ISSN 1063-780X, Plasma Physics Reports, 2008, Vol. 34, No. 7, pp. 538–546. © Pleiades Publishing, Ltd., 2008.
Published in Russian in Fizika Plazmy, 2008, Vol. 34, No. 7, pp. 589–597.

PLASMA INSTABILITY

# Ideal Internal Kink Modes in a Differentially Rotating Cylindrical Plasma[1]

**A. B. Mikhailovskii[a], J. G. Lominadze[b], R. M. O. Galvão[c, d], A. P. Churikov[e], N. N. Erokhin[a], V. D. Pustovitov[a], S. V. Konovalov[a], A. I. Smolyakov[a, f], and V. S. Tsypin[†d]**

[a] Institute of Nuclear Fusion, Russian Research Centre Kurchatov Institute, pl. Kurchatov 1, Moscow, 123182 Russia
[b] Kharadze Abastumani National Astrophysical Observatory, 2a, Kazbegi Ave., Tbilisi 0160, Georgia
[c] Physics Institute, University of São Paulo, Cidade Universitaria, 05508-900, São Paulo, Brazil
[d] Brazilian Center for Physics Research, Rua Xavier Sigaud, 150, 22290-180, Rio de Janeiro, Brazil
[e] Syzran Branch of Samara Technical University, Sovetskaya ul. 45, Syzran, Samara region, 446001 Russia
[f] University of Saskatchewan, Science place 116, Saskatoon S7N 5E2, Canada

Received September 24, 2007; in final form, January 23, 2008

**Abstract**—The Velikhov effect leading to magnetorotational instability (MRI) is incorporated into the theory of ideal internal kink modes in a differentially rotating cylindrical plasma column. It is shown that this effect can play a stabilizing role for suitably organized plasma rotation profiles, leading to suppression of MHD (magnetohydrodynamic) instabilities in magnetic confinement systems. The role of this effect in the problem of the Suydam and the $m = 1$ internal kink modes is elucidated, where $m$ is the poloidal mode number.

PACS numbers: 52.30.Jb, 52.35.Py

**DOI:** 10.1134/S1063780X08070027

## 1. INTRODUCTION

It is known from general stability theory of nonrotating magnetically confined plasmas [1] that, in the presence of magnetic shear, the ideal internal kink modes play a rather important role. These modes are separated into the $m \gg 1$ modes, i.e., the so-called Suydam modes [2], the modes with finite $m > 1$ (see in detail [1]) and the $m = 1$ internal kink mode [3, 4], where $m$ is the azimuthal mode number. The papers [5, 6] brought in important contributions to the theory of the Suydam modes in a rotating plasma by taking into account the Velikhov effect [7, 8] and the finite flow velocity. The goal of the present paper is to further develop the theory of Suydam modes in a differentially rotating cylindrical plasma column, allowing for the Velikhov effect and making a step forward to incorporate this effect into the theory of the $m = 1$ internal kink mode.

We note that by “the Velikhov effect” we mean the term with $d\Omega^2/d\ln r$ in the mode equations of a rotating plasma column, where $\Omega$ is the plasma rotation frequency and $r$ is the radial coordinate. As is well known, this effect is destabilizing if the parameter $d\Omega^2/d\ln r$ is negative, exceeds a threshold value, and is stabilizing for positive $d\Omega^2/d\ln r$.

The growth rates of the Suydam modes in a nonrotating plasma were originally calculated in [9] for the case of stellarator geometry, its results were simplified in [10] for the case of slab geometry, and the growth rates for the case of cylindrical geometry were obtained in [11]. The analysis of the $m = 1$ internal kink mode presented in [3] was carried out in the framework of the energy principle. One of its important results was to clearly demonstrate that the radial dependence of the perturbed radial displacement of the $m = 1$ internal kink mode has a step-function character. The growth rate of this mode was first analytically calculated in [4]. Finally, the theory of ideal internal kink modes in a nonrotating cylindrical plasma column was summarized in Chapter 5 of book [1].

We assume that the equilibrium plasma velocity along the cylinder vanishes. The modes in a plasma moving along the cylinder have been considered, in particular, in [5, 6]. Note also that generalization of the Suydam stability criterion in the case of strong shear of the perpendicular plasma velocity has been studied in [12, 13]. These papers will be commented on below (see Section 7).

One of the most effective tools for studying MHD (magnetohydrodynamic) modes in a rotating plasma is the Frieman–Rotenberg (FR) approach [14]. One of the most important advantages of this approach is the use of the so-called FR variable describing the sum of perturbed kinetic and magnetic pressures. It was shown in

[1] The text was submitted by the authors in English.
[†] Deceased.

[5, 15] that, in terms of this variable and the perturbed radial plasma displacement, the perturbed plasma motion equations reduce to a pair of first-order differential equations with antisymmetric crossed matrix elements (see [5, 15] for details). For this reason, these equations can be properly called the Hameiri–Bondeson–Iacono–Bhattacharjee (HBIB) equations.

In Section 2 we explain our basic equations and the equilibrium. Section 3 is addressed to transformations of the perturbed basic equations. In Section 4 we derive the mode equation. In Section 5 the Suydam modes are analyzed, including the Velikhov effect. In Section 6 this effect on the $m = 1$ internal kink mode is studied. Discussions are given in Section 7.

