Relativistic motion and trajectories in static fields

Equations of Motion

$$\frac{d\boldsymbol{p}}{dt} = q[\boldsymbol{E} + c \quad \times \boldsymbol{B}] \qquad \boldsymbol{p} = mc\gamma \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\boldsymbol{v} = c \qquad \boldsymbol{E} = -\nabla\Phi \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}$$

Relativistic relations

$$W = \gamma mc^2 + q\Phi \qquad \gamma = \sqrt{1 + \left(\frac{p}{mc}\right)^2}$$

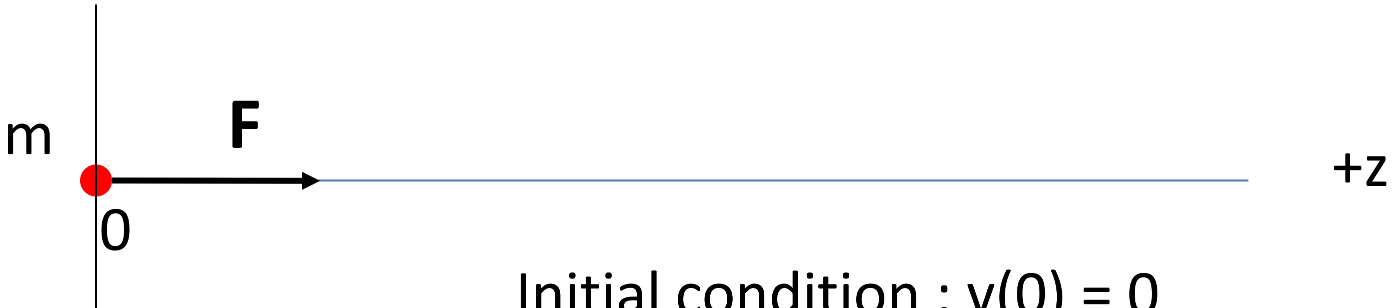
$$\gamma mc^2 = mc^2 + T$$

$$pc = \sqrt{T^2 - \left(mc^2\right)^2} \qquad pv = T\left[\frac{T + 2mc^2}{T + mc^2}\right]$$

W = Total energy T = Kinetic Energy

p = Relativisticc linear momentum v = Speed

Linear motion with constant force

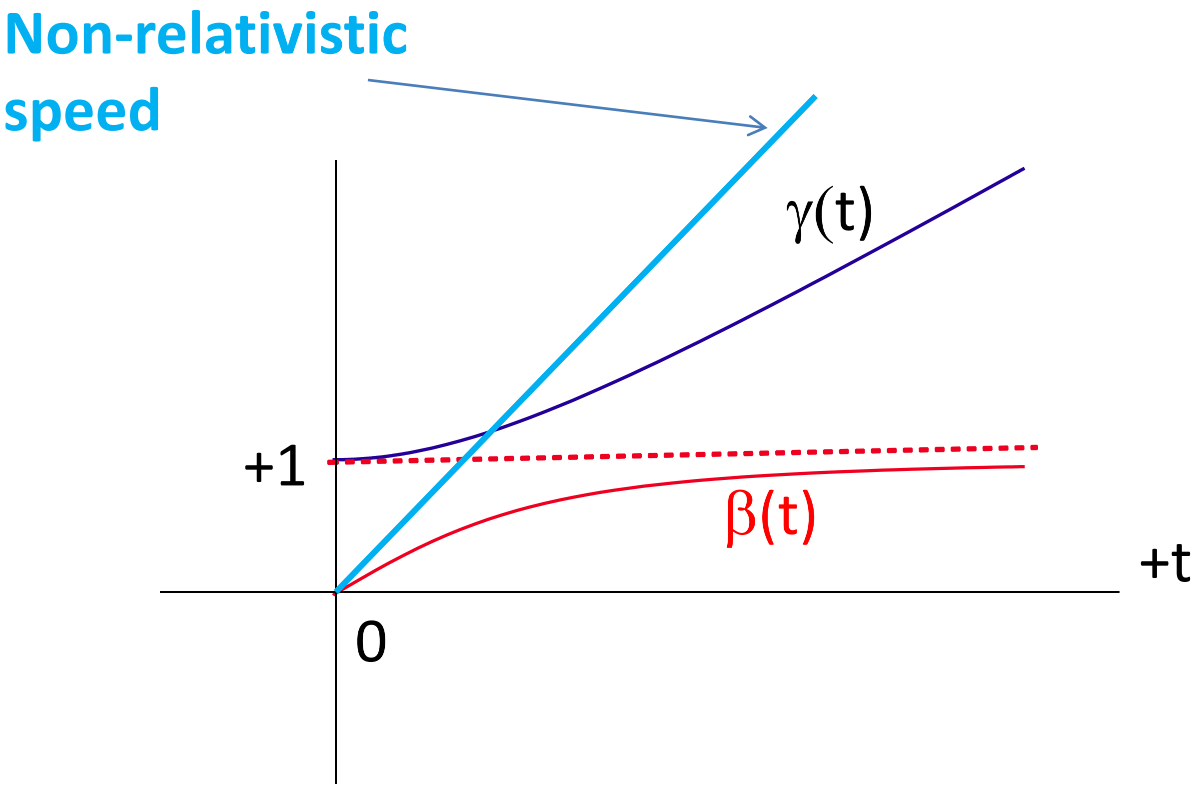


Initial condition : v(0) = 0

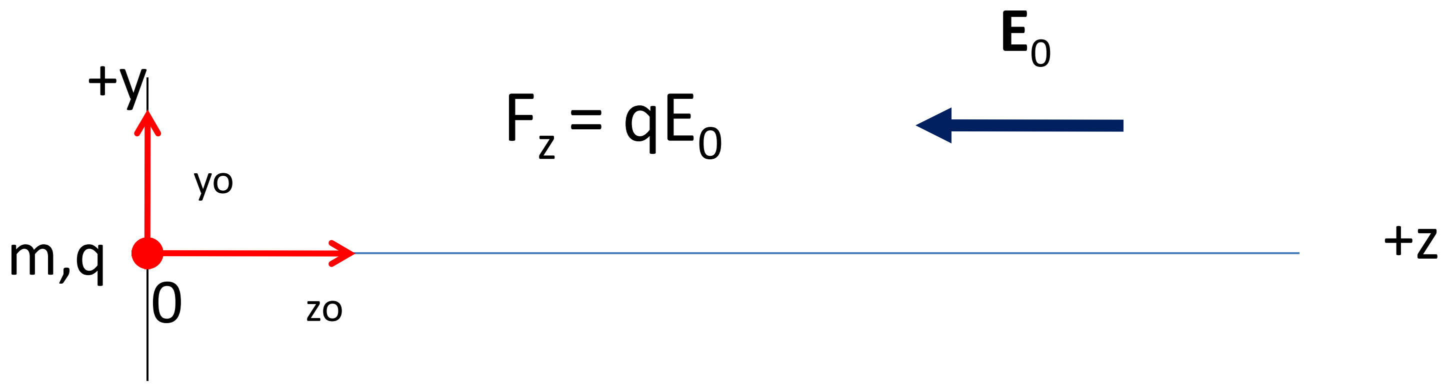
$$\dot{p} = F \qquad p = Ft = \gamma\beta mc = mc\frac{\beta}{\sqrt{1-\beta^2}}$$

$$\beta = \frac{t/\tau}{\sqrt{1+(t/\tau)^2}} \qquad \gamma = \sqrt{1+(t/\tau)^2} \qquad \tau = mc/F$$

Time evolution of speed and γmc^2



Linear motion with constant force



Initial conditions

$$v_y(0) = c\beta_y(0) = c\beta_{y0} \qquad v_z(0) = c\beta_z(0) = c\beta_{z0}$$

$$\gamma_0 = \frac{1}{\sqrt{1-(\beta_{y0}^2 + \beta_{z0}^2)}} \qquad \gamma_{z0} = \frac{1}{\sqrt{1-\beta_{z0}^2}}$$

$$p_{z0} = mc\gamma_0\beta_{z0} \qquad p_{y0} = mc\gamma_0\beta_{y0}$$

$$\dot{p}_z = qE_0 \quad \rightarrow \quad p_z = mc\gamma\beta_z = qE_0 t \quad \rightarrow \quad \gamma\beta_z = \frac{qE_0 t}{mc}$$

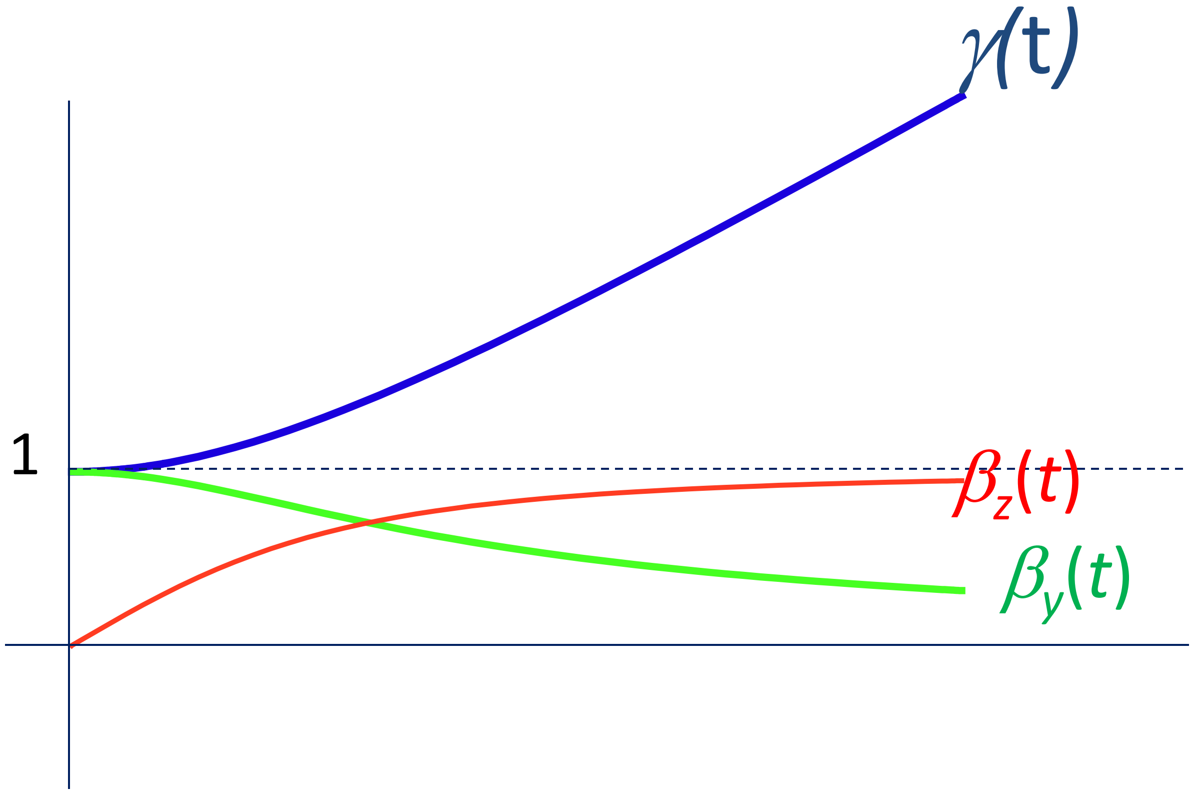
$$\dot{p}_y = 0 \quad \rightarrow \quad mc\gamma\beta_y = mc\gamma_0\beta_{y0} \quad \rightarrow \quad \beta_y = \frac{\gamma_0\beta_{y0}}{\gamma}$$

$$\gamma = \frac{1}{\sqrt{1-(\beta_z^2+\beta_y^2)}} = \frac{1}{\sqrt{1-\beta_z^2-\left(\frac{\gamma_0\beta_{y0}}{\gamma}\right)^2}} = \frac{\gamma_0}{\gamma_{z0}}\frac{1}{\sqrt{1-\beta_z^2}}$$

$$\frac{\beta_z}{\sqrt{1-\beta_z^2}} = \frac{\gamma_{z0}}{\gamma_0}\frac{qE_0 t}{mc} = t/\tau \quad \rightarrow \quad \beta_z = \frac{t/\tau}{\sqrt{1+\left(t/\tau\right)^2}} \qquad \tau = \frac{mc\gamma_0}{\gamma_{z0} qE_0}$$

