TESE DE

DOUTORADO

TEORIA DE PERTURBAÇÕES EM UNIVERSOS ANISOTRÓPICOS: MODELO DE KASNER

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RIO DE JANEIRO, FEVEREIRO DE 1998

- Ao CNPq, pela bolsa recebida.

Resumo

Teoria de Perturbações em Universos Anisotrópicos: Modelo de Kasner

Um método de perturbação gauge-invariante é aplicado ao modelo cosmológico de Kasner, como um modelo de tipo Bianchi-I específico, o qual é tomado como um paradigma para os modelos anisotrópicos. Os casos de perturbação tensorial, escalar e vetorial são analisados no contexto das equações Quase-Maxwellianas e as diferenças em relação à solução de Friedmann-Robertson-Walker são apresentadas e discutidas.

Summary

Perturbation Theory in Anisotropic Universes: Kasner Model

A gauge-invariant perturbation method is applied to the Kasner cosmological model, as a specific Bianchi-I type model, which is assumed as a paradigm for anisotropic models. The tensorial, scalar and vectorial perturbation cases are analysed in the framework of Quasi-Maxwellian equations and the differences in relation to the Friedmann-Robertson-Walker solution are presented and discussed.

Índice

Notação e Convenções

- **Índices Gregos:** 0,1,2,3 (denotam índices do 4-espaço).
- **Índices Latinos:** 1,2,3 (denotam índices do 3-espaço).
- ($\hat{}$): denota quantidades pertencentes ao 3-espaço.
- **Convenções de Simetria:**
  1. *Simetria:* $(\alpha \;\; \beta) \equiv \alpha\,\beta + \beta\,\alpha$
  2. *Antissimetria:* $[\alpha \;\; \beta] \equiv \alpha\,\beta - \beta\,\alpha$
- **Derivada Simples (total ou parcial):** $X_{\alpha\beta,\gamma} = \frac{dX_{\alpha\beta}}{dx^{\gamma}}$
- **Derivada Covariante na 4-geometria:** $X^{\alpha}{}_{\beta;\gamma} \equiv X^{\alpha}{}_{\beta,\gamma} + \Gamma^{\alpha}_{\varepsilon\gamma}\, X^{\varepsilon}_{\beta} - \Gamma^{\varepsilon}_{\beta\gamma}\, X^{\alpha}{}_{\varepsilon}$
- **Derivada Projetada na direção temporal:** $(X^{\alpha}{}_{\beta})^{\bullet} \equiv X^{\alpha}{}_{\beta;\gamma}\, V^{\gamma}$
- **Derivada Covariante na 3-geometria:** $\hat{\nabla}_{\alpha}\, X_{\mu} \equiv h_{\alpha}{}^{\beta}\, h_{\mu}{}^{\nu}\, X_{\nu;\beta}$
- **Assinatura da Métrica:** (+ ,- ,- ,-)
- **Símbolo de Christoffel:** $\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\,(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda})$
- **Tensor de Riemann:** $R^{\mu}{}_{\alpha\beta\gamma} = \Gamma^{\mu}_{\alpha\beta,\gamma} - \Gamma^{\mu}_{\alpha\gamma,\beta} + \Gamma^{\mu}_{\lambda\gamma}\,\Gamma^{\lambda}_{\alpha\beta} - \Gamma^{\mu}_{\lambda\beta}\,\Gamma^{\lambda}_{\alpha\gamma}$

- **Tensor de Levi-Civita:**

$$\varepsilon^{\alpha\beta\mu\nu} = \begin{cases} +1 & \text{para permutações pares de } (0,1,2,3) \\ -1 & \text{para permutações ímpares de } (0,1,2,3) \\ 0 & \text{para índices repetidos} \end{cases}$$

- **Escolha das Constantes Universais:** $8\,\pi\,k/c^2 = 1$

- As demais definições e notações são apresentadas ao longo deste trabalho, à medida que se tornarem necessárias.

Introdução

É o objetivo deste trabalho estudar-se perturbações de modelos cosmológicos com anisotropia. A motivação para que uma tal análise se faça vem da possibilidade de que, na evolução inicial do Universo, tenha havido uma fase anisotrópica, com o *shear* passando de um valor inicial não nulo, na fase anisotrópica inicial, em um espaço-tempo que sofre um processo de isotropização (conforme analisado em [1, 2]) até atingir o valor zero, no que seria a atual fase isotrópica, descrita pelo modelo cosmológico de Friedmann-Robertson-Walker (FRW).

Modelos apresentando anisotropia têm sido estudados, em seus diversos aspectos, na literatura científica [3, 4, 5]. Matarrese *et al.* [6, 7] estudaram a evolução não linear de perturbações escalares de fluidos irrotacionais sem colisão e, posteriormente [8], os modelos cosmológicos não homogêneos com poeira irrotacional e constante cosmológica maior do que zero. Mutoh *et al.* [9] trata um modelo similar, com a parte magnética do tensor de Weyl nula (o qual denomina "universo silencioso"). No presente trabalho, o modelo cosmológico de Kasner será tomado como um paradigma dos modelos anisotrópicos no contexto da Teoria de Perturbações cosmológicas.

O método empregado para a análise das perturbações cosmológicas em um *background* anisotrópico é, basicamente, a abordagem gauge-invariante já utilizada por Novello *et al.* em [10, 11, 12] para o modelo de Friedmann-Robertson-Walker, na qual uma base análoga à de harmônicos esféricos construída nos moldes de Lifshitz e Khalatnikov [13] é usada para escrever-se as variáveis perturbadas, no contexto das equações Quase-Maxwellianas. Entretanto, a presença do *shear* e da parte magnética do tensor de Weyl introduz, como veremos, modificações na aplicação do método que remetem ao trabalho de Bardeen [14]. Este trabalho é, portanto, desenvolvido nas seguintes linhas: o Capítulo 1 apresenta os modelos cosmológicos de tipo Bianchi-I, em especial o modelo de Kasner, bem como suas propriedades. Uma formulação Hamiltoniana para o *background* de Kasner é também exibida, como um primeiro passo para a sua quantização.

O Capítulo 2 trata da construção das bases em termos das quais serão escritas as quantidades perturbadas, para os casos escalar, vetorial e tensorial. As bases são escritas de forma explícita e suas propriedades específicas são analisadas e discutidas. No Capítulo 3, fazemos uma rápida apresentação das equações Quase-Maxwellianas, as quais são usadas para construir-se um sistema dinâmico envolvendo apenas variáveis adequadas no sentido de Stewart [15, 16]. Estas últimas são, neste ponto, analisadas e novas variáveis são obtidas para substituir o *shear*, a parte magnética do tensor de Weyl e a expansão, as quais — sendo diferentes de zero no *background* — não se constituem em verdadeiras perturbações segundo o critério de Stewart [15, 16]. O sistema dinâmico em termos do novo conjunto de variáveis é então obtido, com o auxílio das equações Quase-Maxwellianas.

O Capítulo 4 apresenta a análise das perturbações tensoriais de uma maneira completa, exibindo-se um sistema dinâmico fechado ao seu final. Os Capítulos 5 e 6 apresentam um esboço de análise dos casos de perturbações escalares e vetoriais, respectivamente. Segue-se a Conclusão, onde os resultados obtidos, bem como possíveis extensões deste trabalho são discutidos. Há ainda quatro Apêndices, que exibem resultados auxiliares relevantes para este trabalho: o formalismo da 3-geometria é discutido no Apêndice A, enquanto os Apêndices B, C e D contêm demonstrações auxiliares utilizadas ao longo do trabalho.

Capítulo 1

Universos de Tipo Bianchi-I

1.1 Introdução

Como um paradigma dos modelos cosmológicos não isotrópicos, estudaremos os universos de tipo Bianchi-I. Trataremos, portanto, as geometrias que são soluções das equações de Einstein sem termo cosmológico e que têm uma hipersuperfície 3-dimensional, com seção $t = constante$ para o tempo global t. Em um sistema de coordenadas Gaussiano $(t,\, x,\, y,\, z)$, o elemento de linha dos modelos de Bianchi-I é escrito na forma

$$ds^2 = dt^2 - A^2(t)\, dx^2 - B^2(t)\, dy^2 - C^2(t)\, dz^2, \tag{1.1}$$

onde A, B e C são funções somente de t. A equação acima nos possibilita escrever de imediato tanto o tensor métrico $g_{\mu\nu}$ quanto a sua inversa, $g^{\mu\nu}$, em forma matricial:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -A^2(t) & 0 & 0 \\ 0 & 0 & -B^2(t) & 0 \\ 0 & 0 & 0 & -C^2(t) \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/A^2(t) & 0 & 0 \\ 0 & 0 & -1/B^2(t) & 0 \\ 0 & 0 & 0 & -1/C^2(t) \end{pmatrix}, \tag{1.2}$$

e define-se daí uma classe de observadores privilegiados, para os quais vale $V_\alpha = \delta_\alpha{}^0$.

As conexões associadas são facilmente calculadas neste sistema de coordenadas como:

$$\Gamma^1_{01} = \Gamma^1_{10} = \dot{A}/A \quad \Gamma^0_{11} = A\dot{A} \quad \Gamma^i_{jk} = 0 \;\; \forall i, j, k$$

$$\Gamma^2_{02} = \Gamma^2_{20} = \dot{B}/B \quad \Gamma^0_{22} = B\dot{B} \quad \Gamma^0_{jk} = 0 \;\; \forall j \neq k \tag{1.3}$$

$$\Gamma^3_{03} = \Gamma^3_{30} = \dot{C}/C \quad \Gamma^0_{33} = C\dot{C} \quad \Gamma^i_{0k} = 0 \;\; \forall i \neq k,$$

onde denotamos $()^\bullet \equiv d/dt$. Já as componentes não nulas dos tensores de Riemann e Ricci, bem como o escalar de curvatura se escrevem como

$$R^0{}_{101} = -A\,\ddot{A} \quad R^1_{212} = -\tfrac{\dot{A}}{A}\,B\,\dot{B}$$

$$R^0{}_{202} = -B\,\ddot{B} \quad R^1_{313} = -\tfrac{\dot{A}}{A}\,C\,\dot{C} \qquad 1.4)$$

$$R^0{}_{303} = -C\,\ddot{C} \quad R^2_{323} = -\tfrac{\dot{B}}{B}\,C\,\dot{C},$$

$$R^0{}_0 = \left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right) \qquad R^2{}_2 = \frac{\ddot{B}}{B} + \frac{\dot{B}}{B}\left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)$$

$$R^1{}_1 = \frac{\ddot{A}}{A} + \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) \quad R^3{}_3 = \frac{\ddot{C}}{C} + \frac{\dot{C}}{C}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right), \tag{1.5}$$

$$R = 2\left[\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right) + \left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right)\right], \tag{1.6}$$

onde os pontos representam, novamente, a derivada simples em relação a t.

O tensor de Weyl, definido pela expressão

$$W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - \frac{1}{2}\left(R_{\alpha[\mu}\, g_{\nu]\beta} + R_{\beta[\nu}\, g_{\mu]\alpha}\right) + \frac{1}{6} R\, g_{\alpha[\mu}\, g_{\nu]\beta}, \tag{1.7}$$

tem as seguintes componentes não nulas:

$$\begin{aligned}
W_{0101} &= \frac{A^2}{3}\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} - 2\,\frac{\dot{B}\dot{C}}{BC}\right) \\
W_{0202} &= \frac{B^2}{3}\left(\frac{\dot{A}\dot{B}}{AB} - 2\,\frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right) \\
W_{0303} &= \frac{C^2}{3}\left(-2\,\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right) \\
W_{1212} &= -\frac{(AB)^2}{3}\left(-2\,\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right) \\
W_{1313} &= -\frac{(AC)^2}{3}\left(\frac{\dot{A}\dot{B}}{AB} - 2\,\frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right) \\
W_{2323} &= -\frac{(BC)^2}{3}\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} - 2\,\frac{\dot{B}\dot{C}}{BC}\right).
\end{aligned} \tag{1.8}$$

E, definindo a operação dual [17] como:

$$W^*_{\alpha\mu\beta\nu} = \frac{1}{2}\,\eta_{\alpha\mu}{}^{\gamma\lambda}\, W_{\gamma\lambda\beta\nu},$$

a qual é aplicada indiferentemente a qualquer dos dois pares de índices do tensor de Weyl (a prova desta afirmação pode ser encontrada no Apêndice B), obtemos como única componente não nula:

$$W^{*}_{0123} = -\frac{1}{3}\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} - 2\,\frac{\dot{B}\dot{C}}{BC}\right), \tag{1.9}$$

onde $\eta_{\alpha\beta\mu\nu}$ é um verdadeiro tensor, completamente antissimétrico, escrito em termos do pseudo-tensor de Levi-Civita, $\varepsilon_{\alpha\beta\mu\nu}$, como:

$$\eta_{\alpha\beta\mu\nu} \equiv \sqrt{-g}\;\varepsilon_{\alpha\beta\mu\nu}$$

$$\eta^{\alpha\beta\mu\nu} \equiv -\tfrac{1}{\sqrt{-g}}\;\varepsilon^{\alpha\beta\mu\nu},$$

com: $g \equiv det(g_{\mu\nu})$. Algumas propriedades de $\eta_{\alpha\beta\mu\nu}$ serão muito úteis em cálculos subseqüentes, a saber:

$$\eta^{\alpha\beta\mu\nu}\,\eta_{\gamma\lambda\varepsilon\tau} = -\delta^{\alpha\beta\mu\nu}_{\gamma\lambda\varepsilon\tau} = -\begin{vmatrix} \delta^{\alpha}_{\gamma} & \delta^{\alpha}_{\lambda} & \delta^{\alpha}_{\varepsilon} & \delta^{\alpha}_{\tau} \\ \delta^{\beta}_{\gamma} & \delta^{\beta}_{\lambda} & \delta^{\beta}_{\varepsilon} & \delta^{\beta}_{\tau} \\ \delta^{\mu}_{\gamma} & \delta^{\mu}_{\lambda} & \delta^{\mu}_{\varepsilon} & \delta^{\mu}_{\tau} \\ \delta^{\nu}_{\gamma} & \delta^{\nu}_{\lambda} & \delta^{\nu}_{\varepsilon} & \delta^{\nu}_{\tau} \end{vmatrix} \tag{1.10}$$

Contraindo índices sucessivamente, na relação acima, vem de imediato que:

$$\begin{aligned} \eta^{\alpha\beta\mu\nu}\,\eta_{\gamma\lambda\varepsilon\nu} &= -\delta^{\alpha\beta\mu}_{\gamma\lambda\varepsilon} \\ \eta^{\alpha\beta\mu\nu}\,\eta_{\gamma\lambda\mu\nu} &= -2\,\delta^{\alpha\beta}_{\gamma\lambda} \\ \eta^{\alpha\beta\mu\nu}\,\eta_{\gamma\beta\mu\nu} &= -6\,\delta^{\alpha}_{\gamma} \\ \eta^{\alpha\beta\mu\nu}\,\eta_{\alpha\beta\mu\nu} &= -24. \end{aligned} \tag{1.11}$$

De posse dos resultados acima, passamos ao cálculo de todas as quantidades relevantes do modelo de Bianchi-I: cinemáticas, geométricas e as associadas à matéria. Isto será feito nas seções que se seguem.

1.2 Quantidades Cinemáticas em Bianchi-I

As quantidades cinemáticas para o modelo de Bianchi-I podem ser obtidas a partir da decomposição da derivada covariante da velocidade, $V_{\alpha;\beta}$, em suas partes irredutíveis:

$$V_{\alpha;\beta} = \sigma_{\alpha\beta} + \frac{\theta}{3}\, h_{\alpha\beta} + \omega_{\alpha\beta} + a_{\alpha}\, V_{\beta}, \tag{1.12}$$

onde temos:

Expansão:

$$\theta = V^{\alpha}{}_{;\alpha} \tag{1.13}$$

"Shear"ou Deformação:

$$\sigma_{\alpha\beta} = \frac{1}{2}\, h^{\mu}_{(\alpha}\, h^{\nu}_{\beta)}\, V_{\mu;\nu} - \frac{\theta}{3}\, h_{\alpha\beta} \tag{1.14}$$

Rotação:

$$\omega_{\alpha\beta} = \frac{1}{2}\, h^{\mu}_{[\alpha}\, h^{\nu}_{\beta]}\, V_{\mu;\nu} \tag{1.15}$$

Acceleração:

$$a_{\alpha} = V_{\alpha;\beta}\, V^{\beta}. \tag{1.16}$$

Usando as definições (1.13) a (1.16) acima, para o observador $V^{\mu} = \delta^{\mu}{}_{0}$, escrevemos as quantidades cinemáticas para o modelo de Bianchi-I como:

(i) Expansão:

$$\theta = \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) \tag{1.17}$$

(ii) "Shear":

$$\sigma^1{}_1 = \frac{\dot{A}}{A} - \frac{\theta}{3},$$

$$\sigma^2{}_2 = \frac{\dot{B}}{B} - \frac{\theta}{3}, \tag{1.18}$$

$$\sigma^3{}_3 = \frac{\dot{C}}{C} - \frac{\theta}{3},$$

onde temos: $\sigma^\alpha{}_\alpha = 0$.

(iii) Rotação:

$$\omega_{\alpha\beta} = 0, \ \ \forall \ \alpha, \beta \tag{1.19}$$

(iv) Aceleração:

$$a_\alpha = 0, \ \ \forall \ \alpha. \tag{1.20}$$

E, aplicando (1.17) a (1.20) à definição (1.12), temos que ela se reduz a:

$$V_{\alpha;\beta} = \sigma_{\alpha\beta} + \frac{\theta}{3}\, h_{\alpha\beta}, \tag{1.21}$$

para os universos de Bianchi-I. Além disso, as relações (1.17) e (1.18) podem ser substituídas nos resultados (1.3) para dar:

$$\begin{aligned} \Gamma^\alpha_{0\beta} &= \left(\sigma^\alpha{}_\beta + \tfrac{\theta}{3}\, h^\alpha{}_\beta\right) \qquad\qquad \Gamma^i_{jk} = 0, \ \ \forall \ i, j, k \\ \Gamma^0_{\alpha\beta} &= -g_{\alpha\mu}\, \Gamma^\mu_{0\beta} = -\left(\sigma_{\alpha\beta} + \tfrac{\theta}{3}\, h_{\alpha\beta}\right), \end{aligned} \tag{1.22}$$

um resultado que nos será muito útil. E, de (1.2), temos ainda uma outra relação importante para cálculos futuros:

$$g_{\mu\nu,0} \ = \ 2\, g_{\mu\alpha}\, \Gamma^\alpha_{0\nu} \tag{1.23}$$

$$(g_{\mu\nu})^{\bullet} \; = \; g_{\mu\nu;\alpha} V^{\alpha} = g_{\mu\nu;0} = 0, \;\; \forall \;\; \mu, \nu. \tag{1.24}$$

1.3 Quantidades Geométricas em Bianchi-I

O tensor conforme de Weyl, definido pela expressão (1.7), pode ser decomposto com respeito a um observador arbitrário movendo-se com velocidade V_{μ} nas partes elétrica (denotada por $E_{\mu\nu}$) e magnética (denotada por $H_{\mu\nu}$), as quais são definidas por analogia com o campo eletromagnético:

$$\begin{aligned} E_{\alpha\beta} &= -W_{\alpha\mu\beta\nu} \, V^{\mu} \, V^{\nu}, \\ H_{\alpha\beta} &= -W^{*}_{\alpha\mu\beta\nu} \, V^{\mu} \, V^{\nu}. \end{aligned} \tag{1.25}$$

Das definições acima constatam-se imediatamente as propriedades de simetria dos índices livres, traço nulo e ortogonalidade em relação ao observador com velocidade V^{μ}, isto é:

$$\begin{aligned} E_{\alpha\beta} &= E_{\beta\alpha} \qquad & H_{\alpha\beta} &= H_{\beta\alpha} \\ E_{\alpha\beta} \, g^{\alpha\beta} &= 0 \qquad & H_{\alpha\beta} \, g^{\alpha\beta} &= 0 \\ E_{\alpha\beta} \, V^{\alpha} &= 0 \qquad & H_{\alpha\beta} \, V^{\alpha} &= 0. \end{aligned} \tag{1.26}$$

Fazendo uso das definições (1.25), obtemos para o modelo de Bianchi-I:

$$H_{\alpha\beta} = 0, \;\; \forall \;\; \alpha, \beta \tag{1.27}$$

$$\begin{aligned} E^{1}{}_{1} &= -\tfrac{1}{3} \tfrac{\ddot{A}}{A} + \tfrac{1}{6} \left(\tfrac{\ddot{B}}{B} + \tfrac{\ddot{C}}{C} \right) - \tfrac{1}{3} \tfrac{\dot{B}\dot{C}}{BC} + \tfrac{1}{6} \tfrac{\dot{A}}{A} \left(\tfrac{\dot{B}}{B} + \tfrac{\dot{C}}{C} \right) \\ E^{2}{}_{2} &= -\tfrac{1}{3} \tfrac{\ddot{B}}{B} + \tfrac{1}{6} \left(\tfrac{\ddot{A}}{A} + \tfrac{\ddot{C}}{C} \right) - \tfrac{1}{3} \tfrac{\dot{A}\dot{C}}{AC} + \tfrac{1}{6} \tfrac{\dot{B}}{B} \left(\tfrac{\dot{A}}{A} + \tfrac{\dot{C}}{C} \right) \\ E^{3}{}_{3} &= -\tfrac{1}{3} \tfrac{\ddot{C}}{C} + \tfrac{1}{6} \left(\tfrac{\ddot{A}}{A} + \tfrac{\ddot{B}}{B} \right) - \tfrac{1}{3} \tfrac{\dot{A}\dot{B}}{AB} + \tfrac{1}{6} \tfrac{\dot{C}}{C} \left(\tfrac{\dot{A}}{A} + \tfrac{\dot{B}}{B} \right), \end{aligned} \tag{1.28}$$

onde é facilmente constatado que $E^{\alpha}{}_{\alpha} = 0$.

1.4 Quantidades Associadas à Matéria para Bianchi-I

O tensor energia-momento de um fluido geral será escrito em sua forma costumeira:

$$T_{\mu\nu} = \rho\, V_{\mu} V_{\nu} - p\, h_{\mu\nu} + q_{(\mu} V_{\nu)} + \pi_{\mu\nu}, \tag{1.29}$$

em termos da densidade ρ, pressão p, fluxo de calor q_{μ} e pressão anisotrópica $\pi_{\mu\nu}$. O escalar T é dado por:

$$T \equiv \rho - 3\, p = R, \tag{1.30}$$

onde a última equivalência acima é facilmente provada a partir das equações de Einstein.

1.5 O Modelo de Kasner: Um Caso Especial

É interessante estudar-se um subgrupo dos universos de Bianchi-I, o assim chamado *modelo de Kasner* [18]. Trata-se de um modelo anisotrópico, que representa uma região do espaço- tempo na qual a participação da matéria na criação de curvatura é desprezível [19][1]. Pode-se dizer então que a curvatura é auto-sustentada, de uma forma que está ligada à não-linearidade da gravitação.

As equações de campo para o modelo de Kasner são, portanto, escritas como

$$R_{\alpha\beta} = 0,$$

o que — das equações (1.5) — nos fornece o seguinte sistema de equações diferenciais:

[1] Na verdade, é possível construir-se uma "Solução de Kasner Generalizada" (ver, a esse respeito, a Referência [19]). Entretanto, verifica-se que os termos associados à matéria são de uma ordem mais baixa do que os termos associados à geometria. Veremos, mais adiante, que este resultado justifica fisicamente o interesse em estudar-se perturbações de tipo escalar e vetorial.

$$
\begin{aligned}
\frac{\ddot{A}}{A} &+ \frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}=0 \\
\frac{\ddot{A}}{A} &+ \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)=0 \\
\frac{\ddot{B}}{B} &+ \frac{\dot{B}}{B}\left(\frac{\dot{A}}{A}+\frac{\dot{C}}{C}\right)=0 \\
\frac{\ddot{C}}{C} &+ \frac{\dot{C}}{C}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)=0.
\end{aligned}
\tag{1.31}
$$

A solução do sistema acima é obtida escolhendo-se, para as funções $A(t)$, $B(t)$ e $C(t)$ que descrevem a métrica, o seguinte *Ansatz*:

$$
A(t)=t^{p_1}, \quad B(t)=t^{p_2}, \quad C(t)=t^{p_3}, \tag{1.32}
$$

com p_i ($i=1,2,3$) constantes. As equações (1.31) se reduzem, então, a

$$
\begin{aligned}
p_1+p_2+p_3 &= 1 \\
(p_1)^2+(p_2)^2+(p_3)^2 &= 1,
\end{aligned}
\tag{1.33}
$$

o que caracteriza os modelos de Kasner. Considerando-se o ordenamento $p_1 < p_2 < p_3$ [20], temos os seguintes limites de validade para as constantes p_i:

$$
-\tfrac{1}{3} \leq p_1 \leq 0, \qquad 0 \leq p_2 \leq \tfrac{2}{3}, \qquad \tfrac{2}{3} \leq p_3 \leq 1
$$

De posse das relações (1.33), é fácil ver que há apenas um grau de liberdade na escolha das constantes p_i. Vamos, portanto, escrever p_2 e p_3 em termos da terceira constante p_1. Da primeira das equações (1.33), temos que:

$$p_3 = 1 - (p_1 + p_2).$$

e, substituindo este resultado na segunda relação em (1.33) e resolvendo para p_2 em termos de p_1, obtemos que:

$$(p_1)^2 + (p_2)^2 + [1 - (p_1 + p_2)]^2 = 1 \Longrightarrow$$

$$\Longrightarrow (p_1)^2 + (p_2)^2 + 1 - 2\,(p_1 + p_2) + (p_1)^2 + 2\,p_1\,p_2 + (p_2)^2 = 1 \Longrightarrow$$

$$\Longrightarrow 2\,(p_1)^2 + 2\,(p_2)^2 + 2\,p_1\,p_2 - 2\,(p_1 + p_2) = 0 \Longrightarrow$$

$$\Longrightarrow (p_2)^2 + (p_1 - 1)\,p_2 + (p_1 - 1)\,p_1 = 0,$$

donde, resolvendo a equação de segundo grau acima em p_2, obtemos que:

$$p_2 = \frac{1}{2}\left[(1 - p_1) \pm \sqrt{(1 - p_1)\,(3\,p_1 + 1)}\right]. \qquad (1.34)$$

E daí escrevemos afinal (substituindo (1.34) na primeira das relações (1.33)):

$$p_3 = \frac{1}{2}\left[(1 - p_1) \mp \sqrt{(1 - p_1)\,(3\,p_1 + 1)}\right]. \qquad (1.35)$$

Analisando a equação de segundo grau para p_2 acima, verificamos que o seu delta ($\Delta \equiv (1 - p_1)\,(3\,p_1 + 1)$) tem os seguintes sinais:

(i) $\Delta = 0$, se $p_1 = 1$ ou $p_1 = -1/3$;

(ii) $\Delta > 0$, se:

$$p_1 < 1 \text{ e } p_1 > -1/3 \Rightarrow -1/3 < p_1 < 1$$

ou

$$p_1 > 1 \text{ e } p_1 < -1/3 \Rightarrow \text{não ocorrem simultaneamente}$$

(iii) $\Delta < 0$, se:

$$p_1 < 1 \text{ e } p_1 < -1/3 \Rightarrow p_1 < -1/3$$

ou

$$p_1 > 1 \text{ e } p_1 > -1/3 \Rightarrow p_1 > 1$$

Portanto, supondo-se que escolhamos p_1 como **real**, temos as seguintes possibilidades:

$$(p_2, p_3) \text{ reais se: } -1/3 \leq p_1 \leq 1;$$

$$(p_2, p_3) \text{ complexos se: } p_1 < -1/3 \text{ ou } p_1 > 1.$$

Diante de tais resultados, podemos reescrever as quantidades cinemáticas e geométricas em termos das constantes p_i — o que faremos a seguir.

(I) Quantidades Cinemáticas:

A expansão, dada pela equação (1.17), se reduz a (usando-se a primeira das equações (1.33)):

$$\theta = \frac{1}{t}(p_1 + p_2 + p_3) = \frac{1}{t}, \tag{1.36}$$

o que nos possibilita escrever imediatamente que

$$\theta^{\bullet} = -\frac{1}{t^2} = -\theta^2. \tag{1.37}$$

As componentes não nulas do "shear", equação (1.18), são reescritas, com o uso de (1.36), como:

$$\sigma^1{}_1 = \tfrac{1}{3t}\,(3\,p_1 - 1) = \tfrac{\theta}{3}\,(3\,p_1 - 1)$$

$$\sigma^2{}_2 = \tfrac{1}{3t}\,(3\,p_2 - 1) = \tfrac{\theta}{3}\,(3\,p_2 - 1) \tag{1.38}$$

$$\sigma^3{}_3 = \tfrac{1}{3t}\,(3\,p_3 - 1) = \tfrac{\theta}{3}\,(3\,p_3 - 1)$$

Temos, além disso, de (1.36) e (1.37), os seguintes resultados importantes para futuros cálculos :

$$2\,\sigma^2 \equiv \sigma^\alpha{}_\beta\,\sigma^\beta{}_\alpha = \tfrac{2}{3}\,\theta^2$$

$$(\sigma^\mu{}_\nu)_{,0} = -\theta\,\sigma^\mu{}_\nu \tag{1.39}$$

$$(\sigma^\mu{}_\nu)^\bullet = \sigma^\mu{}_{\nu;\alpha}\,V^\alpha = -\theta\,\sigma^\mu{}_\nu,$$

os quais são identidades válidas somente para o modelo de Kasner.

As componentes do tensor de rotação $\omega_{\alpha\beta}$ e do vetor aceleração a_α são, por (1.19) e (1.20), nulas.

(II) Partes Elétrica e Magnética do Tensor de Weyl:

A equação (1.27) nos garante que a parte magnética do tensor de Weyl, $H_{\alpha\beta}$, tem todas as componentes nulas. A parte elétrica, $E_{\alpha\beta}$, é reescrita — usando-se as equações (1.28), (1.32) e (1.36) — como:

$$\begin{aligned} E^i{}_i &= -\frac{1}{3}\,p_i\,(p_i - 1)\,\theta^2 - \frac{1}{6}\,p_i\,(p_i - 1)\,\theta^2 - \frac{1}{3}\,p_i\,(p_i - 1)\,\theta^2 + \frac{1}{6}\,p_i\,(1 - p_i)\,\theta^2 \\ &= -\frac{2}{3}\,p_i\,(p_i - 1)\,\theta^2 - \frac{1}{3}\,p_1\,(p_i - 1)\,\theta^2 \\ &= -p_i\,(p_i - 1)\,\theta^2, \end{aligned} \tag{1.40}$$

donde escrevemos então:

$$E^1{}_1 = p_1\,(1-p_1)\,\theta^2$$

$$E^2{}_2 = p_2\,(1-p_2)\,\theta^2 \tag{1.41}$$

$$E^3{}_3 = p_3\,(1-p_3)\,\theta^2.$$

E, a exemplo do que foi feito para o *shear* , obtemos os seguintes resultados adicionais (válidos especificamente para o modelo de Kasner):

$$(E^\mu{}_\nu)_{,0} = -2\,\theta\,E^\mu{}_\nu \tag{1.42}$$

$$\begin{aligned}
(E^\mu{}_\nu)^\bullet &= E^\mu{}_{\nu;\alpha}\,V^\alpha = E^\mu{}_{\nu;0}\\
&= E^\mu{}_{\nu,0} + \Gamma^\mu_{0\alpha}\,E^\alpha{}_\nu - \Gamma^\alpha_{0\nu}\,E^\mu{}_\alpha\\
&= -2\,\theta\,E^\mu{}_\nu + \left(\sigma^\mu{}_\alpha + \frac{\theta}{3}\,h^\mu{}_\alpha\right)\,E^\alpha{}_\nu - \left(\sigma^\alpha{}_\nu + \frac{\theta}{3}\,h^\alpha{}_\nu\right)\,E^\mu{}_\alpha\\
&= -2\theta\,E^\mu{}_\nu.
\end{aligned} \tag{1.43}$$

(III) Quantidades Associadas à Matéria:

Conforme mencionado anteriormente, no modelo de Kasner a matéria não contribui para a curvatura. Neste caso, todas as quantidades associadas são nulas, ou seja:

$$\rho \equiv p = 0,$$

$$q_\mu = 0, \;\; \forall\;\; \mu, \tag{1.44}$$

$$\pi_{\mu\nu} = 0, \;\; \forall\;\; \mu,\nu.$$

Uma solução importante do modelo de Kasner é aquela em que há um **plano de**

isotropia. Há duas soluções que apresentam tal característica: a de Kasner ($p_1 = p_2 = 2/3$; $p_3 = -1/3$) e a de Milne ($p_1 = p_2 = 0$; $p_3 = 1$). É conveniente escrever-se os resultados obtidos até agora (*i.e.*, quantidades cinemáticas e parte elétrica do tensor de Weyl, equações (1.38) e (1.41)) para cada uma destas soluções:

(i) Solução de Kasner:

$$\begin{cases} \sigma^1{}_1 = \sigma^2{}_2 = \frac{1}{3}\,\theta \\ \\ \sigma^3{}_3 = -\frac{2}{3}\,\theta \end{cases} \tag{1.45}$$

$$\begin{cases} E^1{}_1 = E^2{}_2 = \frac{2}{9}\,\theta^2 \\ \\ E^3{}_3 = -\frac{4}{9}\,\theta^2 \end{cases} \tag{1.46}$$

(ii) Solução de Milne:

$$\begin{cases} \sigma^1{}_1 = \sigma^2{}_2 = -\frac{1}{3}\,\theta \\ \\ \sigma^3{}_3 = \frac{2}{3}\,\theta \end{cases} \tag{1.47}$$

$$E^1{}_1 \equiv E^2{}_2 \equiv E^3{}_3 = 0 \Longrightarrow E^\mu{}_\nu = 0, \quad \forall \quad \mu\,,\ \nu. \tag{1.48}$$

1.6 Formalismo Hamiltoniano para o *background* de Kasner

Como uma análise adicional, apresentaremos nesta seção a aplicação de um algoritmo pelo qual escreveremos uma Hamiltoniana para o *background* de Kasner, como um primeiro passo para a sua quantização. Este assim chamado **método da Hamiltoniana auxi-**

liar, foi aplicado inicialmente ao *background* de Friedmann-Robertson-Walker (conforme a referência [21] e, posteriormente, em [22]).

Tomemos, inicialmente, as equações de Eisntein para o *background* de Kasner, as quais traduzem a dinâmica desta solução cosmológica; são elas:

$$R_{\alpha\beta} = 0, \tag{1.49}$$

ou, explicitando cada uma,

$$\begin{cases} \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} = 0 \\ \\ \frac{\ddot{A}}{A} + \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = 0 \\ \\ \frac{\ddot{B}}{B} + \frac{\dot{B}}{B}\left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) = 0 \\ \\ \frac{\ddot{C}}{C} + \frac{\dot{C}}{C}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = 0, \end{cases} \tag{1.50}$$

onde as derivadas são no tempo t e onde temos ainda a relação adicional:

$$\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = \theta. \tag{1.51}$$

Renomeando os termos em (1.50), vem:

$$\begin{cases} x \equiv \frac{\dot{A}}{A} \Longrightarrow \dot{x} = \frac{\ddot{A}}{A} - \left(\frac{\dot{A}}{A}\right)^2 \Longrightarrow \frac{\ddot{A}}{A} = \dot{x} + x^2 \\ \\ y \equiv \frac{\dot{B}}{B} \Longrightarrow \dot{y} = \frac{\ddot{B}}{B} - \left(\frac{\dot{B}}{B}\right)^2 \Longrightarrow \frac{\ddot{B}}{B} = \dot{y} + y^2 \\ \\ z \equiv \frac{\dot{C}}{C} \Longrightarrow \dot{z} = \frac{\ddot{C}}{C} - \left(\frac{\dot{C}}{C}\right)^2 \Longrightarrow \frac{\ddot{C}}{C} = \dot{z} + z^2, \end{cases} \tag{1.52}$$

onde (x,y,z) não têm nada a ver com as coordenadas espaciais do *background*, em termos das quais é escrito o tensor métrico. Como as novas variáveis, o sistema dinâmico (1.50) é reescrito como:

$$
\begin{cases}
\dot{x} + \dot{y} + \dot{z} + x^2 + y^2 + z^2 = 0 \\
\\
\dot{x} + x\,(x + y + z) = 0 \\
\\
\dot{y} + y\,(x + y + z) = 0 \\
\\
\dot{z} + z\,(x + y + z) = 0,
\end{cases}
\tag{1.53}
$$

com a (1.51) reescrita como:

$$(x + y + z) = \theta. \tag{1.54}$$

Substituindo, agora, as três últimas equações em (1.53) na primeira delas, obtemos a seguinte equação de vínculo:

$$-x^2 - x\,(y + z) - y^2 - y\,(x + z) - z^2 - z\,(x + y) + x^2 + y^2 + z^2 = 0 \Longrightarrow$$

$$\Longrightarrow 2\,(x\,y + x\,z + y\,z) = 0 \Longrightarrow$$

$$x\,y + y\,z + x\,z = 0, \tag{1.55}$$

e o sistema dinâmico (1.53) fica então:

$$\begin{cases} \dot{x} + x^2 + x\,(y+z) = 0 \\ \\ \dot{y} + y^2 + y\,(x+z) = 0 \\ \\ \dot{z} + z^2 + z\,(x+y) = 0 \\ \\ x\,y + x\,z + y\,z = 0. \end{cases} \tag{1.56}$$

A equação de vínculo, (1.55), elimina uma das variáveis, por exemplo, z:

$$z = -\frac{x\,y}{(x+y)}. \tag{1.57}$$

E, substituindo (1.57) nas duas primeiras equações dinâmicas em (1.56), obtemos:

$$\begin{cases} \dot{x} = -\frac{x}{(x+y)}\,(x^2 + x\,y + y^2) \\ \\ \dot{y} = -\frac{y}{(x+y)}\,(x^2 + x\,y + y^2) \end{cases} \tag{1.58}$$

É facilmente demonstrado que a terceira equação dinâmica do sistema (1.56) não fornece nenhuma informação adicional e pode, por isso, ser descartada como redundante. Derivando o vínculo (1 57) no tempo, obtemos:

$$\begin{aligned} \dot{z} &= -\frac{\dot{x}\,y}{(x+y)} - \frac{x\,\dot{y}}{(x+y)} + x\,y\,\frac{(\dot{x}+\dot{y})}{(x+y)^2} \\ \\ &= \frac{1}{(x+y)^2}\left[-\dot{x}\,y\,(x+y) - x\,\dot{y}\,(x+y) + x\,y\,(\dot{x}+\dot{y})\right] \\ \\ &= \frac{1}{(x+y)^2}\left[-\dot{x}\,x\,y - \dot{x}\,y^2 - x^2\,\dot{y} - x\,y\,\dot{y} + \dot{x}\,x\,y + x\,y\,\dot{y}\right] \end{aligned}$$

$$= -\frac{1}{(x+y)^2}\left(\dot{x}\,y^2 + x^2\,\dot{y}\right). \tag{1.59}$$

Substituindo (1.57) e (1.59) na terceira das equações dinâmicas em (1.56), vem:

$$-\tfrac{1}{(x+y)^2}\,(\dot{x}\,y^2 + x^2\,\dot{y}) + \tfrac{x^2\,y^2}{(x+y)^2} - \tfrac{x\,y}{(x+y)}\,(x+y) = 0 \Longrightarrow$$

$$\Longrightarrow -\dot{x}\,y^2 - x^2\,\dot{y} + x^2\,y^2 - x\,y\,(x^2 + x\,y + y^2) = 0 \Longrightarrow$$

$$\Longrightarrow \dot{x}\,y^2 + x^2\,\dot{y} + x\,y\,(x^2 + x\,y + y^2) = 0.$$

Mas, substituindo (1.58) no resultado acima, vem:

$$\begin{aligned}
-\frac{x\,y^2}{(x+y)}\left(x^2 + x\,y + y^2\right) &- \frac{x^2\,y}{(x+y)}\left(x^2 + x\,y + y^2\right) + x\,y\left(x^2 + x\,y + y^2\right) = \\
&= -\frac{x\,y}{(x+y)}\,(x+y)\left(x^2 + x\,y + y^2\right) + x\,y\left(x^2 + x\,y + y^2\right) = \\
&\equiv 0,
\end{aligned}$$

como esperávamos.

Isto nos deixa, então, com o sistema dinâmico fechado bidimensional (1.58), com a terceira variável z dada em termos de x e y, conforme o vínculo (1.57). Vamos, neste ponto, fazer uma nova mudança de variáveis com o objetivo de desacoplar o sistema — o que produziria uma simplificação bastante significativa em nossos cálculos posteriores. Para tanto, introduzimos uma nova variável temporal, denotada por η, de forma que o sistema dinâmico se reduza a uma forma consideravelmente mais simples:

$$\begin{cases} x' = x \\ \\ y' = y, \end{cases} \tag{1.60}$$

onde $(\quad)' \equiv \frac{d}{d\eta}$. A relação entre a nova coordenada temporal e a antiga é facilmente obtida substituindo-se a variável x definida em (1.58) na primeira das equações (1.60) acima:

$$x' = -\dot{x}\,\frac{(x+y)}{(x^2+x\,y+y^2)} \Longrightarrow$$

$$\Longrightarrow \frac{dx}{d\eta} = -\frac{dx}{dt}\,\frac{(x+y)}{(x^2+x\,y+y^2)} \Longrightarrow$$

$$d\eta = -\frac{(x^2+x\,y+y^2)}{(x+y)}\,dt. \tag{1.61}$$

Isto posto, a pergunta a ser feita é: o novo sistema dinâmico (1.60) admite uma formulação Hamiltoniana? A resposta é não, o que torna necessário aplicar-se o método da Hamiltoniana auxiliar, nos moldes de [21, 22]. Começaremos, portanto, definindo um novo conjunto de variáveis (Q,P), da seguinte maneira:

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \alpha(\eta) & \beta(\eta) \\ \gamma(\eta) & \delta(\eta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \equiv \mathcal{S} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1.62}$$

onde α, β, γ e δ são funções da nova coordenada temporal η, a serem determinadas futuramente. O determinante da matriz de transformação $\mathcal{S}$ é dado por:

$$\Delta \equiv det(\mathcal{S}) = \alpha\delta - \beta\gamma \neq 0, \tag{1.63}$$

e $\mathcal{S}^{-1}$, a inversa de $\mathcal{S}$, fica portanto definida. Então, temos a seguinte dinâmica para (Q,P):

$$\begin{pmatrix} Q' \\ P' \end{pmatrix} = \mathcal{H} \begin{pmatrix} Q \\ P \end{pmatrix}, \tag{1.64}$$

donde vem imediatamente de (1.60) e (1.62) que:

$$\mathcal{H} = \mathcal{S}\, I\, \mathcal{S}^{-1} + \mathcal{S}'\, \mathcal{S}^{-1}, \tag{1.65}$$

com I denotando a matriz identidade, e onde temos, por (1.60):

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = I \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Se (Q,P) são, de fato, variáveis canônicas, então $\mathcal{H}$ não deve ter traço e, neste caso, temos:

$$\begin{aligned} Tr\ \mathcal{H} &= Tr\ \left(\mathcal{S}\, I\, \mathcal{S}^{-1} + \mathcal{S}'\, \mathcal{S}^{-1}\right) \\ &= Tr\ \left(\mathcal{S}\, I\, \mathcal{S}^{-1}\right) + Tr\ \left(\mathcal{S}'\, \mathcal{S}^{-1}\right) \\ &= Tr\ I + \frac{\Delta'}{\Delta} \\ &= \frac{\Delta'}{\Delta} + 2, \end{aligned}$$

donde vem então

$$\frac{\Delta'}{\Delta} = -2 \Longrightarrow \Delta' = -2\,\Delta \Longrightarrow \Delta(\eta) = \Delta_0\, e^{-2\eta}, \tag{1.66}$$

onde Δ_0 é uma constante de integração.

A Hamiltoniana quadrática mais geral para o nosso sistema é escrita como [10, 11]:

$$H = \frac{h_1}{2} Q^2 + \frac{h_2}{2} P^2 + 2\, h_3\, Q P, \tag{1.67}$$

cuja forma matricial é então:

$$\mathcal{H} = \begin{pmatrix} 2\, h_3 & h_2 \\ -h_1 & -2\, h_3 \end{pmatrix}, \tag{1.68}$$

com h_i dados em termos de α, β, γ e δ. A condição para a existência de H é:

$$\frac{\partial Q'}{\partial Q} + \frac{\partial P'}{\partial P} = 0. \tag{1.69}$$

Substituindo (1.66) em (1.63), vem:

$$\alpha\, \delta - \beta\, \gamma = \Delta_0\, e^{-2\,\eta}. \tag{1.70}$$

E, como simplificação, tomamos:

$$\beta \equiv \gamma = 0, \tag{1.71}$$

donde vem de (1.70):

$$\delta = \Delta_0\, e^{-2\,\eta}\, \alpha^{-1}, \tag{1.72}$$

e então temos:

$$\mathcal{S} = \begin{pmatrix} \alpha & 0 \\ 0 & \Delta_0\, e^{-2\,\eta}\, \alpha^{-1} \end{pmatrix}, \tag{1.73}$$

o que nos dá, de (1.62),

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \Delta_0\, e^{-2\,\eta}\, \alpha^{-1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{1.74}$$

Daí vem:

$$\mathcal{S}' = \begin{pmatrix} \alpha' & 0 \\ 0 & -\Delta_0\, e^{-2\,\eta}\, \alpha^{-1} \left[2 + \frac{\alpha'}{\alpha}\right] \end{pmatrix}, \tag{1.75}$$

$$\mathcal{S}^{-1} = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & (\Delta_0)^{-1}\, e^{2\,\eta}\, \alpha \end{pmatrix}, \tag{1.76}$$

e então, de (1.65), vem:

$$\begin{aligned}
\mathcal{H} \;&=\; \begin{pmatrix} \alpha & 0 \\ 0 & \Delta_0\, e^{-2\,\eta}\, \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & (\Delta_0)^{-1}\, e^{2\,\eta}\, \alpha \end{pmatrix} + \\
&\quad + \begin{pmatrix} \alpha' & 0 \\ 0 & -\Delta_0\, e^{-2\,\eta}\, \alpha^{-1} \left[2 + \frac{\alpha'}{\alpha}\right] \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & (\Delta_0)^{-1}\, e^{2\,\eta}\, \alpha \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{\alpha'}{\alpha} & 0 \\ 0 & -\left[2 + \frac{\alpha'}{\alpha}\right] \end{pmatrix} \\
&= \begin{pmatrix} \left[\frac{\alpha'}{\alpha} + 1\right] & 0 \\ 0 & -\left[\frac{\alpha'}{\alpha} + 1\right] \end{pmatrix},
\end{aligned} \tag{1.77}$$

e a condição de existência da Hamiltoniana, (1.69), é realmente satisfeita. Comparando o resultado acima com (1.67), obtemos então:

$$\begin{cases} h_1 \equiv h_2 = 0, \\[2ex] h_3 = \frac{1}{2}\left(\frac{\alpha'}{\alpha} + 1\right), \end{cases} \tag{1.78}$$

donde escrevemos:

$$H(Q, P) = \left(\frac{\alpha'}{\alpha} + 1\right) Q\, P. \tag{1.79}$$

Para reescrever a Hamiltoniana do sistema em uma forma mais adequada, efetuaremos uma transformação canônica $(Q,P) \longrightarrow (\tilde{Q},\tilde{P})$ [23], donde temos então que:

$$\begin{cases} \tilde{Q} = \frac{\partial \tilde{H}}{\partial \tilde{P}}, \\ \\ \tilde{P} = -\frac{\partial \tilde{H}}{\partial \tilde{Q}}, \end{cases}$$

e temos a função geradora da transformação, F, em uma das seguintes quatro formas:

$$\begin{cases} F_1(Q,\tilde{Q},\eta) \\ \\ F_2(Q,\tilde{P},\eta) \\ \\ F_3(P,\tilde{Q},\eta) \\ \\ F_4(P,\tilde{P},\eta). \end{cases}$$

Escolhendo a segunda forma para a função geradora da transformação, com dependência em Q, $\tilde{P}$ e η, temos:

$$\begin{cases} P = \frac{\partial F_2}{\partial Q} \\ \\ \tilde{Q} = \frac{\partial F_2}{\partial \tilde{P}} \\ \\ \tilde{H} = H + \frac{\partial F_2}{\partial \eta}, \end{cases}$$

e gostaríamos de escrever $\tilde{H}$ como a Hamiltoniana de um oscilador harmônico, ou seja,

$$\tilde{H} = \mu \tilde{P}^2 + \nu \tilde{Q}^2,$$

com μ e ν funções de η a serem determinadas. Neste caso, tomaremos F_2 na seguinte forma:

$$F_2(Q, \tilde{P}, \eta) = \varepsilon \, \tilde{P}^a \, ln \, Q, \tag{1.80}$$

onde ε e a são — pelo menos a princípio — funções do tempo η. Com isso, escrevemos:

$$\begin{cases} P = \varepsilon \, \tilde{P}^a \, Q^{-1} \\ \\ \tilde{Q} = a \, \varepsilon \, \tilde{P}^{(a-1)} \, ln \, Q \\ \\ \tilde{H} = \left(\frac{\alpha'}{\alpha} + 1\right) \, Q \, P + \varepsilon' \, \tilde{P}^a \, ln \, Q. \end{cases} \tag{1.81}$$

Daí vem que:

$$\begin{cases} \tilde{P}^a = \varepsilon^{-1} \, Q \, P \Longrightarrow \tilde{P} = \varepsilon^{1/a} \, Q^{1/a} \, P^{1/a} \\ \\ \tilde{Q} = a \, \varepsilon \, \tilde{P}^{(a-1)} \, ln \, Q \Longrightarrow \tilde{Q} = a \, \varepsilon^{1/a} \, Q^{(a-1)/a} \, P^{(a-1)/a} \, ln \, Q, \end{cases}$$

e, com isso, escrevemos:

$$\begin{cases} Q = exp \left[(a \, \varepsilon)^{-1} \, \tilde{Q} \, \tilde{P}^{(1-a)} \right] \\ \\ P = \varepsilon \, \tilde{P}^a \, exp \left[- (a \, \varepsilon)^{-1} \, \tilde{Q} \, \tilde{P}^{(1-a)} \right], \end{cases} \tag{1.82}$$

donde

$$Q \, P = \varepsilon \, \tilde{P}^a, \tag{1.83}$$

donde escrevemos a Hamiltoniana transformada como:

$$\tilde{H} = \left(\frac{\alpha'}{\alpha} + 1 \right) \varepsilon \, \tilde{P}^a + a \, \frac{\varepsilon'}{\varepsilon} \, \tilde{Q} \, \tilde{P}. \tag{1.84}$$

Se, agora, escolhermos:

$$
\begin{cases} a = 2 \\ \\ \varepsilon \equiv e^{\underline{t\epsilon}}, \end{cases} \tag{1.85}
$$

ficamos com:

$$
\begin{cases} Q = exp\left[\frac{1}{2\varepsilon}\,\tilde{Q}\,\tilde{P}^{-1}\right] \\ \\ P = \varepsilon\,\tilde{P}^2\,exp\left[-\frac{1}{2\varepsilon}\,\tilde{Q}\,\tilde{P}^{-1}\right], \end{cases} \tag{1.86}
$$

donde vem:

$$
\tilde{H} = \varepsilon\left(\frac{\alpha'}{\alpha} + 1\right)\tilde{P}^2, \tag{1.87}
$$

que é a Hamiltoniana de uma partícula livre, sem potencial, de "massa" dada por

$$
m = \frac{1}{2\,\varepsilon}\left(\frac{\alpha'}{\alpha} + 1\right). \tag{1.88}
$$

Se escrevermos a Hamiltoniana final (1.87) em termos das coordenadas (x,y,η) iniciais, teremos:

$$
\tilde{H}(x, y, \eta) = \Delta_0\left(\frac{\alpha'}{\alpha} + 1\right)e^{-2\eta}\,x\,y. \tag{1.89}
$$

Isto conclui os cálculos associados ao formalismo Hamiltoniano para o *background* de Kasner.

Capítulo 2

Construção das Bases

2.1 Introdução

Seguindo o formalismo estabelecido por [13] e utilizado na literatura [10, 11, 12, 17, 24], na análise do modelo de Friedmann-Robertson-Walker (FRW), é conveniente construir-se uma base em termos da qual as quantidades perturbadas possam ser escritas e que possam ser fatoradas das equações Quase-Maxwellianas, de forma a explicitar a dependência temporal das perturbações. No caso, já estudado anteriormente, do modelo FRW, é utilizada uma base de harmônicos esféricos, determinados em termos de polinômios homogêneos, construídos em um espaço 4-dimensional e Euclideano, e que satisfazem a Equação de Laplace neste espaço. Em nosso caso, a base a ser construída não será escrita em termos de harmônicos esféricos, já que estamos tratando um modelo com anisotropia. Entretanto, podemos obter uma base específica para o modelo de Bianchi-I, de uma maneira análoga àquela feita para FRW. Isto será realizado nas seções que seguem.

Temos, como no caso do modelo de FRW, três tipos diferentes de base, adequadas à descrição de diferentes tipos de perturbações, e que enumeraremos a seguir:

1. *Base Escalar:*

   Descreve as perturbações da geometria que estão ligadas à matéria (mais especificamente, à densidade de energia); este tipo de perturbação será referido daqui em diante como *Perturbação Escalar.*

2. *Base Vetorial:*

   Descreve as perturbações associadas ao estado de movimento do fluido (mantendo a densidade de energia não perturbada); a este tipo de perturbação nos referiremos como *Perturbação Vetorial.*

3. *Base Tensorial:*

   Descreve as perturbações da geometria que não estão associadas à matéria (em outras palavras, descreve as ondas gravitacionais); este tipo de perturbação será referido como *Perturbação Tensorial.*

Torna-se necessário, neste ponto, tecermos alguns comentários a respeito dos casos escalar e vetorial. Dado que pelo menos um dos casos específicos dos modelos de Bianchi-I (a solução de Kasner) é uma solução exata das equações de Eisntein para o vazio (e, como tal, não possui matéria), teríamos uma dificuldade conceitual aparente em definir perturbações escalares ou vetoriais para este caso específico. Entretanto, é possível construir-se uma solução generalizada para o modelo específico de Kasner (conforme mencionado no Capítulo 1 e na Referência [19]) com termos associados à matéria, os quais verifica-se serem de ordem mais baixa do que os termos restantes, associados à geometria. Logo, podemos argumentar que, embora os termos de matéria não contribuam para o modelo não perturbado, podem vir a ser importantes na métrica perturbada. Neste caso, a análise dos casos escalar e vetorial torna-se interessante, mesmo quando a matéria não contribui para a geometria do *background.*

Vamos, então, proceder à construção dos três tipos de base para o modelo de Bianchi-I. Imporemos, em todos os casos, condições análogas àquelas especificadas para o caso das perturbações no modelo de Friedmann-Robertson-Walker.

2.2 Construção de uma Base Escalar

Começaremos por escrever a equação de Laplace para a base $\left\{\hat{Q}(x, y, z)\right\}$:

$$\hat{\nabla}^2 \hat{Q} = n^2 \hat{Q}, \tag{2.1}$$

onde n^2 é uma função do tempo t e onde o símbolo ($\hat{}$) denota o fato de que os $\hat{Q}$ são quantidades pertencentes à 3-superfície. Calculando o Laplaciano para a métrica de Bianchi-I (vide Apêndice A, equação (A.12)), a equação acima reduz-se a:

$$\left(g^{11}\frac{\partial^2}{\partial x^2} + g^{22}\frac{\partial^2}{\partial y^2} + g^{33}\frac{\partial^2}{\partial z^2}\right)\hat{Q} + n^2 \hat{Q} = 0,$$

a qual é facilmente integrada, dando:

$$\hat{Q}(x, y, z) = N \exp\left\{-i\left(n_1 x + n_2 y + n_3 z\right)\right\} \equiv N \exp -i\, n_j\, x^j, \tag{2.2}$$

onde N e n_j ($j = 1, 2, 3$) são constantes arbitrárias (assim escolhidas por uma questão de simplicidade nos cálculos) e a relação

$$n^2 \equiv -h^{\alpha\beta}\, n_\alpha\, n_\beta, \tag{2.3}$$

é válida.

De posse de uma base escalar explicitada, procedemos à construção de um vetor espacial e de um tensor simétrico de traço nulo, os quais serão usados para escrever as quantidades perturbadas:

$$\hat{Q}_\alpha \equiv \hat{\nabla}_\alpha \hat{Q} = h_\alpha{}^\beta \hat{Q}_{,\beta} = \frac{\partial \hat{Q}}{\partial x^\beta} h_\alpha{}^\beta = -i\, h_\alpha{}^\beta n_\beta \hat{Q}$$
$$\hat{Q}_{\alpha\beta} \equiv f(t)\, \hat{\nabla}_\beta \hat{Q}_\alpha + g(t)\, h_{\alpha\beta} \hat{Q}, \tag{2.4}$$

onde f e g são funções do tempo que serão determinadas a seguir.

É fácil ver-se de (2.4) que $\hat{Q}_\alpha$ tem apenas componentes espaciais, já que

$$\hat{Q}_0 = h_0{}^\beta \hat{Q}_\beta = 0.$$

Além disso, temos que a quantidade $(\hat{\nabla}_\beta \hat{Q}_\alpha)$ é simétrica. A prova desta asserção é direta:

$$\begin{aligned}
\hat{\nabla}_\beta \hat{Q}_\alpha &= h_\beta{}^\mu h_\alpha{}^\nu \left(h_\nu{}^\varepsilon \hat{Q}_\varepsilon \right)_{;\mu} \\
&= h_\beta{}^\mu h_\alpha{}^\nu \left[h_\nu{}^\varepsilon \hat{Q}_{\varepsilon;\mu} + (h_\nu{}^\varepsilon)_{;\mu} \hat{Q}_\varepsilon \right] \\
&= h_\beta{}^\mu h_\alpha{}^\nu \hat{Q}_{\varepsilon;\mu} + \\
&\quad + h_\beta{}^\mu h_\alpha{}^\nu \left(h^\varepsilon{}_{\nu,\mu} + \Gamma^\varepsilon_{\tau\mu} h_\nu{}^\tau - \Gamma^\tau_{\mu\nu} h_\tau{}^\varepsilon \right) \hat{Q}_\varepsilon .
\end{aligned} \tag{2.5}$$

O segundo termo no lado direito da relação acima reduz-se a:

$$h_\beta{}^\mu h_\alpha{}^\nu \left[(\delta_\nu{}^\varepsilon)_{,\mu} - (V^\varepsilon V_\nu)_{,\mu} \right],$$

o qual é obviamente nulo. Os termos seguintes também se anulam, já que as conexões que neles aparecem têm todos os índices espaciais (conforme a equação (1.3)). Ficamos então com:

$$\hat{\nabla}_\beta \hat{Q}_\alpha = h_\beta{}^\mu h_\alpha{}^\nu \hat{Q}_{\nu;\mu}$$

$$
\begin{aligned}
&= h_\beta{}^\mu\, h_\alpha{}^\nu \left(\hat{Q}_{\nu,\mu} - \Gamma^\varepsilon_{\mu\nu}\, \hat{Q}_\varepsilon\right) \\
&= h_\beta{}^\mu\, h_\alpha{}^\nu\, \hat{Q}_{\nu,\mu} \\
&= h_\beta{}^\mu\, h_\alpha{}^\nu \left(h_\nu{}^\gamma\, \hat{Q}_{,\gamma}\right)_{,\mu} \\
&= h_\beta{}^\mu\, h_\alpha{}^\nu \left[h_\nu{}^\gamma\, \hat{Q}_{,\gamma,\mu} - (h_\nu{}^\gamma)_{,\mu}\, \hat{Q}_{,\gamma}\right] \\
&= h_\beta{}^\mu\, h_\alpha{}^\nu\, \hat{Q}_{,\nu,\mu} \\
&= h_\beta{}^\mu\, h_\alpha{}^\nu\, \hat{Q}_{,\mu,\nu} \\
&= \hat{\nabla}_\alpha\, \hat{Q}_\beta,
\end{aligned}
$$

onde novamente os termos $(h_\nu{}^\gamma)_{,\mu}$ e o de conexão totalmente espacial são nulos. Isto prova a simetria de $(\hat{\nabla}_\beta\, \hat{Q}_\alpha)$ e, por extensão, a da quantidade tensorial $\hat{Q}_{\alpha\beta}$ (já que o segundo termo em $h_{\alpha\beta}$, é obviamente simétrico).

Vamos passar, agora, à determinação das funções $f(t)$ e $g(t)$ que utilizamos na definição de $\hat{Q}_{\alpha\beta}$, a segunda das equações (2.4). Impondo que o traço de $\hat{Q}_{\alpha\beta}$ seja nulo, temos:

$$
\hat{Q}^\alpha{}_\alpha = h^{\alpha\beta}\, \hat{Q}_{\alpha\beta} = 0
$$

$$
f(t)\, h^{\alpha\beta}\, \hat{\nabla}_\beta\, \hat{Q}_\alpha + g(t)\, h^{\alpha\beta}\, h_{\alpha\beta}\, \hat{Q} = 0
$$

$$f(t)\,\hat{\nabla}^{\alpha}\,\hat{Q}_{\alpha} + 3\,g(t)\,\hat{Q} = 0. \tag{2.6}$$

Vamos calcular a divergência de $\hat{Q}_{\alpha}$. Da primeira das equações (2.4) vem:

$$\begin{aligned}
\hat{\nabla}^{\alpha}\,\hat{Q}_{\alpha} &= h^{\alpha\beta}\,\hat{\nabla}_{\beta}\,\hat{Q}_{\alpha} \\
&= h^{\alpha\beta}\,\hat{\nabla}_{\beta}\left(\hat{\nabla}_{\alpha}\,\hat{Q}\right) \\
&= h^{\alpha\beta}\,h_{\beta}{}^{\mu}\,h_{\alpha}{}^{\nu}\left(h_{\nu}{}^{\gamma}\,\hat{Q}_{,\gamma}\right)_{;\mu} \\
&= h^{\mu\nu}\left[h_{\nu}{}^{\gamma}\,\hat{Q}_{,\gamma;\mu} + (h_{\nu}{}^{\gamma})_{;\mu}\,\hat{Q}_{,\gamma}\right] \\
&= h^{\mu\nu}\left[h_{\nu}{}^{\gamma}\left(\hat{Q}_{,\gamma,\mu} - \Gamma^{\lambda}_{\gamma\mu}\,\hat{Q}_{,\lambda}\right) + (\delta^{\gamma}{}_{\nu;\mu} - V^{\gamma}{}_{;\mu}\,V_{\nu} - V^{\gamma}\,V_{\nu;\mu})\,\hat{Q}_{,\gamma}\right] \\
&= h^{\mu\gamma}\,\hat{Q}_{,\gamma,\mu} + h^{\mu\nu}\left(\delta^{\gamma}{}_{\nu,\mu} - \Gamma^{\lambda}_{\mu\nu}\,\delta^{\gamma}{}_{\lambda} + \Gamma^{\gamma}_{\lambda\mu}\,\delta^{\lambda}{}_{\nu}\right)\hat{Q}_{,\gamma} \\
&= h^{\mu\gamma}\,\hat{Q}_{,\gamma,\mu} \\
&= i^{2}\,h^{\mu\gamma}\,n_{\gamma}\,n_{\mu}\,\hat{Q} \\
&= -h^{\alpha\beta}\,n_{\alpha}\,n_{\beta}\,\hat{Q} \\
&= n^{2}\,\hat{Q},
\end{aligned} \tag{2.7}$$

da equação (2.3).

