

# Statistical mixing and aggregation in Feller diffusion

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Received 29 August 2009

Accepted 6 October 2009

Published 29 October 2009

Online at [stacks.iop.org/JSTAT/2009/P10023](http://stacks.iop.org/JSTAT/2009/P10023)

[doi:10.1088/1742-5468/2009/10/P10023](https://doi.org/10.1088/1742-5468/2009/10/P10023)

**Abstract.** We consider Feller mean-reverting square-root diffusion, which has been applied to model a wide variety of processes with linearly state-dependent diffusion, such as stochastic volatility and interest rates in finance, and neuronal and population dynamics in the natural sciences. We focus on the statistical mixing (or superstatistical) process in which the parameter related to the mean value can fluctuate—a plausible mechanism for the emergence of heavy-tailed distributions. We obtain analytical results for the associated probability density function (both stationary and time-dependent), its correlation structure and aggregation properties. Our results are applied to explain the statistics of stock traded volume at different aggregation scales.

**Keywords:** rigorous results in statistical mechanics, models of financial markets, stochastic processes

**ArXiv ePrint:** [0910.1394](https://arxiv.org/abs/0910.1394)

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**1. Introduction**

Complex non-equilibrium phenomena have lured the attention of a large part of the physical community in recent years. Despite the knotty character of this type of system, by applying particular techniques, it has been possible to make a physical characterization of their leading properties. Among these techniques, let us mention the ‘superstatistical’ approach. It originated from the observation, in the context of nuclear collisions, that deviations from the standard (exponential) probability density function (PDF) could be explained through the fluctuations of an inner parameter, in that case, the cross section [1]. Because a statistics is made up of another statistics, this approach became known as superstatistics, or statistics of statistics. Thereafter, it was further generalized, endowed with a statistical mechanics interpretation [2] and widely applied since then. In fact, for many systems, it is realistic to consider that some of the characteristic parameters may not be strictly constant, but instead fluctuant, either in time or in space, according to a specific PDF, in a scale much larger than the primary stochastic process. The superstatistical approach has been quite successful in accounting for observations in fluid turbulence [3], physiology [4], human activities [5], ecology [6], and also in finance [7]–[9], amid many others. Let us also point out that, in economics and social sciences, albeit ad hoc, statistical mixtures have been taken into account for some decades [10].

In this work, we consider as the primary process the one given by the stochastic differential equation (SDE):

$$dx = -\gamma [x - \theta] dt + \delta \sqrt{x} dW_t \quad (x \geq 0), \tag{1}$$

where  $W_t$  represents a standard Wiener process, with unitary variance, and  $\gamma$ ,  $\theta$  and  $\delta$  are positive real parameters. This SDE, first studied by Feller [11], is well known in mathematical finance. It was employed by Cox, Ingersoll and Ross to model short-term interest rates [12] and later became popular in mimicking stochastic volatility, like in the

Heston model for price dynamics [13]. Mean-reverting square-root diffusion has also been considered in other contexts, such as in modeling neural spiking [14] or in problems of biological diffusion [15].

Still in the context of finance, in the first approximation, equation (1) describes the dynamics of share trading volumes, although the tails of the empirical distributions deviate from the steady solution associated with the SDE (1):

$$P_s(x) = \mathcal{N} x^{2\gamma\theta/\delta^2-1} \exp(-2\gamma x/\delta^2), \quad (2)$$

which is the Gamma PDF [11], with  $\mathcal{N}$  a normalization constant. Fluctuations that can explain the observed deviations within the superstatistical mixing framework [9] have been detected in the parameter directly related to the mean value [8]. The resulting PDF, known as  $q$ -Gamma, is a generalization of the Gamma distribution that can be cast into the form of the  $F$  distribution and which basically turns the exponential tail into a power-law one. It has been shown to be in excellent agreement with empirical observations at different granularity timescales (from 1 min to days) [8, 9, 16, 17].

The study of complex systems often encompasses the analysis of the probability function of the addition of stochastic observables, mostly to appraise the hypothesis of scale invariance. Precisely, in respect of this, the fact that the description of empirical volume PDFs in terms of  $q$ -Gamma distributions applies at different aggregation scales is particularly interesting, especially because, unlike the Gamma distribution, the  $q$ -Gamma is neither closed under convolution nor correlations can be fully neglected.

To understand these observations motivates the present work. Although our initial motivation comes from an econophysical problem, the present results may be of interest for a wider scenario where linear diffusion applies, as soon as parameter fluctuations are ubiquitous.

This paper is organized as follows. We first summarize the pertinent results related to the Fokker–Planck equation (FPE) associated with the stochastic process (1). We apply these results to obtain the PDFs resulting from an accumulation process. Thereafter, we apply the statistical composition procedure, in which the reverting mean is the fluctuating parameter. We obtain joint distributions that allow us to characterize the correlation structure as well as the aggregation properties. Finally, we apply the analytical results to interpret the granular features of real-time series of trading volumes.

## 2. Primary process

The forward Fokker–Planck equation (FPE) associated with equation (1), for the conditional probability  $P \equiv P(x, t|x', t')$ , is

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x} (\gamma[x - \theta]P) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (\delta^2 x P). \quad (3)$$

The propagator of this FPE has been obtained by Feller [11]. Since then, the time-dependent solution has been systematically overlooked. For instance, in stochastic volatility models like the Heston model and its variants, the price is the quantity of interest [13, 18], while the volatility, modeled by equation (1), is only an auxiliary quantity. Wherefore, it is usually integrated out by considering a distribution of the (initial) volatility equal to the stationary solution. Thence, if we want to go further, we must look for the time-dependent solutions. With that goal in mind, we summarize the procedure

for obtaining the propagator, first presented by Feller [11], that embodies the definitions and the partial results which are going to be useful in the following sections.