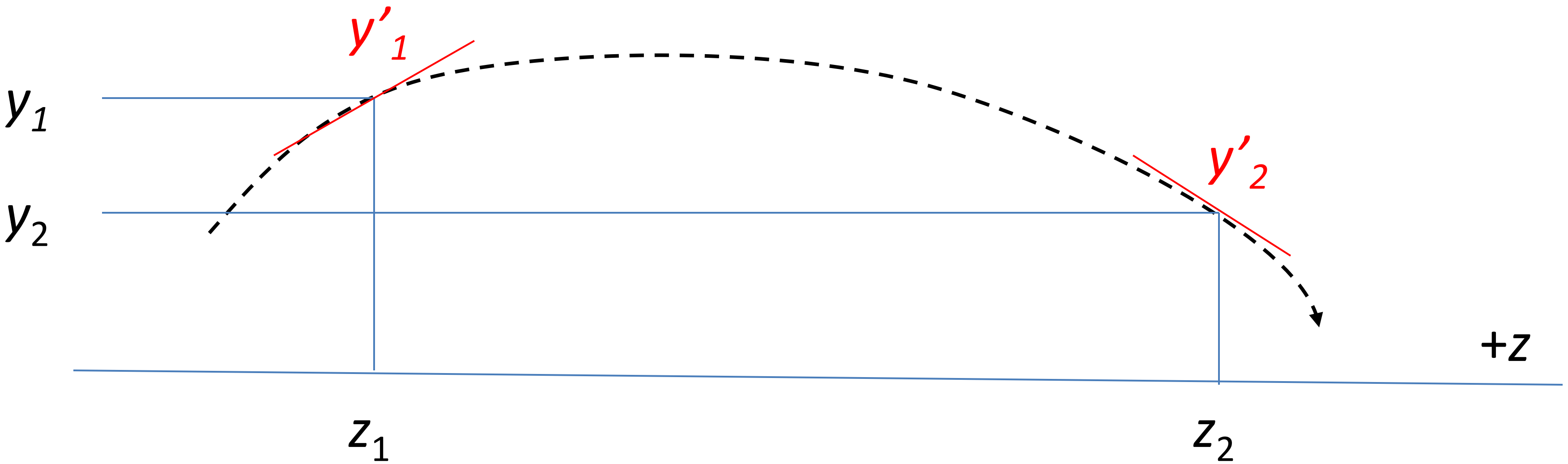
$$\gamma = \frac{\gamma_{z0}}{\gamma_0}\sqrt{1+\left(t/\tau\right)^2} \qquad \beta_y = \frac{\gamma_{z0}\beta_{y0}}{\sqrt{1+\left(t/\tau\right)^2}}$$

Time evolution



Trajectory Equations

Trajectory labeling



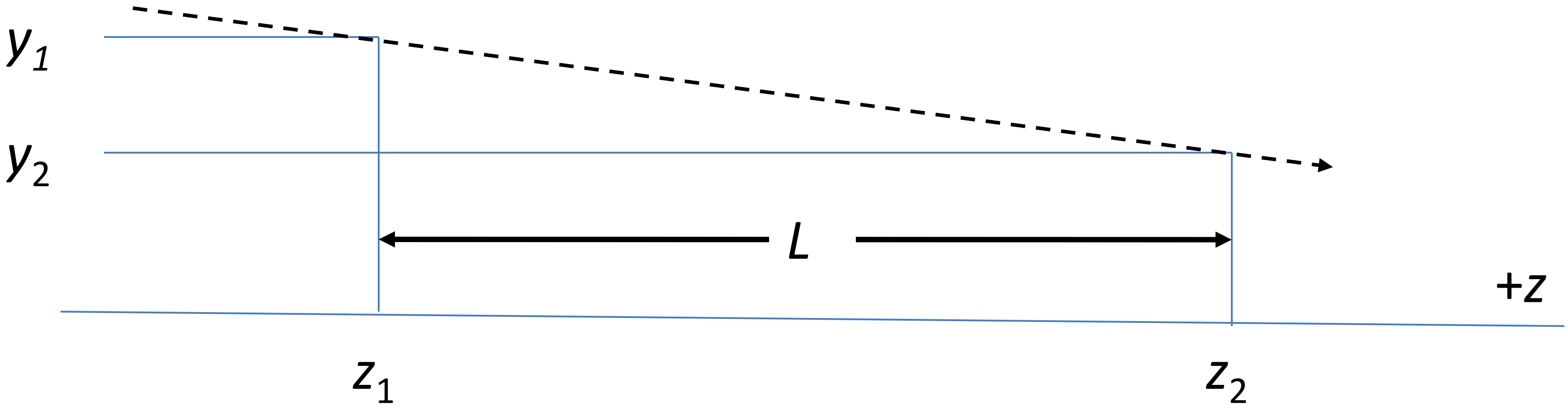
$$y_2 = R_{11}y_1 + R_{12}y'_1$$
$$y'_2 = R_{21}y_1 + R_{22}y'_1$$

Linear transformation

$$\begin{pmatrix} y_2 \\ y'_2 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y'_1 \end{pmatrix}$$

Matrix notation

No-force trajectory



$$\begin{pmatrix} y_2 \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_1' \end{pmatrix}$$

2D Cartesian Trajectories

$$v = \sqrt{v_y^2 + v_z^2} = v_z \sqrt{1 + y'^2} \qquad v_z = \frac{v}{\sqrt{1 + y'^2}} \qquad y' = \frac{dy(z)}{dz}$$

$$\frac{d}{dt} = v_z \frac{d}{dz} = \frac{v}{\sqrt{1 + y'^2}} \frac{d}{dz} \qquad \frac{dp_y}{dt} = -q \frac{\partial \Phi(y,z)}{\partial y}$$

$$\frac{dp_y}{dt} = \frac{v}{\sqrt{1 + y'^2}} \frac{d}{dz} \left[m\gamma\dot{y} \right] = \frac{mc^2\beta}{\sqrt{1 + y'^2}} \frac{d}{dz} \left[\frac{\gamma\beta y'}{\sqrt{1 + y'^2}} \right]$$

2D Trajectory Equation

$$y'' = \frac{1+y'^2}{pv} q[y' \frac{\partial \Phi}{\partial z} - \frac{\partial \Phi}{\partial y}]$$

Conservation of energy : $\gamma mc^2 + q\Phi = \gamma_0 mc^2 + q\Phi_0$

Electric potential : $\Phi = \Phi(y, z)$

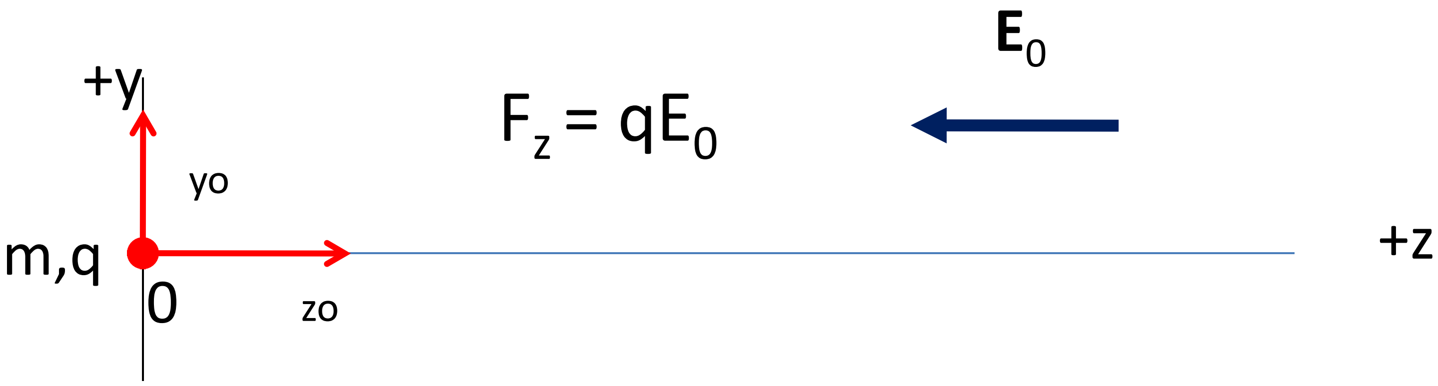
Trajectory slope : $y' = \frac{dy(z)}{dz}$

Linear momentum : $p = mc\gamma\beta$

Speed : $v = \beta c$

$\gamma = (1-\beta^2)^{-1/2}$

2D motion with constant force



Initial conditions

$$\mathrm{v}_y(0) = c\beta_y(0) = c\beta_{y0} \qquad \mathrm{v}_z(0) = c\beta_z(0) = c\beta_{z0}$$

$$\gamma_0 = \frac{1}{\sqrt{1-(\beta_{y0}^2+\beta_{z0}^2)}} \qquad \gamma_{z0} = \frac{1}{\sqrt{1-\beta_{z0}^2}}$$

$$p_{z0} = mc\gamma_0\beta_{z0} \qquad p_{y0} = mc\gamma_0\beta_{y0}$$

$$\Phi(z) = qE_0 z \qquad \gamma + \frac{qE_0}{mc^2} z = \gamma_0 \qquad dz = -\frac{mc^2}{qE_0} d\gamma$$

$$\dot{p}_y = 0 \quad \rightarrow \quad mc\gamma\beta_y = mc\gamma_0\beta_{y0} \quad \rightarrow \quad y'(z) = \frac{\gamma_0\beta_{y0}}{\gamma(z)\beta_z(z)}$$

$$y(z) - y(0) = \gamma_0\beta_{y0}\int_0^z \frac{dz}{\gamma(z)\beta_z(z)} = -\gamma_0\beta_{y0}\frac{mc^2}{qE_0}\int_{\gamma_0}^{\gamma} \frac{d\gamma}{\sqrt{\gamma^2 - \frac{\gamma_0^2}{\gamma_{z0}^2}}}$$

$$y(z) - y(0) = y'(0)\frac{p_{z0}c}{qE_0}\ln\left[\frac{\gamma + \sqrt{\gamma^2 - \frac{\gamma_0^2}{\gamma_{z0}^2}}}{\gamma_0(1+\beta_{z0})}\right]$$