E, substituindo o resultado (2.7) acima em (2.6), vem afinal:

$$f(t)\, n^2\, \hat{Q} + 3\, g(t)\, \hat{Q} = 0 \Longrightarrow$$

$$\Longrightarrow \quad f(t)\, n^2 = -3\, g(t) \Longrightarrow$$

$$\Longrightarrow \quad g(t) = -\frac{1}{3}\, n^2\, f(t).$$

Logo, podemos escolher $f(t)$ e $g(t)$ estará imediatamente fixada. Para simplificar nossos cálculos subseqüentes, tomamos $f(t)$ como uma constante:

$$f(t) = 1,$$

donde vem:

$$g(t) = -\frac{1}{3}\, n^2.$$

Neste caso, o tensor simétrico e de traço nulo construído a partir da base escalar se escreve como:

$$\hat{Q}_{\alpha\beta} = \hat{\nabla}_\beta\, \hat{Q}_\alpha - \frac{1}{3}\, n^2\, \hat{Q}. \tag{2.8}$$

Entretanto, vimos que:

$$\begin{aligned} \hat{\nabla}_\beta\, \hat{Q}_\alpha &= h^\mu{}_\beta\, h^\nu{}_\alpha\, \hat{Q}_{,\mu,\nu} = -h^\mu{}_\beta\, h^\nu{}_\alpha\, n_\mu\, n_\nu\, \hat{Q} \\ &= -n_\alpha\, n_\beta\, \hat{Q}, \end{aligned}$$

de (2.2). Logo, a definição (2.8) acima pode ser também escrita como:

$$\hat{Q}_{\alpha\beta} = -\left(n_\alpha\, n_\beta + \frac{n^2}{3}\, h_{\alpha\beta}\right)\hat{Q}. \tag{2.9}$$

As derivadas projetadas no tempo da base escalar e das quantidades a partir dela construídas são, neste ponto, facilmente obtidas:

$$\left(\hat{Q}\right)^\bullet = \hat{Q}_{,\alpha}\, V^\alpha = -i\, n_\alpha\, V^\alpha\, \hat{Q} = 0, \tag{2.10}$$

$$\begin{aligned}
\left(\hat{Q}_\alpha\right)^\bullet &= \hat{Q}_{\alpha;\beta}\, V^\beta \\
&= \left(\hat{Q}_{\alpha,\beta} - \Gamma^\gamma_{\alpha\beta}\, \hat{Q}_\gamma\right) V^\beta \\
&= \hat{Q}_{,\alpha,\beta}\, V^\beta - \Gamma^\gamma_{\alpha 0}\, \hat{Q}_\gamma \\
&= -n_\alpha\, n_\beta\, V^\beta\, \hat{Q} - \left(\sigma_\alpha{}^\beta + \frac{\theta}{3}\, h_\alpha{}^\beta\right)\hat{Q}_\beta \\
&= -\frac{\theta}{3}\,\hat{Q}_\alpha - \sigma_\alpha{}^\beta\, \hat{Q}_\beta.
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
\left(\hat{Q}_{\alpha\beta}\right)^\bullet &= \left(\hat{\nabla}_\beta\, \hat{Q}_\alpha\right)^\bullet - \frac{1}{3}\left(n^2\, h_{\alpha\beta}\, \hat{Q}\right)^\bullet \\
&= \left[h_\beta{}^\nu\, h_\alpha{}^\nu \left(h_\mu{}^\varepsilon\, \hat{Q}_\varepsilon\right)_{;\nu}\right]_{;\gamma} V^\gamma + \frac{1}{3}\, h^{\mu\nu}{}_{;\gamma}\, V^\gamma\, n_\mu\, n_\nu\, h_{\alpha\beta}\, \hat{Q} + \\
&\quad + \frac{1}{3}\, h^{\mu\nu}\, n_{\mu;\gamma}\, V^\gamma\, n_\nu\, h_{\alpha\beta}\, \hat{Q} + \frac{1}{3}\, h^{\mu\nu}\, n_\mu\, n_{\nu;\gamma}\, V^\gamma\, h_{\alpha\beta}\, \hat{Q} - \\
&\quad - \frac{n^2}{3}\, h_{\alpha\beta;\gamma}\, V^\gamma\, \hat{Q}
\end{aligned}$$

$$
\begin{aligned}
= \;& h^{\nu}{}_{\beta;\gamma}\, V^{\gamma}\, h_{\alpha}{}^{\mu}\, \left(h_{\mu}{}^{\varepsilon}\, \hat{Q}_{\varepsilon}\right)_{;\nu} + h_{\beta}{}^{\nu}\, h^{\mu}{}_{\alpha;\gamma}\, V^{\gamma}\, \left(h_{\mu}{}^{\varepsilon}\, \hat{Q}_{\varepsilon}\right)_{;\nu} + \\
& + h_{\beta}{}^{\nu}\, h_{\alpha}{}^{\mu}\, \left(h_{\mu}{}^{\varepsilon}\, \hat{Q}_{\varepsilon}\right)_{;\nu;\gamma} V^{\gamma} + \frac{1}{3}\, \left(g^{\mu\nu}{}_{;\gamma} - V^{\mu}{}_{;\gamma}\, V^{\nu} - V^{\mu}\, V^{\nu}{}_{;\gamma}\right) V^{\gamma}\, n_{\mu}\, n_{\nu}\, h_{\alpha\beta}\, \hat{Q} + \\
& + \frac{1}{3}\, h^{\mu\nu}\, \left(n_{\mu,\gamma} - \Gamma^{\tau}_{\mu\gamma}\, n_{\tau}\right) V^{\gamma}\, n_{\nu}\, h_{\alpha\beta}\, \hat{Q} + \frac{1}{3}\, h^{\mu\nu}\, n_{\mu}\, \left(n_{\nu,\gamma} - \Gamma^{\tau}_{\nu\gamma}\, n_{\tau}\right) V^{\gamma}\, h_{\alpha\beta}\, \hat{Q} - \\
& - \frac{n^{2}}{3}\, \left(g_{\alpha\beta;\gamma} - V_{\alpha;\gamma}\, V_{\beta} - V_{\alpha}\, V_{\beta;\gamma}\right) V^{\gamma}\, \hat{Q}
\end{aligned}
$$

$$
\begin{aligned}
= \;& \left(h^{\nu}{}_{\beta,\gamma} + \Gamma^{\nu}_{\tau\gamma}\, h_{\beta}{}^{\tau} - \Gamma^{\tau}_{\beta\gamma}\, h_{\tau}{}^{\nu}\right) V^{\gamma}\, h_{\alpha}{}^{\mu}\, \left(h_{\mu}{}^{\varepsilon}\, \hat{Q}_{\varepsilon}\right)_{;\nu} + \\
& + h_{\beta}{}^{\nu}\, \left(h^{\mu}{}_{\alpha,\gamma} + \Gamma^{\mu}_{\tau\gamma}\, h_{\alpha}{}^{\tau} - \Gamma^{\tau}_{\alpha\gamma}\, h_{\tau}{}^{\mu}\right) V^{\gamma}\, \left(h_{\mu}{}^{\varepsilon}\, \hat{Q}_{\varepsilon}\right)_{;\nu} + \\
& + h_{\beta}{}^{\nu}\, h_{\alpha}{}^{\mu}\, \left[\left(h_{\mu}{}^{\varepsilon}\, \hat{Q}_{\varepsilon}\right)_{;\nu}\right]_{;\gamma} V^{\gamma} - \frac{1}{3}\, h^{\mu\nu}\, \left(\sigma_{\mu}{}^{\tau} + \frac{\theta}{3}\, h_{\mu}{}^{\tau}\right) n_{\tau}\, n_{\nu} h_{\alpha\beta}\, \hat{Q} - \\
& - \frac{1}{3}\, h^{\mu\nu}\, n_{\mu}\, \left(\sigma_{\nu}{}^{\tau} + \frac{\theta}{3}\, h_{\nu}{}^{\tau}\right) n_{\tau}\, h_{\alpha\beta}\, \hat{Q}
\end{aligned}
$$

$$
\begin{aligned}
= \;& \left(\sigma_{\tau}{}^{\nu} + \frac{\theta}{3}\, h_{\tau}{}^{\nu}\right) h_{\beta}{}^{\tau}\, h_{\alpha}{}^{\mu}\, \left(h_{\mu}{}^{\varepsilon}\, \hat{Q}_{\varepsilon}\right)_{;\nu} - \left(\sigma_{\beta}{}^{\tau} + \frac{\theta}{3}\, h_{\beta}{}^{\tau}\right) h_{\tau}{}^{\nu}\, h_{\alpha}{}^{\mu}\, \left(h_{\mu}{}^{\varepsilon}\, \hat{Q}_{\varepsilon}\right)_{;\nu} + \\
& + h_{\beta}{}^{\nu}\, \left(\sigma_{\tau}{}^{\mu} + \frac{\theta}{3}\, h_{\tau}{}^{\mu}\right) h_{\alpha}{}^{\tau}\, \left(h_{\mu}{}^{\varepsilon}\, \hat{Q}_{\varepsilon}\right)_{;\nu} - \\
& - h_{\beta}{}^{\nu}\, \left(\sigma_{\alpha}{}^{\tau} + \frac{\theta}{3}\, h_{\alpha}{}^{\tau}\right) h_{\tau}{}^{\mu}\, \left(h_{\mu}{}^{\varepsilon}\, \hat{Q}_{\varepsilon}\right)_{;\nu} + \\
& + h_{\beta}{}^{\nu}\, h_{\alpha}{}^{\mu}\, \left(h_{\mu}{}^{\varepsilon}\, \hat{Q}_{\varepsilon}\right)_{;\nu;\gamma} V^{\gamma} - \frac{1}{3}\, h^{\mu\nu}\, n_{(\mu}\, \left(\sigma_{\nu)}{}^{\tau} + \frac{\theta}{3}\, h_{\nu)}{}^{\tau}\right) n_{\tau}\, h_{\alpha\beta}\, \hat{Q}
\end{aligned}
$$

$$
\begin{aligned}
= \;& h_{\beta}{}^{\nu}\, h_{\alpha}{}^{\mu}\, \left[\left(h_{\mu}{}^{\varepsilon}\, \hat{Q}_{\varepsilon}\right)_{;\gamma;\nu} - R^{\tau}{}_{\mu\nu\gamma}\, \left(h_{\tau}{}^{\varepsilon}\, \hat{Q}_{\varepsilon}\right)\right] V^{\gamma} - \\
& - \frac{2}{3}\, \left(\sigma^{\mu\nu} + \frac{\theta}{3}\, h^{\mu\nu}\right) n_{\mu}\, n_{\nu}\, h_{\alpha\beta}\, \hat{Q}
\end{aligned}
$$

$$
\begin{aligned}
= \;& h_{\beta}{}^{\nu}\, h_{\alpha}{}^{\mu}\, \left\{\left[\left(h_{\mu}{}^{\varepsilon}\, \hat{Q}_{\varepsilon}\right)_{;\gamma} V^{\gamma}\right]_{;\nu} - \left(h_{\mu}{}^{\varepsilon}\, \hat{Q}_{\varepsilon}\right)_{;\gamma} V^{\gamma}{}_{;\nu}\right\} - \\
& - \frac{2}{3}\, \left(\sigma^{\mu\nu} + \frac{\theta}{3}\, h^{\mu\nu}\right) n_{\mu}\, n_{\nu}\, h_{\alpha\beta}\, \hat{Q}
\end{aligned}
$$

$$
= \; h_{\beta}{}^{\nu}\, h_{\alpha}{}^{\mu}\, \left[\left(h_{\mu}{}^{\varepsilon}\, \hat{Q}_{\varepsilon}\right)^{\bullet}\right]_{;\nu} - h_{\beta}{}^{\nu}\, h_{\alpha}{}^{\mu}\, \left(\sigma_{\nu}{}^{\gamma} + \frac{\theta}{3}\, h_{\nu}{}^{\gamma}\right) \left(h_{\mu}{}^{\varepsilon}\, \hat{Q}_{\varepsilon}\right)_{;\gamma} -
$$

$$-\frac{2}{3}\,\sigma^{\mu\nu}\,n_\mu\,n_\nu\,h_{\alpha\beta}\,\hat{Q}+\frac{2}{9}\,\theta\,n^2\,h_{\alpha\beta}\,\hat{Q}$$

$$\begin{aligned}
= \; & h_\beta{}^\nu\,h_\alpha{}^\mu\,\left[(h_\mu{}^\varepsilon)^\bullet\,\hat{Q}_\varepsilon+h_\mu{}^\varepsilon\,\left(\hat{Q}_\varepsilon\right)^\bullet\right]_{;\nu}- \\
& -\left(\sigma_\beta{}^\nu+\frac{\theta}{3}\,h_\beta{}^\nu\right)\,\left[(h_\mu{}^\varepsilon)_{;\gamma}\,\hat{Q}_\varepsilon+h_\mu{}^\varepsilon\,\hat{Q}_{\varepsilon;\gamma}\right]- \\
& -\frac{2}{3}\,\sigma^{\mu\nu}\,n_\mu\,n_\nu\,h_{\alpha\beta}\,\hat{Q}+\frac{2}{9}\,\theta\,n^2\,h_{\alpha\beta}\,\hat{Q}
\end{aligned}$$

$$\begin{aligned}
= \; & h_\beta{}^\nu\,h_\alpha{}^\mu\,\left[-h_\mu{}^\varepsilon\,\left(\frac{\theta}{3}\,h_\varepsilon{}^\tau\,\hat{Q}_\tau+\sigma_\varepsilon{}^\tau\,\hat{Q}_\tau\right)\right]_{;\nu}- \\
& -\left(\sigma_\beta{}^\nu+\frac{\theta}{3}\,h_\beta{}^\nu\right)\,h_\alpha{}^\mu\,\left[h^\varepsilon{}_{\mu,\nu}\,\hat{Q}_\varepsilon+\Gamma^\varepsilon_{\tau\nu}\,h_\mu{}^\tau\,\hat{Q}_\varepsilon-\Gamma^\tau_{\mu\nu}\,h_\tau{}^\varepsilon\,\hat{Q}_\varepsilon\right]- \\
& -\left(\sigma_\beta{}^\nu+\frac{\theta}{3}\,h_\beta{}^\nu\right)\,h_\alpha{}^\mu\,\hat{Q}_{\mu;\nu}- \\
& -\frac{2}{3}\,\sigma^{\mu\nu}\,n_\mu\,n_\nu\,h_{\alpha\beta}\,\hat{Q}+\frac{2}{9}\,\theta\,n^2\,h_{\alpha\beta}\,\hat{Q}
\end{aligned}$$

$$\begin{aligned}
= \; & -h_\beta{}^\nu\,h_\alpha{}^\mu\,\left[\frac{\theta}{3}\,\left(h^\varepsilon{}_{\mu,\nu}+\Gamma^\varepsilon_{\tau\nu}\,h_\mu{}^\tau+\Gamma^\tau_{\mu\nu}\,h_\tau{}^\varepsilon\right)\,\hat{Q}_\varepsilon+\frac{\theta}{3}\,h_\mu{}^\varepsilon\,\hat{Q}_{\varepsilon;\nu}+\right. \\
& +\left(h^\varepsilon{}_{\mu.\nu}+\Gamma^\varepsilon_{\gamma\nu}\,h_\mu{}^\gamma-\Gamma^\gamma_{\mu\nu}\,h_\gamma{}^\varepsilon\right)\,\sigma_\varepsilon{}^\tau\,\hat{Q}_\tau+ \\
& \left.+h_\mu{}^\varepsilon\,\left(\sigma^\tau{}_{\varepsilon,\nu}+\Gamma^\tau_{\gamma\nu}\,\sigma_\varepsilon{}^\gamma-\Gamma^\gamma_{\varepsilon\nu}\,\sigma_\gamma{}^\tau\right)\,\hat{Q}_\tau-h_\mu{}^\varepsilon\,\sigma_\varepsilon{}^\tau\,\hat{Q}_{\tau;\nu}\right]- \\
& -\left(\sigma_\beta{}^\tau+\frac{\theta}{3}\,h_\beta{}^\tau\right)\,h_\tau{}^\nu\,h_\alpha{}^\mu\,\hat{Q}_{\mu;\nu}- \\
& -\frac{2}{3}\,\sigma^{\mu\nu}\,n_\mu\,n_\nu\,h_{\alpha\beta}\,\hat{Q}+\frac{2}{9}\,\theta\,n^2\,h_{\alpha\beta}\,\hat{Q}
\end{aligned}$$

$$\begin{aligned}
= \; & -\frac{\theta}{3}\,h_\beta{}^\nu\,h_\alpha{}^\mu\,\hat{Q}_{\mu;\nu}-\sigma_\alpha{}^\mu\,h_\beta{}^\nu\,h_\mu{}^\tau\,\hat{Q}_{\tau;\nu}-\left(\sigma_\beta{}^\tau+\frac{\theta}{3}\,h_\beta{}^\tau\right)\,h_\tau{}^\nu\,h_\alpha{}^\mu\,\hat{Q}_{\mu;\nu}- \\
& -\frac{2}{3}\,\sigma^{\mu\nu}\,n_\mu\,n_\nu\,h_{\alpha\beta}\,\hat{Q}+\frac{2}{9}\,\theta\,n^2\,h_{\alpha\beta}\,\hat{Q}
\end{aligned}$$

$$\begin{aligned}
= \; & -\frac{\theta}{3}\,\hat{\nabla}_\beta\,\hat{Q}_\alpha-\sigma_\alpha{}^\mu\,\hat{\nabla}_\mu\,\hat{Q}_\beta-\sigma_\beta{}^\tau\,\hat{\nabla}_\tau\,\hat{Q}_\alpha- \\
& -\frac{\theta}{3}\,\hat{\nabla}_\beta\,\hat{Q}_\alpha-\frac{2}{3}\,\sigma^{\mu\nu}\,n_\mu\,n_\nu\,h_{\alpha\beta}\,\hat{Q}+\frac{2}{9}\,\theta\,n^2\,h_{\alpha\beta}\,\hat{Q}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{3}\,\theta\,\hat{\nabla}_\beta\,\hat{Q}_\alpha + \frac{2}{9}\,\theta\,n^2\,h_{\alpha\beta}\,\hat{Q} - \\
&\quad -\sigma_{(\alpha}{}^{\mu}\,\hat{\nabla}_\mu\,\hat{Q}_{\beta)} - \frac{2}{3}\,\sigma^{\mu\nu}\,n_\mu\,n_\nu\,h_{\alpha\beta}\,\hat{Q} \\
&= -\frac{2}{3}\,\theta\left[\hat{\nabla}_\beta\,\hat{Q}_\alpha - \frac{1}{3}\,n^2\,h_{\alpha\beta}\,\hat{Q}\right] - \sigma_{(\alpha}{}^{\mu}\,\hat{\nabla}_\mu\,\hat{Q}_{\beta)} - \frac{2}{3}\,\sigma^{\mu\nu}\,n_\mu\,n_\nu\,h_{\alpha\beta}\,\hat{Q} \\
&= -\frac{2}{3}\,\theta\,\hat{Q}_{\alpha\beta} - \sigma_{(\alpha}{}^{\mu}\,\hat{\nabla}_\mu\,\hat{Q}_{\beta)} - \frac{2}{3}\,\sigma^{\mu\nu}\,n_\mu\,n_\nu\,h_{\alpha\beta}\,\hat{Q}.
\end{aligned}
$$

Mas, das equações (2.4), podemos escrever:

$$
n_\mu\,n_\nu\,\hat{Q} = -\hat{Q}_{\mu\nu} - \frac{1}{3}\,n^2\,h_{\mu\nu}\,\hat{Q} \Longrightarrow
$$

$$
\Longrightarrow \quad \sigma^{\mu\nu}\,n_\mu\,n_\nu\,\hat{Q} = -\sigma^{\mu\nu}\,\hat{Q}_{\mu\nu} - \frac{1}{3}\,n^2\,\sigma^{\mu\nu}\,h_{\mu\nu}\,\hat{Q} \Longrightarrow
$$

$$
\Longrightarrow \quad \sigma^{\mu\nu}\,n_\mu\,n_\nu\,\hat{Q} = -\sigma^{\mu\nu}\,\hat{Q}_{\mu\nu}, \tag{2.12}
$$

e ainda:

$$
\begin{aligned}
\sigma_{(\alpha}{}^{\mu}\,\hat{\nabla}_\mu\,\hat{Q}_{\beta)} &= \sigma_{(\alpha}{}^{\mu}\left[\hat{Q}_{\beta)\mu} + \frac{1}{3}\,n^2\,h_{\beta)\mu}\,\hat{Q}\right] \\
&= \sigma_{(\alpha}{}^{\mu}\,\hat{Q}_{\beta)\mu} + \frac{2}{3}\,n^2\,\sigma_{\alpha\beta}\,\hat{Q}.
\end{aligned} \tag{2.13}
$$

Então vem finalmente que:

$$
\left(\hat{Q}_{\alpha\beta}\right)^{\bullet} = -\frac{2}{3}\,\theta\,\hat{Q}_{\alpha\beta} - \sigma_{(\alpha}{}^{\mu}\,\hat{Q}_{\beta)\mu} - \frac{2}{3}\,n^2\,\sigma_{\alpha\beta}\,\hat{Q} + \frac{2}{3}\,h_{\alpha\beta}\,\sigma^{\mu\nu}\,\hat{Q}_{\mu\nu}. \tag{2.14}
$$

É fácil ver que o traço de $\left(\hat{Q}_{\alpha\beta}\right)^{\bullet}$ também é nulo:

$$
\begin{aligned}
\left(\hat{Q}^{\alpha}{}_{\alpha}\right)^{\bullet} &= -\frac{2}{3}\,\theta\,\hat{Q}^{\alpha}{}_{\alpha} - 2\,\sigma^{\alpha\mu}\,\hat{Q}_{\alpha\mu} - \\
&\quad -\frac{2}{3}\,n^{2}\,\sigma^{\alpha}{}_{\alpha}\,\hat{Q} + \frac{2}{3}\,h^{\alpha}{}_{\alpha}\,\sigma^{\mu\nu}\,\hat{Q}_{\mu\nu} \\
&= -2\,\sigma^{\alpha\mu}\,\hat{Q}_{\alpha\mu} + 2\,\sigma^{\mu\nu}\,\hat{Q}_{\mu\nu} \\
&= 0.
\end{aligned}
$$

A divergência de $\hat{Q}_{\alpha}$ é dada pela relação (2.7). A divergência de $\hat{Q}_{\alpha\beta}$ é:

$$
\begin{aligned}
\hat{\nabla}^{\mu}\,\hat{Q}_{\mu\nu} &= h^{\mu\alpha}\,\hat{\nabla}_{\alpha}\,\hat{Q}_{\mu\nu} \\
&= h^{\mu\alpha}\,\hat{\nabla}_{\alpha}\left[\hat{\nabla}_{\nu}\,\hat{Q}_{\mu} - \frac{1}{3}\,n^{2}\,h_{\mu\nu}\,\hat{Q}\right] \\
&= h^{\mu\alpha}\,\hat{\nabla}_{\alpha}\left(\hat{\nabla}_{\nu}\,\hat{Q}\right) - \frac{1}{3}\,h^{\mu\alpha}\,\hat{\nabla}_{\alpha}\left(n^{2}\,h_{\mu\nu}\,\hat{Q}\right) \\
&= h^{\mu\alpha}\left[\hat{\nabla}_{\nu}\left(\hat{\nabla}_{\alpha}\,\hat{Q}\right) + \hat{R}^{\beta}{}_{\mu\alpha\nu}\,\hat{Q}_{\beta}\right] - \\
&\quad -\frac{1}{3}\,h^{\mu\alpha}\left[\left(\hat{\nabla}_{\alpha}\,n^{2}\right)h_{\mu\nu}\,\hat{Q} + n^{2}\left(\hat{\nabla}_{\alpha}\,h_{\mu\nu}\right)\hat{Q} + n^{2}\,h_{\mu\nu}\left(\hat{\nabla}_{\alpha}\,\hat{Q}\right)\right] \\
&= \hat{\nabla}_{\nu}\left[h^{\mu\alpha}\,\hat{\nabla}_{\alpha}\,\hat{Q}_{\mu}\right] - \left(\hat{\nabla}_{\nu}\,h^{\mu\alpha}\right)\left(\hat{\nabla}_{\alpha}\,\hat{Q}_{\mu}\right) + \\
&\quad +h^{\mu\alpha}\,\hat{R}^{\beta}{}_{\mu\alpha\nu}\,\hat{Q}_{\beta} - \frac{1}{3}\,h_{\nu}{}^{\alpha}\left(\hat{\nabla}_{\alpha}\,n^{2}\right)\hat{Q} - \\
&\quad -\frac{1}{3}\,n^{2}\,h^{\mu\alpha}\left(\hat{\nabla}_{\alpha}\,h_{\mu\nu}\right)\hat{Q} - \frac{1}{3}\,n^{2}\,h_{\nu}{}^{\alpha}\,\hat{Q}_{\alpha}.
\end{aligned}
$$

Mas, de (A.8) e (A.10), temos que o segundo, terceiro e quinto termos no lado direito da equação acima se anulam; além disso, podemos calcular o quarto termo como:

$$
\begin{aligned}
\hat{\nabla}_\nu \left(n^2\right) &= h_\nu{}^\mu \left(n^2\right)_{;\mu} \\
&= -h_\nu{}^\mu \left(h^{\alpha\beta}\, n_\alpha\, n_\beta\right)_{;\mu} \\
&= -h_\nu{}^\mu \left[\left(h^{\alpha\beta}{}_{,\mu} + \Gamma^\alpha_{\gamma\mu}\, h^{\gamma\beta} - \Gamma^\beta_{\gamma\mu}\, h^{\alpha\gamma}\right) n_\alpha\, n_\beta + \right.\\
&\quad + h^{\alpha\beta} \left(n_{\alpha,\mu} - \Gamma^\gamma_{\alpha\mu}\, n_\gamma\, n_\beta\right) + \\
&\quad \left. + h^{\alpha\beta}\, n_\alpha \left(n_{\beta,\mu} - \Gamma^\gamma_{\beta\mu}\, n_\gamma\right)\right] \\
&= -h_\nu{}^\mu \left(g^{\alpha\beta}{}_{,\mu} - V^\alpha{}_{,\mu}\, V^\beta - V^\beta{}_{,\mu}\, V^\alpha\right) n_\alpha\, n_\beta \\
&= 0,
\end{aligned}
$$

donde escrevemos então que:

$$
\begin{aligned}
\hat{\nabla}^\mu\, \hat{Q}_{\mu\nu} &= \hat{\nabla}_\nu \left[\hat{\nabla}^\mu\, \hat{Q}_\mu\right] - \frac{1}{3}\, n^2\, \hat{Q}_\nu \\
&= \hat{\nabla}_\nu \left[n^2\, \hat{Q}\right] - \frac{1}{3}\, n^2\, \hat{Q} \\
&= n^2 \left(\hat{\nabla}_\nu\, \hat{Q}\right) + \left(\hat{\nabla}_\nu\, n^2\right) \hat{Q} - \frac{1}{3}\, n^2\, \hat{Q} \\
&= \left(n^2 - \frac{1}{3}\, n^2\right) \hat{Q}_\nu \\
&= \frac{2}{3}\, n^2\, \hat{Q}_\nu.
\end{aligned}
\tag{2.15}
$$

Por último, vamos calcular os Laplacianos de $\hat{Q}_\alpha$ e $\hat{Q}_{\alpha\beta}$:

$$
\begin{aligned}
\hat{\nabla}^2 \hat{Q}_\alpha &= \hat{\nabla}^\mu \hat{\nabla}_\mu \hat{Q}_\alpha \\
&= \hat{\nabla}^\mu \left(\hat{\nabla}_\alpha \hat{Q}_\mu\right) \\
&= h^{\mu\nu} \hat{\nabla}_\nu \left(\hat{\nabla}_\alpha \hat{Q}_\mu\right) \\
&= h^{\mu\nu} \left[\hat{\nabla}_\alpha \left(\hat{\nabla}_\nu \hat{Q}_\mu\right) + \hat{R}^\beta{}_{\mu\nu\alpha} \hat{Q}_\beta\right] \\
&= \hat{\nabla}_\alpha \left(h^{\mu\nu} \hat{\nabla}_\nu \hat{Q}_\mu\right) - \left(\hat{\nabla}_\alpha h^{\mu\nu}\right) \left(\hat{\nabla}_\nu \hat{Q}_\mu\right) + h^{\mu\nu} \hat{R}^\beta{}_{\mu\nu\alpha} \hat{Q}_\beta = \hat{\nabla}_\alpha \left(\hat{\nabla}^\mu \hat{Q}_\mu\right) \\
&= \hat{\nabla}_\alpha \left(n^2 \hat{Q}\right) \\
&= n^2 \left(\hat{\nabla}_\alpha \hat{Q}\right) - \left(\hat{\nabla}_\alpha n^2\right) \hat{Q} \\
&= n^2 \hat{Q}_\alpha, \qquad (2.16)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\nabla}^2 \hat{Q}_{\alpha\beta} &= \hat{\nabla}^2 \left[\hat{\nabla}_\beta \hat{Q}_\alpha - \frac{1}{3} n^2 h_{\alpha\beta} \hat{Q}\right] \\
&= \hat{\nabla}^\mu \left[\hat{\nabla}_\mu \left(\hat{\nabla}_\beta \hat{Q}_\alpha\right)\right] - \frac{1}{3} n^2 h_{\alpha\beta} \left(\hat{\nabla}^2 \hat{Q}\right) \\
&= h^{\mu\nu} \hat{\nabla}_\nu \left[\hat{\nabla}_\beta \left(\hat{\nabla}_\mu \hat{Q}_\alpha\right) + \hat{R}^\gamma{}_{\alpha\mu\beta} \hat{Q}_\gamma\right] - \frac{1}{3} n^4 h_{\alpha\beta} \hat{Q}
\end{aligned}
$$

$$
\begin{aligned}
&= h^{\mu\nu}\,\hat{\nabla}_{\nu}\left[\hat{\nabla}_{\beta}\left(\hat{\nabla}_{\mu}\hat{Q}_{\alpha}\right)\right]+h^{\mu\nu}\left(\hat{\nabla}_{\nu}\hat{R}^{\gamma}{}_{\alpha\mu\beta}\right)\hat{Q}_{\gamma}-\frac{1}{3}n^{4}h_{\alpha\beta}\hat{Q} \\
&= h^{\mu\nu}\left[\hat{\nabla}_{\beta}\left(\hat{\nabla}_{\nu}\left(\hat{\nabla}_{\mu}\hat{Q}_{\alpha}\right)\right)+\hat{R}^{\gamma}{}_{\mu\nu\beta}\left(\hat{\nabla}_{\gamma}\hat{Q}_{\alpha}\right)+\right. \\
&\quad \left.+\hat{R}^{\gamma}_{\alpha\nu\beta}\left(\hat{\nabla}_{\mu}\hat{Q}_{\gamma}\right)+h_{\nu}\;\;h_{\tau}{}^{\gamma}h_{\alpha}{}^{\lambda}h_{\mu}{}^{\varphi}h_{\beta}{}^{\sigma}\left(\hat{R}^{\tau}{}_{\lambda\varphi\sigma}\right)_{;\varepsilon}\hat{Q}_{\gamma}\right]- \\
&\quad -\frac{1}{3}n^{4}h_{\alpha\beta}\hat{Q} \\
&= \hat{\nabla}_{\beta}\left(\hat{\nabla}^{2}\hat{Q}_{\alpha}\right)+h^{\mu\nu}h_{\tau}{}^{\gamma}h_{\alpha}{}^{\lambda}h_{\mu}{}^{\varphi}h_{\beta}{}^{\gamma}\left(\hat{R}^{\tau}{}_{\lambda\varphi\gamma,\nu}+\Gamma^{\tau}_{\sigma\nu}\hat{R}^{\sigma}{}_{\lambda\varphi\gamma}-\right. \\
&\quad \left.-\Gamma^{\sigma}_{\lambda\nu}\hat{R}^{\tau}{}_{\sigma\varphi\gamma}-\Gamma^{\sigma}_{\varphi\nu}\hat{R}^{\tau}{}_{\lambda\sigma\gamma}-\Gamma^{\sigma}_{\gamma\nu}\hat{R}^{\tau}{}_{\lambda\varphi\sigma}\right)\hat{Q}_{\gamma}-\frac{1}{3}n^{4}h_{\alpha\beta}\hat{Q} \\
&= \hat{\nabla}_{\beta}\left(n^{2}\hat{Q}_{\alpha}\right)-\frac{1}{3}n^{4}h_{\alpha\beta}\hat{Q} \\
&= n^{2}\left(\hat{\nabla}_{\beta}\hat{Q}_{\alpha}\right)-\frac{1}{3}n^{4}h_{\alpha\beta}\hat{Q} \\
&= n^{2}\left[\hat{\nabla}_{\beta}\hat{Q}_{\alpha}-\frac{1}{3}n^{2}h_{\alpha\beta}\hat{Q}\right] \\
&= n^{2}\hat{Q}_{\alpha\beta},
\end{aligned}
\tag{2.17}
$$

o que conclui, portanto, os cálculos referentes à base escalar.

2.3 Construção de uma Base Vetorial

Passaremos, agora, à obtenção de uma base vetorial, a qual denotaremos por $\left\{\hat{P}_{\alpha}(x,y,z,t)\right\}$. Como fizemos na seção anterior, escreveremos a equação de Laplace que uma tal base deverá satisfazer:

$$\hat{\nabla}^2 \hat{P}_\alpha = m^2 \hat{P}_\alpha, \tag{2.18}$$

onde m^2 é uma função de t e onde, novamente, o símbolo $(\hat{\ })$ identifica quantidades pertencentes à 3-superfície. Usando, então, a relação (A.12), escrevemos:

$$\left(g^{11} \frac{\partial^2}{\partial x^2} + g^{22} \frac{\partial^2}{\partial y^2} + g^{33} \frac{\partial^2}{\partial z^2} \right) \hat{P}_\alpha = m^2 \hat{P}_\alpha. \tag{2.19}$$

As equações acima são facilmente integradas, obtendo-se:

$$\hat{P}_\alpha = \mathcal{P}^0_\alpha \, e^{-i\, m_j\, x^j}, \tag{2.20}$$

onde $\mathcal{P}^0_\alpha$ é, a princípio, uma função de todas as coordenadas (t,x,y,z), enquanto os m_j dependem apenas de t, já que uma dependência nas coordenadas espaciais causaria problemas na base (conforme já vimos no caso da base escalar, na seção anterior). As derivadas parciais espacial e temporal se escrevem, de (2.20), como:

$$\begin{aligned}
\hat{P}_{\alpha,k} &= \mathcal{P}^0_{\alpha,k} \, e^{-i\, m_j\, x^j} - i\, m_k \, \mathcal{P}^0_\alpha \, e^{-i\, m_j\, x^j} \\
&= \left(\frac{\mathcal{P}^0_{\alpha,k}}{\mathcal{P}^0_\alpha} - i\, m_k \right) \mathcal{P}^0_\alpha \, e^{-i\, m_j\, x^j} \\
&= \left(\frac{\mathcal{P}^0_{\alpha,k}}{\mathcal{P}^0_\alpha} - i\, m_k \right) \hat{P}_\alpha; \qquad (2.21)
\end{aligned}$$

$$\hat{P}_{\alpha,0} = \mathcal{P}^0_{\alpha,0} \, e^{-i\, m_j\, x^j} - i\, m_{k,0} \, x^k \, \mathcal{P}^0_\alpha \, e^{-i\, m_j\, x^j}.$$

Entretanto, o segundo termo acima apresenta problemas, visto que há uma coordenada espacial: isto é resolvido escolhendo-se:

$$m_{k,0} = 0 \Longrightarrow m_k \equiv c^{\underline{te}}.$$

para qualquer $k = 1, 2, 3$. Neste caso, vem então:

$$\hat{P}_{\alpha,0} = \left(\frac{1}{\mathcal{P}^0_\beta} \frac{d\mathcal{P}^0_\tau}{dt} \right) \hat{P}_\alpha. \tag{2.22}$$

Entretanto, calculando a derivada segunda espacial, obtemos:

$$\hat{P}_{\alpha,k,l} = \left[\frac{1}{\mathcal{P}^0_\tau} \frac{d^2\mathcal{P}^0_\tau}{dx^k\, dx^l} - i \frac{1}{\mathcal{P}^0_\tau} m_{(k} \mathcal{P}^0_{\alpha,l)} - m_k\, m_l \right] \hat{P}_\alpha,$$

onde o segundo termo é complexo. Para evitarmos problemas, tomamos então os $\mathcal{P}^0_\alpha$ como dependentes somente de t e ficamos portanto com os seguintes resultados:

$$\hat{P}_{\alpha,0} = -i\, m_k\, \hat{P}_\alpha, \tag{2.23}$$

$$\hat{P}_{\alpha,k,l} = -m_k\, m_l\, \hat{P}_\alpha. \tag{2.24}$$

Substituindo (2.24) acima em (2.19), obtemos então:

$$m^2 = -h^{\alpha\beta}\, m_\alpha\, m_\beta, \tag{2.25}$$

e podemos passar às propriedades de nossa base vetorial.

1ª) *Propriedade:*

A base vetorial é uma quantidade do tri-espaço, ou seja, não possui uma componente temporal:

$$V^\alpha\, \hat{P}_\alpha = 0, \tag{2.26}$$

e a condição de preservação desta propriedade no tempo é escrita como:

$$\left(V^\alpha\, \hat{P}_\alpha \right)^\bullet = 0 \Longrightarrow$$

$$\Longrightarrow \quad a^{\alpha}\,\hat{P}_{\alpha} + V^{\alpha}\,\left(\hat{P}_{\alpha}\right)^{\bullet} = 0 \Longrightarrow$$

$$\Longrightarrow \quad V^{\alpha}\,\hat{P}_{\alpha;\beta}\,V^{\beta} = 0 \Longrightarrow$$

$$\Longrightarrow \quad V^{\alpha}\,V^{\beta}\,\left[\hat{P}_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta}\,\hat{P}_{\gamma}\right] = 0 \Longrightarrow$$

$$\Longrightarrow \quad V^{\alpha}\,V^{\beta}\,\hat{P}_{\alpha,\beta} = 0 \Longrightarrow$$

$$\Longrightarrow \quad V^{\alpha}\,\hat{P}_{\alpha,0} = 0 \Longrightarrow$$

$$\Longrightarrow \quad V^{\alpha}\,\left(\frac{1}{\mathcal{P}^{0}_{\beta}}\,\frac{d\mathcal{P}^{0}_{\beta}}{dt}\right)\,\hat{P}_{\alpha} \equiv 0,$$

a qual é identicamente satisfeita.

2ª *Propriedade:*

Não é possível formarem-se quantidades escalares a partir de $\hat{P}_{\alpha}$, ou seja:

$$\hat{\nabla}^{\alpha}\,\hat{P}_{\alpha} = 0,$$

a qual, ao ser expandida, nos dá:

$$\begin{aligned}
\hat{\nabla}^{\alpha}\,\hat{P}_{\alpha} &= h^{\alpha\beta}\,\hat{\nabla}_{\beta}\,\hat{P}_{\alpha} \\
&= h^{\alpha\beta}\,h_{\beta}{}^{\gamma}\,h_{\alpha}{}^{\varepsilon}\,\left(h_{\varepsilon}{}^{\tau}\,\hat{P}_{\tau}\right)_{;\gamma}
\end{aligned}$$

$$
\begin{aligned}
&= h^{\varepsilon\gamma} \left(h_{\varepsilon}{}^{\tau} \hat{P}_{\tau} \right)_{;\gamma} \\
&= h^{\alpha\beta} \left[h_{\beta}{}^{\gamma} \hat{P}_{\gamma;\alpha} - \left(V_{\beta;\alpha} V^{\gamma} + V_{\beta} V^{\gamma}{}_{;\alpha} \right) \hat{P}_{\gamma} \right] \\
&= h^{\alpha\beta} \left[\hat{P}_{\gamma,\alpha} - \Gamma^{\beta}_{\gamma\alpha} \hat{P}_{\beta} \right] \\
&= h^{\alpha\gamma} \hat{P}_{\gamma,\alpha} \\
&= -i \, h^{\alpha\beta} m_{\alpha} \hat{P}_{\beta}.
\end{aligned}
$$

Daí, a segunda condição é dada por:

$$
h^{\alpha\beta} m_{\alpha} \hat{P}_{\beta} = 0, \tag{2.27}
$$

ou, reescrevendo,

$$
g^{11} m_1 \hat{P}_1 + g^{22} m_2 \hat{P}_2 + g^{33} m_3 \hat{P}_3 = 0. \tag{2.28}
$$

Isto nos dá:

$$
t^{-2p_1} m_1 \hat{P}_x + t^{-2p_2} m_2 \hat{P}_y + t^{-2p_3} m_3 \hat{P}_z = 0.
$$

E constatamos duas possibilidades:

(i) $p_1 \neq p_2 \neq p_3 \Longrightarrow$ Condição satisfeita se cada um dos três termos acima se anular separadamente, o que equivale a escolher um dos m_i como nulo; isto causaria problemas na definição da base, que não dependeria de uma das coordenadas espaciais.

(ii) *Plano de isotropia:* $p_1 = p_2$.

Tomando $\hat{P}_z = 0$, teríamos que:

$$t^{-2p_1}\left[m_1\,\hat{P}_x + m_2\,\hat{P}_y\right] = 0 \Longrightarrow$$

$$\Longrightarrow\quad \hat{P}_y = -\left(\frac{m_1}{m_2}\right)\,\hat{P}_x, \tag{2.29}$$

donde existe somente uma componente independente na base vetorial, no caso específico do plano de isotropia:

$$\hat{P}_\alpha = \left(\mathcal{P}^0_x\ ,\ -\left(\frac{m_1}{m_2}\right)\,\mathcal{P}^0_x\ ,\ 0\right)\,e^{-i\,m_j\,x^j}. \tag{2.30}$$

A condição de preservação da segunda propriedade no tempo é dada por:

$$\left(\hat{\nabla}^\alpha\,\hat{P}_\alpha\right) = 0 \Longrightarrow$$

$$\Longrightarrow\quad \left(\hat{\nabla}^\alpha\,\hat{P}_\alpha\right)_{;\beta}\,V^\beta = 0 \Longrightarrow$$

$$\Longrightarrow\quad \left(h^{\alpha\gamma}\,\hat{\nabla}_\gamma\,\hat{P}_\alpha\right)_{;\beta}\,V^\beta = 0 \Longrightarrow$$

$$\Longrightarrow\quad \left(-i\,h^{\alpha\gamma}\,m_\alpha\,\hat{P}_\gamma\right)_{;\beta}\,V^\beta = 0 \Longrightarrow$$

$$\Longrightarrow\quad \left(h^{\alpha\gamma}\,m_\alpha\,\hat{P}_\gamma\right)_{;\beta}\,V^\beta = 0 \Longrightarrow$$

$$\Longrightarrow\quad -\left(V^\alpha{}_{;\beta}\,V^\gamma + V^\alpha\,V^\gamma{}_{;\beta}\right)\,m_\alpha\,V^\beta\,\hat{P}_\gamma + h^{\alpha\gamma}\,m_{\alpha;\beta}\,V^\beta\,\hat{P}_\gamma +$$
$$+h^{\alpha\gamma}\,m_\alpha\,\hat{P}_{\gamma;\beta}\,V^\beta = 0 \Longrightarrow$$

$$\Longrightarrow \quad h^{\alpha\gamma}\left(m_{\alpha,\beta} - \Gamma^{\lambda}_{\alpha\beta}\, m_{\lambda}\right) V^{\beta}\, \hat{P}_{\gamma} + h^{\alpha\gamma}\, m_{\alpha}\left(\hat{P}_{\gamma}\right)^{\bullet} = 0 \Longrightarrow$$

$$\Longrightarrow \quad h^{\alpha\beta}\, m_{\alpha}\left(\hat{P}_{\beta}\right)^{\bullet} - h^{\alpha\beta}\, \Gamma^{\lambda}_{0\alpha}\, m_{\lambda}\, \hat{P}_{\beta} = 0 \Longrightarrow$$

$$\Longrightarrow \quad h^{\alpha\beta}\, m_{\alpha}\left(\hat{P}_{\beta}\right)^{\bullet} - \sigma^{\alpha\beta}\, m_{\alpha}\, \hat{P}_{\beta} - \frac{1}{3}\,\theta\, h^{\alpha\beta}\, m_{\alpha}\, \hat{P}_{\beta} = 0,$$

e, dado que o terceiro termo é nulo pela propriedade de divergência nula da base vetorial, vem:

$$h^{\alpha\beta}\, m_{\alpha}\left(\hat{P}_{\beta}\right)^{\bullet} - \sigma^{\alpha\beta}\, m_{\alpha}\, \hat{P}_{\beta} = 0.$$

Se calcularmos a derivada projetada no tempo de $\hat{P}_{\alpha}$, obtemos:

$$\begin{aligned}
\left(\hat{P}_{\alpha}\right)^{\bullet} &= \hat{P}_{\alpha;\beta}\, V^{\beta} \\
&= \left(\hat{P}_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta}\, \hat{P}_{\gamma}\right) V^{\beta} \\
&= \hat{P}_{\alpha,0} - \Gamma^{\beta}_{0\alpha}\, \hat{P}_{\beta} \\
&= \left[\left(\frac{1}{\mathcal{P}^{0}_{\tau}}\, \frac{d\mathcal{P}^{0}_{\tau}}{dt}\right) - \frac{1}{3}\,\theta\right] \hat{P}_{\alpha} - \sigma_{\alpha}{}^{\beta}\, \hat{P}_{\beta}.
\end{aligned} \tag{2.31}$$

E, então, a condição de preservação da segunda propriedade fica:

$$h^{\alpha\beta}\, m_{\alpha}\left[\left(\frac{1}{\mathcal{P}^{0}_{\tau}}\, \frac{d\mathcal{P}^{0}_{\tau}}{dt}\right) \hat{P}_{\beta} - \sigma_{\beta}{}^{\gamma}\, \hat{P}_{\gamma} - \frac{1}{3}\,\theta\, \hat{P}_{\beta}\right] - \sigma^{\alpha\beta}\, m_{\alpha}\, \hat{P}_{\beta} = 0 \Longrightarrow$$

$$\Longrightarrow \left(\frac{1}{\mathcal{P}^0_\tau}\frac{d\mathcal{P}^0_\tau}{dt}\right) h^{\alpha\beta}\, m_\alpha\, \hat{P}_\beta - 2\,\sigma^{\alpha\beta}\, m_\alpha\, \hat{P}_\beta - \frac{1}{3}\,\theta\, h^{\alpha\beta}\, m_\alpha\, \hat{P}_\beta = 0 \Longrightarrow$$

$$\Longrightarrow \sigma^{\alpha\beta}\, m_\alpha\, \hat{P}_\beta = 0 \Longrightarrow$$

$$\Longrightarrow \sigma^{11}\left[m_1\, \hat{P}_x + m_2\, \hat{P}_y\right] = 0 \Longrightarrow$$

$$\Longrightarrow m_1\, \hat{P}_x + m_2\, \hat{P}_y \equiv 0,$$

de (2.29). Logo, também é preservada a propriedade da divergência nula da base vetorial.

A partir do vetor $\left\{\hat{P}_\alpha\right\}$, podemos construir as seguintes quantidades:

(1)Tensor simétrico de segunda ordem e traço nulo:

$$\hat{P}_{\alpha\beta} \equiv \hat{\nabla}_{(\alpha}\, \hat{P}_{\beta)}, \tag{2.32}$$

a qual pode ser reescrita como:

$$\begin{aligned}
\hat{P}_{\alpha\beta} &= \hat{\nabla}_{(\alpha}\, \hat{P}_{\beta)} \\
&= h_{(\alpha}{}^{\gamma}\, h_{\beta)}{}^{\lambda}\left(h_\lambda{}^{\tau}\, \hat{P}_\tau\right)_{;\gamma} \\
&= h_{(\alpha}{}^{\gamma}\, h_{\beta)}{}^{\lambda}\left[-\left(V_{\lambda;\tau}\, V^{\tau} + V_\lambda\, V^{\tau}{}_{;\tau}\right)\, \hat{P}_\tau + h_\lambda{}^{\tau}\, \hat{P}_{\tau;\gamma}\right] \\
&= h_{(\alpha}{}^{\gamma}\, h_{\beta)}{}^{\tau}\, \hat{P}_{\tau;\gamma} \\
&= h_{(\alpha}{}^{\gamma}\, h_{\beta)}{}^{\lambda}\, \hat{P}_{\lambda;\gamma}
\end{aligned}$$

$$= \; -i\, h_{(\alpha}{}^{\gamma}\, h_{\beta)}{}^{\lambda}\, m_{\gamma}\, \hat{P}_{\tau}$$

$$= \; -i\, m_{(\alpha}\, \hat{P}_{\beta)}. \tag{2.33}$$

É fácil calcular as componentes de $\hat{P}_{\alpha\beta}$ para exibi-la na forma de uma matriz:

$$\hat{P}_{11} = -2\, i\, m_1\, \hat{P}_x,$$

$$\hat{P}_{22} = -2\, i\, m_2\, \hat{P}_y = 2\, i\, m_1\, \hat{P}_x,$$

$$\hat{P}_{33} = -2\, i\, m_3\, \hat{P}_z = 0,$$

$$\hat{P}_{12} = -i\, \left(m_1\, \hat{P}_x + m_2\, \hat{P}_y\right)$$

$$\hat{P}_{13} = -i\, \left(m_1\, \hat{P}_z + m_3\, \hat{P}_x\right) = -i\, m_3\, \hat{P}_x,$$

$$\hat{P}_{23} = -i\, \left(m_2\, \hat{P}_z + m_3\, \hat{P}_y\right) = i\, \tfrac{m_1 m_3}{m_2}\, \hat{P}_x,$$

$$\hat{P}_{\alpha\beta} = -i\, \hat{P}_x \begin{pmatrix} 2\, m_1 & 0 & m_3 \\ 0 & -2\, m_1 & -m_1\, m_2/m_3 \\ m_3 & -m_1\, m_3/m_2 & 0 \end{pmatrix}. \tag{2.34}$$

Duas quantidades envolvendo o tensor $\hat{P}_{\alpha\beta}$ serão importantes em cálculos futuros e, como tais, serão aqui obtidas. São elas: $\sigma_{(\alpha}{}^{\gamma}\, \hat{P}_{\beta)\gamma}$ e $\sigma^{\alpha\beta}\, \hat{P}_{\alpha\beta}$:

$$\sigma_{(\alpha}{}^{\gamma}\, \hat{P}_{\beta)\gamma} = \begin{cases} \left(\sigma_1{}^1 + \sigma_3{}^3\right)\, \hat{P}_{\alpha\beta}, & \text{para } \hat{P}_{13} \text{ e } \hat{P}_{23}, \\ \\ 2\, \sigma_1{}^1\, \hat{P}_{\alpha\beta}, & \text{para as outras componentes } \alpha,\, \beta. \end{cases} \tag{2.35}$$

$$
\begin{aligned}
\sigma^{\alpha\beta}\,\hat{P}_{\alpha\beta} &= \sigma^{11}\,\hat{P}_{11} + \sigma^{22}\,\hat{P}_{22} + \sigma^{33}\,\hat{P}_{33} \\
&= -2\,i\,\left[\sigma^{11}\,m_1\,\hat{P}_x + \sigma^{22}\,m_2\,\hat{P}_y\right] \\
&= -2\,i\,\sigma^{11}\,\hat{P}_x\left[m_1 + m_2\left(-\frac{m_1}{m_2}\right)\right] \\
&= 0. \qquad (2.36)
\end{aligned}
$$

A derivada projetada no tempo de $\hat{P}_{\alpha\beta}$ é dada por (usando a relação (2.33)):

$$
\begin{aligned}
\left(\hat{P}_{\alpha\beta}\right)^{\bullet} &= \left(\hat{\nabla}_{(\alpha}\,\hat{P}_{\beta)}\right)^{\bullet} \\
&= -i\,\left(m_{(\alpha}\,\hat{P}_{\beta)}\right)^{\bullet} \\
&= -i\,\left[\left(m_{(\alpha}\right)^{\bullet} + m_{(\alpha}\,\left(\hat{P}_{\beta)}\right)^{\bullet}\right] \\
&= -i\,\left\{\left[m_{(\alpha,\gamma} - \Gamma^{\lambda}_{\gamma(\alpha}\,m_{\lambda}\right]\,V^{\gamma}\,\hat{P}_{\beta)} + m_{(\alpha}\,\left(\frac{1}{\mathcal{P}^0_\tau}\,\frac{d\mathcal{P}^0_\tau}{dt}\right)\,\hat{P}_{\beta)} - \right. \\
&\quad \left. -m_{(\alpha}\,\sigma_{\beta)}{}^{\gamma}\,\hat{P}_{\gamma} - \frac{1}{3}\,\theta\,m_{(\alpha}\,\hat{P}_{\beta)}\right\} \\
&= -i\,\left\{-\left(\sigma_{(\alpha}{}^{\lambda} + \frac{1}{3}\,\theta\,h_{(\alpha}{}^{\lambda}\right)\,m_{\lambda}\,\hat{P}_{\beta)} - m_{(\alpha}\,\sigma_{\beta)}{}^{\gamma}\,\hat{P}_{\gamma} + \right. \\
&\quad \left. + \left(\frac{1}{\mathcal{P}^0_\tau}\,\frac{d\mathcal{P}^0_\tau}{dt}\right)\,m_{(\alpha}\,\hat{P}_{\beta)} - \frac{1}{3}\,\theta\,m_{(\alpha}\,\hat{P}_{\beta)}\right\} \\
&= -i\,\left\{-m_{\gamma}\,\sigma_{(\alpha}{}^{\gamma}\,\hat{P}_{\beta)} - \frac{2}{3}\,\theta\,m_{(\alpha}\,\hat{P}_{\beta)} - \right. \\
&\quad \left. -m_{(\alpha}\,\sigma_{(\beta}{}^{\gamma}\,\hat{P}_{\gamma} + \left(\frac{1}{\mathcal{P}^0_\tau}\,\frac{d\mathcal{P}^0_\tau}{dt}\right)\,m_{(\alpha}\,\hat{P}_{\beta)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= -\sigma^{\gamma}{}_{(\alpha}\left(-i\, m_{\gamma}\, \hat{P}_{\beta)}\right) - \frac{2}{3}\,\theta\left(-i\, m_{(\alpha}\, \hat{P}_{\beta)}\right) - \\
&\quad -\sigma^{\gamma}{}_{(\beta}\left(-i\, m_{\alpha)}\, \hat{P}_{\gamma}\right) + \left(\frac{1}{\mathcal{P}^{0}_{\tau}}\,\frac{d\mathcal{P}^{0}_{\tau}}{dt}\right)\left(-i\, m_{(\alpha}\, \hat{P}_{\beta)}\right) \\
&= \left[\left(\frac{1}{\mathcal{P}^{0}_{\tau}}\,\frac{d\mathcal{P}^{0}_{\tau}}{dt}\right) - \frac{2}{3}\,\theta\right]\hat{P}_{\alpha\beta} - \sigma^{\gamma}{}_{\alpha}\left(-i\, m_{\gamma}\, \hat{P}_{\beta}\right) - \\
&\quad -\sigma^{\gamma}{}_{\beta}\left(-i\, m_{\gamma}\, \hat{P}_{\alpha}\right) - \sigma^{\gamma}{}_{\beta}\left(-i\, m_{\alpha}\, \hat{P}_{\gamma}\right) - \sigma^{\gamma}{}_{\alpha}\left(-i\, m_{\beta}\, \hat{P}_{\gamma}\right) \\
&= \left[\left(\frac{1}{\mathcal{P}^{0}_{\tau}}\,\frac{d\mathcal{P}^{0}_{\tau}}{dt}\right) - \frac{2}{3}\,\theta\right]\hat{P}_{\alpha\beta} - \sigma^{\gamma}{}_{\alpha}\,\hat{P}_{\beta\gamma} - \sigma^{\gamma}{}_{\beta}\,\hat{P}_{\alpha\gamma} \\
&= \left[\left(\frac{1}{\mathcal{P}^{0}_{\tau}}\,\frac{d\mathcal{P}^{0}_{\tau}}{dt}\right) - \frac{2}{3}\,\theta\right]\hat{P}_{\alpha\beta} - \sigma^{\gamma}{}_{(\alpha}\,\hat{P}_{\beta)\gamma}. \qquad (2.37)
\end{aligned}
$$

E o Laplaciano de $\hat{P}_{\alpha\beta}$ é também facilmente calculado como:

$$
\begin{aligned}
\hat{\nabla}^{2}\,\hat{P}_{\alpha\beta} &= \hat{\nabla}^{2}\,\hat{\nabla}_{(\beta}\,\hat{P}_{\alpha)} \\
&= \hat{\nabla}^{\gamma}\,\hat{\nabla}_{\gamma}\,\hat{\nabla}_{(\beta}\,\hat{P}_{\alpha)} \\
&= \hat{\nabla}^{\gamma}\left[\hat{\nabla}_{(\beta}\,\hat{\nabla}_{\gamma}\,\hat{P}_{\alpha)} + \hat{R}^{\lambda}{}_{(\alpha\gamma\beta)}\,\hat{P}_{\lambda}\right] \\
&= \hat{\nabla}_{(\beta}\left(\hat{\nabla}^{2}\,\hat{P}_{\alpha)}\right) + \hat{R}^{\lambda}{}_{\gamma}{}^{\gamma}{}_{(\beta}\,\hat{\nabla}_{\lambda}\,\hat{P}_{\alpha)} + \hat{R}^{\lambda}{}_{(\alpha}{}^{\gamma}{}_{\beta)}\,\hat{\nabla}_{\gamma}\,\hat{P}_{\lambda} \\
&= \hat{\nabla}_{(\beta}\left(m^{2}\,\hat{P}_{\alpha)}\right) \\
&= m^{2}\,\hat{\nabla}_{(\beta}\,\hat{P}_{\alpha)} + \left(\hat{\nabla}_{(\beta}\,m^{2}\right)\,\hat{P}_{\alpha)}
\end{aligned}
$$

$$= \quad m^2 \hat{P}_{\alpha\beta}. \tag{2.38}$$

(2)Pseudovet...

É definido como:

$$\hat{P}^*_\alpha \equiv \eta_\alpha{}^{\beta\mu\nu} V_\beta \left(\hat{\nabla}_\nu \hat{P}_\mu\right), \tag{2.39}$$

e suas componentes podem ser calculadas diretamente como:

$$\begin{aligned}
\hat{P}^*_1 &= \eta_1{}^{023} \left(\hat{\nabla}_3 \hat{P}_y\right) + \eta_1{}^{032} \left(\hat{\nabla}_2 \hat{P}_z\right) \\
&= -i\, \eta_1{}^{023}\, m_3\, \hat{P}_y \\
&= -i\, g_{11}\, m_3\, \eta^{1023}\, \hat{P}_y \\
&= -i\, \eta^{0123}\, g_{11}\, \frac{m_1\, m_3}{m_2}\, \hat{P}_x,
\end{aligned}$$

$$\begin{aligned}
\hat{P}^*_2 &= \eta_2{}^{013} \left(\hat{\nabla}_3 \hat{P}_x\right) + \eta_2{}^{031} \left(\hat{\nabla}_1 \hat{P}_3\right) \\
&= -i\, g_{22}\, \eta^{2013}\, m_3\, \hat{P}_x \\
&= -i\, \eta^{0123}\, g_{22}\, m_3\, \hat{P}_x,
\end{aligned}$$

$$\hat{P}^*_3 = \eta_3{}^{012} \left(\hat{\nabla}_2 \hat{P}_x\right) + \eta_3{}^{021} \left(\hat{\nabla}_1 \hat{P}_y\right)$$

$$= -i\, g_{33} \left[\eta^{3012}\, m_2\, \hat{P}_x + \eta^{3021}\, m_1\, \hat{P}_y\right]$$

$$= -i\, g_{33}\, \eta^{0123} \left(m_1\, \hat{P}_y - m_2\, \hat{P}_x\right)$$

$$= -i\, g_{33}\, \eta^{0123}\, \frac{1}{m_2} \left[(m_1)^2 + (m_2)^2\right] \hat{P}_x,$$

e daí temos então (escrevendo o pseudovetor na forma de uma matriz-linha):

$$\hat{P}^*_\alpha = \begin{pmatrix} g_{11}\,\frac{m_1\, m_3}{m_2} & g_{22}\, m_3 & -\frac{g_{33}}{m_2}\left[(m_1)^2 + (m_2)^2\right] \end{pmatrix}. \tag{2.40}$$

Entretanto, as mesmas propriedades que se aplicam a $\hat{P}_\alpha$ devem também se aplicar a $\hat{P}^*_\alpha$, já que ambos serão usados para construir-se quantidades perturbadas. Da definição do pseudovetor, equação (2.39), vemos que a primeira propriedade é obviamente satisfeita e o pseudovetor é — a exemplo do vetor — uma quantidade pertencente à tri-superfície. A segunda propriedade, da divergência nula, é escrita como:

$$\hat{\nabla}^\alpha\, \hat{P}^*_\alpha = 0 \Longrightarrow$$

$$\Longrightarrow\ h^{\alpha\beta}\, \hat{P}^*_{\beta,\alpha} = 0 \Longrightarrow$$

$$\Longrightarrow\ g^{11}\, \frac{d\hat{P}^*_x}{dx} + g^{22}\, \frac{d\hat{P}^*_y}{dy} + g^{33}\, \frac{d\hat{P}^*_z}{dz} = 0 \Longrightarrow$$

$$\Longrightarrow\ \frac{m_1\, m_3}{m_2}\, \frac{d\hat{P}_x}{dx} + m_3\, \frac{d\hat{P}_x}{dy} - \frac{1}{m_2}\left[(m_1)^2 + (m_2)^2\right] \frac{d\hat{P}_x}{dz} = 0 \Longrightarrow$$

$$\Longrightarrow\ -i \left\{(m_1)^2\, \frac{m_3}{m_2} + m_2\, m_3 - \frac{1}{m_2}\left[(m_1)^2 + (m_2)^2\right]\right\} \hat{P}_x = 0 \Longrightarrow$$

$$\Longrightarrow \quad \left[(m_1)^2 + (m_2)^2\right](m_3 - 1) = 0,$$

o que nos dá duas condições possíveis para que a divergência do pseudovetor seja, de fato, nula:

$$m_3 = 1$$

$$(m_1)^2 + (m_2)^2 = 0,$$

Escolher o segundo resultado implica em

$$\hat{P}^*_z = 0. \tag{2.41}$$

Logo, é ele que escolheremos, obtendo então:

$$m_2 = \pm i\, m_1. \tag{2.42}$$

Pode-se provar por um cálculo direto que ambas as propriedades do pseudovetor são preservadas:

$$\left(V^\alpha \hat{P}^*_\alpha\right) = 0 \Longrightarrow$$

$$\Longrightarrow \quad a^\alpha \hat{P}^*_\alpha + V^\alpha \left(\hat{P}^*_\alpha\right)^\bullet = 0 \Longrightarrow$$

$$\Longrightarrow \quad V^\alpha \left(\hat{P}^*_\alpha\right)^\bullet = 0 \Longrightarrow$$

$$\Longrightarrow \quad V^\alpha V^\beta \hat{P}^*_{\alpha;\beta} = 0 \Longrightarrow$$

$$\Longrightarrow \quad V^{\alpha} V^{\beta} \left(\hat{P}^{*}_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta} \hat{P}^{*}_{\gamma} \right) = 0 \Longrightarrow$$

$$\Longrightarrow \quad V^{\alpha} V^{\beta} \hat{P}^{*}_{\alpha,\beta} = 0 \Longrightarrow$$

$$\Longrightarrow \quad V^{\alpha} V^{\beta} \left(\eta_{\alpha}{}^{\gamma\lambda\tau} V_{\gamma} \hat{\nabla}_{\tau} \hat{P}_{\lambda} \right)_{,\beta} = 0 \Longrightarrow$$

$$\Longrightarrow \quad \eta^{\alpha\gamma\lambda\tau} V_{\alpha} V_{\gamma} V^{\beta} \left(\hat{\nabla}_{\tau} \hat{P}_{\lambda} \right)_{,\beta} \equiv 0$$

$$\left(\hat{\nabla}^{\alpha} \hat{P}^{*}_{\alpha} \right)^{\bullet} = 0 \Longrightarrow$$

$$\Longrightarrow \quad \left(\hat{\nabla}^{\alpha} \hat{P}^{*}_{\alpha} \right)_{;\beta} V^{\beta} = 0 \Longrightarrow$$

$$\Longrightarrow \quad \left(\hat{\nabla}^{\alpha} \hat{P}^{*}_{\alpha} \right)_{,\beta} V^{\beta} \equiv 0$$

A derivada projetada no tempo do pseudovetor é:

$$\begin{aligned}
\left(\hat{P}^{*}_{\alpha} \right)^{\bullet} &= \hat{P}^{*}_{\alpha;\beta} V^{\beta} \\
&= \left(\hat{P}^{*}_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta} \hat{P}^{*}_{\gamma} \right) V^{\beta} \\
&= \hat{P}^{*}_{\alpha.0} - \Gamma^{\beta}_{0\alpha} \hat{P}^{*}_{\beta} \\
&= \hat{P}^{*}_{\alpha.0} - \left(\sigma_{\alpha}{}^{\beta} + \frac{1}{3} \theta \, h_{\alpha}{}^{\beta} \right) \hat{P}^{*}_{\beta}
\end{aligned}$$

$$
\begin{aligned}
&= \hat{P}^*_{\alpha,0} - \frac{1}{3}\,\theta\,\hat{P}^*_\alpha - \sigma_\alpha{}^\beta\,\hat{P}^*_\beta \\
&= \left(\eta_\alpha{}^{\beta\mu\nu}\,V_\beta\,\hat{\nabla}_\nu\,\hat{P}_\mu\right)_{,0} - \frac{1}{3}\,\theta\,\hat{P}^*_\alpha - \sigma_\alpha{}^\beta\,\hat{P}^*_\beta \\
&= \eta_\alpha{}^{\beta\mu\nu}\,V_\beta\,\left(\hat{\nabla}_\nu\,\hat{P}_\mu\right)_{,0} - \frac{1}{3}\,\theta\,\hat{P}^*_\alpha - \sigma_\alpha{}^\beta\,\hat{P}^*_\beta.
\end{aligned}
$$

O primeiro termo pode ser calculado como:

$$
\begin{aligned}
\eta_\alpha{}^{\beta\mu\nu}\,V_\beta\,\left(\hat{\nabla}_\nu\,\hat{P}_\mu\right)_{,0} &= \eta_\alpha{}^{\beta\mu\nu}\,V_\beta\,\left[h_\nu{}^\gamma\,h_\mu{}^\lambda\,\left(h_\lambda{}^\tau\,\hat{P}_\tau\right)_{;\gamma}\right]_{,0} \\
&= \eta_\alpha{}^{\beta\mu\nu}\,V_\beta\,h_\nu{}^\gamma\,h_\mu{}^\lambda\,h_\lambda{}^\tau\,\left(\hat{P}_{\tau;\gamma}\right)_{,0} \\
&= \eta_\alpha{}^{\beta\mu\nu}\,V_\beta\,h_\nu{}^\gamma\,h_\mu{}^\lambda\,\left[\hat{P}_{\lambda,\gamma} - \Gamma^\tau_{\lambda\gamma}\,\hat{P}_\tau\right]_{,0} \\
&= \eta_\alpha{}^{\beta\mu\nu}\,V_\beta\,h_\nu{}^\gamma\,h_\mu{}^\lambda\,\left(-i\,m_\gamma\,\hat{P}_\lambda\right)_{,0} \\
&= -i\,\eta_\alpha{}^{\beta\mu\nu}\,V_\beta\,m_\nu\,h_\mu{}^\lambda\,\hat{P}_{\lambda,0} \\
&= -i\left(\frac{1}{\mathcal{P}^0_\tau}\,\frac{d\mathcal{P}^0_\tau}{dt}\right)\eta_\alpha{}^{\beta\mu\nu}\,V_\beta\,m_\nu\,\hat{P}_\mu \\
&= \left(\frac{1}{\mathcal{P}^0_\tau}\,\frac{d\mathcal{P}^0_\tau}{dt}\right)\eta_\alpha{}^{\beta\mu\nu}\,V_\beta\,\hat{\nabla}_\nu\,\hat{P}_\mu \\
&= \left(\frac{1}{\mathcal{P}^0_\tau}\,\frac{d\mathcal{P}^0_\tau}{dt}\right)\hat{P}^*_\alpha,
\end{aligned}
$$

donde vem então o seguinte resultado:

$$\left(\hat{P}^*_\alpha\right)^\bullet = \left[\left(\frac{1}{\mathcal{P}^0_\tau}\frac{d\mathcal{P}^0_\tau}{dt}\right) - \frac{1}{3}\theta\right]\hat{P}^*_\alpha - \sigma_\alpha{}^\beta\,\hat{P}^*_\beta. \tag{2.43}$$

Finalmente, o Laplaciano do pseudovetor é escrito como:

$$\begin{aligned}
\hat{\nabla}^2\,\hat{P}^*_\beta &= \hat{\nabla}^2\left(\eta_\beta{}^{\gamma\mu\nu}\,V_\gamma\,\hat{\nabla}_\nu\,\hat{P}_\mu\right)\\
&= \eta_\beta{}^{\gamma\mu\nu}\,V_\gamma\,\hat{\nabla}^\alpha\left(\hat{\nabla}_\alpha\,\hat{\nabla}_\nu\,\hat{P}_\mu\right)\\
&= \eta_\beta{}^{\gamma\mu\nu}\,V_\gamma\,\hat{\nabla}^\alpha\left[\hat{\nabla}_\nu\,\hat{\nabla}_\alpha\,\hat{P}_\mu + \hat{R}^\lambda{}_{\mu\alpha\nu}\,\hat{P}_\lambda\right]\\
&= \eta_\beta{}^{\gamma\mu\nu}\,V_\gamma\,\hat{\nabla}^\alpha\,\hat{\nabla}_\nu\left(\hat{\nabla}_\alpha\,\hat{P}_\mu\right)\\
&= \eta_\beta{}^{\gamma\mu\nu}\,V_\gamma\left[\hat{\nabla}_\nu\,\hat{\nabla}^\alpha\left(\hat{\nabla}_\alpha\,\hat{P}_\mu\right) + \hat{R}^\lambda{}_\alpha{}^\alpha{}_\nu\left(\hat{\nabla}_\lambda\,\hat{P}_\mu\right) + \hat{R}^\lambda{}_\mu{}^\alpha{}_\nu\left(\hat{\nabla}_\alpha\,\hat{P}_\lambda\right)\right]\\
&= \eta_\beta{}^{\gamma\mu\nu}\,V_\gamma\,\hat{\nabla}_\nu\left[\hat{\nabla}^2\,\hat{P}_\mu\right]\\
&= \eta_\beta{}^{\gamma\mu\nu}\,V_\gamma\,\hat{\nabla}_\nu\left(m^2\,\hat{P}_\mu\right)\\
&= m^2\left(\eta_\beta{}^{\gamma\mu\nu}\,V_\gamma\,\hat{\nabla}_\nu\,\hat{P}_\mu\right) + \eta_\beta{}^{\gamma\mu\nu}\,V_\gamma\left(\hat{\nabla}_\nu\,m^2\right)\hat{P}_\mu\\
&= m^2\,\hat{P}^*_\beta. \qquad (2.44)
\end{aligned}$$

(3)Pseudotensor simétrico de traço nulo:

Esta quantidade é definida analogamente ao tensor simétrico $\hat{P}_{\alpha\beta}$:

$$\hat{P}^{*}_{\alpha\beta} \equiv \hat{\nabla}_{(\alpha}\, \hat{P}^{*}_{\beta)}, \tag{2.45}$$

a qual pode ser reescrita como:

$$
\begin{aligned}
\hat{P}^{*}_{\alpha\beta} &= h_{(\alpha}{}^{\gamma}\, h_{\beta)}{}^{\lambda}\, \left(h_{\gamma}{}^{\tau}\, \hat{P}^{*}_{\tau}\right)_{;\lambda} \\
&= h_{(\alpha}{}^{\gamma}\, h_{\beta)}{}^{\lambda}\, \left[-\left(V_{\gamma;\lambda}\, V^{\tau} + V_{\gamma}\, V^{\tau}{}_{;\lambda}\right)\, \hat{P}^{*}_{\tau} + h_{\gamma}{}^{\tau}\, \hat{P}^{*}_{\tau;\lambda}\right] \\
&= h_{(\alpha}{}^{\gamma}\, h_{\beta)}{}^{\lambda}\, \hat{P}^{*}_{\gamma;\lambda} \\
&= h_{(\alpha}{}^{\gamma}\, h_{\beta)}{}^{\lambda}\, \left[\hat{P}^{*}_{\gamma,\lambda} - \Gamma^{\tau}_{\gamma\lambda}\, \hat{P}^{*}_{\tau}\right] \\
&= h_{(\alpha}{}^{\gamma}\, h_{\beta)}{}^{\lambda}\, \hat{P}^{*}_{\gamma,\lambda} \\
&= h_{(\alpha}{}^{\gamma}\, h_{\beta)}{}^{\lambda}\, \left(\eta_{\gamma}{}^{\tau\mu\nu}\, V_{\tau}\, \hat{\nabla}_{\nu}\, \hat{P}_{\mu}\right)_{,\lambda} \\
&= \eta_{\gamma}{}^{\tau\mu\nu}\, V_{\tau}\, h_{(\alpha}{}^{\gamma}\, h_{\beta)}{}^{\lambda}\, \left(\hat{\nabla}_{\nu}\, \hat{P}_{\mu}\right)_{,\lambda} \\
&= \left[\eta_{(\alpha}{}^{\tau\mu\nu} - \eta_{\gamma}{}^{\tau\mu\nu}\, V_{(\alpha}\, V^{\gamma}\right]\, V_{\tau}\, h_{\beta)}{}^{\lambda}\, \left(\hat{\nabla}_{\nu}\, \hat{P}_{\mu}\right)_{,\lambda} \\
&= \eta_{(\alpha}{}^{\gamma\mu\nu}\, V_{\gamma}\, h_{\beta)}{}^{\lambda}\, \left(\hat{\nabla}_{\nu}\, \hat{P}_{\mu}\right)_{,\lambda} \\
&= \eta_{(\alpha}{}^{\gamma\mu\nu}\, V_{\gamma}\, h_{\beta)}{}^{\lambda}\, \left[h_{\nu}{}^{\tau}\, h_{\mu}{}^{\varepsilon}\, \left(h_{\varepsilon}{}^{\lambda}\, \hat{P}_{\lambda}\right)_{;\tau}\right]_{,\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&= \eta_{(\alpha}{}^{\gamma\mu\nu} V_\gamma h_{\beta)}{}^\lambda \left[\left(h_\mu{}^\tau \hat{P}_\tau\right)_{;\nu}\right]_{,\lambda} \\
&= \eta_{(\alpha}{}^{\gamma\mu\nu} V_\gamma h_{\beta)}{}^\lambda \left[-\left(V_{\mu;\nu} V^\tau + V_\mu V^\tau{}_{;\lambda}\right) \hat{P}_\tau + h_\mu{}^\tau \hat{P}_{\tau;\nu}\right]_{,\lambda} \\
&= \eta_{(\alpha}{}^{\gamma\mu\nu} V_\gamma h_{\beta)}{}^\lambda \left\{-\left(\sigma_{\mu\nu} + \frac{1}{3}\theta\, h_{\mu\nu}\right) V^\tau \hat{P}_\tau - \right. \\
&\quad \left. -\left(\sigma_\lambda{}^\tau + \frac{1}{3}\theta\, h_\lambda{}^\tau\right) V_\mu \hat{P}_\tau + h_\mu{}^\tau \left[\hat{P}_{\tau,\nu} - \Gamma^\varepsilon_{\tau\nu} \hat{P}_\varepsilon\right]_{,\lambda}\right\} \\
&= \eta_{(\alpha}{}^{\gamma\mu\nu} V_\gamma h_{\beta)}{}^\lambda \left\{-\left(\sigma_\lambda{}^\tau + \frac{1}{3}\theta\, h_\lambda{}^\tau\right) V_\mu \hat{P}_{\tau,\lambda} + h_\mu{}^\tau \hat{P}_{\tau,\nu,\lambda}\right\} \\
&= \eta_{(\alpha}{}^{\gamma\mu\nu} V_\gamma h_{\beta)}{}^\lambda \left(-m_\nu m_\lambda \hat{P}_\mu\right) \\
&= -\eta_{(\alpha}{}^{\gamma\mu\nu} V_\gamma h_{\beta)}{}^\lambda m_\nu m_\lambda \hat{P}_\mu \\
&= -\eta_{(\alpha}{}^{\gamma\mu\nu} V_\gamma m_{\beta)} m_\nu \hat{P}_\mu \\
&= -i\, \eta_{(\alpha}{}^{\gamma\mu\nu} V_\gamma m_{\beta)} \left(-i\, m_\nu \hat{P}_\mu\right) \\
&= -i\, m_{(\beta} \left(\eta_{\alpha)}{}^{\gamma\mu\nu} V_\gamma \hat{\nabla}_\nu \hat{P}_\mu\right) \\
&= -i\, m_{(\alpha} \hat{P}^*_{\beta)}. \qquad (2.46)
\end{aligned}
$$

O traço de $\hat{P}^*_{\alpha\beta}$ é obviamente nulo, por (2.45). A sua divergência pode ser calculada como:

$$
\hat{\nabla}^\alpha \hat{P}^*_{\alpha\beta} = h^{\alpha\gamma} \hat{\nabla}_\gamma \hat{P}^*_{\alpha\beta}
$$

$$
\begin{aligned}
&= h^{\alpha\gamma}\,\hat{\nabla}_\gamma\,\hat{\nabla}_{(\beta}\,\hat{P}^*_{\alpha)} \\
&= h^{\alpha\gamma}\,\hat{\nabla}_\gamma\left[\hat{\nabla}_\beta\,\hat{P}^*_\alpha + \hat{\nabla}_\alpha\,\hat{P}^*_\beta\right] \\
&= \hat{\nabla}^2\,\hat{P}^*_\beta + h^{\alpha\gamma}\left[\hat{\nabla}_\gamma\,\hat{\nabla}_\beta\,\hat{P}^*_\alpha\right] \\
&= \hat{\nabla}^2\,\hat{P}^*_\beta + h^{\alpha\gamma}\left[\hat{\nabla}_\beta\,\hat{\nabla}_\gamma\,\hat{P}^*_\alpha + \hat{R}^\lambda{}_{\alpha\gamma\beta}\,\hat{P}^*_\lambda\right] \\
&= \hat{\nabla}^2\,\hat{P}^*_\beta + h^{\alpha\gamma}\,\hat{\nabla}_\beta\,\hat{\nabla}_\gamma\,\hat{P}^*_\alpha \\
&= \hat{\nabla}^2\,\hat{P}^*_\beta + \hat{\nabla}_\beta\left(h^{\alpha\gamma}\,\hat{\nabla}_\gamma\,\hat{P}^*_\alpha\right) - \left(\hat{\nabla}_\beta\,h^{\alpha\gamma}\right)\hat{\nabla}_\gamma\,\hat{P}^*_\alpha \\
&= \hat{\nabla}^2\,\hat{P}^*_\beta + \hat{\nabla}_\beta\left(\hat{\nabla}^\alpha\,\hat{P}^*_\alpha\right) \\
&= \hat{\nabla}^2\,\hat{P}^*_\beta \\
&= m^2\,\hat{P}^*_\beta, \qquad (2.47)
\end{aligned}
$$

pela equação (2.44).