## 2. BASIC EQUATIONS AND THE EQUILIBRIUM

### *2.1. Basic Plasmadynamic Equations*

We describe the plasma behavior using the standard system of equations of one-fluid MHD: the equation of motion

$$\rho d\mathbf{V}/dt = -\nabla p + \mathbf{j} \times \mathbf{B}, \tag{2.1}$$

the continuity equation

$$d\rho/dt + \rho \nabla \cdot \mathbf{V} = 0 \tag{2.2}$$

and the adiabatic equation of state

$$d(p\rho^{-\Gamma})/dt = 0. \tag{2.3}$$

These equations are complemented by the Maxwell equations,

$$\partial \mathbf{B}/\partial t = -\nabla \times \mathbf{E}, \tag{2.4}$$

$$\nabla \times \mathbf{B} = \mathbf{j}, \tag{2.5}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.6}$$

and the generalized Ohm law,

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0. \tag{2.7}$$

Here, $\rho$ and $\mathbf{V}$ are the mass density and the velocity of the plasma, respectively; $\Gamma$ is the adiabatic exponent; $\mathbf{B}$, $\mathbf{j}$, and $p$ are the magnetic field, the electric current density, and the plasma pressure, respectively; $\mathbf{E}$ is the electric field; and

$$d/dt = \partial/\partial t + \mathbf{V} \cdot \nabla. \tag{2.8}$$

Using Eqs. (2.4) and (2.7), one obtains the frozen-in condition in the form

$$\partial \mathbf{B}/\partial t - [\nabla \times [\mathbf{V} \times \mathbf{B}]] = 0. \tag{2.9}$$

The vector Eq. (2.7) has three projections, while Eq. (2.9) has only two. As the third projection of (2.7), we take

$$\mathbf{E} \cdot \mathbf{B} = 0. \tag{2.10}$$

Thus, our starting equations are Eqs. (2.1)–(2.6), (2.9), and (2.10).

### *2.2. Equilibrium*

We assume that the plasma has cylindrical symmetry characterized by radial coordinate $r$, poloidal coordinate $\theta$, and longitudinal coordinate $z$. The equilibrium parameters are functions of $r$. There is an equilibrium magnetic field $\mathbf{B}_0$ with projections $B_{0\theta}$ and $B_{0z}$ dependent on $r$. The plasma is characterized by equilibrium mass density $\rho_0$ and the equilibrium pressure $p_0$, both dependent on $r$.

The plasma rotates in the poloidal direction with radially dependent angular frequency $\Omega$. Accordingly, there is equilibrium electric field $\mathbf{E}_0 = (E_0, 0, 0)$ related to rotation frequency $\Omega$ by

$$E_0 = -\Omega r B_{0z}. \tag{2.11}$$

In addition, there are the poloidal and longitudinal equilibrium electric currents $j_{0\theta}$, $j_{0z}$ related to the magnetic field derivatives by

$$j_{0\theta} = -\frac{\partial B_{0z}}{\partial r}, \tag{2.12}$$

$$j_{0z} = \frac{1}{r}\frac{\partial}{\partial r}(r B_{0\theta}). \tag{2.13}$$

The equilibrium plasma pressure gradient, $p_0'$, is related to the rotation frequency and the equilibrium magnetic field derivatives by

$$p_0' = \rho_0 r \Omega^2 - \left(\frac{1}{2}\frac{\partial B_0^2}{\partial r} + \frac{B_{0\theta}^2}{r}\right). \tag{2.14}$$

It is also assumed that the plasma moves along the cylinder with the velocity $\mathbf{V}_{0z} = V_{0z}\mathbf{e}_z$.

### *2.3. Description of Perturbations*

The time and spatial dependence of the perturbations is taken in the eikonal form,

$$F(\mathbf{r}, t) = \exp(-i\omega t + im\theta + ik_z z)F(r). \tag{2.15}$$

Here, $F$ is any perturbed function, $\omega$ is the mode frequency, $m$ is the poloidal wavenumber, and $k_z$ is the longitudinal wavenumber. Perturbed magnetic field $\tilde{\mathbf{B}}$ is characterized by the components

$$\tilde{\mathbf{B}} = (\tilde{B}_r, \tilde{B}_\theta, \tilde{B}_z), \tag{2.16}$$

where the tilde denotes the perturbed values. Perturbed velocity $\tilde{\mathbf{V}}$ is expressed in terms of perturbed plasma displacement $\boldsymbol{\xi}$ defined by [15]

$$\tilde{\mathbf{V}} = \partial \boldsymbol{\xi}/\partial t + (\mathbf{V}_{0z} \cdot \nabla)\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla)\mathbf{V}_{0z}, \tag{2.17}$$

$$\tilde{V}_r = -i\tilde{\omega} X, \tag{2.18}$$

$$\tilde{V}_\theta = -i\tilde{\omega}\xi_\theta - \frac{d\Omega}{d\ln r}X, \tag{2.19}$$

$$\tilde{V}_z = -i\tilde{\omega}\xi_z - V'_{0z}X. \tag{2.20}$$

Here, $X \equiv \xi_r$ and $\tilde{\omega} = \omega - m\Omega - k_z V_{0z}$ is the Doppler-shifted mode frequency.

The linearized version of (2.9) yields

$$\tilde{B}_r = ik_\parallel B_0 X, \tag{2.21}$$

$$\tilde{B}_\theta = ik_z Y - \frac{\partial}{\partial r}(B_{0\theta}X), \tag{2.22}$$

$$\tilde{B}_z = -\frac{im}{r}Y - \frac{1}{r}\frac{\partial}{\partial r}(rB_{0z}X). \tag{2.23}$$

Here, $k_\parallel = k_z h_z + mh_\theta/r$ is the parallel wave number, $h_z = B_{0z}/B_0$, $h_\theta = B_{0\theta}/B_0$,

$$Y = \xi_\theta B_{0z} - \xi_z B_{0\theta}. \tag{2.24}$$

Physically, the function $Y$ describes the binormal plasma displacement.

