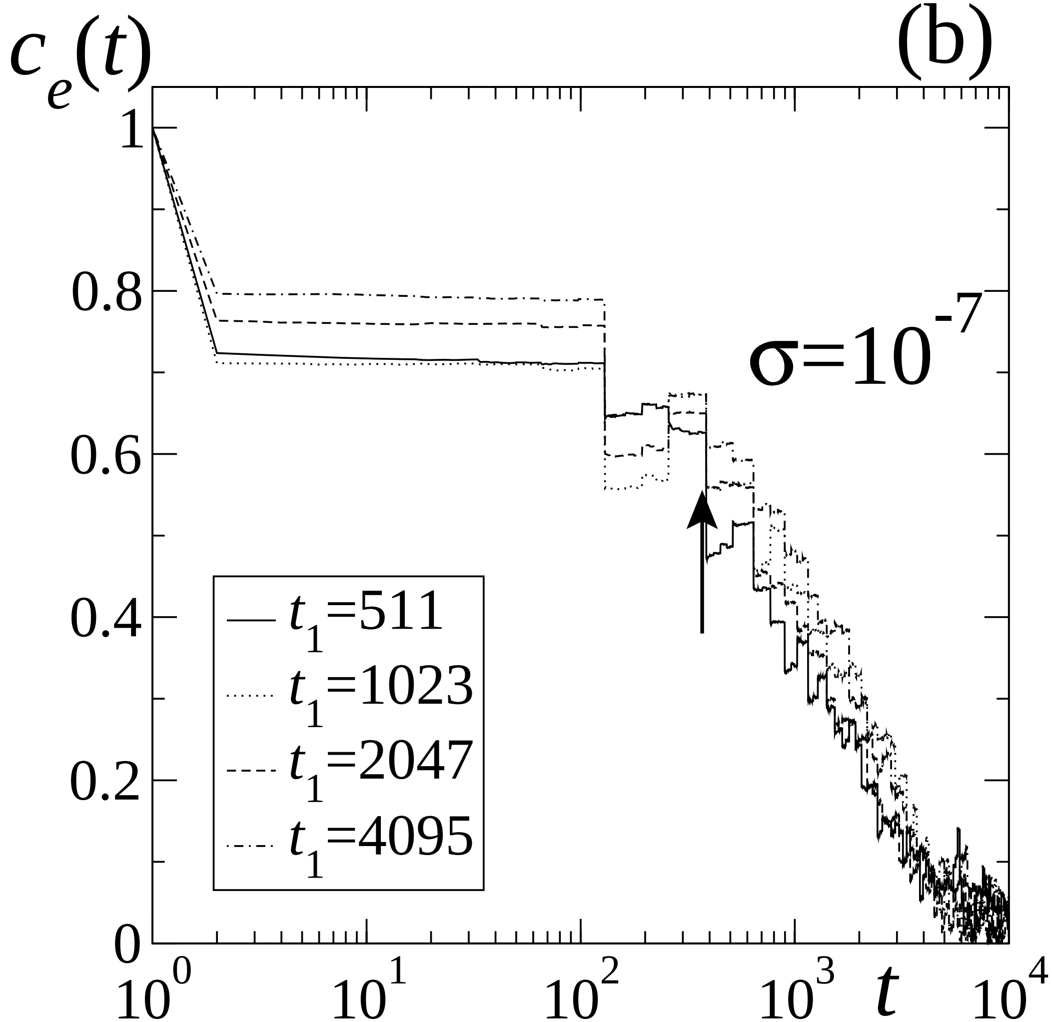
$C(t=t_2-t_1)=(<x(t_2)x(t_1))>-<x(t_2)><x(t_1)>)/(\sigma_1\sigma_2)$

$$\sigma_i = \sqrt{\left\langle x_{t_i}^2 \right\rangle - \left\langle x_{t_i} \right\rangle^2}, \quad i = 1,2$$