O Laplaciano de $\hat{P}^*_{\alpha\beta}$ é calculado de modo exatamente idêntico à relação (2.38), dando imediatamente:

$$
\hat{\nabla}^2\,\hat{P}^*_{\alpha\beta} = m^2\,\hat{P}^*_{\alpha\beta}. \qquad (2.48)
$$

A derivada projetada no tempo do pseudotensor simétrico é dada por (o cálculo é, novamente, idêntico ao da derivada projetada no tempo de $\hat{P}_{\alpha\beta}$):

$$\left(\hat{P}^{*}_{\alpha\beta}\right)^{\bullet}=\left[\left(\frac{1}{\mathcal{P}^{0}_{\tau}}\,\frac{d\mathcal{P}^{0}_{\tau}}{dt}\right)-\frac{2}{3}\,\theta\right]\,\hat{P}^{*}_{\alpha\beta}-\sigma_{(\alpha}{}^{\gamma}\,\hat{P}^{*}_{\beta)\gamma}. \tag{2.49}$$

Finalmente, encerraremos esta seção apresentando explicitamente cada uma das componentes do pseudotensor simétrico $\hat{P}^{*}_{\alpha\beta}$ e calcularemos as quantidades $\left(\sigma^{\alpha\beta}\,\hat{P}^{*}_{\alpha\beta}\right)$ e $\left(\sigma_{(\alpha}{}^{\gamma}\,\hat{P}^{*}_{\beta)\gamma}\right)$:

$$\begin{aligned}\hat{P}^{*}_{11} &= 2\,\eta_{1}{}^{023}\left(m_{1}\,m_{2}\,\hat{P}_{z}-m_{1}\,m_{3}\,\hat{P}_{y}\right)\\ &= -2\,\eta^{0123}\,g_{11}\,\frac{1}{m_{2}}\,(m_{1})^{2}\,m_{3}\,\hat{P}_{x},\end{aligned}$$

$$\begin{aligned}\hat{P}^{*}_{22} &= 2\,\eta_{2}{}^{013}\left(m_{2}\,m_{1}\,\hat{P}_{z}-m_{2}\,m_{3}\,\hat{P}_{x}\right)\\ &= -2\,\eta^{0123}\,g_{11}\,m_{2}\,m_{3}\,\hat{P}_{x},\end{aligned}$$

$$\begin{aligned}\hat{P}^{*}_{33} &= 2\,\eta_{3}{}^{012}\left(m_{3}\,m_{1}\,\hat{P}_{y}-m_{3}\,m_{2}\,\hat{P}_{x}\right)\\ &= 2\,\eta^{0123}\,g_{33}\,m_{3}\,\frac{m_{3}}{m_{2}}\,\left[(m_{1})^{2}+(m_{2})^{2}\right]\,\hat{P}_{x}\\ &= 0,\end{aligned}$$

$$\begin{aligned}\hat{P}^{*}_{12} &= \eta_{1}{}^{023}\,m_{2}\,\left(m_{2}\,\hat{P}_{z}-m_{3}\,\hat{P}_{y}\right)+\\ &\quad+\eta_{2}{}^{013}\,m_{1}\,\left(m_{1}\,\hat{P}_{z}-m_{3}\,\hat{P}_{x}\right)\\ &= -2\,\eta^{0123}\,g_{11}\,m_{1}\,m_{3}\,\hat{P}_{x},\end{aligned}$$

$$\begin{aligned}
\hat{P}^*_{13} &= \eta_1{}^{023}\, m_3\, \left(m_2\, \hat{P}_z - m_3\, \hat{P}_2\right) + \\
&\quad + \eta_3{}^{012}\, m_1\, \left(m_1\, \hat{P}_y - m_2\, \hat{P}_x\right) \\
&= -\eta^{0123}\, g_{11}\, \frac{1}{m_2}\, m_1\, (m_3)^2\, \hat{P}_x,
\end{aligned}$$

$$\begin{aligned}
\hat{P}^*_{23} &= \eta_2{}^{013}\, m_3\, \left(m_1\, \hat{P}_z - m_3\, \hat{P}_x\right) + \\
&\quad + \eta_3{}^{012}\, m_2\, \left(m_1\, \hat{P}_y - m_2\, \hat{P}_x\right) \\
&= -\eta^{0123}\, g_{11}\, (m_3)^2\, \hat{P}_x.
\end{aligned}$$

E, portanto, podemos escrever o pseudotensor na forma matricial como:

$$\hat{P}^*_{\alpha\beta} = -\eta^{0123}\, \hat{P}_x \begin{pmatrix} 2\, g_{11}\, \frac{(m_1)^2 m_3}{m_2} & 2\, g_{11}\, m_1\, m_3 & g_{11}\, \frac{m_1 (m_3)^2}{m_2} \\ 2\, g_{11}\, m_1\, m_3 & 2\, g_{11}\, m_2\, m_3 & g_{11}\, (m_3)^2 \\ g_{11}\, \frac{m_1 (m_3)^2}{m_2} & g_{11}\, (m_3)^2 & 0 \end{pmatrix}. \tag{2.50}$$

As quantidades $\left(\sigma^{\alpha\beta}\, \hat{P}^*_{\alpha\beta}\right)$ e $\left(\sigma_{(\alpha}{}^{\gamma}\, \hat{P}^*_{\beta)\gamma}\right)$ são obtidas de maneira análoga às suas contrapartes (2.35) e (2.36):

$$\begin{aligned}
\left(\sigma^{\alpha\beta}\, \hat{P}^*_{\alpha\beta}\right) &= \sigma^{11}\, \left(\hat{P}^*_{11} + \hat{P}^*_{22}\right) \\
&= 2\, \sigma^{11}\, \frac{m_3}{m_2}\, \left[(m_1)^2 + (m_2)^2\right] \\
&= 0,
\end{aligned} \tag{2.51}$$

de (2.42).

$$\left(\sigma_{(\alpha}{}^{\gamma}\,\hat{P}^{*}_{\beta)\gamma}\right) = \begin{cases} (\sigma_1{}^1 + \sigma_3{}^3)\;\hat{P}^{*}_{\alpha\beta}, & \text{para } \hat{P}^{*}_{13} \text{ e } \hat{P}^{*}_{23}. \\ \\ 2\,\sigma_1{}^1\,\hat{P}^{*}_{\alpha\beta}, & \text{para as demais componentes.} \end{cases} \tag{2.52}$$

Como uma observação final, vamos simplificar os nossos cálculos tomando a função $\mathcal{P}^0_\tau$ como sendo uma **constante**. Esta escolha não influencia a generalidade dos cálculos relativos ao sistema dinâmico perturbado, que será apresentado no Capítulo 6 deste trabalho. Escrevemos então:

$$\left(\frac{1}{\mathcal{P}^0_\tau}\,\frac{d\mathcal{P}^0_\tau}{dt}\right) \equiv 0. \tag{2.53}$$

2.4 Construção de uma Base Tensorial

Como primeiro passo, definimos um tensor misto $\hat{U}^{\mu}{}_{\nu}(t, x, y, z)$, escrito na forma matricial, com componentes:

$$\hat{U}^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha & \psi & \varphi \\ 0 & \eta & \beta & \varepsilon \\ 0 & \chi & \zeta & \gamma \end{pmatrix}, \tag{2.54}$$

onde $(\alpha,\ \psi,\ \varphi,\ \zeta,\ \beta,\ \varepsilon,\ \chi,\ \eta,\ \gamma)$ são funções do tempo e das coordenadas espaciais, a princípio. O símbolo ($\hat{}$) denota novamente o fato de que os tensores $\hat{U}$ são projetados sobre a 3-superfície. A escolha de um tensor misto foi feita com vistas a uma simplificação adicional nos cálculos que se seguem, dado que, neste caso, podemos tomar a derivada simples de $\hat{U}^{\mu}{}_{\nu}$ em relação ao tempo como nula. Conforme mencionado anteriormente e, visto que os $\hat{U}^{\mu}{}_{\nu}$ são por definição quantidades pertencentes à 3-superfície, os índices são levantados e abaixados com o projetor $h_{\alpha\beta}$:

$$\hat{U}^{\mu}{}_{\nu} = h^{\mu\alpha}\,\hat{U}_{\alpha\nu}, \tag{2.55}$$

com $\hat{U}_{\alpha\nu}$ sendo um tensor simétrico: $\hat{U}_{\alpha\nu} = \hat{U}_{\nu\alpha}$.

Se o tensor $\hat{U}^{\mu}{}_{\nu}$ constitui, de fato, uma base, ele deve satisfazer à equação de onda:

$$\hat{\nabla}^2 \hat{U}^{\mu}{}_{\nu} = k^2 \hat{U}^{\mu}{}_{\nu}, \tag{2.56}$$

onde k^2 é uma função de t. Fazendo uso dos resultados do Apêndice A (equação (A.12)), a equação acima é dada por:

$$\left(g^{11} \frac{\partial^2}{\partial x^2} + g^{22} \frac{\partial^2}{\partial y^2} + g^{33} \frac{\partial^2}{\partial z^2}\right) \hat{U}^{\mu}{}_{\nu} + k^2 \hat{U}^{\mu}{}_{\nu} = 0, \tag{2.57}$$

a qual deve ser satisfeita para todas as componentes de $\hat{U}^{\mu}{}_{\nu}$. Temos, então, o seguinte sistema de equações diferenciais parciais de segunda ordem:

$$g^{11} \frac{\partial^2 f}{\partial x^2} + g^{22} \frac{\partial^2 f}{\partial y^2} + g^{33} \frac{\partial^2 f}{\partial z^2} - k^2 f = 0,$$

onde $f(t, x, y, z) \equiv \{\alpha, \psi, \varphi, \eta, \beta, \varepsilon, \chi, \zeta, \gamma\}$.

O sistema de equações acima é facilmente integrável; uma possível solução para o tensor $\hat{U}^{\mu}{}_{\nu}$ é escrita como:

$$\hat{U}^{\mu}{}_{\nu} = \mathcal{U}^{\mu}{}_{\nu}\, e^{-i k_j x^j}, \tag{2.58}$$

onde $\mathcal{U}^{\mu}{}_{\nu}$ são tensores que, a princípio, dependem apenas de t e os k_j são constantes arbitrárias relacionadas à função de onda k^2 e ao tensor métrico por[1]:

[1] Da equação (2.60) é fácil constatar-se que, no caso mais geral em que os k_j são também funções do tempo, teríamos:

$$\hat{U}^{\mu}{}_{\nu,0} = \mathcal{U}^{\mu}{}_{\nu,0}\, e^{-i k_j x^j} - i\, k_{j,0}\, x^j\, \hat{U}^{\mu}{}_{\nu},$$

o que introduziria um termo adicional, proporcional a uma coordenada espacial e que não poderia ser tratado, a não ser tomando-se

$$k_{j,0} = 0 \Longrightarrow k_j = cte,$$

conforme havíamos escolhido anteriormente.

$$k^2 = -h^{jl}\,k_j\,k_l = \left[\left(\frac{k_1}{A(t)}\right)^2 + \left(\frac{k_2}{B(t)}\right)^2 + \left(\frac{k_3}{C(t)}\right)^2\right]. \tag{2.59}$$

Da equação (2.58) obtemos as seguintes derivadas parciais simples:

$$\hat{U}^{\mu}{}_{\nu,0} = \mathcal{U}^{\mu}{}_{\nu,0}\, e^{-i\,k_j\,x^j}, \tag{2.60}$$

$$\hat{U}^{\mu}{}_{\nu,l} = -i\,k_l\,\hat{U}^{\mu}{}_{\nu}, \tag{2.61}$$

as quais serão úteis nos cálculos que se seguem.

Para que o tensor $\hat{U}^{\mu}{}_{\nu}$ seja, de fato, uma base tensorial adequada, ele deve satisfazer às propriedades enunciadas anteriormente em [13, 17, 25], e que passamos a listar aqui:

Propriedade 1 — O tensor $\hat{U}^{\mu}{}_{\nu}$ é um objeto da 3-geometria, isto é, ele é ortogonal a V_α:

$$V_\mu\,\hat{U}^{\mu}{}_{\nu} = 0. \tag{2.62}$$

Note-se, da equação (2.54), que as componentes de $\hat{U}^{\mu}{}_{\nu}$ já foram escolhidas tendo-se esta propriedade em mente.

Propriedade 2 — Não é possível formar-se quantidades escalares a partir de $\hat{U}^{\mu}{}_{\nu}$. Em outras palavras, o traço deste tensor deve ser nulo:

$$h^{\mu\nu}\,\hat{U}_{\mu\nu} = \hat{U}^{\mu}{}_{\mu} = 0. \tag{2.63}$$

Ou, reescrevendo nas componentes de $\hat{U}^{\mu}{}_{\nu}$,

$$\alpha + \beta + \gamma = 0. \tag{2.64}$$

Propriedade 3 — Não é possível formar-se quantidades vetoriais a partir de $\hat{U}^{\mu}{}_{\nu}$. Ou seja, a sua divergência deve ser nula:

$$\hat{\nabla}^{\mu}\,\hat{U}_{\mu\nu} = 0. \tag{2.65}$$

Usando a definição (A.11) do Apêndice A, escrevemos a condição acima como

$$\begin{aligned}
\hat{\nabla}^{\mu}\,\hat{U}_{\mu\nu} &= h^{\gamma\mu}\,\hat{\nabla}_{\gamma}\,\hat{U}_{\mu\nu} \\
&= h^{\gamma\mu}\,h^{\alpha}{}_{\gamma}\,h^{\beta}{}_{\mu}\,h^{\lambda}{}_{\nu}\,\left(h^{\varepsilon}{}_{\beta}\,h^{\tau}{}_{\lambda}\,\hat{U}_{\varepsilon\tau}\right)_{;\alpha} \\
&= h^{\alpha\beta}\,h^{\lambda}{}_{\nu}\,\left(h^{\varepsilon}{}_{\beta}\,h^{\tau}{}_{\lambda}\,\hat{U}_{\varepsilon\tau}\right)_{;\alpha} \\
&= h^{\alpha\beta}\,h^{\lambda}{}_{\nu}\,\Big[h^{\varepsilon}{}_{\beta}\,h^{\tau}{}_{\lambda}\,\hat{U}_{\varepsilon\tau;\alpha} + h^{\varepsilon}{}_{\beta}\,\left(h^{\tau}{}_{\lambda}\right)_{;\alpha}\,\hat{U}_{\varepsilon\tau} + \\
&\quad + \left(h^{\varepsilon}{}_{\beta}\right)_{;\alpha}\,h^{\tau}{}_{\lambda}\,\hat{U}_{\varepsilon\tau}\Big] \\
&= h^{\alpha\varepsilon}\,h^{\tau}{}_{\nu}\,\hat{U}_{\varepsilon\tau;\alpha} + h^{\alpha\varepsilon}\,h^{\lambda}{}_{\nu}\,\left(h^{\tau}{}_{\lambda}\right)_{;\alpha}\,\hat{U}_{\varepsilon\tau} + \\
&\quad + \left(h^{\varepsilon}{}_{\beta}\right)_{;\alpha}\,h^{\alpha\beta}\,h^{\tau}{}_{\nu}\,\hat{U}_{\varepsilon\tau} \\
&= h^{\alpha\varepsilon}\,h^{\tau}{}_{\nu}\,\Big[\hat{U}_{\varepsilon\tau,\alpha} - \Gamma^{\lambda}_{\varepsilon\alpha}\,\hat{U}_{\lambda\tau} - \Gamma^{\lambda}_{\tau\alpha}\,\hat{U}_{\varepsilon\lambda}\Big] + \\
&\quad + h^{\lambda}{}_{\nu}\,\left(h^{\tau}{}_{\lambda}\right)_{;\alpha}\,\hat{U}^{\alpha}{}_{\tau} + \left(h^{\varepsilon}{}_{\beta}\right)_{;\alpha}\,h^{\alpha\beta}\,\hat{U}_{\varepsilon\nu}
\end{aligned}$$

Entretanto, temos que o segundo e o terceiro termos acima são nulos:

$$h^{\alpha\varepsilon}\,h^{\tau}{}_{\nu}\,\Gamma^{\lambda}_{\varepsilon\alpha}\,\hat{U}_{\lambda\tau} \equiv h^{\alpha\varepsilon}\,h^{\tau}{}_{\nu}\,\Gamma^{\lambda}_{\tau\alpha}\,\hat{U}_{\varepsilon\lambda} = 0,$$

já que todos os índices das conxões são espaciais e temos $\Gamma^{i}_{jk} = 0\ \ \forall\ \ i, j, k$, no *background* de Bianchi-I. Além disto, o quarto e o quinto termos acima se escrevem como:

$$
\begin{aligned}
h^{\lambda}{}_{\nu}\,(h^{\tau}{}_{\lambda})_{;\alpha}\,\hat{U}^{\alpha}{}_{\tau} &= h^{\lambda}{}_{\nu}\left[h^{\tau}{}_{\lambda,\alpha} + \Gamma^{\tau}_{\beta\alpha}\,h^{\beta}{}_{\lambda} - \Gamma^{\beta}_{\lambda\alpha}\,h^{\tau}{}_{\beta}\right]\hat{U}^{\alpha}{}_{\tau} \\
&= h^{\lambda}{}_{\nu}\,(\delta^{\tau}{}_{\lambda,\alpha} - V^{\tau}{}_{,\alpha}\,V_{\lambda} - V^{\tau}\,V_{\lambda,\alpha})\,\hat{U}^{\alpha}{}_{\tau} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
(h^{\varepsilon}{}_{\beta})_{;\alpha}\,h^{\alpha\beta}\,\hat{U}_{\varepsilon\nu} &= \left[h^{\varepsilon}{}_{\beta,\alpha} + \Gamma^{\varepsilon}_{\lambda\alpha}\,h^{\lambda}{}_{\beta} - \Gamma^{\lambda}_{\beta\alpha}\,h^{\varepsilon}{}_{\lambda}\right]\,h^{\alpha\beta}\,\hat{U}_{\varepsilon\nu} \\
&= (\delta^{\varepsilon}{}_{\beta,\alpha} - V^{\varepsilon}{}_{,\alpha}\,V_{\beta} - V^{\varepsilon}\,V_{\beta,\alpha})\,h^{\alpha\beta}\,\hat{U}_{\varepsilon\nu} \\
&= 0,
\end{aligned}
$$

onde, novamente, usamos o fato de que conexões com todos os três índices espaciais são nulas no *background* de Bianchi-I, além do fato de que tanto a base quanto o projetor são ortogonais à velocidade. Neste caso, a propriedade da divergência nula se reduz a:

$$h^{\alpha\varepsilon}\,h^{\tau}{}_{\nu}\,\hat{U}_{\varepsilon\tau,\alpha} = 0$$

$$h^{\alpha\varepsilon}\,h^{\tau}{}_{\nu}\left(-i\,k_{\alpha}\,\hat{U}_{\varepsilon\tau}\right) = 0$$

$$-i\,k_{\alpha}\,\hat{U}^{\alpha}{}_{\nu} = 0$$

$$k_\alpha\, \hat{U}^\alpha{}_\nu = 0, \tag{2.66}$$

onde usamos as relações (2.55) e (2.61) para obter:

$$\begin{aligned}
\hat{U}_{\mu\nu,l} &= \left(h_{\mu\beta}\, \hat{U}^\beta{}_\nu\right)_{,l} \\
&= \left(h_{\mu\beta}\right)_{,l} \hat{U}^\beta{}_\nu + h_{\mu\beta}\, \hat{U}^\beta{}_{\nu,l} \\
&= \left(g_{\mu\beta,l} - V_{\mu,l}\, V_\beta - V_\mu\, V_{\beta,l}\right) \hat{U}^\beta{}_\nu - i\, h_{\mu\beta}\, k_l\, \hat{U}^\beta{}_\nu \\
&= -i\, k_l\, \hat{U}_{\mu\nu}.
\end{aligned} \tag{2.67}$$

Reescrevendo a condição de divergência nula, equação (2.66), em termos das componentes de $\hat{U}^\mu{}_\nu$, vem:

$$k_1\, \alpha + k_2\, \eta + k_3\, \chi = 0, \tag{2.68}$$

$$k_1\, \psi + k_2\, \beta + k_3\, \zeta = 0, \tag{2.69}$$

$$k_1\, \varphi + k_2\, \varepsilon + k_3\, \gamma = 0. \tag{2.70}$$

É necessário, ainda, que as três propriedades anteriormente enunciadas sejam preservadas ao longo do tempo. Para isto, vamos calcular as derivadas projetadas no tempo das quantidades exibidas nas Propriedades enunciadas acima e impor que elas sejam nulas.

(1) Preservação da Ortogonalidade:

$$\left(\hat{U}^{\mu}{}_{\nu}\, V^{\nu}\right)^{\bullet} = 0 \Longrightarrow$$

$$\Longrightarrow \quad \hat{U}^{\mu}{}_{\nu;\alpha}\, V^{\alpha}\, V^{\nu} + \hat{U}^{\mu}{}_{\nu}\, a^{\nu} = 0 \Longrightarrow$$

$$\Longrightarrow \quad \left(\hat{U}^{\mu}{}_{\nu,\alpha} + \Gamma^{\mu}_{\beta\alpha}\, \hat{U}^{\beta}{}_{\nu} - \Gamma^{\beta}_{\nu\alpha}\, \hat{U}^{\mu}{}_{\beta}\right) V^{\alpha}\, V^{\nu} = 0 \Longrightarrow$$

$$\Longrightarrow \quad \hat{U}^{\mu}{}_{\nu,\alpha}\, V^{\alpha}\, V^{\nu} = 0 \Longrightarrow$$

$$\Longrightarrow \quad \hat{U}^{\mu}{}_{\nu,0}\, V^{\nu} = 0 \Longrightarrow$$

$$\Longrightarrow \quad \mathcal{U}^{\mu}{}_{\nu,0}\, V^{\nu}\, e^{-i k_j x^j} = 0 \Longrightarrow$$

$$\Longrightarrow \quad \mathcal{U}^{\mu}{}_{\nu,0} = 0. \tag{2.71}$$

Ou seja, a ortogonalidade é preservada se o tensor $\mathcal{U}^{\mu}{}_{\nu}$ não depender da coordenada t.

(2) Preservação do Traço Nulo:

$$\left(\hat{U}^{\mu}{}_{\mu}\right)^{\bullet} = 0 \Longrightarrow$$

$$\Longrightarrow \quad \hat{U}^{\mu}{}_{\mu;\alpha}\, V^{\alpha} = 0 \Longrightarrow$$

$$\Longrightarrow \quad \left(\hat{U}^{\mu}{}_{\mu,\alpha} + \Gamma^{\mu}_{\beta\alpha}\, \hat{U}^{\beta}{}_{\mu} - \Gamma^{\beta}_{\mu\alpha}\, \hat{U}^{\mu}{}_{\beta}\right) V^{\alpha} = 0 \Longrightarrow$$

$$\Longrightarrow \quad \Gamma^{\mu}_{\beta 0}\,\hat{U}^{\beta}{}_{\mu} - \Gamma^{\beta}_{\mu 0}\,\hat{U}^{\mu}{}_{\beta} \equiv 0.$$

E, portanto, verifica-se que a propriedade de traço nulo é imediatamente preservada.

(3) Preservação da Divergência Nula:

$$\left(k_{\mu}\,\hat{U}^{\mu}{}_{\nu}\right)^{\bullet} = 0 \Longrightarrow$$

$$\Longrightarrow \quad k_{\mu;\alpha}\,\hat{U}^{\mu}{}_{\nu}\,V^{\alpha} + k_{\mu}\,\hat{U}^{\mu}{}_{\nu;\alpha}\,V^{\alpha} = 0 \Longrightarrow$$

$$\Longrightarrow \quad \left(k_{\mu,\alpha} - \Gamma^{\beta}_{\mu\alpha}\,k_{\beta}\right)\,\hat{U}^{\mu}{}_{\nu}\,V^{\alpha} + k_{\mu}\,\left(\hat{U}^{\mu}{}_{\nu,\alpha} + \Gamma^{\mu}_{\beta\alpha}\,\hat{U}^{\beta}{}_{\nu} - \Gamma^{\beta}_{\nu\alpha}\,\hat{U}^{\mu}{}_{\beta}\right)\,V^{\alpha} = 0 \Longrightarrow$$

$$\Longrightarrow \quad -k_{\beta}\,\Gamma^{\beta}_{\mu 0}\,\hat{U}^{\mu}{}_{\nu} + k_{\mu}\,\hat{U}^{\mu}{}_{\nu,0} + k_{\mu}\,\Gamma^{\mu}_{\beta 0}\,\hat{U}^{\beta}{}_{\nu} - \Gamma^{\beta}_{\nu 0}\,k_{\mu}\,\hat{U}^{\mu}{}_{\beta} = 0 \Longrightarrow$$

$$\Longrightarrow \quad k_{\mu}\,\hat{U}^{\mu}{}_{\nu,0} = 0 \Longrightarrow$$

$$\Longrightarrow \quad k_{\mu}\,\mathcal{U}^{\mu}{}_{\nu,0}\,e^{-i\,k_j\,x^j} = 0 \Longrightarrow$$

$$\Longrightarrow \quad \mathcal{U}^{\mu}{}_{\nu,0} = 0.$$

E daí, novamente, a condição de divergência nula é preservada se a relação (2.71) for válida.

A derivada projetada no tempo da base tensorial $\hat{U}^{\mu}{}_{\nu}$ é obtida como:

$$\left(\hat{U}^{\mu}{}_{\nu}\right)^{\bullet} \;=\; \hat{U}^{\mu}{}_{\nu;\alpha}\,V^{\alpha}$$

$$
\begin{aligned}
&= \hat{U}^{\mu}{}_{\nu,0} + \Gamma^{\mu}_{0\alpha}\,\hat{U}^{\alpha}{}_{\nu} - \Gamma^{\alpha}_{0\nu}\,\hat{U}^{\mu}{}_{\alpha} \\
&= \mathcal{U}^{\mu}{}_{\nu,0}\,e^{-i\,k_j\,x^j} + \Gamma^{\mu}_{0\alpha}\,\hat{U}^{\alpha}{}_{\nu} - \Gamma^{\alpha}_{0\nu}\,\hat{U}^{\mu}{}_{\alpha} \\
&= \left(\sigma^{\mu}{}_{\alpha} + \frac{\theta}{3}\,h^{\mu}{}_{\alpha}\right)\hat{U}^{\alpha}{}_{\nu} - \left(\sigma^{\alpha}{}_{\nu} + \frac{\theta}{3}\,h^{\alpha}{}_{\nu}\right)\hat{U}^{\mu}{}_{\alpha} \\
&= \sigma^{\mu}{}_{\alpha}\,\hat{U}^{\alpha}{}_{\nu} + \frac{\theta}{3}\,\hat{U}^{\mu}{}_{\nu} - \sigma^{\alpha}{}_{\nu}\,\hat{U}^{\mu}{}_{\alpha} - \frac{\theta}{3}\,\hat{U}^{\mu}{}_{\nu} \\
&= \sigma^{\mu}{}_{\alpha}\,\hat{U}^{\alpha}{}_{\nu} - \sigma^{\alpha}{}_{\nu}\,\hat{U}^{\mu}{}_{\alpha},
\end{aligned}
\tag{2.72}
$$

onde usamos as relações (1.22) e (2.71).

Para completarmos a definição de nossa base tensorial, é necessário que sejamos capazes de escrever quantidades pseudo-tensoriais. Para tanto, definimos o **dual** $\hat{U}^{*\mu}{}_{\nu}$, a exemplo do que foi feito anteriormente, para o caso de Friedmann-Robertson- Walker [13, 17, 24, 25] como:

$$
\hat{U}^{*}_{\mu\nu} = \frac{1}{2}\,h^{\alpha}{}_{(\mu}\,h^{\beta}{}_{\nu)}\,\eta_{\beta}{}^{\lambda\varepsilon\gamma}\,V_{\lambda}\,\left(\hat{\nabla}_{\varepsilon}\,\hat{U}_{\gamma\alpha}\right). \tag{2.73}
$$

Calculando $\hat{U}^{*\mu}{}_{\nu}$, obtemos:

$$
\begin{aligned}
\hat{U}^{*\mu}{}_{\nu} &= h^{\mu\alpha}\,\hat{U}^{*}_{\alpha\nu} \\
&= \frac{1}{2}\,h^{\mu\alpha}\,h^{\tau}{}_{(\alpha}\,h^{\beta}{}_{\nu)}\,\eta_{\beta}{}^{\lambda\varepsilon\gamma}\,V_{\lambda}\,\left(\hat{\nabla}_{\varepsilon}\,\hat{U}_{\gamma\tau}\right) \\
&= \frac{1}{2}\,\left(h^{\mu\alpha}\,h^{\beta}{}_{\nu} + h^{\alpha}{}_{\nu}\,h^{\beta\mu}\right)\,\eta_{\beta}{}^{\lambda\varepsilon\gamma}\,V_{\lambda}\,\left(\hat{\nabla}_{\varepsilon}\,\hat{U}_{\gamma\alpha}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\left[h^{\mu\alpha}\left(\eta_{\nu}{}^{\lambda\varepsilon\gamma}\,V_{\lambda} - \eta_{\beta}{}^{\lambda\varepsilon\gamma}\,V_{\lambda}\,V^{\beta}\,V_{\nu}\right) + \right.\\
&\left. + h^{\alpha}{}_{\nu}\left(\eta^{\mu\lambda\varepsilon\gamma}\,V_{\lambda} - \eta_{\beta}{}^{\lambda\varepsilon\gamma}\,V_{\lambda}\,V^{\beta}\,V^{\mu}\right)\right]\left(\hat{\nabla}_{\varepsilon}\,\hat{U}_{\gamma\alpha}\right)
\end{aligned}
$$

$$
= \frac{1}{2}\left[h^{\mu\alpha}\,\eta_{\nu}{}^{\lambda\varepsilon\gamma} + h^{\alpha}{}_{\nu}\,\eta^{\mu\lambda\varepsilon\gamma}\right] V_{\lambda}\left(\hat{\nabla}_{\varepsilon}\,\hat{U}_{\alpha\gamma}\right)
$$

$$
= \frac{1}{2}\left[\eta^{\mu\lambda\varepsilon\gamma}\,V_{\lambda}\left(\hat{\nabla}_{\varepsilon}\,\hat{U}_{\gamma\nu}\right) + \eta_{\nu}{}^{\lambda\varepsilon\gamma}\,V_{\lambda}\left(\hat{\nabla}_{\varepsilon}\,\hat{U}^{\mu}{}_{\gamma}\right)\right],
$$

já que $\hat{\nabla}_{\varepsilon}\,h^{\alpha\beta} = 0,\ \forall\ \alpha,\beta,\varepsilon$ (conforme provado no Apêndice A, equação (A.8)). Temos, então, que:

$$
\begin{aligned}
\hat{U}^{*\mu}{}_{\nu} &= \frac{1}{2}\left[\eta^{\mu\lambda\varepsilon}{}_{\alpha}\,V_{\lambda}\,h^{\gamma\alpha}\left(\hat{\nabla}_{\varepsilon}\,\hat{U}_{\gamma\nu}\right) + \right.\\
&\left. + \eta_{\nu}{}^{\lambda\varepsilon\gamma}\,V_{\lambda}\left(\hat{\nabla}_{\varepsilon}\,\hat{U}^{\mu}{}_{\gamma}\right)\right]\\
&= \frac{1}{2}\left[\eta^{\mu\lambda\varepsilon}{}_{\alpha}\,V_{\lambda}\left(\hat{\nabla}_{\varepsilon}\,\hat{U}^{\alpha}{}_{\nu}\right) + \right.\\
&\left. + \eta_{\nu}{}^{\lambda\varepsilon\gamma}\,V_{\lambda}\left(\hat{\nabla}_{\varepsilon}\,\hat{U}^{\mu}{}_{\gamma}\right)\right]\\
&= \frac{1}{2}\left[-\eta_{\gamma}{}^{\lambda\varepsilon\mu}\,V_{\lambda}\left(\hat{\nabla}_{\varepsilon}\,\hat{U}^{\gamma}{}_{\nu}\right) + \right.\\
&\left. + \eta_{\nu}{}^{\lambda\varepsilon\gamma}\,V_{\lambda}\left(\hat{\nabla}_{\varepsilon}\,\hat{U}^{\mu}{}_{\gamma}\right)\right] \qquad (2.74)
\end{aligned}
$$

Entretanto, temos que:

$$
\begin{aligned}
\left(\hat{\nabla}_{\varepsilon}\,\hat{U}^{\gamma}{}_{\nu}\right) &= h^{\alpha}{}_{\varepsilon}\,h^{\gamma}{}_{\tau}\,h^{\rho}{}_{\nu}\left(h^{\tau}{}_{\omega}\,h^{\varphi}{}_{\rho}\,\hat{U}^{\omega}{}_{\varphi}\right)_{;\alpha}\\
&= h^{\alpha}{}_{\varepsilon}\,h^{\gamma}{}_{\tau}\,h^{\rho}{}_{\nu}\left[h^{\tau}{}_{\omega}\,h^{\varphi}{}_{\rho}\,\hat{U}^{\omega}{}_{\varphi;\alpha} - \left(V^{\tau}{}_{;\alpha}\,V_{\omega} + V^{\tau}\,V_{\omega;\alpha}\right)h^{\varphi}{}_{\rho}\,\hat{U}^{\omega}{}_{\varphi} - \right.\\
&\left. - \left(V^{\varphi}{}_{;\alpha}\,V_{\rho} + V^{\varphi}\,V_{\rho;\alpha}\right)h^{\tau}{}_{\omega}\,\hat{U}^{\omega}{}_{\varphi}\right]
\end{aligned}
$$

$$= h^{\alpha}{}_{\varepsilon}\, h^{\gamma}{}_{\omega}\, h^{\varphi}{}_{\nu} \left[\hat{U}^{\omega}{}_{\varphi,\alpha} + \Gamma^{\omega}_{\tau\alpha}\, \hat{U}^{\tau}{}_{\varphi} - \Gamma^{\tau}_{\varphi\alpha}\, \hat{U}^{\omega}{}_{\tau}\right]$$

$$= h^{\alpha}{}_{\varepsilon}\, h^{\gamma}{}_{\omega}\, h^{\varphi}{}_{\nu}\, \hat{U}^{\omega}{}_{\varphi,\alpha},$$

já que os termos de conexão se anulam no *background* de Bianchi-I. Utilizando-se, neste ponto, a relação (2.61), obtém-se:

$$\begin{aligned}\left(\hat{\nabla}_{\varepsilon}\, \hat{U}^{\gamma}{}_{\nu}\right) &= -i\, k_{\varepsilon}\, h^{\gamma}{}_{\omega}\, h^{\varphi}{}_{\nu}\, \hat{U}^{\omega}{}_{\varphi} \\ &= -i\, k_{\varepsilon}\, \hat{U}^{\gamma}{}_{\nu}. \end{aligned} \tag{2.75}$$

Substituindo-se a relação (2.75) em (2.74), vem finalmente que:

$$\begin{aligned}\hat{U}^{*\mu}{}_{\nu} &= \frac{1}{2}\left[-\eta_{\gamma}{}^{\lambda\varepsilon\mu}\, V_{\lambda}\left(-i\, k_{\varepsilon}\, \hat{U}^{\gamma}{}_{\nu}\right) - \eta_{\nu}{}^{\lambda\varepsilon\gamma}\, V_{\lambda}\left(-i\, k_{\varepsilon}\, \hat{U}^{\mu}{}_{\gamma}\right)\right] \\ &= -\frac{i}{2}\, k_{\varepsilon}\left[\eta_{\gamma}{}^{\lambda\varepsilon\mu}\, V_{\lambda}\, \hat{U}^{\gamma}{}_{\nu} - \eta_{\nu}{}^{\lambda\varepsilon\gamma}\, V_{\lambda}\, \hat{U}^{\mu}{}_{\gamma}\right] \\ &= -\frac{i}{2}\, k_{\varepsilon}\left(\eta_{\gamma}{}^{\lambda\varepsilon\mu}\, V_{\lambda}\, \hat{U}^{\gamma}{}_{\nu} + \eta^{\gamma\lambda\varepsilon}{}_{\nu}\, V_{\lambda}\, \hat{U}^{\mu}{}_{\gamma}\right). \end{aligned} \tag{2.76}$$

Vamos, agora, escrever as propriedades do dual. A exemplo do tensor $\hat{U}^{\mu}{}_{\nu}$ original, ele também possui as propriedades de ortogonalidade, traço e divergência nulas. É fácil provar ainda que tais propriedades são preservadas no tempo, mediante o cálculo de suas derivadas projetadas em t, conforme veremos a seguir:

1) Ortogonalidade:

$$V_\mu \, \hat{U}^{*\mu}{}_\nu \;=\; -\frac{i}{2} k_\varepsilon \, V_\mu \, \left(\eta_\gamma{}^{\lambda\varepsilon\mu} \, V_\lambda \, \hat{U}^\gamma{}_\nu + \eta^{\gamma\lambda\varepsilon}{}_\nu \, \hat{U}^\mu{}_\gamma \right)$$

$$\equiv \; 0 \tag{2.77}$$

$$\begin{aligned}
\left(V_\mu \, \hat{U}^{*\mu}{}_\nu \right)^\bullet &= a_\mu \, \hat{U}^{*\mu}{}_\nu + V_\mu \, \left(\hat{U}^{*\mu}{}_\nu \right)^\bullet \\
&= V_\mu \, \hat{U}^{*\mu}{}_{\nu;\alpha} \, V^\alpha \\
&= \left(\hat{U}^{*\mu}{}_{\nu,\alpha} + \Gamma^\mu_{\beta\alpha} \, \hat{U}^{*\beta}{}_\nu - \Gamma^\beta_{\nu\alpha} \, \hat{U}^{*\mu}{}_\beta \right) V^\alpha \, V_\mu \\
&= \hat{U}^{*\mu}{}_{\nu,0} \, V_\mu + \Gamma^0_{\beta 0} \, \hat{U}^{*\beta}{}_\nu - \Gamma^\beta_{\nu 0} \, \hat{U}^{*\mu}{}_\beta \, V_\mu \\
&= \hat{U}^{*\mu}{}_{\nu,0} \, V_\mu \\
&= -\frac{i}{2} k_\varepsilon \, V_\mu \, e^{-i k_j x^j} \, \left(\eta_\gamma{}^{\lambda\varepsilon\mu} \, V_\lambda \, \mathcal{U}^\gamma{}_{\nu,0} + \eta^{\gamma\lambda\varepsilon}{}_\nu \, V_\lambda \, \mathcal{U}^\mu{}_{\gamma,0} \right) \\
&= 0,
\end{aligned}$$

e a propriedade é, de fato, preservada.

2) Traço Nulo:

$$
\begin{aligned}
\hat{U}^{*\mu}{}_{\mu} &= -\frac{i}{2}\, k_{\varepsilon} \left(\eta_{\gamma}{}^{\lambda\varepsilon\mu}\, V_{\lambda}\, \hat{U}^{\gamma}{}_{\nu} + \eta^{\gamma\lambda\varepsilon}{}_{\mu}\, V_{\lambda}\, \hat{U}^{\mu}{}_{\gamma} \right) \\
&= -\frac{i}{2}\, k_{\varepsilon} \left(\eta^{\gamma\lambda\varepsilon\mu}\, V_{\lambda}\, \hat{U}_{\gamma\mu} + \eta^{\gamma\lambda\varepsilon\mu}\, V_{\lambda}\, \hat{U}_{\gamma\mu} \right) \\
&= -i\, k_{\varepsilon}\, \eta^{\gamma\lambda\varepsilon\mu}\, V_{\lambda} \hat{U}_{\gamma\mu} \\
&= -i\, k_{\varepsilon}\, \eta^{\gamma\mu\lambda\varepsilon}\, V_{\lambda}\, \hat{U}_{\gamma\mu} \\
&= 0.
\end{aligned}
$$

E, verificando a condição de preservação da propriedade no tempo, obtemos:

$$
\begin{aligned}
\left(\hat{U}^{*\mu}{}_{\mu} \right)^{\bullet} &= \hat{U}^{*\mu}{}_{\mu;\alpha}\, V^{\alpha} \\
&= \hat{U}^{*\mu}{}_{\mu,0} + \Gamma^{\mu}_{\nu\alpha}\, \hat{U}^{\nu}{}_{\mu}\, V^{\alpha} - \Gamma^{\nu}_{\mu\alpha}\, \hat{U}^{\mu}{}_{\nu}\, V^{\alpha} \\
&= \Gamma^{\mu}_{\nu 0}\, \hat{U}^{\nu}{}_{\mu} - \Gamma^{\nu}_{\mu 0}\, \hat{U}^{\mu}{}_{\nu} \\
&= 0,
\end{aligned}
$$

usando-se a propriedade (2.71).

3) Divergência Nula:

$$\begin{aligned}
\hat{\nabla}^{\mu}\, \hat{U}^{*}_{\mu\nu} &= \hat{\nabla}^{\mu} \left(h_{\mu\alpha}\, \hat{U}^{*\alpha}{}_{\nu}\right) \\
&= \left(\hat{\nabla}^{\mu}\, h_{\mu\alpha}\right) \hat{U}^{*\alpha}{}_{\nu} + h_{\mu\alpha} \left(\hat{\nabla}^{\mu}\, \hat{U}^{*\alpha}{}_{\nu}\right) \\
&= \hat{\nabla}_{\alpha}\, \hat{U}^{*\alpha}{}_{\nu},
\end{aligned}$$

de (A.8). Usando, agora, os resultados (1.22) e (2.76), vem que:

$$\begin{aligned}
\hat{\nabla}_{\alpha}\, \hat{U}^{*\alpha}{}_{\nu} &= h^{\beta}{}_{\alpha}\, h^{\alpha}{}_{\gamma}\, h^{\mu}{}_{\nu} \left(h^{\gamma}{}_{\varepsilon}\, h^{\tau}{}_{\mu}\, \hat{U}^{*\varepsilon}{}_{\tau}\right)_{;\beta} \\
&= h^{\beta}{}_{\gamma}\, h^{\mu}{}_{\nu} \left(h^{\gamma}{}_{\varepsilon}\, h^{\tau}{}_{\mu}\, \hat{U}^{*\varepsilon}{}_{\tau}\right)_{;\beta} \\
&= h^{\beta}{}_{\gamma}\, h^{\mu}{}_{\nu} \Big[h^{\gamma}{}_{\varepsilon}\, h^{\tau}{}_{\mu}\, \hat{U}^{*\varepsilon}{}_{\tau;\beta} + \\
&\quad + (h^{\gamma}{}_{\varepsilon})_{;\beta}\; h^{\tau}{}_{\mu}\, \hat{U}^{*\varepsilon}{}_{\tau} + h^{\gamma}{}_{\varepsilon}\; (h^{\tau}{}_{\mu})_{;\beta}\; \hat{U}^{*\varepsilon}{}_{\tau}\Big] \\
&= h^{\beta}{}_{\varepsilon}\, h^{\tau}{}_{\nu}\, \hat{U}^{*\varepsilon}{}_{\tau;\beta} - (V^{\gamma}{}_{;\beta}\, V_{\varepsilon} + V^{\gamma}\, V_{\varepsilon;\beta})\; h^{\beta}{}_{\gamma}\, h^{\tau}{}_{\nu}\, \hat{U}^{*\varepsilon}{}_{\tau} - \\
&\quad - (V^{\tau}{}_{;\beta}\, V_{\mu} + V^{\tau}\, V_{\mu;\beta})\; h^{\beta}{}_{\varepsilon}\, h^{\mu}{}_{\nu}\, \hat{U}^{*\varepsilon}{}_{\tau} \\
&= h^{\beta}{}_{\varepsilon}\, h^{\tau}{}_{\nu} \left[\hat{U}^{*\varepsilon}{}_{\tau,\beta} + \Gamma^{\varepsilon}_{\alpha\beta}\, \hat{U}^{*\alpha}{}_{\tau} - \Gamma^{\alpha}_{\tau\beta}\, \hat{U}^{*\varepsilon}{}_{\alpha}\right] \\
&= h^{\beta}{}_{\varepsilon}\, h^{\tau}{}_{\nu}\, \hat{U}^{*\varepsilon}{}_{\tau,\beta} \\
&= -\frac{i}{2}\, k_{\alpha}\, h^{\beta}{}_{\varepsilon}\, h^{\tau}{}_{\nu} \left(\eta_{\gamma}{}^{\lambda\alpha\varepsilon}\, V_{\lambda}\, \hat{U}^{\gamma}{}_{\tau,\beta} + \eta^{\gamma\lambda\alpha}{}_{\tau}\, V_{\lambda}\, \hat{U}^{\varepsilon}{}_{\gamma,\beta}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2} k_\alpha k_\beta h^\beta{}_\varepsilon h^\tau{}_\nu \left(\eta_\gamma{}^{\lambda\alpha\varepsilon} V_\lambda \hat{U}^\gamma{}_\tau + \eta^{\gamma\lambda\alpha}{}_\tau V_\lambda \hat{U}^\varepsilon{}_\gamma\right) \\
&= -\frac{1}{2} k_\alpha k_\varepsilon \left(\eta_\gamma{}^{\lambda\alpha\varepsilon} V_\lambda \hat{U}^\gamma{}_\nu + \eta^{\gamma\lambda\alpha}{}_\nu V_\lambda \hat{U}^\varepsilon{}_\gamma\right) \\
&= 0,
\end{aligned} \tag{2.78}
$$

já que o primeiro termo apresenta uma quantidade simétrica nos índices (α, ε) multiplicada por outra que é antissimétrica nesses mesmos índices. Além disto, o segundo termo acima se anula devido à condição de divergência nula da base tensorial, equação (2.66).

A preservação da propriedade acima pode ser imediatamente calculada como:

$$
\begin{aligned}
\left(\hat{\nabla}_\mu \hat{U}^{*\mu}{}_\nu\right)^\bullet &= \left(\hat{\nabla}_\mu \hat{U}^{*\mu}{}_\nu\right)_{;\alpha} V^\alpha \\
&= \left[\left(\hat{\nabla}_\mu \hat{U}^{*\mu}{}_\nu\right)_{,\alpha} - \Gamma^\beta_{\nu\alpha} \left(\hat{\nabla}_\mu \hat{U}^{*\mu}{}_\beta\right)\right] V^\alpha \\
&= \frac{\partial}{\partial t}\left(\hat{\nabla}_\mu \hat{U}^{*\mu}{}_\nu\right) - \Gamma^\beta_{\nu 0} \left(\hat{\nabla}_\mu \hat{U}^{*\mu}{}_\beta\right) \\
&\equiv 0,
\end{aligned}
$$

da condição da divergência nula do dual.

Outras quantidades úteis para nossos cálculos são a derivada projetada no tempo do tensor dual — $\left(\hat{U}^{*\mu}{}_\nu\right)^\bullet$ — e o duplo dual — $\hat{U}^{**\mu}{}_\nu$. Ambas são obtidas a seguir:

(i) Derivada Projetada no Tempo do Dual:

$$
\begin{aligned}
\left(\hat{U}^{*\mu}{}_{\nu}\right)^{\bullet} &= \hat{U}^{*\mu}{}_{\nu;\alpha}\, V^{\alpha} \\
&= \left[\hat{U}^{*\mu}{}_{\nu,\alpha} + \Gamma^{\mu}_{\beta\alpha}\, \hat{U}^{*\beta}{}_{\nu} - \Gamma^{\beta}_{\nu\alpha}\, \hat{U}^{*\mu}{}_{\beta}\right] V^{\alpha} \\
&= \hat{U}^{*\mu}{}_{\nu,0} + \Gamma^{\mu}_{\beta 0}\, \hat{U}^{*\beta}{}_{\nu} - \Gamma^{\beta}_{\nu 0}\, \hat{U}^{*\mu}{}_{\beta} \\
&= -\frac{i}{2}\, k_{\varepsilon}\, \left[\eta_{\gamma}{}^{\lambda\varepsilon\mu}\, V_{\lambda}\, \hat{U}^{\gamma}{}_{\nu,0} + \eta^{\gamma\lambda\varepsilon}{}_{\nu}\, V_{\lambda}\, \hat{U}^{\mu}{}_{\gamma,0}\right] + \\
&\quad + \left(\sigma^{\mu}{}_{\beta} + \frac{\theta}{3}\, h^{\mu}{}_{\beta}\right)\, \hat{U}^{*\beta}{}_{\nu} - \left(\sigma_{\nu}{}^{\beta} + \frac{\theta}{3}\, h_{\nu}{}^{\beta}\right)\, \hat{U}^{*\mu}{}_{\beta} \\
&= \sigma^{\mu}{}_{\beta}\, \hat{U}^{*\beta}{}_{\nu} - \sigma_{\nu}{}^{\beta}\, \hat{U}^{*\mu}{}_{\beta} + \frac{\theta}{3}\, \hat{U}^{*\mu}{}_{\nu} - \frac{\theta}{3}\, \hat{U}^{*\mu}{}_{\nu} \\
&= \sigma^{\mu}{}_{\alpha}\, \hat{U}^{*\alpha}{}_{\nu} - \sigma_{\nu}{}^{\alpha}\, \hat{U}^{*\mu}{}_{\alpha}. \qquad (2.79)
\end{aligned}
$$

(ii) Duplo Dual:

$$
\begin{aligned}
\hat{U}^{**\mu}{}_{\nu} &= h^{\mu\alpha}\, \hat{U}^{**}_{\alpha\nu} \\
&= h^{\mu\alpha}\, \left[\frac{1}{2}\, h^{\tau}{}_{(\alpha}\, h^{\beta}{}_{\nu)}\, \eta_{\beta}{}^{\lambda\varepsilon\gamma}\, V_{\lambda}\, \left(\hat{\nabla}_{\varepsilon}\, \hat{U}^{*}_{\gamma\tau}\right)\right] \\
&= \frac{1}{2}\, h^{\mu(\alpha}\, h^{\beta)}{}_{\nu}\, \eta_{\beta}{}^{\lambda\varepsilon\gamma}\, V_{\lambda}\, \left(\hat{\nabla}_{\varepsilon}\, \hat{U}^{*}_{\gamma\alpha}\right) \\
&= \frac{1}{2}\, h^{\mu(\alpha}\, h^{\beta)}{}_{\nu}\, \eta_{\beta}{}^{\lambda\varepsilon\gamma}\, V_{\lambda}\, \left[\hat{\nabla}_{\varepsilon}\, \left(\frac{1}{2}\, h^{\tau}{}_{(\gamma}\, h^{\rho}{}_{\alpha)}\, \eta_{\rho}{}^{\omega\varphi\zeta}\, V_{\omega}\, \left(\hat{\nabla}_{\varphi}\, \hat{U}_{\zeta\tau}\right)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}\, h^{\mu(\alpha}\, h^{\beta)}{}_{\nu}\, h^{\tau}{}_{(\gamma}\, h^{\rho}{}_{\alpha)}\, \eta_{\beta}{}^{\lambda\varepsilon\gamma}\, \eta_{\rho}{}^{\omega\varphi\zeta}\, V_{\lambda}\, \hat{\nabla}_{\varepsilon}\left[V_{\omega}\left(\hat{\nabla}_{\varphi}\, \hat{U}_{\zeta\tau}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}\left(h^{\mu\rho}\, h^{\beta}{}_{\nu}\, h^{\tau}{}_{\gamma} + h^{\mu\tau}\, h^{\beta}{}_{\nu}\, h^{\rho}{}_{\gamma} + h^{\mu\beta}\, h^{\rho}{}_{\nu}\, h^{\tau}{}_{\gamma} + h^{\mu\beta}\, h^{\tau}{}_{\nu}\, h^{\rho}{}_{\gamma}\right) \; . \\
&\quad .\; \eta_{\beta}{}^{\lambda\varepsilon\gamma}\, \eta_{\rho}{}^{\omega\varphi\zeta}\, V_{\lambda}\left[\left(\hat{\nabla}_{\varepsilon}\, V_{\omega}\right)\left(\hat{\nabla}_{\varphi}\, \hat{U}_{\zeta\tau}\right) + V_{\omega}\, \hat{\nabla}_{\varepsilon}\left(\hat{\nabla}_{\varphi}\, \hat{U}_{\zeta\tau}\right)\right]
\end{aligned}
$$

$$
= \frac{1}{4}\left(h^{\mu(\beta}\, h^{\rho)}{}_{\nu}\, h^{\tau}{}_{\gamma} + h^{\mu(\beta}\, h^{\tau)}{}_{\nu}\, h^{\rho}{}_{\gamma}\right)\, \eta_{\beta}{}^{\lambda\varepsilon\gamma}\, \eta_{\rho}{}^{\omega\varphi\zeta}\, V_{\lambda}\, V_{\omega}\, \hat{\nabla}_{\varepsilon}\left(\hat{\nabla}_{\varphi}\, \hat{U}_{\tau\zeta}\right)
$$

$$
\begin{aligned}
&= \frac{1}{4}\Big[\eta_{\nu}{}^{\lambda\varepsilon\gamma}\, \eta^{\mu\omega\varphi\zeta}\, V_{\lambda}\, V_{\omega}\, \hat{\nabla}_{\varepsilon}\left(\hat{\nabla}_{\varphi}\, \hat{U}_{\gamma\zeta}\right) + \eta_{\nu}{}^{\lambda\varepsilon\gamma}\, \eta_{\gamma}{}^{\omega\varphi\zeta}\, V_{\lambda}\, V_{\omega}\, \hat{\nabla}_{\varepsilon}\left(\hat{\nabla}_{\varphi}\, \hat{U}^{\mu}{}_{\zeta}\right) + \\
&\quad + \eta^{\mu\lambda\varepsilon\gamma}\, \eta_{\nu}{}^{\omega\varphi\zeta}\, V_{\lambda}\, V_{\omega}\, \hat{\nabla}_{\varepsilon}\left(\hat{\nabla}_{\varphi}\, \hat{U}_{\zeta\gamma}\right) + \eta^{\mu\lambda\varepsilon\gamma}\, \eta_{\gamma}{}^{\omega\varphi\zeta}\, V_{\lambda}\, V_{\omega}\, \hat{\nabla}_{\varepsilon}\left(\hat{\nabla}_{\varphi}\, \hat{U}_{\zeta\nu}\right)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}\Big[\eta_{\nu\lambda\varepsilon\gamma}\, \eta^{\mu\omega\varphi\zeta}\, V^{\lambda}\, V_{\omega}\, \hat{\nabla}^{\varepsilon}\left(\hat{\nabla}_{\varphi}\, \hat{U}^{\gamma}{}_{\zeta}\right) + \eta_{\nu\lambda\varepsilon\gamma}\, \eta^{\omega\varphi\zeta\gamma}\, V^{\lambda}\, V_{\omega}\, \hat{\nabla}^{\varepsilon}\left(\hat{\nabla}_{\varphi}\, \hat{U}^{\mu}{}_{\zeta}\right) + \\
&\quad + \eta^{\mu\lambda\varepsilon\gamma}\, \eta_{\nu\omega\varphi\zeta}\, V_{\lambda}\, V^{\omega}\, \hat{\nabla}_{\varepsilon}\left(\hat{\nabla}^{\varphi}\, \hat{U}^{\zeta}{}_{\gamma}\right) + \eta^{\mu\lambda\varepsilon\gamma}\, \eta_{\omega\varphi\zeta\gamma}\, V_{\lambda}\, V^{\omega}\, \hat{\nabla}_{\varepsilon}\left(\hat{\nabla}^{\varphi}\, \hat{U}^{\zeta}{}_{\nu}\right)\Big] .
\end{aligned}
$$

