

# Spectra of primordial fluctuations in two-perfect-fluid regular bounces

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We introduce analytic solutions for a class of two components bouncing models, where the bounce is triggered by a negative energy density perfect fluid. The equation of state of the two components are constant in time, but otherwise unrelated. By numerically integrating regular equations for scalar cosmological perturbations, we find that the (would be) growing mode of the Newtonian potential before the bounce never matches with the the growing mode in the expanding stage. For the particular case of a negative energy density component with a stiff equation of state we give a detailed analytic study, which is in complete agreement with the numerical results. We also perform analytic and numerical calculations for long wavelength tensor perturbations, obtaining that, in most cases of interest, the tensor spectral index is independent of the negative energy fluid and given by the spectral index of the growing mode in the contracting stage. We compare our results with previous investigations in the literature.

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## INTRODUCTION

Cosmological models with a bounce [1] - a contraction which reverses into an expansion - may solve the horizon problem [2] in a non-inflationary way, i.e. as the Pre Big-Bang [3] and Ekpyrotic [4] scenarios. In order to make bouncing models real competitors of inflation, or at least complementary to it, while addressing the singularity problem unavoidably present in such models, the spectrum of density perturbations which results from the bounce should be understood as it is in inflationary theories. Unfortunately, the physics of cosmological perturbations during a bounce is much more subtle, because of the reversal of growing and decaying modes before and after the bounce, so that even though the bounce duration itself may be very short, usual matching conditions [5, 6] should be used with particular care and verified.

These subtleties have generated many works on the subject, in particular after the proposal of the Ekpyrotic scenario [4], which is based on a very slow contraction and needs a bounce, as in the Pre Big-Bang model studied in the Einstein frame. Unanimous conclusions on the resulting spectrum of metric fluctuations in the expanding stage are still to come. Among the ongoing controversies, one concerned the fate of cosmological perturbations in a hydrodynamical radiation bounce triggered by a negative energy density scalar field [7], generalized afterwards in [8]. This bounce - and its generalization - has been sug-

gested as a simple toy model, which has the advantage of providing analytic solutions for the background, although having a component which violates the null energy condition. Note that without assuming spatial curvature and demanding general relativity to hold, such a negative energy component is required at the level of an effective theory in order for a bounce to take place [9].

The initial result in this class of two-fluid models was that the spectrum of the Newtonian potential after the bounce was the same as that of the growing mode before the bounce. Such a result was obtained evolving numerically [7] and analytically [7, 8] a set of regular equations. This result was later challenged in [10, 11], generalized in Ref. [12] in which the scalar perturbations are evolved through a bounce characterized by a single physical scale, arguing that the growing mode before the bounce matched only with the decaying mode after the bounce, a possibility which has been already found [5, 6, 13, 14, 15]. However, the analysis of [10, 11], demanding the most general possible situation (the case at hand in the present work being a subset), needs to rely on a set of singular equations, a fact that could cast doubts on its accuracy had they use them directly; these authors however obtained the solution in the form of a Born-like series containing only convergent integrals; Non-singular equations have also been evolved in different contexts, e.g. with a double scalar field bounce [16] or one with a non-local dilaton potential stemming from string theory

[17]. Note that for tensor modes, we already know that a matching between growing modes before and after the bounce occurs [8, 18].

In this paper we consequently reanalyze the behaviour of cosmological perturbations during the radiation bounce, obtaining results in agreement with these later studies [10, 11], and in contrast with our previous findings for scalar perturbations in [7, 8]. In section II we present the background bouncing models containing two perfect fluids with linear and unrelated equations of state. We also show how to describe the negative energy perfect fluid in terms of a K-essence scalar field. In section III we propose a set of regular equations for linear perturbations of the above background models, one for the Newtonian potential and the other for the velocity potential of the fluid responsible for the bounce, which can be related to the linear perturbations of the K-essence scalar field yielding simpler regular equations suitable for the numerical analysis presented in section IV (another set involving the density contrast instead of the velocity potential is given in the appendix). In section V we justify some of the numerical results through an analytical study of approximate solutions and their matchings. We end in section VI with discussions and conclusions.

## BACKGROUND

We shall consider a class of bouncing universes filled by two non-interacting perfect fluids with parameters of state  $w_+, w_-$ , constant in time [8], relating the energy densities  $\epsilon_\pm$  to the pressures  $p_\pm$  through  $p_\pm = w_\pm \epsilon_\pm$ . The Einstein energy constraint for homogeneous and isotropic solutions is

$$H^2 = \left( \frac{da}{adt} \right)^2 = \ell_{\text{Pl}}^2 \left[ \frac{\rho_+}{a^{3(1+w_+)}} - \frac{\rho_-}{a^{3(1+w_-)}} \right], \quad (1)$$

where  $8\pi G = M_{\text{pl}}^{-2} = 3\ell_{\text{Pl}}^2$ ,  $\rho_+, \rho_-$  being constants. Eq. (1) is obtained using the background FLRW metric

$$ds^2 = a^2(\eta) (d\eta^2 - \delta_{ij} dx^i dx^j), \quad (2)$$

and by assuming energy conservation for each single fluid separately in order to make explicit its dependence on the scale factor. We restrict ourselves to  $w_- > w_+$ . It is clear that the negative energy density fluid, or, in other words,  $\rho_-$ , is important only close to the bounce, in agreement with what is required from a phenomenological model.

By introducing a new coordinate time  $\tau$ :

$$d\tau = \frac{dt}{a^\beta}, \quad \text{with} \quad \beta = \frac{3}{2}(2w_+ - w_- + 1), \quad (3)$$

we can solve Eq. (1) for the scale factor as

$$a(\tau) = a_0 \left( 1 + \frac{\tau^2}{\tau_0^2} \right)^\alpha, \quad (4)$$

with

$$\alpha = \frac{1}{3(w_- - w_+)}, \quad (5)$$

$$a_0 = \left( \frac{\rho_-}{\rho_+} \right)^\alpha, \quad (6)$$

$$\tau_0^2 = \frac{4\alpha^2 \rho_-}{\ell_{\text{Pl}}^2 \rho_+^2}. \quad (7)$$

Note that the new coordinate time  $\tau$  makes it possible to generalize the solution, obtained in terms of the usual conformal time found in [8] to arbitrary values of  $w_+, w_-$ . Note also that this new coordinate time allowed to get general solutions for the scale factor in a universe filled by dust plus dark energy, the latter having an arbitrary constant equation of state [19].