Using (2.2) and (2.3), perturbed plasma density $\tilde{\rho}$ and pressure $\tilde{p}$ are expressed in terms of $\boldsymbol{\xi}$ by

$$\tilde{\rho} = -X\rho'_0 - \rho_0 u, \tag{2.25}$$

$$\tilde{p} = -Xp'_0 - \rho_0 c_s^2 u. \tag{2.26}$$

Here, $c_s^2 = \Gamma p_0/\rho_0$ is the square of the sound velocity,

$$u = \tau + i\left(\frac{m}{r}\xi_\theta + k_z\xi_z\right), \tag{2.27}$$

$$\tau \equiv \frac{1}{r}\frac{\partial}{\partial r}(rX). \tag{2.28}$$

One can introduce the third variable characterizing the displacement vector $\boldsymbol{\xi}$:

$$Z = \boldsymbol{\xi}\cdot\mathbf{B}_0 \equiv \xi_\theta B_{0\theta} + \xi_z B_{0z}. \tag{2.29}$$

Physically, the function $Z$ describes the plasma displacement along the equilibrium magnetic field. In terms of $Y$ and $Z$, the expressions for $\xi_\theta$ and $\xi_z$ are of the form

$$\xi_\theta = (h_\theta Z + h_z Y)/B_0, \tag{2.30}$$

$$\xi_z = (h_z Z - h_\theta Y)/B_0, \tag{2.31}$$

while

$$u = \tau + \frac{i}{B_0}(k_\parallel Z + k_b Y), \tag{2.32}$$

where $k_b = h_z m/r - h_\theta k_z$ is the binormal wave number.

## 3. TRANSFORMATIONS OF PERTURBED BASIC EQUATIONS

### *3.1. Reduction of Perturbed Basic Equations*

The $(\theta, z)$ projections of (2.1) take the form

$$\rho_0\left(-i\tilde{\omega}\tilde{V}_\theta + \frac{\kappa^2}{2\Omega}\tilde{V}_r\right) = -\frac{im}{r}\tilde{p} + [\mathbf{j}\times\mathbf{B}]^{\sim}_\theta, \tag{3.1}$$

$$\rho_0(-i\tilde{\omega}\tilde{V}_z + \tilde{V}_r V'_{0z}) = -ik_z\tilde{p} + [\mathbf{j}\times\mathbf{B}]^{\sim}_z. \tag{3.2}$$

Here, $\kappa^2 = (2\Omega/r)d(r^2\Omega)/dr \equiv 4\Omega^2 + d\Omega^2/d\ln r$ and the tilde denotes the linearly perturbed part.

Using (2.18)–(2.20), we obtain from (3.1) and (3.2) the parallel equation of motion

$$\begin{aligned}&\rho_0(-\tilde{\omega}^2 Z - i2\tilde{\omega}\Omega h_\theta B_0 X)\\ &= -ik_\parallel B_0\tilde{p} - \tilde{B}_r[\mathbf{j}_0\times\mathbf{B}_0]_r.\end{aligned} \tag{3.3}$$

By means of (2.21), (2.26), we find from (3.3) the following expression for $Z$:

$$Z = \frac{-iB_0}{\alpha_s\tilde{\omega}^2}\left[\lambda_{ZX}X + c_s^2 k_\parallel\left(\tau + \frac{ik_b Y}{B_0}\right)\right]. \tag{3.4}$$

Here, $\alpha_s = 1 - k_\parallel^2 c_s^2/\tilde{\omega}^2$ and,

$$\lambda_{ZX} = 2\tilde{\omega}\Omega h_\theta + k_\parallel r\Omega^2. \tag{3.5}$$

In addition, by means of (2.18)–(2.20), (3.1), and (3.2), we find the binormal projection of the equation of motion,

$$\rho_0\left(-\tilde{\omega}^2\frac{Y}{B_0} - i2\tilde{\omega}\Omega h_\theta X\right) = -ik_b\tilde{p} + (\mathbf{j}\times\mathbf{B})^{\sim}_b. \tag{3.6}$$

Here,

$$\begin{aligned}(\mathbf{j}\times\mathbf{B})^{\sim}_b &= h_z(\mathbf{j}\times\mathbf{B})^{\sim}_\theta - h_\theta(\mathbf{j}\times\mathbf{B})^{\sim}_z\\ &= -B_0 k^2 Y + iB_0^2 k_b\tau\\ &+ i[k_b(\rho_0 r\Omega^2 - p'_0) + 2k_z h_\theta B_0^2/r]X,\end{aligned} \tag{3.7}$$

where $k^2 = m^2/r^2 + k_z^2$.

Let us introduce the Frieman–Rotenberg variable $p_*$ defined by [14]:

$$p_* = \tilde{p} + \mathbf{B}_0\cdot\tilde{\mathbf{B}}. \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6), we arrive at

$$\begin{aligned}&\rho_0\left[(D_0 - k_y^2 v_A^2)\frac{Y}{B_0} - i2\tilde{\omega}\Omega h_\theta X\right]\\ &= -ik_b p_* + ik_b\mathbf{B}_0\cdot\tilde{\mathbf{B}} + iB_0^2 k_b\tau\\ &+ i\left[k_b(\rho_0 r\Omega^2 - p'_0) + \frac{2k_z h_\theta}{r}B_0^2\right]X,\end{aligned} \tag{3.9}$$

where

$$D_0 = \tilde{\omega}^2\alpha_A, \quad (3.10)$$

$\alpha_A = 1 - k_\parallel^2 v_A^2/\tilde{\omega}^2$, $v_A^2 = B_0^2/4\pi\rho_0$ is the Alfvén velocity squared. According to (2.22) and (2.23),

$$\begin{aligned}\mathbf{B}_0\cdot\tilde{\mathbf{B}} &= -iB_0k_bY - B_0^2\tau \\ &+ (p_0' + 2B_\theta^2/r - \rho_0 r\Omega^2)X.\end{aligned} \quad (3.11)$$

Substituting (3.11) into (3.9), one has

$$\begin{aligned}\rho_0\left(D_0\frac{Y}{B_0} - i2\tilde{\omega}\Omega h_\theta X\right) \\ = -ik_b p_* + \frac{2ih_z h_\theta B_0^2}{r}k_\parallel X.\end{aligned} \quad (3.12)$$