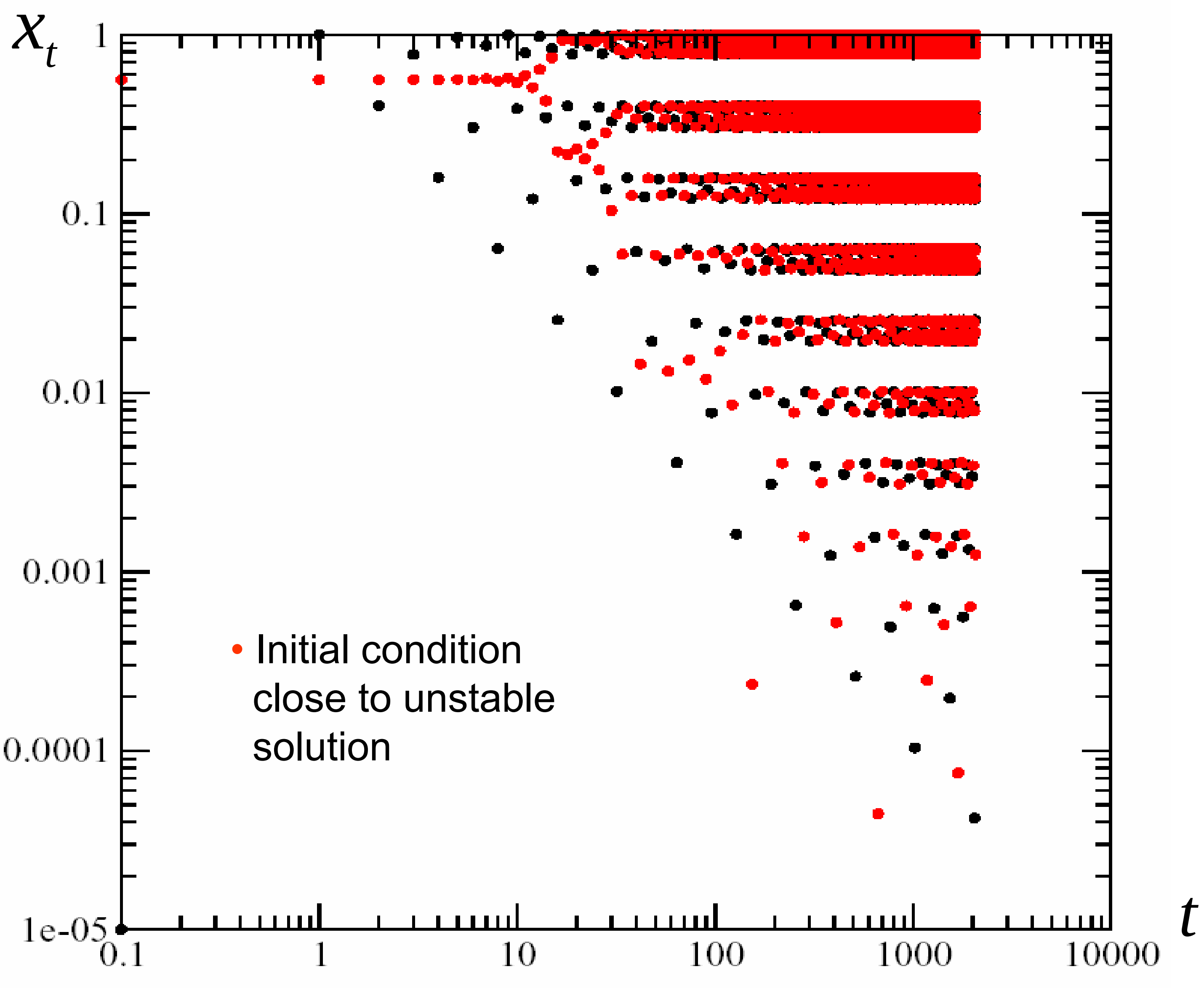
Two-step relaxation



Two-step relaxation



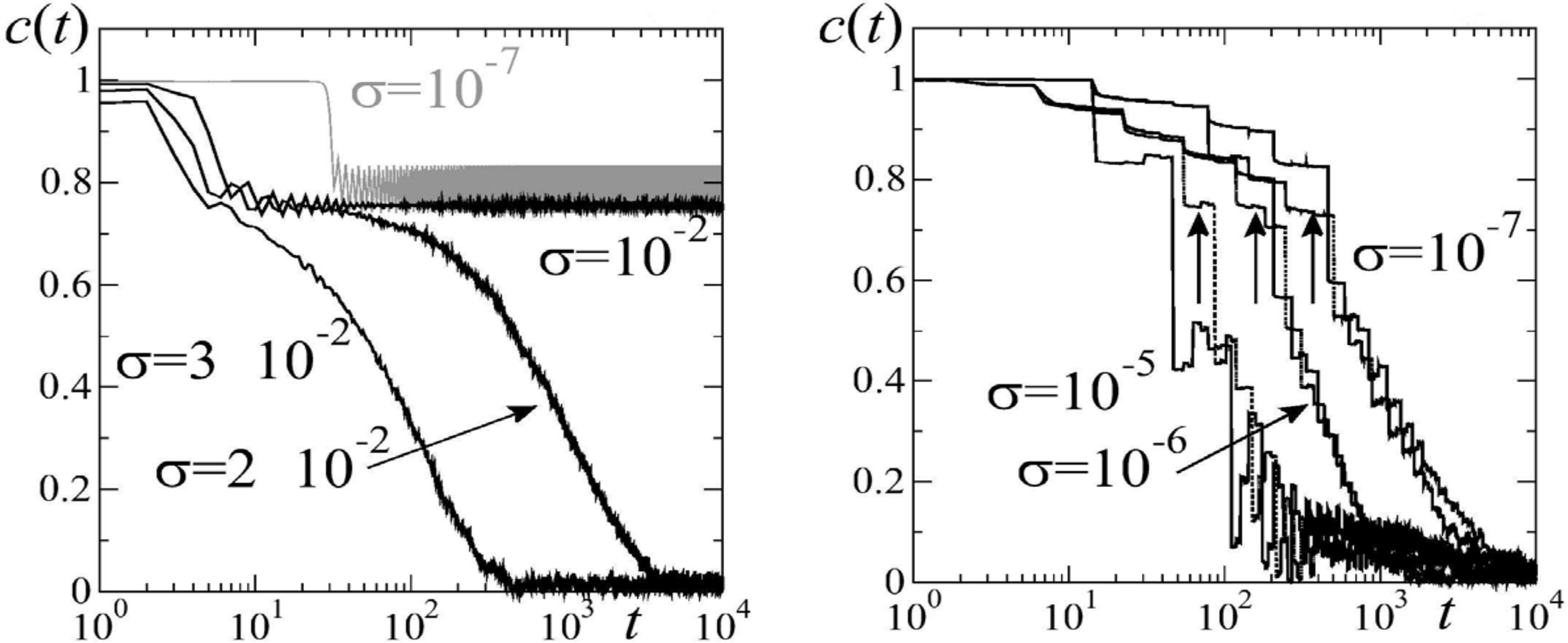