One can also describe fluids in terms of velocity potentials [20]. In the case of a perfect fluid, the velocity potential action is very simple, and identical to an action for a K-essence scalar field [21] (which, incidentally, can also be used to produce a bounce with no curvature and only one scalar degree of freedom [22]), namely

$$\mathcal{S} = \int \left[ \pm \frac{1}{2} (\nabla_\mu \phi \nabla^\mu \phi)^{(1+w_\pm)/(2w_\pm)} \right] \sqrt{-g} d^4x, \quad (8)$$

where the  $\pm$  sign is chosen according to whether the fluid has positive or negative energy density. The energy-momentum tensor for  $\phi$  reads

$$T_{\mu\nu} = \pm \left[ \frac{(1+w_\pm)}{2w_\pm} \nabla_\mu \phi \nabla_\nu \phi (\nabla_\beta \phi \nabla^\beta \phi)^{(1-w_\pm)/(2w_\pm)} - \frac{1}{2} g_{\mu\nu} (\nabla_\beta \phi \nabla^\beta \phi)^{(1+w_\pm)/(2w_\pm)} \right], \quad (9)$$

and the field equation of motion (stemming from the energy-momentum conservation) is given by

$$\nabla^\mu \nabla_\mu \phi + \frac{(1-w_\pm)}{w_\pm} \frac{\nabla_\mu \phi \nabla_\nu \phi \nabla_\alpha \nabla_\beta \phi g^{\mu\alpha} g^{\nu\beta}}{\nabla_\rho \phi \nabla^\rho \phi} = 0. \quad (10)$$

In the homogeneous background (2), the energy density and pressure read

$$\epsilon_{\pm(0)} \equiv T_{(0)0}^0 = \pm \frac{1}{2w_\pm} \left( \frac{\varphi'}{a} \right)^{(1+w_\pm)/w_\pm} \quad (11)$$

$$p_{\pm(0)} \equiv -\frac{T_{(0)i}^i}{3} = w_\pm T_{(0)0}^0 = w_\pm \epsilon_{\pm(0)}, \quad (12)$$

and the equation of motion (10) reduces to

$$\varphi'' + (3w_\pm - 1)\mathcal{H}\varphi' = 0, \quad (13)$$

where  $\varphi$ ,  $\epsilon_{\pm(0)}$  and  $p_{\pm(0)}$  are the homogeneous parts of  $\phi$ ,  $\epsilon_\pm$  and  $p_\pm$ , respectively. In (11) and the following

equations, a prime represents a derivative with respect to the conformal time  $\eta$  of the metric (17).

From the equation for  $\varphi$  one obtains  $\varphi' = C_\varphi/a^{(3w_\pm-1)}$ , where  $C_\varphi$  is a constant related to  $\rho_\pm$  through

$$\rho_\pm = \frac{C_\varphi^{(1+w_\pm)/w_\pm}}{2w_\pm}. \quad (14)$$

In order to make contact with the notation of Ref. [7], which is contained in the general case we treat here ( $\alpha = 1/2$ ,  $w_- = 1$ ), we will choose to represent the negative energy perfect fluid by this K-essence scalar field and let the positive energy fluid with its original hydrodynamical representation, thus leading to the action

$$\mathcal{S} = - \int \left[ \frac{1}{16\pi G} R + \epsilon_+ + \frac{1}{2} (\nabla_\mu \phi \nabla^\mu \phi)^{(1+w_-)/(2w_-)} \right] \sqrt{-g} d^4x, \quad (15)$$

where  $R$  is the curvature scalar that takes into account gravity. Eq. (15) reduces to that of Ref. [7] if  $w_- = 1$ .

The full Einstein equations then reads

$$G_{\mu\nu} = 8\pi G(T_{\mu\nu}^+ + T_{\mu\nu}^-), \quad (16)$$

where  $G_{\mu\nu}$  is the Einstein tensor,  $T_{\mu\nu}^\pm$  are the energy-momentum tensors of the positive and negative energy perfect fluid respectively, with  $T_{\mu\nu}^+$  written in the hydrodynamical representation, and  $T_{\mu\nu}^-$  expressed in terms of  $\phi$  given by Eq. (9).

### Regular equations for cosmological perturbations.

A set of regular equations is a necessary tool for a numerical analysis of bounce physics. We shall generalize the treatment of Ref. [7] to the generalized class of bouncing models found in Sect. II. The most general

form of scalar metric perturbations on the background given by Eq. (4) reads, in the longitudinal gauge,

$$ds^2 = a^2(\eta) [(1 + 2\Phi)d\eta^2 - (1 - 2\Phi)\delta_{ij}dx^i dx^j], \quad (17)$$

where  $\Phi$  is the gauge invariant Bardeen potential [23, 24]. For the matter fields we have

$$\phi = \varphi(\eta) + \delta\phi(\mathbf{x}, \eta) \quad \text{and} \quad \epsilon_\pm = \varepsilon_\pm(\eta) + \delta\epsilon_\pm(\mathbf{x}, \eta), \quad (18)$$

where  $\epsilon_\pm \equiv \pm T_{0\pm}^0$ .

From Eq. (9) at first order we obtain

$$\delta_- \equiv \frac{\delta\epsilon_-}{\varepsilon_-} = \frac{1 + w_-}{w_-} \left( \frac{\delta\phi'}{\varphi'} - \Phi \right). \quad (19)$$

One can also check from Eq. (9) that  $\delta p_- \equiv \delta T_i^i/3 = w_- \delta\epsilon_-$ .