Transfer matrix

$$\begin{bmatrix} y(z) \\ y'(z) \end{bmatrix} = \begin{bmatrix} 1 & L_{eff} \\ 0 & \dfrac{p_0}{p_z} \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

$$L_{eff}(z) = \frac{p(0)c}{qE_0} \ln\left[\frac{\gamma + \sqrt{\gamma^2 - \dfrac{\gamma_0^2}{\gamma_{z0}^2}}}{\gamma_0 (1 - \beta_{z0})} \right]$$

$$\gamma(z) = \gamma_0 - \frac{qE_0}{mc^2} z$$

Non relativistic trajectory

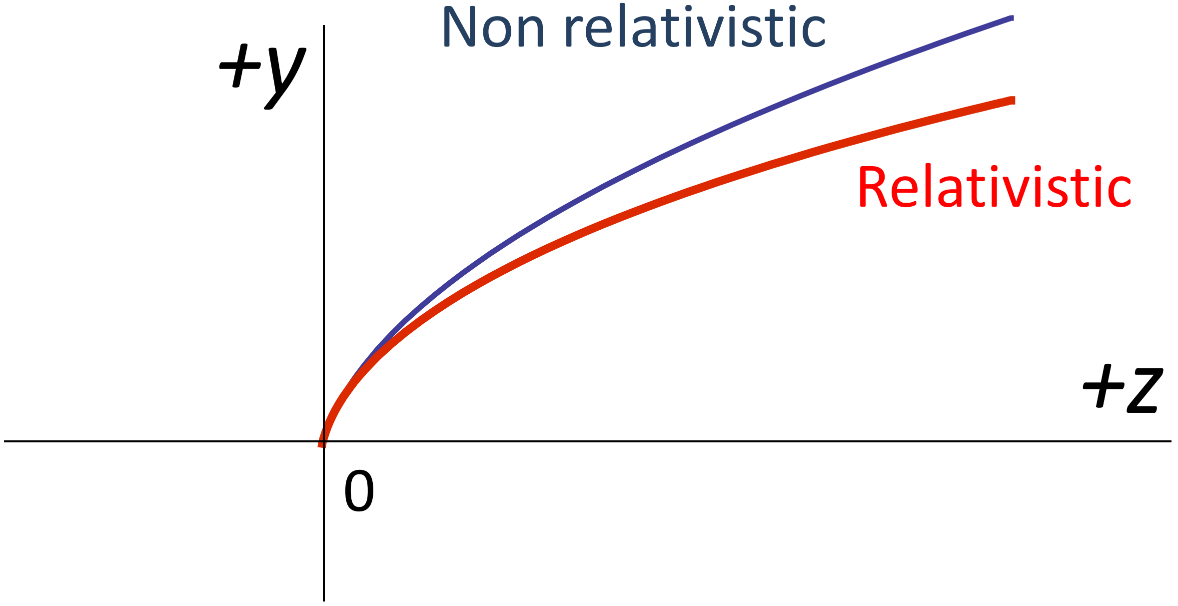
$$\dot{p}_y = 0 \qquad v_z(z)y'(z) = v_{z0}y'(0) = \text{constant}$$

$$y'(z) = \frac{v_{z0}y'(0)}{v_z(z)} \qquad v_z^2 = v_{z0}^2 + 2az$$

$$\mathrm{y}(\mathrm{z}) - \mathrm{y}(0) = v_{z0}y'(0)\int_0^z \frac{dz}{v_z(z)} = y'(0)\int_0^z \frac{dz}{\sqrt{1+\frac{2a}{v_{z0}^2}z}}$$

$$\mathrm{y}(\mathrm{z}) - \mathrm{y}(0) = \frac{y'(0)mv_{z0}^2}{qE_0}\left[\sqrt{1+\frac{2a}{v_{z0}^2}z} - 1\right]$$

Trajectory in a constant electric field



Paraxial trajectories (y” << 1)

$$y'' = \frac{q}{pv}[y' \frac{\partial \Phi}{\partial z} - \frac{\partial \Phi}{\partial y}]$$

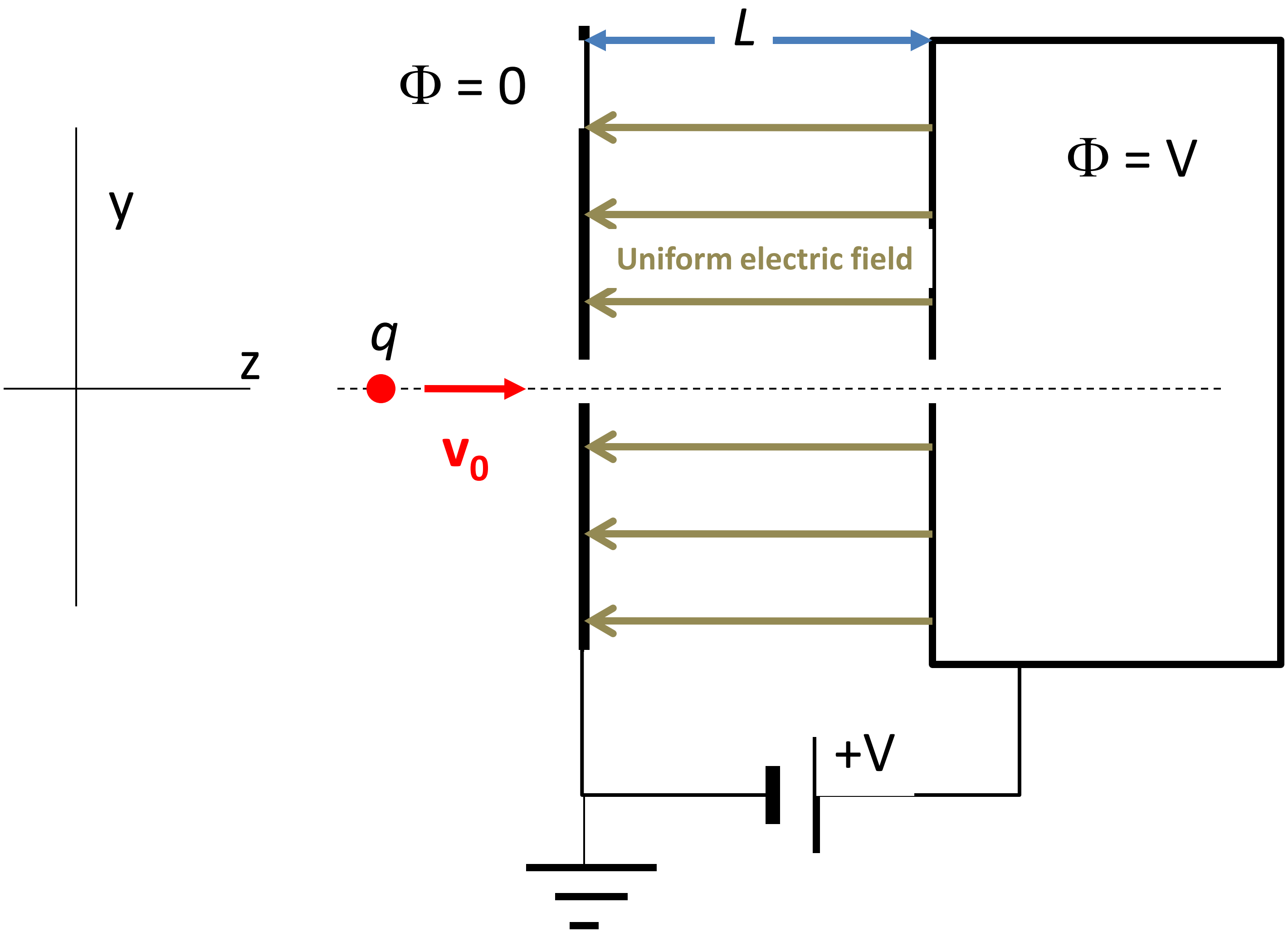
Linearization of electric potential near mirror symmetry plane y = 0

$$\Phi(y,z) = \Phi(0,z) + \frac{y^2}{2}\left[\frac{\partial^2 \Phi}{\partial z^2}\right]_{y=0} + \cdots$$

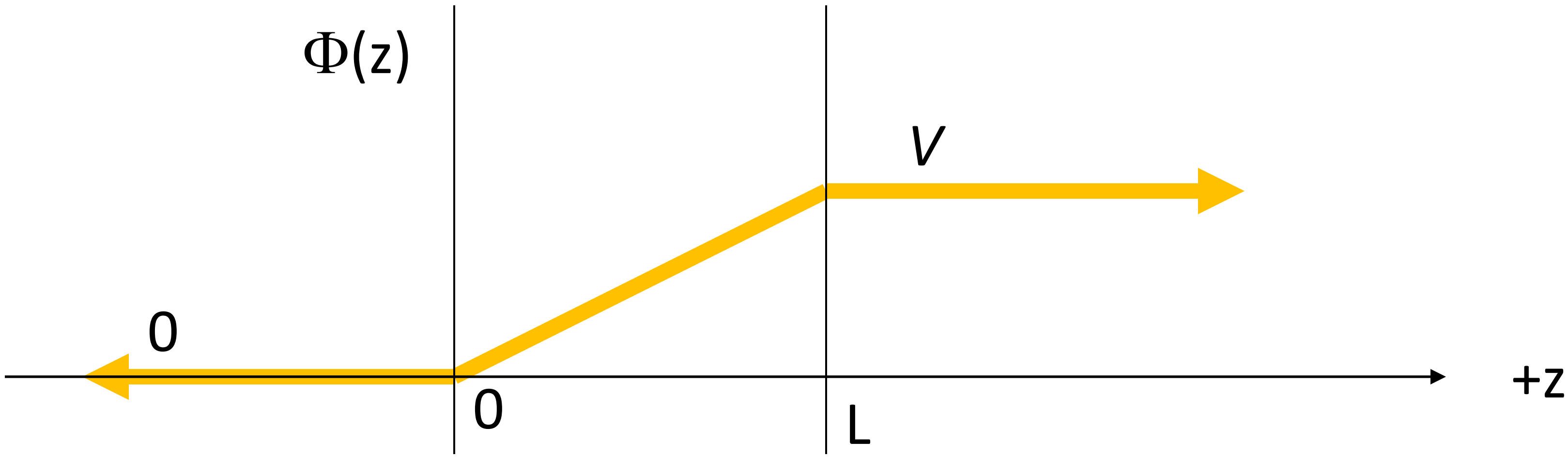
$$\frac{\partial \Phi(y,z)}{\partial y} = y\left[\frac{\partial^2 \Phi}{\partial z^2}\right]_{y=0} + \cdots$$