Calculando termo a termo, obtemos:

$$
\begin{aligned}
1^{\underline{o}}\; termo \;\equiv\; T_1 &= -\frac{1}{4}\, \delta^{\mu\omega\varphi\zeta}_{\nu\lambda\varepsilon\gamma}\, V^{\lambda}\, V_{\omega}\, \hat{\nabla}^{\varepsilon}\left(\hat{\nabla}_{\varphi}\, \hat{U}^{\gamma}{}_{\zeta}\right) \\
&= -\frac{1}{4}\begin{vmatrix} \delta^{\mu}_{\nu} & \delta^{\mu}_{\lambda} & \delta^{\mu}_{\varepsilon} & \delta^{\mu}_{\gamma} \\ \delta^{\omega}_{\nu} & \delta^{\omega}_{\lambda} & \delta^{\omega}_{\varepsilon} & \delta^{\omega}_{\gamma} \\ \delta^{\varphi}_{\nu} & \delta^{\varphi}_{\lambda} & \delta^{\varphi}_{\varepsilon} & \delta^{\varphi}_{\gamma} \\ \delta^{\zeta}_{\nu} & \delta^{\zeta}_{\lambda} & \delta^{\zeta}_{\varepsilon} & \delta^{\zeta}_{\gamma} \end{vmatrix}\, V^{\lambda}\, V_{\omega}\, \hat{\nabla}^{\varepsilon}\left(\hat{\nabla}_{\varphi}\, \hat{U}^{\gamma}{}_{\zeta}\right) \\
&= -\frac{1}{4}\left(\delta^{\mu}_{\gamma}\, \delta^{\varphi}_{\nu}\, \delta^{\zeta}_{\varepsilon} - \delta^{\mu}_{\gamma}\, \delta^{\varphi}_{\varepsilon}\, \delta^{\zeta}_{\nu}\right)\, \hat{\nabla}^{\varepsilon}\left(\hat{\nabla}_{\varphi}\, \hat{U}^{\gamma}{}_{\zeta}\right) \\
&= \frac{1}{4}\left[\hat{\nabla}^{\varphi}\left(\hat{\nabla}_{\varphi}\, \hat{U}^{\mu}{}_{\nu}\right) - \hat{\nabla}^{\varepsilon}\left(\hat{\nabla}_{\nu}\, \hat{U}^{\mu}{}_{\varepsilon}\right)\right]
\end{aligned}
$$

$$= \frac{1}{4}\left[\hat{\nabla}^2 \hat{U}^\mu{}_\nu - h^{\alpha\varepsilon}\,\hat{\nabla}_\alpha\left(\hat{\nabla}_\nu \hat{U}^\mu{}_\varepsilon\right)\right]$$

$$= \frac{1}{4}\left\{k^2 \hat{U}^\mu{}_\nu - h^{\alpha\varepsilon}\left[\hat{\nabla}_\nu\left(\hat{\nabla}_\alpha \hat{U}^\mu{}_\varepsilon\right) + \hat{R}^\mu{}_{\gamma\nu\alpha}\,\hat{U}^\gamma{}_\varepsilon - \hat{R}^\gamma{}_{\varepsilon\nu\alpha}\,\hat{U}^\mu{}_\gamma\right]\right\}$$

$$\begin{aligned} = {} & \frac{k^2}{4}\hat{U}^\mu{}_\nu - \frac{1}{4}\left[\hat{\nabla}_\nu\left(h^{\alpha\varepsilon}\,\hat{\nabla}_\alpha \hat{U}^\mu{}_\varepsilon\right) - \left(\hat{\nabla}_\nu h^{\alpha\varepsilon}\right)\left(\hat{\nabla}_\alpha \hat{U}^\mu{}_\varepsilon\right)\right] + \\ & + \frac{1}{4}\hat{R}^{\mu\varepsilon}{}_{\nu\gamma}\,\hat{U}^\gamma{}_\varepsilon + \frac{1}{4}\hat{R}^\gamma{}_\nu\,\hat{U}^\mu{}_\gamma . \end{aligned}$$

Entretanto, sabemos que as componentes dos tensores de Riemann com todos os índices projetados no 3-espaço são escritas basicamente como:

$$\begin{aligned} \hat{R}^\varepsilon{}_{\mu\nu\alpha}\,l_\varepsilon &= \hat{\nabla}_{[\mu}\,\hat{\nabla}_{\nu]}\,l_\alpha \\ &= l_{\alpha,[\mu,\nu]} + \textit{termos de conexão com índices espaciais}, \end{aligned}$$

onde l_α é um vetor arbitrário. Ora, o primeiro termo na relação acima é obviamente nulo; e os termos de conexão, sendo calculados no *background*, o são igualmente. Daí, o primeiro termo se reduz a:

$$T_1 = \frac{k^2}{4}\,\hat{U}^\mu{}_\nu .$$

$$2^{\underline{o}}\ \textit{termo} \equiv T_2 = \frac{1}{4}\,\delta^{\omega\varphi\zeta}_{\nu\lambda\varepsilon}\,V^\lambda\,V_\omega\,\hat{\nabla}^\varepsilon\left(\hat{\nabla}_\varphi \hat{U}^\mu{}_\zeta\right)$$

$$
\begin{aligned}
&= \frac{1}{4} \begin{vmatrix} \delta^{\omega}_{\nu} & \delta^{\omega}_{\lambda} & \delta^{\omega}_{\varepsilon} \\ \delta^{\varphi}_{\nu} & \delta^{\varphi}_{\lambda} & \delta^{\varphi}_{\varepsilon} \\ \delta^{\zeta}_{\nu} & \delta^{\zeta}_{\lambda} & \delta^{\zeta}_{\varepsilon} \end{vmatrix} V^{\lambda} V_{\omega} \hat{\nabla}^{\varepsilon} \left(\hat{\nabla}_{\varphi} \hat{U}^{\mu}{}_{\zeta} \right) \\
&= \frac{1}{4} \left[\hat{\nabla}^{\varphi} \left(\hat{\nabla}_{\varphi} \hat{U}^{\mu}{}_{\nu} \right) - \hat{\nabla}^{\varepsilon} \left(\hat{\nabla}_{\nu} \hat{U}^{\mu}{}_{\varepsilon} \right) \right] \\
&= \frac{1}{4} \left[\hat{\nabla}^{2} \hat{U}^{\mu}{}_{\nu} - \hat{\nabla}_{\nu} \left(\hat{\nabla}^{\varepsilon} \hat{U}^{\mu}{}_{\varepsilon} \right) + termos\ \hat{R}\ nulos \right] \\
&= \frac{1}{4} k^{2} \hat{U}^{\mu}{}_{\nu}.
\end{aligned}
$$

$3^{\underline{0}}$ *termo* $\equiv T_3 =$ *igual a* T_1, *trocando os índices* (φ,ε) *e* (ζ,γ):

$$
T_3 = \frac{1}{4} k^2 \hat{U}^{\mu}{}_{\nu}.
$$

$$
4^{\underline{0}}\ termo \equiv T_4 = \frac{1}{4} \delta^{\mu\lambda\varepsilon}_{\omega\varphi\zeta} V_{\lambda} V^{\omega} \hat{\nabla}_{\varepsilon} \left(\hat{\nabla}^{\varphi} \hat{U}^{\zeta}{}_{\nu} \right) = \frac{1}{4} k^2 \hat{U}^{\mu}{}_{\nu},
$$

pelos mesmos argumentos acima. Temos, então, finalmente, o seguinte resultado para o duplo dual:

$$
\hat{U}^{**\mu}{}_{\nu} = k^2 \hat{U}^{\mu}{}_{\nu}. \tag{2.80}
$$

Isto conclui os cálculos relativos à base tensorial $\hat{U}^{\mu}{}_{\nu}$. Podemos então, neste ponto, obter as componentes de $\hat{U}^{\mu}{}_{\nu}$ explicitamente. Para tanto, lançaremos mão das condições de traço e divergência nulas, equações (2.64) e (2.68)-(2.70), bem como da condição de preservação da ortogonalidade, equação (2.71). Por esta última condição, constata-se que as componentes $\{\alpha,\psi,\varphi,\eta,\beta,\ \varepsilon,\chi,\zeta,\gamma\}$ são, obrigatoriamente, constantes. Além disto, temos, de (1.32) e (2.54), que:

$$\hat{U}^2{}_1 \equiv \eta = g_{11}\, g^{22}\, \hat{U}^1{}_2 \equiv t^{2\,(p_1-p_2)}\, \psi$$

$$\hat{U}^3{}_1 \equiv \chi = g_{11}\, g^{33}\, \hat{U}^1{}_3 \equiv t^{2\,(p_1-p_3)}\, \varphi \tag{2.81}$$

$$\hat{U}^3{}_2 \equiv \zeta = g_{22}\, g^{33}\, \hat{U}^2{}_3 \equiv t^{2\,(p_2-p_3)}\, \varepsilon.$$

Substituindo (2.81) em (2.68)-(2.70) e usando (2.64), as condições de divergência nula se escrevem como:

$$k_1\,\alpha + k_2\, t^{2\,(p_1-p_2)}\,\psi + k_3\, t^{2\,(p_1-p_3)}\,\varphi = 0$$

$$k_1\,\psi + k_2\,\beta + k_3\, t^{2\,(p_2-p_3)}\,\varepsilon = 0 \tag{2.82}$$

$$k_1\,\varphi + k_2\,\varepsilon - k_3\,(\alpha+\beta) = 0.$$

Entretanto, sabendo-se que os k_i e as componentes da base são constantes, concluímos que cada termo nas relações acima deve sê-lo também. Neste caso, temos as seguintes condições que devem ser necessariamente satisfeitas para que as relações (2.82) sejam consistentes:

$$k_2\, t^{2\,(p_1-p_2)}\,\psi = constante \implies p_1 = p_2 \;\; ou \;\; \psi = 0$$

$$k_3\, t^{2\,(p_1-p_3)}\,\varphi = constante \implies p_1 = p_3 \;\; ou \;\; \varphi = 0 \tag{2.83}$$

$$k_3\, t^{2\,(p_2-p_3)}\,\varepsilon = constante \implies p_2 = p_3 \;\; ou \;\; \varepsilon = 0,$$

já que fazer qualquer um dos k_i igual a zero seria inaceitável — eliminando uma das dependências espaciais da base, como já vimos anteriormente. Logo, não há a possibilidade de obtermos uma base tensorial adequada e consistente para o caso de um *background* com anisotropia total. Isto se traduz, como vimos acima, no fato de que a anisotropia total implica em um "excesso" de condições sobre esta base. Contudo, há uma base tensorial consistente no caso específico de um modelo de Kasner com um **plano de isotropia** — caso das soluções de Milne e Kasner, as quais são dadas, conforme vimos no Capítulo 1, por:

- *Solução de Milne:* $(p_1,p_2,p_3) = (0,0,1)$
- *Solução de Kasner:* $(p_1,p_2,p_3) = (2/3,2/3,-1/3)$

Os resultados que se seguem foram obtidos especificamente para a solução de Kasner. Temos, primeiramente, de (2.81), que:

$$
\begin{aligned}
\hat{U}^2{}_1 &\equiv \eta = \hat{U}^1{}_2 \equiv \psi \\
\hat{U}^3{}_1 &\equiv \chi = \tfrac{1}{\theta^2}\,\hat{U}^1{}_3 \equiv \tfrac{1}{\theta^2}\,\varphi = 0 \qquad (2.84) \\
\hat{U}^3{}_2 &\equiv \zeta = \tfrac{1}{\theta^2}\,\hat{U}^2{}_3 \equiv \tfrac{1}{\theta^2}\,\varepsilon = 0.
\end{aligned}
$$

Com os resultados acima, a terceira condição — de divergência nula, dada pela equação (2.70) — é escrita como:

$$k_3\,\gamma = 0 \Longrightarrow \gamma = 0,$$

onde, novamente, escolher-se k_3 como zero na relação acima eliminaria a dependência da base tensorial na coordenada z — uma complicação adicional desnecessária e que leva a problemas de adequação do tensor $\hat{U}^\mu{}_\nu$ que daí resultaria. Da propriedade de traço nulo, equação (2.64), obtemos então que:

$$\alpha + \beta = 0 \Longrightarrow \beta = -\alpha,$$

e, lançando mão de (2.68)-(2.70), vem:

$$
\begin{cases}
k_1\,\psi + k_2\,\beta = 0 \Longrightarrow \psi = -\frac{k_2}{k_1}\,\beta = \frac{k_2}{k_1}\,\alpha \\
\\
k_1\,\alpha + k_2\,\psi = 0 \Longrightarrow \psi = -\frac{k_1}{k_2}\,\alpha.
\end{cases}
$$

Os resultados acima são consistentes se:

$$
\frac{k_2}{k_1}\,\alpha = -\frac{k_1}{k_2}\,\alpha \Longrightarrow (k_2)^2 = -\,(k_1)^2\,,
$$

donde obtemos:

$$
k_2 = \pm i\,k_1. \tag{2.85}
$$

E, portanto, a base tensorial específica para a solução de Kasner é dada por:

$$
\hat{U}^{\mu}{}_{\nu} = e^{-i\,k_j\,x^j} \begin{pmatrix} \alpha & \pm i\,\alpha & 0 \\ \pm i\,\alpha & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.86}
$$

onde temos:

$$
k^2 = (k_3)^2\,\left[C(t)\right]^{-2} = (k_3)^2\,\left(\frac{t}{t_0}\right)^{2/3}. \tag{2.87}
$$

Temos, então, os seguintes resultados úteis:

$$
\left(\sigma^{\alpha}{}_{\mu}\,\hat{U}^{\mu}{}_{\beta} - \sigma^{\mu}{}_{\beta}\,\hat{U}^{\alpha}{}_{\mu}\right) = 0,\;\; \forall\;\; (\alpha,\beta) \Longrightarrow \left(\hat{U}^{\mu}{}_{\nu}\right)^{\bullet} = 0, \tag{2.88}
$$

$$\left(\sigma^{\alpha}{}_{\mu}\,\hat{U}^{\mu}{}_{\beta}+\sigma^{\mu}{}_{\beta}\,\hat{U}^{\alpha}{}_{\mu}\right) = \begin{cases} \frac{2}{3}\,\theta\,\hat{U}^{1}{}_{1} & (\alpha=\beta=1) \\ \frac{2}{3}\,\theta\,\hat{U}^{2}{}_{2} & (\alpha=\beta=2) \\ 0 & (\alpha=\beta=3) \\ \frac{2}{3}\,\theta\,\hat{U}^{1}{}_{2} & (\alpha=1 \quad \beta=2) \\ 0 & (\alpha=1 \quad \beta=3) \\ 0 & (\alpha=2 \quad \beta=3)\,, \end{cases}$$

$$= \frac{2}{3}\,\theta\,\hat{U}^{\alpha}{}_{\beta}. \tag{2.89}$$

$$\sigma^{\alpha}{}_{\beta}\,\hat{U}^{\beta}{}_{\alpha} = \sigma^{1}{}_{1}\,\alpha+\sigma^{2}{}_{2}\,\beta+\sigma^{3}{}_{3}\,\gamma = \sigma^{1}{}_{1}\,(\alpha+\beta) = 0, \tag{2.90}$$

$$\left(E^{\alpha}{}_{\mu}\,\hat{U}^{\mu}{}_{\beta}+E^{\mu}{}_{\beta}\,\hat{U}^{\alpha}{}_{\mu}\right) = \frac{4}{9}\,\theta^{2}\,\hat{U}^{\alpha}{}_{\beta}, \tag{2.91}$$

$$E^{\alpha}{}_{\beta}\,\hat{U}^{\beta}{}_{\alpha} = 0, \tag{2.92}$$

$$\eta^{\alpha\beta\mu\nu}\,V_{\beta}\,\sigma_{\mu\gamma}\,\hat{U}^{\gamma}{}_{\nu} = 0. \tag{2.93}$$

As componentes do tensor dual são calculadas a partir da equação (2.76), como:

$$\begin{aligned} \hat{U}^{*1}{}_{1} &= -\frac{i}{2}\,k_{\varepsilon}\left(\eta_{\gamma}{}^{0\varepsilon 1}\,\hat{U}^{\gamma}{}_{1}+\eta^{\gamma 0\varepsilon}{}_{1}\,\hat{U}^{1}{}_{\gamma}\right) \\ &= -\frac{i}{2}\left[\eta_{2}{}^{031}\,k_{3}\,\hat{U}^{2}{}_{1}+\eta_{3}{}^{021}\,k_{2}\,\hat{U}^{3}{}_{1}+\right. \\ &\quad \left.+\eta^{203}{}_{1}\,k_{3}\,\hat{U}^{1}{}_{2}+\eta^{302}{}_{1}\,k_{2}\,\hat{U}^{1}{}_{3}\right] \end{aligned}$$

$$
\begin{aligned}
&= -\frac{i}{2}\left[\eta^{2031}\, g_{11}\, k_3\, \hat{U}^1{}_2 + \eta^{3021}\, g_{11}\, k_2\, \hat{U}^1{}_3 + \right. \\
&\qquad \left. + \eta^{2031}\, g_{11}\, k_3\, \hat{U}^1{}_2 + \eta^{3021}\, g_{11}\, k_2\, \hat{U}^1{}_3\right] \\
&= i\, g_{11}\, k_3\, \hat{U}^1{}_2,
\end{aligned}
$$

$$\hat{U}^{*2}{}_2 = -\hat{U}^{*1}{}_1,$$

$$\hat{U}^{*3}{}_3 = -\tfrac{i}{2}\, k_\varepsilon \left(\eta_\gamma{}^{\lambda\varepsilon 3}\, V_\lambda\, \hat{U}^\gamma{}_3 + \eta^{\gamma\lambda\varepsilon}{}_3\, V_\lambda\, \hat{U}^3{}_\gamma\right) = 0$$

$$\hat{U}^{*1}{}_2 \equiv \hat{U}^{*2}{}_1 = -i\, g_{11}\, k_3\, \hat{U}^1{}_1$$

$$\hat{U}^{*1}{}_3 \equiv \hat{U}^{*2}{}_3 = 0.$$

Daí temos:

$$\hat{U}^{*\mu}{}_\nu = -g_{11}\, k_3\, e^{-i\, k_j\, x^j} \begin{pmatrix} \pm\alpha & i\,\alpha & 0 \\ i\,\alpha & \mp\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.94}$$

donde constatamos facilmente de (2.86) que:

$$
\begin{aligned}
\hat{U}^{*\mu}{}_\nu &= -g_{11}\, k_3\, e^{-i\, k_j\, x^j} \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \pm i\,\alpha & 0 \\ \pm i\,\alpha & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= -g_{11}\, k_3 \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \hat{U}^\mu{}_\nu.
\end{aligned} \tag{2.95}
$$

Com isto e o resultado (2.94), obtemos os seguintes resultados (os cálculos são exibidos no Apêndice C):

$$\sigma^{\alpha}{}_{\beta}\,\hat{U}^{*\beta}{}_{\alpha} = \sigma^{1}{}_{1}\,\hat{U}^{*1}{}_{1} + \sigma^{2}{}_{2}\,\hat{U}^{*2}{}_{2} + \sigma^{3}{}_{3}\,\hat{U}^{*3}{}_{3} = 0, \tag{2.96}$$

$$\left(\sigma^{\alpha}{}_{\mu}\,\hat{U}^{*\mu}{}_{\beta} - \sigma^{\mu}{}_{\beta}\,\hat{U}^{*\alpha}{}_{\mu}\right) = 0, \;\; \forall \;\; (\alpha,\beta) \Longrightarrow \left(\hat{U}^{*\mu}{}_{\nu}\right)^{\bullet} = 0, \tag{2.97}$$

$$\left(\sigma^{\alpha}{}_{\mu}\,\hat{U}^{*\mu}{}_{\beta} + \sigma^{\mu}{}_{\beta}\,\hat{U}^{*\mu}{}_{\alpha}\right) = 2\,\sigma^{1}{}_{1}\,\hat{U}^{*\alpha}{}_{\beta} = \frac{2}{3}\theta\,\hat{U}^{*\alpha}{}_{\beta}, \tag{2.98}$$

$$\left[\eta^{\lambda\gamma\alpha\beta}\,V_{\gamma}\,\sigma^{\mu}{}_{\lambda}\,h_{\varepsilon\nu}\,\hat{U}^{*\varepsilon}{}_{\alpha;\beta} + \eta^{\lambda\gamma\alpha\beta}\,V_{\gamma}\,\sigma_{\nu\lambda}\,h_{\varepsilon}{}^{\mu}\,\hat{U}^{*\varepsilon}{}_{\alpha;\beta}\right] = \frac{2}{3}\,k^{2}\,\theta\,\hat{U}^{\mu}{}_{\nu}, \tag{2.99}$$

$$\left[\eta^{\mu\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta} + \eta_{\nu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h^{\mu}{}_{\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta}\right] = 2\,k^{2}\,\hat{U}^{\mu}{}_{\nu}, \tag{2.100}$$

$$\left[\eta^{\mu\gamma\alpha\beta}\,V_{\gamma}\,\sigma_{\nu\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta} + \eta_{\nu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,\sigma^{\mu}{}_{\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta}\right] = \frac{2}{3}\,\theta\,k^{2}\,\hat{U}^{\mu}{}_{\nu}, \tag{2.101}$$

$$\left[\eta^{\gamma\tau\alpha\beta}\,V_{\tau}\,\sigma_{\gamma\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta}\right] = 0, \tag{2.102}$$

$$\eta^{\alpha\beta\mu\nu}\,V_{\beta}\,\sigma_{\mu\gamma}\,\hat{U}^{*\gamma}{}_{\nu} = 0. \tag{2.103}$$

Repetindo os cálculos efetuados acima para a solução de Milne, obtemos novamente de (2.81) que:

$$\begin{aligned} &\hat{U}^{2}{}_{1} \equiv \eta = \hat{U}^{1}{}_{2} \equiv \psi \\ &\hat{U}^{3}{}_{1} \equiv \chi = t^{-2}\,\hat{U}^{1}{}_{3} \equiv t^{-2}\,\varphi = 0 \\ &\hat{U}^{3}{}_{2} \equiv \zeta = t^{-2}\,\hat{U}^{2}{}_{3} \equiv t^{-2}\,\varepsilon = 0, \end{aligned} \tag{2.104}$$

e (2.70) nos dá de novo:

$$k_{3}\,\gamma = 0 \Longrightarrow \gamma = o,$$

onde tornamos a considerar necessariamente $k_3 \neq 0$. A equação (2.64) nos dá então:

$$\beta = -\alpha,$$

como antes — o que, de (2.68) - (2.70), acarreta

$$k_2 = \pm i\, k_1, \tag{2.105}$$

e então a base tensorial específica para a solução de Milne é também dada pela equação (2.86), com k^2 escrito como:

$$k^2 = (k_3)^2 \left(\frac{t}{t_0}\right)^{-2} = (k_3\, t_0)^2\, t^{-2}. \tag{2.106}$$

E os resultados úteis se seguem de maneira análoga (vide cálculos correspondentes no Apêndice C) como:

$$\left(\sigma^{\alpha}{}_{\mu}\, \hat{U}^{\mu}{}_{\beta} - \sigma^{\mu}{}_{\beta}\, \hat{U}^{\alpha}{}_{\mu}\right) = 0, \quad \forall \quad (\alpha, \beta) \Longrightarrow \left(\hat{U}^{\mu}{}_{\nu}\right)^{\bullet} = 0, \tag{2.107}$$

$$\left(\sigma^{\alpha}{}_{\mu}\, \hat{U}^{\mu}{}_{\beta} + \sigma^{\mu}{}_{\beta}\, \hat{U}^{\alpha}{}_{\mu}\right) = -\frac{2}{3}\, \theta\, \hat{U}^{\alpha}{}_{\beta}, \tag{2.108}$$

$$\sigma^{\alpha}{}_{\beta}\, \hat{U}^{\beta}{}_{\alpha} = 0, \tag{2.109}$$

$$\left(E^{\alpha}{}_{\mu}\, \hat{U}^{\mu}{}_{\beta} + E^{\mu}{}_{\beta}\, \hat{U}^{\alpha}{}_{\mu}\right) = 0, \tag{2.110}$$

$$E^{\alpha}{}_{\beta}\, \hat{U}^{\beta}{}_{\alpha} = 0, \tag{2.111}$$

$$\eta^{\alpha\beta\mu\nu}\, V_{\beta}\, \sigma_{\mu\gamma}\, \hat{U}^{\gamma}{}_{\nu} = 0. \tag{2.112}$$

Para finalizar os cálculos referentes à base tensorial, o tensor dual é idêntico ao dado pela equação (2.94); temos também que:

$$\sigma^{\alpha}{}_{\beta}\, \hat{U}^{*\beta}{}_{\alpha} = 0, \tag{2.113}$$

$$\left(\sigma^{\alpha}{}_{\mu}\, \hat{U}^{*\mu}{}_{\beta} - \sigma^{\mu}{}_{\beta}\, \hat{U}^{*\alpha}{}_{\mu}\right) = 0 \Longrightarrow \left(\hat{U}^{*\mu}{}_{\nu}\right)^{\bullet} = 0, \tag{2.114}$$

$$\left(\sigma^{\alpha}{}_{\mu}\, \hat{U}^{*\mu}{}_{\beta} + \sigma^{\mu}{}_{\beta}\, \hat{U}^{*\alpha}{}_{\mu}\right) = -\frac{2}{3}\, \theta\, \hat{U}^{*\alpha}{}_{\beta}, \tag{2.115}$$

$$\left[\eta^{\lambda\gamma\alpha\beta}\, V_{\gamma}\, \sigma^{\mu}{}_{\lambda}\, h_{\varepsilon\nu}\, \hat{U}^{*\varepsilon}{}_{\alpha;\beta} + \eta^{\lambda\gamma\alpha\beta}\, V_{\gamma}\, \sigma_{\nu\lambda}\, h_{\varepsilon}{}^{\mu}\, \hat{U}^{*\varepsilon}{}_{\alpha;\beta}\right] = -\frac{2}{3}\, \theta\, k^2\, \hat{U}^{\mu}{}_{\nu}, \tag{2.116}$$

$$\left[\eta^{\mu\gamma\alpha\beta}\, V_\gamma\, h_{\nu\lambda}\, \hat{U}^{*\lambda}{}_{\alpha;\beta} + \eta_\nu{}^{\gamma\alpha\beta}\, V_\gamma\, h^\mu{}_\lambda\, \hat{U}^{*\lambda}{}_{\alpha;\beta}\right] = 2\, k^2\, \hat{U}^\mu{}_\nu, \tag{2.117}$$

$$\left[\eta^{\mu\gamma\alpha\beta}\, V_\gamma\, \sigma_{\nu\lambda}\, \hat{U}^{*\lambda}{}_{\alpha;\beta} + \eta_\nu{}^{\gamma\alpha\beta}\, V_\gamma\, \sigma^\mu{}_\lambda\, \hat{U}^{*\lambda}{}_{\alpha;\beta}\right] = -\frac{2}{3}\, \theta\, k^2\, \hat{U}^\mu{}_\nu, \tag{2.118}$$

$$\left[\eta^{\gamma\tau\alpha\beta}\, V_\tau\, \sigma_{\gamma\lambda}\, \hat{U}^{*\lambda}{}_{\alpha;\beta}\right] = 0, \tag{2.119}$$

$$\eta^{\alpha\beta\mu\nu}\, V_\beta\, \sigma_{\mu\gamma}\, \hat{U}^{*\gamma}{}_\nu = 0. \tag{2.120}$$

Capítulo 3

O Formalismo Quase-Maxwelliano na Teoria de Perturbações Cosmológicas

3.1 Introdução

Historicamente a teoria das perturbações cosmológicas tem sido estudada de duas maneiras diferentes. A primeira destas abordagens é baseada nas equações de Einstein originais (conforme o trabalho de Lifshitz e Khalatnikov [13], por exemplo). A segunda abordagem é baseada nas equações Quase-Maxwellianas (QM) [26], a qual é equivalente à primeira, mas apresenta certas vantagens em relação a esta. Estas vantagens tornam-se mais evidentes no caso em que a variação do tensor de Weyl é uma quantidade importante a ser considerada. Hawking [24] foi um dos primeiros a fazer uso das equações QM na descrição das perturbações cosmológicas em 1966, assim como Jordan *et al.*, em 1961 [27]. Entretanto, a descrição de Hawking apresentava vários erros que foram corrigidos nos trabalhos de Novello e Salim (1982/1983) [28, 17], e, em trabalho posterior destes com Heintzmann [25].

Entretanto, desde o artigo original de Lifshitz e Khalatnikov, tem sido uma prática comum examinar-se variações de quantidades não observáveis, tais como o tensor métrico, ($\delta g_{\mu\nu}$). Isto levou a problemas ao efetuar-se a distinção entre as verdadeiras perturbações e aquelas associadas a meras transformações de coordenadas — o assim chamado *problema de gauge* da teoria de perturbações cosmológicas. Muitas foram as tentativas de solução deste problema. Uma abordagem consiste em escrever-se quantidades gauge-independentes em termos de ($\delta g_{\mu\nu}$) e de suas derivadas, o que foi feito, entre outros, por Hawking (1966) [24], Jones (1976) [29], Olson (1976) [30], D'Eath (1976) [31], Bardeen (1980) [14], Brandenberger *et al.* (1983) [32], Abbott e Traschen [33], Huang e Vishniac (1990) [34] e Mukhanov *et al.* (1992) [35]. Todos estes trabalhos resolvem o problema de gauge das perturbações; contudo, as variáveis em termos das quais os sistemas dinâmicos são escritos nestes trabalhos, apesar de gauge-independentes, têm uma interpretação física extremamente complexa — em especial, as variáveis de Bardeen [14], que obtiveram um maior sucesso, sendo amplamente usadas na literatura [36, 37, 38].

A segunda abordagem (Novello, Salim, Motta da Silva, Klippert e Jorás [10, 11] e Novello, Salim, Motta da Silva e Klippert [12]) consiste em partir de quantidades fisicamente observáveis e que sejam, além disto, verdadeiras perturbações — gauge-independentes, portanto.

Esta abordagem resolve o problema de gauge das perturbações cosmológicas, com a vantagem de fazê-lo em termos de quantidades observáveis, com uma interpretação física direta. Resta, apenas, definir o que seriam as "verdadeiras"perturbações originais. Isto, no entanto, é bastante fácil se o **lema de Stewart** (conforme referências [15] e [16]) for considerado. Segundo este lema, qualquer quantidade que seja nula em um dado *background* a ser perturbado constitui-se, após a perturbação, em uma quantidade gauge-independente (e, portanto, em uma verdadeira perturbação). Tudo o que se tem a fazer, então, é escrever-se o sistema dinâmico de equações QM perturbadas em termos somente destas "boas"observáveis, as quais constituem neste caso um **conjunto mínimo fechado**, em termos do qual todas as outras quantidades são descritas. Note-se que esta abordagem resolve o problema de gauge evitando lidar-se com aquelas quantidades que sejam consideradas "ruins" no sentido definido pelo lema de Stewart. Para analisar-se tais quantidades, ainda seria necessário efetuar-se uma escolha de gauge, a exemplo de Lifshitz e Khalatnikov. Novello *et al.* usaram este método para resolver o problema das perturbações no modelo de Friedmann-Robertson-Walker (FRW) [10, 11, 12], em termos das partes elétrica e magnética do tensor de Weyl, do *shear* e da rotação — todas quantidades nulas no *background* de FRW.

Neste capítulo, mostraremos as equações Quase-Maxwellianas e as escreveremos para o *background* de Kasner. Em seguida, aplicaremos a elas o método descrito acima, perturbando as equações QM e escrevendo-as em termos das quantidades gauge-independentes disponíveis. Contudo, como estamos tratando aqui um modelo anisotrópico, veremos que é necessário definir-se quantidades gauge-independentes "auxiliares", escritas em termos das quantidades observáveis não nulas deste *background* (*shear*, parte elétrica do tensor de Weyl e expansão), e que irão substituí-las no sistema dinâmico a ser obtido. Veremos ainda que, embora um tal recurso seja algo semelhante à abordagem de Bardeen [14] no sentido de envolver quantidades gauge-invariantes "artificiais", elas nem de longe apresentam a complexidade algébrica destas, já que são construídas a partir das equações QM originais, não perturbadas, para o modelo de Kasner.

3.2 As Equações Quase-Maxwellianas

As equações Quase-Maxwellianas (QM) são obtidas a partir das identidades de Bianchi (ver, a este respeito, as referências [10, 17, 39]) e constituem-se em um sistema dinâmico que descreve a propagação da gravitação. Elas são completamente equivalentes às equações de Einstein, mas são escritas em uma forma similar à das equações de Maxwell do eletromagnetismo. Nesta seção, portanto, obteremos o sistema completo de doze equações associadas ao formalismo Quase-Maxwelliano.

Temos, para todo espaço Riemanniano, a seguinte relação identicamente satisfeita para o tensor de curvatura de Riemann, $R_{\alpha\mu\beta\nu}$ [40]:

$$R^{\alpha\beta}{}_{\mu\nu;\lambda} + R^{\alpha\beta}{}_{\nu\lambda;\mu} + R^{\alpha\beta}{}_{\lambda\mu;\nu} = 0, \tag{3.1}$$

a qual conhecemos por **identidade de Bianchi**. Ela pode ser escrita de um modo mais compacto multiplicando-se a equação (3.1) pelo tensor de Levi-Civita $\eta^{\mu\nu\rho\sigma}$ e contraindo-se os índices α e μ na equação resultante:

$$R^{\alpha\beta\mu\nu}{}_{;\nu} = R^{\mu[\alpha;\beta]}. \tag{3.2}$$

As identidades de Bianchi acima podem ser novamente reescritas em termos do tensor de Weyl, usando-se a definição (1.7) em (3.2) acima:

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = \frac{1}{2}\, R^{\mu[\alpha;\beta]} - \frac{1}{12}\, g^{\mu[\alpha}\, R^{;\beta]}. \tag{3.3}$$

O lado direito da equação acima pode ser redefinido se usarmos as equações de Einstein,

$$T_{\mu\nu} = -R_{\mu\nu} + \frac{1}{2}\, g_{\mu\nu}\, R, \tag{3.4}$$

para obter o seguinte resultado:

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = -\frac{1}{2}\, T^{\mu[\alpha;\beta]} + \frac{1}{6}\, g^{\mu[\alpha}\, T^{;\beta]}. \tag{3.5}$$

Usando-se agora a decomposição do tensor de Weyl em suas partes elétrica e magnética, equações (1.25), bem como a decomposição do tensor momento-energia $T_{\mu\nu}$ em suas partes irredutíveis, equação (1.29), projetamos a equação (3.5), multiplicando-a por produtos de V_μ e $h_{\mu\nu}$. Há quatro projeções independentes para a equação (3.5). São elas:

$$W^{\alpha\beta\mu\nu}{}_{;\nu}\, V_\beta\, V_\mu\, h_\alpha{}^\sigma,$$

$$W^{\alpha\beta\mu\nu}{}_{;\nu}\, \eta^{\sigma\lambda}{}_{\alpha\beta}\, V_\mu\, V_\lambda,$$

$$W^{\alpha\beta\mu\nu}{}_{;\nu}\, h_\nu{}^{(\sigma}\, \eta^{\tau)\lambda}{}_{\alpha\beta}\, V_\lambda;$$

$$W^{\alpha\beta\mu\nu}{}_{;\nu}\, V_\beta\, h_{\mu(\tau}\, h_{\sigma)\alpha}.$$

Um cálculo algébrico relativamente simples, porém bastante longo, nos dá então as quatro equações Quase-Maxwellianas:

Primeira Projeção:

$$\begin{aligned} h^{\alpha\beta}\, h^{\mu\nu}\, E_{\beta\mu;\nu} \;&+\; \eta^\alpha{}_{\beta\mu\nu}\, V^\beta\, H^{\nu\gamma}\, \sigma^\mu{}_\gamma + 3\, H^{\alpha\beta}\, \omega_\beta - \\ &-\; \frac{1}{3}\, h^{\alpha\beta}\, \rho_{,\beta} - \frac{1}{3}\, \theta\, q^\alpha + \frac{1}{2}\, (\sigma^\alpha{}_\beta - 3\, \omega^\alpha{}_\beta)\, q^\beta - \\ &-\; \frac{1}{2}\, \pi^{\alpha\beta}\, a_\beta - \frac{1}{2}\, h^{\alpha\beta}\, \pi_\beta{}^\mu{}_{;\mu} = 0. \end{aligned} \tag{3.6}$$

Segunda Projeção:

$$\begin{aligned} h^{\alpha\beta}\, h^{\mu\nu}\, H_{\beta\mu;\nu} \;&-\; \eta^\alpha{}_{\beta\mu\nu}\, V^\beta\, E^{\nu\gamma}\, \sigma^\mu{}_\gamma - 3\, E^{\alpha\beta}\, \omega_\beta - (\rho + p)\, \omega^\alpha + \\ &+\; \frac{1}{2}\, \eta^{\alpha\beta\mu\nu}\, V_\nu\, q_{\beta;\mu} - \frac{1}{2}\, \eta^{\alpha\beta\mu\nu}\, (\sigma_{\gamma\mu} + \omega_{\gamma\mu})\, \pi^\mu{}_\beta\, V_\nu = 0. \end{aligned} \tag{3.7}$$

Terceira Projeção:

$$h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (H^{\mu\nu})^{\bullet} \;+\; \theta\, H^{\alpha\beta} - \frac{1}{2}\, H_{\mu}{}^{(\alpha}\, h^{\beta)}{}_{\nu}\, V^{\mu;\nu} + \eta^{\alpha\mu\gamma\varepsilon}\, \eta^{\beta\nu\lambda\tau}\, V_{\gamma}\, V_{\lambda}\, H_{\varepsilon\tau} \left(\sigma_{\mu\nu} + \frac{\theta}{3}\, h_{\mu\nu}\right) -$$

$$-\; a_{\mu}\, E_{\nu}{}^{(\alpha}\, \eta^{\beta)\gamma\mu\nu}\, V_{\gamma} + \frac{1}{2}\, E_{\mu}{}^{\nu}{}_{;\gamma}\, h_{\nu}{}^{(\alpha}\, \eta^{\beta)\lambda\gamma}{}_{\mu}\, V_{\lambda} + \frac{3}{4}\, q^{(\alpha}\, \omega^{\beta)} - \frac{1}{2}\, h^{\alpha\beta}\, q^{\mu}\, \omega_{\mu} -$$

$$-\; \frac{1}{4}\, \sigma_{\mu}{}^{(\alpha}\, \eta^{\beta)\nu\mu\lambda}\, V_{\lambda}\, q_{\nu} - \frac{1}{4}\, h^{\mu(\alpha}\, \eta^{\beta)\nu\gamma\lambda}\, V_{\lambda}\, \pi_{\mu\nu;\gamma} = 0. \qquad (3.8)$$

Quarta Projeção:

$$h_{\mu}{}^{\alpha}\, h_{\nu}{}^{\beta}\, (E^{\mu\nu})^{\bullet} \;+\; \theta\, E^{\alpha\beta} - \frac{1}{2}\, E_{\mu}{}^{(\alpha}\, h^{\beta)}{}_{\nu}\, V^{\mu;\nu} + \eta^{\alpha\mu\gamma\varepsilon}\, \eta^{\beta\nu\lambda\tau}\, V_{\gamma}\, V_{\lambda}\, E_{\varepsilon\tau} \left(\sigma_{\mu\nu} + \frac{\theta}{3}\, h_{\mu\nu}\right) +$$

$$+\; a_{\mu}\, H_{\nu}{}^{(\alpha}\, \eta^{\beta)\gamma\mu\nu}\, V_{\gamma} - \frac{1}{2}\, H_{\mu}{}^{\nu}{}_{;\gamma}\, h_{\nu}{}^{(\alpha}\, \eta^{\beta)\lambda\gamma\mu}\, V_{\lambda} - \frac{1}{6}\, h^{\alpha\beta}\, (q^{\mu}{}_{;\mu} - q^{\mu}\, a_{\mu} - \pi^{\mu\nu}\, \sigma_{\mu\nu}) +$$

$$+\; \frac{1}{2}\, (\rho + p)\, \sigma^{\alpha\beta} - \frac{1}{2}\, q^{(\alpha}\, a^{\beta)} + \frac{1}{4}\, h^{\mu(\alpha}\, h^{\beta)\nu}\, q_{\mu;\nu} - \frac{1}{2}\, h_{\mu}{}^{\alpha}\, h_{\nu}{}^{\beta}\, (\pi^{\mu\nu})^{\bullet} -$$

$$-\; \frac{1}{4}\, \pi_{\mu}{}^{(\alpha}\, \sigma^{\beta)\mu} + \frac{1}{4}\, \pi_{\mu}{}^{(\alpha}\, \omega^{\beta)\mu} - \frac{1}{6}\, \theta\, \pi^{\alpha\beta} = 0. \qquad (3.9)$$

Temos, ainda, duas leis de conservação que representam as projeções da divergência do tensor $T_{\mu\nu}$ (nula, conforme se constata facilmente das equações de Einstein e das identidades de Bianchi contraídas) sobre os subespaços paralelo e ortogonal:

Primeira Lei de Conservação:

$$T^{\mu\nu}{}_{;\nu}\, V_{\mu} = 0 \Longrightarrow$$

$$\Longrightarrow \rho^{\bullet} + (\rho + p)\,\theta + (q^{\mu})^{\bullet}\, V_{\mu} + q^{\mu}{}_{;\mu} - \pi^{\mu\nu}\,\sigma_{\mu\nu} - \frac{1}{3}\,\theta\,\pi^{\mu\nu}\,h_{\mu\nu} = 0. \qquad (3.10)$$

<u>*Segunda Lei de Conservação:*</u>

$$T^{\mu\nu}{}_{;\nu}\,h_{\mu\alpha} = 0 \Longrightarrow$$

$$\Longrightarrow (\rho + p)\,a_{\alpha} \;-\; p_{,\mu}\,h^{\mu}{}_{\alpha} + (q_{\mu})^{\bullet}\,h^{\mu}{}_{\alpha} + \theta\,q_{\alpha} +$$

$$+\;\; q^{\mu}\,\sigma_{\mu\alpha} + \frac{1}{3}\,\theta\,q^{\mu}\,h_{\mu\alpha} + q^{\mu}\,\omega_{\mu\alpha} + \pi_{\alpha}{}^{\mu}{}_{;\mu}$$

$$+\;\; \pi^{\mu\nu}\,\sigma_{\mu\nu}\,V_{\alpha} + \frac{1}{3}\,\theta\,\pi^{\mu\nu}\,h_{\mu\nu}\,V_{\alpha} = 0. \qquad (3.11)$$

A definição do tensor de curvatura de Riemann, escrita para um vetor S_{α} qualquer como:

$$S_{\mu;[\alpha;\beta]} = -R^{\nu}{}_{\mu\alpha\beta}\,S_{\nu},$$

é o ponto de partida para que obtenhamos as restantes seis equações que completam o sistema QM com o qual trabalharemos aqui. Destas, três são equações dinâmicas para as quantidades cinemáticas (expansão, *shear* e rotação); as outras três são equações de vínculo envolvendo as quantidades cinemáticas. Os passos algébricos para a sua obtenção podem ser encontrados nas referências [17, 40]. Aqui, limitar-nos-emos a apresentá-las em sua forma final:

<u>*Equação de Raychaudhuri:*</u>

$$\theta^{\bullet} + \frac{1}{3}\,\theta^{2} + \sigma^{\alpha\beta}\,\sigma_{\alpha\beta} + \omega^{\alpha\beta}\,\omega_{\alpha\beta} - a^{\alpha}{}_{;\alpha} + \frac{1}{2}\,(\rho + 3\,p) = 0. \qquad (3.12)$$

Equação Dinâmica para o "shear":

$$h^{\mu}{}_{\alpha}\, h^{\nu}{}_{\beta}\, (\sigma_{\mu\nu})^{\bullet} - \frac{1}{3}\, h_{\alpha\beta}\, (\omega^{\mu\nu}\, \omega_{\mu\nu} + \sigma^{\mu\nu}\, \sigma_{\mu\nu} - a^{\mu}{}_{;\mu}) + a_{\alpha}\, a_{\beta} -$$

$$- \frac{1}{2}\, h^{\mu}{}_{\alpha}\, h^{\nu}{}_{\beta}\, a_{(\mu;\nu)} + \frac{2}{3}\,\theta\, \sigma_{\alpha\beta} + \sigma_{\alpha\mu}\, \sigma^{\mu}{}_{\beta} + \omega_{\alpha\mu}\, \omega^{\mu}{}_{\beta} +$$

$$+ E_{\alpha\beta} + \frac{1}{2}\, \pi_{\alpha\beta} - \frac{1}{6}\, (\rho + 3\, p)\, h_{\alpha\beta} = 0. \tag{3.13}$$

Equação Dinâmica para a Rotação:

$$h^{\mu}{}_{\alpha}\, h^{\nu}{}_{\beta}\, (\omega_{\mu\nu})^{\bullet} - \frac{1}{2}\, h^{\mu}{}_{\alpha}\, h^{\nu}{}_{\beta}\, a_{[\mu;\nu]} + \frac{2}{3}\,\theta\, \omega_{\alpha\beta} + \sigma_{\mu[\alpha}\, \omega^{\mu}{}_{\beta]} = 0. \tag{3.14}$$

Primeira Equação de Vínculo:

$$\frac{2}{3}\,\theta_{,\mu}\, h^{\mu}{}_{\alpha} - (\sigma^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu})_{;\mu}\; h^{\nu}{}_{\alpha} - a^{\mu}\, (\sigma_{\mu\alpha} + \omega_{\mu\alpha}) + q_{\mu}\, h^{\mu}{}_{\alpha} = 0. \tag{3.15}$$

Segunda Equação de Vínculo:

$$\omega^{\alpha}{}_{;\alpha} + 2\, \omega^{\alpha}\, a_{\alpha} = 0. \tag{3.16}$$

Terceira Equação de Vínculo:

$$H_{\alpha\beta} + \frac{1}{2}\, h_{\alpha}{}^{\mu}\, h_{\beta}{}^{\nu}\, \eta_{\mu}{}^{\gamma\lambda\tau}\, V_{\tau}\, (\sigma_{\nu\gamma} + \omega_{\nu\gamma})_{;\lambda} - a_{(\alpha}\, \omega_{\beta)} = 0. \tag{3.17}$$

3.3 Equações Quase-Maxwellianas para o Modelo de Kasner

Considerando que o modelo de Kasner tem somente três quantidades não nulas — a parte elétrica do tensor de Weyl, o *shear* e a expansão — é fácil ver que o sistema de equações QM, (3.6) a (3.17), reduz-se no *background* a apenas três equações: as dinâmicas para $E_{\alpha\beta}$, $\sigma_{\alpha\beta}$ e θ. Todas as demais são identicamente nulas, como se pode facilmente constatar por uma inspeção direta. Neste caso, temos então:

Equação Dinâmica do "shear":

$$(\sigma_{\alpha\beta})^{\bullet} + E_{\alpha\beta} + \frac{2}{3}\,\theta\,\sigma_{\alpha\beta} - \frac{1}{3}\,h_{\alpha\beta}\,\sigma^{\mu\nu}\,\sigma_{\mu\nu} + \sigma_{\alpha\mu}\,\sigma^{\mu}{}_{\beta} = 0. \tag{3.18}$$

Quarta Projeção:

$$(E_{\alpha\beta})^{\bullet} + 3\,\theta\,E_{\alpha\beta} + \frac{3}{2}\,\sigma^{\mu}{}_{(\alpha}\,E_{\beta)\mu} - h_{\alpha\beta}\,\sigma^{\mu\nu}\,E_{\mu\nu} = 0. \tag{3.19}$$

Equação de Raychaudhuri:

$$\theta^{\bullet} + \frac{1}{3}\,\theta^2 + 2\,\sigma^2 = 0. \tag{3.20}$$

3.4 Equações Quase-Maxwellianas Perturbadas

Neste ponto já podemos perturbar as equações QM. Para tanto, escreveremos todas as quantidades perturbadas na forma usual [17, 10]:

$$A_{(perturbada)} = A_{(background)} + (\delta A)\,.$$

Entretanto, o fato de existirem quantidades tensoriais não nulas no *background* de

Kasner exige que façamos uma modificação do método, diferentemente do que foi feito no caso do modelo FRW [10, 11, 12]. Dado que temos três quantidades não nulas no modelo de Kasner, vamos substituí-las por três outras quantidades "artificiais", construídas a partir das quantidades originais, mas que sejam nulas no *background* (constituindo-se, portanto, em "boas" quantidades de trabalho, no sentido do lema de Stewart [15]). Tal procedimento é similar àquele adotado, entre outros, por Bardeen [14].

Uma opção direta se apresenta ao examinarmos as equações QM escritas para o *background* de Kasner, (3.18) a (3.20) — o que faremos a seguir.

Se tomarmos a equação (3.18) e substituirmos o termo em $(\sigma_{\alpha\beta})^{\bullet}$, que sabemos ser dado em termos de $\sigma_{\alpha\beta}$ (pelo terceiro resultado em (1.39)), definimos a seguinte quantidade tensorial auxiliar:

$$X_{\alpha\beta} \equiv E_{\alpha\beta} - \frac{1}{3}\,\theta\,\sigma_{\alpha\beta} - \frac{2\sigma^2}{3}\,h_{\alpha\beta} + \sigma_{\alpha\mu}\,\sigma^{\mu}{}_{\beta}. \tag{3.21}$$

Tomando, agora, a equação (3.19) e substituindo o termo $(E_{\alpha\beta})^{\bullet}$ pelo resultado (1.43), obtemos a seguinte nova quantidade tensorial:

$$Y_{\alpha\beta} \equiv \theta\,E_{\alpha\beta} + \frac{3}{2}\,\sigma_{\mu(\alpha}\,E^{\mu}{}_{\beta)} - h_{\alpha\beta}\,\sigma^{\mu\nu}\,E_{\mu\nu}. \tag{3.22}$$

Por último, vamos considerar a equação de Raychaudhuri, (3.20). Se substituirmos nela os resultados (1.37) e o primeiro resultado em (1.39), obtemos a terceira quantidade necessária — desta vez, uma escalar:

$$W \equiv 2\sigma^2 - \frac{2}{3}\,\theta^2. \tag{3.23}$$

É fácil provar-se que as novas quantidades — $X_{\alpha\beta}$, $Y_{\alpha\beta}$ e W — são, de fato, "boas" quantidades de trabalho, pois são todas elas nulas no *background* de Kasner. Da primeira das equações (1.39), vemos imediatamente que

$$W = 0; \tag{3.24}$$

quanto às outras, faremos uso de um cálculo direto, a seguir.

$$X_{\alpha\beta} = 0, \;\; \forall \;\; \alpha \neq \beta.$$

$$\begin{aligned}
X_{11} &= E_{11} - \frac{1}{3}\,\theta\,\sigma_{11} - \frac{2\sigma^2}{3}\,g_{11} + \sigma_{11}\,\sigma^1{}_1 \\
&= g_{11}\left[E^1{}_1 - \frac{1}{3}\,\theta\,\sigma^1{}_1 - \frac{2\sigma^2}{3} + \sigma^1{}_1\,\sigma^1{}_1\right] \\
&= -t^{2p_1}\left[p_1\,(1-p_1)\,\theta^2 - \left(\frac{1}{3}\,\theta\right)^2\,(3\,p_1 - 1) - \frac{2}{9}\,\theta^2 + \frac{1}{9}\,\theta^2\,(3\,p_1 - 1)^2\right] \\
&= -\theta^2\,t^{2p_1}\left[p_1 - (p_1)^2 - \frac{1}{3}\,p_1 + \frac{1}{9} - \frac{2}{9} + (p_1)^2 - \frac{2}{3}\,p_1 + \frac{1}{9}\right] \\
&= -t^{2(p_1-1)}\left[p_1 - p_1 + \frac{2}{9} - \frac{2}{9}\right] \\
&= 0,
\end{aligned}$$

e X_{22} e X_{33} têm o mesmo resultado.

$$Y_{\alpha\beta} = 0, \;\; \forall \;\; \alpha \neq \beta$$

$$\begin{aligned}
Y_{11} &= \theta\,E_{11} + 3\,\sigma_{11}\,E^1{}_1 - g_{11}\,\sigma^\mu{}_\nu\,E^\nu{}_\mu \\
&= g_{11}\left[\theta\,E^1{}_1 + 3\,\sigma^1{}_1\,E^1{}_1 - \sigma^\mu{}_\nu\,E^\nu{}_\mu\right] \\
&= E_{11}\left[\theta + 3\,\sigma^1{}_1\right] - g_{11}\,\left(\sigma^1{}_1\,E^1{}_1 + \sigma^2{}_2\,E^2{}_2 + \sigma^3{}_3\,E^3{}_3\right)
\end{aligned}$$

$$= \quad g_{11} \left\{ E^{1}{}_{1} \left[\theta + 3\,\sigma^{1}{}_{1} \right] - (\sigma^{\mu}{}_{1}\, E^{\nu}{}_{\mu}) \right\},$$

e, analogamente, temos:

$$Y^{22} = g_{22} \left\{ E^{2}{}_{2} \left[\theta + 3\,\sigma^{2}{}_{2} \right] - (\sigma^{\mu}{}_{\nu}\, E^{\nu}{}_{\mu}) \right\}$$

$$Y^{33} = g_{33} \left\{ E^{3}{}_{3} \left[\theta + 3\,\sigma^{3}{}_{3} \right] - (\sigma^{\mu}{}_{\nu}\, E^{\nu}{}_{\mu}) \right\}.$$

Calculando, agora, a quantidade escalar $(\sigma^{\mu}{}_{\nu}\, E^{\nu}{}_{\mu})$, obtemos:

$$\begin{aligned}
(\sigma^{\mu}{}_{\nu}\, E^{\nu}{}_{\mu}) &= \frac{1}{3}\,\theta^{3} \sum_{i=1}^{3} p_i\, (3\,p_i - 1)\,(1 - p_i) \\
&= \frac{1}{3}\,\theta^{3} \sum_{i=1}^{3} p_i \left[3\,p_i - 1 - 3\,(p_i)^2 + p_i \right] \\
&= \frac{1}{3}\,\theta^{3} \sum_{i=1}^{3} p_i \left[-3\,(p_i)^2 + 4\,p_i - p_i \right] \\
&= \frac{1}{3}\,\theta^{3} \left\{ -3 \left[(p_1)^3 + (p_2)^3 + (p_3)^3 \right] + 3 \right\} \\
&= \theta^{3} \left\{ 1 - \left[(p_1)^3 + (p_2)^3 + (p_3)^3 \right] \right\},
\end{aligned}$$

a qual não é invariável para qualquer solução (p_1, p_2, p_3). Vamos, portanto, calcular estas quantidades para diferentes soluções de Kasner.

1)Solução de Kasner $\longrightarrow$ $(p_1, p_2, p_3) = (2/3, 2/3, -1/3)$

$$
\begin{aligned}
(\sigma^{\mu}{}_{\nu}\, E^{\nu}{}_{\mu}) &= \theta^3 \left\{1 - \left[2\left(\frac{2}{3}\right)^3 + \left(-\frac{1}{3}\right)^3\right]\right\} \\
&= \theta^3 \left\{1 - \left[2\,\frac{8}{27} - \frac{1}{27}\right]\right\} \\
&= \theta^3 \left[1 - \frac{15}{27}\right] \\
&= \theta^3\, \frac{12}{27} \\
&= \frac{4}{9}\,\theta^3.
\end{aligned}
$$

Logo, temos, para cada componente de $Y_{\alpha\beta}$,

$$
\begin{aligned}
Y_{11} &= \theta^3\, g_{11} \left\{\frac{2}{3}\left(1 - \frac{2}{3}\right)[1 + (2-1)] - \frac{4}{9}\right\} \\
&= \theta^3\, g_{11} \left[\frac{2}{9}\,2 - \frac{4}{9}\right] \\
&= 0,
\end{aligned}
$$

$$
Y_{22} \equiv Y_{11} = 0,
$$

$$
\begin{aligned}
Y_{33} &= \theta^3\, g_{33} \left\{-\frac{1}{3}\left(1 + \frac{1}{3}\right)[1 + (-1-1)] - \frac{4}{9}\right\} \\
&= \theta^3\, g_{33} \left[\frac{4}{9} - \frac{4}{9}\right]
\end{aligned}
$$

$$= \quad 0,$$

2)Solução de Milne $\longrightarrow (p_1,p_2,p_3) = (0,0,1)$

A variável $Y_{\mu\nu}$ é trivialmente nula, já que a parte elétrica do tensor de Weyl é zero para esta solução particular, conforme a equação (1.48).

3)Um Caso Arbitrário

Supondo um caso sem plano de isotropia, escolhemos agora um valor arbitrário para p_1:

$$p_1 = 1/2,$$

donde, das equações (1.34) e (1.35), obtemos imediatamente o seguinte resultado:

$$p_2 = \left(1 \pm \sqrt{5}\right)/4,$$

$$p_3 = \left(1 \mp \sqrt{5}\right)/4,$$

o qual é também uma possível solução para a métrica de Kasner, dado que as equações (1.33) são válidas. Repetindo os cálculos feitos para os dois casos anteriores, vem:

$$\begin{aligned}
(\sigma^{\mu}{}_{\nu} E^{\nu}{}_{\mu}) &= \theta^3 \left\{1 - \left[\frac{1}{8} + \frac{1}{64}\left[\left(1 \pm \sqrt{5}\right)^3 + \left(1 \mp \sqrt{5}\right)^3\right]\right]\right\} \\
&= \theta^3 \left\{1 - \left[\frac{1}{8} + \frac{1}{64}\left(16 \pm 8\sqrt{5} + 16 \mp 8\sqrt{5}\right)\right]\right\} \\
&= \theta^3 \left\{1 - \left[\frac{1}{8} + \frac{1}{2}\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \theta^3 \left[1 - \frac{5}{8}\right] \\
&= \frac{3}{8}\,\theta^3.
\end{aligned}
$$

$$
\begin{aligned}
Y_{11} &= \frac{1}{4}\,\theta^2 \left[\theta + \theta\left(\frac{3}{2} - 1\right)\right] - \frac{3}{8}\,\theta^3 \\
&= \theta^3 \left\{\frac{1}{4}\left[1 + \frac{1}{2}\right] - \frac{3}{8}\right\} \\
&= \theta^3 \left(\frac{3}{8} - \frac{3}{8}\right) \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
Y_{22} &= \theta^3 \left\{\frac{1}{4}\left(1 \pm \sqrt{5}\right)\left(1 - \frac{1}{4}\left(1 \pm \sqrt{5}\right)\right)\left[1 + \frac{1}{4}\left(-1 \pm 3\sqrt{5}\right)\right] - \frac{3}{8}\right\} \\
&= \frac{3}{8}\,\theta^3 \left\{\frac{1}{8}\left(-2 \mp 2\sqrt{5} \pm 2\sqrt{5} + 10\right) - 1\right\} \\
&= \frac{3}{8}\,\theta^3\,(1 - 1) \\
&= 0,
\end{aligned}
$$

$$
Y_{33} = \theta^3 \left\{\frac{1}{4}\left(1 \mp \sqrt{5}\right)\left(1 - \frac{1}{4}\left(1 \mp \sqrt{5}\right)\right)\left[1 + \frac{3}{4}\left(1 \mp \sqrt{5}\right) - 1\right] - \frac{3}{8}\right\}
$$

$$
\begin{aligned}
&= \theta^3 \left\{ \frac{3}{64} \left(1 \mp \sqrt{5}\right)^2 \left(3 \pm \sqrt{5}\right) - \frac{3}{8} \right\} \\
&= \frac{3}{8} \theta^3 \left\{ \frac{1}{8} \left(3 \pm \sqrt{5}\right) \left(6 \mp 2\sqrt{5}\right) - 1 \right\} \\
&= \frac{3}{8} \theta^3 \left\{ \frac{1}{8} \left(18 \mp 6\sqrt{5} \pm 6\sqrt{5} - 10\right) - 1 \right\} \\
&= \frac{3}{8} \theta^3 \left(1 - 1\right) \\
&= 0.
\end{aligned}
$$

Conclui-se daí que, dado qualquer conjunto de p_i que satisfaça à condição de Kasner, a variável $Y_{\alpha\beta}$ será sempre nula — e, portanto, uma "boa"quantidade de trabalho. Nosso próximo passo será, portanto, o de obter um sistema dinâmico fechado nas quantidades de trabalho disponíveis. Este procedimento é descrito na próxima seção.

3.5 Obtenção de um Sistema Dinâmico

Devemos, agora, escrever a partir das equações QM um sistema dinâmico que envolva somente o conjunto das "boas"variáveis, a saber:

$$
\mathcal{M} = \left\{ X_{\alpha\beta} \, , \, Y_{\alpha\beta} \, , \, W \, , \, H_{\alpha\beta} \, , \, \pi_{\alpha\beta} \, , \, \rho \, , \, q_\alpha \, , \, a_\alpha \, , \, \omega_\alpha \right\} . \tag{3.25}
$$

Destas oito variáveis, apenas as seis últimas são observáveis do sistema — e, como tais, já possuem uma dinâmica descrita pelas equações QM. Note-se que tomaremos daqui por diante, para uma maior simplificação de nossos cálculos, a relação entre a pressão p e a densidade de matéria ρ como sendo

$$p = \lambda\, \rho, \tag{3.26}$$

com $\lambda = c^{\underline{te}}$, mesmo após a perturbação, a exemplo do feito anteriormente no estudo das perturbações em FRW [10, 11, 12]. Isto explica o não aparecimento da variável p no conjunto mínimo fechado $\mathcal{M}$ acima[1]. Entretanto, um exame de $\mathcal{M}$ mostra que as três primeiras variáveis, construídas a partir de quantidades gauge-dependentes, não têm, ainda, uma dinâmica explícita. Nosso procedimento, então, será o seguinte:

1. Derivar, projetando no tempo, as definições de $X_{\alpha\beta}$, $Y_{\alpha\beta}$ e W, substituindo os termos em $(E_{\alpha\beta})^{\bullet}$, $(\sigma_{\alpha\beta})^{\bullet}$ e $\theta^{\bullet}$ pelas respectivas equações Quase-Maxwellianas.

2. Reescrever os termos gauge-dependentes nas novas variáveis $X_{\alpha\beta}$, $Y_{\alpha\beta}$ e W.

3. Reescrever as equações QM restantes em termos das variáveis do conjunto $\mathcal{M}$.

Isto posto, procedemos ao primeiro e segundo passos, obtendo as equações dinâmicas para as novas variáveis de trabalho, escritas em termos das variáveis de $\mathcal{M}$:

<u>*I)Equação Dinâmica para $X_{\alpha\beta}$:*</u>

$$\begin{aligned}
h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (X_{\alpha\beta})^{\bullet} &= h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu} \left\{ (E_{\alpha\beta})^{\bullet} - \frac{1}{3}\,\theta\, (\sigma_{\alpha\beta})^{\bullet} - \frac{1}{3}\,\theta^{\bullet}\, \sigma_{\alpha\beta} - \frac{2\sigma^2}{3}\, (h_{\alpha\beta})^{\bullet} - \right. \\
&\left. -\frac{1}{3}\, h_{\alpha\beta}\, \sigma^{\gamma\lambda}\, (\sigma_{\gamma\lambda})^{\bullet} - \frac{1}{3}\, h_{\alpha\beta}\, \sigma_{\gamma\lambda} \left(\sigma^{\gamma\lambda}\right)^{\bullet} + (\sigma_{\alpha\gamma})^{\bullet}\, \sigma^{\gamma}{}_{\beta} + \sigma_{\alpha\gamma}\, (\sigma^{\gamma}{}_{\beta})^{\bullet} \right\} \\
&= \left[h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (E_{\alpha\beta})^{\bullet}\right] - \frac{1}{3}\,\theta \left[h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (\sigma_{\alpha\beta})^{\bullet}\right] - \frac{1}{3}\,\sigma_{\mu\nu}\, [\theta^{\bullet}] - \frac{2\sigma^2}{3}\, h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (h_{\alpha\beta})^{\bullet} - \\
&-\frac{1}{3}\, h_{\mu\nu}\, \sigma^{\alpha\beta} \left[h^{\gamma}{}_{\alpha}\, h^{\lambda}{}_{\beta}\, (\sigma_{\gamma\lambda})^{\bullet}\right] - \frac{1}{3}\, h_{\mu\nu}\, \sigma_{\alpha\beta} \left[h^{\alpha}{}_{\gamma}\, h^{\beta}{}_{\lambda} \left(\sigma^{\gamma\lambda}\right)^{\bullet}\right] + \\
&+\sigma^{\lambda}{}_{\nu} \left[h^{\alpha}{}_{\mu}\, h^{\gamma}{}_{\lambda}\, (\sigma_{\alpha\gamma})^{\bullet}\right] + \sigma_{\mu\lambda} \left[h^{\lambda}{}_{\gamma}\, h^{\beta}{}_{\nu}\, (\sigma^{\gamma}{}_{\beta})^{\bullet}\right]
\end{aligned}$$

[1] Esta relação será sempre considerada como válida ao longo deste trabalho.

$$
\begin{aligned}
= \; & \left[h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (E_{\alpha\beta})^{\bullet}\right] - \frac{1}{3}\,\theta\, \left[h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (\sigma_{\alpha\beta})^{\bullet}\right] - \frac{1}{3}\,\sigma_{\mu\nu}\, [\theta^{\bullet}] - \\
& -\frac{2}{3}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, \left[h^{\gamma}{}_{\alpha}\, h^{\lambda}{}_{\beta}\, (\sigma_{\gamma\lambda})^{\bullet}\right] + \sigma^{\gamma}{}_{(\mu}\, \left[h^{\alpha}{}_{\nu)}\, h^{\beta}{}_{\gamma}\, (\sigma_{\alpha\beta})^{\bullet}\right] + \\
& + termos\ em\ (h)^{\bullet}.
\end{aligned}
$$

Os termos em $(h_{\alpha\beta})^{\bullet}$ não contribuem para a perturbação que será efetuada posteriormente, já que a métrica perturbada ainda é Riemanniana e, portanto, temos que a derivada covariante do tensor métrico perturbado é nula. Termos deste tipo serão sempre desconsiderados em nossos cálculos daqui por diante.