Using Eq. (10) to obtain a linear equation for  $\delta\phi$ , and the perturbed Einstein equations in order to obtain an equation for the Bardeen potential  $\Phi$ , after eliminating  $\delta\rho_+$ , yields the coupled set of regular equations for the modes of wavenumber  $k$ , namely

$$\Phi_k'' + 3\mathcal{H}(1 + w_+) \Phi_k' + \left[ w_+ k^2 + 2\mathcal{H}' + (\mathcal{H}^2 - K)(1 + 3w_+) + \frac{3}{2}\mathcal{H}^2\Omega_- F \right] \Phi_k = \frac{3}{2}\mathcal{H}^2\Omega_- F \frac{\delta\phi_k'}{\varphi'}, \quad (20)$$

and

$$\delta\phi_k'' + \mathcal{H}(3w_- - 1)\delta\phi_k' + k^2 w_- \delta\phi_k = (1 + 3w_-)\varphi'\Phi_k', \quad (21)$$

which we wrote in full generality by including a possibly non-vanishing curvature of the homogeneous spatial section  $K$ , and we have set  $F \equiv (w_+ - w_-)(1 + w_-)/w_-$ , and  $\Omega_- = \varepsilon_- \ell_{\text{Pl}}^2 a^2/\mathcal{H}^2$ .

Another way one can write Eq. (20) using the background equations of motion, which will be useful when discussing the possible spectra, reads:

$$\begin{aligned} \Phi_k'' + 3\mathcal{H}(1 + w_+) \Phi_k' + \left[ w_+ k^2 + \frac{(w_- - 1)}{w_-} \mathcal{H}' + (1 + 3w_+) \frac{(w_- - 1)}{2w_-} \mathcal{H}^2 - (1 + 3w_+) \frac{(3w_- + 1)}{2w_-} K \right] \Phi_k = \\ = \frac{3}{4w_-} \ell_{\text{Pl}}^2 (2w_- \rho_-)^{1/(1+w_-)} F \frac{\delta\phi_k'}{a^2}. \end{aligned} \quad (22)$$

The advantage of Eqs. (20,21) is their general use in

bounces with two hydrodynamical fluids, constituting a

set of coupled regular equations for numerical analysis of bounce physics. We should like to use this opportunity to mention the fact that attempts to write down two *uncoupled* equations for two separate parts of the Newtonian potential, as suggested in Ref. [7] are incorrect [25, 26, 27], as well as its use [8].

Another set of regular equations for general two-fluid models solely in terms of hydrodynamical variables (without using the K-essence scalar field to describe the  $w_-$  perfect fluid) can be obtained using, together with the perturbed Einstein equations, the perturbed energy-

momentum tensor conservation equations

$$\delta'_{-k} + (1 + w_-)(\theta'_{-k} - 3\Phi'_k) = 0, \quad (23)$$

$$\theta'_{-k} + \mathcal{H}(1 - 3w_-)\theta_{-k} - k^2\Phi_k = \frac{w_-k^2\delta_{-k}}{1 + w_-}, \quad (24)$$

where  $(\delta u_{k-})_i = a\partial_i\theta_{-k}/k^2$  [ $(\delta u_{k-})_i$  is the perturbed  $w_-$  three-velocity mode], and  $\delta_- \equiv \delta\rho_-/\rho_-$ . The result reads

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$$\Phi''_k + 3\mathcal{H}(1 + w_+)\Phi'_k + \left[ w_+k^2 + 2\mathcal{H}' + (\mathcal{H}^2 - K)(1 + 3w_+) + \frac{3}{2}\mathcal{H}^2\Omega_-F \right] \Phi_k = \frac{3}{2}\mathcal{H}^2\Omega_-F \frac{\theta'_{-k} + (1 - 3w_-)\mathcal{H}\theta_{-k}}{k^2}, \quad (25)$$

$$\theta''_{-k} + \mathcal{H}(1 - 3w_-)\theta'_{-k} + [k^2w_- + (1 - 3w_-)\mathcal{H}']\theta_{-k} = k^2(1 + 3w_-)\Phi'_k, \quad (26)$$

where  $F \equiv (w_+ - w_-)(1 + w_-)/w_-$  and  $\Omega_- = \rho_- \ell_{\text{Pl}}^2 a^2 / \mathcal{H}^2$ , as before.

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The relation between  $\theta$  and  $\delta\phi$  is given by,

$$\theta_{-k} = k^2 \frac{\delta\phi_k}{\varphi'}, \quad (27)$$

from which the system (20,21) can be recovered from (25,26) straightforwardly.

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## NUMERICAL RESULTS

The system (20,21) written in terms of  $\Phi_k$  and  $Y_k = \delta\varphi\ell_{\text{Pl}}$  which evolve in the variable  $x = \tau/\tau_0$  for the family of bounces described by (4) read

$$\begin{aligned} \frac{d^2\Phi_k}{dx^2} + (1 + \alpha + \alpha\beta)\frac{2x}{x^2 + 1}\frac{d\Phi_k}{dx} + \left[ w_+\tilde{k}^2(1 + x^2)^{2\alpha(\beta-1)} + \frac{4\alpha(1 - 3\alpha + \alpha\beta)}{(1 + x^2)^2(2 - 3\alpha + 2\alpha\beta)} \right] \Phi_k \\ = -\frac{2\sqrt{2}\alpha(\alpha\beta + 1)(1 - \alpha + \alpha\beta)}{2 - 3\alpha + 2\alpha\beta}(1 + x^2)^{\alpha(\beta-3)}\frac{dY_k}{dx}, \end{aligned} \quad (28)$$

for the metric perturbation, and

$$\frac{d^2Y_k}{dx^2} + (2 + \alpha\beta - 3\alpha)\frac{2x}{x^2 + 1}\frac{dY_k}{dx} + w_-\tilde{k}^2(1 + x^2)^{2\alpha(\beta-1)}Y_k = \frac{2\sqrt{2}}{\alpha}(1 + x^2)^{3\alpha-2-\alpha\beta}\frac{d\Phi_k}{dx}. \quad (29)$$

with  $\tilde{k} = k\tau_0 a_0^{\beta-1}$  for the scalar field part.