Falling into the attractor



Two-step relaxation

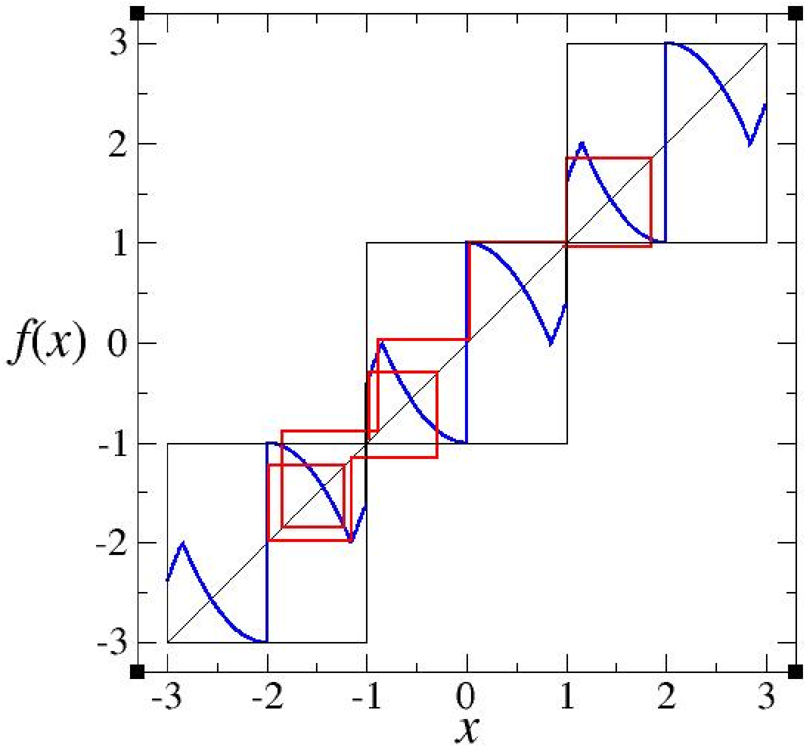
Initial conditions placed:

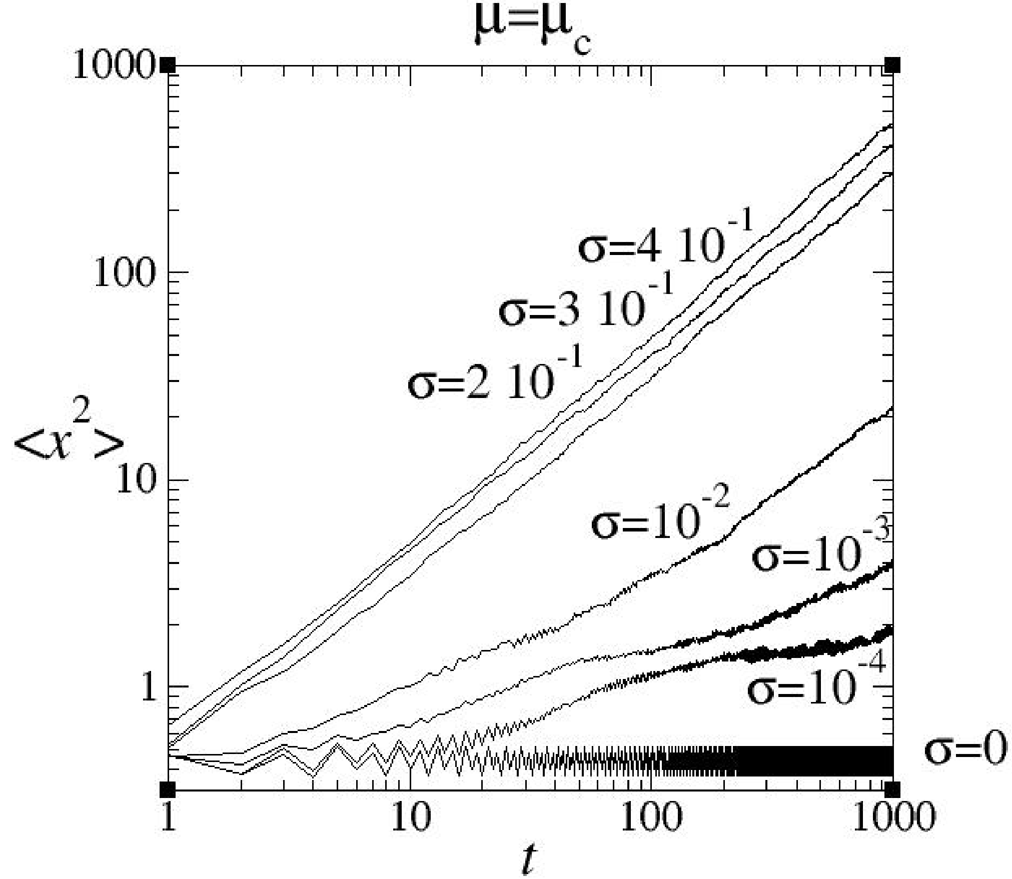
outside the attractor | ~ inside the attractor



5000 trajectories initially within an interval of width ~ 10^{-7}

Diffusivity at the edge of chaos





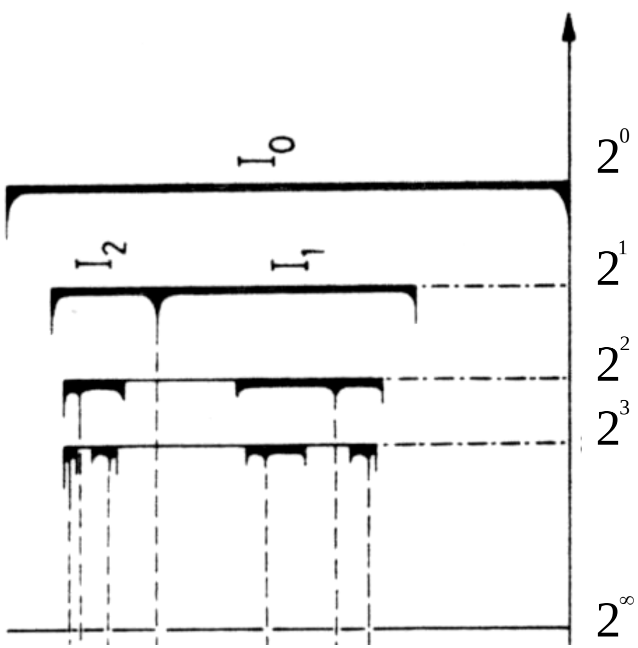
Crossover time / contact with bifurcation gap

$$t_x = \sigma^{1-r} - 1, \quad r \cong 0.6332$$

Largest number of bands at noise amplitude σ

$$t_x = 2^N - 1 \text{ with } N \approx -\ln\sigma / \ln\kappa, \quad \kappa \cong 6.619$$