Example of Electrostatic Focusing

2D Electrostatic Focusing

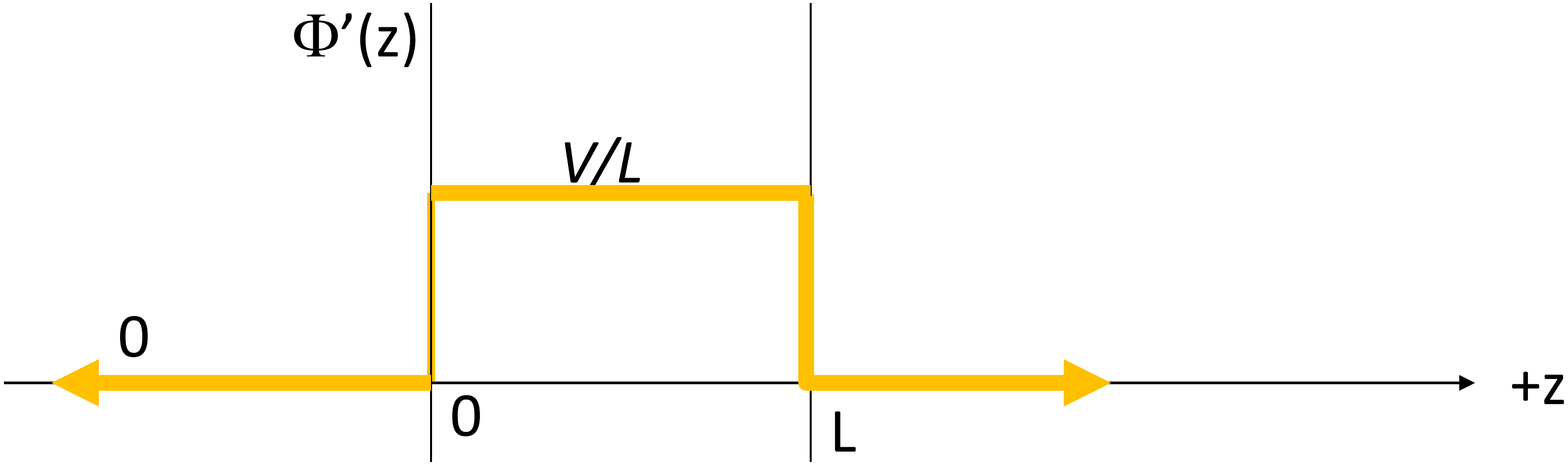


Electric Potential



$$\Phi(0,z) = \frac{V}{L} z[\theta(z) - \theta(z-L)] + V\theta(z-L)$$

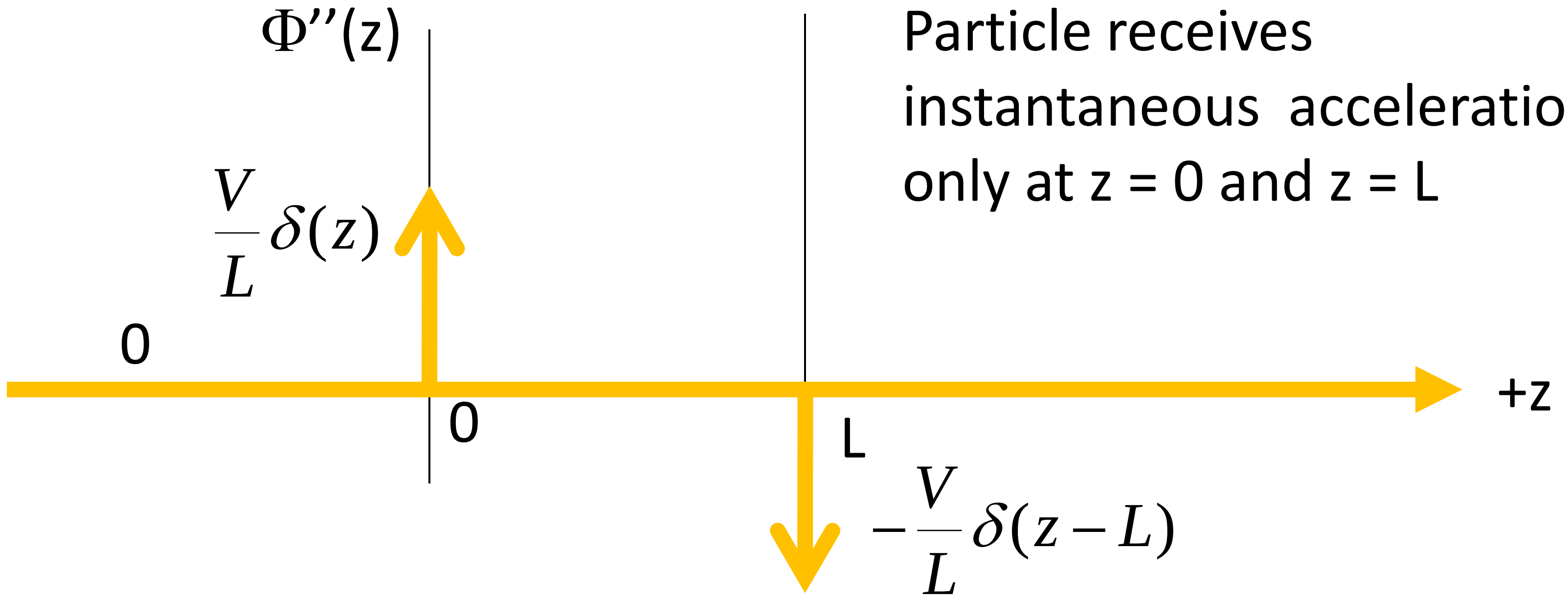
Electric Potential: First z-derivative



$$\Phi'(0,z) = \frac{V}{L}[\theta(z) - \theta(z-L)]$$

$$E_z = -\Phi'(0,z)$$

Electric Potential: Second z-derivative



Particle receives instantaneous acceleration only at z = 0 and z = L

$$\Phi''(0,z) = \frac{V}{L}[\delta(z) - \delta(z-L)]$$

Trajectory equations

At z = 0 and z = L

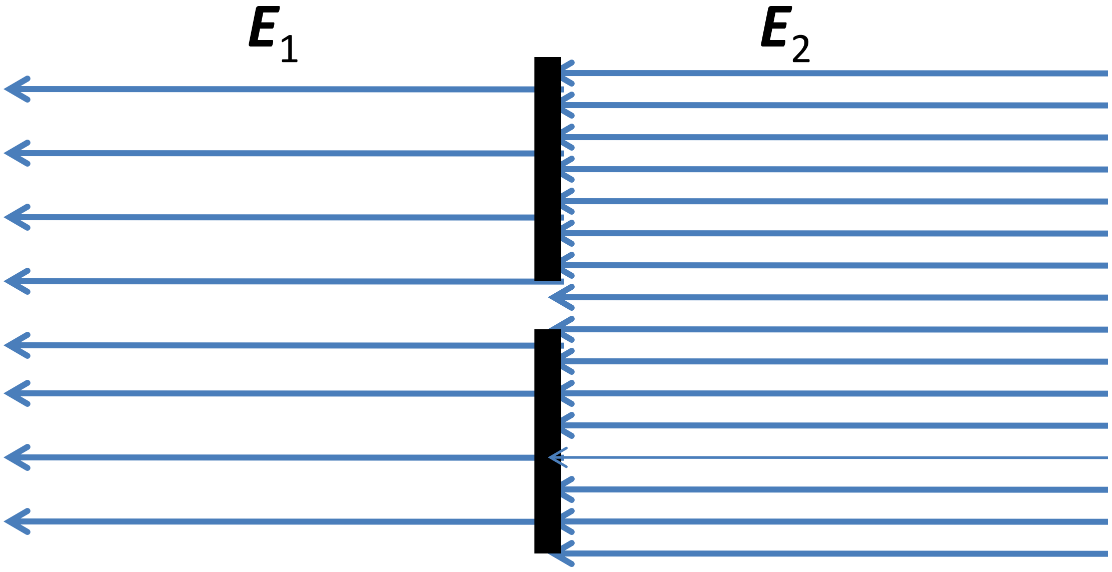
$$y'' = \frac{q}{pv}\left[-\frac{\partial\Phi}{\partial y}\right] = \frac{q}{pv}\left[-y\Phi''\right] = -\frac{q}{pv}\frac{V}{L}y\left[\delta(z) - \delta(z-L)\right]$$

From z = 0 to z = L

$$y'' = \frac{y'}{pv}\frac{\partial\Phi}{\partial z} = \frac{y'}{pv}\frac{qV}{L}\left[\theta(z) - \theta(z-L)\right]$$

Constant force

Focal power of an aperture slot



$$\Phi(z) = \left[\theta(-z)E_1 + \theta(z)E_2\right]z$$

$$\Phi'(z) = \left[\theta(-z)E_1 + \theta(z)E_2\right]$$

$$\Phi''(z) = \delta(z)\left(E_2 - E_1\right)$$

$$y'' = \frac{q}{pv}[y''\Phi' + y\Phi''] = \frac{q}{pv}[y''\Phi' + y\delta(z)\left(E_2 - E_1\right)]$$

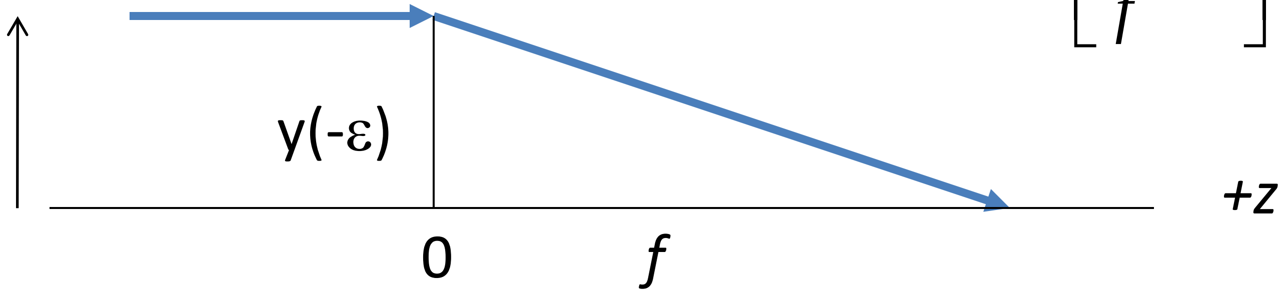
$$y'(\varepsilon) - y'(-\varepsilon) = \int_{-\varepsilon}^{\varepsilon} \frac{q}{pv}[y''\Phi' + y\delta(z)\left(E_2 - E_1\right)]$$

$$y'(\varepsilon) - y'(-\varepsilon) = \frac{qy(-\varepsilon)\left(E_2 - E_1\right)}{p(-\varepsilon)v(-\varepsilon)} \qquad \varepsilon \to 0$$

Focal length

For y'(-ε) = 0

$$R = \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix}$$



$$\text{Slope} = -\frac{y(-\varepsilon)}{f} = \frac{y(-\varepsilon)q}{p(-\varepsilon)v(-\varepsilon)}(E_2 - E_1)$$

$$\frac{1}{f} = -\frac{q(E_2 - E_1)}{p(-\varepsilon)v(-\varepsilon)}$$

Change in slope at z = 0

$$\Delta y'\Big|_{z=0} = \underbrace{\lim}_{\varepsilon\to 0}\left[y'(\varepsilon) - y'(-\varepsilon)\right]$$

$$\Delta y'\Big|_{z=0} = \frac{qV}{L}\underbrace{\lim}_{\varepsilon\to 0}\left[-\int_{-\varepsilon}^{\varepsilon}\frac{y}{pv}\Phi'' dz\right] = -\frac{qV}{L}\underbrace{\lim}_{\varepsilon\to 0}\left[\int_{-\varepsilon}^{\varepsilon}\frac{y}{pv}\delta(z)dz\right]$$

$$\Delta y'\Big|_{z=0} = -\frac{qV}{L}\frac{y(0)}{p(0)v(0)}$$

$$\begin{bmatrix} y_2(\varepsilon) \\ y_2'(\varepsilon) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\dfrac{qV}{Lp(-\varepsilon)v(-\varepsilon)} & 1 \end{bmatrix}\begin{bmatrix} y_1(-\varepsilon) \\ y_1'(-\varepsilon) \end{bmatrix}$$

Change in slope at z = L

$$\Delta y'\Big|_{z=L} = \underbrace{\lim}_{\varepsilon\to 0}\left[y'(L+\varepsilon) - y'(L-\varepsilon)\right]$$

$$\Delta y'\Big|_{z=L} = \frac{qV}{L}\underbrace{\lim}_{\varepsilon\to 0}\left[\int_{L-\varepsilon}^{L+\varepsilon}\frac{y}{pv}\delta(z-L)dz\right]$$

$$\Delta y'\Big|_{z=L} = \frac{qV}{L}\frac{y(L)}{p(L)v(L)}$$

$$\begin{bmatrix} y_2(L+\varepsilon) \\ y_2'(L+\varepsilon) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\dfrac{qV}{Lp(-\varepsilon)v(-\varepsilon)} & 1 \end{bmatrix}\begin{bmatrix} y_1(L-\varepsilon) \\ y_1'(L-\varepsilon) \end{bmatrix}$$