In what follows below, we have solved, numerically, the set consisting of Eqs. (28) and (29), setting unnormalized vacuum initial conditions (the fact that we do not bother about the normalization here is because we are merely interested in the transmitted spectrum), reading

$$\begin{aligned} \Phi_{k,\text{ini}} &= \frac{x^{-3\alpha(1+w_+)}}{\sqrt{\tilde{k}^3}} \exp \left[ -i \frac{\tilde{k}\sqrt{w_+}}{1 + 2\alpha(\beta-1)} x^{2\alpha(\beta-1)+1} \right], \\ Y_{k,\text{ini}} &= \frac{x^{\alpha(1-3w_-)}}{\sqrt{\tilde{k}}} \exp \left[ -i \frac{\tilde{k}\sqrt{w_-}}{1 + 2\alpha(\beta-1)} x^{2\alpha(\beta-1)+1} \right]. \end{aligned} \quad (30)$$


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Fig. 1 shows the time evolution for the spectrum  $P_\Phi/k^{n_S-1}$ , for that particular case for which the theoretic-

cal value for the scalar spectral index  $n_S$  in the expanding

stage is known and given by Eq. (51), i.e. for  $w_- = 1$  and  $w_+ = 10^{-2}$ . The plots for three different wavenumbers show how the spectrum for the (growing mode of the) Newtonian potential in the expanding stage is the one of curvature perturbations in the contracting stage. Fig. 2 shows the same plot (near the bounce only) for a different value of  $w_-$ , namely  $w_- = 1/4$ , again rescaled with the theoretical prediction.

### ANALYTIC STUDY OF THE STIFF MATTER CASE

In this section, we will analytically justify the spectra of the class of bounces which are driven by negative energy stiff matter. Note that for  $w_- = 1$  in the equation for the Newtonian potential  $\Phi_k$  (22), the ‘‘mass’’ term on the left hand side vanishes for flat spatial sections,  $K = 0$ , and infinite wavelength,  $k = 0$ . In this case, one can obtain a solution around the bounce which can be matched with the solutions far from it to obtain the spectra for large wavelengths. This section is a generalization of what was done in Ref. [7], which concentrated on the radiation-stiff matter case only.

#### The relevant phases in the perturbations evolution

Let us first consider the asymptotic limit  $\eta \rightarrow -\infty$ , where  $x \propto \eta^{3(1-w_+)/ (1+3w_+)}$  and  $a \propto \eta^{2/(1+3w_+)}$  (the positive energy fluid dominates). Taking into account the initial conditions (30), which in terms of  $\eta$  read

$$\begin{aligned} \Phi_{k, \text{ini}} &\propto \frac{\exp(-ik\eta)}{\eta^{3(1+w_+)/ (1+3w_+)} \sqrt{k^3}}, \\ X_{k, \text{ini}} &\propto \frac{\exp(-ik\eta)}{\eta^{2/(1+3w_+)} \sqrt{k}}, \end{aligned} \quad (31)$$

one can see that the source terms in Eqs. (22 and (21) are negligible provided  $w_- < 7/3$ . Defining the variables  $u_k \equiv a^{3(1+w_+)/2} \Phi_k$  and  $v_k \equiv a^{(3w_- - 1)/2} \delta\phi_k$ , one obtains the equations:

$$u_k'' + \left[ w_+ k^2 - 6 \frac{(1+w_+)}{(1+3w_+)^2 \eta^2} \right] u_k = 0, \quad (32)$$

and

$$v_k'' + \left[ k^2 - \frac{(a^{(3w_- - 1)/2})''}{a^{(3w_- - 1)/2}} \right] v_k = 0. \quad (33)$$

The solutions of these equations are

$$u_k = \sqrt{\eta} \left[ \Phi_{(1)} H_\nu^{(1)}(\sqrt{w_+ k \eta}) + \Phi_{(2)} H_\nu^{(2)}(\sqrt{w_+ k \eta}) \right], \quad (34)$$

from which one derives  $\Phi_k$ , and

$$\begin{aligned} \delta\phi_k &= a^{(1-3w_-)/2} \sqrt{\eta} [X_{(1)} H_\mu^{(1)}(\sqrt{w_- k \eta}) \\ &+ X_{(2)} H_\mu^{(2)}(\sqrt{w_- k \eta})]. \end{aligned} \quad (35)$$

where  $\nu = \frac{5+3w_+}{2(1+3w_+)}$  and  $\mu = \frac{3+3w_+-6w_-}{2(1+3w_+)}$ . The coefficients  $\Phi_{(i)}$  and  $X_{(i)}$  are time-independent and only depend on  $k$ .

Restricting now to the case  $w_- = 1$ , and taking into account the initial conditions (31), one obtains that  $\Phi_{(1)} = X_{(1)} = 0$ ,  $\Phi_{(2)} = 1/k$ , and  $X_{(2)} = 1$ . In the region where  $k\eta \ll 1$  (we are considering long wavelengths) but still far from the bounce, where the source terms can still be neglected, one has

$$\Phi_{< k} = A_1 + \frac{A_2}{\eta^{(5+3w_+)/ (1+3w_+)}} + O(k^2 \eta^2), \quad (36)$$

and

$$\delta\phi_{< k} = B_1 + \frac{B_2}{\eta^{3(1-w_+)/ (1+3w_+)}} + O(k^2 \eta^2), \quad (37)$$

where  $A_1 = k^{3(1-w_+)/ [2(1+3w_)]}$ ,  $A_2 = k^{(-7-9w_+)/ [2(1+3w_)]}$ ,  $B_1 = k^{3(1-w_+)/ [2(1+3w_)]}$  and  $B_2 = k^{3(w_+-1)/ [2(1+3w_)]}$ .