Sampling the ‘energy landscape‘ at noise level σ



2^N bands of widths $w_1, \ldots, w_{2^N}$, $N = 0, 1, \ldots$

as $N \to \infty$ the widths $w_i \to 0$

$t < t_x$ moving on a plateau with $\xi_t = \exp_q(\lambda_q t)$

$t \sim t_x$ crossover to

$t > t_x$ a chaotic regime with $\xi_t = \exp(\lambda_1 t)$

Entropy at noise level σ

$$S_c(\mu_c, \sigma) = 2^N \sigma\, s, \quad \text{where} \quad s = -\int p(\xi) \ln p(\xi)\, d\xi$$

At contact with the bifurcation gap

$$2^N = t_x + 1 \quad \text{and} \quad \sigma = (t_x + 1)^{1/1-r}$$

one obtains

$$S_c(\mu_c, \sigma)/s = (1 + t_x)^{-r/1-r}$$

or

$$\boxed{t_x = (s/S_c)^{(1-r)/r}}$$

} scaling relations

Diverging relaxation time as entropy vanishes

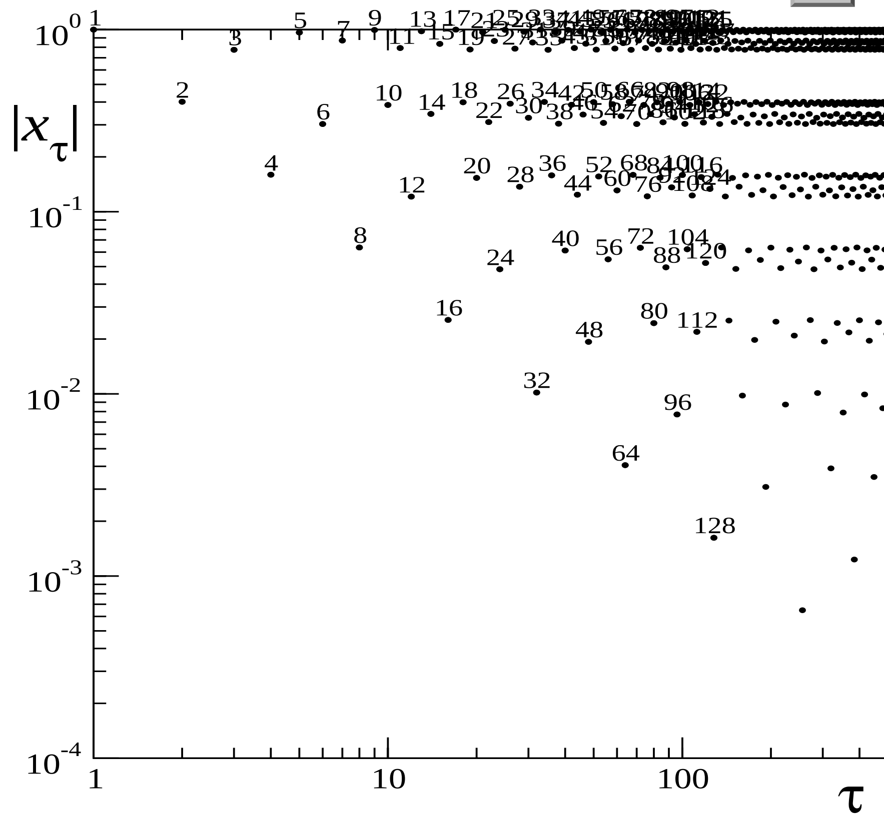
$$\boxed{t_x \to \infty \quad \text{and} \quad S_c \to 0 \quad \text{as} \quad \sigma \to 0}$$