Overall Transfer Matrix

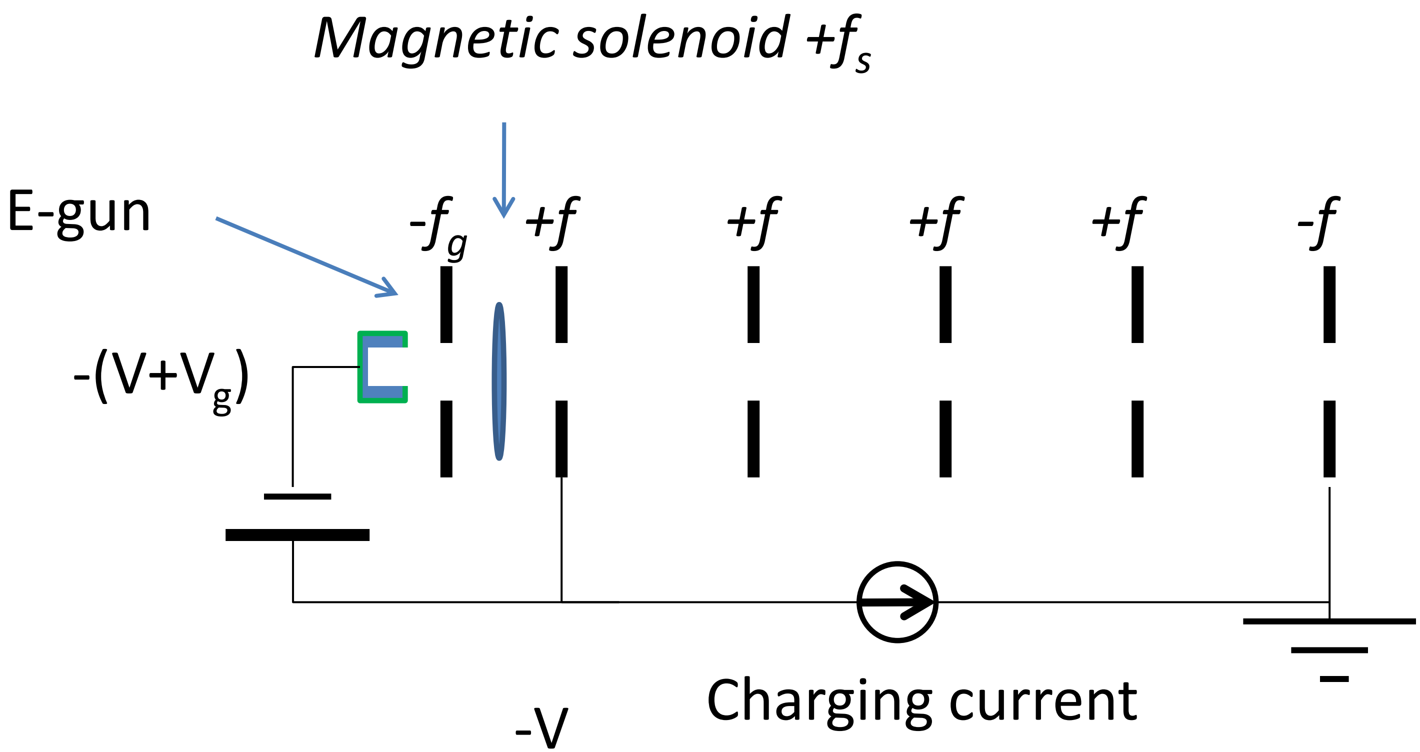
$$R=\begin{bmatrix}1 & 0\\ -\alpha(L+\varepsilon) & 1\end{bmatrix}\begin{bmatrix}1 & L_{eff}\\ 0 & \dfrac{p(\varepsilon)}{p(L-\varepsilon)}\end{bmatrix}\begin{bmatrix}1 & 0\\ \alpha(-\varepsilon) & 1\end{bmatrix}\qquad \varepsilon\to 0$$

$$L_{eff}(z)=\frac{p(\varepsilon)c}{qE_0}\ln\left[\frac{\gamma+\sqrt{\gamma^2-\dfrac{\gamma_0^2}{\gamma_{z0}^2}}}{\gamma_0(1-\beta_{z0})}\right]$$

$$\gamma(L)=\gamma_0-\frac{qE_0}{mc^2}L \qquad\qquad \alpha(u)=\frac{qV}{Lp(u)v(u)}$$

Electrostatic focusing in the UH/CBPF accelerator

Accerator focusing



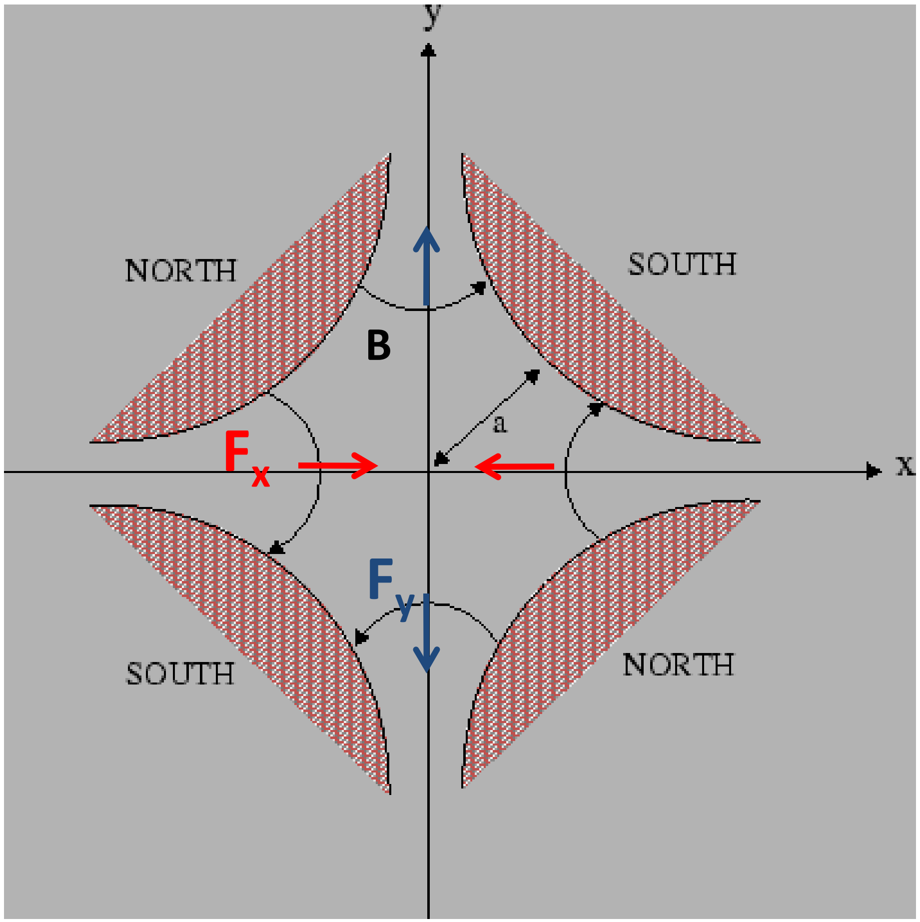
In addition to overall acceleration focusing there are the following discrete focusing elements

Magnetic focusing

Magnetic quadrupole focusing

$$\mathbf{A} = -\mathbf{e}_z \frac{B_0}{a} xy \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

$$B_x = B_0 \frac{y}{a} \qquad B_y = B_0 \frac{x}{a}$$



Trajectory equation

$$\frac{dp_x}{dt} = v_z \frac{dp_x}{dz} = \frac{c\beta\gamma m}{\sqrt{1+x'^2}} \frac{d}{dz} \frac{c\beta x'}{\sqrt{1+x'^2}}$$

$$\frac{dp_x}{dt} = \frac{c^2\beta^2\gamma m}{\sqrt{1+x'^2}} [\frac{x''}{\sqrt{1+x'^2}} - \frac{x'^2 x''}{\left(\sqrt{1+x'^2}\right)^3}$$

$$\frac{dp_x}{dt} = \frac{c^2\beta^2\gamma m}{\left(1+x'^2\right)^2} x''$$

Paraxial trajectory x'^{2} << 1

$$\frac{dp_x}{dt} = \frac{c^2\beta^2\gamma m}{\left(1+x'^2\right)^2} x'' = -|q|c\,\frac{\beta}{\sqrt{1+x'^2}}\,B_0\,\frac{x}{a}$$

$$x'^2 << 1$$

$$x'' + k^2 x = 0 \qquad\qquad k^2 = \frac{|q|B_0}{mc\gamma\beta a}$$

$$x(z) = A\cos kz + B\sin kz$$

Initial conditions and Matrix representation

$$x(0) = x_1 \qquad\qquad x'(0) = x_1'$$

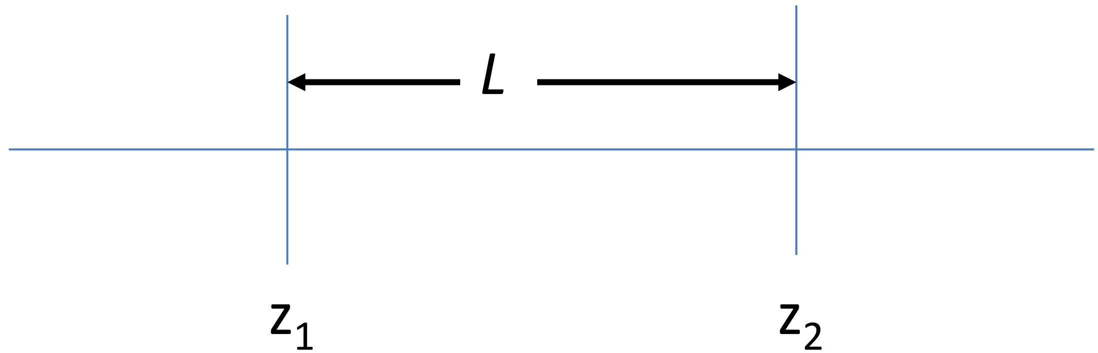
$$x(z) = x_1 \cos kz + \frac{x_1'}{k} \sin kz$$

$$x'(z) = -kx_1 \sin kz + x_1' \cos kz$$

In general

$$\begin{bmatrix} x_2 \\ x_2' \end{bmatrix} = \begin{bmatrix} \cos kL & \frac{1}{k} \sin kL \\ -k \sin kL & \cos kL \end{bmatrix} \begin{bmatrix} x_1 \\ x_1' \end{bmatrix} \qquad\qquad L = z_2 - z_1$$

x and y transport matrix for a magnetic quadrupole of length L



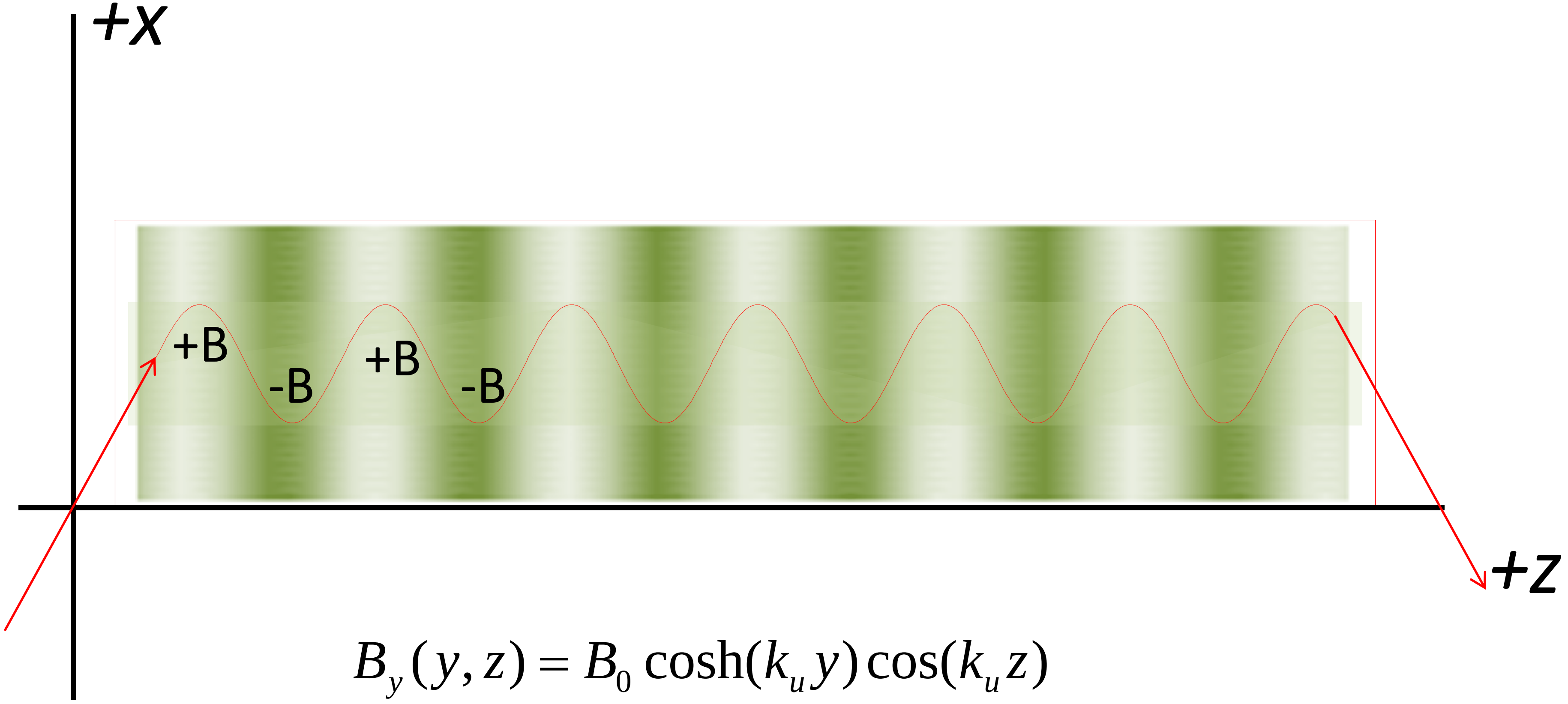
In generalIn general

$$R_x = \begin{bmatrix} \cos kL & \frac{1}{k}\sin kL \\ -k\sin kL & \cos kL \end{bmatrix} \qquad R_y = \begin{bmatrix} \cosh kL & \frac{1}{k}\sinh kL \\ k\sinh kL & \cosh kL \end{bmatrix}$$