Now we have to propagate this solution through the bounce and match it with the solution in the expanding phase. During the bounce, the source terms in Eqs. (22) and (21) cannot be neglected but now it is the terms proportional to  $k^2$  that are negligible. For  $w_- = 1$ , Eq. (22) simplifies and, upon returning to the variables  $Y \equiv \ell_{\text{pl}} \delta\phi$  and  $x = \tau/\tau_0$ , one obtains

$$\frac{d^2 \Phi_k}{dx^2} + \frac{8\alpha x}{x^2 + 1} \frac{d\Phi_k}{dx} = -\frac{\sqrt{2}}{(1+x^2)} \frac{dY_k}{dx}, \quad (38)$$

$$\frac{d^2 Y_k}{dx^2} + \frac{2x}{x^2 + 1} \frac{dY_k}{dx} = \frac{8\sqrt{2}\alpha}{(1+x^2)} \frac{d\Phi_k}{dx}, \quad (39)$$

where, as  $w_- = 1$ ,  $\alpha = 1/[3(1-w_+)]$ . The solutions of these equations read

$$\Phi^{\text{Bounce}} = \tilde{A} + \tilde{B} f_1(x) + \tilde{C} f_2(x), \quad (40)$$

and

$$Y^{\text{Bounce}} = \tilde{D} + \tilde{B} f_3(x) + \tilde{C} f_4(x), \quad (41)$$

with  $\tilde{A}$ ,  $\tilde{B}$ ,  $\tilde{C}$  and  $\tilde{D}$  arbitrary constants. The bounce functions  $f_i(x)$  are found to be

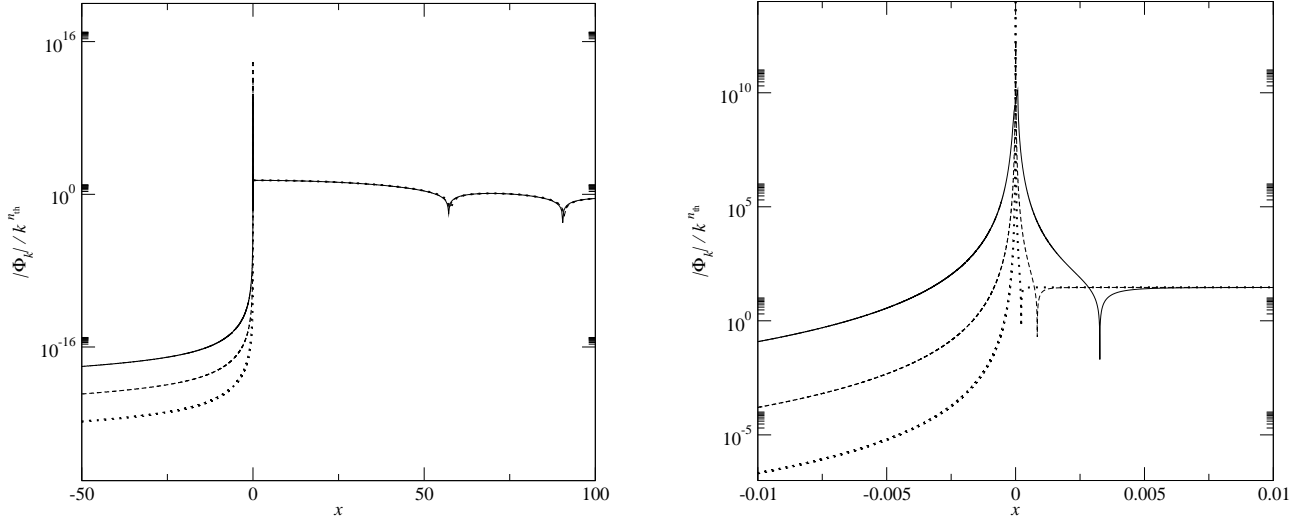


FIG. 1: Example of the time dependence of the Newtonian potential for three different wavelengths ( $\tilde{k} = 10^{-5}, 10^{-6}$  and  $10^{-7}$  respectively) as function of  $x = \eta/\eta_0$ . This example, for which  $w_- = 1$  and  $w_+ = 10^{-2}$  is typical of most cases for which there is a constant mode, as found in [10, 11]. Note that the amplitude at the bounce can be much larger than that of the constant mode that dominates later. The Bardeen potential is here rescaled by the predicted spectrum, which in this case is given by Eq. (51)

$$f_1(x) \equiv \frac{x}{(1+x^2)^{4\alpha}}, \quad f_3(x) \equiv -\frac{\sqrt{2}}{(1+x^2)^{4\alpha}} \quad (42)$$

$$f_2(x) \equiv -8\alpha \int^x d\tilde{x} \left\{ \frac{\tilde{x}}{(1+\tilde{x}^2)^{1+4\alpha}} \left[ (\tilde{x}^2+1) F\left(\frac{1}{2}, -4\alpha+1, \frac{3}{2}, -\tilde{x}^2\right) + \tilde{x} F\left(-\frac{1}{2}, -4\alpha, \frac{1}{2}, -\tilde{x}^2\right) \right] \right\} \quad (43)$$

$$f_4(x) \equiv \int^x d\tilde{x} \left[ \frac{1}{(1+\tilde{x}^2)^{1+4\alpha}} F\left(-\frac{1}{2}, -4\alpha, \frac{1}{2}, -\tilde{x}^2\right) \right], \quad (44)$$

where  $F$  denotes the hypergeometric function. For  $x \gg 1$ , the solutions can be written as

$$\Phi^{\text{Bounce}} \approx \tilde{A} + \frac{\tilde{B}}{x^{8\alpha-1}} + \frac{\tilde{C}}{x^2} = \tilde{A} + \frac{\tilde{B}}{\eta^{(5+3w_+)/ (1+3w_+)}} + \frac{\tilde{C}}{\eta^{6(1-w_+)/ (1+3w_+)}} \quad (45)$$

and

$$Y^{\text{Bounce}} \approx \tilde{D} - \frac{\sqrt{2}\tilde{B}}{x^{8\alpha}} + \frac{\tilde{C}}{x} = \tilde{D} + \frac{\tilde{B}}{\eta^{8/(1+3w_+)}} + \frac{\tilde{C}}{\eta^{3(1-w_+)/ (1+3w_+)}} \quad (46)$$

These solutions coincide with those obtained in Ref. [7] for  $w_+ = 1/3$ .

If we now compare Eqs. (45,46) with Eqs. (36,37), one can obtain that  $\tilde{A} = k^{3(1-w_+)/[2(1+3w_)]}$ ,  $\tilde{B} = k^{(-7-9w_+)/[2(1+3w_)]}$ ,  $\tilde{D} = k^{3(1-w_+)/[2(1+3w_)]}$  and  $\tilde{C} = k^{3(w_+-1)/[2(1+3w_)]}$ . One can also see this by noting that the third term in Eq. (45) is the first contribution of the source term to  $\Phi$ , which of course must have the  $k$ -dependence of  $B_2$  in (37), while the second term of Eq. (46) is the first contribution of the source term to

$\delta\phi$ , which must have the  $k$ -dependence of  $A_2$  in (36).