It follows from (3.12) that

$$\frac{Y}{B_0} = \frac{i}{D_0}\left(\frac{k_b p_*}{\rho_0} - \lambda_{YX}X\right), \quad (3.13)$$

where

$$\lambda_{YX} = 2h_z\left(\tilde{\omega}\Omega + \frac{k_\parallel}{r}h_\theta v_A^2\right). \quad (3.14)$$

By means of (3.13) and (3.14), Eq. (3.11) reduces to

$$\begin{aligned}\mathbf{B}_0\cdot\tilde{\mathbf{B}} &= -B_0^2\tau + \frac{k_b}{D_0}(k_b v_A^2 p_* - B_0^2\lambda_{YX}X) \\ &+ \left(p_0' + \frac{2B_0^2h_\theta^2}{r} - \rho_0 r\Omega^2\right)X.\end{aligned} \quad (3.15)$$

Substituting (3.13) into (3.3) yields

$$\begin{aligned}Z &= \frac{-iB_0}{\alpha_s\tilde{\omega}^2}\left[\left(\lambda_{ZX} + \frac{k_\parallel k_b c_s^2}{D_0}\lambda_{YX}\right)X \right.\\ &\left. + c_s^2 k_\parallel\tau - \frac{k_b^2 k_\parallel c_s^2}{D_0\rho_0}p_*\right].\end{aligned} \quad (3.16)$$

With (3.13) and (3.16), Eq. (2.33) takes the form

$$u = \frac{1}{\alpha_s}\left[\tau - \frac{k_b^2 p_*}{D_0\rho_0} + \left(\frac{k_\parallel\lambda_{ZX}}{\tilde{\omega}^2} + \frac{k_b\lambda_{YX}}{D_0}\right)X\right]. \quad (3.17)$$

By means of (3.17) we obtain from (2.26)

$$\begin{aligned}\tilde{p} &= -\frac{\rho_0 c_s^2}{\alpha_s}\left(\tau - \frac{k_b^2 p_*}{D_0\rho_0}\right) \\ &- \left[p_0' + \frac{\rho_0 c_s^2}{\alpha_s}\left(\frac{k_\parallel\lambda_{ZX}}{\tilde{\omega}^2} + \frac{k_b\lambda_{YX}}{D_0}\right)\right]X.\end{aligned} \quad (3.18)$$

## 4. DERIVATION OF MODE EQUATION

With (3.16) and (3.18), Eq. (3.8) reduces to

$$D\tau = C_1X - C_2p_*. \quad (4.1)$$

Here,

$$D = D_0(1 + \beta/\alpha_s), \quad (4.2)$$

$$C_1 = D_0\left[\frac{2h_\theta^2}{r} - \frac{r\Omega^2}{v_A^2} - \frac{\beta k_\parallel}{\alpha_s\tilde{\omega}^2}\lambda_{ZX}\right] - k_b\left(1 + \frac{\beta}{\alpha_s}\right)\lambda_{YX}, \quad (4.3)$$

$$C_2 = [D_0 - k_b^2 v_A^2(1 + \beta/\alpha_s)]/B_0^2. \quad (4.4)$$

The $r$th projection of (2.1) is represented in the form

$$\begin{aligned}\rho_0\left[-\left(\tilde{\omega}^2 - \frac{d\Omega^2}{d\ln r}\right)X + 2i\Omega\tilde{\omega}\xi_\theta\right] - \tilde{\rho}r\Omega^2 \\ = -\tilde{p}' + (\mathbf{j}+\mathbf{B})_r^{\sim}.\end{aligned} \quad (4.5)$$

It can be seen that

$$(\mathbf{j}\times\mathbf{B})_r^{\sim} = -k_\parallel^2B_0^2X - \frac{\partial}{\partial r}(\mathbf{B}_0\cdot\tilde{\mathbf{B}}) - \frac{2}{r}h_\theta B_0\tilde{B}_\theta. \quad (4.6)$$

By means of (3.17), Eq. (2.25) reduces to

$$\begin{aligned}\tilde{\rho} &= -\frac{\rho_0}{\alpha_s}\left(\tau - \frac{k_b^2 p_*}{D_0\rho_0}\right) \\ &- \left[\rho_0' + \frac{\rho_0}{\alpha_s}\left(\frac{k_\parallel\lambda_{ZX}}{\tilde{\omega}^2} + \frac{k_b\lambda_{YX}}{D_0}\right)\right]X.\end{aligned} \quad (4.7)$$

Similarly, after substituting (3.13) and (3.16), Eq. (2.30) yields

$$\begin{aligned}\xi_\theta = i\left\{-\frac{h_\theta k_\parallel c_s^2}{\alpha_s\tilde{\omega}^2}\tau + \left(h_z + \frac{h_\theta k_b k_\parallel c_s^2}{\alpha_s\tilde{\omega}^2}\right)\frac{k_b}{D_0\rho_0}p_* \right.\\ \left. - \left[\frac{\lambda_{YX}}{D_0}\left(h_z + \frac{h_\theta k_b k_\parallel c_s^2}{\alpha_s\tilde{\omega}^2}\right) + \frac{h_\theta\lambda_{ZX}}{\alpha_s\tilde{\omega}^2}\right]X\right\}.\end{aligned} \quad (4.8)$$

Finally, it follows from (2.22) and (3.13) that

$$\begin{aligned}\tilde{B}_\theta &= -B_{0\theta}\tau - \frac{k_z B_0 k_b}{D_0}p_* \\ &- \left[r\frac{\partial}{\partial r}\left(\frac{h_\theta B_0}{r}\right) - \frac{k_z B_0}{D_0}\lambda_{YX}\right]X.\end{aligned} \quad (4.9)$$