Overall 4x4 transport matrix

$$\begin{bmatrix} x_2 \\ x_2' \\ y_2 \\ y_2' \end{bmatrix} = \begin{bmatrix} \cos kL & \frac{1}{k}\sin kL & 0 & 0 \\ -kL\sin kL & \cos kL & 0 & 0 \\ 0 & 0 & \cosh kL & \frac{1}{k}\sinh kL \\ 0 & 0 & k\sinh kL & \cosh kL \end{bmatrix} \begin{bmatrix} x_1 \\ x_1' \\ y_1 \\ y_1' \end{bmatrix}$$

Electron trajectory in a planar undulator



$$B_y(y,z) = B_0 \cosh(k_u y)\cos(k_u z)$$

$$B_z(y,z) = B_0 \sinh(k_u y)\sin(k_u z)$$

$$B_x = 0$$

Equations of motion

$$\frac{d\boldsymbol{p}}{dt} = qc \quad \times \boldsymbol{B} \qquad \boldsymbol{p} = mc\gamma \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\boldsymbol{v} = c \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$P = \mathrm{v} \cdot \frac{d\boldsymbol{p}}{dt} = mc^2 \frac{d\gamma}{dt} = qc \quad \cdot \quad \times \boldsymbol{B} = 0 = \qquad \rightarrow \qquad \gamma = \text{constant}$$

2D velocities

$$\dot{p}_x = q(\mathrm{v} \times \mathrm{B})_x = -qc\beta_z B_y \quad \rightarrow \quad \dot{\beta}_x = -\frac{q\beta_z B_y}{m\gamma}$$

$$\dot{\beta}_x = v_z \beta'_x \quad \rightarrow \quad {}_x = \beta_{x0} - \frac{q}{mc\gamma}\int_0^z B_y dz$$

$$\beta_z = \sqrt{\beta^2 - \beta_x^2} = \beta\sqrt{1 - \left(\frac{\beta_x}{\beta}\right)^2}$$

Transverse velocity on plane y = 0

$$\beta_x = \beta_{x0} - \frac{q}{mc\gamma}\int_0^z B_y dz = \beta_{x0} - \frac{qB_0}{mc\gamma}\int_0^z \cos(k_u z) dz$$

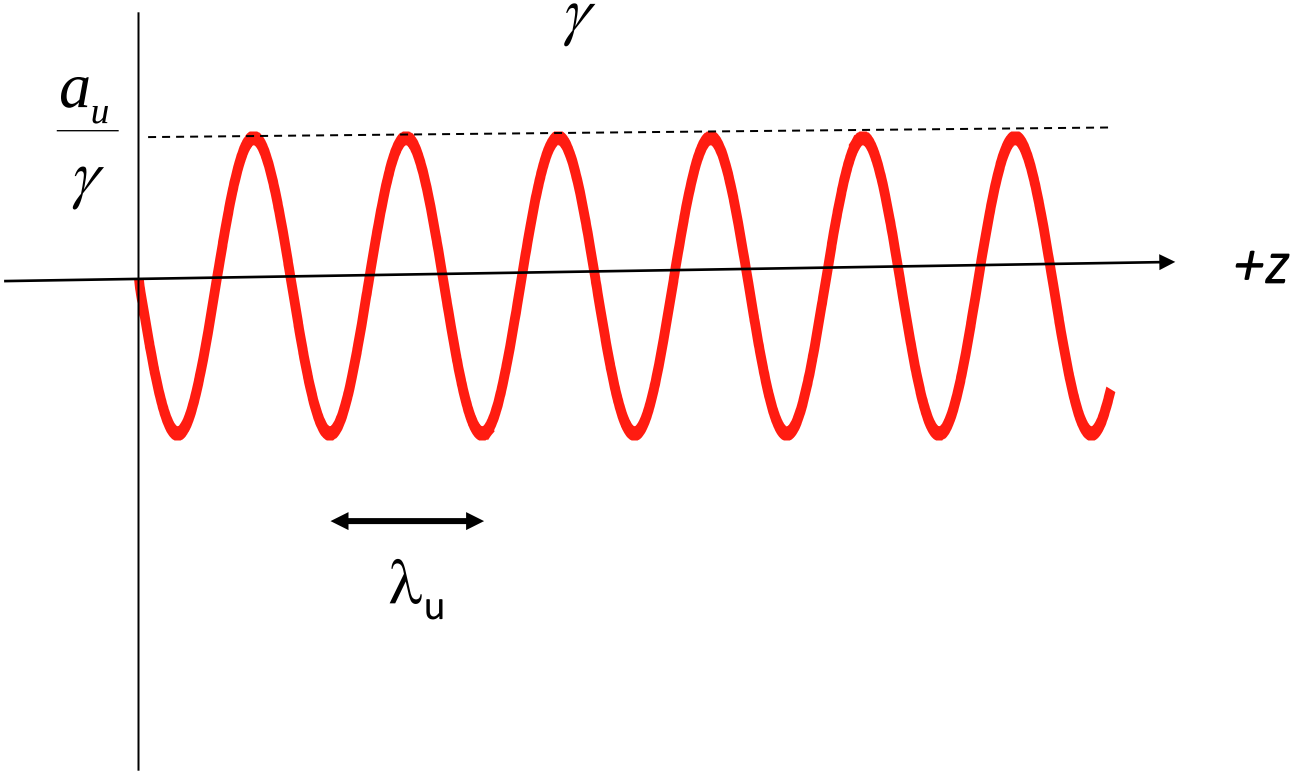
$$\beta_x = \beta_{x0} - \frac{qB_0}{mc\gamma k_u}\sin(k_u z)$$

With $\beta_{x0} = 0$ the electron trajectory does not drift with respect to the z - axis

$$\beta_x = -\frac{qB_0}{mc\gamma k_u}\sin(k_u z) = -\frac{a_u}{\gamma}\sin(k_u z)$$

$$\text{Undulator parameter } a_u = \frac{qB_0}{mck_u} = 93.4 B_0 \lambda_u$$

$$\beta_x = -\frac{a_u}{\gamma}\sin k_u z$$



Longitudinal velocity

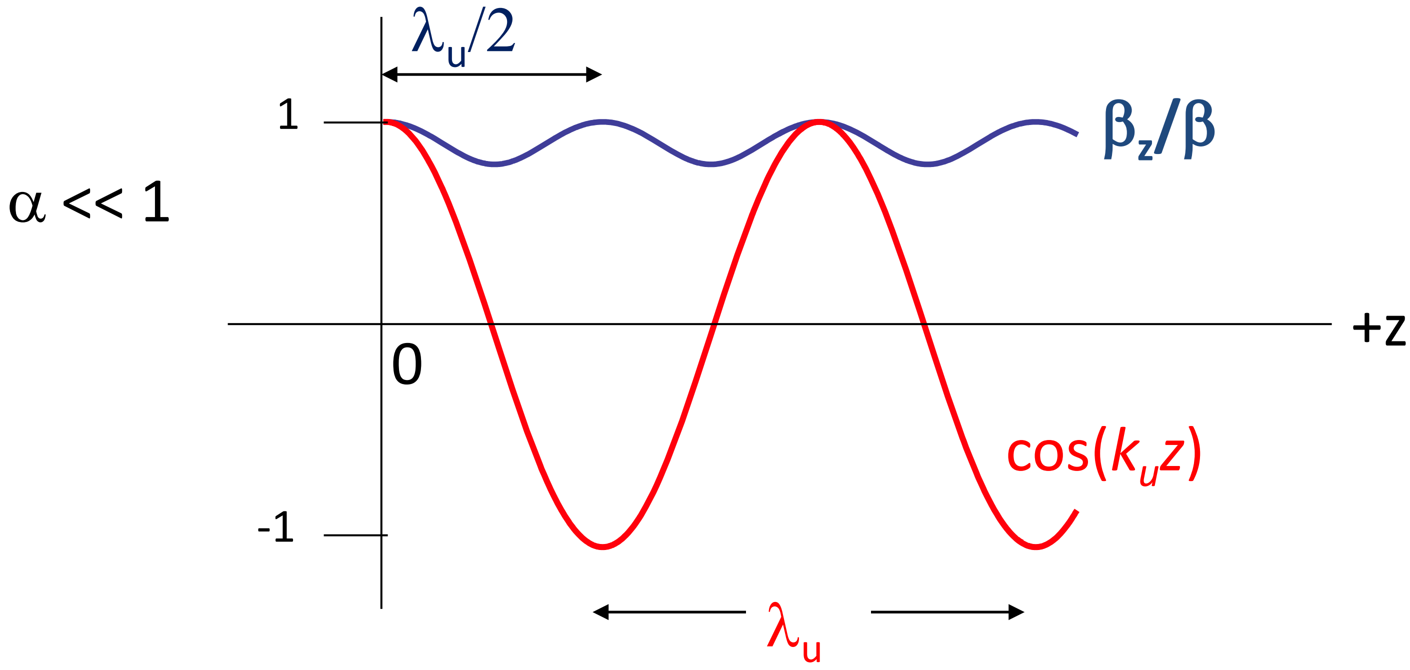
$$\beta_z = \sqrt{\beta^2 - \beta_x^2} = \sqrt{\beta^2 - \left(\frac{a_u}{\gamma}\sin(k_u z)\right)^2}$$

$$\frac{\beta_z}{\beta} = \sqrt{1 - \left(\frac{a_u}{\beta\gamma}\sin(k_u z)\right)^2}$$

$$\frac{\beta_z}{\beta} = \sqrt{1 - \left(\alpha\sin(k_u z)\right)^2}$$

β_z oscillates about its average value close to twice in one undulator period

$$\beta_z = \overline{\beta}_z + \frac{\beta\alpha^2}{4}\sin 2k_u z + O(\alpha^4)$$