We now have to match Eqs. (45,46) with the solutions in the expanding phase which are far from the bounce and where the source terms are negligible, i.e.:

$$\Phi_{>k} = \bar{A}_1 + \frac{\bar{A}_2}{\eta^{(5+3w_+)/ (1+3w_+)}} + O(k^2\eta^2), \quad (47)$$

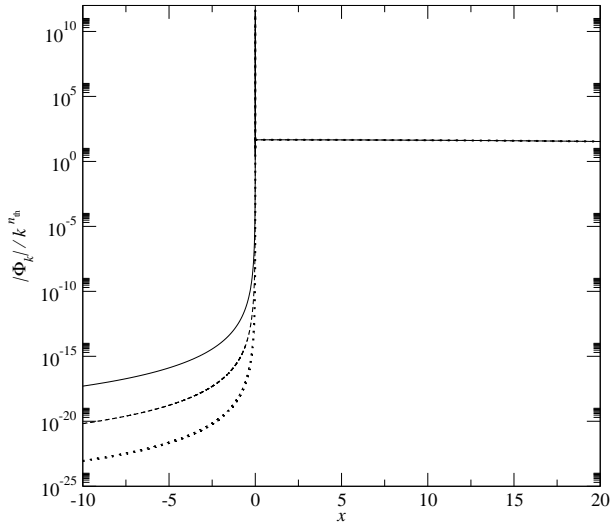


FIG. 2: Same as FIG. 1 with a different value for  $w_+$ , namely  $w_+ = 1/4$ . Again, the Bardeen potential is rescaled with the spectrum found by [10, 11], which we thus independently confirm.

and

$$\delta\phi_{>k} = \bar{B}_1 + \frac{\bar{B}_2}{\eta^{3(w_+-1)/(1+3w_+)}} + O(k^2\eta^2), \quad (48)$$

yielding, for the constant part of  $\Phi$ , which determines the spectrum,

$$\begin{aligned} \bar{A}_1 &\propto k^{3(1-w_+)/[2(1+3w_+)]} + k^{3(w_+-1)/[2(1+3w_+)]} \\ &\approx k^{3(w_+-1)/[2(1+3w_+)]}, \end{aligned} \quad (49)$$

because  $w_+ < 1$  and we assume  $k \ll 1$ . Note that  $\Phi$  gets the spectrum of  $\delta\phi$ .

If we now calculate the power spectrum

$$\mathcal{P}_k \equiv \frac{k^3}{2\pi^2} |\Phi_k|^2 \equiv A_s k^{n_s-1}, \quad (50)$$

we obtain

$$n_s - 1 = \frac{12w_+}{1 + 3w_+}, \quad (51)$$

as obtained in the numerical calculations and in Refs. [10, 11]. This result was also obtained by considerations on the matching conditions in [5, 6, 13], which predict the spectral index in the expanding stage as the one of curvature perturbations in the contracting stage. Note incidentally that it coincides with the spectrum obtained in Ref. [28], where the bounce is not caused by a negative energy stiff matter but by quantum effects: the background and the spectrum have the same behavior.

## GRAVITATIONAL WAVES

The equation for the Fourier transforms of the amplitude of the two polarization degrees of gravitational waves in cosmology is

$$\frac{d^2 h_k}{dt^2} + 3H \frac{dh_k}{dt} + \frac{k^2}{a^2} h_k = 0. \quad (52)$$

Once we introduce  $v \equiv a^{(3-\beta)/2} h$  and use the same coordinate variable  $\tau$  as introduced in Eq. (2), the above equation becomes:

$$\ddot{v}_k + \left[ k^2 a^{2(\beta-1)} + \frac{(\beta-3)\ddot{a}}{2a} - \frac{(\beta-1)(\beta-3)}{4} \left( \frac{\dot{a}}{a} \right)^2 \right] v_k = 0, \quad (53)$$

where  $\dot{f} \equiv df/d\tau$ . The very existence of an asymptotic vacuum demands the condition  $2\alpha(1-\beta) < 1$ , or, in other words, if  $w_+ > -1/3$ , which we shall therefore assume. From here on, for convenience, we also define  $\delta \equiv 1 + 2\alpha(\beta-1) > 0$ . We also restrict to  $w_+ < 1$  in order to have the constant mode as the growing mode in the expanding stage.

The equation which we numerically evolve is:

$$\frac{d^2 v_k}{dx^2} + \left[ \tilde{k}^2 (1+x^2)^{2\alpha(\beta-1)} + \frac{\alpha(\beta-3)}{(1+x^2)^2} \{1 - [1 + \alpha(\beta-3)]x^2\} \right] v_k = 0 \quad (54)$$

with, as usual,  $x = \tau/\tau_0$ ,  $\tilde{k} = ka_0^{\beta-1}\tau_0$ .

Finally, the initial conditions corresponding to the adiabatic vacuum are taken to be

$$\mu_{k,\text{ini}} = \frac{\sqrt{3}\ell_{\text{Pl}}}{\sqrt{k}} \exp(-ik\eta) \implies v_{k,\text{ini}} = \sqrt{3\tau_0}\ell_{\text{Pl}} \frac{x^{\alpha(1-\beta)}}{\sqrt{\tilde{k}}} \exp(-ik\eta), \quad (55)$$

where  $\mu_k = a h_k$ , and we get rid of the prefactor since we are mostly interested in the spectral index anyway (just like for the scalar case, the normalisation here is essentially irrelevant). Recall also that  $k\eta = \tilde{k}x^\delta/\delta$ .