Laplace-transforming equation (3), one gets

$$\frac{\partial \tilde{P}}{\partial t} = - \left( \gamma + \frac{\delta^2}{2} w \right) \frac{\partial \tilde{P}}{\partial w} - \gamma \theta w \tilde{P}, \tag{4}$$

where  $\tilde{P} \equiv \tilde{P}(w, t|x', t')$ . With the initial condition  $P(x, t|x', t') = \delta(x - x')$ , whose Laplace transform is  $\tilde{P}(w, t|x', t') = \exp(-w x')$ , the solution of equation (4), which can be obtained by the method of characteristics, is

$$\tilde{P}(w, t|x', t') = \frac{\exp(-Aw/(1+Bw))}{(1+Bw)^\beta} = \frac{\exp(-(A/B)[1 - (1/(1+Bw))])}{(1+Bw)^\beta}, \tag{5}$$

where we have defined

$$\begin{aligned} \Theta &\equiv \exp(-\gamma[t - t']), \\ A &\equiv x'\Theta, \\ B &\equiv B_0[1 - \Theta] \equiv \frac{\delta^2}{2\gamma}[1 - \Theta], \\ \beta &\equiv \frac{2\gamma\theta}{\delta^2} = \frac{\theta}{B_0}. \end{aligned}$$

In the long-time limit  $\Delta t \equiv (t - t') \gg 1/\gamma$  (hence  $\Theta \rightarrow 0$ ), equation (5) becomes

$$\tilde{P}(w, t|x', t') = \frac{1}{(1+B_0w)^\beta}, \tag{6}$$

whose inverse Laplace transform gives us the steady solution in  $x$  space, i.e.

$$P_s(x) = \mathcal{L}^{-1} \left( \frac{1}{(1+B_0w)^\beta} \right) = \frac{x^{\beta-1} \exp(-x/B_0)}{B_0^\beta \Gamma(\beta)}, \tag{7}$$

for nonnegative  $x$  and zero otherwise. This is the Gamma (or Erlang) distribution,  $\Gamma_{\beta, B_0}$  [19].

For any  $\Delta t$ , the conditional PDF  $P(x, t|x', t')$  can be obtained by first expanding the exponential in equation (5) and then performing the mappings  $\beta \rightarrow n + \beta$  and  $B_0 \rightarrow B$  in equation (7). That is

$$P(x, t|x', t') = \mathcal{L}^{-1}(\tilde{P}(w, t|x', t')) \tag{8}$$

$$= \sum_{n \geq 0} \frac{\exp(-A/B)(A/B)^n}{n!} \mathcal{L}^{-1} \left( \frac{1}{(1+Bw)^{n+\beta}} \right) \tag{9}$$

$$= \sum_{n \geq 0} \frac{\exp(-A/B)(A/B)^n}{n!} \frac{x^{n+\beta-1} \exp(-x/B)}{B^{n+\beta} \Gamma(n+\beta)} \tag{10}$$

$$= \frac{x^{(\beta-1/2)} \exp(-(A+x)/B)}{BA^{(\beta-1/2)}} I_{\beta-1} \left( \frac{2\sqrt{Ax}}{B} \right) \tag{11}$$

$$= \left( \frac{x}{x'\Theta} \right)^{(\beta-1)/2} \frac{\exp(-(x+x'\Theta)/B_0[1-\Theta])}{B_0[1-\Theta]} I_{\beta-1} \left( \frac{2\sqrt{xx'\Theta}}{B_0[1-\Theta]} \right), \tag{12}$$

where  $I_n(x)$  is the  $n$ th-order modified Bessel function of the first kind [20].

In the limit  $\gamma\Delta t \gg 1$ , equation (12) tends to the stationary PDF (7), which is also obtained by performing the integration  $\int dx' P(x, t|x', t')P_s(x')$ . Still in the steady state, the two-time joint PDF is

$$P(x, t; x', t') = P(x, t|x', t')P_s(x') \quad (13)$$

$$= \frac{(xx')^{(\beta-1)/2} \exp(-(x+x')/B_0[1-\Theta])}{\Gamma(\beta)B_0^{\beta+1}[1-\Theta]\Theta^{(\beta-1)/2}} I_{\beta-1} \left( \frac{2\sqrt{xx'\Theta}}{B_0[1-\Theta]} \right), \quad (14)$$

which in the long-term limit ( $\Theta \rightarrow 0$ ), of course, becomes the product of the stationary PDFs,  $P_s(x)P_s(x')$ .

## 2.1. Aggregation

Once the propagator  $P(x, t|x', t')$  is known, given by equation (12), and assuming stationarity and Markovianity, we can determine the  $N$ -time joint PDF:

$$P(x_1; \dots; x_N) \equiv P(x_1, t_1; \dots; x_N, t_N) = \prod_{i=1}^{N-1} P(x_{i+1}, t_{i+1}|x_i, t_i)P_s(x_1). \quad (15)$$

In equation (15), we can consider elements generated by equation (1) that are equally spaced in time, such that  $t_i = (i-1)\Delta t$ , for  $1 \leq i \leq N$ . Then, we can evaluate the resulting stationary PDF of  $X = \sum_{i=1}^N x_i$ :

$$P_N(X) = \int \dots \int dx_1 \dots dx_N \delta \left( X - \sum x_i \right) P(x_1; \dots; x_N). \quad (16)$$

For arbitrary  $N \geq 2$ , when  $\Delta t \gg 1/\gamma$  ( $\Theta \rightarrow 0$ ),  $P_N(X)$  must tend to the  $N$ -fold convolution of the Gamma distribution (7). Since it is closed under convolution, we have

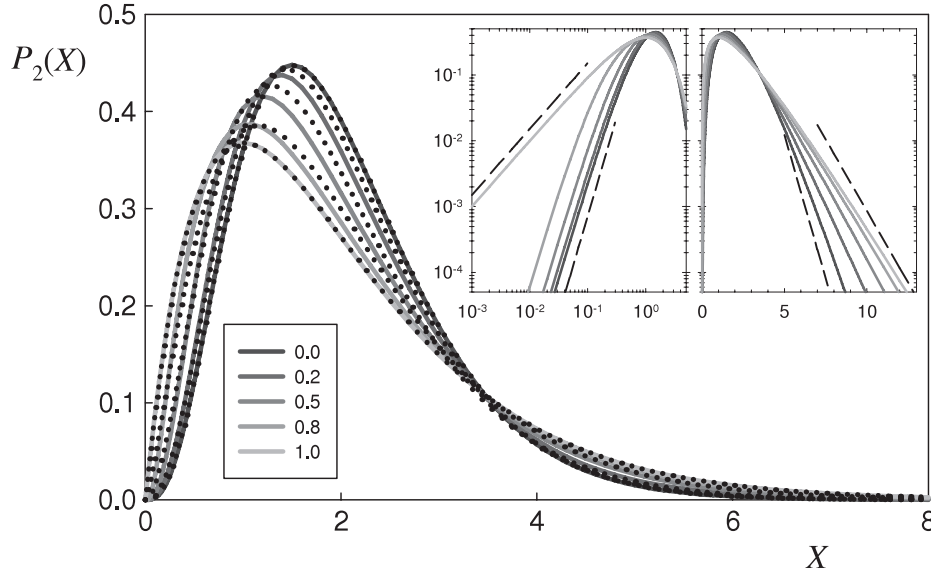
$$\lim_{\Delta t \rightarrow \infty} P_N(X) = \frac{X^{N\beta-1} \exp(-X/B_0)}{B_0^{N\beta} \Gamma(N\beta)}. \quad (17)$$