Substituindo os termos entre colchetes na equação acima pelas equações QM correspondentes, (3.9), (3.12) e (3.13), e coletando no lado esquerdo todos os termos escritos nas "boas" variáveis de trabalho, obtemos:

$$
\begin{aligned}
h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (X_{\alpha\beta})^{\bullet} \; & - \; \frac{1}{2}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu)}{}^{\lambda}\, H_{\beta\lambda;\alpha} - \frac{1}{6}\, h_{\mu\nu}\, h^{\alpha\beta}\, q_{\alpha;\beta} + \\
& + \; \frac{1}{4}\, h^{\alpha}{}_{(\mu}\, h^{\beta}{}_{\nu)}\, q_{\alpha;\beta} - \frac{1}{9}\,\theta\, h_{\mu\nu}\, h^{\alpha\beta}\, a_{\alpha;\beta} + \frac{1}{6}\,\theta\, h^{(\alpha}{}_{\mu}\, h^{\beta)}{}_{\nu}\, a_{\alpha;\beta} + \\
& + \; \sigma_{\mu\nu}\, h^{\alpha\beta}\, a_{\alpha;\beta} + \frac{2}{3}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, a_{\alpha;\beta} - \frac{1}{2}\, h^{\alpha}{}_{(\mu}\, \sigma^{\beta}{}_{\nu)}\, a_{(\alpha;\beta)} + \frac{1}{3}\,\sigma_{\mu\nu}\,\rho - \\
& - \; \frac{1}{2}\, h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (\pi_{\alpha\beta})^{\bullet} - \frac{1}{6}\,\theta\,\pi_{\mu\nu} - \frac{1}{6}\, h_{\mu\nu}\,\sigma^{\alpha\beta}\,\pi_{\alpha\beta} + \frac{1}{4}\,\sigma^{\alpha}{}_{(\mu}\,\pi_{\nu)\alpha} \\
\\
& = \; -\frac{2}{3}\,\theta\, E_{\mu\nu} + \frac{1}{2}\, E^{\alpha}{}_{(\mu}\,\sigma_{\nu)\alpha} - \frac{1}{3}\, h_{\mu\nu}\,\sigma^{\alpha\beta}\, E_{\alpha\beta} + \\
& \quad\; +\frac{1}{3}\,\theta^{2}\,\sigma_{\mu\nu} + \left(2\sigma^{2}\right)\,\sigma_{\mu\nu} + \theta\,\left(\frac{2\sigma^{2}}{3}\right)\, h_{\mu\nu} - \\
& \quad\; -\theta\,\sigma_{\mu\alpha}\,\sigma^{\alpha}{}_{\nu} + \frac{2}{3}\, h_{\mu\nu}\,\sigma^{\alpha\beta}\,\sigma_{\alpha\gamma}\,\sigma^{\gamma}{}_{\beta} - 2\,\sigma^{\alpha\beta}\,\sigma_{\alpha\mu}\,\sigma_{\beta\nu} \\
\\
& \equiv \; LD.
\end{aligned}
$$

Resta-nos, então, escrever LD — o lado direito da equação acima — em termos das quantidades $X_{\alpha\beta}$, $Y_{\alpha\beta}$ e W, usando as equações (3.21), (3.22) e (3.23):

$$
\begin{aligned}
LD \;\equiv\;& -\frac{2}{3}\,\theta\,E_{\mu\nu} + \frac{1}{2}\,E^{\alpha}{}_{(\mu}\,\sigma_{\nu)\alpha} - \frac{1}{3}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,E_{\alpha\beta} + \frac{1}{3}\,\theta^{2}\,\sigma_{\mu\nu} + \left(2\sigma^{2}\right)\,\sigma_{\mu\nu} + \\
& +\theta\,\left(\frac{2\sigma^{2}}{3}\right)\,h_{\mu\nu} - \theta\,\sigma_{\mu\alpha}\,\sigma^{\alpha}{}_{\nu} + \frac{2}{3}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,\sigma_{\alpha\gamma}\,\sigma^{\gamma}{}_{\beta} - 2\,\sigma^{\alpha\beta}\,\sigma_{\alpha\mu}\,\sigma_{\beta\nu}
\end{aligned}
$$

$$
\begin{aligned}
=\;& -\frac{5}{3}\,\theta\,X_{\mu\nu} + \theta\,E_{\mu\nu} + \frac{1}{2}\,E^{\alpha}{}_{(\mu}\,\sigma_{\nu)\alpha} - \frac{1}{3}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,E_{\alpha\beta} - \\
& -\frac{2}{9}\,\theta^{2}\,\sigma^{\mu\nu} + \left(2\sigma^{2}\right)\,\sigma_{\mu\nu} - \frac{2}{3}\,\theta\,\left(\frac{2\sigma^{2}}{3}\right)\,h_{\mu\nu} + \\
& +\frac{2}{3}\,\theta\,\sigma_{\mu\alpha}\,\sigma^{\alpha}{}_{\nu} + \frac{2}{3}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,\sigma_{\alpha\gamma}\,\sigma^{\gamma}{}_{\beta} - 2\,\sigma^{\alpha\beta}\,\sigma_{\alpha\mu}\,\sigma_{\beta\nu}
\end{aligned}
$$

$$
\begin{aligned}
=\;& -\frac{5}{3}\,\theta\,X_{\mu\nu} + Y_{\mu\nu} - E^{\alpha}{}_{(\mu}\,\sigma_{\nu)\alpha} + \frac{2}{3}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,E_{\alpha\beta} - \frac{2}{9}\,\theta^{2}\,\sigma_{\mu\nu} + \\
& + \left(2\sigma^{2}\right)\,\sigma_{\mu\nu} - \frac{2}{3}\,\theta\,\left(\frac{2\sigma^{2}}{3}\right)\,h_{\mu\nu} + \frac{2}{3}\,\theta\,\sigma_{\mu\alpha}\,\sigma^{\alpha}{}_{\nu} + \\
& +\frac{2}{3}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,\sigma_{\alpha\gamma}\,\sigma^{\gamma}{}_{\beta} - 2\,\sigma^{\alpha\beta}\,\sigma_{\alpha\mu}\,\sigma_{\beta\nu}
\end{aligned}
$$

$$
\begin{aligned}
=\;& -\frac{5}{3}\,\theta\,X_{\mu\nu} - \sigma^{\alpha}{}_{(\mu}\,X_{\nu)\alpha} + Y_{\mu\nu} + \frac{1}{3}\,\left[2\sigma^{2} - \frac{2}{3}\,\theta^{2}\right]\,\sigma_{\mu\nu} + \\
& +\frac{2}{3}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,E_{\alpha\beta} - \frac{2}{3}\,\theta\,\left(\frac{2\sigma^{2}}{3}\right)\,h_{\mu\nu} + \frac{2}{3}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,\sigma_{\alpha\gamma}\,\sigma^{\gamma}{}_{\beta}
\end{aligned}
$$

$$
\begin{aligned}
=\;& -\frac{5}{3}\,\theta\,X_{\mu\nu} - \sigma^{\alpha}{}_{(\mu}\,X_{\nu)\alpha} + Y_{\mu\nu} + \frac{1}{3}\sigma_{\mu\nu}\,W + \\
& +\frac{2}{3}\,h_{\mu\nu}\,\left[\sigma^{\alpha\beta}\,E_{\alpha\beta} - \theta\,\left(\frac{2\sigma^{2}}{3}\right) + \sigma^{\alpha\beta}\,\sigma_{\alpha\gamma}\,\sigma^{\gamma}{}_{\beta}\right]
\end{aligned}
$$

$$
\begin{aligned}
=\;& -\frac{5}{3}\,\theta\,X_{\mu\nu} - \sigma^{\alpha}{}_{(\mu}\,X_{\nu)\alpha} + \frac{2}{3}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,X_{\alpha\beta} + Y_{\mu\nu} + \frac{1}{3}\,\sigma_{\mu\nu}\,W + \\
& +\frac{2}{3}\,h_{\mu\nu}\,\left[\sigma^{\alpha\beta}\,\left(\frac{1}{3}\,\theta\,\sigma_{\alpha\beta} + \frac{2\sigma^{2}}{3} - \sigma_{\alpha\gamma}\,\sigma^{\gamma}{}_{\beta}\right) - \theta\,\left(\frac{2\sigma^{2}}{3}\right) + \sigma^{\alpha\beta}\,\sigma_{\alpha\gamma}\,\sigma^{\gamma}{}_{\beta}\right]
\end{aligned}
$$

$$
= -\frac{5}{3}\,\theta\,X_{\mu\nu} - \sigma^{\alpha}{}_{(\mu}\,X_{\nu)\alpha} + \frac{2}{3}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,X_{\alpha\beta} + Y_{\mu\nu} + \frac{1}{3}\,\sigma_{\mu\nu}\,W.
$$

E, portanto, a equação dinâmica para $X_{\alpha\beta}$ se escreve como:

$$
\begin{aligned}
h^{\alpha}{}_{\mu}\, h^{\nu}{}_{\nu}\, (X_{\alpha\beta})^{\bullet} \;&+\; \frac{5}{3}\,\theta\, X_{\mu\nu} + \sigma^{\alpha}{}_{(\mu}\, X_{\nu)\alpha} - \frac{2}{3}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, X_{\alpha\beta} - Y_{\mu\nu} - \\
&-\; \frac{1}{3}\,\sigma_{\mu\nu}\, W - \frac{1}{2}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu)}{}^{\lambda}\, H_{\beta\lambda;\alpha} - \frac{1}{6}\, h_{\mu\nu}\, h^{\alpha\beta}\, q_{\alpha;\beta} + \\
&+\; \frac{1}{4}\, h^{\alpha}{}_{(\mu}\, h^{\beta}{}_{\nu)}\, q_{\alpha;\beta} - \frac{1}{9}\,\theta\, h_{\mu\nu}\, h^{\alpha\beta}\, a_{\alpha;\beta} + \frac{1}{6}\,\theta\, h^{(\alpha}{}_{\mu}\, h^{\beta)}{}_{\nu}\, a_{\alpha;\beta} + \\
&+\; \sigma_{\mu\nu}\, h^{\alpha\beta}\, a_{\alpha;\beta} + \frac{2}{3}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, a_{\alpha;\beta} - \frac{1}{2}\, h^{\alpha}{}_{(\mu}\, \sigma^{\beta}{}_{\nu)}\, a_{(\alpha;\beta)} + \\
&+\; \frac{1}{3}\,\sigma_{\mu\nu}\,\rho - \frac{1}{2}\, h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (\pi_{\alpha\beta})^{\bullet} - \frac{1}{6}\,\theta\,\pi_{\mu\nu} - \frac{1}{6}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, \pi_{\alpha\beta} + \\
&+\; \frac{1}{4}\,\sigma^{\alpha}{}_{(\mu}\, \pi_{\nu)\alpha} \\
&=\; 0. \qquad (3.27)
\end{aligned}
$$

<u>II)Equação Dinâmica para $Y_{\alpha\beta}$:</u>

Repetindo o mesmo procedimento executado acima, vem:

$$
\begin{aligned}
h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (Y_{\alpha\beta})^{\bullet} \;=\;& \theta\, \left[h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (E_{\alpha\beta})^{\bullet}\right] + E_{\mu\nu}\, [\theta^{\bullet}] + \frac{3}{2}\, h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, \sigma_{\gamma(\alpha}\, \left(E_{\beta)}{}^{\gamma}\right)^{\bullet} + \\
&+\frac{3}{2}\, h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, E^{\gamma}{}_{(\alpha}\, \left(\sigma_{\beta)\gamma}\right)^{\bullet} - h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, h_{\alpha\beta}\, \sigma^{\gamma}{}_{\lambda}\, \left(E^{\lambda}{}_{\gamma}\right)^{\bullet} - \\
&-h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, h_{\alpha\beta}\, E^{\gamma}{}_{\lambda}\, \left(\sigma^{\lambda}{}_{\gamma}\right)^{\bullet} - h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, \sigma^{\gamma\lambda}\, E_{\gamma\lambda}\, (h_{\alpha\beta})^{\bullet} \\
\\
=\;& \theta\, \left[h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (E_{\alpha\beta})^{\bullet}\right] + E_{\mu\nu}\, [\theta^{\bullet}] + \frac{3}{2}\, \sigma^{\gamma}{}_{(\mu}\, \left[h^{\beta}{}_{\nu)}\, h^{\lambda}{}_{\gamma}\, (E_{\beta\lambda})^{\bullet}\right] + \\
&+\frac{3}{2}\, E^{\gamma}{}_{(\mu}\, \left[h^{\alpha}{}_{\nu)}\, h^{\beta}{}_{\gamma}\, (\sigma_{\alpha\beta})^{\bullet}\right] - h_{\mu\nu}\, \sigma^{\gamma\lambda}\, \left[h^{\alpha}{}_{\gamma}\, h^{\beta}{}_{\lambda}\, (E_{\alpha\beta})^{\bullet}\right] - \\
&-h_{\mu\nu}\, E^{\gamma\lambda}\, \left[h^{\alpha}{}_{\gamma}\, h^{\beta}{}_{\lambda}\, (\sigma_{\alpha\beta})^{\bullet}\right] + \{termos\ em (h)^{\bullet}\}
\end{aligned}
$$

Como anteriormente, os termos em $(h_{\alpha\beta})^{\bullet}$ não contribuem para as perturbações e serão descartados daqui para a frente. Substituindo, agora, os termos restantes pelas equações Quase-Maxwellianas não perturbadas dinâmicas para $E_{\alpha\beta}$, $\sigma_{\alpha\beta}$ e θ — (3.9), (3.12) e (3.13), respectivamente — obtemos diretamente que:

$$
\begin{aligned}
h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (Y_{\alpha\beta})^{\bullet} \;&+\; \frac{1}{2}\,\theta\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu)}{}^{\lambda}\, H_{\lambda\alpha;\beta} + \frac{3}{4}\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, \sigma_{\nu)}{}^{\lambda}\, H_{\lambda\alpha;\beta} + \\
&+\; \frac{3}{4}\, \eta_{\gamma}{}^{\lambda\alpha\beta}\, V_{\lambda}\, \sigma_{(\mu}{}^{\gamma}\, h_{\nu)}{}^{\tau}\, H_{\tau\alpha;\beta} - h_{\mu\nu}\, \eta^{\gamma\tau\alpha\beta}\, V_{\tau}\, \sigma_{\gamma}{}^{\lambda}\, H_{\lambda\alpha;\beta} - \\
&-\; \frac{1}{2}\,\theta\, h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (\pi_{\alpha\beta})^{\bullet} - \frac{3}{4}\, \sigma^{\alpha}{}_{(\mu}\, h_{\nu)}{}^{\beta}\, (\pi_{\alpha\beta})^{\bullet} + \frac{1}{2}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, (\pi_{\alpha\beta})^{\bullet} + \\
&+\; \frac{1}{6}\,\theta\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, \pi_{\alpha\beta} - \frac{1}{4}\,\theta\, \sigma^{\alpha}{}_{(\mu}\, \pi_{\nu)\alpha} + \frac{1}{2}\, \sigma_{\mu\nu}\, \sigma^{\alpha\beta}\, \pi_{\alpha\beta} - \\
&-\; \frac{3}{4}\, \sigma^{\alpha}{}_{\mu}\, \sigma^{\beta}{}_{\nu}\, \pi_{\alpha\beta} - \frac{3}{8}\, \sigma^{\alpha\beta}\, \sigma_{\alpha(\mu}\, \pi_{\nu)\beta} + \frac{3}{4}\, E^{\alpha}{}_{(\mu}\, \pi_{\nu)\alpha} + \\
&+\; \frac{1}{4}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, \sigma^{\gamma}{}_{(\alpha}\, \pi_{\beta)\gamma} - \frac{1}{2}\, h_{\mu\nu}\, E^{\alpha\beta}\, \pi_{\alpha\beta} - \frac{1}{6}\,\theta\, h_{\mu\nu}\, h^{\alpha\beta}\, q_{\alpha;\beta} + \\
&+\; \frac{1}{4}\,\theta\, h^{\alpha}{}_{(\mu}\, h_{\nu)}{}^{\beta}\, q_{\alpha;\beta} + \frac{3}{8}\, h^{\alpha}{}_{(\mu}\, \sigma_{\nu)}{}^{\beta}\, q_{(\alpha;\beta)} - \frac{1}{2}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, q_{\alpha;\beta} - \\
&-\; \frac{1}{2}\, \sigma_{\mu\nu}\, h^{\alpha\beta}\, q_{\alpha;\beta} - E_{\mu\nu}\, a^{\alpha}{}_{;\alpha} + E_{\mu\nu}\, h^{\alpha\beta}\, a_{\alpha;\beta} - \\
&-\; \frac{3}{4}\, h^{\alpha}{}_{(\mu}\, E_{\nu)}{}^{\beta}\, a_{(\alpha;\beta)} + h_{\mu\nu}\, E^{\alpha\beta}\, a_{\alpha;\beta} + \\
&+\; \left\{ \frac{1}{2}\,\theta\, (1+\lambda)\, \sigma_{\mu\nu} + \frac{1}{2}\, (1+3\lambda)\, E_{\mu\nu} + \frac{3}{2}\, (1+\lambda)\, \sigma^{\alpha}{}_{\mu}\, \sigma_{\nu\alpha} - \right. \\
&\left. -\frac{1}{2}\, (1+\lambda)\, \left(2\sigma^{2}\right)\, h_{\mu\nu} \right\} \rho
\end{aligned}
$$

$$
\begin{aligned}
&=\; -\frac{4}{3}\,\theta^{2}\, E_{\mu\nu} + \frac{2}{3}\,\theta\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, E_{\alpha\beta} - 3\, \sigma_{\mu\nu}\, \sigma^{\alpha\beta}\, E_{\alpha\beta} - \theta\, \sigma^{\alpha}{}_{(\mu}\, E_{\nu)\alpha} + \\
&\quad +\frac{9}{2}\, E_{\alpha\beta}\, \sigma^{\alpha}{}_{\mu}\, \sigma^{\beta}{}_{\nu} + \frac{3}{4}\, \sigma_{\alpha\beta}\, \sigma^{\alpha}{}_{(\mu}\, E_{\nu)}{}^{\beta} - 3\, E^{\alpha}{}_{\mu}\, E^{\nu\alpha} - \\
&\quad -2\, h_{\mu\nu}\, \sigma^{\alpha}{}_{\beta}\, \sigma^{\beta}{}_{\gamma}\, E^{\gamma}{}_{\alpha} + h_{\mu\nu}\, E^{\alpha\beta}\, E_{\alpha\beta}
\end{aligned}
$$

$$
\equiv\; LD
$$

Entretanto, o termo em ρ no lado esquerdo da equação acima pode ser reescrito em uma maneira mais compacta como:

$$
\left\{\frac{1}{2}\theta\,(1+\lambda)\,\sigma_{\mu\nu}+\frac{1}{2}\,(1+3\lambda)\,E_{\mu\nu}+\frac{3}{2}\,(1+\lambda)\,\sigma^{\alpha}{}_{\mu}\,\sigma_{\nu\alpha}-\frac{1}{2}\,(1+\lambda)\,\left(2\sigma^2\right)\,h_{\mu\nu}\right\}=
$$

$$
\begin{aligned}
&= \frac{1}{2}\left\{(1+3\lambda)\,E_{\mu\nu}+3\,(1+\lambda)\,\left[\frac{1}{3}\theta\,\sigma_{\mu\nu}-\frac{2\sigma^2}{3}\,h_{\mu\nu}+\sigma^{\alpha}{}_{\mu}\,\sigma_{\nu\alpha}\right]\right\}\\
&= \frac{1}{2}\left\{(1+3\lambda)\,E_{\mu\nu}+3\,(1+\lambda)\,\left[\frac{2}{3}\theta\,\sigma_{\mu\nu}+X_{\mu\nu}-E_{\mu\nu}\right]\right\}\\
&= \frac{1}{2}\left\{3\,(1+\lambda)\,X_{\mu\nu}+(1+3\lambda-3-3\lambda)\,E_{\mu\nu}+3\,(1+\lambda)\,\frac{2}{3}\theta\,\sigma_{\mu\nu}\right\}\\
&= \frac{3}{2}\,(1+\lambda)\,X_{\mu\nu}-E_{\mu\nu}+\theta\,(1+\lambda)\,\sigma_{\mu\nu}.
\end{aligned}
$$

Além disto, podemos reescrever o lado direito da equação dinâmica de $Y_{\alpha\beta}$ em termos das "boas" variáveis de trabalho:

$$
\begin{aligned}
LD \;\equiv\; & -\frac{4}{3}\theta^2\,E_{\mu\nu}+\frac{2}{3}\theta\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,E_{\alpha\beta}-3\,\sigma_{\mu\nu}\,\sigma^{\alpha\beta}\,E_{\alpha\beta}-\theta\,\sigma^{\alpha}{}_{(\mu}\,E_{\nu)\alpha}+\\
& +\frac{9}{2}\,E_{\alpha\beta}\,\sigma^{\alpha}{}_{\mu}\,\sigma^{\beta}{}_{\nu}+\frac{3}{4}\,\sigma_{\alpha\beta}\,\sigma^{\alpha}_{(\mu}\,E_{\nu)}{}^{\beta}-3\,E^{\alpha}{}_{\mu}\,E_{\nu\alpha}-\\
& -2\,h_{\mu\nu}\,\sigma^{\alpha}{}_{\beta}\,\sigma^{\beta}{}_{\gamma}\,E^{\gamma}{}_{\alpha}+h_{\mu\nu}\,E^{\alpha\beta}\,E_{\alpha\beta}\\
=\; & -2\,\theta\,Y_{\mu\nu}+\frac{2}{3}\theta^2\,E_{\mu\nu}-\frac{4}{3}\theta\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,E_{\alpha\beta}-3\,\sigma_{\mu\nu}\,\sigma^{\alpha\beta}\,E_{\alpha\beta}+\\
& +2\,\theta\,\sigma^{\alpha}{}_{(\mu}\,E_{\nu)\alpha}+\frac{9}{2}\,E_{\alpha\beta}\,\sigma^{\alpha}{}_{\mu}\,\sigma^{\beta}{}_{\nu}+\frac{3}{4}\,\sigma_{\alpha\beta}\,\sigma^{\alpha}{}_{(\mu}\,E_{\nu)}{}^{\beta}-3\,E^{\alpha}{}_{\mu}\,E_{\nu\alpha}-\\
& -2\,h_{\mu\nu}\,\sigma^{\alpha}{}_{\beta}\,\sigma^{\beta}{}_{\gamma}\,E^{\gamma}{}_{\alpha}+h_{\mu\nu}\,E^{\alpha\beta}\,\sigma_{\alpha\beta}\\
=\; & -2\,\theta\,Y_{\mu\nu}+\frac{3}{2}\,\sigma^{\alpha}{}_{(\mu}\,Y_{\nu)\alpha}+\frac{2}{3}\theta^2\,E_{\mu\nu}-\frac{4}{3}\theta\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,E_{\alpha\beta}+\frac{1}{2}\,\theta\,\sigma^{\alpha}{}_{(\mu}\,E_{\nu)\alpha}-
\end{aligned}
$$

$$-\frac{3}{2}\,\sigma_{\alpha\beta}\,\sigma^{\alpha}{}_{(\mu}\,E_{\nu)}{}^{\beta} - 3\,E^{\alpha}{}_{\mu}\,E_{\nu\alpha} - 2\,h_{\mu\nu}\,\sigma^{\alpha}{}_{\beta}\,\sigma^{\beta}{}_{\gamma}\,E^{\gamma}{}_{\alpha} + h_{\mu\nu}\,E^{\alpha\beta}\,E_{\alpha\beta}$$

$$\begin{aligned}
= \quad & -2\,\theta\,Y_{\mu\nu} + \frac{3}{2}\,\sigma^{\alpha}{}_{(\mu}\,Y_{\nu)\alpha} - h_{\mu\nu}\,\sigma^{\alpha\beta}\,Y_{\alpha\beta} + \\
& + \frac{2}{3}\,\theta^2\,E_{\mu\nu} - \frac{1}{3}\,\theta\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,E_{\alpha\beta} + \frac{1}{2}\,\theta\,\sigma^{\alpha}{}_{(\mu}\,E_{\nu)\alpha} - \frac{3}{2}\,\sigma_{\alpha\beta}\,\sigma^{\alpha}{}_{(\mu}\,E_{\nu)}{}^{\beta} - \\
& - 3\,E^{\alpha}{}_{\mu}\,E_{\nu\alpha} + h_{\mu\nu}\,\sigma^{\alpha}{}_{\beta}\,\sigma^{\beta}{}_{\gamma}\,E^{\gamma}{}_{\alpha} + h_{\mu\nu}\,E^{\alpha\beta}\,E_{\alpha\beta}
\end{aligned}$$

$$\begin{aligned}
= \quad & -2\,\theta\,Y_{\mu\nu} + \frac{3}{2}\,\sigma^{\alpha}{}_{(\mu}\,Y_{\nu)\alpha} - h_{\mu\nu}\,\sigma^{\alpha\beta}\,Y_{\alpha\beta} - \frac{3}{2}\,E^{\alpha}{}_{(\mu}\,X_{\nu)\alpha} + \frac{2}{3}\,\theta^2\,E_{\mu\nu} - \\
& - \frac{1}{3}\,\theta\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,E_{\alpha\beta} - \left(2\sigma^2\right)\,E_{\mu\nu} + h_{\mu\nu}\,\sigma^{\alpha}{}_{\beta}\,\sigma^{\beta}{}_{\gamma}\,E^{\gamma}{}_{\alpha} + h_{\mu\nu}\,E^{\alpha\beta}\,E_{\alpha\beta}
\end{aligned}$$

$$\begin{aligned}
= \quad & -2\,\theta\,Y_{\mu\nu} + \frac{3}{2}\,\sigma^{\alpha}{}_{(\mu}\,Y_{\nu)\alpha} - h_{\mu\nu}\,\sigma^{\alpha\beta}\,Y_{\alpha\beta} - \\
& - \frac{3}{2}\,E^{\alpha}{}_{(\mu}\,X_{\nu)\alpha} + h_{\mu\nu}\,E^{\alpha\beta}\,X_{\alpha\beta} - \left[\left(2\sigma^2\right) - \frac{2}{3}\,\theta^2\right]\,E_{\mu\nu}
\end{aligned}$$

$$\begin{aligned}
= \quad & -2\,\theta\,Y_{\mu\nu} + \frac{3}{2}\,\sigma^{\alpha}{}_{(\mu}\,Y_{\nu)\alpha} - h_{\mu\nu}\,\sigma^{\alpha\beta}\,Y_{\alpha\beta} - \\
- \quad & \frac{3}{2}\,E^{\alpha}{}_{(\mu}\,X_{\nu)\alpha} + h_{\mu\nu}\,E^{\alpha\beta}\,X_{\alpha\beta} - E_{\mu\nu}\,W.
\end{aligned}$$

Logo, a equação dinâmica para $Y_{\alpha\beta}$ se escreve finalmente como:

$$\begin{aligned}
h^{\alpha}{}_{\mu}\,h^{\beta}{}_{\nu}\,\left(Y_{\alpha\beta}\right)^{\bullet} \quad + \quad & 2\,\theta\,Y_{\mu\nu} - \frac{3}{2}\,\sigma^{\alpha}{}_{(\mu}\,Y_{\nu)\alpha} + h_{\mu\nu}\,\sigma^{\alpha\beta}\,Y_{\alpha\beta} + \\
+ \quad & \frac{3}{2}\,E^{\alpha}{}_{(\mu}\,X_{\nu)\alpha} - h_{\mu\nu}\,E^{\alpha\beta}\,X_{\alpha\beta} + \frac{1}{2}\,\theta\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu)}{}^{\lambda}\,H_{\lambda\alpha;\beta} + \\
+ \quad & \frac{3}{4}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,\sigma_{\nu)}{}^{\lambda}\,H_{\lambda\alpha;\beta} + \frac{3}{4}\,\eta^{\gamma\lambda\alpha\beta}\,V_{\lambda}\,\sigma_{\gamma(\mu}\,h_{\nu)}{}^{\tau}\,H_{\tau\alpha;\beta} - h_{\mu\nu}\,\eta^{\gamma\tau\alpha\beta}\,V_{\tau}\,\sigma_{\gamma}{}^{\lambda}\,H_{\lambda\alpha;\beta} - \\
- \quad & \frac{1}{2}\,\theta\,h^{\alpha}{}_{\mu}\,h^{\beta}{}_{\nu}\,\left(\pi_{\alpha\beta}\right)^{\bullet} - \frac{3}{4}\,\sigma^{\alpha}{}_{(\mu}\,h_{\nu)}{}^{\beta}\,\left(\pi_{\alpha\beta}\right)^{\bullet} + \frac{1}{2}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,\left(\pi_{\alpha\beta}\right)^{\bullet} + \\
+ \quad & \frac{1}{6}\,\theta\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,\pi_{\alpha\beta} - \frac{1}{4}\,\theta\,\sigma^{\alpha}{}_{(\mu}\,\pi_{\nu)\alpha} + \frac{1}{2}\,\sigma_{\mu\nu}\,\sigma^{\alpha\beta}\,\pi_{\alpha\beta} - \\
- \quad & \frac{3}{4}\,\sigma^{\alpha}{}_{\mu}\,\sigma^{\beta}{}_{\nu}\,\pi_{\alpha\beta} - \frac{3}{8}\,\sigma^{\alpha\beta}\,\sigma_{\alpha(\mu}\,\pi_{\nu)\beta} + \frac{3}{4}\,E^{\alpha}{}_{(\mu}\,\pi_{\nu)\alpha} + \\
+ \quad & \frac{1}{4}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,\sigma^{\gamma}{}_{(\alpha}\,\pi_{\beta)\gamma} - \frac{1}{2}\,h_{\mu\nu}\,E^{\alpha\beta}\,\pi_{\alpha\beta} - \frac{1}{6}\,\theta\,h_{\mu\nu}\,h^{\alpha\beta}\,q_{\alpha;\beta} +
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{4}\,\theta\, h^{\alpha}{}_{(\mu}\, h^{\beta}{}_{\nu)}\, q_{\alpha;\beta} + \frac{3}{8}\, h^{\alpha}{}_{(\mu}\, \sigma_{\nu)}{}^{\beta}\, q_{(\alpha;\beta)} - \frac{1}{2}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, q_{\alpha;\beta} - \\
&- \frac{1}{2}\,\sigma_{\mu\nu}\, h^{\alpha\beta}\, q_{\alpha;\beta} - E_{\mu\nu}\, a^{\alpha}{}_{;\alpha} + E_{\mu\nu}\, h^{\alpha\beta}\, a_{\alpha;\beta} - \frac{3}{4}\, h^{\alpha}{}_{(\mu}\, E_{\nu)}{}^{\beta}\, a_{(\alpha;\beta)} + \\
&+ h_{\mu\nu}\, E^{\alpha\beta}\, a_{\alpha;\beta} + E_{\mu\nu}\, W - \left[E_{\mu\nu} - \tfrac{3}{2}\,(1+\lambda)\,\sigma_{\mu\nu}\right]\rho + \\
&+ \frac{3}{2}\,(1+\lambda)\, X_{\mu\nu}\,\rho = 0,
\end{aligned} \tag{3.28}
$$

onde o último termo em (3.28) é de segunda ordem de perturbação e, como tal, pode ser desconsiderado em nossos cálculos posteriores.

II)Equação Dinâmica para W:

Repetindo os passos executados na obtenção das equações dinâmicas para $X_{\alpha\beta}$ e $Y_{\alpha\beta}$, vem:

$$
\begin{aligned}
W^{\bullet} &= 2\,\sigma^{\alpha}{}_{\beta}\,\left(\sigma^{\beta}{}_{\alpha}\right)^{\bullet} - \frac{4}{3}\,\theta\,\theta^{\bullet} \\
&= 2\,\sigma^{\alpha}{}_{\beta}\,\left(\sigma_{\alpha\gamma}\, h^{\gamma\beta}\right)^{\bullet} - \frac{4}{3}\,\theta\,\theta^{\bullet} \\
&= 2\,\sigma^{\alpha}{}_{\beta}\, h^{\beta\gamma}\,\left(\sigma_{\alpha\gamma}\right)^{\bullet} - \frac{4}{3}\,\theta\,\theta^{\bullet} + 2\,\sigma^{\alpha}{}_{\beta}\,\sigma_{\alpha\gamma}\,\left(h^{\beta\gamma}\right)^{\bullet}.
\end{aligned}
$$

e, como das outras vezes, o último termo será desconsiderado por não contribuir para a métrica perturbada. Neste caso, temos então que:

$$
\begin{aligned}
W^{\bullet} &= 2\,\sigma^{\mu\nu}\,\left[h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\,\left(\sigma_{\alpha\beta}\right)^{\bullet}\right] - \frac{4}{3}\,\theta\,\left[\theta^{\bullet}\right] \\
&= 2\,\sigma^{\alpha\beta}\, a_{\alpha;\beta} - \frac{4}{3}\,\theta\, a^{\alpha}{}_{;\alpha} - \sigma^{\alpha\beta}\,\pi_{\alpha\beta} + \\
&\quad + \frac{2}{3}\,\theta\,(1+3\lambda)\,\rho - 2\,\sigma^{\alpha\beta}\,\left(E_{\alpha\beta} + \sigma_{\alpha\mu}\,\sigma^{\mu}{}_{\beta}\right) + \frac{4}{9}\,\theta^{3}.
\end{aligned}
$$

Donde escrevemos:

$$
\begin{aligned}
W^{\bullet} &= -2\,\sigma^{\alpha\beta}\,a_{\alpha;\beta} + \frac{4}{3}\,\theta\,a^{\alpha}{}_{;\alpha} + \sigma^{\alpha\beta}\,\pi_{\alpha\beta} - \frac{2}{3}\theta\,(1+3\lambda)\,\rho \\
&= -2\,\sigma^{\alpha\beta}\,(E_{\alpha\beta} + \sigma_{\alpha\mu}\,\sigma^{\mu}{}_{\beta}) + \frac{2}{3}\,\theta\,\sigma^{\alpha\beta}\,\sigma_{\alpha\beta} - \frac{2}{3}\,\theta\,W \\
&= -\frac{2}{3}\,\theta\,W - 2\,\sigma^{\alpha\beta}\,\left(E_{\alpha\beta} + \sigma_{\alpha\mu}\,\sigma^{\mu}{}_{\beta} - \frac{1}{3}\,\theta\,\sigma_{\alpha\beta}\right) \\
&= -\frac{2}{3}\,\theta\,W - 2\,\sigma^{\alpha\beta}\,\left(X_{\alpha\beta} + \frac{2\sigma^2}{3}\,h_{\alpha\beta}\right) \\
& -\frac{2}{3}\,\theta\,W - 2\,\sigma^{\alpha\beta}\,X_{\alpha\beta}.
\end{aligned}
$$

E a equação dinâmica para W se escreve como:

$$
\begin{aligned}
W^{\bullet} &+ \frac{2}{3}\,\theta\,W + 2\,\sigma^{\alpha\beta}\,X_{\alpha\beta} + \sigma^{\alpha\beta}\,\pi_{\alpha\beta} - \\
&- 2\,\sigma^{\alpha\beta}\,a_{\alpha;\beta} + \frac{4}{3}\,\theta\,h^{\alpha\beta}\,a_{\alpha;\beta} - \frac{2}{3}\,\theta\,(1+3\lambda)\,\rho = \\
&= 0.
\end{aligned}
\tag{3.29}
$$

Obtidas as equações dinâmicas para $X_{\alpha\beta}$, $Y_{\alpha\beta}$ e W, podemos, agora, desconsiderar as equações dinâmicas para $E_{\alpha\beta}$, $\sigma_{\alpha\beta}$ e θ. Podemos, então, passar à tarefa de reescrever as equações Quase-Maxwellianas não perturbadas restantes em termos das variáveis do conjunto $\mathcal{M}$, equação (3.25), o que é feito a seguir.

A primeira equação a ser reescrita será a equação dinâmica para $H_{\alpha\beta}$, a terceira projeção, equação (3.8). Coletando os termos nas "boas" variáveis no lado esquerdo da equação, temos:

$$
\begin{aligned}
h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (H_{\alpha\beta})^{\bullet} \;&+\; \theta\, H_{\mu\nu} - \frac{1}{2}\, V^{\alpha;\beta}\, h_{\beta(\mu}\, H_{\nu)\alpha} + \\
&+\; \eta_{\mu}{}^{\alpha\gamma\varepsilon}\, \eta_{\nu}{}^{\beta\lambda\tau}\, V_{\gamma}\, V_{\lambda}\, \sigma_{\alpha\beta}\, H_{\varepsilon\tau} + \frac{1}{3}\, \theta\, \eta_{\mu}{}^{\alpha\gamma\varepsilon}\, \eta_{\nu}{}^{\beta\lambda\tau}\, V_{\gamma}\, V_{\lambda}\, h_{\alpha\beta}\, H_{\varepsilon\tau} - \\
&-\; \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, E_{\nu)\beta}\, a_{\alpha} + \frac{3}{4}\, q_{(\mu}\, \omega_{\nu)} - \frac{1}{2}\, h_{\mu\nu}\, h^{\alpha\beta}\, q_{\alpha}\, \omega_{\beta} - \\
&-\; \frac{1}{4}\, \eta_{(\mu}{}^{\alpha\beta\gamma}\, V_{\gamma}\, \sigma_{\nu)\beta}\, q_{\alpha} - \frac{1}{4}\, \eta_{(\mu}{}^{\alpha\beta\gamma}\, V_{\gamma}\, h_{\nu)\lambda}\, \pi_{\lambda\alpha;\beta} \\
&=\; \frac{1}{2}\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu)}{}^{\lambda}\, E_{\lambda\alpha;\beta} \equiv LD.
\end{aligned}
$$

Reescrevendo o o termo LD acima em termos das "boas"quantidades, temos (usando a definição de $X_{\alpha\beta}$):

$$
\begin{aligned}
LD \;&\equiv\; \frac{1}{2}\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu)}{}^{\lambda}\, E_{\lambda\alpha;\beta} \\
&=\; \frac{1}{2}\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu)}{}^{\lambda} \left[X_{\lambda\alpha} + \frac{1}{3}\, \theta\, \sigma_{\lambda\alpha} + \frac{2\sigma^{2}}{3}\, h_{\lambda\alpha} - \sigma_{\lambda\varepsilon}\, \sigma^{\varepsilon}{}_{\alpha} \right]_{;\beta} \\
&=\; \frac{1}{2}\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu)}{}^{\lambda}\, X_{\lambda\alpha;\beta} + \frac{1}{6}\, \theta\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu)}{}^{\lambda}\, \sigma_{\lambda\alpha;\beta} + \\
&\quad + \frac{1}{6}\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, \sigma_{\nu)\alpha}\, \theta_{,\beta} + \frac{1}{2} \left(2\sigma^{2}\right) \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu)}{}^{\lambda}\, h_{\lambda\alpha\beta} + \\
&\quad + \frac{1}{2}\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu)\alpha} \left(2\sigma^{2}\right)_{,\beta} - \frac{1}{2}\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu)}{}^{\lambda} \left(\sigma_{\lambda\varepsilon}\, \sigma^{\varepsilon}{}_{\alpha}\right)_{;\beta}
\end{aligned}
$$

Todavia, o quinto termo acima é nulo, conforme provamos abaixo:

$$
\begin{aligned}
\frac{1}{2}\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu)\alpha} \left(2\sigma^{2}\right)_{,\beta} \;&=\; \frac{1}{2}\, \eta_{(\mu}{}^{\gamma}{}_{\nu)}{}^{\beta}\, V_{\gamma} \left(2\sigma^{2}\right)_{,\beta} - \frac{1}{2}\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\nu)}\, V_{\gamma}\, V_{\alpha} \left(2\sigma^{2}\right)_{,\beta} \\
&=\; 0.
\end{aligned}
$$

Além disto, o segundo termo em LD pode ser reescrito de um modo mais adequado, utilizando-se a terceira equação de vínculo, (3.17):

$$\frac{1}{2}\,\eta_{\alpha}{}^{\gamma\lambda\tau}\,V_{\tau}\,h_{\beta}{}^{\nu}\,\sigma_{\nu\gamma;\lambda} = -H_{\alpha\beta} + a_{(\alpha}\,\omega_{\beta)} - \\ -\frac{1}{2}\,\eta_{\alpha}{}^{\gamma\lambda\tau}\,V_{\tau}\,h_{\beta}{}^{\nu}\,\omega_{\nu\gamma;\lambda}.$$

Simetrizando, agora, nos índices (α,β), vem:

$$\frac{1}{2}\,\eta_{(\alpha}{}^{\gamma\lambda\tau}\,V_{\tau}\,h_{\beta)}{}^{\nu}\,\sigma_{\nu\gamma;\lambda} = -2\,H_{\alpha\beta} + a_{(\alpha}\,\omega_{\beta)} - \\ -\frac{1}{2}\,\eta_{(\alpha}{}^{\gamma\lambda\tau}\,V_{\tau}\,h_{\beta)}{}^{\nu}\,\omega_{\nu\gamma;\lambda}.$$

Multiplicando por $\theta/3$, e rearranjando índices, obtemos finalmente o segundo termo em LD:

$$\frac{1}{6}\,\theta\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu)}{}^{\lambda}\,\sigma_{\lambda\alpha;\beta} = -\frac{2}{3}\,\theta\,H_{\mu\nu} + \frac{2}{3}\,\theta\,a_{(\mu}\,\omega_{\nu)} - \\ -\frac{1}{6}\,\theta\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu)}{}^{\lambda}\,\omega_{\lambda\alpha;\beta}.$$

Logo, o lado direito da equação dinâmica de $H_{\alpha\beta}$ se escreve como:

$$LD = \frac{1}{2}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu)}{}^{\lambda}\,X_{\lambda\alpha;\beta} - \frac{2}{3}\,\theta\,H_{\mu\nu} + \frac{2}{3}\,\theta\,a_{(\mu}\,\omega_{\nu)} - \frac{1}{6}\,\theta\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu)}{}^{\lambda}\,\omega_{\lambda\alpha;\beta} + \\ +\frac{1}{6}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,\sigma_{\nu)\alpha}\,\theta_{,\beta} - \frac{1}{2}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu)}{}^{\lambda}\,(\sigma_{\lambda\varepsilon}\,\sigma^{\varepsilon}{}_{\alpha})_{;\beta}\,. \tag{3.30}$$

e devemos analisar os termos

$$\frac{1}{6}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,\sigma_{\nu)\alpha}\,\theta_{,\beta} - \frac{1}{2}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu)}{}^{\lambda}\,(\sigma_{\lambda}{}^{\varepsilon}\,\sigma_{\varepsilon\alpha})_{;\beta}\,,$$

da equação (3.30) acima. O primeiro destes dois termos pode ser obtido com o auxílio da primeira equação de vínculo, (3.15):

$$
\begin{aligned}
\frac{1}{6}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,\theta_{,\beta} \;=\;& \frac{1}{4}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,\omega^{\lambda}{}_{\beta;\lambda} + \\
&+\frac{1}{4}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,\sigma_\beta{}^{\lambda}\,a_\lambda + \frac{1}{4}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,a^{\lambda}\,\omega_{\beta\lambda} - \\
&-\frac{1}{4}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,q_\beta + \frac{1}{4}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,\sigma^{\lambda}{}_{\beta;\lambda}.
\end{aligned}
\tag{3.31}
$$

Entretanto, não há nenhuma outra equação em nosso sistema que apresente um termo em $\sigma^{\lambda}{}_{\beta;\lambda}$. Sabemos, porém, que esta quantidade é nula no *background*. Por uma questão de simplificação de nossos cálculos, vamos portanto considerá-la como sendo também nula na perturbação. Da mesma forma, a quantidade $(\sigma_{\lambda\varepsilon}\,\sigma^{\varepsilon}{}_{\alpha})_{;\beta}$, que também aparece no lado direito de (3.30), também é nula no *background* e será — a exemplo da anterior — considerada como nula mesmo após a perturbação. Neste caso, após descartarmos todos os termos em segunda ordem de perturbação e usarmos as relações (3.30) e (3.31), a equação dinâmica para $H_{\alpha\beta}$ se escreve como:

$$
\begin{aligned}
h_\mu{}^{\alpha}\,h_\nu{}^{\beta}\,(H_{\alpha\beta})^{\bullet} \;&+\; \frac{4}{3}\,\theta\,H_{\mu\nu} - \frac{1}{2}\,\sigma^{\alpha}{}_{(\mu}\,H_{\nu)\alpha} + \eta_\mu{}^{\alpha\gamma\varepsilon}\,\eta_\nu{}^{\beta\lambda\tau}\,V_\gamma\,V_\lambda\,\sigma_{\alpha\beta}\,H_{\varepsilon\tau} + \\
&+\; \frac{1}{3}\,\theta\,\eta_\mu{}^{\alpha\gamma\varepsilon}\,\eta_\nu{}^{\beta\lambda\tau}\,V_\gamma\,V_\lambda\,h_{\alpha\beta}\,H_{\varepsilon\tau} - \eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,E_{\nu)\beta}\,a_\alpha - \\
&-\; \frac{1}{4}\,\eta_{(\mu}{}^{\alpha\beta\gamma}\,V_\gamma\,\sigma_{\nu)\beta}\,q_\alpha - \frac{1}{4}\,\eta_{(\mu}{}^{\alpha\beta\gamma}\,V_\gamma\,h_{\nu)}{}^{\lambda}\,\pi_{\lambda\alpha;\beta} - \\
&-\; \frac{1}{2}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,h_{\nu)}{}^{\lambda}\,X_{\lambda\alpha;\beta} + \frac{1}{6}\,\theta\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,h_{\nu)}{}^{\lambda}\,\omega_{\lambda\alpha;\beta} - \\
&-\; \frac{1}{4}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,\omega^{\lambda}{}_{\beta;\lambda} - \frac{1}{4}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,\sigma_\beta{}^{\lambda}\,a_\lambda + \\
&+\; \frac{1}{4}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\sigma_{\nu)\alpha}\,q_\beta = 0.
\end{aligned}
\tag{3.32}
$$

Resta-nos, apenas, reescrever as duas únicas equações Quase-Maxwellianas que não foram ainda utilizadas na construção deste sistema dinâmico: a primeira e a segunda projeções, equações (3.6) e (3.7). Reescrevendo o primeiro termo de (3.6) em termos das "boas" quantidades de trabalho, vem:

$$
\begin{aligned}
h^{\alpha\beta}\, h^{\mu\nu}\, E_{\mu;\nu} &= h^{\alpha\beta}\, h^{\mu\nu} \left[X_{\beta\mu;\nu} + \frac{1}{3}\, (\theta\, \sigma_{\beta\mu})_{;\nu} + \right. \\
& \left. + \frac{1}{3}\, \left(2\sigma^2\, h_{\beta\mu}\right)_{;\nu} - (\sigma_{\beta\gamma}\, \sigma^{\gamma}{}_{\mu})_{;\nu} \right] \\
&= h^{\alpha\beta}\, h^{\mu\nu}\, X_{\beta\mu;\nu} + \frac{1}{2}\, \sigma^{\alpha}{}_{\mu} \left[\frac{2}{3}\, h^{\mu\nu}\, \theta_{,\nu} \right] + \frac{1}{3}\, \theta\, h^{\alpha\beta}\, (\sigma_{\beta}{}^{\nu})_{;\nu} + \\
& + \frac{1}{3}\, \theta\, \sigma^{\alpha\mu}\, (V^{\mu}{}_{;\nu}\, V^{\nu} + V^{\mu}\, V^{\nu}{}_{;\nu}) + \frac{1}{3}\, h^{\alpha\beta}\, W_{,\beta} + \frac{2}{3}\, \theta \left[\frac{2}{3}\, h^{\alpha\beta}\, \theta_{,\beta} \right] - \\
& - \frac{1}{3} \left(W + \frac{2}{3}\, \theta^2 \right) \left[\left(\sigma^{\alpha\beta} + \frac{1}{3}\, \theta\, h^{\alpha\beta} \right) V_{\beta} + (\sigma^{\mu}{}_{\mu} + \theta)\, h^{\alpha\beta}\, V_{\beta} \right] - \\
& - h^{\alpha\beta}\, h^{\mu\nu}\, (\sigma_{\beta\gamma}\, \sigma^{\gamma}{}_{\mu})_{;\nu} \\
&= h^{\alpha\beta}\, h^{\mu\nu}\, X_{\beta\mu;\nu} + \frac{1}{2}\, \sigma^{\alpha}{}_{\mu} \left[h^{\mu\nu}\, \sigma^{\beta}{}_{\nu;\beta} + h^{\mu\nu}\, \omega^{\beta}{}_{\nu;\beta} + \sigma^{\mu\nu}\, a_{\nu} - h^{\mu\nu}\, q_{\nu} \right] + \\
& + \frac{1}{3}\, \theta\, h^{\alpha\beta}\, \sigma^{\mu}{}_{\beta;\mu} + \frac{1}{3}\, \theta\, \sigma^{\alpha\beta}\, a_{\beta} + \frac{1}{3}\, \theta^2\, \sigma^{\alpha\beta}\, V_{\beta} + \frac{1}{3}\, h^{\alpha\beta}\, W_{,\beta} + \\
& + \frac{2}{3}\, \theta \left[h^{\alpha\beta}\, \sigma^{\gamma}{}_{\beta;\gamma} + h^{\alpha\beta}\, \omega^{\gamma}{}_{\beta;\gamma} + \sigma^{\alpha\beta}\, a_{\beta} - h^{\alpha\beta}\, q_{\beta} \right] - \\
& - \frac{2}{9}\, \theta^2 \left(\sigma^{\alpha\beta} + \frac{1}{3}\, \theta\, h^{\alpha\beta} \right) V_{\beta} - \frac{2}{9}\, \theta^3\, h^{\alpha\beta}\, V_{\beta} \\
& - h^{\alpha\beta}\, h^{\mu\nu}\, (\sigma_{\beta\gamma}\, \sigma^{\gamma}{}_{\mu})_{;\nu} \\
&= h^{\alpha\beta}\, h^{\mu\nu}\, X_{\beta\mu;\nu} + \frac{1}{2}\, \sigma^{\alpha\beta}\, \sigma^{\gamma}{}_{\beta;\gamma} + \frac{1}{2}\, \sigma^{\alpha\beta}\, \omega^{\gamma}{}_{\beta;\gamma} + \frac{1}{2}\, \sigma^{\alpha}{}_{\gamma}\, \sigma^{\gamma\beta}\, a_{\beta} - \\
& - \frac{1}{2}\, \sigma^{\alpha\beta}\, q_{\beta} + \theta\, \sigma^{\alpha\beta}\, a_{\beta} + \frac{1}{3}\, h^{\alpha\beta}\, W_{,\beta} + \theta\, h^{\alpha\beta}\, \sigma^{\gamma}{}_{\beta;\gamma} + \frac{2}{3}\, \theta\, h^{\alpha\beta}\, \omega^{\gamma}{}_{\beta;\gamma} - \\
& - \frac{2}{3}\, \theta\, h^{\alpha\beta}\, q_{\beta} - h^{\alpha\beta}\, h^{\mu\nu}\, (\sigma_{\beta\gamma}\, \sigma^{\gamma}{}_{\mu})_{;\nu} + \\
& + \left(\frac{1}{9}\, \theta^2\, \sigma^{\alpha\beta} - \frac{8}{27}\, \theta^3\, h^{\alpha\beta} \right) V_{\beta} \\
&= h^{\alpha\beta}\, h^{\mu\nu}\, X_{\beta\mu;\nu} + \frac{1}{2} \left(\sigma^{\alpha\beta} + \frac{4}{3}\, \theta\, h^{\alpha\beta} \right) \omega^{\mu}{}_{\beta;\mu} + \frac{1}{2} \left(\sigma^{\alpha}{}_{\gamma}\, \sigma^{\gamma\beta} + 2\, \theta\, \sigma^{\alpha\beta} \right) a_{\beta} - \\
& - \frac{1}{2} \left(\sigma^{\alpha\beta} + \frac{4}{3}\, \theta\, h^{\alpha\beta} \right) q_{\beta} + \frac{1}{3}\, h^{\alpha\beta}\, W_{,\beta} + \left\{ \frac{1}{9}\, \theta^2 \left(\sigma^{\alpha\beta} - \frac{8}{3}\, \theta\, h^{\alpha\beta} \right) V_{\beta} + \right.
\end{aligned}
$$

$$+\frac{1}{2}\left(\sigma^{\alpha\beta}+2\,\theta\,h^{\alpha\beta}\right)\,\sigma^{\gamma}{}_{\beta;\gamma}-h^{\alpha\beta}\,h^{\mu\nu}\,\left(\sigma_{\beta\gamma}\,\sigma^{\gamma}{}_{\mu}\right)_{;\nu}\Big\}. \tag{3.33}$$

Entretanto, os dois últimos termos entre chaves serão considerados como nulos mesmo após a perturbação, enquanto que o primeiro termo entre chaves, em V_β, não é definido para perturbações tensoriais. Neste caso, a primeira projeção se escreve como:

$$\begin{aligned}h^{\alpha\beta}\,h^{\mu\nu}\,X_{\beta\mu;\nu} &+ \eta^{\alpha\beta\mu\nu}\,V_\beta\,\sigma^{\gamma}{}_{\mu}\,H_{\nu\gamma}-\frac{1}{2}h^{\alpha\beta}\,h^{\mu\nu}\,\pi_{\beta\mu;\nu}+\\ &+ \frac{1}{2}\left(\sigma^{\alpha\beta}+\frac{4}{3}\theta\,h^{\alpha\beta}\right)\,\omega^{\mu}{}_{\beta;\mu}+\frac{1}{2}\left(\sigma^{\alpha}{}_{\gamma}\,\sigma^{\beta\gamma}+2\,\theta\,\sigma^{\alpha\beta}\right)\,a_\beta-\theta\,h^{\alpha\beta}\,q_\beta-\\ &- \frac{1}{3}\,h^{\alpha\beta}\,\rho_{,\beta}+\frac{1}{3}\,h^{\alpha\beta}\,W_{,\beta}.\end{aligned} \tag{3.34}$$

Passando, agora, à segunda projeção — equação (3.7) — escrevemos:

$$\begin{aligned}-\eta^{\alpha\beta\mu\nu}\,V_\beta\,\left(\sigma^{\gamma}{}_{\mu}\,E_{\nu\gamma}\right) &= -\eta^{\alpha\beta\mu\nu}\,V_\beta\,\sigma^{\gamma}{}_{\mu}\,\left[X_{\nu\gamma}+\frac{1}{3}\,\theta\,\sigma_{\nu\gamma}+\frac{2\sigma^2}{3}\,h_{\nu\gamma}-\sigma_{\nu}{}^{\lambda}\,\sigma_{\lambda\gamma}\right]\\ &= -\eta^{\alpha\beta\mu\nu}\,V_\beta\,\sigma^{\gamma}{}_{\mu}\,X_{\nu\gamma}-\eta^{\alpha\beta\mu\nu}\,V_\beta\,\left[\frac{1}{3}\,\theta\,\left(\sigma^{\gamma}{}_{\mu}\,\sigma_{\nu\gamma}\right)+\frac{2\sigma^2}{3}\,\sigma_{\mu\nu}-\left(\sigma^{\gamma}{}_{\mu}\,\sigma_{\gamma}{}^{\lambda}\,\sigma_{\lambda\nu}\right)\right].\end{aligned}$$

Entretanto, $\sigma_{\mu\nu}$, $(\sigma_\mu{}^\gamma\,\sigma_{\gamma\nu})$ e $(\sigma_\mu{}^\gamma\,\sigma_{\lambda\nu})$ são simétricos nos índices (μ,ν). Logo, multiplicados como estão por $\eta^{\alpha\beta\mu\nu}$, eles se anulam imediatamente e temos então que:

$$\begin{aligned}h^{\alpha\beta}\,h^{\mu\nu}\,H_{\beta\mu;\nu} &- \eta^{\alpha\beta\mu\nu}\,V_\beta\,\sigma^{\gamma}{}_{\mu}\,X_{\nu\gamma}+\\ &+\frac{1}{2}\,\eta^{\alpha\beta\mu\nu}\,V_\beta\,\sigma^{\gamma}{}_{\mu}\,\pi_{\nu\gamma}-3\,E^{\alpha\beta}\,\omega_\beta+\frac{1}{2}\,\eta^{\alpha\beta\mu\nu}\,V_\beta\,q_{\mu;\nu}=0.\end{aligned} \tag{3.35}$$

Isto encerra, portanto, a tarefa de escrever-se um sistema dinâmico em termos somente das variáveis do conjunto $\mathcal{M}$, equação (3.25). Entretanto, resta ainda um caso específico a ser estudado: o da solução de Milne. Ela apresenta características ligeiramente diversas do caso mais geral, o qual acabamos de analisar e será, por conseguinte, estudada na última seção deste capítulo, a seguir.

3.6 Um Caso Específico: A Solução de Milne

Dos resultados (1.47) e (1.48) obtidos para esta solução específica do modelo cosmológico de Kasner, verifica-se de imediato que a parte elétrica do tensor de Weyl $E_{\mu\nu}$ é nula, constituindo-se portanto em uma "boa" variável de trabalho, embora o *shear* e a expansão continuem não nulos e ainda devam ser substituídos por outras quantidades que sejam zero no *background*.

As equações Quase-Maxwellianas escritas para o *background* da solução de Milne passam, então, a ser escritas como:

$$(\sigma_{\alpha\beta})^{\bullet} + \frac{2}{3}\,\theta\,\sigma_{\alpha\beta} - \frac{2\sigma^2}{3}\,h_{\alpha\beta} + \sigma_{\alpha\mu}\,\sigma^{\mu}{}_{\beta} = 0, \tag{3.36}$$

$$\theta^{\bullet} + \frac{1}{3}\,\theta^2 + 2\sigma^2 = 0. \tag{3.37}$$

e, partindo delas, bem como fazendo uso das relações (1.37) e (1.39) — válidas para qualquer solução do modelo de Kasner — redefinimos novas quantidades: a escalar W, dada pela equação (3.23), e uma nova quantidade tensorial, $Z_{\alpha\beta}$, de forma a enfatizar a sua especificidade para a solução de Milne:

$$Z_{\alpha\beta} \equiv -\frac{1}{3}\,\theta\,\sigma_{\alpha\beta} - \frac{2\sigma^2}{3}\,h_{\alpha\beta} + \sigma_{\alpha\mu}\,\sigma^{\mu}{}_{\beta}, \tag{3.38}$$

$$W \equiv 2\sigma^2 - \frac{2}{3}\,\theta^2.$$

A segunda destas quantidades é nula, como já foi discutido anteriormente neste capítulo (vide equação (3.24)). A variável $Z_{\alpha\beta}$ pode ser imediatamente calculada, de (3.38), para dar:

$$Z_{\alpha\beta} = 0, \quad \forall \quad \alpha \neq \beta,$$

$$
\begin{aligned}
Z_{11} &= -\frac{1}{3}\,\theta\,\sigma_{11} - \frac{2\sigma^2}{3}\,g_{11} + \sigma_{11}\,\sigma^1{}_1 \\
&= -g_{11}\,\left[\frac{1}{3}\,\theta\,\sigma^1{}_1 + \frac{1}{3}\,\left(2\sigma^2\right) - \sigma^1{}_1\,\sigma^1{}_1\right] \\
&= \left[-\frac{1}{9}\,\theta^2 + \frac{1}{3}\,.\frac{2}{3}\,\theta^2 - \left(\frac{1}{3}\,\theta\right)^2\right] \\
&= \theta^2\,\left[-\frac{1}{9} + \frac{2}{9} - \frac{1}{9}\right] \\
&= 0 \equiv Z_{22},
\end{aligned}
$$

$$
\begin{aligned}
Z_{33} &= -g_{33}\,\left[\frac{1}{3}\,\theta\,\sigma^3{}_3 + \frac{1}{3}\,\left(2\sigma^2\right) - \sigma^3{}_3\,\sigma^3{}_3\right] \\
&= t^2\,\left[\frac{2}{9}\,\theta^2 + \frac{2}{9}\,\theta^2 - \left(\frac{2}{3}\,\theta\right)^2\right] \\
&= t^2\,\theta^2\,\left[\frac{4}{9} - \frac{4}{9}\right] \\
&= 0.
\end{aligned}
$$

O conjunto mínimo fechado de "boas"observáveis é definido, pois, da seguinte forma:

$$
\mathcal{M} = \left\{Z_{\alpha\beta}, W, H_{\alpha\beta}, E_{\alpha\beta}, \pi_{\alpha\beta}, \rho, q_\alpha, a_\alpha, \omega_\alpha\right\}, \tag{3.39}
$$

onde repetiremos os passos e procedimentos já definidos para o caso mais geral. Em primeiro lugar, obteremos as equações dinâmicas para $Z_{\alpha\beta}$ e W, de maneira análoga à realizada anteriormente.

I)Equação Dinâmica para $Z_{\alpha\beta}$:

$$
\begin{aligned}
h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (Z_{\alpha\beta})^{\bullet} &= -\frac{1}{3}\,\theta\, \left[h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (\sigma_{\alpha\beta})^{\bullet}\right] - \frac{1}{3}\,\sigma_{\mu\nu}\, [\theta^{\bullet}] - \\
& -\frac{2}{3}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, \left[h^{\gamma}{}_{\alpha}\, h^{\lambda}{}_{\beta}\, (\sigma_{\gamma\lambda})^{\bullet}\right] + \sigma^{\gamma}{}_{(\mu}\, \left[h^{\alpha}{}_{\nu)}\, h^{\beta}{}_{\gamma}\, (\sigma_{\alpha\beta})^{\bullet}\right].
\end{aligned}
$$

Substituindo os colchetes pelas equações QM (3.12) e (3.13) e coletando no lado esquerdo todos os termos escritos nas variáveis de $\mathcal{M}$, vem:

$$
\begin{aligned}
h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (Z_{\alpha\beta})^{\bullet} &+ \frac{1}{2}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu)}{}^{\lambda}\, H_{\lambda\alpha;\beta} + \frac{2}{3}\,\theta\, E_{\mu\nu} - \frac{1}{2}\,\sigma^{\alpha}{}_{(\mu}\, E_{\nu)\alpha} + \\
&+ \frac{1}{3}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, E_{\alpha\beta} - \frac{1}{6}\, h_{\mu\nu}\, h^{\alpha\beta}\, q_{\alpha;\beta} + \frac{1}{4}\, h^{\alpha}{}_{(\mu}\, h^{\beta}{}_{\nu)}\, q_{\alpha;\beta} - \\
&- frac19\,\theta\, h_{\mu\nu}\, h^{\alpha\beta}\, a_{\alpha;\beta} + \frac{1}{6}\,\theta\, h^{(\alpha}{}_{\mu}\, h^{\beta)}{}_{\nu}\, a_{\alpha;\beta} + \sigma_{\mu\nu}\, h^{\alpha\beta}\, a_{\alpha;\beta} + \\
&+ \frac{2}{3}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, a_{\alpha;\beta} - \frac{1}{2}\, h^{\alpha}{}_{(\mu}\, \sigma^{\beta}{}_{\nu)}\, a_{(\alpha;\beta)} + \frac{1}{3}\,\sigma_{\mu\nu}\,\rho - \\
&- \frac{1}{2}\, h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (\pi_{\alpha\beta})^{\bullet} - \frac{1}{6}\,\theta\,\pi_{\mu\nu} - \frac{1}{6}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\,\pi_{\alpha\beta} + \frac{1}{4}\,\sigma^{\alpha}{}_{(\mu}\,\pi_{\nu)\alpha} \\
&= \frac{1}{3}\,\theta^{2}\,\sigma_{\mu\nu} + \left(2\sigma^{2}\right)\,\sigma_{\mu\nu} + \theta\,\left(\frac{2\sigma^{2}}{3}\right)\, h_{\mu\nu} - \theta\,\sigma_{\mu\alpha}\,\sigma^{\alpha}{}_{\nu} + \\
&\quad +\frac{2}{3}\, h_{\mu\nu}\,\sigma^{\alpha\beta}\,\sigma_{\alpha\gamma}\,\sigma^{\gamma}{}_{\beta} - 2\,\sigma^{\alpha\beta}\,\sigma_{\alpha\mu}\,\sigma_{\beta\nu} \\
&\equiv LD,
\end{aligned}
$$

e, usando as definições (3.23) e (3.38), escrevemos LD acima em termos de $Z_{\alpha\beta}$ e W:

$$
\begin{aligned}
LD \equiv\; & \frac{1}{3}\,\theta^{2}\,\sigma_{\mu\nu} + \left(2\sigma^{2}\right)\,\sigma_{\mu\nu} + \theta\,\left(\frac{2\sigma^{2}}{3}\right)\, h_{\mu\nu} - \theta\,\sigma_{\mu\alpha}\,\sigma^{\alpha}{}_{\nu} + \\
& +\frac{2}{3}\, h_{\mu\nu}\,\sigma^{\alpha\beta}\,\sigma_{\alpha\gamma}\,\sigma^{\gamma}{}_{\beta} - 2\,\sigma^{\alpha\beta}\,\sigma_{\alpha\mu}\,\sigma_{\beta\nu}
\end{aligned}
$$

$$= -\frac{5}{3}\,\theta\, Z_{\alpha\beta} - \sigma^{\alpha}{}_{(\mu}\, Z_{\nu)\alpha} + \frac{2}{3}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, Z_{\alpha\beta} + \frac{1}{3}\, \sigma_{\mu\nu}\, W,$$

o que nos dá a seguinte equação dinâmica para $Z_{\alpha\beta}$:

$$\begin{aligned}
h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (Z_{\alpha\beta})^{\bullet} &+ \frac{5}{3}\,\theta\, Z_{\mu\nu} + \sigma^{\alpha}{}_{(\mu}\, Z_{\nu)\alpha} - \frac{2}{3}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, Z_{\alpha\beta} + \\
&+ \frac{1}{2}\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu)}{}^{\lambda}\, H_{\lambda\alpha;\beta} + \frac{2}{3}\,\theta\, E_{\mu\nu} - \frac{1}{2}\,\sigma^{\alpha}{}_{(\mu}\, E_{\nu)\alpha} + \frac{1}{3}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, E_{\alpha\beta} - \\
&- \frac{1}{6}\, h_{\mu\nu}\, h^{\alpha\beta}\, q_{\alpha;\beta} + \frac{1}{4}\, h^{\alpha}{}_{(\mu}\, h^{\beta}{}_{\nu)}\, q_{\alpha;\beta} - \frac{1}{9}\,\theta\, h_{\mu\nu}\, h^{\alpha\beta}\, a_{\alpha;\beta} + \frac{1}{6}\,\theta\, h^{(\alpha}{}_{\mu}\, h^{\beta)}{}_{\nu}\, a_{\alpha;\beta} + \\
&+ \sigma_{\mu\nu}\, h^{\alpha\beta}\, a_{\alpha;\beta} + \frac{2}{3}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, a_{\alpha;\beta} - \frac{1}{2}\, h^{\alpha}{}_{(\mu}\, \sigma^{\beta}{}_{\nu)}\, a_{(\alpha;\beta)} + \frac{1}{3}\, \sigma_{\mu\nu}\, \rho - \\
&- \frac{1}{3}\, \sigma_{\mu\nu}\, W - \frac{1}{2}\, h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (\pi_{\alpha\beta})^{\bullet} - \frac{1}{6}\,\theta\, \pi_{\mu\nu} - \frac{1}{6}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, \pi_{\alpha\beta} + \\
&+ \frac{1}{4}\, \sigma^{\alpha}{}_{(\mu}\, \pi_{\nu)\alpha} = 0. \qquad (3.40)
\end{aligned}$$

II)Equação Dinâmica para W:

$$\begin{aligned}
W^{\bullet} &= 2\,\sigma^{\mu\nu}\left[h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (\sigma_{\alpha\beta})^{\bullet}\right] - \frac{4}{3}\,\theta\,[\theta^{\bullet}] \\
&= 2\,\sigma^{\alpha\beta}\, a_{\alpha;\beta} - \frac{4}{3}\,\theta\, a^{\alpha}{}_{;\alpha} - \sigma^{\alpha\beta}\, \pi_{\alpha\beta} + \\
&\quad + \frac{2}{3}\,\theta\,(1+3\lambda)\,\rho - 2\,\sigma^{\alpha\beta}\, E_{\alpha\beta} - 2\,\sigma^{\alpha\beta}\, \sigma_{\alpha\mu}\, \sigma^{\mu}{}_{\beta} + \frac{4}{9}\,\theta^{3}.
\end{aligned}$$

Entretanto, os dois últimos termos acima podem ser calculados como:

$$\begin{aligned}
\left[-2\,\sigma^{\alpha\beta}\, \sigma_{\alpha\mu}\, \sigma^{\mu}{}_{\beta} + \frac{4}{9}\,\theta^{3}\right] &= -2\,\sigma^{\alpha\beta}\, \sigma_{\alpha\mu}\, \sigma^{\mu}{}_{\beta} + \frac{2}{3}\,\theta\,\left[-W + \left(2\sigma^{2}\right)\right] \\
&= -\frac{2}{3}\,\theta\, W - 2\,\sigma^{\alpha\beta}\left[\sigma_{\alpha\mu}\, \sigma^{\mu}{}_{\beta} - \frac{1}{3}\,\theta\, \sigma_{\alpha\beta}\right]
\end{aligned}$$

$$= \; -\frac{2}{3}\,\theta\,W + 2\,\sigma^{\alpha\beta}\left[Z_{\alpha\beta} + \frac{2\sigma^2}{3}\,h_{\alpha\beta}\right]$$

$$= \; -\frac{2}{3}\,\theta\,W + 2\,\sigma^{\alpha\beta}\,Z_{\alpha\beta},$$

donde vem afinal a equação dinâmica de W para a solução de Milne:

$$\begin{aligned} W^{\bullet} \; &+ \; \frac{2}{3}\,\theta\,W - 2\,\sigma^{\alpha\beta}\,Z_{\alpha\beta} + 2\,\sigma^{\alpha\beta}\,E_{\alpha\beta} - \\ &- \; 2\,\sigma^{\alpha\beta}\,a_{\alpha;\beta} + \frac{4}{3}\,\theta\,a^{\alpha}{}_{;\alpha} - \frac{2}{3}\,(1+3\lambda)\,\rho + \sigma^{\alpha\beta}\,\pi_{\alpha\beta} = 0. \end{aligned} \tag{3.41}$$

Com os resultados (3.40) e (3.41) obtidos, podemos passar às demais equações dinâmicas do sistema.