### Analytic Approximations

We are first interested in determining the matching point between the short and long wavelength approximations. The potential in terms of the conformal time is

$$\frac{a''}{a} = a^{2(1-\beta)} \left[ \frac{\ddot{a}}{a} + (1-\beta) \left( \frac{\dot{a}}{a} \right)^2 \right] \equiv V(\eta), \quad (56)$$

Expliciting this in the  $\tau$  variable, this is

$$V = 2\alpha a_0^{2(1-\beta)} f(\tau), \quad (57)$$

where

$$f(\tau) \equiv - \{ [1 + 2\alpha(\beta - 2)]\tau^2 - \tau_0^2 \} \tau_0^{4\alpha(\beta-1)} (\tau^2 + \tau_0^2)^{-2[1+\alpha(\beta-1)]} \quad (58)$$

so the matching point at which  $k^2 \sim |a''/a|$ , i.e

$$x_M = \left\{ \frac{\tilde{k}}{\sqrt{2\alpha[1 + 2\alpha(\beta - 2)]}} \right\}^{-1/\delta} \gg 1, \quad (59)$$

where the last inequality stems from the requirement that there is an asymptotic vacuum, i.e.  $\delta > 0$ .

The zeros of the first derivative of  $V$  are determined by the equation

$$\tau \{ [2\alpha(4\alpha - 3 - 6\alpha\beta + 2\beta + 2\alpha\beta^2) + 1]\tau^2 + [2\alpha(3 - 2\beta) - 3]\tau_0^2 \} = 0. \quad (60)$$

We will here treat the simplest case where the potential  $V$  has only one extremal point, at  $\tau = 0$ , hence imposing that the coefficients of  $\tau^2$  and  $\tau_0^2$  have the same sign.

Asymptotically far from the bounce, Eq. (54) becomes:

$$\frac{d^2 v_k}{dx^2} + \left[ \tilde{k}^2 x^{4\alpha(\beta-1)} - \frac{\gamma(1+\gamma)}{x^2} \right] v_k = 0, \quad (61)$$

where  $\gamma \equiv \alpha(\beta - 3)$ ; the above equation admits a solution in terms of the Hankel function, in accordance with the vacuum initial conditions (55):

$$v_{k,1} = A\sqrt{x} H_\nu^{(2)} \left( \frac{\tilde{k}x^\delta}{\delta} \right), \quad (62)$$

supposed to be valid up to  $x_M$  of Eq. (59), with

$$A^2 = \frac{3\pi\tau_0}{2\delta} \ell_{\text{Pl}}^2 \exp^{-i\pi(\nu + \frac{1}{2})},$$

and

$$\nu \equiv \frac{\gamma + \frac{1}{2}}{\delta}.$$

Note incidentally at this point that the matching time (59) gives an argument for the Hankel function which

does not depend on  $\tilde{k}$ . We shall henceforth call  $H_\nu^{(2)}(\tilde{k}x_M^\delta/\delta) = H_\nu^{(2)}(1/\delta) \equiv h_\nu^M$ .

On the other hand, for long wavelengths close to the bounce, Eq. (54) simplifies to

$$\frac{d^2 v_k}{dx^2} + \frac{\gamma[1 - (1+\gamma)x^2]}{(1+x^2)^2} v_k = 0. \quad (63)$$

In this limit, setting  $v_k = \sqrt{1+x^2}u$  and  $z = ix$ , one gets the Legendre equation

$$(1-z^2) \frac{d^2 u}{dz^2} - 2z \frac{du}{dz} + \left[ \gamma(\gamma+1) - \frac{(1+\gamma)^2}{1-z^2} \right] u = 0, \quad (64)$$

which in this case has, as the two independent solutions, a power law and a hypergeometric function. Summarizing, we obtain, in this second regime, the general solution

$$\begin{aligned} v_{k,2} &= (1+x^2)^{-\gamma/2} \left[ B + C x {}_2F_1 \left( \frac{1}{2}, -\gamma, \frac{3}{2}, -x^2 \right) \right] \\ &\sim x^{-\gamma} \left[ B + C \frac{\sqrt{\pi} \Gamma(-\frac{1}{2} - \gamma)}{2\Gamma(-\gamma)} \right] + \frac{C x^{\gamma+1}}{1+2\gamma} + \dots \end{aligned} \quad (65)$$



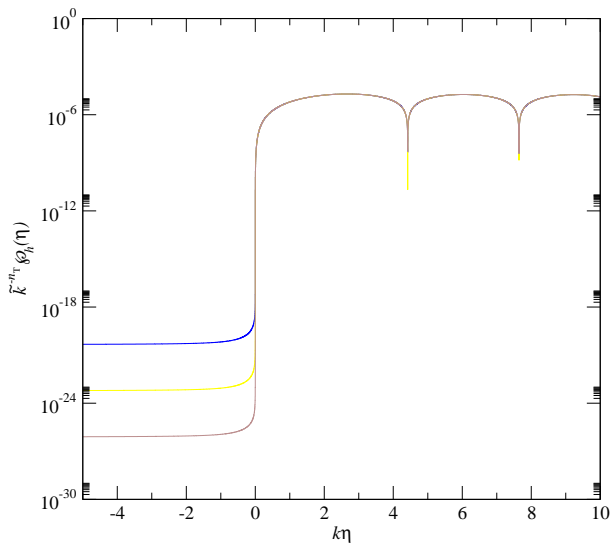


FIG. 3: Example of the time dependence of the tensor perturbation for three different wavelengths ( $\tilde{k} = 10^{-5}$ ,  $10^{-6}$  and  $10^{-7}$  respectively) as function of  $k\eta$ . This example, for which  $w_- = 1/3$  and  $w_+ = 10^{-2}$  is typical, again, and in complete agreement with our analytical prediction. All other cases lead to identical figures, except for the actual numbers. The cases for which the potential does not satisfy the condition that the potential has only one extremum at  $\tau = 0$  cannot be compared with theoretical expectation, in the lack of it, and so are not shown.