In the opposite limit  $\Delta t \ll 1/\gamma$  ( $\Theta \rightarrow 1$ ):

$$\lim_{\Delta t \rightarrow 0} P_N(X) = \frac{1}{N} P_1 \left( \frac{X}{N} \right) = \frac{X^{\beta-1} \exp(-X/NB_0)}{(NB_0)^\beta \Gamma(\beta)}. \quad (18)$$

That is, in both limits, the sum of adjacent variables is Gamma-distributed, although with different values of the parameters. For intermediate degrees of correlation, following the behavior of the Bessel factor for small and large values (power law and exponential, respectively)<sup>3</sup>, the PDF of the sum is expected to remain close in shape to the Gamma distribution, i.e. growing as a power law at the origin and decaying asymptotically with an exponential tail. The extreme cases suggest that, as the number  $N$  of added variables accrues, the power-law exponent increases as well: the less, the more correlated the aggregated variables are. Concomitantly, the exponential tail decays more slowly: the slower, the larger the correlations.

<sup>3</sup> The asymptotic behaviors of the modified Bessel function are:  $I_n(x) \rightarrow (x/2)^n/\Gamma(n+1)$ , for  $x \ll \sqrt{n+1}$  and  $I_n(x) \sim e^x/\sqrt{2\pi x}$ , for large  $x \gg n^2 - 1/4$ .



**Figure 1.** PDF of the aggregation of two consecutive values, given by equation (19), for  $(\beta, B_0) = (2, 0.5)$  and different values of  $\Theta$  (hence  $\Delta t$ ) indicated in the figure. Dotted lines correspond to fittings with the Gamma distribution (for qualitative assessment) that exactly coincide in the extreme cases  $\Theta = 0, 1$ . Insets show the PDFs in logarithmic scales to better display the tails. Dashed lines are drawn for comparison and correspond to  $X^{\beta-1}$  and  $X^{2\beta-1}$  in the left panel and to  $\exp(-X/B_0)$  and  $\exp(-X/(2B_0))$  in the right-hand panel, which follow the asymptotic behaviors of the extreme cases  $\Theta = 0$  and  $1$ , respectively, while intermediate cases are ruled by  $\exp(-X/[(\sqrt{\Theta} + 1)B_0])$ .

Let us analyze more carefully the particular case  $N = 2$ . Probability distribution (16) is explicitly

$$P_2(X) = \frac{\sqrt{\pi} X^{\beta-1/2} \exp(-X/B_0[1 - \Theta]) I_{\beta-1/2}(\sqrt{\Theta} X/B_0[1 - \Theta])}{\Gamma(\beta) B_0^{\beta+1/2} [1 - \Theta]^{1/2} (2\sqrt{\Theta})^{\beta-1/2}}. \quad (19)$$

Taking into account the asymptotic behavior of the Bessel function, we can make the following observations. For  $\Theta \rightarrow 0$ , i.e. approaching independence,  $P_2(X) \sim X^{\beta-1/2} \exp(-X/B_0[1 - \Theta]) X^{\beta-1/2} = X^{2\beta-1} \exp(-X/B_0[1 - \Theta])$ . For small enough  $X$  the distribution goes to zero as  $P_2(X) \sim X^{2\beta-1}$  and exponentially decays with a characteristic constant value equal to  $B_0[1 - \Theta]$ . This behavior changes when we approach the full-dependence case, the functional form of which is  $P_2(X) \sim X^{\beta-1/2} \exp(-X/B_0[1 - \Theta]) \exp(\sqrt{\Theta} X/B_0[1 - \Theta]) X^{-1/2} = X^{\beta-1} \exp(-X/B_0[1 + \sqrt{\Theta}])$ . Therefore, the limit  $\Theta \rightarrow 1$  yields for small  $X$ ,  $P_2(X) \sim X^{\beta-1}$  which, despite being a power law, has got a different exponent. The large  $X$  decay keeps its exponential form but with a different constant of decay equal to  $B_0[1 + \sqrt{\Theta}]$ . The behavior of equation (19) between the two limiting cases is shown in figure 1.

It is noteworthy that, for arbitrary  $\Delta t$ , the distribution (19) approaches the Gamma distribution (which is the exact distribution in the extreme cases) at least for a few decades. On the one hand, the behavior near the origin is the same as for  $\Delta t \rightarrow \infty$ . On

the other hand, an effective parameter for the exponential decay can be found to adjust the tails since the true asymptotic behavior is attained after many decades only.

## 2.2. Correlations and moments

The joint probability density given by equation (14) allows evaluation of moments and two-time auto-correlation functions. To that end, it is useful to calculate

$$\tilde{C}_{nm} \equiv \langle x^n x'^m \rangle = \int \int dx dx' P(x; x') x^n x'^m,$$

which is explicitly

$$\tilde{C}_{nm} = \frac{\Gamma(\beta + m) \Gamma(\beta + n) B_0^{m+n}}{\Gamma(\beta)^2} [1 - \Theta]^{\beta+m+n} {}_2\mathcal{F}_1(\beta + m, \beta + n, \beta, \Theta). \quad (20)$$

In particular, for any  $\gamma\Delta t$ , the (centered) linear auto-correlation is (see also [18])

$$C_{11} = \tilde{C}_{11} - \tilde{C}_{10}\tilde{C}_{01} = \beta\Theta B_0^2. \quad (21)$$

From equation (20), higher-order correlations behave as

$$\tilde{C}_{nm} \simeq \frac{\Gamma(\beta + m) \Gamma(\beta + n) B_0^{m+n}}{\Gamma(\beta)^2} \left(1 - \frac{mn}{\beta} \Theta\right), \quad (22)$$

for large  $\gamma\Delta t$ , at first order in  $\Theta$ . Meanwhile, for small  $\gamma\Delta t$ , at first order in  $1 - \Theta$ , one has

$$\tilde{C}_{nm} \simeq \frac{\Gamma(\beta + m + n) B_0^{m+n}}{\Gamma(\beta)^2} \left(1 - \frac{mn(1 - \Theta)}{\beta + m + n - 1}\right). \quad (23)$$

Hence, correlations of any order decay like a single exponential function, with characteristic time  $1/\gamma$ .