Using (4.6)–(4.9), we transform (4.5) to

$$p_*' = \rho_0(\lambda_{r*}p_* - \lambda_{r\tau}\tau + \lambda_{rX}X). \quad (4.10)$$

Here,

$$\lambda_{r*} = \frac{1}{\rho_0 D_0}\left[2\Omega\tilde{\omega}k_b\left(h_z + \frac{h_\theta k_b k_\parallel c_s^2}{\alpha_s\tilde{\omega}^2}\right) + \frac{\Omega^2 r k_b^2}{\alpha_s} + \frac{2}{r}h_\theta v_A^2 k_z k_b\right], \quad (4.11)$$

$$\lambda_{r\tau} = 2\Omega\frac{h_\theta k_\parallel c_s^2}{\alpha_s\tilde{\omega}} + \frac{\Omega^2 r}{\alpha_s} - \frac{2}{r}h_\theta^2 v_A^2, \quad (4.12)$$

$$\lambda_{rX} = D_0 - \frac{d\Omega^2}{d\ln r} - \Omega^2\left[\frac{d\ln\rho_0}{d\ln r} + \frac{r}{\alpha_s}\left(k_\parallel\frac{\lambda_{ZX}}{\tilde{\omega}^2} + \frac{k_b\lambda_{YX}}{D_0}\right)\right] - 2\Omega\tilde{\omega}\left[\frac{\lambda_{YX}}{D_0}\left(h_z + \frac{h_\theta k_b k_\parallel c_s^2}{\alpha_s\tilde{\omega}^2}\right) + \frac{h_\theta\lambda_{ZX}}{\alpha_s\tilde{\omega}^2}\right] + 2h_\theta v_A^2\left[\frac{1}{B_0}\frac{\partial}{\partial r}\left(\frac{h_\theta B_0}{r}\right) - \frac{k_z\lambda_{YX}}{rD_0}\right]. \quad (4.13)$$

By means of (4.1), we exclude $\tau$ from (4.10), arriving at

$$Dp'_* = C_3 X - \bar{C}_1 p_*. \quad (4.14)$$

Here,

$$\bar{C}_1 = -\rho_0(D\lambda_{r*} + \lambda_{r\tau}C_2), \quad (4.15)$$

$$C_3 = \rho_0(D\lambda_{rX} - \lambda_{r\tau}C_1). \quad (4.16)$$

Substituting (4.3) and (4.4) into (4.15) and (4.16), one obtains

$$\bar{C}_1 = C_1, \quad (4.17)$$

$$C_3 = \rho_0\left\{D_0\left(1 + \frac{\beta}{\alpha_s}\right)\left[\zeta^{(1)} + \left(\frac{\Omega^2 r}{v_A^2} - \frac{2}{r}h_\theta^2\right)\times\left(\frac{\Omega^2 r}{\alpha_s} - \frac{2}{r}h_\theta^2 v_A^2 + \frac{2\Omega k_\parallel h_\theta c_s^2}{\alpha_s\tilde{\omega}}\right)\right] - \left(1 + \frac{\beta}{\alpha_s}\right)\lambda_{YX}^2 - D_0\frac{\lambda_{ZX}}{\alpha_s\tilde{\omega}^2}\times\left[\Omega^2 r k_\parallel + 2\Omega\tilde{\omega}h_\theta(1+\beta) + \frac{2}{r}k_\parallel h_\theta c_s^2\right]\right\}, \quad (4.18)$$

where

$$\zeta^{(1)} = D_0 - \frac{d\Omega^2}{d\ln r} - \Omega^2\frac{d\ln\rho_0}{d\ln r} + \frac{1}{\rho_0}\frac{d}{d\ln r}\left(\frac{h_\theta B_0}{r}\right)^2. \quad (4.19)$$

Excluding $p_*$ from (4.14) by means of (4.1), we arrive at the mode equation

$$\frac{1}{r}\left(\frac{rD}{C_2}X'\right)' - UX = 0, \quad (4.20)$$

where

$$U = -\frac{1}{D}\left(C_3 - \frac{C_1^2}{C_2}\right) - \left(\frac{D}{rC_2}\right)' + r\left(\frac{C_1}{rC_2}\right)'. \quad (4.21)$$

Now, we allow tokamak ordering (see in detail [1]). Assume $\tilde{\omega}$ = const; then, Eq. (4.20) reduces to

$$\frac{1}{r}[\rho_0 r^3(k_\parallel^2 v_A^2 - \tilde{\omega}^2)X']' - U_0 X = 0, \quad (4.22)$$

where

$$U_0 = m^2 U. \quad (4.23)$$

Expression (4.23) is transformed to

$$U_0 = U_0^{(0)} + U_0^{(r)}, \quad (4.24)$$

where $U_0^{(0)}$ is the standard expression for the function $U_0$ in a nonrotating plasma given by (see [1])

$$U_0^{(0)} = \frac{1}{r^2}(m^2 - 1 + k_z^2 r^2)(mB_{0\theta} + rk_z B_{0z})^2 + 2k_z^2 r p_0' + \frac{2k_z^2}{m^2}(r^2 k_z^2 B_{0z}^2 - m^2 B_{0\theta}^2), \quad (4.25)$$

while $U_0^{(r)}$ is the rotational part of this function defined by

$$U_0^{(r)} = \rho_0(k_z^2 r^2 + h_\theta^2 m^2)d\Omega^2/d\ln r. \quad (4.26)$$

We will use the notation

$$U_0^V = U_0^{(0)} + U_0^{(r)}, \quad (4.27)$$

where the superscript "$V$" in $U_0$ stands for the first letter in the name of the author of [7].