III)Equação Dinâmica de $H_{\alpha\beta}$:

É dada pela terceira projeção, equação (3.8), a qual já está escrita em termos das observáveis do conjunto $\mathcal{M}$. Faz-se necessária somente a eliminação dos termos de segunda ordem de perturbação para escrevermos:

$$\begin{aligned} h^{\alpha}{}_{\mu}\,h^{\beta}{}_{\nu}\,(H_{\alpha\beta})^{\bullet} \; &+ \; \frac{2}{3}\,\theta\,H_{\mu\nu} - \frac{1}{2}\,\sigma^{\alpha}{}_{(\mu}\,H_{\nu)\alpha} + \eta_{\mu}{}^{\alpha\gamma\varepsilon}\,\eta_{\nu}{}^{\beta\lambda\tau}\,V_{\gamma}\,V_{\lambda}\,\sigma_{\alpha\beta}\,H_{\varepsilon\tau} + \\ &+ \; \frac{1}{3}\,\theta\,\eta_{\mu}{}^{\alpha\gamma\varepsilon}\,\eta_{\nu}{}^{\beta\lambda\tau}\,V_{\gamma}\,V_{\lambda}\,h_{\alpha\beta}\,H_{\varepsilon\tau} - \frac{1}{2}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu)}{}^{\lambda}\,E_{\lambda\alpha;\beta} - \\ &- \; \frac{1}{4}\,\eta_{(\mu}{}^{\alpha\beta\gamma}\,V_{\gamma}\,\sigma_{\nu)\lambda}\,q_{\alpha} - \frac{1}{4}\,\eta_{(\mu}{}^{\alpha\beta\gamma}\,V_{\gamma}\,h_{\nu)\lambda}\,\pi_{\lambda\alpha;\beta} = 0. \end{aligned} \tag{3.42}$$

IV)Equação Dinâmica de $E_{\alpha\beta}$:

É dada pela quarta projeção, equação (3.9), a qual também já está escrita em termos das "boas" variáveis de trabalho. Eliminando termos de segunda ordem de perturbação, vem por conseguinte que:

$$
\begin{aligned}
h^\alpha{}_\mu \, h^\beta{}_\nu \, (E_{\alpha\beta})^\bullet \; &+ \; \frac{2}{3}\,\theta\, E_{\mu\nu} - \frac{1}{2}\,\sigma^\alpha{}_{(\mu}\, E_{\nu)\alpha} + \eta_\mu{}^{\alpha\gamma\varepsilon}\, \eta_\nu{}^{\beta\lambda\tau}\, V_\gamma\, V_\lambda\, \sigma_{\alpha\beta}\, E_{\varepsilon\tau} + \\
&+ \; \frac{1}{3}\,\theta\, \eta_\mu{}^{\alpha\gamma\varepsilon}\, \eta_\nu{}^{\beta\lambda\tau}\, V_\gamma\, V_\lambda\, h_{\alpha\beta}\, E_{\varepsilon\tau} + \frac{1}{2}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\, V_\gamma\, h_{\nu)}{}^\lambda\, H_{\lambda\alpha;\beta} - \\
&- \; \frac{1}{6}\, h_{\mu\nu}\, q^\alpha{}_{;\alpha} + \frac{1}{4}\, h_{(\mu}{}^\alpha\, h_{\nu)}{}^\beta\, q_{\alpha;\beta} + \frac{1}{2}\,(1+\lambda)\,\sigma_{\mu\nu}\,\rho - \\
&- \; \frac{1}{2}\, h_\mu{}^\alpha\, h_\nu{}^\beta\, (\pi_{\alpha\beta})^\bullet - \frac{1}{6}\,\theta\,\pi_{\mu\nu} + \frac{1}{6}\, h_{\mu\nu}\,\sigma^{\alpha\beta}\,\pi_{\alpha\beta} - \frac{1}{4}\,\sigma^\alpha{}_{(\mu}\,\pi_{\nu)\alpha} = 0.
\end{aligned}
\tag{3.43}
$$

V)Equação Dinâmica de ρ:

Vem da primeira lei de conservação, equação (3.10):

$$
\rho^\bullet + (1+\lambda)\,\theta\,\rho + q^\alpha{}_{;\alpha} - \sigma^{\alpha\beta}\,\pi_{\alpha\beta} = 0. \tag{3.44}
$$

VI)Equação Dinâmica de q_α:

Vem da segunda lei de conservação, dada pela equação (3.11):

$$
\begin{aligned}
h_\alpha{}^\beta\, q_\beta \; &+ \; \frac{4}{3}\,\theta\, q_\alpha + \sigma_\alpha{}^\beta\, q_\beta - \lambda\, h_\alpha{}^\beta\, \rho_{,\beta} + \\
&+ \; \pi_\alpha{}^\beta{}_{;\beta} + V_\alpha \left(\sigma^{\beta\gamma} + \frac{1}{3}\,\theta\, h^{\beta\gamma}\right) \pi_{\beta\gamma} = 0.
\end{aligned}
\tag{3.45}
$$

VII)Equação Dinâmica de ω_α:

É dada pela equação (3.14), sem qualquer alteração.

Resta-nos, agora, verificar o restante das equações QM. A primeira projeção, equação (3.6), fica, após a eliminação dos termos em segunda ordem de perturbação:

$$
\begin{aligned}
h_\alpha{}^\beta\, h^{\mu\nu}\, E_{\beta\mu;\nu} &+ \eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, \sigma_\mu{}^\gamma\, H_{\nu\gamma} - \frac{1}{3}\,\theta\, h_\alpha{}^\beta\, q_\beta + \\
&+ \frac{1}{2}\,\sigma_\alpha{}^\beta\, q_\beta - \frac{1}{3}\, h_\alpha{}^\beta\, \rho_{,\beta} - \frac{1}{2}\, h_\alpha{}^\beta\, \pi_\beta{}^\mu{}_{;\mu} = 0.
\end{aligned}
\tag{3.46}
$$

A segunda projeção, equação (3.7), se escreve como (após a eliminação dos termos de segunda ordem):

$$
\begin{aligned}
h_\alpha{}^\beta\, h^{\mu\nu}\, H_{\beta\mu;\nu} &- \eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, \sigma_\mu{}^\gamma\, E_{\nu\gamma} + \frac{1}{2}\,\eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, q_{\mu;\nu} + \\
&+ \frac{1}{2}\,\eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, \sigma_\mu{}^\gamma\, \pi_{\nu\gamma} = 0.
\end{aligned}
\tag{3.47}
$$

A primeira equação de vínculo, (3.15), nos dá:

$$
h_\alpha{}^\beta\, q_\beta - \sigma_\alpha{}^\beta\, a_\beta - h_\alpha{}^\beta\, \omega_\beta{}^\mu{}_{;\mu} = \left[h_\alpha{}^\beta\, \sigma_\beta{}^\mu{}_{;\mu} - \frac{2}{3}\, h_\alpha{}^\beta\, \theta_{,\beta}\right]. \tag{3.48}
$$

Podemos calcular o termo entre colchetes acima, obtendo os seguintes resultados:

$$
W_{,\beta} = \left(2\sigma^2\right)_{,\beta} - \frac{4}{3}\,\theta\,\theta_{,\beta},
$$

$$
\begin{aligned}
\left[h_\alpha{}^\beta\, \sigma_\beta{}^\mu{}_{;\mu} - \frac{2}{3}\, h_\alpha{}^\beta\, \theta_{,\beta}\right] &= -\frac{3}{\theta}\, h_\alpha{}^\beta\, Z_\beta{}^\mu{}_{;\mu} - \frac{2}{3}\,\theta\, h_\alpha{}^\beta\, a_\alpha - \frac{1}{\theta}\, h_\alpha{}^\beta\, W_{,\beta} - \\
&-\frac{3}{\theta}\left\{h_\alpha{}^\beta\, \left(\sigma_{\beta\lambda}\,\sigma^{\lambda\mu}\right)_{;\mu} - \frac{1}{3}\,\left(\sigma_\alpha{}^\beta - \frac{1}{3}\, h_\alpha{}^\beta\right)\,\theta_{,\beta}\right\},
\end{aligned}
$$

donde vem então:

$$
\begin{aligned}
h_\alpha{}^\beta\, Z_\beta{}^\mu{}_{;\mu} & - \frac{1}{3}\,\theta\, h_\alpha{}^\beta\, \omega_\beta{}^\mu{}_{;\mu} + \frac{1}{3}\,\theta\, h_\alpha{}^\beta\, q_\beta - \frac{1}{3}\,\theta\, \sigma_\alpha{}^\beta\, a_\beta + \frac{2}{9}\,\theta^2\, h_\alpha{}^\beta\, a_\beta + \\
& + \frac{1}{3}\, h_\alpha{}^\beta\, W_{,\beta} + \left\{ h_\alpha{}^\beta \left(\sigma_{\beta\lambda}\, \sigma^{\lambda\mu}\right)_{;\mu} - \frac{1}{3} \left(\sigma_\alpha{}^\beta - \frac{1}{3}\,\theta\, h_\alpha{}^\beta\right) \theta_{,\beta} \right\} = 0,
\end{aligned}
\tag{3.49}
$$

onde os termos no lado direito da equação (3.49) são também nulos no *background* em questão e, como têm apenas um índice, não são definidos para perturbações tensoriais.

A segunda equação de vínculo, (3.16), nos dá tão somente o resultado:

$$
\omega^\alpha{}_{;\alpha} = 0. \tag{3.50}
$$

Finalmente, a terceira equação de vínculo, (3.17), simetrizada em seus índices livres, nos dá que:

$$
2\, H_{\alpha\beta} + \frac{1}{2}\, h_{(\alpha}{}^\mu\, h_{\beta)}{}^\nu\, \eta_\mu{}^{\gamma\lambda\tau}\, V_\tau\, \omega_{\nu\gamma;\lambda} + \frac{1}{2}\, h_{(\alpha}{}^\mu\, h_{\beta)}{}^\nu\, \eta_\mu{}^{\gamma\lambda\tau}\, V_\tau\, \sigma_{\nu\gamma;\lambda} = 0, \tag{3.51}
$$

onde podemos escrever $\left(\sigma^\lambda{}_{\alpha;\beta}\right)$ em termos de $Z_{\alpha\beta}$ e W como:

$$
\begin{aligned}
\sigma^\lambda{}_{\alpha;\beta} & = -\frac{3}{\theta}\, Z^\lambda{}_{\alpha;\beta} + \frac{3}{4\theta^2}\, \sigma^\lambda{}_\alpha\, \sigma^\varepsilon{}_\tau\, (\sigma^\tau{}_{\varepsilon;\beta}) + \\
& \quad + \frac{3}{\theta}\, \sigma^\varepsilon{}_\alpha \left(\sigma^\lambda{}_\varepsilon\right)_{;\beta} + \frac{3}{\theta}\, \sigma^\lambda{}_\varepsilon\, (\sigma^\varepsilon{}_{\alpha;\beta}),
\end{aligned}
$$

e verifica-se que a derivada covariante do *shear* não pode ser eliminada em termos apenas das quantidades auxiliares, embora ela seja também nula no *background* e, portanto, uma "boa" quantidade de trabalho para o modelo que estamos considerando. Para simplificar os cálculos posteriores, entretanto, vamos considerá-la daqui por diante como nula mesmo após a perturbação.

Isto dá conta de todas as equações Quase-Maxwellianas e podemos, agora, passar à tarefa de analisar as perturbações tensorial, escalar e vetorial — o que será feito nos capítulos que se seguem.

Capítulo 4

Perturbações Tensoriais no Modelo de Kasner

4.1 Introdução

A exemplo do que foi feito para o modelo de Friedmann-Robertson- Walker [10, 17, 25], perturbaremos o sistema dinâmico obtido no capítulo anterior para as variáveis do conjunto $\mathcal{M}$, (3.25). Neste capítulo, analisaremos somente as perturbações associadas às ondas gravitacionais. Apenas as quantidades tensoriais serão consideradas neste caso, o que inicialmente reduz o conjunto básico $\mathcal{M}$ a apenas quatro variáveis:

$$\mathcal{M} = \{X_{\alpha\beta}\ ,\ Y_{\alpha\beta}\ ,\ H_{\alpha\beta}\ ,\ \pi_{\alpha\beta}\}\,.$$

Entretanto, da termodinâmica causal, a equação de evolução da pressão anisotrópica envolve o *shear* [41]:

$$\tau\ (\pi_{ij})^{\bullet} + \pi_{ij} = \xi\,\sigma_{ij},$$

onde τ denota o parâmetro de relaxamento e ξ é o parâmetro de viscosidade. No caso mais geral, τ e ξ são funções das variáveis de equilíbrio (como, por exemplo, a densidade

ρ e a temperatura T). Novamente, como no caso FRW [10], vamos nos limitar ao caso em que o parâmetro de relaxação pode ser considerado como negligível, o que nos leva a:

$$\pi_{ij} = \xi\, \sigma_{ij}, \tag{4.1}$$

com a viscosidade dependendo do tempo, via variáveis de equilíbrio do sistema. Isto coloca-nos ante o seguinte problema: como é possível que uma "boa" variável de trabalho seja escrita em termos de uma outra, que é gauge-dependente? A resposta está na viscosidade, a qual — sendo zero no *background* — seria também uma quantidade gauge-independente após a perturbação. Entretanto, esta quantidade não está definida para perturbações tensoriais e, portanto, devemos tomar

$$\xi = 0, \tag{4.2}$$

com

$$(\delta\pi_{ij}) = \xi\,(\delta\sigma_{ij}) = 0, \tag{4.3}$$

após a perturbação, para preservarmos a consistência do sistema dinâmico. Com isto, a pressão anisotrópica é removida do conjunto mínimo $\mathcal{M}$ de "boas" variáveis de trabalho, reduzindo-o a apenas três quantidades:

$$\mathcal{M} = \{X_{\alpha\beta}\ ,\ Y_{\alpha\beta}\ ,\ H_{\alpha\beta}\}\,.$$

Podemos, então, proceder ao passo seguinte, que é o de expandir as quantidades perturbadas em termos da base tensorial $\hat{U}^i{}_j$, analisada no Capítulo 2:

$$(\delta X^i{}_j) = X(t)\,\hat{U}^i{}_j$$

$$(\delta Y^i{}_j) = Y(t)\,\hat{U}^i{}_j \tag{4.4}$$

$$(\delta H^i{}_j) = H(t)\,\hat{U}^{\hat{i}}{}_j.$$

As equações do sistema dinâmico, em sua forma perturbada, são listadas a seguir para o caso tensorial. Foram descartados todos os termos escalares e vetoriais (não definidos para este tipo de perturbação), bem como a quantidade $(\delta\pi_{ij})$, por conta da relação (4.3). A equação dinâmica de $X_{\alpha\beta}$, (3.27), é dada por:

$$\begin{aligned} h^{\mu}{}_{\alpha}\, h^{\beta}{}_{\nu}\, (\delta X^{\alpha}{}_{\beta})^{\bullet} \;\; + \;\; & \frac{5}{3}\,\theta\,(\delta X^{\mu}{}_{\nu}) + \sigma^{\mu}{}_{\alpha}\,(\delta X^{\alpha}{}_{\nu}) + \sigma^{\alpha}{}_{\nu}\,(\delta X^{\mu}{}_{\alpha}) - \\ & -\frac{2}{3}\,h^{\mu}{}_{\nu}\,\sigma^{\alpha}{}_{\beta}\left(\delta X^{\beta}{}_{\alpha}\right) - (\delta Y^{\mu}{}_{\nu}) + \frac{1}{2}\,\eta^{\mu\gamma}{}_{\alpha}{}^{\beta}\,V_{\gamma}\,h_{\nu}{}^{\lambda}\,(\delta H^{\alpha}{}_{\lambda})_{;\beta} + \\ & +\frac{1}{2}\,\eta_{\nu}{}^{\gamma}{}_{\alpha}{}^{\beta}\,V_{\gamma}\,h^{\mu\lambda}\,(\delta H^{\alpha}{}_{\lambda})_{;\beta} = 0. \end{aligned} \tag{4.5}$$

A equação dinâmica de $Y_{\alpha\beta}$, (3.28), é escrita para perturbações tensoriais como:

$$\begin{aligned} h_{\alpha}{}^{\mu}\, h_{\nu}{}^{\beta}\, (\delta Y^{\alpha}{}_{\beta})^{\bullet} \;\; + \;\; & 2\theta\,(\delta Y^{\mu}{}_{\nu}) - \frac{3}{2}\,\sigma^{\mu}{}_{\alpha}\,(\delta Y^{\alpha}{}_{\nu}) - \frac{3}{2}\,\sigma^{\alpha}{}_{\nu}\,(\delta Y^{\mu}{}_{\alpha}) + h^{\mu}{}_{\nu}\,\sigma^{\alpha}{}_{\beta}\left(\delta Y^{\beta}{}_{\alpha}\right) + \\ + \;\; & \frac{3}{2}\,E^{\mu}{}_{\alpha}\,(\delta X^{\alpha}{}_{\nu}) + \frac{3}{2}\,E^{\alpha}{}_{\nu}\,(\delta X^{\mu}{}_{\alpha}) - h^{\mu}{}_{\nu}\,E^{\alpha}{}_{\beta}\left(\delta X^{\beta}{}_{\alpha}\right) + \\ + \;\; & \frac{1}{2}\,\theta\,\eta^{\mu\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu\lambda}\left(\delta H^{\lambda}{}_{\alpha}\right)_{;\beta} + \frac{1}{2}\,\theta\,\eta_{\nu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h^{\mu}{}_{\lambda}\left(\delta H^{\lambda}{}_{\alpha}\right)_{;\beta} + \\ + \;\; & \frac{3}{4}\,\eta^{\mu\gamma\alpha\beta}\,V_{\gamma}\,\sigma_{\nu\lambda}\left(\delta H^{\lambda}{}_{\alpha}\right)_{;\beta} + \frac{3}{4}\,\eta_{\nu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,\sigma^{\mu}{}_{\lambda}\left(\delta H^{\lambda}{}_{\alpha}\right)_{;\beta} + \\ + \;\; & \frac{3}{4}\,\eta^{\gamma\lambda\alpha\beta}\,V_{\lambda}\,\sigma_{\gamma}{}^{\mu}\,h_{\nu\tau}\,(\delta H^{\tau}{}_{\alpha})_{;\beta} + \frac{3}{4}\,\eta^{\gamma\lambda\alpha\beta}\,V_{\lambda}\,\sigma_{\gamma\nu}\,h^{\mu}{}_{\tau}\,(\delta H^{\tau}{}_{\alpha})_{;\beta} - \\ - \;\; & h^{\mu}{}_{\nu}\,\eta^{\gamma\tau\alpha\beta}\,V_{\tau}\,\sigma_{\gamma\lambda}\left(\delta H^{\lambda}{}_{\alpha}\right)_{;\beta} \\ = \;\; & 0. \end{aligned} \tag{4.6}$$

A equação dinâmica de W, (3.29), escrita para perturbações tensoriais fica:

$$2\,\sigma^{\alpha}{}_{\beta}\left(\delta X^{\beta}{}_{\alpha}\right) = 0. \tag{4.7}$$

e a equação dinâmica de $H_{\alpha\beta}$, (3.32), para este caso, é:

$$\begin{aligned}
h^{\mu}{}_{\alpha}\,h^{\beta}{}_{\nu}\,(\delta H^{\alpha}{}_{\beta})^{\bullet} &+ \frac{4}{3}\,\theta\,(\delta H^{\mu}{}_{\nu}) - \frac{1}{2}\,\sigma^{\mu}{}_{\alpha}\,(\delta H^{\alpha}{}_{\nu}) - \frac{1}{2}\,\sigma^{\alpha}{}_{\nu}\,(\delta H^{\mu}{}_{\alpha}) + \\
&+ \eta^{\mu\alpha\gamma}{}_{\varepsilon}\,\eta_{\nu}{}^{\beta\lambda\tau}\,V_{\gamma}\,V_{\lambda}\,\sigma_{\alpha\beta}\,(\delta H^{\varepsilon}{}_{\tau}) + \frac{1}{3}\,\theta\,\eta^{\mu\alpha\gamma}{}_{\varepsilon}\,\eta_{\nu}{}^{\beta\lambda\tau}\,V_{\gamma}\,V_{\lambda}\,h_{\alpha\beta}\,(\delta H^{\varepsilon}{}_{\tau}) - \\
&- \frac{1}{2}\,\eta^{\mu\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu\lambda}\left(\delta X^{\lambda}{}_{\alpha}\right)_{;\beta} - \frac{1}{2}\,\eta_{\nu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h^{\mu}{}_{\lambda}\left(\delta X^{\lambda}{}_{\alpha}\right)_{;\beta} \\
&= 0.
\end{aligned} \tag{4.8}$$

Além disto, a primeira e a segunda projeções, equações (3.34) e (3.35), reduzem-se no caso das perturbações tensoriais a:

$$h^{\alpha}{}_{\beta}\,h^{\mu\nu}\left(\delta X^{\beta}{}_{\mu}\right)_{;\nu} + \eta^{\alpha\beta\mu\nu}\,V_{\beta}\,\sigma_{\mu\gamma}\,(\delta H^{\gamma}{}_{\nu}) = 0, \tag{4.9}$$

$$h^{\alpha}{}_{\beta}\,h^{\mu\nu}\left(\delta H^{\beta}{}_{\mu}\right)_{;\nu} - \eta^{\alpha\beta\mu\nu}\,V_{\beta}\,\sigma_{\mu\gamma}\,(\delta X^{\gamma}{}_{\nu}) = 0. \tag{4.10}$$

Resta-nos, agora, a tarefa de resolver o sistema dinâmico formado pelas equações (4.5) a (4.10). Faremos isto para os dois modelos de Kasner com plano de isotropia: a solução de Kasner propriamente dita e a de Milne, a qual — como já vimos — tem um sistema dinâmico diferente do encontrado para o caso mais geral.

4.2 Solução de Kasner

Se considerarmos o caso especial da solução de Kasner, com:

$$
\begin{aligned}
p_1 &= p_2 = \tfrac{2}{3} \\
p_3 &= -\tfrac{1}{3},
\end{aligned}
\tag{4.11}
$$

temos, da equação (4.7) e da definição (4.4), que

$$X(t)\,\left(\sigma^{\alpha}{}_{\beta}\,\hat{U}^{\beta}{}_{\alpha}\right) = 0,$$

onde, pelo resultado (2.90), é fácil verificar-se que a relação acima é identicamente satisfeita para esta solução.

A primeira projeção (4.9) nos dá:

$$X(t)\,h^{\alpha}{}_{\beta}\,h^{\mu\nu}\,\hat{U}^{\beta}{}_{\mu;\nu} + H(t)\,\eta^{\alpha\beta\mu\nu}\,V_{\beta}\,\sigma_{\mu\gamma}\,\hat{U}^{*\gamma}{}_{\nu} = 0.$$

E, das relações (2.66) e (2.103), temos que ela também é identicamente satisfeita. Da mesma forma, a segunda projeção, equação (4.10), nos dá que:

$$H(t)\,h^{\alpha}{}_{\beta}\,h^{\mu\nu}\,\hat{U}^{*\beta}{}_{\mu;\nu} - X(t)\,\eta^{\alpha\beta\mu\nu}\,V_{\beta}\,\sigma_{\mu\gamma}\,\hat{U}^{\gamma}{}_{\nu} = 0,$$

onde as relações (2.78) e (2.93) nos dão — novamente — que também esta equação é identicamente satisfeita.

A equação dinâmica de X, (4.5), fica:

$$
\begin{aligned}
X^{\bullet}\,\hat{U}^{\mu}{}_{\nu} \;&+\; X\,h^{\mu}{}_{\alpha}\,h^{\beta}{}_{\nu}\,\left(\hat{U}^{\alpha}{}_{\beta}\right)^{\bullet} + \frac{5}{3}\,\theta\,X\,\hat{U}^{\mu}{}_{\nu} + \\
&+\; X\,\left(\sigma^{\mu}{}_{\alpha}\,\hat{U}^{\alpha}{}_{\nu} + \sigma^{\alpha}{}_{\nu}\,\hat{U}^{\mu}{}_{\alpha}\right) - \frac{2}{3}\,X\,h^{\mu}{}_{\nu}\,\left(\sigma^{\alpha}{}_{\beta}\,\hat{U}^{\beta}{}_{\alpha}\right) - Y\,\hat{U}^{\mu}{}_{\nu} + \\
&+\; \frac{1}{2}\,H\,\left(\eta^{\mu\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta} + \eta_{\nu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h^{\mu}{}_{\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta}\right)
\end{aligned}
$$

$$= \quad 0. \tag{4.12}$$

E, usando os resultados (2.88) a (2.90) e (2.100) em (4.12) acima, temos (após fatorarmos a base $\hat{U}^{\mu}{}_{\nu}$):

$$X^{\bullet} + \frac{7}{3}\,\theta\,X - Y + k^2\,H = 0. \tag{4.13}$$

A equação dinâmica de Y, (4.6), é dada por:

$$\begin{aligned}
Y^{\bullet}\,\hat{U}^{\mu}{}_{\nu} \quad &+ \quad Y\,h^{\mu}{}_{\alpha}\,h^{\beta}{}_{\nu}\,\left(\hat{U}^{\alpha}{}_{\beta}\right)^{\bullet} + 2\,\theta\,Y\,\hat{U}^{\mu}{}_{\nu} - \\
&- \quad \frac{3}{2}\,Y\,\left(\sigma^{\mu}{}_{\alpha}\,\hat{U}^{\alpha}{}_{\nu} + \sigma^{\alpha}{}_{\nu}\,\hat{U}^{\mu}{}_{\alpha}\right) + Y\,h^{\mu}{}_{\nu}\,\left(\sigma^{\alpha}{}_{\beta}\,\hat{U}^{\beta}{}_{\alpha}\right) + \\
&+ \quad \frac{3}{2}\,X\,\left(E^{\mu}{}_{\alpha}\,\hat{U}^{\alpha}{}_{\nu} + E^{\alpha}{}_{\nu}\,\hat{U}^{\mu}{}_{\alpha}\right) - X\,h^{\mu}{}_{\nu}\,\left(E^{\alpha}{}_{\beta}\,\hat{U}^{\beta}{}_{\alpha}\right) + \\
&+ \quad \frac{1}{2}\,\theta\,H\,\left[\eta^{\mu\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta} + \eta_{\nu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h^{\mu}{}_{\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta}\right] + \\
&+ \quad \frac{3}{4}\,H\,\left[\eta^{\mu\gamma\alpha\beta}\,V_{\gamma}\,\sigma_{\nu\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta} + \eta_{\nu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,\sigma^{\mu}{}_{\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta}\right] + \\
&+ \quad \frac{3}{4}\,H\,\left[\eta^{\gamma\lambda\alpha\beta}\,V_{\lambda}\,\left(\sigma_{\gamma}{}^{\mu}\,h_{\nu\tau} + \sigma_{\gamma\nu}\,h^{\mu}{}_{\tau}\right)\,\hat{U}^{*\tau}{}_{\alpha;\beta}\right] - \\
&- \quad H\,h^{\mu}{}_{\nu}\,\left(\eta^{\gamma\tau\alpha\beta}\,V_{\tau}\,\sigma_{\gamma\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta}\right) \\
&= \quad 0,
\end{aligned} \tag{4.14}$$

e, usando os resultados (2.88) a (2.92) e (2.99) a (2.102) em (4.14), escrevemos afinal:

$$Y^{\bullet} + \theta\,Y + \frac{2}{3}\,\theta^2\,X + 2\,\theta\,k^2\,H = 0. \tag{4.15}$$

Finalmente, a equação dinâmica de H, (4.8), é dada por:

$$H^{\bullet}\,\hat{U}^{*\mu}{}_{\nu} \quad + \quad H\,h^{\mu}{}_{\alpha}\,h^{\beta}{}_{\nu}\,\left(\hat{U}^{*\alpha}{}_{\beta}\right)^{\bullet} + \frac{4}{3}\,\theta\,H\,\hat{U}^{*\mu}{}_{\nu} - \frac{1}{2}\,H\,\left(\sigma^{\mu}{}_{\alpha}\,\hat{U}^{*\alpha}{}_{\nu} + \sigma^{\alpha}{}_{\nu}\,\hat{U}^{*\mu}{}_{\alpha}\right) +$$

$$
\begin{aligned}
&+ \quad H\left(\eta^{\mu\alpha\gamma}{}_{\varepsilon}\,\eta_{\nu}{}^{\beta\lambda\tau}\,V_{\gamma}\,V_{\lambda}\,\sigma_{\alpha\beta}\,\hat{U}^{*\varepsilon}{}_{\tau}\right)+\frac{1}{3}\,\theta\,H\left(\eta^{\mu\alpha\gamma}{}_{\varepsilon}\,\eta_{\nu}{}^{\beta\lambda\tau}\,V_{\gamma}\,V_{\lambda}\,h_{\alpha\beta}\,\hat{U}^{*\varepsilon}{}_{\tau}\right)- \\
&- \quad \frac{1}{2}\,X\left[\eta^{\mu\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta}+\eta_{\nu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h^{\mu}{}_{\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta}\right] \\
&= \quad 0. \qquad (4.16)
\end{aligned}
$$

O quinto, o sexto e o sétimo termos acima podem ser calculados como:

$$
\begin{aligned}
\left(\eta^{\mu\alpha\gamma}{}_{\varepsilon}\,\eta_{\nu}{}^{\beta\lambda\tau}\,V_{\gamma}\,V_{\lambda}\,\sigma_{\alpha\beta}\,\hat{U}^{*\varepsilon}{}_{\tau}\right) &= \eta_{\varphi\alpha\gamma\varepsilon}\,\eta^{\omega\beta\lambda\tau}\,V^{\gamma}\,V_{\lambda}\,h^{\mu\varphi}\,h_{\nu\omega}\,\sigma^{\alpha}{}_{\beta}\,\hat{U}^{*\varepsilon}{}_{\tau} \\
&= -\delta^{\omega\beta\lambda\tau}_{\varphi\alpha\gamma\varepsilon}\,V^{\gamma}\,V_{\lambda}\,h^{\mu\varphi}\,h_{\nu\omega}\,\sigma^{\alpha}{}_{\beta}\,\hat{U}^{*\varepsilon}{}_{\tau} \\
&= -\begin{vmatrix}
\delta^{\omega}_{\varphi} & \delta^{\omega}_{\alpha} & \delta^{\omega}_{\gamma} & \delta^{\omega}_{\varepsilon} \\
\delta^{\beta}_{\varphi} & \delta^{\beta}_{\alpha} & \delta^{\beta}_{\gamma} & \delta^{\beta}_{\varepsilon} \\
\delta^{\lambda}_{\varphi} & \delta^{\lambda}_{\alpha} & \delta^{\lambda}_{\gamma} & \delta^{\lambda}_{\varepsilon} \\
\delta^{\tau}_{\varphi} & \delta^{\tau}_{\alpha} & \delta^{\tau}_{\gamma} & \delta^{\tau}_{\varepsilon}
\end{vmatrix}\,V^{\gamma}\,V_{\lambda}\,h^{\mu\varphi}\,h_{\nu\omega}\,\sigma^{\alpha}{}_{\beta}\,\hat{U}^{*\varepsilon}{}_{\tau} \\
&= -\left(\sigma^{\alpha}{}_{\nu}\,\hat{U}^{*\mu}{}_{\alpha}+\sigma^{\mu}{}_{\alpha}\,\hat{U}^{*\alpha}{}_{\nu}\right) \\
&= -\frac{2}{3}\,\theta\,\hat{U}^{*\mu}{}_{\nu},
\end{aligned}
$$

de (2.98).

$$
\begin{aligned}
\left(\eta^{\mu\alpha\gamma}{}_{\varepsilon}\,\eta_{\nu}{}^{\beta\lambda\tau}\,V_{\gamma}\,V_{\lambda}\,h_{\alpha\beta}\,\hat{U}^{*\varepsilon\tau}\right) &= \eta^{\mu}{}_{\beta}{}^{\gamma}{}_{\varepsilon}\,\eta_{\nu}{}^{\beta\lambda\tau}\,V_{\gamma}\,V_{\lambda}\,\hat{U}^{*\varepsilon}{}_{\tau}- \\
&\quad -\eta^{\mu\alpha\gamma}{}_{\varepsilon}\,\eta_{\nu}{}^{\beta\lambda\tau}\,V_{\gamma}\,V_{\lambda}\,V_{\alpha}\,V_{\beta}\,\hat{U}^{*\varepsilon}{}_{\tau} \\
&= -\delta^{\alpha\lambda\tau}_{\beta\gamma\varepsilon}\,V^{\gamma}\,V_{\lambda}\,h^{\mu\beta}\,h_{\nu\alpha}\,\hat{U}^{*\varepsilon}{}_{\tau}
\end{aligned}
$$

$$= \; - \begin{vmatrix} \delta^{\alpha}_{\beta} & \delta^{\alpha}_{\gamma} & \delta^{\alpha}_{\varepsilon} \\ \delta^{\lambda}_{\beta} & \delta^{\lambda}_{\gamma} & \delta^{\lambda}_{\varepsilon} \\ \delta^{\tau}_{\beta} & \delta^{\tau}_{\gamma} & \delta^{\tau}_{\varepsilon} \end{vmatrix} V^{\gamma}\, V_{\lambda}\, h^{\mu\beta}\, h_{\nu\alpha}\, \hat{U}^{*\varepsilon}{}_{\tau}$$

$$= \; \hat{U}^{*\mu}{}_{\nu},$$

$$\left[\eta^{\mu\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu\lambda}\, \hat{U}^{\lambda}{}_{\alpha;\beta} + \eta_{\nu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h^{\mu}{}_{\lambda}\, \hat{U}^{\lambda}{}_{\alpha;\beta}\right]$$

$$= \; \left(\eta^{\alpha\gamma\beta\mu}\, V_{\gamma}\, h_{\nu\lambda} + \eta^{\alpha\gamma\beta}{}_{\nu}\, V_{\gamma}\, h^{\mu}{}_{\lambda}\right) \left[\hat{U}^{\lambda}{}_{\alpha,\beta} + \Gamma^{\lambda}_{\tau\beta}\, \hat{U}^{\tau}{}_{\alpha} - \Gamma^{tau}_{\alpha\beta}\, \hat{U}^{\lambda}{}_{\tau}\right]$$

$$= \; -i\, k_{\beta}\, \left(\eta^{\alpha\gamma\beta\mu}\, V_{\gamma}\, h_{\nu\lambda} + \eta^{\alpha\gamma\beta}{}_{\nu}\, V_{\gamma}\, h^{\mu}{}_{\lambda}\right) \hat{U}^{\lambda}{}_{\alpha}$$

$$= \; -i\, k_{\beta}\, \left(\eta_{\alpha}{}^{\gamma\beta\mu}\, V_{\gamma}\, \hat{U}^{\alpha}{}_{\nu} + \eta^{\alpha\gamma\beta}{}_{\nu}\, V_{\gamma}\, \hat{U}^{\mu}{}_{\alpha}\right)$$

$$= \; 2\, \hat{U}^{\mu}{}_{\nu},$$

onde foram usadas as relações (2.61) e (2.76). Usando-se, então, as relações (2.88)-(2.93) e (2.96)-(2.103) e fatorando-se $\hat{U}^{*\mu}{}_{\nu}$, escrevemos afinal:

$$H^{\bullet}(t) + \frac{2}{3}\,\theta\, H(t) - X(t) = 0. \tag{4.17}$$

E, com isto, o sistema dinâmico para perturbações tensoriais da solução de Kasner com plano de isotropia $p_1 = p_2$ se reduz às equações (4.13), (4.15) e (4.17) ou, em forma matricial, como:

$$\begin{pmatrix} X^\bullet \\ Y^\bullet \\ H^\bullet \end{pmatrix} = \begin{pmatrix} -\frac{7}{3}\theta & 1 & -k^2 \\ -\frac{2}{3}\theta^2 & -\theta & -2\theta k^2 \\ 1 & 0 & -\frac{2}{3}\theta \end{pmatrix} \begin{pmatrix} X \\ Y \\ H \end{pmatrix}, \tag{4.18}$$

onde X, Y e H são funções do tempo a serem determinadas.

4.3 Solução de Milne

No caso específico da solução de Milne, o conjunto mínimo fechado de observáveis reduz-se a:

$$\mathcal{M} = \{Z_{\alpha\beta}, H_{\alpha\beta}, E_{\alpha\beta}\},$$

onde a pressão anisotrópica foi retirada do conjunto pelos mesmos argumentos exibidos em (4.1)-(4.3). Expandindo as três quantidades restantes em $\mathcal{M}$ na base tensorial, escrevemos então:

$$(\delta Z^i{}_j) = Z(t)\,\hat{U}^i{}_j$$

$$(\delta E^i{}_j) = E(t)\,\hat{U}^i{}_j \tag{4.19}$$

$$(\delta H^i{}_j) = H(t)\,\hat{U}^{*i}{}_j.$$

A equação dinâmica para $Z_{\alpha\beta}$, (3.40), é dada por:

$$\begin{aligned} h^\mu{}_\alpha h^\beta{}_\nu\,(\delta Z^\alpha{}_\beta)^\bullet \;&+\; \frac{5}{3}\theta\,(\delta Z^\mu{}_\nu) + \sigma^\mu{}_\alpha\,(\delta Z^\alpha{}_\nu) + \sigma^\alpha{}_\nu\,(\delta Z^\mu{}_\alpha) - \\ &-\; \frac{2}{3}h^\mu{}_\nu\,\sigma^\alpha{}_\beta\left(\delta Z^\beta{}_\alpha\right) + \frac{1}{2}\eta^{\mu\gamma\alpha\beta}\,V_\gamma\,h_{\nu\lambda}\left(\delta H^\lambda{}_\alpha\right)_{;\beta} + \\ &+\; \frac{1}{2}\eta_\nu{}^{\gamma\alpha\beta}\,V_\gamma\,h^\mu{}_\lambda\left(\delta H^\lambda{}_\alpha\right)_{;\beta} + \frac{2}{3}\theta\,(\delta E^\mu{}_\nu) - \frac{1}{2}\sigma^\mu{}_\alpha\,(\delta E^\alpha{}_\nu) - \end{aligned}$$

$$- \frac{1}{2} \sigma^{\alpha}{}_{\nu} \left(\delta E^{\mu}{}_{\alpha}\right) + \frac{1}{3} h^{\mu}{}_{\nu} \sigma^{\alpha}{}_{\beta} \left(\delta E^{\beta}{}_{\alpha}\right) = 0. \tag{4.20}$$

A equação dinâmica para W, (3.41), reduz-se a um vínculo:

$$\sigma^{\alpha}{}_{\beta} \left[\left(\delta Z^{\beta}{}_{\alpha}\right) - \left(\delta E^{\beta}{}_{\alpha}\right)\right] = 0. \tag{4.21}$$

A equação dinâmica de $H_{\alpha\beta}$, (3.42), é escrita como:

$$\begin{aligned} h^{\mu}{}_{\alpha} h^{\beta}{}_{\nu} \left(\delta H^{\alpha}{}_{\beta}\right)^{\bullet} &+ \frac{2}{3} \theta \left(\delta H^{\mu}{}_{\nu}\right) - \frac{1}{2} \sigma^{\mu}{}_{\alpha} \left(\delta H^{\alpha}{}_{\nu}\right) - \frac{1}{2} \sigma^{\alpha}{}_{\nu} \left(\delta H^{\mu}{}_{\alpha}\right) + \\ &+ \eta^{\mu\alpha\gamma}{}_{\varepsilon} \eta_{\nu}{}^{\beta\lambda\tau} V_{\gamma} V_{\lambda} \sigma_{\alpha\beta} \left(\delta H^{\varepsilon}{}_{\tau}\right) + \\ &+ \frac{1}{3} \theta \eta^{\mu\alpha\gamma}{}_{\varepsilon} \eta_{\nu}{}^{\beta\lambda\tau} V_{\gamma} V_{\lambda} h_{\alpha\beta} \left(\delta H^{\varepsilon}{}_{\tau}\right) - \\ &- \frac{1}{2} \eta^{\mu\gamma\alpha\beta} V_{\gamma} h_{\nu\lambda} \left(\delta E^{\lambda}{}_{\alpha}\right)_{;\beta} - \frac{1}{2} \eta_{\nu}{}^{\gamma\alpha\beta} V_{\gamma} h^{\mu}{}_{\lambda} \left(\delta E^{\lambda}{}_{\alpha}\right)_{;\beta} = 0. \end{aligned} \tag{4.22}$$

A equação dinâmica para $E_{\alpha\beta}$, (3.43), fica:

$$\begin{aligned} h^{\mu}{}_{\alpha} h^{\beta}{}_{\nu} \left(\delta E^{\alpha}{}_{\beta}\right)^{\bullet} &+ \frac{2}{3} \theta \left(\delta E^{\mu}{}_{\nu}\right) - \frac{1}{2} \sigma^{\mu}{}_{\alpha} \left(\delta E^{\alpha}{}_{\nu}\right) - \frac{1}{2} \sigma^{\alpha}{}_{\nu} \left(\delta E^{\mu}{}_{\alpha}\right) + \\ &+ \eta^{\mu\alpha\gamma}{}_{\varepsilon} \eta_{\nu}{}^{\beta\lambda\tau} V_{\gamma} V_{\lambda} \sigma_{\alpha\beta} \left(\delta E^{\varepsilon}{}_{\tau}\right) + \frac{1}{3} \theta \eta \mu\alpha\gamma_{\varepsilon} \eta_{\nu}{}^{\beta\lambda\tau} V_{\gamma} V_{\lambda} h_{\alpha\beta} \left(\delta E^{\varepsilon}{}_{\tau}\right) + \\ &+ \frac{1}{2} \eta^{\mu\gamma\alpha\beta} V_{\gamma} h_{\nu\lambda} \left(\delta H^{\lambda}{}_{\alpha}\right)_{;\beta} + \frac{1}{2} \eta_{\nu}{}^{\gamma\alpha\beta} V_{\gamma} h^{\mu}{}_{\lambda} \left(\delta H^{\lambda}{}_{\alpha}\right)_{;\beta} = 0. \end{aligned} \tag{4.23}$$

As equações dinâmicas de ρ, q_{α} e ω_{α} — (3.44)-(3.45) e (3.14) — são identicamente nulas para perturbações tensoriais em Milne, o que nos deixa com os vínculos do sistema. A primeira projeção, dada pela equação (3.46), é:

$$h_{\alpha\beta} h^{\mu\nu} \left(\delta E^{\beta}{}_{\mu}\right)_{;\nu} + \eta_{\alpha}{}^{\beta\mu\nu} V_{\beta} \sigma_{\mu\gamma} \left(\delta H^{\gamma}{}_{\nu}\right) = 0. \tag{4.24}$$

A segunda projeção, (3.47), se escreve como:

$$h_{\alpha\beta}\, h^{\mu\nu} \left(\delta H^{\beta}{}_{\mu}\right)_{;\nu} - \eta_{\alpha}{}^{\beta\mu\nu}\, V_{\beta}\, \sigma_{\mu\gamma}\, (\delta E^{\gamma}{}_{\nu}) = 0. \tag{4.25}$$

A primeira equação de vínculo, (3.48), nos dá:

$$h_{\alpha\beta}\, h^{\mu\nu} \left(\delta Z^{\beta}{}_{\mu}\right)_{;\nu} = 0. \tag{4.26}$$

A segunda equação de vínculo, (3.50), é também identicamente nula. A última equação de vínculo, (3.51), após a simplificação discutida na seção 3.6, nos dá afinal:

$$(\delta H^{\alpha}{}_{\beta}) = 0, \tag{4.27}$$

e o sistema dinâmico reduz-se às equações (4.20)- (4.27). Podemos, agora, efetuar a decomposição na base tensorial, equação (4.19), em cada uma das equações do sistema. Temos então da equação dinâmica de Z, (4.20), que:

$$\begin{aligned}
Z^{\bullet}\, \hat{U}^{\mu}{}_{\nu} \;&+\; Z\, h^{\mu}{}_{\alpha}\, h^{\beta}{}_{\nu} \left(\hat{U}^{\alpha}{}_{\beta}\right)^{\bullet} + \frac{5}{3}\theta\, Z\, \hat{U}^{\mu}{}_{\nu} + Z \left(\sigma^{\mu}{}_{\alpha}\, \hat{U}^{\alpha}{}_{\nu} + \sigma^{\alpha}{}_{\nu}\, \hat{U}^{\mu}{}_{\alpha}\right) - \\
&-\; \frac{2}{3} Z\, h^{\mu}{}_{\nu}\, \sigma^{\alpha}{}_{\beta}\, \hat{U}^{\beta}{}_{\alpha} + \frac{1}{2} H \left[\eta^{\mu\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu\lambda}\, \hat{U}^{*\lambda}{}_{\alpha;\beta} + \eta_{\nu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h^{\mu}{}_{\lambda}\, \hat{U}^{*\lambda}{}_{\alpha;\beta}\right] + \\
&+\; \frac{2}{3}\theta\, E\, \hat{U}^{\mu}{}_{\nu} - \frac{1}{2} E \left(\sigma^{\mu}{}_{\alpha}\, \hat{U}^{\alpha}{}_{\nu} + \sigma^{\alpha}{}_{\nu}\, \hat{U}^{\mu}{}_{\alpha}\right) + \\
&+\; \frac{1}{3} E\, h^{\mu}{}_{\nu} \left(\sigma^{\alpha}{}_{\beta}\, \hat{U}^{\beta}{}_{\alpha}\right) = 0.
\end{aligned} \tag{4.28}$$

Usando, na equação acima, os resultados (2.107)-(2.109) e (2.117) para a solução de Milne e fatorando $\hat{U}^{\mu}{}_{\nu}$, obtemos a equação em sua forma final:

$$Z^{\bullet} + \theta\, Z + \theta\, E + k^{2}\, H = 0. \tag{4.29}$$

O vínculo (4.21) é identicamente satisfeito, por (2.109). Passando à equação dinâmica de H, (4.30), vem:

$$
\begin{aligned}
H^{\bullet}\,\hat{U}^{*\mu}{}_{\nu} &+ H\,h^{\mu}{}_{\alpha}\,h^{\beta}{}_{\nu}\left(\hat{U}^{*\alpha}{}_{\beta}\right)^{\bullet} + \frac{2}{3}\,\theta\,H\,\hat{U}^{*\mu}{}_{\nu} - \frac{1}{2}\,H\,\left(\sigma^{\mu}{}_{\alpha}\,\hat{U}^{*\alpha}{}_{\nu} + \sigma^{\alpha}{}_{\nu}\,\hat{U}^{*\mu}{}_{\alpha}\right) + \\
&+ H\,\left(\eta^{\mu\alpha\gamma}{}_{\varepsilon}\,\eta_{\nu}{}^{\beta\lambda\tau}\,V_{\gamma}\,V_{\lambda}\,\sigma_{\alpha\beta}\,\hat{U}^{*\varepsilon}{}_{\tau}\right) + \\
&+ \frac{1}{3}\,\theta\,H\,\left(\eta^{\mu\alpha\gamma}{}_{\varepsilon}\,\eta_{\nu}{}^{\beta\lambda\tau}\,V_{\gamma}\,V_{\lambda}\,h_{\alpha\beta}\,\hat{U}^{*\varepsilon}{}_{\tau}\right) - \\
&- \frac{1}{2}\,E\,\left(\eta^{\mu\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu\lambda}\,\hat{U}^{\lambda}{}_{\alpha;\beta} + \eta_{\nu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h^{\mu}{}_{\lambda}\,\hat{U}^{\lambda}{}_{\alpha;\beta}\right) = 0. \qquad (4.30)
\end{aligned}
$$

Usando os resultados (2.113)-(2.115) e calculando os três últimos termos do lado direito da equação acima (de um modo análogo àquele adotado para a solução de Kasner na seção anterior), escrevemos — após fatorar o dual $\hat{U}^{*\mu}{}_{\nu}$:

$$H^{\bullet} + 2\,\theta\,H - E = 0. \qquad (4.31)$$

A equação (4.23), após o uso de (2.107)-(2.109) e (2.117), bem como a fatoração da base $\hat{U}^{\mu}{}_{\nu}$, se escreve como:

$$E^{\bullet} + \frac{4}{3}\,\theta\,E + k^{2}\,H = 0. \qquad (4.32)$$

Passando aos vínculos restantes, é fácil ver que as equações (4.24)- (4.26) são identicamente satisfeitas para perturbações tensoriais, devido às propriedades da base tensorial e de seu dual (especialmente a da divergência nula). A equação (4.27) reduz-se a:

$$H = 0, \qquad (4.33)$$

o que nos dá — da equação (4.31):

$$E = 0. \qquad (4.34)$$

Substituindo (4.33) e (4.34) em (4.29), vem:

$$Z^{\bullet} + \theta\, Z = 0,$$

o que, após uma integração simples, dá:

$$Z(t) = Z_0\, t^{-1}. \tag{4.35}$$

Isto conclui os nossos cálculos para a solução de Milne.

Capítulo 5

Perturbações Escalares no Modelo de Kasner

5.1 Introdução

Analisaremos aqui o caso das perturbações ligadas à matéria, as quais envolvem praticamente todas as variáveis do conjunto mínimo fechado de observáveis $\mathcal{M}$, equação (3.25). As quantidades de trabalho, escritas em termos da base escalar apresentada no Capítulo 2, são:

$$
\begin{array}{ll}
(\delta X_{\alpha\beta}) = X(t)\,\hat{Q}_{\alpha\beta} & (\delta Y_{\alpha\beta}) = Y(t)\,\hat{Q}_{\alpha\beta} \\
(\delta\pi_{\alpha\beta}) = \pi(t)\,\hat{Q}_{\alpha\beta} & (\delta q_{\alpha}) = q(t)\,\hat{Q}_{\alpha} \\
(\delta a_{\alpha}) = \psi(t)\,\hat{Q}_{\alpha} & (\delta\rho) = R(t)\,\hat{Q} \\
(\delta W) = W(t)\,\hat{Q} & (\delta p) = \lambda\,(\delta\rho)\,.
\end{array}
\tag{5.1}
$$

Como estamos tratando aqui de perturbações escalares, não consideraremos perturbações associadas à vorticidade [10, 36]. Portanto, da terceira equação de vínculo, (3.17), temos que a parte magnética do tensor de Weyl não é definida para perturbações escalares. A relação termodinâmica entre o *shear* e a pressão anisotrópica, (4.1), continua válida. Entretanto, visto que a viscosidade ξ é nula no *background*, podemos considerar a sua perturbação $(\delta\xi)$ como uma "boa" variável de trabalho, o que nos daria então:

$$(\delta\pi_{\mu\nu}) = \sigma_{\mu\nu}\,(\delta\xi) = \sigma_{\mu\nu}\,\xi(t)\,\hat{Q}, \tag{5.2}$$

e, da definição (5.1) acima, bem como da relação (2.9), teríamos:

$$\xi(t)\,\sigma_{\mu\nu}\,\hat{Q} = -\pi(t)\left(n_\mu\,n_\nu + \frac{n^2}{3}\,h_{\mu\nu}\right)\hat{Q},$$

donde, multiplicando ambos os lados da equação acima por $\sigma^{\mu\nu}$, vem afinal:

$$\xi(t) = -\frac{1}{(2\sigma^2)}\,(\sigma^{\mu\nu}\,n_\mu\,n_\nu)\,\pi(t). \tag{5.3}$$

Com isto, o conjunto mínimo fechado de "boas" observáveis $\mathcal{M}$ se escreve como:

$$\mathcal{M} = \{X_{\alpha\beta}, Y_{\alpha\beta}, W, \pi_{\alpha\beta}, q_\alpha, a_\alpha, \rho\}\,. \tag{5.4}$$

E, além disto, dado que a parte espacial da velocidade, V_k, é também zero no *background* devido à nossa escolha inicial de observadores co-moventes, podemos escrever:

$$(\delta V_k) = V(t)\,\hat{Q}_k, \tag{5.5}$$

embora não seja ela uma variável de trabalho adequada, já que depende da escolha do observador. Isto posto, podemos passar à análise propriamente dita dos sistemas dinâmicos obtidos para perturbações escalares. Apresentaremos a seguir o sistema dinâmico para o caso geral do modelo de Kasner, seguindo com a análise de um caso especial, a saber, o da solução de Milne.

5.2 Pertubações Escalares — O Caso Geral

Todos os resultados discutidos na seção precedente são obviamente aplicáveis ao caso geral. Procedemos, então, à listagem das equações relevantes. A equação dinâmica de

$X_{\alpha\beta}$, (3.27), fica — após ser perturbada:

$$
\begin{aligned}
h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (\delta X_{\alpha\beta})^{\bullet} \;&+\; \frac{5}{3}\,\theta\,(\delta X_{\mu\nu}) + \sigma^{\alpha}{}_{(\mu}\left(\delta X_{\nu)\alpha}\right) - \frac{2}{3}\, h_{\mu\nu}\,\sigma^{\alpha\beta}\,(\delta X_{\alpha\beta}) - \\
&-\; (\delta Y_{\mu\nu}) - \frac{1}{3}\,\sigma_{\mu\nu}\,(\delta W) - \frac{1}{6}\, h_{\mu\nu}\, h^{\alpha\beta}\,(\delta q_{\alpha})_{;\beta} + \\
&+\; \frac{1}{4}\, h^{\alpha}{}_{(\mu}\, h^{\beta}{}_{\nu)}\,(\delta q_{\alpha})_{;\beta} - \frac{1}{9}\,\theta\, h_{\mu\nu}\, h^{\alpha\beta}\,(\delta a_{\alpha})_{;\beta} + \\
&+\; \frac{1}{6}\,\theta\, h^{(\alpha}{}_{\mu}\, h^{\beta)}{}_{\nu}\,(\delta a_{\alpha})_{;\beta} + \sigma_{\mu\nu}\, h^{\alpha\beta}\,(\delta a_{\alpha})_{;\beta} + \\
&+\; \frac{2}{3}\, h_{\mu\nu}\,\sigma^{\alpha\beta}\,(\delta a_{\alpha})_{;\beta} - \frac{1}{2}\, h^{\alpha}{}_{(\mu}\,\sigma^{\beta}{}_{\nu)}\left(\delta a_{(\alpha}\right)_{;\beta)} + \\
&+\; \frac{1}{3}\,\sigma_{\mu\nu}\,(\delta\rho) - \frac{1}{2}\, h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\,(\delta\pi_{\alpha\beta})^{\bullet} - \frac{1}{6}\,\theta\,(\delta\pi_{\mu\nu}) - \\
&-\; \frac{1}{6}\, h_{\mu\nu}\,\sigma^{\alpha\beta}\,(\delta\pi_{\alpha\beta}) + \frac{1}{4}\,\sigma^{\alpha}{}_{(\mu}\left(\delta\pi_{\nu)\alpha}\right) = 0.
\end{aligned}
\tag{5.6}
$$

A equação dinâmica para $Y_{\alpha\beta}$, (3.28), é escrita na forma perturbada como:

$$
\begin{aligned}
h^{\alpha}{}_{\mu}\, h^{\beta}{}_{nu}\, (\delta Y_{\alpha\beta})^{\bullet} \;&+\; 2\,\theta\,(\delta Y_{\mu\nu}) - \frac{3}{2}\,\sigma^{\alpha}{}_{(\mu}\left(\delta Y_{\nu)\alpha}\right) + \\
&+\; h_{\mu\nu}\,\sigma^{\alpha\beta}\,(\delta Y_{\alpha\beta}) + \frac{3}{2}\, E^{\alpha}{}_{(\mu}\left(\delta X_{\nu)\alpha}\right) - \\
&-\; h_{\mu\nu}\, E^{\alpha\beta}\,(\delta X_{\alpha\beta}) - \frac{1}{6}\,\theta\, h_{\mu\nu}\, h^{\alpha\beta}\,(\delta q_{\alpha})_{;\beta} +
\end{aligned}
$$

$$
\begin{aligned}
+ & \frac{1}{4}\,\theta\, h^{\alpha}{}_{(\mu}\, h^{\beta}{}_{\nu)}\, \left(\delta q_{\alpha}\right)_{;\beta} + \frac{3}{8}\, h^{\alpha}{}_{(\mu}\, \sigma^{\beta}{}_{\nu)}\, \left(\delta q_{(\alpha}\right)_{;\beta)} - \\
- & \frac{1}{2}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, \left(\delta q_{\alpha}\right)_{;\beta} - \frac{1}{2}\, \sigma_{\mu\nu}\, h^{\alpha\beta}\, \left(\delta q_{\alpha}\right)_{;\beta} - E_{\mu\nu}\, \left(\delta a^{\alpha}\right)_{;\alpha} + \\
+ & E_{\mu\nu}\, h^{\alpha\beta}\, \left(\delta a_{\alpha}\right)_{;\beta} - \\
- & \frac{3}{4}\, h^{\alpha}{}_{(\mu}\, E_{\nu)}{}^{\beta}\, \left(\delta_{(\alpha}\right)_{;\beta)} + h_{\mu\nu}\, E^{\alpha\beta}\, \left(\delta a_{\alpha}\right)_{;\beta} + \\
+ & E_{\mu\nu}\, (\delta W) - \left[E_{\mu\nu} - (1+\lambda)\, \theta\, \sigma_{\mu\nu}\right] (\delta \rho) - \\
- & \frac{1}{2}\,\theta\, h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, \left(\delta \pi_{\alpha\beta}\right)^{\bullet} - \frac{3}{4}\, \sigma^{\alpha}{}_{(\mu}\, h^{\beta}{}_{\nu)}\, \left(\delta \pi_{\alpha\beta}\right)^{\bullet} + \\
+ & \frac{1}{2}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, \left(\delta \pi_{\alpha\beta}\right)^{\bullet} + \frac{1}{6}\,\theta\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, \left(\delta \pi_{\alpha\beta}\right) - \\
- & \frac{3}{8}\, \sigma^{\alpha\beta}\, \sigma_{\alpha(\mu}\, \left(\delta \pi_{\nu)\alpha}\right) + \frac{3}{4}\, E^{\alpha}{}_{(\mu}\, \left(\delta \pi_{\nu)\alpha}\right) + \\
+ & \frac{1}{4}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, \sigma^{\gamma}{}_{(\alpha}\, \left(\delta \pi_{\beta)\gamma}\right) - \frac{1}{2}\, h_{\mu\nu}\, E^{\alpha\beta}\, \left(\delta \pi_{\alpha\beta}\right) = 0.
\end{aligned}
\tag{5.7}
$$

Perturbando (3.29), a equação dinâmica para W, obtemos:

$$
\begin{aligned}
(\delta W)^{\bullet} & + \frac{2}{3}\,\theta\, (\delta W) + 2\, \sigma^{\alpha\beta}\, \left(\delta X_{\alpha\beta}\right) + \sigma^{\alpha\beta}\, \left(\delta \pi_{\alpha\beta}\right) - \\
& - 2\, \sigma^{\alpha\beta}\, \left(\delta a_{\alpha}\right)_{;\beta} + \frac{4}{3}\,\theta\, h^{\alpha\beta}\, \left(\delta a_{\alpha}\right)_{;\beta} - \frac{2}{3}\, (1+3\lambda)\, \theta\, (\delta \rho) = 0.
\end{aligned}
$$

(5.8)

A equação dinâmica de $H_{\alpha\beta}$, (3.32), escreve-se após a perturbação como uma equação de vínculo:

$$
\begin{aligned}
\frac{1}{2}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,h_{\nu)}{}^\lambda\,(\delta X_{\lambda\alpha})_{;\beta} \;&+\; \frac{1}{4}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,h_{\nu)}{}^\lambda\,(\delta\pi_{\lambda\alpha})_{;\beta} - \\
&-\;\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,E_{\nu)\alpha}\,(\delta a_\beta) + \frac{1}{4}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,\sigma_\beta{}^\lambda\,(\delta a_\lambda) \\
&=\; 0.
\end{aligned} \tag{5.9}
$$

A primeira lei de conservação, (3.10) perturbada, nos dá a equação dinâmica de ρ:

$$(\delta\rho)^\bullet + (1+\lambda)\,\theta\,(\delta\rho) + (\delta q^\alpha)_{;\alpha} - \sigma^{\alpha\beta}\,(\delta\pi_{\alpha\beta}) = 0, \tag{5.10}$$

e a segunda lei de conservação, (3.11), nos dá após a perturbação:

$$
\begin{aligned}
h_\alpha{}^\beta\,(\delta q_\beta)^\bullet \;&+\; \frac{4}{3}\,\theta\,h_\alpha{}^\beta\,(\delta q_\beta) + \sigma_\alpha{}^\beta\,(\delta q_\beta) - \lambda\,h_\alpha{}^\beta\,(\delta\rho)_{,\beta} + \\
&+\; \left(\delta\pi_\alpha{}^\beta\right)_{;\beta} + V_\alpha\,\sigma^{\beta\gamma}\,(\delta\pi_{\beta\gamma}) = 0.
\end{aligned} \tag{5.11}
$$

A equação dinâmica para a rotação, (3.14), é identicamente nula, bem como a segunda e a terceira equações de vínculo, (3.16) e (3.17). Já a primeira projeção, equação (3.34), perturbada, nos dá:

$$
\begin{aligned}
h_\alpha{}^\beta\, h^{\mu\nu}\,(\delta X_{\beta\mu})_{;\nu} \;&-\; \frac{1}{2}\, h_\alpha{}^\beta\, h^{\mu\nu}\,(\delta\,\pi_{\beta\mu})_{;\nu} + \frac{1}{2}\left(\sigma_{\alpha\gamma}\,\sigma^{\beta\gamma} + 2\,\theta\,\sigma_\alpha{}^\beta\right)(\delta a_\beta) - \\
&-\; \theta\, h_\alpha{}^\beta\,(\delta q_\beta) - \frac{1}{3}\, h_\alpha{}^\beta\,(\delta\rho)_{,\beta} + \frac{1}{3}\, h_\alpha{}^\beta\,(\delta W)_{,\beta} + \\
&+\; \frac{1}{9}\,\theta^2\left(\sigma_\alpha{}^\beta - \frac{8}{3}\,\theta\, h_\alpha{}^\beta\right)(\delta V_\beta) = 0, \qquad (5.12)
\end{aligned}
$$

e a segunda projeção, (3.35), é dada por:

$$
\begin{aligned}
\eta_\alpha{}^{\beta\mu\nu}\, V_\beta\,\sigma_\mu{}^\gamma\,(\delta X_{\nu\gamma}) \;&-\; \frac{1}{2}\,\eta_\alpha{}^{\beta\mu\nu}\, V_\beta\,\sigma_\mu{}^\gamma\,(\delta\pi_{\nu\gamma}) - \\
&-\; \frac{1}{2}\,\eta_\alpha{}^{\beta\mu\nu}\, V_\beta\,(\delta q_\mu)_{;\nu} = 0. \qquad (5.13)
\end{aligned}
$$

O sistema dinâmico para o caso geral das perturbações escalares é, pois, dado pelas equações (5.6)-(5.13). Escrevendo cada uma destas equações em termos da base escalar $\hat{Q}$, obtemos os resultados a seguir. A equação de vínculo (5.13) dá:

$$
\left[X(t) - \frac{1}{2}\,\pi(t)\right]\,\eta_\alpha{}^{\beta\mu\nu}\, V_\beta\,\sigma_\mu{}^\gamma\,\hat{Q}_{\nu\gamma} - \frac{1}{2}\, q(t)\,\eta_\alpha{}^{\beta\mu\nu}\, V_\beta\,\hat{Q}_{\mu;\nu} = 0.
$$

O último termo do lado esquerdo na equação acima pode ser calculado como:

$$
\begin{aligned}
\eta_\alpha{}^{\beta\mu\nu}\, V_\beta\,\hat{Q}_{\mu;\nu} \;&=\; \eta_\alpha{}^{\beta\mu\nu}\, V_\beta\left(\hat{\nabla}_\mu\,\hat{Q}\right)_{;\nu} \\
&=\; \eta_\alpha{}^{\beta\mu\nu}\, V_\beta\left[h_\mu{}^\gamma\,\hat{Q}_{,\gamma}\right]_{;\nu}
\end{aligned}
$$

$$
\begin{aligned}
&= \eta_{\alpha}{}^{\beta\mu\nu}\, V_{\beta} \left[-\left(V_{\mu;\nu}\, V^{\gamma} + V_{\mu}\, V^{\gamma}{}_{;\nu}\right) \hat{Q}_{,\gamma} + h_{\mu}{}^{\gamma} \left(\hat{Q}_{,\gamma}\right)_{;\nu} \right] \\
&= \eta_{\alpha}{}^{\beta\mu\nu}\, V_{\beta} \left[\hat{Q}_{,\mu,\nu} - \Gamma^{\gamma}_{\mu\nu}\, \hat{Q}_{,\gamma} \right] \\
&= 0,
\end{aligned}
\tag{5.14}
$$

enquanto o primeiro termo nos dá (usando as relações (2.8), (2.5) e (2.2), respectivamente):

$$
\begin{aligned}
\eta_{\alpha}{}^{\beta\mu\nu}\, V_{\beta}\, \sigma_{\mu}{}^{\gamma}\, \hat{Q}_{\nu\gamma} &= \eta_{\alpha}{}^{\beta\mu\nu}\, V_{\beta}\, \sigma_{\mu}{}^{\gamma} \left[\left(\hat{\nabla}_{\gamma}\, \hat{Q}_{\nu} \right) - \frac{n^2}{3}\, h_{\gamma\nu}\, \hat{Q} \right] \\
&= \eta_{\alpha}{}^{\beta\mu\nu}\, V_{\beta}\, \sigma_{\mu}{}^{\gamma} \left(\hat{\nabla}_{\gamma}\, \hat{Q}_{\nu} \right) - \frac{n^2}{3}\, \eta_{\alpha}{}^{\beta\mu\nu}\, V_{\beta}\, \sigma_{\mu\nu}\, \hat{Q} \\
&= \eta_{\alpha}{}^{\beta\mu\nu}\, V_{\beta}\, \sigma_{\mu}{}^{\gamma}\, h_{\gamma}{}^{\lambda}\, h_{\nu}{}^{\varepsilon}\, \hat{Q}_{,\lambda,\varepsilon} \\
&= \eta_{\alpha}{}^{\beta\mu\nu}\, V_{\beta}\, \sigma_{\mu}{}^{\lambda}\, \hat{Q}_{,\lambda,\nu} \\
&= \eta_{\alpha}{}^{\beta\mu\nu}\, V_{\beta}\, \sigma_{\mu}{}^{\gamma} \left(-n_{\gamma}\, n_{\nu}\, \hat{Q} \right) \\
&= -\eta_{\alpha}{}^{\beta\mu\nu}\, V_{\beta}\, \sigma_{\mu}{}^{\gamma}\, n_{\gamma}\, n_{\nu}\, \hat{Q} \\
&= -\sigma^{\alpha}{}_{[\mu}\, n_{\nu]}\, n_{\alpha}\, \hat{Q}.
\end{aligned}
\tag{5.15}
$$

Daí vem então:

$$
\left[X(t) - \frac{1}{2}\, \pi(t) \right]\, \sigma^{\alpha}{}_{[\mu}\, n_{\nu]}\, n_{\alpha}\, \hat{Q} = 0,
$$

o que nos dá duas possibilidades de solução, dado que a base $\hat{Q}$ é obviamente não nula. Investigando a primeira delas, ou seja, a de que a antissimetria seja nula, obtemos:

$$(\sigma^{\alpha}{}_{\mu}\, n_{\nu} - \sigma^{\alpha}{}_{\mu}\, n_{\mu})\; n_{\alpha} = 0,$$

a qual é imediatamente válida para todo $\mu = \nu$. Se tivermos, no entanto, $\mu \neq \nu$, obtemos três condições,

$$\mu = 1 \;\; , \;\; \nu = 2 \Longrightarrow (\sigma^{1}{}_{1} - \sigma^{2}{}_{2})\; n_1\, n_2 = 0,$$

$$\mu = 1 \;\; , \;\; \nu = 3 \Longrightarrow (\sigma^{1}{}_{1} - \sigma^{3}{}_{3})\; n_1\, n_3 = 0,$$

$$\mu = 2 \;\; , \;\; \nu = 3 \Longrightarrow (\sigma^{2}{}_{2} - \sigma^{3}{}_{3})\; n_2\, n_3 = 0,$$

que só são satisfeitas simultaneamente se houver um plano de isotropia e se, simultaneamente, um dos n_i for nulo. Como esta segunda exigência levaria a problemas no problemas no estabelecimento da base escalar (já que neste caso, ela não dependeria de uma das coordenadas espaciais), devemos considerar como válida a outra possibilidade de solução, qual seja,

$$\pi(t) = 2\, X(t), \tag{5.16}$$

o que — usando-se a relação (5.3) — nos dá também que:

$$\xi(t) = -\frac{2}{(2\sigma^2)}\; (\sigma^{\mu\nu}\, n_{\mu}\, n_{\nu})\; X(t). \tag{5.17}$$

O vínculo (5.9), proveniente da equação dinâmica de H, nos dá:

$$\psi(t)\; \left[\eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, E_{\nu)\alpha}\, \hat{Q}_{\beta} - \frac{1}{4}\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, \sigma_{\nu)\alpha}\, \sigma_{\beta}{}^{\lambda}\, \hat{Q}_{\lambda}\right] =$$

$$= \frac{1}{2}\left[X(t) + \frac{1}{2}\pi(t)\right]\eta_{(\mu}{}^{\gamma\alpha\beta} V_\gamma h_{\nu)}{}^\lambda \hat{Q}_{\lambda\alpha;\beta},$$

onde, substituindo o resultado (5.16), vem:

$$\psi(t)\left[\eta_{(\mu}{}^{\gamma\alpha\beta} V_\gamma E_{\nu)\alpha}\hat{Q}_\beta - \frac{1}{4}\eta_{(\mu}{}^{\gamma\alpha\beta} V_\gamma \sigma_{\nu)\alpha}\sigma_\beta{}^\lambda \hat{Q}_\lambda\right] =$$

$$= X(t)\,\eta_{(\mu}{}^{\gamma\alpha\beta} V_\gamma h_{\nu)}{}^\lambda \hat{Q}_{\lambda\alpha;\beta}.$$

Entretanto, podemos calcular os termos acima separadamente como (utilizando as relações (2.4), (2.9) e (1.3)):

$$\begin{aligned}
\eta_{(\mu}{}^{\gamma\alpha\beta} V_\gamma h_{\nu)}{}^\lambda \hat{Q}_{\lambda\alpha;\beta} &= \eta_{(\mu}{}^{\gamma\alpha\beta} V_\gamma h_{\nu)}{}^\lambda\Big\{-\Big[\Big(n_\lambda n_\alpha + \frac{n^2}{3} h_{\lambda\alpha}\Big)\hat{Q}\Big]_{,\beta} + \\
&\quad + \Gamma^\tau_{\lambda\beta}\hat{Q}_{\tau\alpha} - \Gamma^\tau_{\alpha\beta}\hat{Q}_{\tau\lambda}\Big\} \\
&= i\,\eta_{(\mu}{}^{\gamma\alpha\beta} V_\gamma h_{\nu)}{}^\lambda\Big(n_\lambda n_\alpha + \frac{n^2}{3} h_{\lambda\alpha}\Big) n_\beta \hat{Q} \\
&= i\Big[\eta_{(\mu}{}^{\gamma\alpha\beta} V_\gamma n_{\nu)} n_\alpha n_\beta + \frac{n^2}{3}\eta_{(\mu}{}^{\gamma\alpha\beta} V_\gamma h_{\nu)\alpha} n_\beta\Big]\hat{Q} \\
&= i\,\frac{n^2}{3}\Big[\eta_{(\mu}{}^{\gamma}{}_{\nu)}{}^{\beta} V_\gamma - \eta_{(\mu}{}^{\gamma\alpha\beta} V_\gamma V_\alpha V_{\nu)}\Big] n_\beta \hat{Q} \\
&= 0,
\end{aligned}$$

$$\eta_{(\mu}{}^{\gamma\alpha\beta} V_\gamma E_{\nu)\alpha}\hat{Q} = \eta_{(\mu}{}^{\gamma\alpha\beta} V_\gamma E_{\nu)\alpha}\left(-i\,h_\beta{}^\tau n_\tau \hat{Q}\right)$$

$$= \quad -i\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,E_{\nu)\alpha}\,n_\beta\,\hat{Q}.$$

$$\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,\sigma_\beta{}^\lambda\,\hat{Q}_\lambda \quad = \quad -i\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,\sigma_\beta{}^\lambda\,h_\lambda{}^\tau\,n_\tau\,\hat{Q}$$

$$= \quad -i\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,\sigma_\beta{}^\gamma\,n_\lambda\,\hat{Q},$$

o que nos dá:

$$-i\,\psi(t)\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\left[E_{\nu)\alpha}\,n_\beta - \frac{1}{4}\,\sigma_{\nu)\alpha}\,\sigma_\beta{}^\lambda\,n_\lambda\right]\,\hat{Q} = 0 \implies$$

$$\implies \psi(t)\,\left[\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\left(E_{\nu)\alpha}\,n_\beta - \frac{1}{4}\,\sigma_{\nu)\alpha}\,\sigma_\beta{}^\lambda\,n_\lambda\right)\right] = 0.$$

Usando a definição de $X_{\alpha\beta}$ no *background* — relação (3.21) — podemos escrever $E_{\nu)\alpha}$ em termos de θ e do *shear* acima:

$$\psi(t)\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\left[\frac{1}{3}\,\theta\,\sigma_{\nu)\alpha}\,n_\beta + \frac{2n^2}{3}\,h_{\nu)\alpha}\,n_\beta - \sigma_{\nu)\lambda}\,\sigma^\lambda{}_\alpha\,n_\beta - \right.$$
$$\left. -\frac{1}{4}\,\sigma_{\nu)\alpha}\,\sigma_\beta{}^\lambda\,n_\lambda\right] = 0.$$

O segundo termo entre colchetes acima é nulo quando multiplicado por $\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma$, o que nos dá então:

$$\psi(t)\,\left\{\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\left[\left(\frac{1}{3}\,\theta\,\sigma_{\nu)\alpha} - \sigma_{\nu)\lambda}\,\sigma^\lambda{}_\alpha\right)\,n_\beta - \frac{1}{4}\,\sigma_{\nu)\alpha}\,\sigma_\beta{}^\lambda\,n_\lambda\right]\right\} = 0.$$

Tomar o termo entre chaves acima como sendo nulo, implicaria em restrições excessivas tanto sobre a base escalar quanto sobre o próprio *background*. Logo, tomaremos o resultado

$$\psi(t) = 0, \tag{5.18}$$

como válido. Note-se que uma tal escolha é adicionalmente vantajosa por fixar uma variável — a aceleração — que não possui uma dinâmica.