However, a different behavior occurs if  $\gamma$  fluctuates, within the framework that we will be consider in section 3. In particular, a power-law decay of the correlation function is obtained if one assumes that the PDF of  $\gamma$  decays exponentially [21]. In such a scenario it should be stressed that our stationary distributions remain the same because they do not depend on the parameter  $\gamma$ .

From equation (20), the statistical raw moments can be directly obtained as

$$\langle x^n \rangle = \tilde{C}_{n0} = \frac{\Gamma(\beta + n)}{\Gamma(\beta)} B_0^n. \quad (24)$$

Hence the first centered moments are

$$\langle (x - \langle x \rangle)^2 \rangle = \beta B_0^2, \quad \langle (x - \langle x \rangle)^3 \rangle = 2\beta B_0^3.$$

Concerning aggregation, for  $N = 2$ , from equation (19) and using the properties in [20], statistical moments  $\langle X^n \rangle$  are given by

$$\langle X^n \rangle = \frac{\Gamma(2\beta + n)}{\Gamma(2\beta)} B_0^n [1 - \Theta]^{\beta+n} {}_2\mathcal{F}_1\left(\beta + \frac{n}{2}, \beta + \frac{1}{2}(n + 1), \beta + \frac{1}{2}, \Theta\right). \quad (25)$$



The first raw moments are

$$\begin{aligned}\langle X \rangle &= 2\beta B_0, \\ \langle X^2 \rangle &= 2\beta(2\beta + 1 + \Theta)B_0^2, \\ \langle X^3 \rangle &= 4\beta(\beta + 1)(2\beta + 1 + 3\Theta)B_0^3\end{aligned}\quad (26)$$

from where the first centered moments are

$$\langle (X - \langle X \rangle)^2 \rangle = 2\beta(1 + \Theta)B_0^2, \quad (27)$$

$$\langle (X - \langle X \rangle)^3 \rangle = 4\beta(1 + 3\Theta)B_0^3. \quad (28)$$

Notice the increase in the moments with  $\Theta$  due to the longer exponential tails.

For arbitrary  $N$ , in the limit of vanishing  $\Theta$ , the raw moments can also be obtained from the moment generating function  $M(z) = (1 - B_0 z)^{-N\beta}$  for any  $N \geq 1$ .

### 3. Statistical mixing

Let us now reckon that, instead of being constant,  $B_0$  evolves stochastically, independent of  $x$ , along the times series of the primary process, for which the time lag between successive points is  $\Delta t$ . In agreement with previous observations [8],  $\eta = B_0^{-1}$  can be assumed Gamma-distributed with parameters  $(\alpha, 1/\kappa_0)$ . Moreover, we consider the update scale of  $B_0$  (within which the parameter remains basically constant) much larger than the timescale  $1/\gamma$ , so that the average over different samples is obtained by performing the mixture:

$$P^{(M)}(\dots) = \int d\eta P(\dots|\eta)P(\eta). \quad (29)$$

In particular, we can determine the mixed joint distribution  $P^{(M)}(x, t; x', t')$ , which, after integration, is

$$\begin{aligned}P^{(M)}(x, t; x', t') &= \frac{\Gamma(\alpha + 2\beta)}{\Gamma(\alpha)\Gamma(\beta)^2} \frac{\kappa^\alpha (xx')^{\beta-1} [1 - \Theta]^\beta}{(\kappa + x + x')^{\alpha+2\beta}} \\ &\times {}_2\mathcal{F}_1\left(\frac{\alpha}{2} + \beta, \frac{1}{2} + \frac{\alpha}{2} + \beta, \beta, \frac{4xx'\Theta}{(\kappa + x + x')^2}\right),\end{aligned}\quad (30)$$

where  $\kappa = \kappa_0[1 - \Theta]$ . Subsequent integration over  $x'$  allows us to obtain the global distribution of  $x$  (locally stationary) in the mixing process:

$$P^{(M)}(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\kappa_0^\alpha x^{\beta-1}}{(\kappa_0 + x)^{\alpha+\beta}}. \quad (31)$$

This PDF is a generalization of the Gamma distribution (recovered in the limit  $\alpha \rightarrow \infty$ , while  $\kappa_0/\alpha$  is kept finite), known as  $q$ -Gamma. After suitable rescaling, it can also be cast into the form of an  $F$  distribution for which non-integer degrees of freedom are permitted.

It is worth noticing that, even in the limit  $\Delta t \gg 1/\gamma$  (independence):

$$\lim_{\Delta t \rightarrow \infty} P^{(M)}(x; x') = \frac{\Gamma(\alpha + 2\beta)}{\Gamma(\alpha)\Gamma(\beta)^2} \frac{\kappa_0^\alpha (xx')^{\beta-1}}{(\kappa_0 + x + x')^{\alpha+2\beta}} \quad (32)$$



is different from the product  $P^{(M)}(x)P^{(M)}(x')$ . Nonetheless, this does not necessarily correspond to a case of correlated variables  $x$  and  $x'$ . Bearing in mind the multivariate Student- $t$  distribution [22], we can understand our equation (32) as a bivariate  $F$  distribution of uncorrelated variables.