Ergodicity failure as $\sigma \to 0$

Attractor position subsequences

$$x_\tau = \left| g^{(\tau)}(0) \right|,$$

$$\tau = (2k+1)2^n, \; k, n = 0, 1, \ldots$$



A waiting time for each subsequence

$$t = (2k+1)(2^n - 1); \;\; t_w = 2k+1;$$

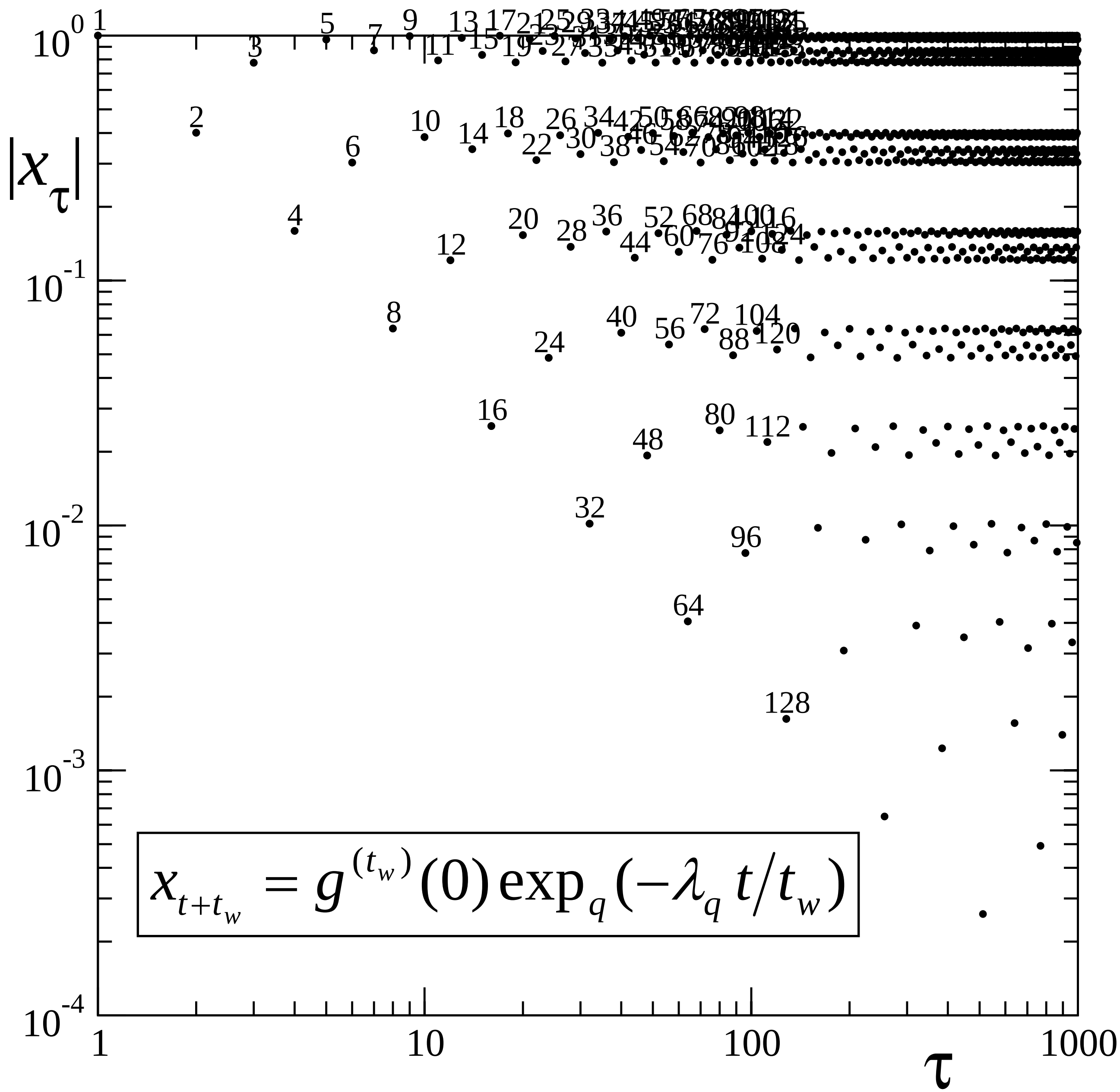
$$\tau = t_w + t; \;\; t/t_w = 2^n - 1$$

Aging scaling relation

$$x_{t+t_w} = g^{(t_w)}(0) g^{(t)}(0) \quad \text{or} \quad x_{t+t_w} = g^{(t_w)}(0) \exp_q(-\lambda_q \, t/t_w)$$