O vínculo restante, equação (5.12), fixa-nos tão somente a quantidade (δV_k):

$$\left[X(t) - \frac{1}{2}\,\pi(t)\right]\, h_\alpha{}^\beta\, h^{\mu\nu}\, \hat{Q}_{\beta\mu;\nu} + \frac{1}{2}\,\psi(t)\, \left(\sigma_{\alpha\gamma}\,\sigma^{\beta\gamma} + 2\,\theta\,\sigma_\alpha{}^\beta\right)\, \hat{Q}_\beta -$$

$$-\theta\, q(t)\, h_\alpha{}^\beta\, \hat{Q}_\beta - \frac{1}{3}\, [R(t) - W(t)]\, h_\alpha{}^\beta\, \hat{Q}_{,\beta} +$$

$$+\frac{1}{9}\,\theta^2\, V(t)\, \left(\sigma_\alpha{}^\beta - \frac{8}{3}\,\theta\, h_\alpha{}^\beta\right)\, \hat{Q}_\beta = 0.$$

De (5.16) e (5.18), o primeiro e o segundo termos acima são imediatamente nulos. E, do primeiro dos resultados (2.4), escrevemos (após fatorar a base $\hat{Q}$):

$$V(t)\, \left\{\frac{1}{9}\,\theta^2\, \left[\left(\sigma_\alpha{}^\beta\, n_\beta\right) - \frac{8}{3}\,\theta\, n_\alpha\right]\right\} = \left\{\theta\, q(t) + \frac{1}{3}\, [R(t) - W(t)]\right\}\, n_\alpha. \tag{5.19}$$

Podemos, agora, passar à análise das equações dinâmicas. A equação dinâmica de X, (5.6), nos dá — usando os resultados (5.16) e (5.18) diretamente:

$$\begin{aligned}
\frac{4}{3}\,\theta\, X(t)\, \hat{Q}_{\mu\nu} \quad &+ \quad \frac{3}{2}\, X(t)\, \sigma^\alpha{}_{(\mu}\, \hat{Q}_{\nu)\alpha} - X(t)\, h_{\mu\nu}\, \left(\sigma^{\alpha\beta}\, \hat{Q}_{\alpha\beta}\right) - Y(t) + \\
&+ \quad \frac{1}{3}\, [R(t) - W(t)]\, \sigma_{\mu\nu}\, \hat{Q} - \frac{1}{6}\, q(t)\, h_{\mu\nu}\, h^{\alpha\beta}\, \hat{Q}_{\alpha;\beta} + \\
&+ \quad \frac{1}{4}\, q(t)\, h^\alpha{}_{(\mu}\, h^\beta{}_{\nu)}\, \hat{Q}_{\alpha;\beta} = 0.
\end{aligned}$$

Temos, porém, de (2.9), que:

$$
\begin{aligned}
\sigma^{\alpha}{}_{(\mu}\,\hat{Q}_{\nu)\alpha} &= -\sigma^{\alpha}{}_{(\mu}\left(n_{\nu)}\,n_{\alpha} + \frac{n^2}{3}\,h_{\nu)\alpha}\right)\hat{Q} \\
&= -\left(\sigma^{\alpha}{}_{(\mu}\,n_{\nu)}\,n_{\alpha} + \frac{2n^2}{3}\,\sigma_{\mu\nu}\right)\hat{Q}, \qquad (5.20)
\end{aligned}
$$

$$
\begin{aligned}
h_{\mu\nu}\,h^{\alpha\beta}\,\hat{Q}_{\alpha;\beta} &= h_{\mu\nu}\,h^{\alpha\beta}\left(\hat{Q}_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta}\,\hat{Q}_{\gamma}\right) \\
&= -i\,h_{\mu\nu}\,h^{\alpha\beta}\,n_{\alpha}\,\hat{Q}_{,\beta} \\
&= -h_{\mu\nu}\,h^{\alpha\beta}\,n_{\alpha}\,n_{\beta}\,\hat{Q} \\
&= n^2\,h_{\mu\nu}\,\hat{Q}, \qquad (5.21)
\end{aligned}
$$

$$
\begin{aligned}
h^{\alpha}{}_{(\mu}\,h^{\beta}{}_{\nu)}\,\hat{Q}_{\alpha;\beta} &= h^{\alpha}{}_{(\mu}\,h^{\beta}{}_{\nu)}\left(\hat{Q}_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta}\,\hat{Q}_{\gamma}\right) \\
&= h^{\alpha}{}_{(\mu}\,h^{\beta}{}_{\nu)}\left(-n_{\alpha}\,n_{\beta}\,\hat{Q}\right) \\
&= -n_{(\mu}\,n_{\nu)}\,\hat{Q} \\
&= -2\,n_{\mu}\,n_{\nu}\,\hat{Q}, \qquad (5.22)
\end{aligned}
$$

e, fazendo uso também das relações (2.9) e (2.12), vem:

$$\left[\frac{4}{3}\,\theta\,X(t) - Y(t) - \frac{1}{2}\,q(t)\right]\,\hat{Q}_{\mu\nu} \;\; + \;\; \Big\{X(t)\,\Big[\Big(\sigma^{\alpha\beta}\,n_\alpha\,n_\beta\Big)\,h_{\mu\nu} -$$

$$- \;\; \frac{3}{2}\,\left(\sigma^{\alpha}{}_{(\mu}\,n_{\nu)}\,n_\alpha + \frac{2n^2}{3}\,\sigma_{\mu\nu}\right)\Big] + \frac{1}{3}\,[R(t) - W(t)]\,\sigma_{\mu\nu}\Big\}\,\hat{Q} = 0. \tag{5.23}$$

A equação (5.11) nos dá dois resultados independentes. São eles:

$$\sigma^{\alpha\beta}\,(\delta\pi_{\alpha\beta}) = 0, \tag{5.24}$$

$$h_\alpha{}^\beta\,(\delta q_\beta)^\bullet \;\; + \;\; \frac{4}{3}\,\theta\,h_\alpha{}^\beta\,(\delta q_\beta) + \sigma_\alpha{}^\beta\,(\delta q_\beta) -$$

$$- \;\; \lambda\,h_\alpha{}^\beta\,(\delta\rho)_{,\beta} + \left(\delta\pi_\alpha{}^\beta\right)_{;\beta} = 0. \tag{5.25}$$

A relação (5.24) nos dá — usando-se o resultado (2.12):

$$\pi(t)\,\left(\sigma^{\alpha\beta}\,\hat{Q}_{\alpha\beta}\right) = -\pi(t)\,\left(\sigma^{\alpha\beta}\,n_\alpha\,n_\beta\right)\,\hat{Q} = 0 \;\;\Longrightarrow$$

$$\Longrightarrow \;\; \pi(t)\,\left(\sigma^{\alpha\beta}\,n_\alpha\,n_\beta\right) = 0,$$

onde temos, de (1.38), que:

$$\left(\sigma^{\alpha\beta}\,n_\alpha\,n_\beta\right) \;=\; \sigma^{11}\,(n_1)^2 + \sigma^{22}\,(n_2)^2 + \sigma^{33}\,(n_3)^2$$

$$= \;\; \frac{1}{3}\,\theta\,\left[(3p_1 - 1)\,(n_1)^2 + (3p_2 - 1)\,(n_2)^2 + (3p_3 - 1)\,(n_3)^2\right]$$

$$= -\frac{1}{3}\theta\left[t^{-2p_1}(3p_1-1)(n_1)^2 + t^{-2p_2}(3p_2-1)(n_2)^2 + \right.$$
$$\left. + \; t^{-2p_3}(3p_3-1)(n_3)^2\right],$$

a qual é diferente de zero em geral. Logo, ficamos com o resultado

$$\pi(t) = 0, \tag{5.26}$$

o que — de (5.16) e (5.17) — implica em:

$$\pi(t) \equiv \xi(t) \equiv X(t) = 0. \tag{5.27}$$

E ainda, de (5.23), temos:

$$\left[Y(t) + \frac{1}{2}q(t)\right]\hat{Q}_{\mu\nu} = \frac{1}{3}\left[R(t) - W(t)\right]\sigma_{\mu\nu}\hat{Q}. \tag{5.28}$$

Já da relação (5.25), obtemos uma equação dinâmica para o fluxo de calor:

$$q^{\bullet}(t)\,\hat{Q}_\alpha + q(t)\,h_\alpha{}^\beta\left(\hat{Q}_\alpha\right)^{\bullet} + \frac{4}{3}\theta\,q(t)\,\hat{Q}_\alpha +$$
$$+ \; q(t)\,\sigma_\alpha{}^\beta\,\hat{Q}_\beta - \lambda\,R(t)\,h_\alpha{}^\beta\,\hat{Q}_{,\beta} = 0,$$

a qual — fazendo uso de (2.4) e (2.10) — se escreve como:

$$q^{\bullet}(t)\,\hat{Q}_\alpha - q(t)\,h_\alpha{}^\beta\left(\frac{1}{3}\theta\,\hat{Q}_\beta + \sigma_\beta{}^\gamma\,\hat{Q}_\gamma\right) + \frac{4}{3}\theta\,q(t)\,\hat{Q}_\alpha +$$
$$+ \; q(t)\,\sigma_\alpha{}^\beta\,\hat{Q}_\beta - \lambda\,R(t)\,\hat{Q}_\alpha = 0 \Longrightarrow$$

$$\Longrightarrow q^{\bullet}(t)\,\hat{Q}_\alpha + \theta\,q(t)\,\hat{Q}_\alpha -$$

$$- \quad q(t)\,\sigma_\alpha{}^\gamma\,\hat{Q}_\gamma + q(t)\,\sigma_\alpha{}^\beta\,\hat{Q}_\beta - \lambda\,R(t)\,\hat{Q}_\alpha = 0 \Longrightarrow$$

$$\Longrightarrow \quad [q^\bullet(t) + \theta\,q(t) - \lambda\,R(t)]\;\hat{Q}_\alpha = 0.$$

e, fatorando $\hat{Q}_\alpha$, obtemos afinal:

$$q^\bullet(t) + \theta\,q(t) - \lambda\,R(t) = 0. \tag{5.29}$$

Com os resultados (5.27), a equação dinâmica para Y, (5.7), se escreve de uma maneira mais simples:

$$\begin{aligned}
Y^\bullet(t)\,\hat{Q}_{\mu\nu} \quad &+ \quad Y(t)\,h^\alpha{}_\mu\,h^\beta{}_\nu\,\left(\hat{Q}_{\alpha\beta}\right)^\bullet + 2\,\theta\,Y(t)\,\hat{Q}_{\mu\nu} - \frac{3}{2}\,Y(t)\,\sigma^\alpha{}_{(\mu}\,\hat{Q}_{\nu)\alpha} + \\
&+ \quad Y(t)\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,\hat{Q}_{\alpha\beta} - \frac{1}{6}\,\theta\,q(t)\,h_{\mu\nu}\,h^{\alpha\beta}\,\hat{Q}_{\alpha;\beta} + \\
&+ \quad \frac{1}{4}\,\theta\,q(t)\,h^\alpha{}_{(\mu}\,h^\beta{}_{\nu)}\,\hat{Q}_{\alpha;\beta} + \frac{3}{8}\,q(t)\,h^\alpha{}_{(\mu}\,\sigma^\beta\,h_{\nu)}\,\hat{Q}_{(\alpha;\beta)} - \\
&- \quad \frac{1}{2}\,q(t)\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,\hat{Q}_{\alpha;\beta} - \frac{1}{2}\,q(t)\,\sigma_{\mu\nu}\,h^{\alpha\beta}\,\hat{Q}_{\alpha;\beta} + \\
&+ \quad E_{\mu\nu}\,W(t)\,\hat{Q} - [E_{\mu\nu} - (1+\lambda)\,\theta\,\sigma_{\mu\nu}]\;R(t)\,\hat{Q} = 0.
\end{aligned}$$

A relação (2.14) nos dá:

$$\begin{aligned}
h^\alpha{}_\mu\,h^\beta{}_\nu\,\left(\hat{Q}_{\alpha\beta}\right)^\bullet \quad = \quad h^\alpha{}_\mu\,h^\beta{}_\nu\,\Big[&-\frac{2}{3}\,\theta\,\hat{Q}_{\alpha\beta} - \sigma^\gamma{}_{(\alpha}\,\hat{Q}_{\beta)\gamma} - \\
&-\frac{2n^2}{3}\,\sigma_{\alpha\beta}\,\hat{Q} + \frac{2}{3}\,h_{\alpha\beta}\,\sigma^{\gamma\lambda}\,\hat{Q}_{\gamma\lambda}\Big]
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{3}\,\theta\,\hat{Q}_{\mu\nu} - \sigma^{\alpha}{}_{(\mu}\,\hat{Q}_{\nu)\alpha} - \frac{2n^2}{3}\,\sigma_{\mu\nu}\,\hat{Q} + \\
&+ \frac{2}{3}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,\hat{Q}_{\alpha\beta},
\end{aligned}
$$

donde escrevemos:

$$
\begin{aligned}
Y^{\bullet}(t)\,\hat{Q}_{\mu\nu} \;&+\; \frac{4}{3}\,\theta\,Y(t)\,\hat{Q}_{\mu\nu} - \frac{5}{2}\,Y(t)\,\sigma^{\alpha}{}_{(\mu}\,\hat{Q}_{\nu)\alpha} - \frac{2n^2}{3}\,Y(t)\,\sigma_{\mu\nu}\,\hat{Q} + \\
&+ \frac{5}{3}\,Y(t)\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,\hat{Q}_{\alpha\beta} - \frac{1}{6}\,\theta\,q(t)\,h_{\mu\nu}\,h^{\alpha\beta}\,\hat{Q}_{\alpha;\beta} + \frac{1}{4}\,\theta\,q(t)\,h^{\alpha}{}_{(\mu}\,h^{\beta}{}_{\nu)}\,\hat{Q}_{\alpha;\beta} + \\
&+ \frac{3}{8}\,q(t)\,h^{\alpha}{}_{(\mu}\,\sigma^{\beta}{}_{\nu)}\,\hat{Q}_{(\alpha;\beta)} - \frac{1}{2}\,q(t)\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,\hat{Q}_{\alpha;\beta} - \frac{1}{2}\,q(t)\,\sigma_{\mu\nu}\,h^{\alpha\beta}\,\hat{Q}_{\alpha;\beta} + \\
&+ E_{\mu\nu}\,W(t)\,\hat{Q} - \left[E_{\mu\nu} - (1+\lambda)\;\theta\,\sigma_{\mu\nu}\right]\,R(t)\,\hat{Q} = 0.
\end{aligned}
$$

E, calculando

$$
\begin{aligned}
h^{\alpha}{}_{(\mu}\,\sigma^{\beta}{}_{\nu)}\,\hat{Q}_{(\alpha;\beta)} &= h^{\alpha}{}_{(\mu}\,\sigma^{\beta}{}_{\nu)}\,\left(\hat{Q}_{(\alpha,\beta)} - \Gamma^{\gamma}_{(\alpha\beta)}\,\hat{Q}_{\gamma}\right) \\
&= h^{\alpha}{}_{(\mu}\,\sigma^{\beta}{}_{\nu)}\,\hat{Q}_{(\alpha,\beta)} \\
&= -i\,h^{\alpha}{}_{(\mu}\,\sigma^{\beta}{}_{\nu)}\,\left[n_{(\alpha}\,\hat{Q}_{,\beta)}\right] \\
&= -h^{\alpha}{}_{(\mu}\,\sigma^{\beta}{}_{\nu)}\,\left[n_{(\alpha}\,n_{\beta)}\,\hat{Q}\right] \\
&= -2\,h^{\alpha}{}_{(\mu}\,\sigma^{\beta}{}_{\nu)}\,n_{\alpha}\,n_{\beta}\,\hat{Q}
\end{aligned}
$$

$$= \quad -2\,\sigma^{\alpha}{}_{(\mu}\,n_{\nu)}\,n_{\alpha}\,\hat{Q}, \tag{5.30}$$

$$h_{\mu\nu}\,\sigma^{\alpha\beta}\,\hat{Q}_{\alpha;\beta} \quad = \quad h_{\mu\nu}\,\sigma^{\alpha\beta}\,\left(\hat{Q}_{\alpha;\beta} - \Gamma^{\gamma}_{\alpha\beta}\,\hat{Q}_{\gamma}\right)$$

$$= \quad -h_{\mu\nu}\,\left(\sigma^{\alpha\beta}\,n_{\alpha}\,n_{\beta}\right)\,\hat{Q}, \tag{5.31}$$

$$\sigma_{\mu\nu}\,h^{\alpha\beta}\,\hat{Q}_{\alpha;\beta} \quad = \quad \sigma_{\mu\nu}\,h^{\alpha\beta}\,\left(\hat{Q}_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta}\,\hat{Q}_{\gamma}\right)$$

$$= \quad \sigma_{\mu\nu}\,h^{\alpha\beta}\,\hat{Q}_{\alpha,\beta}$$

$$= \quad -\sigma_{\mu\nu}\,h^{\alpha\beta}\,n_{\alpha}\,n_{\beta}\,\hat{Q}$$

$$= \quad n^{2}\,\sigma_{\mu\nu}\,\hat{Q}, \tag{5.32}$$

bem como fazendo uso das relações (2.11) e (5.20)-(5.22), escrevemos:

$$\left[Y^{\bullet}(t) + \frac{4}{3}\,\theta\,Y(t) + \frac{1}{2}\,\theta\,q(t)\right]\,\hat{Q}_{\mu\nu} = -\left\{Y(t)\,\left[n^{2}\,\sigma_{\mu\nu} + \frac{5}{2}\,\sigma^{\alpha}{}_{(\mu}\,n_{\nu)}\,n_{\alpha} - \right.\right.$$

$$\left. - \quad \frac{5}{3}\,h_{\mu\nu}\,\left(\sigma^{\alpha\beta}\,n_{\alpha}\,n_{\beta}\right)\right] + E_{\mu\nu}\,W(t) - \left[E_{\mu\nu} - (1+\lambda)\,\theta\,\sigma_{\mu\nu}\right]\,R(t) -$$

$$\left. - \quad \frac{1}{2}\,q(t)\,\left[n^{2}\,\sigma_{\mu\nu} + \frac{3}{2}\,\sigma^{\alpha}{}_{(\mu}\,n_{\nu)}\,n_{\alpha} - h_{\mu\nu}\,\left(\sigma^{\alpha\beta}\,n_{\alpha}\,n_{\beta}\right)\right]\right\}\,\hat{Q}. \tag{5.33}$$

A equação dinâmica de W, (5.8), é dada por:

$$W^{\bullet}(t)\,\hat{Q} + W(t)\,\left(\hat{Q}\right)^{\bullet} + \frac{2}{3}\,\theta\,W(t)\,\hat{Q} - \frac{2}{3}\,(1+3\lambda)\,\theta\,R(t)\,\hat{Q} = 0.$$

Usando o resultado (2.10) e fatorando a base $\hat{Q}$, obtemos por fim:

$$W^{\bullet}(t) + \frac{2}{3}\,\theta\,W(t) - \frac{2}{3}\,(1+3\lambda)\,\theta\,R(t) = 0. \qquad (5.34)$$

A última equação a ser escrita em termos da base escalar é (5.10), que fornece a dinâmica da densidade de matéria. Ela dá:

$$R^{\bullet}(t)\,\hat{Q} + R(t)\,\left(\hat{Q}\right)^{\bullet} + (1+\lambda)\,\theta\,R(t)\,\hat{Q} + q(t)\,\hat{Q}^{\alpha}{}_{;\alpha} = 0,$$

e, usando os resultados (2.7) e (2.10) e fatorando a base $\hat{Q}$, obtemos:

$$R^{\bullet}(t) + (1+\lambda)\,\theta\,R(t) + n^2\,q(t) = 0. \qquad (5.35)$$

Ficamos, então, com as equações (5.18), (5.19), (5.27), (5.28), (5.29), (5.33), (5.34) e (5.35) para descrever o sistema dinâmico para as perturbações escalares. Alguns comentários são relevantes neste ponto:

- As variáveis $X(t)$, $\pi(t)$, $\xi(t)$ e $\psi(t)$ são nulas após a perturbação, conforme as equações (5.18) e (5.27).
- A variável $V(t)$, inadequada à nossa análise, é dada em termos de $q(t)$, $R(t)$ e $W(t)$, conforme o vínculo (5.19).
- A equação dinâmica de $Y(t)$, (5.33), e o vínculo (5.28) — que envolve $Y(t)$, $q(t)$, $W(t)$ e $R(t)$ — não são fatoráveis em termos de $\hat{Q}_{\mu\nu}$.
- As equações dinâmicas (5.29) e (5.35) constituem um sistema dinâmico fechado nas variáveis fluxo de calor e densidade de matéria, respectivamente. A equação

dinâmica para a variável auxiliar $W(t)$, (5.34), pode ser integrada de imediato, uma vez que a função $R(t)$ seja conhecida.

Diferentemente do caso das perturbações tensoriais, os resultados acima não apresentam mudanças significativas para o caso especial da solução de Kasner com plano de isotropia. Isto ocorre basicamente porque todas as componentes do tensor construído a partir da base escalar, $\hat{Q}_{\alpha\beta}$, são não nulas a princípio. Entretanto, a solução de Milne — por apresentar um conjunto de variáveis de trabalho diferente do exibido no caso mais geral — será analisada em separado na próxima seção.

5.3 A Solução de Milne

Lidamos aqui com as variáveis do conjunto $\mathcal{M}$ dado pela equação (3.39). Como no caso geral, as variáveis ω_α e $H_{\alpha\beta}$ não estão definidas para perturbações escalares; a pressão anisotrópica satisfaz, do mesmo modo, às relações (5.2) e (5.3). A relação (5.5) continua válida e podemos, então, escrever as variáveis de $\mathcal{M}$ em termos da base escalar:

$$
\begin{array}{ll}
(\delta E_{ij}) = E(t)\,\hat{Q}_{ij} & (\delta Z_{ij}) = Z(t)\,\hat{Q}_{ij} \\
(\delta \pi_{ij}) = \pi(t)\,\hat{Q}_{ij} & (q_j) = q(t)\,\hat{Q}_j \\
(\delta a_j) = \psi(t)\,\hat{Q}_j & (\delta\rho) = R(t)\,\hat{Q} \\
(\delta W) = W(t)\,\hat{Q} & (\delta p) = \lambda\,(\delta\rho)\,.
\end{array}
\tag{5.36}
$$

A equação dinâmica para Z, (3.40), é perturbada como:

$$
\begin{aligned}
h^\alpha{}_\mu\, h^\beta{}_\nu\, (\delta Z_{\alpha\beta})^\bullet \; & + \; \frac{5}{3}\theta\,(\delta Z_{\mu\nu}) + \sigma^\alpha{}_{(\mu}\left(\delta Z_{\nu)\alpha}\right) - \frac{2}{3}h_{\mu\nu}\,\sigma^{\alpha\beta}\,(\delta Z_{\alpha\beta}) + \\
& + \; \frac{2}{3}\theta\,(\delta E_{\mu\nu}) - \frac{1}{2}\sigma^\alpha{}_{(\mu}\left(\delta E_{\nu)\alpha}\right) + \frac{1}{3}h_{\mu\nu}\,\sigma^{\alpha\beta}\,(\delta E_{\alpha\beta}) - \\
& - \; \frac{1}{6}h_{\mu\nu}\,h^{\alpha\beta}\,(\delta q_\alpha)_{;\beta} + \frac{1}{4}h^\alpha{}_{(\mu}\,h^\beta{}_{\nu)}\,(\delta q_\alpha)_{;\beta} - \frac{1}{9}\theta\,h_{\mu\nu}\,h^{\alpha\beta}\,(\delta a_\alpha)_{;\beta} +
\end{aligned}
$$

$$+\ \frac{1}{6}\,\theta\,h^{(\alpha}{}_{\mu}\,h^{\beta)}{}_{\nu}\,(\delta a_{\alpha})_{;\beta}+\sigma_{\mu\nu}\,h^{\alpha\beta}\,(\delta a_{\alpha})_{;\beta}+\frac{2}{3}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,(\delta a_{\alpha})_{;\beta}-$$

$$-\ \frac{1}{2}\,h^{\alpha}{}_{(\mu}\,\sigma^{\beta}_{\nu)}\,(\delta a_{\alpha})_{;\beta}+\frac{1}{3}\,\sigma_{\mu\nu}\,(\delta\rho)-\frac{1}{3}\,\sigma_{\mu\nu}\,(\delta W)-$$

$$-\ \frac{1}{2}\,h^{\alpha}{}_{\mu}\,h^{\beta}{}_{\nu}\,(\delta\pi_{\alpha\beta})^{\bullet}-\frac{1}{6}\,\theta\,(\delta\pi_{\mu\nu})-\frac{1}{6}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,(\delta\pi_{\alpha\beta})+$$

$$+\ \frac{1}{4}\,\sigma^{\alpha}{}_{(\mu}\,\Big(\delta\pi_{\nu)\alpha}\Big)=0. \tag{5.37}$$

A equação dinâmica de W, (3.41), fica:

$$(\delta W)^{\bullet}\ +\ \frac{2}{3}\,\theta\,(\delta W)-2\,\sigma^{\alpha\beta}\,(\delta Z_{\alpha\beta})+2\,\sigma^{\alpha\beta}\,(\delta E_{\alpha\beta})-2\,\sigma^{\alpha\beta}\,(\delta a_{\alpha})_{;\beta}+$$

$$+\ \frac{4}{3}\,\theta\,(\delta a^{\alpha})_{;\alpha}-\frac{2}{3}\,(1+3\lambda)\,\theta\,(\delta\rho)+\sigma^{\alpha\beta}\,(\delta\pi_{\alpha\beta})=0, \tag{5.38}$$

e a equação dinâmica original de H, (3.42), transforma-se neste caso em um vínculo:

$$\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu)}{}^{\lambda}{}_{;}\,(\delta E_{\lambda\alpha})_{;\beta}\ +\ \frac{1}{2}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,\sigma_{\nu)\beta}\,(\delta q_{\alpha})+$$

$$+\ \frac{1}{2}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu)}{}^{\lambda}\,(\delta\pi_{\lambda\alpha})_{;\beta}=0. \tag{5.39}$$

A equação dinâmica de E, (3.43), é perturbada como:

$$h^{\alpha}{}_{\mu}\,h^{\beta}{}_{\nu}\,(\delta E_{\alpha\beta})^{\bullet}\ +\ \frac{2}{3}\,\theta\,(\delta E_{\mu\nu})-\frac{1}{2}\,\sigma^{\alpha}{}_{(\mu}\,\Big(\delta E_{\nu)\alpha}\Big)+\eta_{\mu}{}^{\alpha\gamma\varepsilon}\,\eta_{\nu}{}^{\beta\lambda\tau}\,V_{\gamma}\,V_{\lambda}\,\sigma_{\alpha\beta}\,(\delta E_{\varepsilon\tau})+$$

$$+ \frac{1}{3}\theta\,\eta_\mu{}^{\alpha\gamma\varepsilon}\,\eta_\nu{}^{\beta\lambda\tau}\,V_\gamma\,V_\lambda\,h_{\alpha\beta}\,(\delta E_{\varepsilon\tau}) - \frac{1}{6}\,h_{\mu\nu}\,(\delta q^\alpha)_{;\alpha} + \frac{1}{4}\,h^\alpha{}_{(\mu}\,h^\beta{}_{\nu)}\,(\delta q_\alpha)_{;\beta} +$$

$$+ \frac{1}{2}\,(1+\lambda)\,\sigma_{\mu\nu}\,(\delta\rho) - \frac{1}{2}\,h^\alpha{}_\mu\,h^\beta{}_\nu\,(\delta\pi_{\alpha\beta})^\bullet - \frac{1}{6}\,\theta\,(\delta\pi_{\mu\nu}) +$$

$$+ \frac{1}{6}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,(\delta\pi_{\alpha\beta}) - \frac{1}{4}\,\sigma^\alpha{}_{(\mu}\left(\delta\pi_{\nu)\alpha}\right) = 0. \tag{5.40}$$

As equações dinâmicas de ρ e q_α, (3.44) e (3.45), são escritas diretamente como:

$$(\delta\rho)^\bullet + (1+\lambda)\,\theta\,(\delta\rho) + (\delta q^\alpha)_{;\alpha} - \sigma^{\alpha\beta}\,(\delta\pi_{\alpha\beta}) = 0, \tag{5.41}$$

$$h_\alpha{}^\beta\,(\delta q_\beta)^\bullet + \frac{4}{3}\theta\,(\delta q_\alpha) + \sigma_\alpha{}^\beta\,(\delta q_\beta) - \lambda\,h_\alpha{}^\beta\,(\delta\rho)_{,\beta} +$$

$$+ \left(\delta\pi_\alpha{}^\beta\right)_{;\beta} + V_\alpha\,\sigma^{\beta\gamma}\,(\delta\pi_{\beta\gamma}) = 0. \tag{5.42}$$

A equação dinâmica da vorticidade, a segunda e a terceira equações de vínculo, (3.14), (3.50) e (3.51), são identicamente nulas e resta-nos apresentar a perturbação da primeira e segunda projeções e da equação de vínculo restante, (3.46)-(3.48):

$$h_\alpha{}^\beta\,h^{\mu\nu}\,(\delta E_{\beta\mu})_{;\nu} - \frac{1}{3}\theta\,h_\alpha{}^\beta\,(\delta q_\beta) + \frac{1}{2}\,\sigma_\alpha{}^\beta\,(\delta q_\beta) -$$

$$- \frac{1}{3}\,h_\alpha{}^\beta\,(\delta\rho)_{,\beta} - \frac{1}{2}\,h_\alpha{}^\beta\,(\delta\pi_\beta{}^\mu)_{;\mu} = 0, \tag{5.43}$$

$$\eta_\alpha{}^{\beta\mu\nu}\,V_\beta\,\sigma_\mu{}^\gamma\,(\delta E_{\nu\gamma}) - \frac{1}{2}\,\eta_\alpha{}^{\beta\mu\nu}\,V_\beta\,(\delta q_\mu)_{;\nu} -$$

$$- \frac{1}{2}\,\eta_\alpha{}^{\beta\mu\nu}\,V_\beta\,\sigma_\mu{}^\gamma\,(\delta\pi_{\nu\gamma}) = 0, \tag{5.44}$$

$$h_\alpha{}^\beta \, (\delta Z_\beta{}^\mu)_{;\mu} \; + \; \frac{1}{3}\,\theta\, h_\alpha{}^\beta \, (\delta q_\beta) - \frac{1}{3}\,\theta\,\sigma_\alpha{}^\beta \, (\delta a_\beta) +$$

$$+ \; \frac{2}{9}\,\theta^2\, h_\alpha{}^\beta \, (\delta a_\beta) + \frac{1}{3}\, h_\alpha{}^\beta \, (\delta W)_{,\beta} = 0, \tag{5.45}$$

onde fizemos, na equação (3.48), a escolha de considerarmos os termos entre chaves como sendo nulos mesmo após a perturbação, a fim de manter uma simplificação relativa nos cálculos que se seguem.

Substituindo, portanto, as variáveis escritas em termos da base escalar de (5.36) na equação (5.42), obtemos duas equações distintas:

$$\pi(t)\,\sigma^{\alpha\beta}\,\hat{Q}_{\alpha\beta} = 0, \tag{5.46}$$

e

$$q^\bullet(t)\,\hat{Q}_\alpha \; + \; q(t)\,h_\alpha{}^\beta\,\left(\hat{Q}_\beta\right)^\bullet + \frac{4}{3}\,\theta\, q(t)\,\hat{Q}_\alpha + q(t)\,\sigma_\alpha{}^\beta\,\hat{Q}_\beta -$$

$$- \;\; \lambda\, R(t)\, h_\alpha{}^\beta\,\hat{Q}_{,\beta} + \pi(t)\,\hat{Q}_\alpha{}^\beta{}_{;\beta} = 0. \tag{5.47}$$

A relação (5.46) dá o mesmo resultado (5.26), ou seja, a perturbação associada à pressão anisotrópica é, novamente, nula. A equação (5.47) também é idêntica ao caso geral, relação (5.29). A equação (5.41) também não sofre mudanças, obtendo-se então novamente o resultado (5.35).

A equação de vínculo (5.44) nos dá:

$$E(t)\,\eta_\alpha{}^{\beta\mu\nu}\,V_\beta\,\sigma_\mu{}^\gamma\,\hat{Q}_{\nu\gamma} - \frac{1}{2}\,q(t)\,\eta_\alpha{}^{\beta\mu\nu}\,V_\beta\,\hat{Q}_{\mu;\nu} = 0,$$

e, usando as relações (5.14) e (5.15), obtemos:

$$E(t)\,\sigma^{\alpha}{}_{[\mu}\,n_{\nu]}\,n_{\alpha}\,\hat{Q} = 0,$$

a qual — a exemplo do caso geral — só é válida se:

$$E(t) = 0. \tag{5.48}$$

O vínculo (5.43) é escrito como:

$$\frac{1}{3}\,\theta\,q(t)\,\hat{Q}_{\alpha} - \frac{1}{2}\,q(t)\,\sigma_{\alpha}{}^{\beta}\,\hat{Q}_{\beta} + \frac{1}{3}\,R(t)\,\hat{Q}_{\alpha} = 0,$$

donde vem:

$$[R(t) + \theta\,q(t)]\;\hat{Q}_{\alpha} = \frac{3}{2}\,q(t)\,\sigma_{\alpha}{}^{\beta}\,\hat{Q}_{\beta}. \tag{5.49}$$

O vínculo (5.39), proveniente da equação dinâmica de H, é dado por:

$$q(t)\,\left(\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,\sigma_{\nu)\beta}\,\hat{Q}_{\alpha}\right) = 0,$$

a qual, dado que o termo entre parênteses é não nulo, dá como resultado:

$$q(t) = 0. \tag{5.50}$$

Substituindo (5.50) em (5.49) obtemos:

$$R(t) = 0, \tag{5.51}$$

o que faz as equações (5.29) e (5.35), bem como a equação dinâmica de E, (5.40), identicamente nulas.

Tendo em mente os resultados (5.48) e (5.50)-(5.51) acima, o último vínculo restante, equação (5.45), se escreve como:

$$Z(t)\, h_{\alpha}{}^{\beta}\, \hat{Q}_{\beta}{}^{\mu}{}_{;\mu} - \frac{1}{3}\,\theta\, \psi(t)\, \sigma_{\alpha}{}^{\beta}\, \hat{Q}_{\beta} + \frac{2}{9}\,\theta^2\, \psi(t)\, \hat{Q}_{\alpha} + \frac{1}{3}\, W(t)\, h_{\alpha}{}^{\beta}\, \hat{Q}_{;\beta} = 0.$$

donde — usando a primeira das relações (2.4) e (2.14) — obtemos:

$$\left[2n^2\, Z(t) + W(t) + \frac{2}{3}\,\theta^2\, \psi(t)\right]\, \hat{Q}_{\alpha} = \theta\, \psi(t)\, \sigma_{\alpha}{}^{\beta}\, \hat{Q}_{\beta}. \tag{5.52}$$

A dinâmica de W, equação (5.38), fica:

$$\begin{aligned} W^{\bullet}(t)\, \hat{Q} \quad &+ \quad W(t)\, \dot{\hat{Q}} + \frac{2}{3}\,\theta\, W(t)\, \hat{Q} - 2\, Z(t)\, \sigma^{\alpha\beta}\, \hat{Q}_{\alpha\beta} - \\ &- \quad 2\, \psi(t)\, \sigma^{\alpha\beta}\, \hat{Q}_{\alpha;\beta} + \frac{4}{3}\,\theta\, \psi(t)\, \hat{Q}^{\alpha}{}_{;\alpha} = 0, \end{aligned}$$

e — fazendo uso de (2.7), (2.10), (2.13) e (5.31) — vem, após fatorarmos $\hat{Q}$:

$$\begin{aligned} W^{\bullet}(t) \quad &+ \quad \frac{2}{3}\,\theta\, W(t) + 2\, \left(\sigma^{\alpha\beta}\, n_{\alpha}\, n_{\beta}\right)\, Z(t) + \\ &+ \quad 2\, \left[\left(\sigma^{\alpha\beta}\, n_{\alpha}\, n_{\beta}\right) + \frac{2n^2}{3}\,\theta\right]\, \psi(t) = 0. \end{aligned} \tag{5.53}$$

Finalmente, a equação (5.37) — que dá a dinâmica da variável auxiliar Z — se escreve como:

$$\begin{aligned} Z^{\bullet}(t)\, \hat{Q}_{\mu\nu} \quad &+ \quad Z(t)\, h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, \left(\hat{Q}_{\alpha\beta}\right)^{\bullet} + \frac{5}{3}\,\theta\, Z(t)\, \hat{Q}_{\mu\nu} + Z(t)\, \sigma^{\alpha}{}_{(\mu}\, \hat{Q}_{\nu)\alpha} - \\ &- \quad \frac{2}{3}\, Z(t)\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, \hat{Q}_{\alpha\beta} - \frac{1}{9}\,\theta\, \psi(t)\, h_{\mu\nu}\, h^{\alpha\beta}\, \hat{Q}_{\alpha;\beta} + \frac{1}{6}\,\theta\, \psi(t)\, h^{(\alpha}{}_{\mu}\, h^{\beta)}{}_{\nu}\, \hat{Q}_{\alpha;\beta} + \end{aligned}$$

$$+ \quad \psi(t)\, \sigma_{\mu\nu}\, h^{\alpha\beta}\, \hat{Q}_{\alpha;\beta} + \frac{2}{3}\, \psi(t)\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, \hat{Q}_{\alpha;\beta} - \frac{1}{2}\, \psi(t)\, h^{\alpha}{}_{(\mu}\, \sigma^{\beta}{}_{\nu)}\, \hat{Q}_{(\alpha;\beta)} -$$

$$- \quad \frac{1}{3}\, W(t)\, \sigma_{\mu\nu}\, \hat{Q} = 0,$$

e, com o auxílio das equações (2.9), (2.14), (5.21)-(5.22) e (5.30)-(5.31), obtemos então:

$$\begin{aligned}
\left[Z^{\bullet}(t) + \theta\, Z(t) + \frac{1}{3}\, \theta\, \psi(t)\right]\, \hat{Q}_{\mu\nu} \quad = \quad & \left\{ \left[Z(t) + \frac{1}{3}\, W(t) - n^{2}\, \psi(t)\right]\, \sigma_{\mu\nu} + \right. \\
& + \frac{2}{3}\, \left(\sigma^{\alpha\beta}\, n_{\alpha}\, n_{\beta}\right)\, \psi(t)\, h_{\mu\nu} - \\
& \left. - \left(\sigma^{\alpha}{}_{(\mu}\, n_{\nu)}\, n_{\alpha}\right)\, \psi(t) \right\} \hat{Q}. \qquad (5.54)
\end{aligned}$$

Obtemos, assim, um sistema dinâmico fechado nas quantidades Z e W — equações (5.53)-(5.54) — mais uma equação de vínculo, (5.52), que fornece a aceleração ψ em termos de Z e W. Entretanto, as equações (5.52) e (5.54) não são completamente fatoráveis na base escalar.

Capítulo 6

Perturbações Vetoriais no Modelo de Kasner

6.1 Introdução

As perturbações ligadas ao estado de movimento do fluido, sem perturbar-se a densidade de energia, são descritas, a princípio, pelos seguintes conjuntos mínimos fechados de observáveis:

$$\mathcal{M} = \{X_{\alpha\beta}\, , Y_{\alpha\beta}\, , H_{\alpha\beta}\, , q_{\alpha}\, , a_{\alpha}\, , \omega_{\alpha}\}\, . \tag{6.1}$$

para a solução de Kasner ($p_1 = p_2 = 2/3$, $p_3 = -1/3$), e

$$\mathcal{M} = \{Z_{\alpha\beta}\, , E_{\alpha\beta}\, , H_{\alpha\beta}\, , q_{\alpha}\, , a_{\alpha}\, , \omega_{\alpha}\}\, , \tag{6.2}$$

para a solução de Milne. A pressão anisotrópica é eliminada do conjunto $\mathcal{M}$, em ambos os casos, pelos argumentos relacionados às equações (4.1), (4.2) e (4.3). Além disto, há ainda uma variável inadequada: V_k, a parte espacial da velocidade (a qual é nula no *background* por uma questão de escolha de referencial e cuja perturbação correspondente irá estar ligada a esta escolha). Iniciaremos pela análise das perturbações da solução de Kasner, escrevendo, então, as quantidades perturbadas correspondentes em termos da base vetorial e das quantidades com ela construídas:

$$
\begin{aligned}
&(\delta X_{\alpha\beta}) = X(t)\,\hat{P}_{\alpha\beta} \quad && (\delta Y_{\alpha\beta}) = Y(t)\,\hat{P}_{\alpha\beta} \\
&(\delta H_{\alpha\beta}) = H(t)\,\hat{P}^{*}_{\alpha\beta} \quad && (\delta q_{\alpha}) = q(t)\,\hat{P}_{\alpha} \\
&(\delta a_{\alpha}) = \psi(t)\,\hat{P}_{\alpha} \quad && (\delta \omega_{\alpha}) = \Omega(t)\,\hat{P}^{*}_{\alpha} \\
&(\delta V_{k}) = V(t)\,\hat{P}_{k}.
\end{aligned}
\tag{6.3}
$$

Da mesma forma, as variáveis perturbadas do conjunto $\mathcal{M}$ para a solução de Milne se esvrevem em termos da base $\hat{P}_{\alpha}$ como:

$$
\begin{aligned}
&(\delta Z_{\alpha\beta}) = Z(t)\,\hat{P}_{\alpha\beta} \quad && (\delta E_{\alpha\beta}) = E(t)\,\hat{P}_{\alpha\beta} \\
&(\delta H_{\alpha\beta}) = H(t)\,\hat{P}^{*}_{\alpha\beta} \quad && (\delta q_{\alpha}) = q(t)\,\hat{P}_{\alpha} \\
&(\delta a_{\alpha}) = \psi(t)\,\hat{P}_{\alpha} \quad && (\delta \omega_{\alpha}) = \Omega(t)\,\hat{P}^{*}_{\alpha} \\
&(\delta V_{k}) = V(t)\,\hat{P}_{k}.
\end{aligned}
\tag{6.4}
$$

Podemos, então, passar ao estudo dos sistemas dinâmicos que são obtidos para ambas as soluções específicas acima; isto será feito nas duas seções a seguir.

6.2 Solução de Kasner

Considerando-se o conjunto $\mathcal{M}$ dado por (6.1) e a equação dinâmica de $X_{\alpha\beta}$, (3.27), temos após a perturbação:

$$
\begin{aligned}
h_\mu{}^\alpha h_\nu{}^\beta \, (\delta X_{\alpha\beta})^\bullet \;\; + & \;\; \frac{5}{3}\,\theta\,(\delta X_{\mu\nu}) + \sigma^\alpha{}_{(\mu}\left(\delta X_{\nu)\alpha}\right) - \frac{2}{3}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,(\delta X_{\alpha\beta}) - (\delta Y_{\mu\nu}) - \\
- & \;\; \frac{1}{2}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,h_{\nu)}{}^\lambda\,(\delta H_{\beta\lambda})_{;\alpha} - \frac{1}{6}\,h_{\mu\nu}\,h^{\alpha\beta}\,(\delta q_\alpha)_{;\beta} + \frac{1}{4}\,h^\alpha{}_{(mu}\,h^\beta{}_{\nu)}\,(\delta q_\alpha)_{;\beta} - \\
- & \;\; \frac{1}{9}\,\theta\,h_{\mu\nu}\,h^{\alpha\beta}\,(\delta a_\alpha)_{;\beta} + \frac{1}{6}\,\theta\,h^{(\alpha}{}_\mu\,h^{\beta)}{}_\nu\,(\delta a_\alpha)_{;\beta} + \sigma_{\mu\nu}\,h^{\alpha\beta}\,(\delta a_\alpha)_{;\beta} + \\
+ & \;\; \frac{2}{3}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,(\delta a_\alpha)_{;\beta} - \frac{1}{2}\,h^\alpha{}_{(\mu}\,\sigma^\beta{}_{\nu)}\left(\delta a_{(\alpha}\right)_{;\beta)} = 0.
\end{aligned}
\tag{6.5}
$$

Usando (6.3), podemos reescrever (6.5) como:

$$
\begin{aligned}
X^\bullet(t)\,\hat{P}_{\mu\nu} \;\; + & \;\; X(t)\,h^\alpha{}_\mu\,h^\beta{}_\nu\left(\hat{P}_{\alpha\beta}\right)^\bullet + \frac{5}{3}\,\theta\,X(t)\,\hat{P}_{\mu\nu} + X(t)\,\sigma^\alpha{}_{(\mu}\,\hat{P}_{\nu)\alpha} - \\
- & \;\; \frac{2}{3}\,X(t)\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,\hat{P}_{\alpha\beta} - Y(t)\,\hat{P}_{\mu\nu} + \frac{1}{2}\,H(t)\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,h_{\nu)}{}^\lambda\,\hat{P}^*_{\lambda\alpha;\beta} - \frac{1}{6}\,q(t)\,h_{\mu\nu}\,h^{\alpha\beta}\,\hat{P}_{\alpha;\beta} + \\
+ & \;\; \frac{1}{4}\,q(t)\,h^\alpha{}_{(\mu}\,h^\beta{}_{\nu)}\,\hat{P}_{\alpha;\beta} - \frac{1}{9}\,\theta\,\psi(t)\,h_{\mu\nu}\,h^{\alpha\beta}\,\hat{P}_{\alpha;\beta} + \frac{1}{6}\,\theta\,\psi(t)\,h^{(\alpha}{}_\mu\,h^{\beta)}{}_\nu\,\hat{P}_{\alpha;\beta} + \psi(t)\,\sigma_{\mu\nu}\,h^{\alpha\beta}\,\hat{P}_{\alpha;} \\
+ & \;\; \frac{2}{3}\,\psi(t)\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,\hat{P}_{\alpha;\beta} - \frac{1}{2}\,\psi(t)\,h^\alpha{}_{(mu}\,\sigma^\beta{}_{\nu)}\,\hat{P}_{(\alpha;\beta)} = 0.
\end{aligned}
$$

(

Usando, agora, os resultados (2.35), (2.37) e os do Apêndice D, a equação acima fica:

$$
X^\bullet(t)\,\hat{P}_{\mu\nu} \;\; + \;\; \frac{2}{3}\,\theta\,X(t)\,\hat{P}_{\mu\nu} - X(t)\,\sigma^\alpha{}_{(\mu}\,\hat{P}_{\nu)\alpha} + \frac{5}{3}\,\theta\,X(t)\,\hat{P}_{\mu\nu} +
$$

$$+ \quad X(t)\,\sigma^{\alpha}{}_{(\mu}\,\hat{P}_{\nu)\alpha} - Y(t)\,\hat{P}_{\mu\nu} + \frac{m^2}{2}\,H(t)\,\hat{P}_{\mu\nu} + \frac{1}{4}\,q(t)\,\hat{P}_{\mu\nu} +$$

$$+ \quad \frac{1}{3}\,\theta\,\psi(t)\,\hat{P}_{\mu\nu} - \frac{1}{2}\,\psi(t)\,\sigma^{\alpha}{}_{(\mu}\,\hat{P}_{\nu)\alpha} = 0,$$

e vem então

$$\left\{X^{\bullet}(t) + \theta\,X(t) - Y(t) + \frac{m^2}{2}\,H(t) + \frac{1}{4}\,q(t) + \frac{1}{3}\,\theta\,\psi(t)\right\}\,\hat{P}_{\mu\nu} - \frac{1}{2}\,\psi(t)\,\sigma^{\alpha}{}_{(\mu}\,\hat{P}_{\nu)\alpha} = 0. \tag{6.7}$$

A equação dinâmica de $Y_{\alpha\beta}$, (3.28), é dada por:

$$\begin{aligned}
h^{\alpha}{}_{\mu}\,h^{\beta}{}_{\nu}\,(\delta Y_{\alpha\beta})^{\bullet} \quad &+ \quad 2\,\theta\,(\delta Y_{\mu\nu}) - \frac{3}{2}\,\sigma^{\alpha}{}_{(\mu}\,\left(\delta Y_{\nu)\alpha}\right) + h_{\mu\nu}\,\sigma^{\alpha\beta}\,(\delta Y_{\alpha\beta}) + \\
&+ \quad \frac{3}{2}\,E^{\alpha}{}_{(\mu}\,\left(\delta X_{\nu)\alpha}\right) - h_{\mu\nu}\,E^{\alpha\beta}\,(\delta X_{\alpha\beta}) + \frac{1}{2}\,\theta\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu)}{}^{\lambda}\,(\delta H_{\lambda\alpha})_{;\beta} + \\
&+ \quad \frac{3}{4}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,\sigma_{\nu)}{}^{\lambda}\,(\delta H_{\lambda\alpha})_{;\beta} + \frac{3}{4}\,\eta^{\gamma\lambda\alpha\beta}\,V_{\lambda}\,\sigma_{\gamma(\mu}\,h_{\nu)}{}^{\tau}\,(\delta H_{\tau\alpha})_{;\beta} - \\
&- \quad h_{\mu\nu}\,\eta^{\gamma\tau\alpha\beta}\,V_{\tau}\,\sigma_{\gamma}{}^{\lambda}\,(\delta H_{\lambda\alpha})_{;\beta} - \frac{1}{6}\,\theta\,h_{\mu\nu}\,h^{\alpha\beta}\,(\delta q_{\alpha})_{;\beta} + \\
&+ \quad \frac{1}{4}\,\theta\,h^{\alpha}{}_{(\mu}\,h^{\beta}{}_{\nu)}\,(\delta q_{\alpha})_{;\beta} + \frac{3}{8}\,h^{\alpha}{}_{(\mu}\,\sigma^{\beta}{}_{\nu)}\,\left(\delta q_{(\alpha}\right)_{;\beta)} - \frac{1}{2}\,h_{\mu\nu}\,\sigma^{\alpha\beta}\,(\delta q_{\alpha})_{;\beta} - \\
&- \quad \frac{1}{2}\,\sigma_{\mu\nu}\,h^{\alpha\beta}\,(\delta q_{\alpha})_{;\beta} - E_{\mu\nu}\,(\delta a^{\alpha})_{;\alpha} + E_{\mu\nu}\,h^{\alpha\beta}\,(\delta a_{\alpha})_{;\beta} - \\
&- \quad \frac{3}{4}\,h^{\alpha}{}_{(\mu}\,E^{\beta}{}_{\nu)}\,\left(\delta a_{(\alpha}\right)_{;\beta)} + h_{\mu\nu}\,E^{\alpha\beta}\,(\delta a_{\alpha})_{;\beta} = 0.
\end{aligned} \tag{6.8}$$

Usando novamente (6.3), a equação acima fica:

$$
\begin{aligned}
Y^{\bullet}(t)\, \hat{P}_{\mu\nu} &+ Y(t)\, h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu} \left(\hat{P}_{\alpha\beta}\right)^{\bullet} + 2\,\theta\, Y(t)\, \hat{P}_{\mu\nu} - \frac{3}{2} Y(t)\, \sigma^{\alpha}{}_{(\mu}\, P_{\nu)\alpha} + Y(t)\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, \hat{P}_{\alpha\beta} + \\
&+ \frac{3}{2} X(t)\, E^{\alpha}{}_{(\mu}\, \hat{P}_{\nu)\alpha} + \frac{1}{2}\theta\, H(t)\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu)}{}^{\lambda}\, \hat{P}^{*}_{\lambda\alpha;\beta} + \frac{3}{4} H(t)\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, \sigma_{\nu)}{}^{\lambda}\, \hat{P}^{*}_{\alpha;\beta} + \\
&+ \frac{3}{4} H(t)\, \eta^{\gamma\lambda\alpha\beta}\, V_{\lambda}\, \sigma_{\gamma(\mu}\, h_{\nu)}{}^{\tau}\, \hat{P}^{*}_{\tau\alpha;\beta} - H(t)\, h_{\mu\nu}\, \eta^{\gamma\tau\alpha\beta}\, V_{\tau}\, \sigma_{\gamma}{}^{\lambda}\, \hat{P}^{*}_{\lambda\alpha;\beta} - \\
&- \frac{1}{6}\theta\, q(t)\, h_{\mu\nu}\, h^{\alpha\beta}\, \hat{P}_{\alpha;\beta} + \frac{1}{4}\theta\, q(t)\, h^{\alpha}{}_{(\mu}\, h^{\beta}{}_{\nu)}\, \hat{P}_{\alpha;\beta} + \frac{3}{8} q(t)\, h^{\alpha}{}_{(\mu}\, \sigma^{\beta}{}_{\nu)}\, \hat{P}_{(\alpha;\beta)} - \\
&- \frac{1}{2} q(t)\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, \hat{P}_{\alpha;\beta} - \frac{1}{2} q(t)\, \sigma_{\mu\nu}\, h^{\alpha\beta}\, \hat{P}_{\alpha;\beta} - \psi(t)\, E_{\mu\nu}\, \hat{P}^{\alpha}{}_{;\alpha} + \\
&+ \psi(t)\, E_{\mu\nu}\, h^{\alpha\beta}\, \hat{P}_{\alpha;\beta} - \frac{3}{4}\psi(t)\, h^{\alpha}{}_{(\mu}\, E^{\beta}{}_{\nu)}\, \hat{P}_{(\alpha;\beta)} + \psi(t)\, h_{\mu\nu}\, E^{\alpha\beta}\, \hat{P}_{\alpha;\beta} = 0. \qquad (6.9)
\end{aligned}
$$

Agora, substituindo as equações (2.37), (2.35), e as relações demonstradas no Apêndice D, obtemos:

$$
\begin{aligned}
Y^{\bullet}(t)\, \hat{P}_{\mu\nu} &- \frac{2}{3}\theta\, Y(t)\, \hat{P}_{\mu\nu} - Y(t)\, \sigma^{\alpha}{}_{(\mu}\, \hat{P}_{\nu)\alpha} + 2\,\theta\, Y(t)\, \hat{P}_{\mu\nu} - \frac{3}{2} Y(t)\, \sigma^{\alpha}{}_{(\mu}\, \hat{P}_{\nu)\alpha} + \\
&+ \theta\, X(t)\, \sigma^{\alpha}{}_{(\mu}\, \hat{P}_{\nu)\alpha} + \frac{m^2}{2}\theta\, H(t)\, \hat{P}_{\mu\nu} + \frac{3m^2}{4} H(t)\, \sigma^{\alpha}{}_{(\mu}\, \hat{P}_{\nu)\alpha} + \\
&+ \frac{3m^2}{4} H(t)\, \sigma^{\alpha}{}_{(\mu}\, \hat{P}_{\nu)\alpha} + \frac{1}{4}\theta\, q(t)\, \hat{P}_{\mu\nu} + \frac{3}{8} q(t)\, \sigma^{\alpha}{}_{(\mu}\, \hat{P}_{\nu)\alpha} - \\
&- \frac{1}{2}\theta\, \psi(t)\, \sigma^{\alpha}{}_{(\mu}\, \hat{P}_{\nu)\alpha} = 0.
\end{aligned}
$$

Isto nos dá então a seguinte equação:

$$\left\{Y^{\bullet}(t) + \frac{4}{3}\,\theta\, Y(t) + \frac{m^2}{2}\,\theta\, H(t) + \frac{1}{4}\,\theta\, q(t)\right\}\,\hat{P}_{\mu\nu} -$$

$$- \quad \left\{\frac{5}{2}\,Y(t) - \theta\, X(t) - \frac{3m^2}{2}\,H(t) - \frac{3}{8}\,q(t) + \frac{1}{2}\,\theta\,\psi(t)\right\}\,\sigma^{\alpha}{}_{(\mu}\,\hat{P}_{\nu)\alpha} = 0. \qquad (6.10)$$

A equação dinâmica de W, (3.29), reduz-se a (após a perturbação):

$$\sigma^{\alpha\beta}\,(\delta X_{\alpha\beta}) - \sigma^{\alpha\beta}\,(\delta a_{\alpha})_{;\beta} + \tfrac{2}{3}\,\theta\, h^{\alpha\beta}\,(\delta a_{\alpha})_{;\beta} = 0 \Longrightarrow$$

$$\Longrightarrow X(t)\,\sigma^{\alpha\beta}\,\hat{P}_{\alpha\beta} - \psi(t)\,\sigma^{\alpha\beta}\,\hat{P}_{\alpha;\beta} + \tfrac{2}{3}\,\theta\,\psi(t)\, h^{\alpha\beta}\,\hat{P}_{\alpha;\beta} \equiv 0,$$

a qual é identicamente nula.