It is now a simple matter to match the solutions (62) and (65) as well as their derivatives to get

$$Ah_\nu^M \tilde{k}^{-1/(2\delta)} = (B + C\Upsilon) \tilde{k}^{\gamma/\delta} + \frac{C}{1 + 2\gamma} \tilde{k}^{(-1-\gamma)/\delta}, \quad (66)$$

and

$$\begin{aligned} & \frac{A}{2} \tilde{k}^{1/(2\delta)} (h_\nu^M + h_{\nu-1}^M - h_{\nu+1}^M) \\ &= (-C\Upsilon - B) \gamma \tilde{k}^{(\gamma+1)/\delta} + \frac{C}{1 + 2\gamma} \tilde{k}^{-\gamma/\delta}, \end{aligned} \quad (67)$$

where we have set

$$\Upsilon \equiv \frac{\sqrt{\pi} \Gamma(-\frac{1}{2} - \gamma)}{2\Gamma(-\gamma)}$$

for notational simplicity. The solution of this system provides  $B$  and  $C$  a functions of the reduced wavenumber  $\tilde{k}$ , and we shall retain in what follows the leading order terms, which is, as we are considering  $w_+ < 1$ ,  $\tilde{k}^{(1+2\gamma)/(2\delta)} = \tilde{k}^{3(w_+-1)/[2(3w_++1)]}$ , yielding for  $h \approx x^\gamma \mu$ ,

$$h \approx \tilde{k}^{3(w_+-1)/[2(3w_++1)]} (\text{const.} + x^{2\gamma+1}). \quad (68)$$

The actual gravitational wave spectrum is

$$\mathcal{P}_h \equiv \frac{2k^3}{\pi^2} |h|^2, \quad (69)$$

so we end up with

$$\mathcal{P}_h \propto \tilde{k}^{n_\tau}, \quad (70)$$

being

$$n_\tau = \frac{12w_+}{1 + 3w_+} = \frac{2\alpha}{1 + 2\alpha(\beta - 1)}. \quad (71)$$

It is worth pointing out at this stage that Eq. (71) gives the same result as in the scalar case [Eq. (51)] for the specific case that we could study analytically. The reason for such similar results stems from the fact that the dominant terms which match through the bounces under investigation are the growing modes of curvature perturbation and gravitational waves, both satisfying the same differential equation in the single fluid regime. The above result (71) was already obtained in previous investigations [8, 28], although for two different subsets of the family of bounces studied here.

## CONCLUSIONS

In all the early universe models which aim at solving the horizon problem with a contraction instead of a superluminal expansion, a deep understanding of the physics at the bounce is crucial (and presently lacking in its full generality). What we have shown here is a step towards the understanding of cosmological perturbations through a bounce triggered by a second perfect fluid (with negative energy density), in the framework of flat spatial section and general relativity.

We have analysed in greater details, both numerically and analytically, this class of two perfect fluid bounces with flat spatial sections, using a completely regular system of equations, concluding indeed that the constant mode of the scalar gravitational potential after the bounce does not acquire a piece of the growing mode before the bounce. Therefore, our conclusions agree with [10, 11], and are in contrast with our previous results for scalar perturbations in [7, 8]. Another important result is the insensitivity of the scalar spectral index from the peculiarities of the bouncing component in the class of models studied in this paper. One interesting result is that when the negative energy fluid has stiff matter equation of state, the background model and the perturbations have the same behaviour as the quantum bouncing cosmological models analyzed in Ref. [28]. Our results are interesting for the predictions of cosmological alternative models. Whereas by a very slow contraction - as in Ekpyrotic/cyclic model - it seems really difficult to generate a nearly scale-invariant spectrum of curvature perturbations without the need of isocurvature perturbations or extra-dimensions, a homogeneous dust contraction [13, 28] seems in agreement with observations and even free from details due to the bouncing component

which were left open from previous investigations which focuses on  $w_- = 1$  [16, 28]. Note however, as far as complete model building is concerned, the assumption of homogeneity may not hold close to the bounce and should thus be verified. This point is however out of the scope of the present article whose aim was to concentrate on the propagation of scalar and tensor perturbations through a regular, although phenomenological, bounce.

We have also performed the analytical and numerical calculations for gravitational waves. In this case, the constant mode of the long wavelength tensor perturbations after the bounce do acquire a piece of the growing mode before the bounce. Also in this case, the slope of the final spectrum does not depend on the negative energy perfect fluid equation of state. This paves the way to a generic behavior for tensor perturbations, as such a phenomenological model thus does not suffer from the drawback (still present for the scalar modes) of relying heavily upon the details of the bounce physics. Both these results agree with the previous investigation [8] for a restricted class of bounces. This can be understood by noticing that the crucial time in the evolution of the perturbations is when the perturbation wavelength becomes comparable with the curvature scale of the background, when, for large wavelengths, the universe is still far from the bounce and hence the effects of the negative energy fluid are negligible. This result was already anticipated in Ref. [28].

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## APPENDIX: REGULAR EQUATIONS FOR GENERAL TWO-FLUID MODELS IN TERMS OF HYDRODYNAMICAL VARIABLES.

Another possible set of regular equations uses the density contrast  $\delta_- \equiv \delta\rho_-/\rho_-$  of the fluid driving the bounce instead of its velocity potential. The equations are:

$$\Phi_k'' + 3\mathcal{H}(1 + w_+) \Phi_k' + (w_+ k^2 + 2\mathcal{H}' + \mathcal{H}^2 + 3w_+ \mathcal{H}^2) \Phi_k = \frac{3}{2} \mathcal{H}^2 \delta_- \Omega_- (w_+ - w_-) \quad (72)$$

and

$$\begin{aligned} \delta_-'' + (1 - 3w_-) \mathcal{H} \delta_-' + \left[ w_- k^2 - \frac{9}{2} \mathcal{H}^2 (w_+ - w_-) (1 + w_-) \Omega_- \right] \delta_- = \\ - (1 + w_-) \{ [k^2 (1 + 3w_+) + 3[2\mathcal{H}' + (1 + 3w_+) \mathcal{H}^2]] \Phi_k + 3[2 + 3(w_+ + w_-)] \mathcal{H} \Phi_k' \} \end{aligned} \quad (73)$$

We note that with this new set the order of the system of linear differential equations is increased with respect to the systems (20,21) or (22,26): Eq. (73) is indeed equivalent to a third order differential equation for  $\delta\phi_k$  [see Eq. (19)]. As a result, solving this last set of equations may lead to spurious solutions and it is therefore better to stick with Eqs. (20,21).

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