Time evolution of the longitudinal position

$$\frac{\beta_z}{\beta} = \sqrt{1-\left(\frac{a_u}{\beta\gamma}\sin(k_u z)\right)^2}$$

$$\theta = k_u z$$

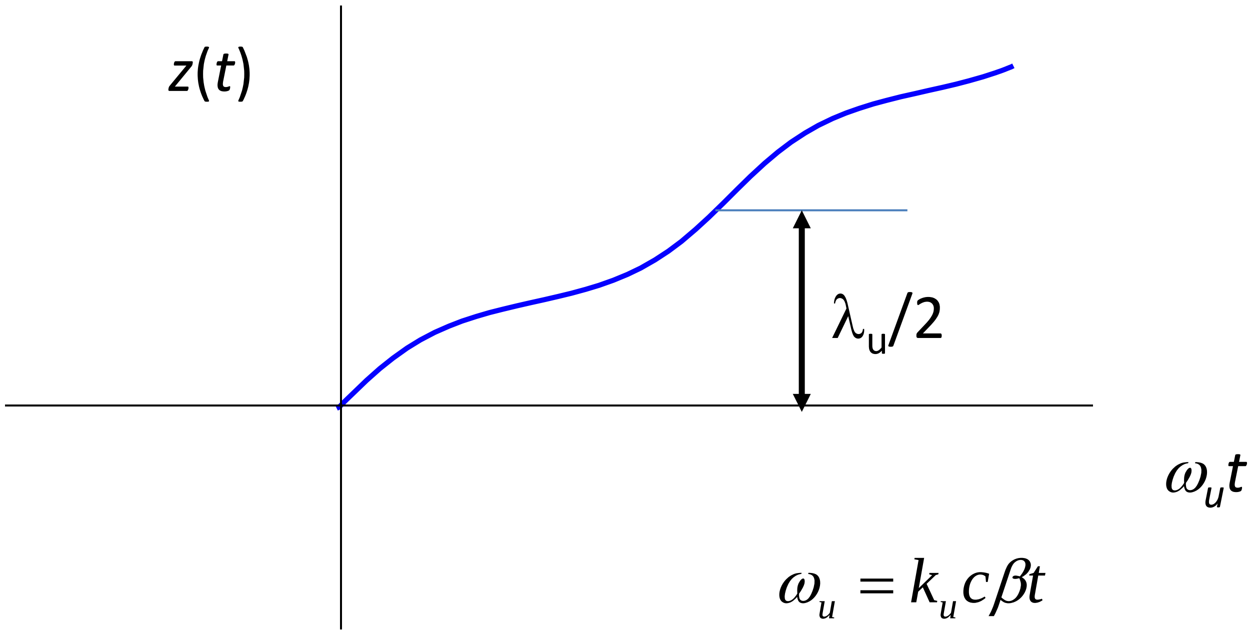
$$\tau = c\beta k_u t$$

Elliptic Integral of the First Kind

$$\frac{d\theta}{d\tau} = \sqrt{1-(\alpha\sin\theta)^2} \quad \rightarrow \quad \tau = \int_o^\phi \frac{d\theta}{\sqrt{1-(\alpha\sin\theta)^2}}$$

$$z(t) = \frac{1}{k_u}\mathrm{am}(\tau,\alpha^2]$$

sn = Jacobi amplitude function



$$\omega_u = k_u c \beta t$$

Trajectory solution

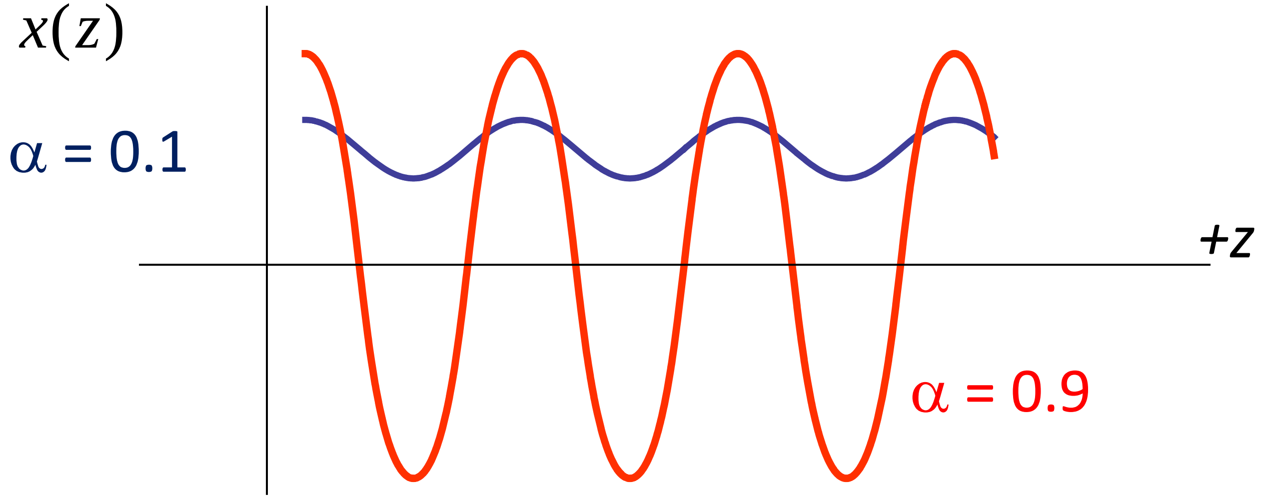
$$\frac{dx(z)}{dz} = \frac{\beta_x}{\beta_z} = -\frac{a_u}{\gamma}\frac{\sin(k_u z)}{\beta_z}$$

$$x(z) - x(0) = -\frac{\alpha}{k_u}\int_0^z \frac{\sin(k_u \mathfrak{z})d\mathfrak{z}}{\sqrt{1-[\alpha\sin(k_u \mathfrak{z})]^2}}$$

$$x(z) - x(0) = \frac{\alpha}{k_u}\ln\left[\frac{\alpha\cos(k_u z) + \sqrt{1-(\alpha\sin(k_u z)^2}}{1+\alpha}\right]$$

$$x(z) - x(0) = \frac{\alpha}{k_u}\ln[1+\alpha\cos(k_u z)] = \frac{\alpha^2}{k_u}\cos(k_u z) \qquad \alpha \to 0$$

Undulator trajectories



$$x(z) = \frac{\alpha}{k_u} \ln\left[\frac{\alpha\cos(k_u z) + \sqrt{1-(\alpha\sin(k_u z))^2}}{1+\alpha}\right] \qquad \alpha = \frac{a_u}{\gamma\beta}$$

Electron motion in a planar unduator

Luis Elias

General Electron Hamiltonian

$$\mathbf{H} = \mathbf{mc}^2\sqrt{1+\left(\frac{\pi_{\mathbf{x}} - \mathbf{qA}_{\mathbf{x}}}{\mathbf{mc}}\right)^2+\left(\frac{\pi_{\mathbf{y}} - \mathbf{qA}_{\mathbf{y}}}{\mathbf{mc}}\right)^2+\left(\frac{\pi_{\mathbf{z}} - \mathbf{qA}_{\mathbf{z}}}{\mathbf{mc}}\right)^2} + \mathbf{q}\Phi$$

$$\boldsymbol{\pi} = \mathbf{mc}\gamma\boldsymbol{\beta} + \mathbf{qA}$$

is the **c**anonical electron linear momentum, ***A*** is the vector potential and Φ is the scalar potential

Trajectory Hamiltonian

We want to change independent variable from time t to longitudinal position z. The canonical conjugate coordinate to z is π_z, consequently we want to use $\Pi_z = -\pi_z$ as the trajectory Hamiltonian and use H as the canonical variable conjugate to time. H is a constant $H = \gamma mc^2$ of motion when it does not depend explicitly on time, we can therefore use the following general trajectory Hamiltonian:

$$\Pi_z = -qA_z - mc\sqrt{\left(\frac{\gamma mc^2 - q\Phi}{mc^2}\right)^2 - 1 - \left(\frac{\pi_x - qA_x}{mc}\right)^2 - \left(\frac{\pi_y - A_y}{mc}\right)^2}$$

Equations of motion

$$x' = \frac{dx}{dz} = \frac{\partial \Pi_z}{\partial \pi_x}, \qquad y' = \frac{dy}{dz} = \frac{\partial \Pi_z}{\partial \pi_y}$$

$$\pi'_x = \frac{d\pi_x}{dz} = -\frac{\partial \Pi_z}{\partial x}, \qquad \pi'_y = \frac{d\pi_y}{dz} = -\frac{\partial \Pi_z}{\partial y}$$

2-D undulator field

Let the 2-D static magnetic undulator fields be:

$$A_x = A_u \cosh(k_u y)\sin(k_u z)$$

$$A_y = A_z = 0$$

$$B_y = \frac{\partial A_x}{\partial z} = k_u A_u \cosh(k_u y)\cos(k_u z)$$

$$B_z = -\frac{\partial A_x}{\partial y} = -k_u A_u \sinh(k_u y)\sin(k_u z)$$

$$B_x = 0$$

$$\Phi = 0$$

Trajectory Hamiltonian

$$\Pi_z = -mc\sqrt{\left(\frac{\gamma mc^2 - q\Phi}{mc^2}\right)^2 - 1 - \left(\frac{\pi_x - qA_x}{mc}\right)^2 - \left(\frac{\pi_y}{mc}\right)^2}$$

$$\Pi_z = -mc\sqrt{\beta^2\gamma^2 - \left(\frac{\pi_x - qA_x}{mc}\right)^2 - \left(\frac{\pi_y}{mc}\right)^2}$$

$$A_x = A_u \cosh(k_u y)\sin(k_u z)$$

$$\pi_x = 0$$

$H = \gamma mc^2$ and π_x are constants of motion. We choose $\pi_x = 0$ as an Initial condition, so that the electron undulates about the z-axis without drifting away from it.

Trajectory Hamiltonian expansion

For electron motion close to the *z*-axis, *y* and π_y are small quantities compared to the period of undulation and *z*-momentum respectively. The trajectory Hamiltonian can be expanded in a Taylor series about (y, $\pi_{y,}$ $\pi_{x,}$ = 0).

$$\Pi_z(y,\pi_y,z,\pi_x) = (\Pi_z)_0 + \left(\frac{\partial \Pi_z}{\partial y}\right)_0 y + \left(\frac{\partial \Pi_z}{\partial \pi_y}\right)_0 \pi_y + \left(\frac{\partial \Pi_z}{\partial \pi_x}\right)_0 \pi_x +$$

$$\frac{1}{2}\left[\begin{array}{l}\left(\frac{\partial^2 \Pi_z}{\partial y^2}\right)_0 y^2 + \left(\frac{\partial^2 \Pi_z}{\partial \pi_y{}^2}\right)_0 \pi_y{}^2 + \left(\frac{\partial^2 \Pi_z}{\partial y \partial \pi_y}\right)_0 y\pi_y + \left(\frac{\partial^2 \Pi_z}{\partial y \partial \pi_x}\right)_0 y\pi_x \\ + \left(\frac{\partial^2 \Pi_z}{\partial \pi_y \partial \pi_x}\right)_0 \pi_y \pi_x + \left(\frac{\partial^2 \Pi_z}{\partial \pi_x{}^2}\right)_0 \pi_x{}^2\end{array}\right] + \ldots$$

$$\Pi_z = -mc\sqrt{\beta^2\gamma^2 - \left(\frac{\pi_x}{mc} - a_u \cosh(k_u y)\sin(k_u z)\right)^2 - \left(\frac{\pi_y}{mc}\right)^2}$$

$$(\Pi_z)_0 = -mc\sqrt{\beta^2\gamma^2 - a_u{}^2 \sin^2(k_u z)}, \qquad a_u = \frac{|q|A_u}{mc}$$

$$\left(\frac{\partial \Pi_z}{\partial \pi_x}\right)_0 = -\frac{\alpha \sin(k_u z)}{\sqrt{1-\alpha^2 \sin^2(k_u z)}}, \qquad \alpha = \frac{a_u}{\beta\gamma}$$

$$\left(\frac{\partial \Pi_z}{\partial y}\right)_0 = \left(\frac{\partial \Pi_z}{\partial \pi_y}\right)_0 = \left(\frac{\partial^2 \Pi_z}{\partial y \partial \pi_y}\right)_0 = \left(\frac{\partial^2 \Pi_z}{\partial y \partial \pi_x}\right)_0 = \left(\frac{\partial^2 \Pi_z}{\partial \pi_y \partial \pi_x}\right)_0 = 0$$

$$\left(\frac{\partial^2 \Pi_z}{\partial y^2}\right)_0 = -\frac{m^2 c^2 a_u{}^2 k_u{}^2 \sin^2(k_u z)}{(\Pi_z)_0} = -\frac{mc a_u{}^2 k_u{}^2 \sin^2(k_u z)}{\beta\gamma\sqrt{1-\alpha^2 \sin^2(k_u z)}}$$

$$\left(\frac{\partial^2 \Pi_z}{\partial \pi_y{}^2}\right)_0 = \frac{1}{(\Pi_z)_0} = \frac{1}{mc\beta\gamma\sqrt{1-\alpha^2 \sin^2(k_u z)}}$$

$$\left(\frac{\partial^2 \Pi_z}{\partial \pi_x{}^2}\right)_0 = \frac{1}{(\Pi_z)_0}\left[1 + \frac{mc}{(\Pi_z)_0} a_u{}^2 \sin^2(k_u z)\right]$$

$$\left(\frac{\partial^2 \Pi_z}{\partial \pi_x{}^2}\right)_0 = \frac{1}{mc\beta\gamma\sqrt{1-\alpha^2 \sin^2(k_u z)}}\left[1 + \frac{a_u{}^2 \sin^2(k_u z)}{\sqrt{1-\alpha^2 \sin^2(k_u z)}}\right]$$