## 5. SUYDAM MODES ALLOWING FOR VELIKHOV EFFECT

### *5.1. Mode Equation*

The Suydam modes are the ideal internal kink modes with $m \gg 1$ localized in a vicinity of the singular point $r = r_0$, for which

$$(mB_{0\theta} + rk_z B_{0z})_{r=r_0} = 0. \quad (5.1)$$

Introducing $x \equiv r - r_0$, for $x \neq 0$ one has

$$mB_{0\theta} + rk_z B_{0z} = -\frac{xmB_{0\theta}}{r_0}S, \quad (5.2)$$

where $S$ is the magnetic shear defined by

$$S = r_0 q'/q, \tag{5.3}$$

$q = rB_{0z}/(RB_{0\theta})$ is the safety factor, $R = L/(2\pi)$, and $L$ is the cylinder length.

By means of Eq. (4.22), we arrive at the following mode equation:

$$(x^2X')' - \left(\frac{m^2}{r_0^2}x^2 + U_0^V\right)X + \frac{\gamma^2 r_0^4}{m^2 S^2 v_{A\theta}^2}X'' = 0. \tag{5.4}$$

Here

$$U_0^V = \frac{2r_0 p_0'}{B_0^2 S^2} + \frac{2r_0^2}{v_A^2 S^2}\frac{d\Omega^2}{d\ln r}, \tag{5.5}$$

$\gamma^2 \equiv -\tilde{\omega}^2$ is the squared growth rate of the perturbations, and $v_{A\theta} = (B_{0\theta}/B_{0z})v_A$ is the poloidal Alfvén velocity.

We are interested in perturbations near the instability boundary, so that squared growth rate $\gamma^2$ is assumed to be a small parameter. Then, for not too small $x$, the term with $\gamma^2$ is unimportant, while Eq. (5.4) reduces to

$$(x^2X')' - \left(\frac{m^2}{r_0^2}x^2 + U_0^V\right)X = 0. \tag{5.6}$$

This is the mode equation in the so-called ideal region. On the other hand, for sufficiently small $x$, the term with $m^2x^2$ in Eq. (5.4) is unimportant, so that this equation transits to

$$(x^2X')' - U_0^V X + \frac{\gamma^2 r_0^4}{m^2 S^2 v_{A\theta}^2}X'' = 0. \tag{5.7}$$

This is the mode equation in the so-called inertial region. We will look for solutions of Eqs. (5.6) and (5.7) and then match the asymptotics of these solutions, obtaining the growth rate of the eigenmodes.

### 5.2. Solution in the Ideal Region

The solution of Eq. (5.6) is of the form

$$X \sim x^{-1/2}K_{i\alpha}(\hat{x}). \tag{5.8}$$

Here $\hat{x} \equiv mx/r_0$, $K_{i\alpha}$ is the Bessel function of the second kind of imaginary argument, and the parameter $\alpha$ is determined by

$$\alpha^2 = -(1/4 + U_0^V) > 0. \tag{5.9}$$

It is assumed in (5.8) that $x > 0$. For $x < 0$, in accordance with (5.6), the solution (5.8) should be substituted by replacement $x \longrightarrow -x$. The asymptotic of solution (5.8) for $\hat{x} \ll 1$ has the form

$$X \sim \hat{x}^{-(s+1)}(1 + \hat{x}^{(2s+1)}\Delta), \tag{5.10}$$

where

$$s = -1/2 + (1/4 + U_0^V)^{1/2} = -1/2 + i\alpha, \tag{5.11}$$

$$\Delta = 2^{-(2s+1)}/f(s), \tag{5.12}$$

$$f(s) = \Gamma(s + 1/2)/\Gamma(-s - 1/2), \tag{5.13}$$

and $\Gamma$ is the gamma function.

### 5.3. Solution in Inertial Region

We introduce a new variable $\xi$ defined by

$$\xi = xmSv_{A\theta}/\gamma r_0^2. \tag{5.14}$$

Then, Eq. (5.7) is transformed to

$$(\xi^2 + 1)d^2X/d\xi^2 + 2\xi dX/d\xi - s(s+1)X = 0. \tag{5.15}$$

The general solution of Eq. (5.15) is of the form

$$X = AX_+(\xi) + BX_-(\xi). \tag{5.16}$$

Here, $A$ and $B$ are arbitrary constants and the functions $X_+$ and $X_-$ are given by

$$X_+ = F\left(-\frac{s}{2}, \frac{1+s}{2}; \frac{1}{2}; -\xi^2\right), \tag{5.17}$$

$$X_- = \xi F\left(\frac{1-s}{2}, 1 + \frac{s}{2}; \frac{3}{2}; -\xi^2\right), \tag{5.18}$$

where $F$ is the hypergeometrical function. The function $X_+$ corresponds to the even solutions and $X_-$ to the odd solutions. The asymptotics of Eqs. (5.17) and (5.18) for $\xi \gg 1$ are

$$X_\pm \sim \hat{x}^{-(s+1)}(1 + \hat{x}^{2s+1}\lambda^{-(2s+1)}/\Delta_\pm); \tag{5.19}$$

This asymptotics, as Eqs. (5.17) and (5.18), is written for $x > 0$. Here,

$$\Delta_+ = \frac{1}{f(s)}\Gamma^2\left(\frac{1+s}{2}\right)/\Gamma^2\left(-\frac{s}{2}\right), \tag{5.20}$$

$$\Delta_- = \frac{1}{f(s)}\Gamma^2\left(1 + \frac{s}{2}\right)/\Gamma^2\left(\frac{1-s}{2}\right), \tag{5.21}$$

the value $\lambda$ is the dimensionless growth rate introduced by $\lambda = \gamma/\omega_A$, where $\omega_A = Sv_{A\theta}/r_0$.

### 5.4. Dispersion Relations for Suydam Modes and Growth Rate near the Instability Boundary

From the requirement of coincidence of Eqs. (5.10) and (5.16), we find the dispersion relations for the Suydam modes near the instability boundary,

$$\lambda^{2s+1} = 1/(\Delta\Delta_\pm). \tag{5.22}$$

The subscript "+": corresponds to even modes and the subscript "–" to odd modes.