### 3.1. Aggregation of mixed variables

Performing the mixing operation given by equation (29) on equation (19), after integration, we obtain

$$P_2^{(M)}(X) = \frac{\Gamma(\alpha + 2\beta)}{\Gamma(\alpha)\Gamma(2\beta)} \frac{[1 - \Theta]^\beta \kappa^\alpha X^{2\beta-1}}{(\kappa + X)^{\alpha+2\beta}} \times {}_2F_1\left(\frac{\alpha}{2} + \beta, \frac{1}{2} + \frac{\alpha}{2} + \beta, \frac{1}{2} + \beta, \frac{\Theta X^2}{(\kappa + X)^2}\right), \quad (33)$$

which, in the independence limit  $\Delta t \gg 1/\gamma$  (but still smaller than any characteristic time of the mixing process), evolves into

$$\lim_{\Delta t \rightarrow \infty} P_2^{(M)}(X) = \frac{\Gamma(\alpha + 2\beta)}{\Gamma(\alpha)\Gamma(2\beta)} \frac{\kappa_0^\alpha X^{2\beta-1}}{(\kappa_0 + X)^{\alpha+2\beta}}. \quad (34)$$

This result can be easily generalized to  $N$  terms by iterated convolution operations:

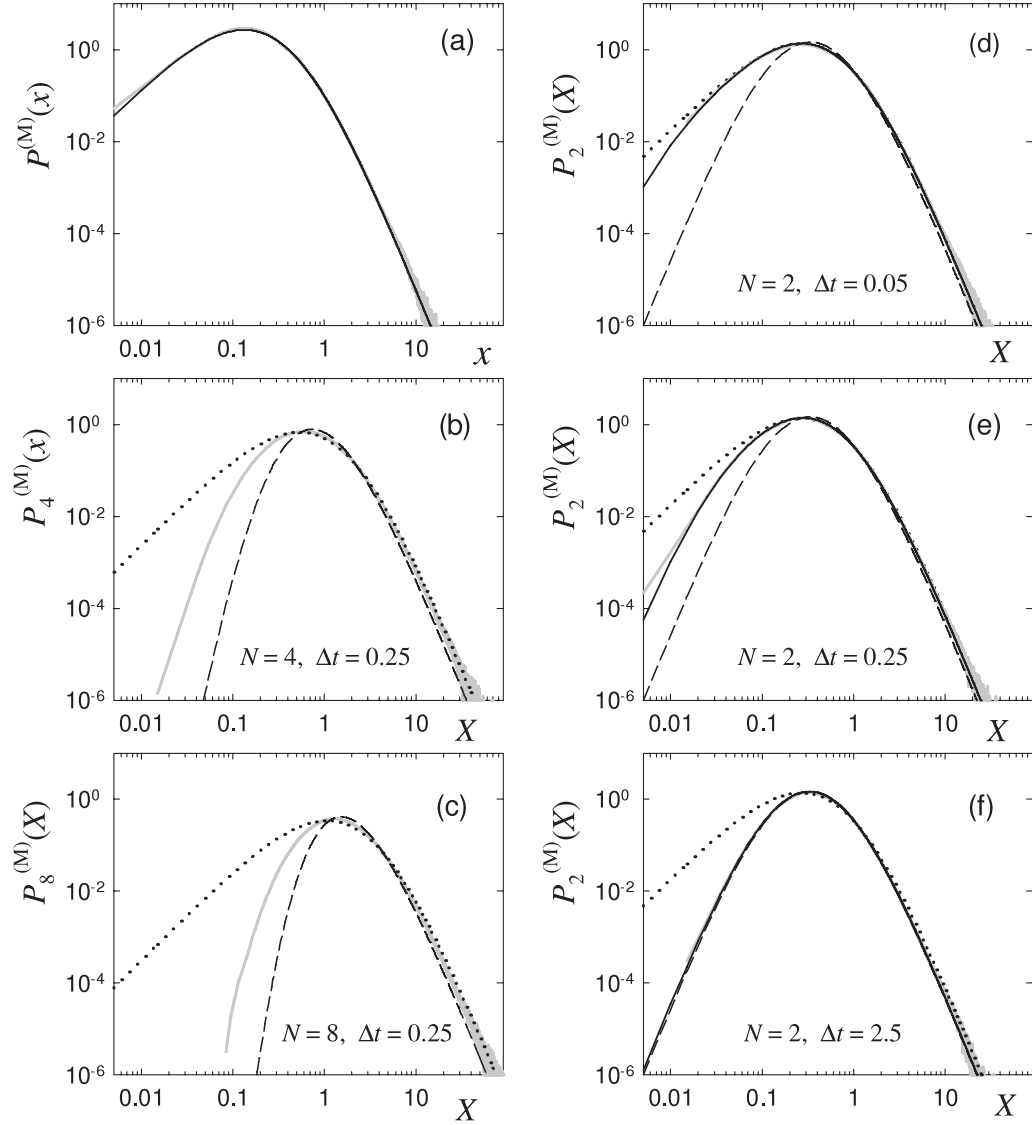
$$\lim_{\Delta t \rightarrow \infty} P_N^{(M)}(X) = \frac{\Gamma(\alpha + N\beta)}{\Gamma(\alpha)\Gamma(N\beta)} \frac{\kappa_0^\alpha X^{N\beta-1}}{(\kappa_0 + X)^{\alpha+N\beta}}. \quad (35)$$

On the other hand, in the opposite limit  $\Delta t \ll 1/\gamma$ :

$$\lim_{\Delta t \rightarrow 0} P_N^{(M)}(X) = \frac{1}{N} P^{(M)}\left(\frac{X}{N}\right) = \frac{\Gamma(\alpha + \beta)}{N\Gamma(\alpha)\Gamma(\beta)} \frac{\kappa_0^\alpha (X/N)^{\beta-1}}{(\kappa_0 + X/N)^{\alpha+\beta}}. \quad (36)$$

Notice that in both limiting cases a  $q$ -Gamma arises, although with different exponents. This is a consequence of the mixing of the Gamma distributions that rule the respective extreme cases. For intermediate instances, the (effective) power-law exponent at the origin follows the same scaling relation as the unmixed Gamma distribution. Meanwhile, the exponent of the tail is insensitive to both  $\Delta t$  and  $N$ , conserving its value  $\alpha + 1$ , which only depends on the degree of inhomogeneities (given by  $\alpha$ ), as was empirically noticed in the application given in [8].