Trajectory equations

$$\frac{dx}{dz} = \frac{\partial \Pi_z}{\partial \pi_x} = \left(\frac{\partial \Pi_z}{\partial \pi_x}\right)_0 + \left(\frac{\partial^2 \Pi_z}{\partial {\pi_x}^2}\right)_0 \pi_x = \left(\frac{\partial \Pi_z}{\partial \pi_x}\right)_0 \quad (\pi_x = 0)$$

$$\frac{dx}{dz} = -\frac{\alpha \sin(k_u z)}{\sqrt{1-\alpha^2 \sin^2(k_u z)}}$$

$$\frac{dy}{dz} = \frac{\partial \Pi_z}{\partial \pi_y} = \left(\frac{\partial^2 \Pi_z}{\partial y^2}\right)_0 \pi_y$$

$$\frac{d\pi_y}{dz} = -\left(\frac{\partial^2 \Pi_z}{\partial {\pi_y}^2}\right)_0 y$$

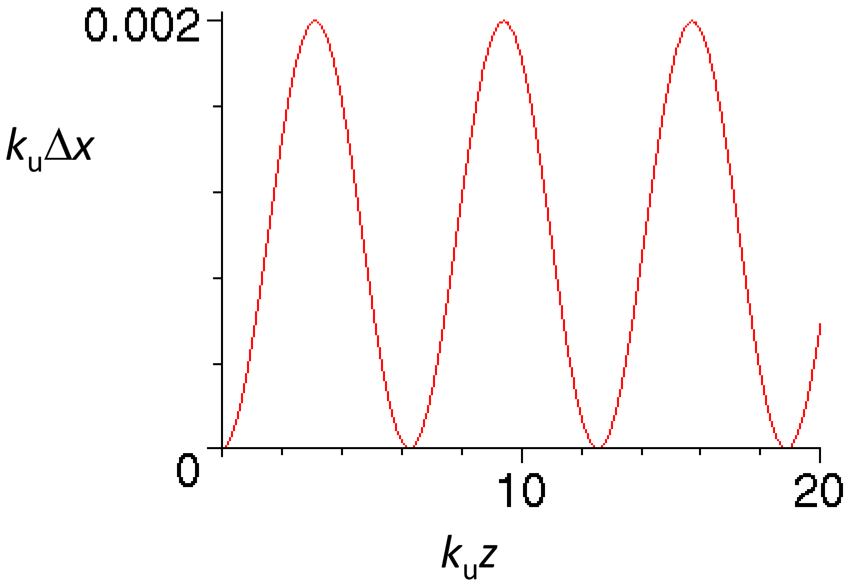
To second order in *y*, π_y and π_x .

X-trajectory

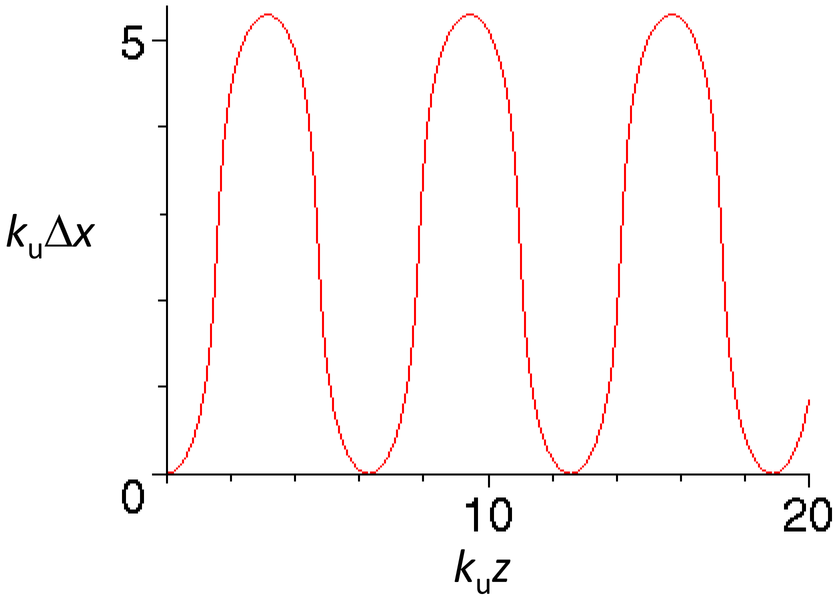
$$\frac{dx}{dz} = -\frac{\alpha \sin(k_u z)}{\sqrt{1-\alpha^2 \sin^2(k_u z)}}$$

$$k_u x(z) = k_u x(0) - \ln\left\{\frac{\alpha \cos k_u z + \sqrt{1-\alpha^2 \sin^2 k_u z}}{\alpha + 1}\right\}$$

$$k_u \Delta x = k_u [x(0) - x(z)] = \ln\left\{\frac{\alpha \cos k_u z + \sqrt{1-\alpha^2 \sin^2 k_u z}}{\alpha + 1}\right\}$$



$\alpha = 0.001$



$\alpha = 0.99$

Betatron focusing

Y-trajectory

Combining the last two terms of the trajectory equations we obtain:

$$\frac{dy}{dz} = \frac{\partial \Pi_z}{\partial \pi_y} = \left(\frac{\partial^2 \Pi_z}{\partial y^2}\right)_0 \pi_y \quad \text{and} \quad \frac{d\pi_y}{dz} = -\left(\frac{\partial^2 \Pi_z}{\partial {\pi_y}^2}\right)_0 y$$

$$\frac{d^2 y}{dz^2} = -\left(\frac{\partial^2 \Pi_z}{\partial y^2}\right)_0 \left(\frac{\partial^2 \Pi_z}{\partial {\pi_y}^2}\right)_0 y$$

The coefficient of *y* depends on *z* through trigonometric functions of $k_u z$. We can remove the short-period undulator Oscillations by averaging over one period of the undulator.

Averaged y-trajectory

$$\left\langle \frac{d^2 y}{dz^2} \right\rangle_{\lambda_u} = -\left\langle \left(\frac{\partial^2 \Pi_z}{\partial y^2} \right)_0 \left(\frac{\partial^2 \Pi_z}{\partial \pi_y{}^2} \right)_0 \right\rangle_{\lambda_u} y$$

$$\left\langle \left(\frac{\partial^2 \Pi_z}{\partial y^2} \right)_0 \left(\frac{\partial^2 \Pi_z}{\partial \pi_y{}^2} \right)_0 \right\rangle_{\lambda_u} = \left\langle \frac{\alpha^2 k_u{}^2 \sin^2(k_u z)}{\sqrt{1-\alpha^2 \sin^2(k_u z)}} \frac{1}{\sqrt{1-\alpha^2 \sin^2(k_u z)}} \right\rangle_{\lambda_u}$$

$$\left\langle \left(\frac{\partial^2 \Pi_z}{\partial y^2} \right)_0 \left(\frac{\partial^2 \Pi_z}{\partial \pi_y{}^2} \right)_0 \right\rangle_{\lambda_u} = \left\langle \frac{\alpha^2 k_u{}^2 \sin^2(k_u z)}{\left(1-\alpha^2 \sin^2(k_u z)\right)} \right\rangle_{\lambda_u}$$

For small α we have:

$$\left\langle \left(\frac{\partial^2 \Pi_z}{\partial y^2}\right)_0 \left(\frac{\partial^2 \Pi_z}{\partial \pi_y{}^2}\right)_0 \right\rangle_{\lambda_u} = \left\langle \alpha^2 k_u{}^2 \sin^2(k_u z)\left(1+\alpha^2 \sin^2(k_u z)\right)\right\rangle_{\lambda_u}$$

$$\left\langle \left(\frac{\partial^2 \Pi_z}{\partial y^2}\right)_0 \left(\frac{\partial^2 \Pi_z}{\partial \pi_y{}^2}\right)_0 \right\rangle_{\lambda_u} = \frac{\alpha^2 k_u{}^2)\left(1+\frac{3}{4}\alpha^2\right)}{2}$$

$$\left\langle \frac{d^2 y}{dz^2} \right\rangle_{\lambda_u} = -\left\langle \left(\frac{\partial^2 \Pi_z}{\partial y^2}\right)_0 \left(\frac{\partial^2 \Pi_z}{\partial \pi_y{}^2}\right)_0 \right\rangle_{\lambda_u} y = -\frac{\alpha^2 k_u{}^2)\left(1+\frac{3}{4}\alpha^2\right)}{2} y$$

$$\frac{d^2 \langle y \rangle}{dz^2} + K_\beta{}^2 \langle y \rangle = 0,$$

From Newton's Second Law

$$\frac{dp_y}{dt} = \gamma\beta^2 mc^2 y'' =$$

$$F_{y=} -|q|v_z B_x = -|q|v_x B_z$$

$$y'' = -\alpha^2 k_u \sinh k_u y (\sin k_u z)^2$$

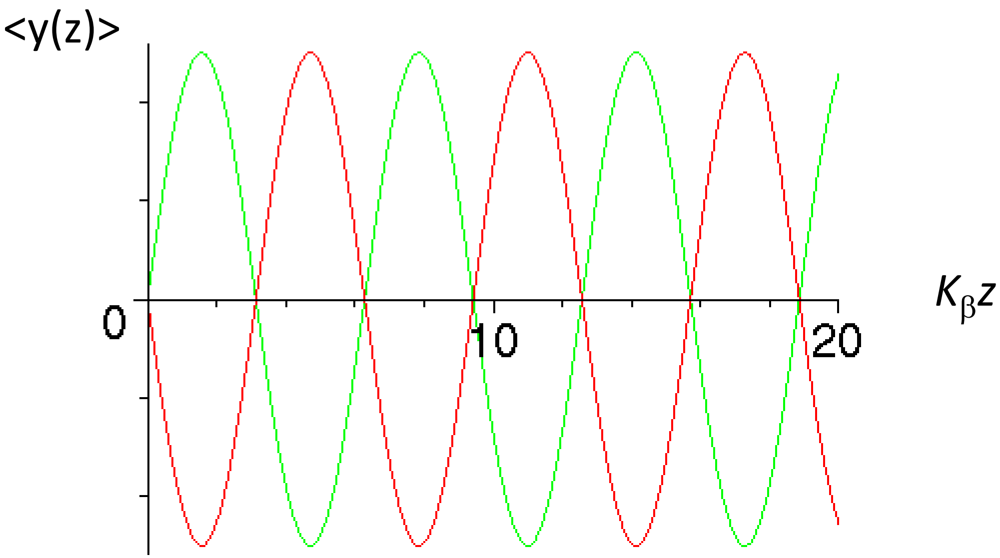
For small $(\mathrm{k_u}\, y)$

$$\sinh(\mathrm{k_u}\, y) \cong k_u y$$

Averaging over one undulator period yields

$$\mathrm{y}'' + \frac{(\mathrm{k_u}\alpha)^2}{2} y = 0 \qquad \mathrm{y}'' + K_\beta^2 y \qquad \alpha = \frac{a_u}{\gamma\beta}$$

Betatron oscillation



$$\alpha = \frac{a_u}{\beta\gamma}$$

$$\lambda_\beta = \frac{2\pi}{K_\beta} = \frac{\sqrt{2}}{\alpha\left(1+\frac{3}{4}\alpha^2\right)}\lambda_u$$

Betatron period

Betatron period for UCF-FEL

$$a_u = 0.14$$

$$\lambda_u = 0.008\,\text{m}$$