For the lowest level of the even modes, Eq. (5.22) yields

$$\lambda = \lambda_0 \equiv 16\exp(-\pi/\alpha - C - \pi/2), \tag{5.23}$$

where $C$ is the Euler constant. This $\lambda$ corresponds to the growth rate:

$$\gamma = S v_{A\theta}\lambda_0/r_0. \tag{5.24}$$

Therefore, as an estimate for the growth rate of the Suydam modes, one can use the formula $\gamma \simeq S v_{A\theta}/r$.

### *5.5. Qualitative Discussion of Suydam Modes*

For $\gamma = 0$, Eq. (5.4) has the solution

$$X \sim x^{-1/2} \tag{5.25}$$

in the condition

$$1/4 + U_0^V = 0. \tag{5.26}$$

This is the stability boundary. In order to determine what the stability/instability regions are, one can use the energy principle [1]. Then, one should constitute the potential energy functional and determine where it is negative. In the case considered, the functional is proportional to

$$W \sim \int [(xX')^2 + U_0^V X^2] dx. \tag{5.27}$$

One can find that $W > 0$ if

$$1/4 + U_0^V > 0. \tag{5.28}$$

This is the Suydam–Velikhov stability criterion. It can be expressed in the following form:

$$\frac{S^2}{4} + \frac{2r_0 p_0'}{B_0^2} + \frac{2r_0^2}{v_A^2}\frac{d\Omega^2}{d\ln r} > 0. \tag{5.29}$$

The term with $S^2$ in Eq. (5.29) describes the stabilization effect by the magnetic shear. The term with $p_0'$ corresponds to the magnetic hill effect. Thus, in nonrotating plasma the Suydam stability criterion describes a competition between the effects of magnetic shear and magnetic hill. The Velikhov effect essentially modifies this competition for

$$\frac{V_0^2}{v_{Ti}^2} \ge \frac{\Delta r}{r_0}, \tag{5.30}$$

where $\Delta r$ is the characteristic radial width of the rotation frequency shear, $V_0$ is the characteristic velocity of plasma rotation, and $v_{Ti}$ is the ion thermal velocity.

It is remarkable that, for $\Delta r \ll r_0$, the Velikhov effect is essential even for subthermal rotation velocity. Then, organizing a favorable rotation frequency profile, $d\Omega^2/d\ln r > 0$, one can suppress the Suydam modes.

## 6. THE VELIKHOV EFFECT ON THE $m = 1$ INTERNAL KINK MODE

Taking $m = 1$ in Eq. (4.22), we obtain

$$\begin{aligned} U_0^V = {} & k_z^2[2rp_0' - B_{0\theta}^2(1-q)(1+3q)] \\ & + \rho_0(k_z^2 r^2 h_\theta^2) d\Omega^2/d\ln r. \end{aligned} \tag{6.1}$$

The simplest problem formulation on the $m = 1$ ideal internal kink mode is the following. One considers an ideally conducting casing at the plasma boundary $r = r_*$ and that the longitudinal current density decreases with increasing radius, so that the safety factor $q$ increases with increasing radius. It is assumed that, at some $r = r_0 < r_*$, the relation $q(r_0) = 1$ is satisfied.

The instability condition of the $m = 1$ mode can be obtained if one takes the radial dependence of $X$ in the form

$$X = \begin{cases} C, & r < r_0, \\ 0, & r > r_0, \end{cases} \tag{6.2}$$

whence $C$ is a constant. This choice of perturbation will be explained below. According to Eq. (6.2), for $r \longrightarrow r_0$ derivative $X'$ is formally infinite; however, it enters integral (5.27) with weight $x^2 \longrightarrow 0$. Therefore, the contribution of the square of this derivative to the above integral can be considered negligibly small. This approximation will also be justified below.

The integral of type Eq. (5.27) characterizes the contribution into the energy functional only from a small vicinity near the singular point. The total energy functional is of the form

$$W \sim |C|^2 \int_0^{r_0} U_0^V r dr, \tag{6.3}$$

where $U_0^V$ is given by Eq. (6.1). In the absence of the Velikhov effect one has $p_0' < 0$, $1 - q > 0$, i.e., both terms in the square brackets of Eq. (6.1) are negative. Consequently, Eq. (6.3) yields

$$W < 0, \tag{6.4}$$

which corresponds to instability.

Evidently, the presence of the term with $d\Omega^2/d\ln r$ in $U_0^V$ should modify this conclusion. Such a modification is obtained using the standard approach that goes back to [4]. Let us turn to an explanation of this approach.

Introducing $\gamma^2$, we represent Eq. (4.22) in the form

$$\frac{d}{dr}\left\{[\rho_0\gamma^2 + (\mathbf{k}\cdot\mathbf{B}_0)^2] r^3 \frac{dX}{dr}\right\} - rU_0^V X = 0. \tag{6.5}$$

Here $\mathbf{k} \cdot \mathbf{B} \equiv (1 - q)B_{0\theta}/r$. In solving Eq. (6.5), one distinguishes the region of the singular layer, where $(\mathbf{k} \cdot \mathbf{B})^2 \le \rho_0\gamma^2$, and the external regions where $(\mathbf{k} \cdot \mathbf{B}) \gg \rho\gamma^2$. For the external regions, one then finds that

$$X \approx C, \quad X' = \frac{C}{r^3(\mathbf{k} \cdot \mathbf{B}_0)^2}\int_0^r rU_0^V dr, \quad r < r_0, \tag{6.6}$$

$$X = -\bar{C}\int_r^{r_*} \frac{dr}{r^3(\mathbf{k} \cdot \mathbf{B}_0)^2},$$
$$X' = \frac{\bar{C}}{r^3(\mathbf{k} \cdot \mathbf{B}_0)^2}, \quad r > r_0, \tag{6.7}$$

where $\bar{C}$ is another constant. Inside the singular layer, one takes (cf. Eq. (5.2))

$$\mathbf{k} \cdot \mathbf{B}_0 = -(r - r_0)q'B_{0\theta}r_0/r. \tag{6.8}$$

Then, Eq. (6.5) reduces to

$$\frac{d}{dx}\left[(1 + x^2)\frac{dX}{dx}\right] = 0, \tag{6.9}$$

where

$$x = \frac{\omega_A(r_0)}{r_0}(r - r_0). \tag{6.10}$$

It follows from Eq. (6.9) that

$$X = \frac{C}{2}\left(1 - \frac{2}{\pi}\arctan x\right). \tag{6.11}$$

Here, the constants are chosen allowing for the fact that for $|x| \longrightarrow \infty$ solution (6.11) turns into Eq. (6.2).