These behaviors are illustrated in figure 2 for numerical implementation of equation (1) with mixing. From panels (d)–(f) we verify that  $P_2^{(M)}(X)$  departs from equation (36) and approaches equation (35), as  $\Delta t$  swells and added variables become independent. As aggregation proceeds, i.e.  $N$  increases, the distribution shrinks below the maximum, departing from the fully dependent case towards the independent one. This is because the exponent of the power law at the origin follows the behavior of the independent case ruled by the  $N\beta$  exponent (a reflection of the behavior at the origin of the unmixed case as exemplified in figure 1) while, as  $X$  approaches the maximum, there is a crossover towards the independence  $\Delta t \rightarrow 0$  limit. The figure evidentially also shows that, in contrast, the tail exponent does not depend on  $N$  nor on  $\Delta t$ . Its changeless value  $\alpha + 1$  indicates that the manifestation of inhomogeneities stays invariant at the different accumulation scales.



**Figure 2.** PDFs of the mixing variable  $x$  (a) and aggregated variables  $X$  for different values of  $N$  and  $\Delta t$  indicated in (b)–(f). The (gray) lines join the points of the histograms built from numerical implementations of equation (1) with mixing. Numerical integration of the stochastic differential equation was performed by means of an Euler algorithm, with time step  $10^{-3}$ , and an update of  $1/B_0$ , drawn from  $\Gamma_{\alpha,1/\kappa_0}$ , was done at each  $\delta t = 100 \gamma^{-1}$ . Parameter values are  $\gamma = 1$ ,  $\alpha = 4$ ,  $\beta = 3$  and  $\kappa_0 = 1/3$ . In panel (a) the full line represents the global PDF given by equation (31). Panels (b)–(f) exhibit the global PDF of the addition of  $N = 2$  (d)–(f), 4 (b) and 8 (c) consecutive variables at each  $\Delta t$ , for the same process. For  $N = 2$ , the full line represents equation (33), the dotted line the limit  $\Delta t \rightarrow 0$  equation (36) and the dashed line the independence limit given by equation (35). All the plots are on the same log–log scale for comparison.

Analogously to the Gamma approximation for the aggregation of variables, discussed at the end of section 2.1, the  $q$ -Gamma distribution plays the same role after mixing. As a consequence, even when not the exact solution, the  $q$ -Gamma suits the PDF generated by the mixed Feller diffusion.

### 3.2. Correlations and moments after mixing

As the term ‘statistics of statistics’ suggests, the global statistical properties correspond to the averaging of the locally stationary statistical properties over the fluctuations. Therefore, central two-time correlation functions of  $x$  are

$$C_{nm}^{(M)} = \int \int dx dx' d\eta P(x, x') P(\eta) (x - \langle x \rangle)^n (x' - \langle x \rangle)^m = \int d\eta P(\eta) C_{nm}. \quad (37)$$

Assuming that  $P(\eta)$  follows a  $\Gamma(\alpha, 1/\kappa_0)$ , from equation (21), we have

$$C_{11}^{(M)} = \langle 1/\eta^2 \rangle \beta \Theta = \frac{\beta \Theta \kappa_0^2}{(\alpha - 1)(\alpha - 2)}, \quad (38)$$

which displays the same exponential decay as the unmixed process.

It is important to introduce two remarks. First, correlation functions and moments must not be computed for time differences  $\Delta t$  greater than the characteristic time in which the fluctuating parameter can be considered constant. Second, although we can feel enticed to compute the overall (mixing) statistical properties by integrating the variables using the  $P^{(M)}(\dots)$  weights, e.g.  $C_{11}^{(M)} = \langle xx' \rangle^{(M)} - \langle x \rangle^{(M)} \langle x' \rangle^{(M)}$ , this is wrong, since one must keep in mind the non-stationary nature of the stochastic process (due to the fluctuations in  $\eta$ ). Consequently, the statistical properties must first be computed locally and only afterward the parameter fluctuations taken into account. Otherwise, spurious results may come forth such as, for instance, centered correlations tending to a constant value different from zero in the long-time limit. For instance, one has

$$\tilde{C}_{11}^{(M)} \equiv \langle xx' \rangle^{(M)} = \frac{\beta(\beta + \Theta)\kappa_0^2}{(\alpha - 1)(\alpha - 2)}, \quad (39)$$

which for  $\Theta = 0$  is equal to  $\int \langle x \rangle \langle x' \rangle P(\eta) d\eta$ , as it should be according to the discussion above.

In particular, moments after mixing must also be computed by averaging over the statistics of  $\eta$ , i.e.  $\int (\dots) P(\eta) d\eta$ , the expressions (24)–(28) obtained in section 2.2 for the locally stationary process.

For  $N = 2$  aggregated variables, by integrating equation (25) over  $\eta$  with a  $\Gamma(\alpha, 1/\kappa_0)$  weight (which leads to  $\langle B_0^n \rangle = \kappa_0^n \Gamma(\alpha - n) / \Gamma(\alpha)$  for  $n < \alpha$ ), one obtains

$$\langle X^n \rangle^{(M)} = \frac{\Gamma(2\beta + n)\Gamma(\alpha - n)}{\Gamma(2\beta)\Gamma(\alpha)} \kappa_0^n [1 - \Theta]^{\beta+n} {}_2F_1 \left( \beta + \frac{n}{2}, \beta + \frac{1}{2}(n + 1), \beta + \frac{1}{2}, \Theta \right).$$

Notice that only moments with order  $n < \alpha$  are defined. The first raw moments (which can also be obtained by performing the mixing directly on equations (26)) are

$$\langle X \rangle^{(M)} = \frac{2\beta\kappa_0}{(\alpha - 1)}, \quad (40)$$

$$\langle X^2 \rangle^{(M)} = \frac{2\beta(2\beta + 1 + \Theta)\kappa_0^2}{(\alpha - 1)(\alpha - 2)}, \quad (41)$$

$$\langle X^3 \rangle^{(M)} = \frac{4\beta(\beta + 1)(2\beta + 1 + 3\Theta)\kappa_0^3}{(\alpha - 1)(\alpha - 2)(\alpha - 3)}. \quad (42)$$

Finally, for the centered moments, by integration of equations (27) and (28) over the fluctuations of  $\eta$ , one finds

$$\begin{aligned} \langle (X - \langle X \rangle)^2 \rangle^{(M)} &= \frac{2\beta(1 + \Theta)\kappa_0^2}{(\alpha - 1)(\alpha - 2)}, \\ \langle (X - \langle X \rangle)^3 \rangle^{(M)} &= \frac{4\beta(1 + 3\Theta)\kappa_0^3}{(\alpha - 1)(\alpha - 2)(\alpha - 3)}. \end{aligned}$$

Notice that, in the limit of  $\alpha, \kappa_0 \rightarrow \infty$  with  $\kappa_0/\alpha = B_0$  constant (that is, when the Gamma distribution of the fluctuating parameter tends to a Dirac  $\delta$  centered at  $1/B_0$ ), one recovers the unmixed moments and correlations.