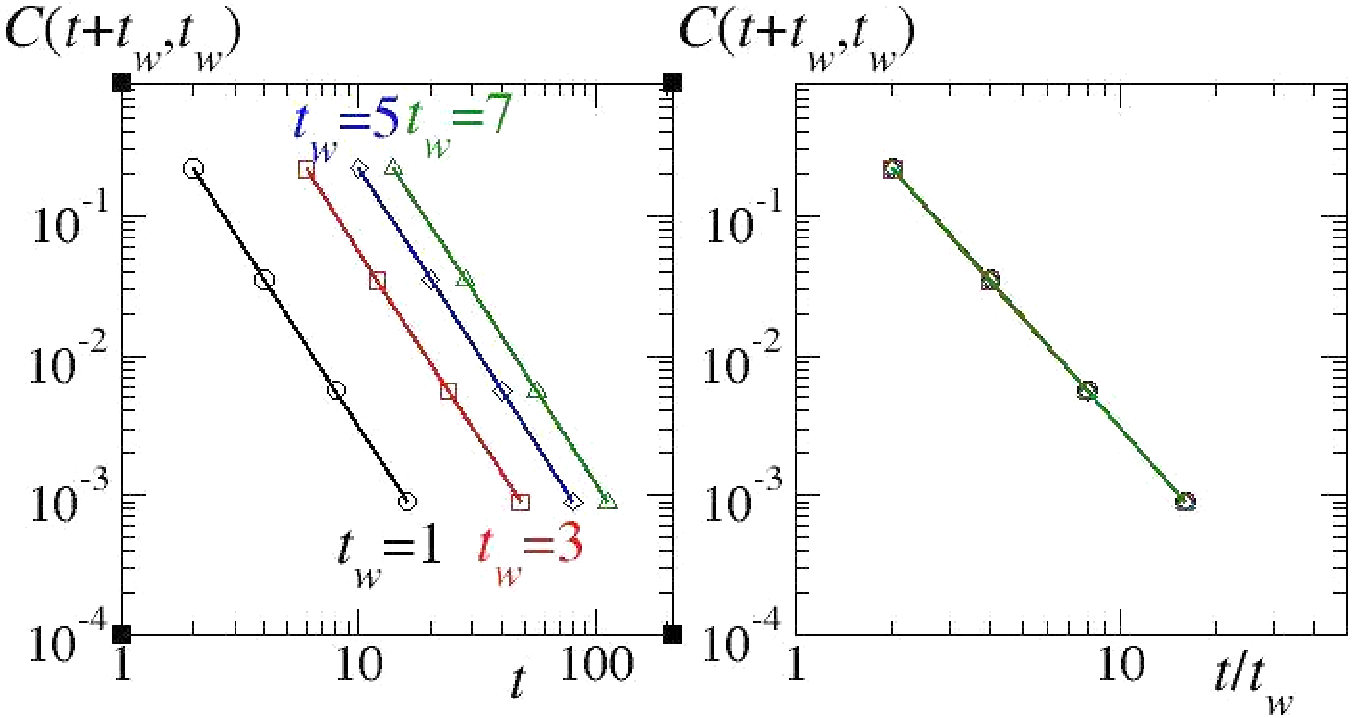
- Time translation invariance for $t > t_x(\sigma)$

- Containment of aging for increasing noise amplitude σ > 0



$|x_\tau|$
τ
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
1
10
100
1000
$x_{t+t_w} = g^{(t_w)}(0)\exp_q(-\lambda_q\, t/t_w)$

‘Aging at the edge of chaos’



$$c(t+t_w,t)=\frac{1}{N}\sum_{n=1}^{N}\phi^{(n)}(t+t_w)\phi^{(n)}(t), \quad \phi(\tau)= f_{\mu_c}^{(\tau)}(0), \quad f_{\mu}(x)=1-\mu x^2,$$

$$t/t_w = 2^n - 1, \quad t_w = 2k+1, \quad k = 0,1,\ldots$$

Main message:

A primary physical circumstance in which *q*-statistics appears to find appropriate application is one where a system is driven towards ergodicity/mixing failure, and relaxation en route to equilibrium becomes severely slowed down.

This situation is exemplified by the dynamics in a supercooled liquid close to glass formation, but also, interestingly, by the slow dynamics in critical systems, and in noise-perturbed unimodal maps at the onset of chaos. Another example is the localization transition in incommensurable systems.