One finds from Eq. (6.11) that for $|x| \gg 1$

$$X' = -\frac{C}{\pi}\frac{r_0\lambda}{(r - r_0)^2}, \tag{6.12}$$

where $\lambda = \gamma/\omega_A(r_0)$ is the dimensionless growth rate. Matching this expression with the second Eq. (6.6) and using Eq. (6.8), we arrive at the dispersion relation

$$\lambda = \lambda_H^V, \tag{6.13}$$

where

$$\lambda_H^V = -\frac{\pi}{S^2 B_{0\theta}^2}\int_0^{r_0} rU_0^V dr. \tag{6.14}$$

As in the case of Suydam modes (see Section 5), organizing a favorable rotation frequency profile, $d\Omega^2/d\ln r > 0$, one can suppress the $m = 1$ internal kink mode. A qualitative estimate of the rotation frequency necessary for such a suppression is given by Eq. (5.30).

## 7. DISCUSSIONS

We have analyzed the effect of differential plasma rotation (the Velikhov effect [7]) on the nonaxisymmetric modes in a cylindrical plasma immersed in a strong magnetic field. We have found that this effect can be essential even for subthermal rotation velocity; see the estimate given by Eq. (5.30). Our analysis, which is relevant to the simplest magnetic field geometry, discovers practically a new trend in studying the rotation effect on the internal modes in toroidal geometry. In this context, it is reasonable to note that rather strong profiles of plasma rotation velocity are revealed in the region of internal transport barriers (ITBs) of toroidal systems (see, e.g., [16] and the bibliography given there). Evidently, detailed numerical calculations should be performed for elucidating the role of the Velikhov effect in the ITB.

The difference of our analysis from that of [5, 6] is that we consider $\tilde{\omega}$ as a finite constant, $\tilde{\omega}(r_0) \neq 0$, which is small compared with $k_z v_A$ in the ideal region. In contrast, $\tilde{\omega}(r_0) = 0$ was taken in [5, 6]. This means that we deal with growth rates greater than those in [5, 6].

Generalization of the Suydam stability criterion for a sheared perpendicular plasma velocity neglecting the Velikhov effect was considered in [12]. The authors of [12] have concluded that the Suydam instability takes place even in the presence of a magnetic well, i.e., for $U_0^V > 0$. This conclusion has been refuted in [13]. In contrast to [12], in the present paper, the case of magnetic hill is considered, $U_0^V < 0$. Therefore, criticism of [13] does not concern the results of our paper.

## ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (RFBR) (grant no. 06-02-16767); the International Science and Technology Center (ISTC) (grant no. G-1217); the NSERC Canada and NATO Collaborative Linkage Grant; the Research Foundation of the State of São Paulo (FAPESP); the National Council of Scientific and Technological Development (CNPq); and the Ministry of Science and Technology, Brazil.

## REFERENCES

1. A. B. Mikhailovskii, *Instabilities in a Confined Plasma* (IOP, Bristol, 1998).
2. B. R. Suydam, in *Proceedings of the Second United Nations International Conference on Peaceful Uses of Atomic Energy, Geneva, 1958*, Vol. 31, p. 157.
3. V. D. Shafranov, Zh. Tekh. Fiz. **40**, 241 (1970) [Sov. Phys. Tech. Phys. **15**, 175 (1970)].

4. M. N. Rosenbluth, R. Y. Dagazian, and P. H. Rutherford, Phys. Fluids **16**, 1864 (1973).
5. A. Bondeson, R. Iacono, and A. Bhattacharjee, Phys. Fluids **30**, 2167 (1987).
6. C. Wang, J. W. S. Blokland, R. Keppens, and J. P. Goedbloed, J. Plas. Phys. **70**, 651 (2004).
7. E. P. Velikhov, Zh. Éksp. Teor. Fiz. **36**, 1398 (1959) [Sov. Phys. JETP **9**, 995 (1959)].
8. I. V. Khal'zov, V. I. Ilgisonis, A. I. Smolyakov, and E. P. Velikhov, Phys. Fluids **18**, 124 107 (2006).
9. R. M. Kulsrud, Phys. Fluids **6**, 904 (1963).
10. A. B. Mikhailovskii, *Electromagnetic Instabilities in an Inhomogeneous Plasma* (Énergoatomizdat, Moscow, 1991; Institute of Physics, Bristol, 1992).
11. T. E. Stringer, Nucl. Fusion **15**, 125 (1975).
12. A. B. Mikhailovskii and S. E. Sharapov, Plasma Phys. Controlled Fusion **42**, 57 (2000).
13. A. V. Timofeev, Plasma Phys. Controlled Fusion **43**, L31 (2001).
14. E. Frieman and N. Rotenberg, Rev. Mod. Phys. **32**, 898 (1960).
15. E. Hameiri, Phys. Fluids **26**, 230 (1981).
16. ITER Physics Expert Group on Confinement and Transport, ITER Physics Expert Group on Confinement Modelling and Database, and ITER Physics Basis Editors, Nucl. Fusion **39**, 2175 (1999).