The case  $\Theta \rightarrow 0$ , i.e. the addition of independently  $F$ -distributed variables ( $N = 2$ ), has been studied before from a pure statistics perspective [23]–[25]. In these cases the resulting distribution was approximated to the  $F$  distribution by imposing statistical moments matching. This procedure could be extended to arbitrary  $\Theta$  by means of the above expressions for the lowest-order moments.

#### 4. Application to traded volume in financial markets

Although the largest part of the work carried out on financial markets is devoted to the (log-)price fluctuations and the volatility, it is recognized the essential role of the traded volume for a trustworthy characterization of a financial market global dynamics portrait [26]. As a matter of fact, the price evolves in time when a certain quantity of equities is negotiated.

So far as we are aware, the first studies on high-frequency traded volume were presented in [27] wherein asymptotic power-law decay of both the PDF and the auto-correlation have been held. Shortly after, another study on the traded volume PDF was presented [16], but at that time the entire span of the traded volume values was taken into account and the PDFs under analysis were very well adjusted by  $q$ -Gamma distributions. This observation holds both for consolidated highly liquid stock markets [9, 16, 17] (NYSE and NASDAQ) and for emerging ones like the Brazilian [8] and the Chinese [28]. This fact indicates universality of the functional form of the distribution function, at least approximately, and therefore of the underlying dynamical mechanism independent of the size of the market.

We shall now investigate the applicability of the dynamical scenario presented in the preceding sections to model stock traded volumes. We have analyzed two different

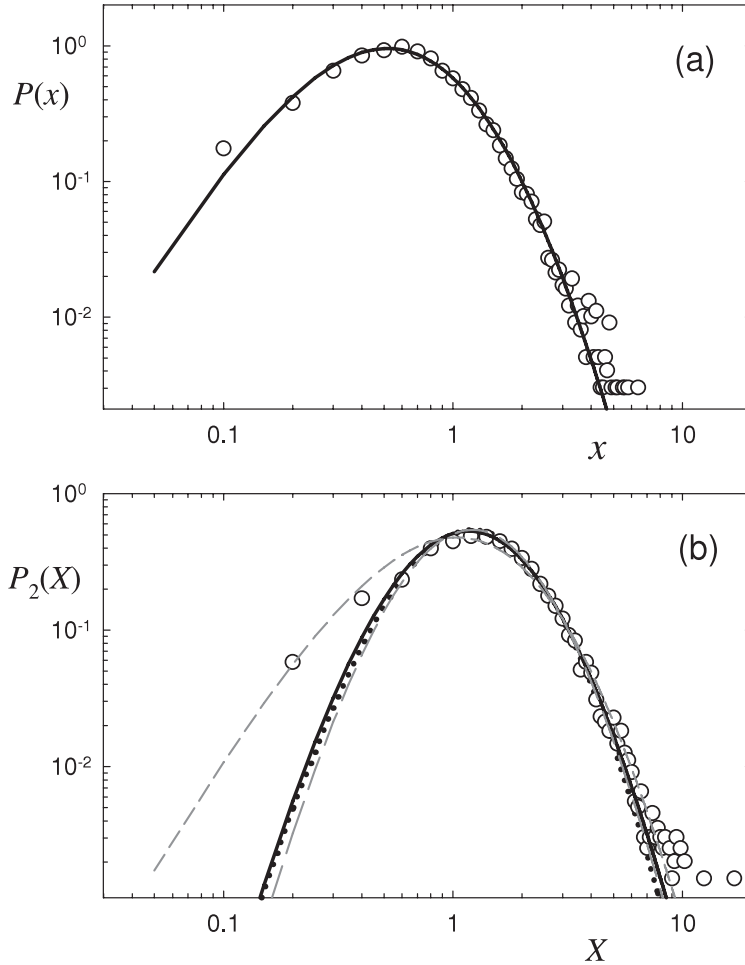
paradigmatic examples: (i) the total volumes traded in the emerging Brazilian stock market BOVESPA (a total of 9970 observations, recorded at intervals of 30 min, spanning the period from 3 January 2005 to 13 September 2007) and (ii) the 1 min records of Pfizer (PFE) traded volume at the New York Stock Exchange between 1 July 2004 and 31 December 2004 in a total of 49 585 registered values. During the respective period each market can be considered in a regular state, i.e. neither a crash nor other extreme behavior was earmarked. Notwithstanding, future work should shed light on the traded volume dynamics and its connection with the theory that predicts log-oscillatory behavior for the price [29, 30] in the advent of a crash. It should be noted that crashes are empirically associated with herding phenomena between agents and huge traded volumes [31]. Accordingly, the theory of log-oscillations must reflect the emergence of a new dynamics enhancing large values of  $\theta$  and vice versa. Our assumption is supported by prior empirical financial studies of daily time series which found an increase in trading volume over the six months prior to a price plunge [32].

We start by determining the values of the set of parameters  $\alpha, \beta, \kappa_0$ , by adjusting the traded volume empirical distribution at the lowest-time resolution of the data, that will be considered the unit timescale, in each case.

For the Brazilian market, whose lowest scale is 30 min, the empirical distribution of 30 min stock volumes (hence  $\Delta t = 1$ ) is depicted in figure 3(a). From the nonlinear regression procedure (minimization of  $\chi^2$  error leading to the optimization of the parameter correlation matrix) we obtained  $(\alpha, \beta, \kappa_0) = (7.11 \pm 0.32, 3.90 \pm 0.04, 1.45 \pm 0.01)$  (the respective Kolmogorov–Smirnov distance,  $D_{KS}$ , to the empirical probability distribution is equal to 0.016,  $\chi^2 = 0.0380$ ,  $R^2 = 0.994$ ). Since the very small volume regime ( $x \lesssim 0.1$ ) is ruled by a different mechanism [8], it was not considered in the fitting procedures.

In order to model the empirical distribution of the 1 h traded volume, through equation (33), we used the values of the parameters resulting from the numerical adjustment of the 30 min traded volume PDF, together with the value of  $\gamma$  obtained from the adjustment of the linear auto-correlation function with an exponential decay. Although the exponential does not describe well the correlations in long-term regimes, it can be considered as a good approximation for the short timescales of interest. From which we appraised  $\gamma = 1.5 \pm 0.03$ , since the characteristic exponential time decay was  $\tau = 20$  min and we adopted 30 min as the time unit.

Once we obtained the values of parameters  $(\alpha, \beta, \kappa_0, \gamma)$ , we compared the 1 h traded volume empirical PDF (symbols) with equation (33) (full curve), as shown in figure 3(b). We observe a fair agreement between them ( $D_{KS} = 0.037$ ), especially recalling that very small volumes should not be considered. However, notice that a simple nonlinear  $q$ -Gamma adjustment (dotted line) provides also a very good description, with parameters  $(\alpha, \beta, \kappa_0) = (7.97 \pm 0.35, 6.97 \pm 0.06, 1.82 \pm 0.01)$  [ $D_{KS} = 0.014$ ,  $\chi^2 = 0.0224$ ,  $R^2 = 0.997$ ] (dotted curve in figure 3(b)). This illustrates once more how the  $q$ -Gamma model, although approximate, appears to hold at different aggregation scales. Furthermore, the fitting we present fails to reject the null hypothesis for Pearson's statistical test with  $P = 0.05$ . Notice also that, although correlations are not completely negligible at these timescales, they do not play an important role in the resulting PDF, which is very close to the one that would be obtained by assuming independence (gray long dashed line).



**Figure 3.** (a) Empirical PDF of the BOVESPA 30 min traded volume  $x$  (circles) and best nonlinear regression result for equation (31) (full line), yielding  $(\alpha, \beta, \kappa_0) = (7.11, 3.90, 1.45)$ . (b) Empirical PDF of the BOVESPA 1 h traded volume,  $X$  (circles). The full line corresponds to equation (33),  $P_2^{(M)}(X = x + x')$ , with  $(\alpha, \beta, \kappa_0)$  as above and  $\gamma = 1.5$ . The dotted line represents a  $q$ -Gamma distribution with parameters  $(\alpha, \beta, \kappa_0) = (7.97, 6.97, 1.82)$  ( $D_{KS} = 0.014$ ) obtained by a nonlinear regression procedure. For comparison, the (gray) short and long dashed lines correspond to the limits of full dependence and independence, given by equations (36) and (34), respectively.

It is remarkable that volumes for a company in a developed market, recorded at high frequency as PFE, display the same qualitative features, despite correlations being stronger at those (high frequency) timescales (not shown; however, as an illustration see [16, 17]). In particular, from  $q$ -Gamma fitting, an almost constant value of  $\alpha$  is observed at the different scales. Moreover, PFE (in 2004), as well as the top 10 NASDAQ and NYSE stock volumes (in 2001), respectively display  $\alpha \simeq 4, 4$  and  $3$ , while  $\alpha \simeq 7-8$  for the Brazilian market, indicating a lower degree of inhomogeneities in the former case. Then  $\alpha$  constitutes an index to detect and quantify the level of inhomogeneities of a market or period.



## 5. Concluding remarks

From the time-dependent PDF of the SDE describing mean-reverting square-root diffusion, we derived the two-point (two-time) joint PDF in the steady state, as well as the PDF of the addition of variables generated according to this dynamical process. We further considered a scenario in which the mean-reverting term presents fluctuations that can be introduced twofold: they correspond to variations of the parameter either over runs or within the same run in a timescale much larger than the scale needed for the system to reach stationarity. Although our survey was mainly inspired by previous empirical findings of inhomogeneities in the traded volume flow corroborating a Gamma–Gamma phenomenological proposal, we uphold that a similar approach might be applied to the study of systems exhibiting inhomogeneous occurrence of Poisson events [33] or simply on problems for which the  $q$ -Gamma distribution has shown to be statistically relevant, like in granular media [34]. We have also discussed paradoxical results related to the non-commutativity of averaging and mixing operations.

In both limits of full independence ( $\Theta \rightarrow 0$ ) and full dependence ( $\Theta \rightarrow 1$ ), the PDF of the sum of consecutive variables is a  $q$ -Gamma distribution. We have shown that, in intermediate situations, for arbitrary degree of correlations, the upshot of aggregation at different scales is also well described by that distribution, although not the exact one. Moreover, while the increase at the origin is ruled by the independence behavior, the tail is governed by the degree of inhomogeneities which manifest at any aggregation scale. Then the  $q$ -Gamma form is approximately preserved.

We would like to stress that the model we have introduced explains why (i) even for  $\Theta \neq 0$ , the exponent  $\beta$  of the associated PDF increases by increasing the number of added variables as well and (ii) the tail exponent, which may be considered an indicator and quantifier of the presence of inhomogeneities, is preserved at different scales.

We have shown that all of this scenario applies in the specific case of traded volume within periods of regular trading behavior (i.e. the absence of bubbles/crashes) which represent the majority of trading history. Nonetheless, for the fallout of stock market crashes, a further survey based on appropriate data that bridges the trading volume dynamics with the occurrence of extreme episodes (namely allowing for the theory of log-oscillations [30]) is of manifest interest for an all-inclusive comprehension of financial markets dynamics. In particular, it would be interesting to understand the relation between the theory of log-oscillations and possible modifications in the description of the degree of inhomogeneities in a market (fluctuations in  $\theta$ ) that are depicted by the parameter  $\alpha$  during most of the trading records.

## Acknowledgments

We acknowledge BOVESPA for furnishing the data. CA is grateful to Brazilian agencies Faperj and CNPq for partial financial support and SMDQ acknowledges funding by the European Union's Marie Curie Fellowship program.

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