A equação dinâmica de $H_{\alpha\beta}$, não perturbada, é dada por (3.32), onde os termos associados ao tensor de rotação devem ser convertidos em tensores antes de podermos proceder à perturbação. Para tanto, usaremos a relação [25]:

$$\omega^{\tau} = \frac{1}{2}\,\eta^{\alpha\beta\rho\tau}\,\omega_{\alpha\beta}\,V_{\rho}. \qquad (6.11)$$

É fácil inverter-se a relação (6.11) acima para escrever-se:

$$\omega_{\alpha\beta} = -\eta_{\alpha\beta\rho\tau}\,V^{\rho}\,\omega^{\tau}. \qquad (6.12)$$

Note-se que, substituindo-se (6.11) em (6.12), reobtemos $\omega_{\alpha\beta}$. Podemos, então, reescrever os termos em (3.32):

$$\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu)}{}^{\lambda}\,\omega_{\lambda\alpha;\beta} \quad = \quad -\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu)}{}^{\lambda}\,(\eta_{\lambda\alpha\varepsilon\tau}\,V^{\varepsilon}\,\omega^{\tau})_{;\beta}$$

$$= \quad -\eta_{(\mu}{}^{\gamma\alpha\beta}\,\eta_{\lambda\alpha\varepsilon\tau}\,V_\gamma\,h_{\nu)}{}^{\lambda}\,\left[\left(\sigma_\beta{}^{\varepsilon}+\frac{1}{3}\,\theta\,h_\beta{}^{\varepsilon}\right)\,\omega^\tau + V^\varepsilon\,\omega^\tau{}_{;\beta}\right]$$

$$= \quad -\delta^{\chi\gamma\beta}_{\lambda\varepsilon\tau}\,V_\gamma\,V^\varepsilon\,h_{\chi(\mu}\,h_{\nu)}{}^{\lambda}\,\omega^\tau{}_{;\beta}$$

$$= \quad -2\,h_{\mu\nu}\,\omega^\alpha{}_{;\alpha} - V_\gamma\,V^\varepsilon\,h_{\varepsilon(\mu}\,h_{\nu)}{}^{\beta}\,\omega^\gamma{}_{;\beta} - V_\gamma\,V^\beta\,h_{\tau(\mu}\,h_{\nu)}{}^{\gamma}\,\omega^\tau{}_{;\beta} +$$

$$+V_\gamma\,V^\beta\,h_{\lambda(\mu}\,h_{\nu)}{}^{\lambda}\,\omega^\gamma{}_{;\beta} + V_\gamma\,V^\varepsilon\,h_{\varepsilon(\mu}\,h_{\nu)}{}^{\gamma}\,\omega^\beta{}_{;\beta} + h_{\tau(\mu}\,h_{\nu)}{}^{\beta}\,\omega^\tau{}_{;\beta}$$

$$= \quad h^\alpha{}_{(\mu}\,h^\beta{}_{(\nu}\,\omega_{\alpha;\beta} - 2\,h_{\mu\nu}\,\omega^\alpha{}_{;\alpha},$$

$$\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,\omega^\lambda{}_{\beta;\lambda} \quad = \quad -\eta_{(\mu}{}^{\gamma\alpha\beta}\,\eta^\lambda{}_{\beta\varepsilon\tau}\,V_\gamma\,\sigma_{\nu)\alpha}\,\left[\left(\sigma_\lambda{}^{\varepsilon}+\frac{1}{3}\,\theta\,h_\lambda{}^{\varepsilon}\right)\,\omega^\tau + V^\varepsilon\,\omega^\tau{}_{;\lambda}\right]$$

$$= \quad \delta^{\lambda\gamma\alpha}_{\rho\varepsilon\tau}\,V_\gamma\,V^\varepsilon\,h^{\lambda\rho}\,h_{\chi(\mu}\,\sigma_{\nu)\alpha}\,\omega^\tau{}_{;\lambda}$$

$$= \quad h^\lambda{}_{(\mu}\,\sigma_{\nu)\alpha}\,\omega^\alpha{}_{;\alpha} - h_{\tau(\mu}\,\sigma_{\nu)}{}^{\lambda}\omega^\tau{}_{;\lambda}$$

$$= \quad h^\lambda{}_{(\mu}\,\sigma_{\nu)}{}^{\beta}\,\omega_{\beta;\lambda} - h^\alpha{}_{(\mu}\,\sigma_{\nu)}{}^{\lambda}\,\omega_{\alpha;\beta}$$

$$= \quad -h^\alpha{}_{(\mu}\,\sigma_{\nu)}{}^{\beta}\,\omega_{[\alpha;\beta]}.$$

E, portanto, a equação dinâmica de $H_{\alpha\beta}$ se escreve, após a perturbação, como:

$$h^\alpha{}_\mu\,h^\beta{}_\nu\,(\delta H_{\alpha\beta})^\bullet \quad + \quad \frac{4}{3}\,\theta\,(\delta H_{\mu\nu}) - \frac{1}{2}\,\sigma^\alpha{}_{(\mu}\,\left(\delta H_{\nu)\alpha}\right) + \eta_\mu{}^{\alpha\gamma\varepsilon}\,\eta_\nu{}^{\beta\lambda\tau}\,V_\gamma\,V_\lambda\,\sigma_{\alpha\beta}\,(\delta H_{\varepsilon\tau}) +$$

$$+ \quad \frac{1}{3}\,\theta\,\eta_\mu{}^{\alpha\gamma\varepsilon}\,\eta_\nu{}^{\beta\lambda\tau}\,V_\gamma\,V_\lambda\,h_{\alpha\beta}\,(\delta H_{\varepsilon\tau}) - \eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,E_{\nu)\beta}\,(\delta a_\alpha) -$$

$$
\begin{aligned}
- &\frac{1}{4}\,\eta_{(\mu}{}^{\alpha\beta\gamma}\,V_\gamma\,\sigma_{\nu)\beta}\,\left(\delta q_\alpha\right) - \frac{1}{2}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,h_{\nu)}{}^\lambda\,\left(\delta X_{\lambda\alpha}\right)_{;\beta} + \\
+ &\frac{1}{6}\,\theta\,h^\alpha{}_{(\mu}\,h^\beta{}_{\nu)}\,\left(\delta\omega_\alpha\right)_{;\beta} - \frac{1}{3}\,\theta\,h_{\mu\nu}\,\left(\delta\omega^\alpha\right)_{;\alpha} + \frac{1}{4}\,h^\alpha{}_{(\mu}\,\sigma^\beta{}_{\nu)}\,\left(\delta\omega_{[\alpha}\right)_{;\beta]} - \\
- &\frac{1}{4}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,\sigma_\beta{}^\lambda\,\left(\delta a_\lambda\right) + \frac{1}{4}\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,\left(\delta q_\beta\right) = 0. \qquad (6.13)
\end{aligned}
$$

E, repetindo o processo executado para as outras equações, escrevemos então:

$$
\begin{aligned}
H^\bullet(t)\,\hat{P}^*_{\mu\nu} \;\; + &\;\; H(t)\,h^\alpha{}_\mu\,h^\beta{}_\nu\,\left(\hat{P}^*_{\alpha\beta}\right)^\bullet + \frac{4}{3}\,\theta\,H(t)\,\hat{P}^*_{\mu\nu} - \frac{1}{2}\,H(t)\,\sigma^\alpha{}_{(\mu}\,\hat{P}^*_{\nu)\alpha} - \\
- &\;\; H(t)\,\sigma^\alpha{}_{(\mu}\,\hat{P}^*_{\nu)\alpha} + \frac{1}{3}\,\theta\,H(t)\,\hat{P}^*_{\mu\nu} - \psi(t)\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,E_{\nu)\beta}\,\hat{P}_\alpha + \frac{1}{2}\,q(t)\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,\hat{P}_\beta - \\
- &\;\; \frac{1}{2}\,X(t)\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,h_{\nu)}{}^\lambda\,\hat{P}_{\lambda\alpha;\beta} + \frac{1}{6}\,\theta\,\Omega(t)\,h^\alpha{}_{(\mu}\,h^\beta{}_{\nu)}\,\hat{P}^*_{\alpha;\beta} - \frac{1}{3}\,\theta\,\Omega(t)\,h_{\mu\nu}\,\hat{P}^{*\alpha}{}_{;\alpha} + \\
+ &\;\; \frac{1}{4}\,\Omega(t)\,h^\alpha{}_{(\mu}\,\sigma^\beta{}_{\nu)}\,\hat{P}^*_{[\alpha;\beta]} - \frac{1}{4}\,\psi(t)\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,\sigma_\beta{}^\lambda\,\hat{P}_\lambda = 0.
\end{aligned}
$$

E, usando os resultados já citados anteriormente, bem como os do Apêndice D, obtemos afinal:

$$
\begin{aligned}
&\left\{H^\bullet(t) + \theta\,H(t) - \frac{1}{2}\,X(t) - \frac{1}{6}\,\theta\,\Omega(t)\right\}\,\hat{P}^*_{\mu\nu} - \\
- &\;\frac{5}{2}\,H(t)\,\sigma^\alpha{}_{(\mu}\,\hat{P}^*_{\nu)\alpha} - \left[\frac{2}{3}\,\psi(t) + \frac{1}{2}\,q(t)\right]\,\left(\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,\hat{P}_\beta\right) - \\
- &\;\frac{1}{4}\,\psi(t)\,\left(\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu)\alpha}\,\sigma_\beta{}^\lambda\,\hat{P}_\lambda\right) = 0. \qquad (6.14)
\end{aligned}
$$

A primeira lei de conservação, equação (3.10), é identicamente nula e a segunda lei, (3.11), é perturbada como:

$$h_\alpha{}^\beta\ (\delta q_\beta)^\bullet + \theta\ (\delta q_\alpha) + \sigma_\alpha{}^\beta\ (\delta q_\beta) + \frac{1}{3}\,\theta\, h_\alpha{}^\beta\ (\delta q_\beta) = 0. \qquad (6.15)$$

Substituindo (6.3), vem:

$$q^\bullet(t)\,\hat{P}_\alpha + q(t)\,h_\alpha{}^\beta\ \left(\hat{P}_\beta\right)^\bullet + \theta\, q(t)\,\hat{P}_\alpha + \sigma_\alpha{}^\beta\, q(t)\,\hat{P}_\beta + \frac{1}{3}\,\theta\, q(t)\,h_\alpha{}^\beta\,\hat{P}_\beta = 0,$$

o que — substituindo-se (2.31) — nos dá, afinal,

$$q^\bullet(t) + \theta\, q(t) = 0. \qquad (6.16)$$

A equação seguinte é a dinâmica de ω_α, (3.14), a qual também precisa ser reescrita em termos do vetor de rotação. Temos, portanto, que — usando-se a definição (6.12):

$$\begin{aligned}
h_\alpha{}^\mu\, h_\beta{}^\nu\ (\omega_{\mu\nu})^\bullet &= h_\alpha{}^\mu\, h_\beta{}^\nu\ \left(-\eta_{\mu\nu}{}^{\gamma\lambda}\, V_\gamma\, \omega_\lambda\right)^\bullet \\
&= -\eta_{\mu\nu}{}^{\gamma\lambda}\, h_\alpha{}^\mu\, h_\beta{}^\nu\, V_\gamma\ (\omega_\lambda)^\bullet - \eta_{\mu\nu}{}^{\gamma\lambda}\, h_\alpha{}^\mu\, h_\beta{}^\nu\, a_\gamma\, \omega_\lambda \\
&= -\eta_{\alpha\beta}{}^{\mu\nu}\, V_\mu\ (\omega_\nu)^\bullet,
\end{aligned}$$

onde o segundo termo acima foi descartado por ser de segunda ordem de perturbação. Continuando, vem ainda que:

$$\sigma_{\mu[\alpha}\,\omega^\mu{}_{\beta]} = -\sigma_{\mu[\alpha}\,\eta^\mu{}_{\beta]}{}^{\gamma\lambda}\, V_\gamma\, \omega_\lambda$$

$$= \eta^{\mu}{}_{[\alpha}{}^{\gamma\lambda} V_\gamma \, \sigma_{\beta]\mu} \, \omega_\lambda$$

$$= -\eta_{[\alpha}{}^{\mu\gamma\lambda} V_\gamma \, \sigma_{\beta]\mu} \, \omega_\lambda,$$

e, perturbando a equação (3.14) reescrita, temos:

$$\begin{aligned} \eta_{\alpha\beta}{}^{\mu\nu} V_\mu \, (\delta\omega_\nu)^\bullet &+ \frac{2}{3} \theta \, \eta_{\alpha\beta}{}^{\mu\nu} V_\mu \, (\delta\omega_\nu) - \eta_{[\alpha}{}^{\mu\gamma\lambda} V_\gamma \, \sigma_{\beta]\mu} \, (\delta\omega_\lambda) + \\ &+ \frac{1}{2} h_\alpha{}^\mu h_\beta{}^\nu \left(\delta a_{[\mu}\right)_{;\nu]} = 0. \end{aligned} \qquad (6.17)$$

Com (6.3), (6.17) acima se escreve como:

$$\begin{aligned} \Omega^\bullet(t) \, \eta_{\alpha\beta}{}^{\mu\nu} V_\mu \, \hat{P}^*_\nu &+ \Omega(t) \, \eta_{\alpha\beta}{}^{\mu\nu} V_\mu \left(\hat{P}^*_\nu\right)^\bullet + \frac{2}{3} \theta \, \Omega(t) \, \eta_{\alpha\beta}{}^{\mu\nu} V_\mu \, \hat{P}^*_\nu - \\ &- \Omega(t) \, \eta_{[\alpha}{}^{\mu\gamma\lambda} V_\gamma \, \sigma_{\beta]\mu} \, \hat{P}^*_\lambda + \frac{1}{2} \psi(t) \, h_\alpha{}^\mu h_\beta{}^\nu \, \hat{P}_{[\mu;\nu]} = 0. \end{aligned}$$

E, usando as relações (2.43) e as do Apêndice D, temos:

$$\Omega^\bullet(t) + \frac{1}{3} \theta \, \Omega(t) + \frac{1}{2} \psi(t) = 0. \qquad (6.18)$$

A segunda equação de vínculo, (3.16), é identicamente nula, por (D.17), e resta-nos, então, perturbar as duas projeções restantes: a primeira projeção, equação (3.33) e a segunda projeção, equação (3.42). A primeira projeção deve ter um dos seus termos reescrito em termos do vetor de rotação:

$$\left(\sigma_\alpha{}^\beta + \frac{1}{3} \theta \, h_\alpha{}^\beta\right) \omega_\beta{}^\mu{}_{;\mu} = -\eta^\mu{}_\beta{}^{\gamma\lambda} \left(\sigma_\alpha{}^\beta + \frac{1}{3} \theta \, h_\alpha{}^\beta\right) [V_{\gamma;\mu} \, \omega_\lambda + V_\gamma \, \omega_{\lambda;\mu}]$$

$$= \quad -\eta^{\mu\beta\gamma\lambda}\, V_\gamma \left(\sigma_{\alpha\beta} + \frac{4}{3}\,\theta\, h_{\alpha\beta}\right)\, \omega_{\lambda;\mu}.$$

Logo, a primeira projeção perturbada se escreve como:

$$h_\alpha{}^\beta\, h^{\mu\nu}\, (\delta X_{\beta\mu})_{;\nu} \quad + \quad \eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, \sigma_\mu{}^\gamma\, (\delta H_{\nu\gamma}) - \eta^{\mu\beta\gamma\lambda}\, V_\gamma \left(\sigma_{\alpha\beta} + \frac{4}{3}\,\theta\, h_{\alpha\beta}\right) (\delta\omega_\lambda)_{;\mu} +$$

$$= \quad \frac{1}{2}\left(\sigma_{\alpha\gamma}\,\sigma^{\beta\gamma} + 2\,\theta\,\sigma_\alpha{}^\beta\right)(\delta a_\beta) - \theta\,(\delta q_\alpha) + \frac{1}{9}\,\theta^2 \left(\sigma_\alpha{}^\beta - \frac{8}{3}\,\theta\, h_\alpha{}^\beta\right)(\delta V_\beta) = 0, \tag{6.19}$$

e, substituindo novamente (6.3) e as relações do Apêndice D, vem afinal:

$$\theta\left[\frac{8}{27}\,\theta^2\, V(t) + q(t) - \frac{4m^2}{3}\,\Omega(t)\right]\hat{P}_\alpha \quad - \quad \frac{1}{2}\,\psi(t)\,\sigma_{\alpha\gamma}\,\sigma^{\beta\gamma}\,\hat{P}_\beta -$$

$$- \quad \left[\frac{1}{9}\,\theta^2\, V(t) - m^2\, H(t) + m^2\,\Omega(t) + \theta\,\psi(t)\right]\sigma_\alpha{}^\beta\,\hat{P}_\beta = 0. \tag{6.20}$$

A segunda projeção perturbada fica:

$$h_\alpha{}^\beta\, h^{\mu\nu}\, (\delta H_{\beta\mu})_{;\nu} \quad - \quad \eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, \sigma_\mu{}^\gamma\, (\delta X_{\nu\gamma}) - 3\, E_\alpha{}^\beta\, (\delta\omega_\beta) +$$

$$+ \quad \frac{1}{2}\,\eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, (\delta q_\mu)_{;\nu} = 0, \tag{6.21}$$

e repetindo os passos executados para todas as outras equações, escrevemos:

$$H(t)\, h_\alpha{}^\beta\, h^{\mu\nu}\, \hat{P}^*_{\beta\mu;\nu} \;-\; X(t)\, \eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, \sigma_\mu{}^\gamma\, \hat{P}_{\nu\gamma} - 3\, \Omega(t)\, E_\alpha{}^\beta\, \hat{P}^*_\beta +$$

$$+ \;\; \frac{1}{2}\, q(t)\, \eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, \hat{P}_{\mu;\nu} = 0,$$

o que nos dá afinal:

$$\left[m^2\, H(t) + \frac{1}{2}\, q(t)\right]\, \hat{P}^*_\alpha \;+\; \left[3\, X(t) - 2\,\theta\, \Omega(t)\right]\, \sigma_\alpha{}^\beta\, \hat{P}^*_\beta = 0. \tag{6.22}$$

6.3 Solução de Milne

O sistema dinâmico para o caso específico da solução de Milne é, como já vimos, escrito para as variáveis do conjunto mínimo fechado da equação (6.2). Temos, então, para a equação dinâmica de $Z_{\alpha\beta}$ — relação (3.40) — o seguinte resultado, após a perturbação:

$$\begin{aligned} h^\alpha{}_\mu\, h^\beta{}_\nu\, (\delta Z_{\alpha\beta})^\bullet \;&+\; \frac{5}{3}\,\theta\, (\delta Z_{\mu\nu}) + \sigma^\alpha{}_{(\mu}\, \Big(\delta Z_{\nu)\alpha}\Big) - \frac{2}{3}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, (\delta Z_{\alpha\beta}) + \frac{2}{3}\,\theta\, (\delta E_{\alpha\beta}) - \\ &-\; \frac{1}{2}\,\sigma^\alpha{}_{(\mu}\, \Big(\delta E_{\nu)\alpha}\Big) + \frac{1}{3}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, (\delta E_{\alpha\beta}) - \frac{1}{6}\, h_{\mu\nu}\, h^{\alpha\beta}\, (\delta q_\alpha)_{;\beta} + \\ &+\; \frac{1}{4}\, h^\alpha{}_{(\mu}\, h^\beta{}_{\nu)}\, (\delta q_\alpha)_{;\beta} - \frac{1}{9}\,\theta\, h_{\mu\nu}\, h^{\alpha\beta}\, (\delta a_\alpha)_{;\beta} + \frac{1}{6}\,\theta\, h^{(\alpha}{}_\mu\, h^{\beta)}{}_\nu\, (\delta a_\alpha)_{;\beta} + \\ &+\; \sigma_{\mu\nu}\, h^{\alpha\beta}\, (\delta a_\alpha)_{;\beta} + \frac{2}{3}\, h_{\mu\nu}\, \sigma^{\alpha\beta}\, (\delta a_\alpha)_{;\beta} - \frac{1}{2}\, h^\alpha{}_{(\mu}\, \sigma^\beta{}_{\nu)}\, (\delta a_\alpha)_{;\beta} = 0. \end{aligned} \tag{6.23}$$

Reescrevendo (6.23) em termos da base vetorial, com o auxílio de (6.4), vem então:

Substituindo (6.4) em (6.25) acima, vem:

$$H^{\bullet}(t)\, P^{*}_{\mu\nu} \quad + \quad H(t)\, h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, \left(\hat{P}^{*}_{\alpha\beta}\right)^{\bullet} + \frac{2}{3}\,\theta\, H(t)\, \hat{P}^{*}_{\mu\nu} - \frac{1}{2}\, H(t)\, \sigma^{\alpha}{}_{(\mu}\, \hat{P}^{*}_{\nu)\alpha} - H(t)\, \sigma^{\alpha}{}_{(\mu}\, \hat{P}^{*}_{\nu)\alpha} +$$

$$+ \quad \frac{1}{3}\,\theta\, H(t)\, \hat{P}^{*}_{\mu\nu} - \frac{1}{2}\, E(t)\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu)}{}^{\lambda}\, \hat{P}_{\lambda\alpha;\beta} - \frac{1}{4}\, q(t)\, \eta_{(\mu}{}^{\alpha\beta\gamma}\, V_{\gamma}\, \sigma_{\nu)\beta}\, \hat{P}_{\alpha} = 0,$$

e, usando (2.49) e (D.1), obtemos:

$$\left\{H^{\bullet}(t) + \frac{1}{3}\,\theta\, H(t) - \frac{1}{2}\, E(t)\right\}\, \hat{P}^{*}_{\mu\nu} -$$

$$- \quad \frac{5}{2}\, H(t)\, \sigma^{\alpha}{}_{(\mu}\, \hat{P}^{*}_{\nu)\alpha} + \frac{1}{4}\, q(t)\, \left(\eta_{(\mu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, \sigma_{\nu)\alpha}\, \hat{P}_{\beta}\right) = 0. \qquad (6.26)$$

A equação (3.43) é perturbada como:

$$h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, (\delta E)^{\bullet} \quad + \quad \frac{2}{3}\,\theta\, (\delta E_{\mu\nu}) - \frac{1}{2}\, \sigma^{\alpha}{}_{(\mu}\, \left(\delta E_{\nu)\alpha}\right) + \eta_{\mu}{}^{\alpha\gamma\varepsilon}\, \eta_{\nu}{}^{\beta\lambda\tau}\, V_{\gamma}\, V_{\lambda}\, \sigma_{\alpha\beta}\, (\delta E_{\varepsilon\tau}) +$$

$$+ \quad \frac{1}{3}\,\theta\, \eta_{\mu}{}^{\alpha\gamma\varepsilon}\, \eta_{\nu}{}^{\beta\lambda\tau}\, V_{\gamma}\, V_{\lambda}\, h_{\alpha\beta}\, (\delta E_{\varepsilon\tau}) - \frac{1}{6}\, h_{\mu\nu}\, (\delta q^{\alpha})_{;\alpha} + \frac{1}{4}\, h^{\alpha}{}_{(\mu}\, h^{\beta}{}_{\nu)}\, (\delta q_{\alpha})_{;\beta} = 0. \qquad (6.27)$$

Se usarmos (6.4), a equação acima fica:

$$E^{\bullet}(t)\, \hat{P}_{\mu\nu} \quad + \quad E(t)\, h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu}\, \left(\hat{P}_{\alpha\beta}\right)^{\bullet} + \frac{2}{3}\,\theta\, E(t)\, \hat{P}_{\mu\nu} - \frac{1}{2}\, E(t)\, \sigma^{\alpha}{}_{(\mu}\, \hat{P}_{\nu)\alpha} - E(t)\, \sigma^{\alpha}{}_{(\mu}\, \hat{P}_{\nu)\alpha} +$$

$$+ \quad \frac{1}{3}\,\theta\, E(t)\, \hat{P}_{\mu\nu} - \frac{1}{6}\, q(t)\, h_{\mu\nu}\, \hat{P}^{\alpha}{}_{;\alpha} + \frac{1}{4}\, q(t)\, h^{\alpha}{}_{(\mu}\, h^{\beta}{}_{\nu)}\, \hat{P}_{\alpha;\beta} = 0,$$

e — usando (2.37), (D.3) e (D.4) — obtemos daí:

$$\left\{E^{\bullet}(t) + \frac{1}{3}\,\theta\,E(t) + \frac{1}{4}\,q(t)\right\}\,\hat{P}_{\mu\nu} - \frac{5}{2}\,E(t)\,\sigma^{\alpha}{}_{(\mu}\,\hat{P}_{\nu)\alpha} = 0. \tag{6.28}$$

A equação dinâmica de ρ, (3.44), é escrita como:

$$\left(\delta q^{\alpha}\right)_{;\alpha} \Longrightarrow q(t)\,\hat{P}^{\alpha}{}_{;\alpha} \equiv 0,$$

é identicamente nula, por (D.3), e a equação dinâmica de q_α é idêntica à do caso da solução de Kasner, (6.15), obtendo-se então a equação (6.16). O mesmo acontece com a equação dinâmica da rotação, (6.17), obtendo-se então (6.18). A segunda equação de vínculo, (3.16) é também identicamente nula, por (D.17), como no caso anterior.

A primeira projeção, equação (3.46), é perturbada como:

$$h_{\alpha}{}^{\beta}\,h^{\mu\nu}\,\left(\delta E_{\beta\mu}\right)_{;\nu} + \eta_{\alpha}{}^{\beta\mu\nu}\,V_{\beta}\,\sigma_{\mu}{}^{\gamma}\,\left(\delta H_{\nu\gamma}\right) - \frac{1}{3}\,\theta\,h_{\alpha}{}^{\beta}\,\left(\delta q_{\beta}\right) - \frac{1}{2}\,\sigma_{\alpha}{}^{\beta}\,\left(\delta q_{\beta}\right) = 0, \tag{6.29}$$

a qual, usando-se (6.4), é reescrita como:

$$E(t)\,h_{\alpha}{}^{\beta}\,h^{\mu\nu}\,\hat{P}_{\beta\mu;\nu} + H(t)\,\eta_{\alpha}{}^{\beta\mu\nu}\,V_{\beta}\,\sigma_{\mu}{}^{\gamma}\,\hat{P}^{*}_{\nu\gamma} - \tfrac{1}{3}\,\theta\,q(t)\,h_{\alpha}{}^{\beta}\,\hat{P}_{\beta} + \tfrac{1}{2}\,q(t)\,\sigma_{\alpha}{}^{\beta}\,\hat{P}_{\beta} = 0 \Longrightarrow$$

$$\Longrightarrow -m^{2}\,H(t)\,\sigma_{\alpha}{}^{\beta}\,\hat{P}_{\beta} - \tfrac{1}{3}\,\theta\,q(t)\,\hat{P}_{\alpha} + \tfrac{1}{2}\,q(t)\,\sigma_{\alpha}{}^{\beta}\,\hat{P}_{\beta} = 0 \Longrightarrow$$

$$\left[m^{2}\,H(t) - \frac{1}{2}\,q(t)\right]\,\sigma_{\alpha}{}^{\beta}\,hatP_{\beta} = -\frac{1}{3}\,\theta\,q(t)\,\hat{P}_{\alpha}, \tag{6.30}$$

onde usamos as relações (D.20) e (D.21).

A segunda projeção, (3.47), é perturbada como:

$$h_{\alpha}{}^{\beta}\,h^{\mu\nu}\,\left(\delta H_{\beta\mu}\right)_{;\nu} - \eta_{\alpha}{}^{\beta\mu\nu}\,V_{\beta}\,\sigma_{\mu}{}^{\gamma}\,\left(\delta E_{\nu\gamma}\right) + \frac{1}{2}\,\eta_{\alpha}{}^{\beta\mu\nu}\,V_{\beta}\,\left(\delta q_{\mu}\right)_{;\nu} = 0, \tag{6.31}$$

e, usando (6.4), (D.26), (D.27) e (D.28), vem então:

$$H(t)\, h_\alpha{}^\beta\, h^{\mu\nu}\, \hat{P}^*_{\beta\mu;\nu} - E(t)\, \eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, \sigma_\mu{}^\gamma\, \hat{P}_{\nu\gamma} + \frac{1}{2}\, q(t)\, \eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, \hat{P}_{\mu;\nu} = 0 \Longrightarrow$$

$$\left[m^2\, H(t) + \frac{1}{2}\, q(t)\right]\, \hat{P}^*_\alpha = -3\, E(t)\, \sigma_\alpha{}^\beta\, \hat{P}_\beta. \tag{6.32}$$

A primeira equação de vínculo, (3.49), precisa ter o seu segundo termo reescrito em termos do vetor de rotação; logo, usando-se a equação (6.12), obtemos para este termo:

$$\begin{aligned}
h_\alpha{}^\beta\, \omega_\beta{}^\mu{}_{;\mu} &= -\eta_\beta{}^{\mu\gamma\lambda}\, \left(V_\gamma\, \omega_\lambda\right)_{;\mu}\, h_\alpha{}^\beta \\
&= -\left(\eta_\alpha{}^{\mu\gamma\lambda} - \eta_\beta{}^{\mu\gamma\lambda}\, V_\alpha\, V_\beta\right)\, \left[\left(\sigma_{\gamma\mu} + \frac{1}{3}\, \theta\, h_{\gamma\mu}\right)\, \omega_\lambda + V_\gamma\, \omega_{\lambda;\mu}\right] \\
&= -\eta_\alpha{}^{\mu\gamma\lambda}\, V_\gamma \omega_{\lambda;\mu} \\
&= \eta_\alpha{}^{\gamma\mu\lambda}\, V_\gamma\, \omega_{\lambda;\mu} \\
&= -\eta_\alpha{}^{\gamma\beta\lambda}\, V_\gamma\, \omega_{\beta;\lambda}.
\end{aligned}$$

Portanto, a primeira equação de vínculo perturbada se escreve para a solução de Milne como:

$$\begin{aligned}
h_\alpha{}^\beta\, \left(\delta Z_\beta{}^\mu\right)_{;\mu} &+ \frac{1}{3}\, \theta\, \eta_\alpha{}^{\gamma\beta\lambda}\, V_\gamma\, \left(\delta\omega_\beta\right)_{;\lambda} + \frac{1}{3}\, \theta\, h_\alpha{}^\beta\, \left(\delta q_\beta\right) - \\
&- \frac{1}{3}\, \theta\, \sigma_\alpha{}^\beta\, \left(\delta a_\beta\right) + \frac{2}{9}\, \theta^2\, h_\alpha{}^\beta\, \left(\delta a_\beta\right) = 0,
\end{aligned} \tag{6.33}$$

e, usando-se (6.4) e (D.20), obtemos:

$$\left[m^2 Z(t) + \frac{1}{3}\theta\,\Omega(t) + \frac{1}{3}\theta\,q(t) + \frac{2}{9}\theta^2\,\psi(t)\right]\hat{P}_\alpha = \frac{1}{3}\theta\,\psi(t)\,\sigma_\alpha{}^\beta\,\hat{P}_\beta. \tag{6.34}$$

A segunda equação de vínculo, (3.50), é identicamente nula por uma inspeção direta e por (D.3). A última equação a ser considerada é a terceira equação de vínculo, (3.51). Reescrita em termos do vetor rotação, conforme feito para o caso anterior da solução de Kasner, temos:

$$2\,(\delta H_{\alpha\beta}) + \frac{1}{2}h^{\mu(\alpha}\,h_{\beta)}{}^\nu\,(\delta\omega^\mu)_{;\nu} - h_{\alpha\beta}\,(\delta\omega^\gamma)_{;\gamma} = 0. \tag{6.35}$$

E, após usarmos as equações (6.4), (D.3) e (D.29), temos enfim:

$$2\,H(t) + \frac{1}{2}\Omega(t) = 0, \tag{6.36}$$

o que conclui os cálculos para o sistema dinâmico para este caso.

Capítulo 7

Conclusão

Neste trabalho, aplicou-se o método gauge-invariante de perturbação, já usado anteriormente para FRW [10, 11, 12], ao modelo cosmológico de Kasner. Verificou-se que a existência de um *shear* não nulo no *background* introduziu complicações em diversos níveis deste estudo. A primeira delas está relacionada à obtenção das bases, notadamente no que concerne ao cálculo das quantidades tensoriais associadas aos três tipos de base. Isto se traduz no que poderíamos denominar um "excesso" de condições para que as bases pudessem ser escritas explicitamente — um passo importante para que as equações Quase-Maxwellianas pudessem ser escritas em uma forma razoável. As consequências deste "excesso" no caso tensorial são duas: a limitação da base a uma matriz 2×2 com apenas uma componente livre e — esta mais importante — a impossibilidade de escrever-se uma base tensorial explícita para o caso mais geral do modelo de Kasner, isto é, aquele em que a anisotropia é completa. Verificou-se somente ser possível resolver-se este problema e explicitar-se uma base tensorial no caso mais restrito das soluções com um *plano de isotropia*, caso das soluções específicas de Kasner e Milne. Somente para estes dois casos foi possível obter-se um sistema dinâmico fechado a partir das equações QM, ainda que restrito a apenas três quantidades. Vale notar que resultados análogos para a base tensorial são também válidos no caso das perturbações da solução de Schwarzschild, conforme trabalho de R.Klippert [42]. Os casos escalar e vetorial compartilham das mesmas características acima, mas a existência de um *shear* não nulo complica as equações QM de tal modo a tornar impossível a fatoração da base. Estes fatos, embora óbvios algebricamente, estão ligados ao *shear* de uma maneira que está longe de ser clara fisicamente.

A segunda complicação está ligada à necessidade de trocar as quantidades gauge-dependentes por outras que — embora construídas a partir destas — fossem nulas no *background* e, portanto, verdadeiras perturbações. A interpretação física destas quantidades não é, obviamente tão direta quanto seria a de uma observável — embora sejam elas muito menos complexas do que as quantidades de Bardeen [14]. Estas quantidades "sintéticas" foram, como vimos, construídas a partir das equações QM para o *background* não perturbado, de uma maneira bastante direta. Entretanto, seria possível construir-se outras quantidades gauge-invariantes igualmente válidas, embora não tão simples quanto as que foram usadas aqui.

Uma possível abordagem a ser considerada futuramente seria, portanto, uma análise mais profunda das consequências da aplicação do método gauge-invariante a modelos anisotrópicos, em especial as consequências sobre a construção das bases. Um estudo adicional do sistema dinâmico fechado obtido para a solução de Kasner no caso das perturbações tensoriais também seria importante. Finalmente, como parte de um programa de trabalho mais amplo, poder-se-ia considerar uma generalização deste método a outros tipos de modelos cosmológicos: o de Gödel (como um paradigma de modelos cosmológicos com rotação) e os modelos tipo Bianchi-V e IX seriam alternativas futuras, dado que a análise para o modelo de Schwarzschild [42] já foi executada. Adicionalmente, uma análise mais profunda da quantização do *background* de Kasner deverá ser realizada, já que uma Hamiltoniana para este modelo foi obtida.

Apêndice A

Separação $3+1$ e Formalismo da 3-Superfície

Neste apêndice apresentamos definições e resultados necessários para este trabalho. Como um primeiro passo para a aplicação do método gauge-invariante à teoria das perturbações cosmológicas, vamos considerar a separação do *manifold* Riemanniano 4-dimensional em uma congruência de curvas tipo tempo, com vetor tangente $\vec{V}$ normalizado a:

$$V^{\mu} V^{\nu} g_{\mu\nu} = +1, \tag{A.1}$$

o qual será identificado com o vetor velocidade — escrito para uma classe de observadores co-moventes ($V_{\mu} = \delta^{0}{}_{\mu}$), mais o espaço das 3-superfícies ortogonais a $\vec{V}$. Temos, então, que a métrica do *background*, $g_{\mu\nu}$, e o vetor $\vec{V}$ pertencente ao 4-espaço induzem um projetor $h_{\mu\nu}$, definido no 3-espaço, que separa qualquer objeto geométrico do 4-espaço em quantidades definidas sobre as 3-superfícies ortogonais a $\vec{V}$. O projetor $h_{\mu\nu}$ será definido como de costume [10, 17, 25]:

$$h_{\mu\nu} = g_{\mu\nu} - V_{\mu} V_{\nu}, \tag{A.2}$$

e satisfaz todas as propriedades usuais de um projetor:

$$h_{\mu\nu} = h_{\nu\mu},$$

$$h_{\mu\nu}\, V^{\mu} = 0, \tag{A.3}$$

$$h_{\mu\nu}\, h^{\mu\lambda} = h_{\nu}{}^{\lambda} = \delta_{\nu}{}^{\lambda} - V^{\lambda}\, V_{\nu}.$$

Dado, portanto, qualquer vetor A_{α} (pertencente ao 4-espaço), podemos definir a sua projeção sobre o 3-espaço como:

$$\hat{A}_{\alpha} = h_{\alpha}{}^{\beta}\, A_{\beta}, \tag{A.4}$$

e o mesmo vale para um tensor $B_{\alpha\beta}$:

$$\hat{B}_{\alpha\beta} = h_{\alpha}{}^{\mu}\, h_{\beta}{}^{\nu}\, B_{\mu\nu}. \tag{A.5}$$

De maneira análoga, definimos o operador derivada covariante projetado na 3-superfície, $\hat{\nabla}_{\alpha}$, como:

$$\hat{\nabla}_{\alpha} \equiv h_{\alpha}{}^{\beta}\, (\quad)_{;\beta},$$

o que nos possibilita escrever as relações que se seguem:

$$\hat{\nabla}_{\alpha}\, A_{\beta} = h_{\alpha}{}^{\mu}\, h_{\beta}{}^{\nu}\, (h_{\nu}{}^{\gamma}\, A_{\gamma})_{;\mu}$$

$$\hat{\nabla}_{\alpha}\, B_{\mu\nu} = h_{\alpha}{}^{\beta}\, h_{\mu}{}^{\gamma}\, h_{\nu}{}^{\lambda}\, (h_{\gamma}{}^{\tau}\, h_{\lambda}{}^{\varepsilon}\, B_{\tau\varepsilon})_{;\beta}, \tag{A.6}$$

onde todos os índices devem ser projetados.

A derivada covariante segunda, projetada na 3-superfície, é portanto escrita de maneira análoga como:

$$
\begin{aligned}
\hat{\nabla}_\alpha\, \hat{\nabla}_\beta\, A_\gamma &= h_\alpha{}^\mu\, h_\beta{}^\nu\, h_\gamma{}^\lambda\, \left[h_\nu{}^\varepsilon\, h_\lambda{}^\tau\, (h_\tau{}^\sigma\, A_\sigma)_{;\varepsilon}\right]_{;\mu}, \\
\hat{\nabla}_\alpha\, \hat{\nabla}_\beta\, B_{\mu\nu} &= h_\alpha{}^\gamma\, h_\beta{}^\lambda\, h_\mu{}^\varepsilon\, h_\nu{}^\tau\, \left[h_\lambda{}^\rho\, h_\varepsilon{}^\varphi\, h_\tau{}^\sigma\, \left(h_\varphi{}^\xi\, h_\sigma{}^\eta\, B_{\xi\eta}\right)_{;\rho}\right]_{;\gamma},
\end{aligned}
\tag{A.7}
$$

onde todos os índices são, por sua vez, projetados.

De posse destas relações, é fácil constatar-se que o projetor $h_{\mu\nu}$ funciona como a métrica que descreve o 3-espaço; em outras palavras, a derivada projetada no 3-espaço de $h_{\mu\nu}$ é nula, como provamos abaixo:

$$
\begin{aligned}
\hat{\nabla}_\alpha\, h_{\mu\nu} &= h_\alpha{}^\beta\, h_\mu{}^\gamma\, h_\nu{}^\lambda\, (h_\gamma{}^\varepsilon\, h_\lambda{}^\tau\, h_{\varepsilon\tau})_{;\beta} \\
&= h_\alpha{}^\beta\, h_\mu{}^\gamma\, h_\nu{}^\lambda\, \left[(h_\gamma{}^\varepsilon)_{;\beta}\, h_\lambda{}^\tau\, h_{\varepsilon\tau} + h_\gamma{}^\varepsilon\, (h_\lambda{}^\tau)_{;\beta}\, h_{\varepsilon\tau} + h_\gamma{}^\varepsilon\, h_\lambda{}^\tau\, (h_{\varepsilon\tau})_{;\beta}\right] \\
&= h_\alpha{}^\beta\, h_\mu{}^\gamma\, h_{\nu\varepsilon}\, \left[(\delta_\gamma{}^\varepsilon)_{;\beta} - V^\varepsilon{}_{;\beta}\, V_\gamma - V^\varepsilon\, V_{\gamma;\beta}\right] + \\
&+ h_\alpha{}^\beta\, h_{\mu\tau}\, h_\nu{}^\lambda\, \left[(\delta_\lambda{}^\tau)_{;\beta} - V^\tau{}_{;\beta}\, V_\lambda - V^\tau\, V_{\lambda;\beta}\right] + \\
&+ h_\alpha{}^\beta\, h_\mu{}^\varepsilon\, h_\nu{}^\tau\, [g_{\varepsilon\tau;\beta} - V_{\varepsilon;\beta}\, V_\tau - V_\varepsilon\, V_{\tau;\beta}] \\
&= h_\alpha{}^\beta\, h_\mu{}^\gamma\, h_{\nu\varepsilon}\, \left[(\delta_\gamma{}^\varepsilon)_{,\beta} + \Gamma^\varepsilon_{\lambda\beta}\, \delta_\gamma{}^\lambda - \Gamma^\lambda_{\gamma\beta}\, \delta_\lambda{}^\varepsilon\right] + \\
&+ h_\alpha{}^\beta\, h_{\mu\tau}\, h_\nu{}^\lambda\, \left[(\delta_\lambda{}^\tau)_{,\beta} + \Gamma^\tau_{\varepsilon\beta}\, \delta_\lambda{}^\varepsilon - \Gamma^\varepsilon_{\lambda\beta}\, \delta_\varepsilon{}^\tau\right] \\
&= h_\alpha{}^\beta\, h_\mu{}^\gamma\, h_\nu{}^\varepsilon\, \left(\Gamma^\varepsilon_{\gamma\beta} - \Gamma^\varepsilon_{\gamma\beta}\right) + h_\alpha{}^\beta\, h_{\mu\tau}\, h_\nu{}^\lambda\, \left(\Gamma^\tau_{\lambda\beta} - \Gamma^\tau_{\lambda\beta}\right) \\
&= 0,
\end{aligned}
\tag{A.8}
$$

onde fizemos uso das propriedades (A.3) e da relação $g_{\alpha\beta;\gamma} = 0$, $\forall \alpha,\ \beta,\ \gamma$. Vemos daí que $h_{\mu\nu}$ sobe e desce índices dos objetos da 3-superfície.

O tensor de Riemann projetado na 3-superfície, $\hat{R}_{\alpha\beta\mu\nu}$, tem uma definição análoga à de sua contraparte do 4-espaço [17]:

$$\hat{\nabla}_\alpha \hat{\nabla}_\beta \hat{A}_\mu - \hat{\nabla}_\beta \hat{\nabla}_\alpha \hat{A}_\mu = \hat{R}^\nu{}_{\mu\alpha\beta} \hat{A}_\nu,$$
$$\hat{\nabla}_\alpha \hat{\nabla}_\beta \hat{B}_{\mu\nu} - \hat{\nabla}_\beta \hat{\nabla}_\alpha \hat{B}_{\mu\nu} = \hat{R}^\lambda{}_{\mu\alpha\beta} \hat{B}_{\lambda\nu} + \hat{R}^\lambda{}_{\nu\alpha\beta} \hat{B}_{\mu\lambda}. \tag{A.9}$$

E, fazendo uso da primeira relação (A.7) na primeira das definições (A.9) acima, obtemos o seguinte resultado importante:

$$\begin{aligned}
\hat{R}^\nu{}_{\mu\alpha\beta} \hat{A}_\nu &= \hat{\nabla}_{[\alpha} \hat{\nabla}_{\beta]} \hat{A}_\mu \\
&= h_{[\alpha}{}^\gamma h_{\beta]}{}^\lambda h_\mu{}^\nu \left[h_\lambda{}^\sigma h_\nu{}^\varepsilon \left(h_\varepsilon{}^\tau \hat{A}_\tau\right)_{;\sigma}\right]_{;\gamma} \\
&= h_{[\alpha}{}^\gamma h_{\beta]}{}^\lambda h_\mu{}^\nu \Big[(h_\lambda{}^\sigma)_{;\gamma} h_\nu{}^\varepsilon \left(h_\varepsilon{}^\tau \hat{A}_\tau\right)_{;\sigma} \\
&\quad + h_\lambda{}^\sigma (h_\nu{}^\varepsilon)_{;\gamma} \left(h_\varepsilon{}^\tau \hat{A}_\tau\right)_{;\sigma} + h_\lambda{}^\sigma h_\nu{}^\varepsilon \left(h_\varepsilon{}^\tau \hat{A}_\tau\right)_{;\sigma;\gamma}\Big] \\
&= h_{[\alpha}{}^\gamma h_{\beta]}{}^\lambda h_\mu{}^\varepsilon \left(h_\varepsilon{}^\tau \hat{A}_\tau\right)_{;\lambda;\gamma} \\
&= h_{[\alpha}{}^\gamma h_{\beta]}{}^\lambda h_\mu{}^\varepsilon \left\{\left[\left(h_\varepsilon{}^\tau \hat{A}_\tau\right)_{;\lambda}\right]_{,\gamma} - \Gamma^\nu_{\varepsilon\gamma} \left(h_\nu{}^\tau \hat{A}_\tau\right)_{;\lambda} - \Gamma^\nu_{\lambda\gamma} \left(h_\varepsilon{}^\tau \hat{A}_\tau\right)_{;\nu}\right\} \\
&= h_{[\alpha}{}^\gamma h_{\beta]}{}^\lambda h_\mu{}^\varepsilon \Big\{\left[\left(h_\varepsilon{}^\tau \hat{A}_\tau\right)_{,\lambda} - \Gamma^\sigma_{\varepsilon\lambda} \left(h_\sigma{}^\tau \hat{A}_\tau\right)\right]_{,\gamma} - \Gamma^\nu_{\varepsilon\gamma} (h_\nu{}^\tau)_{;\lambda} \hat{A}_\tau - \\
&\quad - \Gamma^\nu_{\varepsilon\gamma} h_\nu{}^\tau \hat{A}_{\tau;\lambda} - \Gamma^\nu_{\lambda\gamma} (h_\varepsilon{}^\tau)_{;\nu} \hat{A}_\tau - \Gamma^\nu_{\lambda\gamma} h_\varepsilon{}^\tau \hat{A}_{\tau;\nu}\Big\} \\
&= h_{[\alpha}{}^\gamma h_{\beta]}{}^\lambda h_\mu{}^\varepsilon \Big\{h_\varepsilon{}^\tau \hat{A}_{\tau,\lambda,\gamma} - (\Gamma^\sigma_{\varepsilon\lambda})_{,\gamma} h_\sigma{}^\tau \hat{A}_\tau - \Gamma^\sigma_{\varepsilon\lambda} h_\sigma{}^\tau \hat{A}_{\tau,\gamma} -
\end{aligned}$$

$$-\Gamma^{\nu}_{\varepsilon\gamma}\, h_\nu{}^\tau\, \hat{A}_{\tau;\lambda} - \Gamma^{\nu}_{\lambda\gamma}\, h_\varepsilon{}^\tau\, \hat{A}_{\tau,\nu}\Big\}$$

$$= \; h_\mu{}^\tau\, \hat{A}_{\tau,[\beta,\alpha]} - \Gamma^{\nu}_{[\beta\alpha]}\, h_\mu{}^\tau\, \hat{A}_{\tau,\nu}$$

$$= \; 0, \qquad \text{(A.10)}$$

donde, dado que $\hat{A}_\nu$ é um 3-vetor arbitrário, temos que $\hat{R}_{\alpha\beta\mu\nu}$ tem todas as componentes nulas. Daí vem imediatamente que $\hat{R}_{\alpha\beta}$ e $\hat{R}$, o 3-tensor de Ricci e o 3-escalar de curvatura respectivamente, são também nulos.

Neste ponto, podemos calcular certas quantidades que nos serão úteis ao longo deste trabalho:

(1) Divergência de um 3-tensor $\hat{B}_{\mu\nu}$:

$$\begin{aligned}
\hat{\nabla}^\mu\, \hat{B}_{\mu\nu} &= h^{\mu\alpha}\, \hat{\nabla}_\alpha\, \hat{B}_{\mu\nu} \\
&= h^{\mu\alpha}\, h_\alpha{}^\beta\, h_\mu{}^\gamma\, h_\nu{}^\lambda\, \left(h_\gamma{}^\varepsilon\, h_\lambda{}^\tau\, \hat{B}_{\varepsilon\tau}\right)_{;\beta} \\
&= h^{\gamma\beta}\, h_\nu{}^\lambda\, \left(h_\gamma{}^\varepsilon\, h_\lambda{}^\tau\, \hat{B}_{\varepsilon\tau}\right)_{;\beta} \\
&= h^{\beta\gamma}\, h_\nu{}^\lambda\, \left[(h_\gamma{}^\varepsilon)_{;\beta}\, h_\lambda{}^\tau\, \hat{B}_{\varepsilon\tau} + h_\gamma{}^\varepsilon\, (h_\lambda{}^\tau)_{;\beta}\, \hat{B}_{\varepsilon\tau} + h_\gamma{}^\varepsilon\, h_\lambda{}^\tau\, \hat{B}_{\varepsilon\tau;\beta}\right] \\
&= h^{\beta\varepsilon}\, h_\nu{}^\tau\, \left[\hat{B}_{\varepsilon\tau,\beta} - \Gamma^{\alpha}_{\varepsilon\beta}\, \hat{B}_{\alpha\tau} - \Gamma^{\alpha}_{\tau\beta}\, \hat{B}_{\varepsilon\alpha}\right] \\
&= h^{\beta\varepsilon}\, h_\nu{}^\tau\, \frac{\partial \hat{B}_{\varepsilon\tau}}{\partial x^\beta}. \qquad \text{(A.11)}
\end{aligned}$$

(2) Laplaciano de um 3-tensor $\hat{B}_{\mu\nu}$:

$$
\begin{aligned}
\hat{\nabla}^{2}\hat{B}_{\mu\nu} &= \hat{\nabla}^{\alpha}\hat{\nabla}_{\alpha}\hat{B}_{\mu\nu} \\
&= h^{\alpha\beta}\hat{\nabla}_{\beta}\hat{\nabla}_{\alpha}\hat{B}_{\mu\nu} \\
&= h^{\alpha\beta}h_{\beta}{}^{\gamma}h_{\alpha}{}^{\lambda}h_{\mu}{}^{\varepsilon}h_{\nu}{}^{\tau}\left[h_{\lambda}{}^{\rho}h_{\varepsilon}{}^{\varphi}h_{\tau}{}^{\delta}\left(h_{\varphi}{}^{\phi}h_{\delta}{}^{\xi}\hat{B}_{\phi\xi}\right)_{;\rho}\right]_{;\gamma} \\
&= h^{\gamma\lambda}h_{\mu}{}^{\varepsilon}h_{\nu}{}^{\tau}\left[h_{\lambda}{}^{\rho}h_{\varepsilon}{}^{\varphi}h_{\tau}{}^{\delta}\left(h_{\varphi}{}^{\alpha}h_{\delta}{}^{\beta}\hat{B}_{\alpha\beta}\right)_{;\rho}\right]_{;\gamma} \\
&= -h^{\gamma\lambda}\left(\sigma_{\lambda\gamma}+\frac{\theta}{3}h_{\lambda\gamma}\right)h_{\mu}{}^{\varphi}h_{\nu}{}^{\delta}\left(h_{\varphi}{}^{\alpha}h_{\delta}{}^{\beta}\hat{B}_{\alpha\beta}\right)^{\bullet}- \\
&\quad -h^{\gamma\rho}h_{\mu}{}^{\varepsilon}h_{\nu}{}^{\delta}\left(\sigma_{\varepsilon\gamma}+\frac{\theta}{3}h_{\varepsilon\gamma}\right)V^{\varphi}\left(h_{\varphi}{}^{\alpha}h_{\delta}{}^{\beta}\hat{B}_{\alpha\beta}\right)_{;\rho}- \\
&\quad -h^{\gamma\rho}h_{\mu}{}^{\varphi}h_{\nu}{}^{\tau}\left(\sigma_{\tau\gamma}+\frac{\theta}{3}h_{\tau\gamma}\right)V^{\delta}\left(h_{\varphi}{}^{\alpha}h_{\delta}{}^{\beta}\hat{B}_{\alpha\beta}\right)_{;\rho}+ \\
&\quad +h^{\gamma\rho}h_{\mu}{}^{\varphi}h_{\nu}{}^{\delta}\left(h_{\varphi}{}^{\alpha}h_{\delta}{}^{\beta}\hat{B}_{\alpha\beta}\right)_{;\rho;\gamma} \\
&= -\theta\,h_{\mu}{}^{\varphi}h_{\nu}{}^{\delta}\left(h_{\varphi}{}^{\alpha}h_{\delta}{}^{\beta}\hat{B}_{\alpha\beta}\right)^{\bullet}-h_{\nu}{}^{\delta}\left(\sigma_{\mu}{}^{\rho}+\frac{\theta}{3}h_{\mu}{}^{\rho}\right)V^{\varphi}\left(h_{\varphi}{}^{\alpha}h_{\delta}{}^{\beta}\hat{B}_{\alpha\beta}\right)_{;\rho}- \\
&\quad -h_{\mu}{}^{\varphi}\left(\sigma_{\nu}{}^{\rho}+\frac{\theta}{3}h_{\nu}{}^{\rho}\right)V^{\delta}\left(h_{\varphi}{}^{\alpha}h_{\delta}{}^{\beta}\hat{B}_{\alpha\beta}\right)_{;\rho}+h^{\gamma\rho}h_{\mu}{}^{\varphi}h_{\nu}{}^{\delta}\left(h_{\varphi}{}^{\alpha}h_{\delta}{}^{\beta}\hat{B}_{\alpha\beta}\right)_{;\rho;\gamma} \\
&= -\theta\,h_{\mu}{}^{\varphi}h_{\nu}{}^{\delta}\left[\left(h_{\varphi}{}^{\alpha}\right)^{\bullet}h_{\delta}{}^{\beta}\hat{B}_{\alpha\beta}+h_{\varphi}{}^{\alpha}\left(h_{\delta}{}^{\beta}\right)^{\bullet}\hat{B}_{\alpha\beta}+h_{\varphi}{}^{\alpha}h_{\delta}{}^{\beta}\left(\hat{B}_{\alpha\beta}\right)^{\bullet}\right]- \\
&\quad -h_{\nu}{}^{\delta}\left(\sigma_{\mu}{}^{\rho}+\frac{\theta}{3}h_{\mu}{}^{\rho}\right)V^{\varphi}\left[-\left(V^{\alpha}{}_{;\rho}V_{\varphi}+V^{\alpha}V_{\varphi;\rho}\right)h_{\delta}{}^{\beta}\hat{B}_{\alpha\beta}-h_{\varphi}{}^{\alpha}\left(h_{\delta}{}^{\beta}\hat{B}_{\alpha\beta}\right)_{;\rho}\right]- \\
&\quad -h_{\mu}{}^{\varphi}\left(\sigma_{\nu}{}^{\rho}+\frac{\theta}{3}h_{\nu}{}^{\rho}\right)V^{\delta}\left[\left(h_{\varphi}{}^{\alpha}\hat{B}_{\alpha\beta}\right)_{;\rho}h_{\delta}{}^{\beta}-\left(V^{\beta}{}_{;\rho}V_{\delta}+V^{\beta}V_{\delta;\rho}\right)h_{\varphi}{}^{\alpha}\hat{B}_{\alpha\beta}\right]+ \\
&\quad +h^{\gamma\rho}h_{\mu}{}^{\varphi}h_{\nu}{}^{\delta}\left\{\left[\left(h_{\varphi}{}^{\alpha}h_{\delta}{}^{\beta}\hat{B}_{\alpha\beta}\right)_{;\rho}\right]_{,\gamma}-\Gamma^{\varepsilon}_{\varphi\gamma}\left(h_{\varepsilon}{}^{\alpha}h_{\delta}{}^{\beta}\hat{B}_{\alpha\beta}\right)_{;\rho}\right\}- \\
&\quad -\Gamma^{\varepsilon}_{\delta\gamma}\left(h_{\varphi}{}^{\alpha}h_{\delta}{}^{\beta}\hat{B}_{\alpha\beta}\right)_{;\rho}-\Gamma^{\varepsilon}_{\rho\gamma}\left(h_{\varphi}{}^{\alpha}h_{\delta}{}^{\beta}\hat{B}_{\alpha\beta}\right)_{;\varepsilon}\right\}
\end{aligned}
$$

$$
\begin{aligned}
= \; & -\theta\, h_\mu{}^\alpha\, h_\nu{}^\beta \left(\hat{B}_{\alpha\beta}\right)^\bullet + \left(\sigma_\alpha{}^\beta + \frac{\theta}{3}\, h_\alpha{}^\beta\right) \left(\sigma_{(\mu}{}^\alpha + \frac{\theta}{3}\, h_{(\mu}{}^\alpha\right) \hat{B}_{\nu)\alpha} + \\
& + h^{\gamma\rho}\, h_\mu{}^\varphi\, h_\nu{}^\delta \left\{ \left[\left(h_\varphi{}^\alpha\, h_\delta{}^\beta\, \hat{B}_{\alpha\beta}\right)_{,\rho} - \Gamma^\tau_{\varphi\rho} \left(h_\tau{}^\alpha\, h_\delta{}^\beta\, \hat{B}_{\alpha\beta}\right) - \Gamma^\tau_{\delta\rho} \left(h_\varphi{}^\alpha\, h_\tau{}^\beta\, \hat{B}_{\alpha\beta}\right)\right]_{,\gamma} + \right. \\
& + \Gamma^\varepsilon_{\varphi\gamma} \left[(V^\alpha{}_{;\rho}\, V_\varepsilon + V^\alpha\, V_{\varepsilon;\rho})\, h_\delta{}^\beta\, \hat{B}_{\alpha\beta} - h_\varepsilon{}^\alpha \left(h_\delta{}^\beta\, \hat{B}_{\alpha\beta}\right)_{;\rho}\right] + \\
& + \Gamma^\varepsilon_{\delta\gamma} \left[\left(V^\beta{}_{;\rho}\, V_\varepsilon + V^\beta\, V_{\varepsilon;\rho}\right) h_\varphi{}^\alpha\, \hat{B}_{\alpha\beta} - h_\varepsilon{}^\beta \left(h_\varphi{}^\alpha\, \hat{B}_{\alpha\beta}\right)_{;\rho}\right] - \\
& \left. - \Gamma^0_{\rho\gamma} \left(h_\varphi{}^\alpha\, h_\delta{}^\beta\, \hat{B}_{\alpha\beta}\right)^\bullet \right\} \\
= \; & -\theta\, h_\mu{}^\alpha\, h_\nu{}^\beta \left(\hat{B}_{\alpha\beta}\right)^\bullet + \left(\sigma_\alpha{}^\beta + \frac{\theta}{3}\, h_\alpha{}^\beta\right) \left(\sigma_{(\mu}{}^\alpha + \frac{\theta}{3}\, h_{(\mu}{}^\alpha\right) \hat{B}_{\nu)\alpha} + \\
& + h_\mu{}^\alpha\, h_\nu{}^\beta\, h^{\gamma\rho}\, \hat{B}_{\alpha\beta,\rho,\gamma} - \left(\sigma_\alpha{}^\beta + \frac{\theta}{3}\, h_\alpha{}^\beta\right) \left(\sigma_{(\mu}{}^\alpha + \frac{\theta}{3}\, h_{(\mu}{}^\alpha\right) \hat{B}_{\nu)\alpha} + \\
& + \theta\, h_\mu{}^\alpha\, h_\nu{}^\beta \left(\hat{B}_{\alpha\beta}\right)^\bullet \\
= \; & h^{\gamma\lambda}\, h_\mu{}^\alpha\, h_\nu{}^\beta\, \frac{\partial^2 \hat{B}_{\alpha\beta}}{\partial x^\gamma \partial x^\lambda}. \qquad \text{(A.12)}
\end{aligned}
$$

Apêndice B

Demonstração Auxiliar Referente à Operação Dual

Partindo da definição do dual aplicado ao tensor de Weyl,

$$W^{*}_{\alpha\beta\mu\nu} = \frac{1}{2}\,\eta_{\alpha\beta}{}^{\gamma\lambda}\,W_{\gamma\lambda\mu\nu}, \qquad \text{(B.1)}$$

pretendemos provar que a relação:

$$W^{*}_{\alpha\beta\mu\nu} - W_{\alpha\beta}{}^{*}_{\mu\nu} = 0, \qquad \text{(B.2)}$$

é válida, ou seja, a operação dual é aplicada indiferentemente a qualquer um dos dois pares de índices do tensor de Weyl, conforme enunciado no Capítulo 1 deste trabalho. Começaremos calculando duas quantidades intermediárias:

$$W^{**}_{\alpha\beta\mu\nu} \quad W^{*}_{\alpha\beta}{}^{*}_{\mu\nu},$$

isto é, o duplo dual aplicado sucessivamente ao mesmo par de índices e o aplicado ao outro par, respectivamente. A primeira operação é facilmente obtida, usando-se as relações (B.1) e (1.11):

$$
\begin{aligned}
W^{**}_{\alpha\beta\mu\nu} &= \frac{1}{4}\,\eta_{\alpha\beta}{}^{\tau\varepsilon}\,\eta_{\tau\varepsilon}{}^{\gamma\lambda}\,W_{\gamma\lambda\mu\nu} \\
&= \frac{1}{4}\,\eta_{\alpha\beta\tau\varepsilon}\,\eta^{\gamma\lambda\tau\varepsilon}\,W_{\gamma\lambda\mu\nu} \\
&= \frac{1}{4}\left(-2\,\delta^{\gamma\lambda}_{\alpha\beta}\right)\,W_{\gamma\lambda\mu\nu} \\
&= -\frac{1}{2}\left(\delta^{\gamma}_{\alpha}\,\delta^{\lambda}_{\beta}-\delta^{\gamma}_{\beta}\,\delta^{\lambda}_{\alpha}\right)\,W_{\gamma\lambda\mu\nu} \\
&= -\frac{1}{2}\,\left(W_{\alpha\beta\mu\nu}-W_{\beta\alpha\mu\nu}\right) \\
&= -\frac{1}{2}\,\left(2\,W_{\alpha\beta\mu\nu}\right) \\
&= -W_{\alpha\beta\mu\nu}. \qquad \text{(B.3)}
\end{aligned}
$$

A segunda quantidade, $W^{*}_{\alpha\beta}{}^{*}_{\mu\nu}$, é dada por, usando-se novamente (B.1) e (1.11),

$$
\begin{aligned}
W^{*}_{\alpha\beta}{}^{*}_{\mu\nu} &= \frac{1}{4}\,\eta_{\alpha\beta}{}^{\gamma\varepsilon}\,\eta_{\mu\nu}{}^{\lambda\tau}\,W_{\gamma\varepsilon\lambda\tau} \\
&= \frac{1}{4}\,\eta_{\alpha\beta\gamma\varepsilon}\,\eta^{\varphi\chi\lambda\tau}\,g_{\mu\varphi}\,g_{\nu\chi}\,W^{\gamma\varepsilon}{}_{\lambda\tau} \\
&= -\frac{1}{4}\,\delta^{\varphi\chi\lambda\tau}_{\alpha\beta\gamma\varepsilon}\,g_{\mu\varphi}\,g_{\nu\chi}\,W^{\gamma\varepsilon}{}_{\lambda\tau}
\end{aligned}
$$

$$= -\frac{1}{4} \begin{vmatrix} \delta^{\varphi}_{\alpha} & \delta^{\varphi}_{\beta} & \delta^{\varphi}_{\gamma} & \delta^{\varphi}_{\varepsilon} \\ \delta^{\chi}_{\alpha} & \delta^{\chi}_{\beta} & \delta^{\chi}_{\gamma} & \delta^{\chi}_{\varepsilon} \\ \delta^{\lambda}_{\alpha} & \delta^{\lambda}_{\beta} & \delta^{\lambda}_{\gamma} & \delta^{\lambda}_{\varepsilon} \\ \delta^{\tau}_{\alpha} & \delta^{\tau}_{\beta} & \delta^{\tau}_{\gamma} & \delta^{\tau}_{\varepsilon} \end{vmatrix} g_{\mu\varphi}\, g_{\nu\chi}\, W^{\gamma\varepsilon}{}_{\lambda\tau}$$

$$\begin{aligned} = & -\frac{1}{4} \left[\delta^{\varphi}_{\alpha} \left(\delta^{\chi}_{\beta}\, \delta^{\lambda}_{[\gamma}\, \delta^{\tau}_{\varepsilon]} + \delta^{\chi}_{\gamma}\, \delta^{\lambda}_{[\varepsilon}\, \delta^{\tau}_{\beta]} + \delta^{\chi}_{\varepsilon}\, \delta^{\lambda}_{[\beta}\, \delta^{\tau}_{\gamma]}\right) - \right. \\ & -\delta^{\varphi}_{\beta} \left(\delta^{\chi}_{\alpha}\, \delta^{\lambda}_{[\gamma}\, \delta^{\tau}_{\varepsilon]} + \delta^{\chi}_{\gamma}\, \delta^{\lambda}_{[\varepsilon}\, \delta^{\tau}_{\alpha]} + \delta^{\chi}_{\varepsilon}\, \delta^{\lambda}_{[\alpha}\, \delta^{\tau}_{\gamma]}\right) + \\ & +\delta^{\varphi}_{\gamma} \left(\delta^{\chi}_{\alpha}\, \delta^{\lambda}_{[\beta}\, \delta^{\tau}_{\varepsilon]} + \delta^{\chi}_{\beta}\, \delta^{\lambda}_{[\varepsilon}\, \delta^{\tau}_{\alpha]} + \delta^{\chi}_{\varepsilon}\, \delta^{\lambda}_{[\alpha}\, \delta^{\tau}_{\beta]}\right) - \\ & \left. -\delta^{\varphi}_{\varepsilon} \left(\delta^{\chi}_{\alpha}\, \delta^{\lambda}_{[\beta}\, \delta^{\tau}_{\gamma]} + \delta^{\chi}_{\beta}\, \delta^{\lambda}_{[\gamma}\, \delta^{\tau}_{\alpha]} + \delta^{\chi}_{\gamma}\, \delta^{\lambda}_{[\alpha}\, \delta^{\tau}_{\beta]}\right)\right] g_{\mu\varphi}\, g_{\nu\chi}\, W^{\gamma\varepsilon}{}_{\lambda\tau} \end{aligned}$$

$$\begin{aligned} = & -\frac{1}{4} \left\{-4\, g_{\alpha\mu}\, W^{\gamma}{}_{\nu\gamma\beta} + 4\, g_{\beta\mu}\, W^{\gamma}{}_{\nu\gamma\alpha} + 4\, g_{\alpha\nu}\, W^{\gamma}{}_{\mu\gamma\beta} - 4\, g_{\beta\nu}\, W^{\gamma}{}_{\mu\gamma\alpha} + \right. \\ & \left. +2\, g_{\alpha\mu}\, g_{\beta\nu}\, W^{\gamma\lambda}{}_{\gamma\lambda} - 2\, g_{\alpha\nu}\, g_{\beta\mu}\, W^{\gamma\lambda}{}_{\gamma\lambda} - 4\, W_{\alpha\beta\mu\nu}\right\} \end{aligned}$$

$$= -W_{\alpha\beta\mu\nu} + g_{\alpha[\mu}\, W^{\gamma}{}_{\nu]\gamma\beta} - g_{\beta[\mu}\, W^{\gamma}{}_{\nu]\gamma\alpha} - \frac{1}{2}\, g_{\alpha[\mu}\, g_{\nu]\beta}\, W^{\gamma\lambda}{}_{\gamma\lambda}. \tag{B.4}$$

Entretanto, de (1.7), vem:

$$\begin{aligned} W^{\gamma}{}_{\alpha\gamma\beta} &= R^{\gamma}{}_{\alpha\gamma\beta} - \frac{1}{2} \left(R^{\gamma}{}_{[\alpha}\, g_{\beta]\gamma} + R_{\gamma[\beta}\, g_{\alpha]}{}^{\gamma}\right) + \frac{1}{6}\, g^{\gamma}{}_{[\alpha}\, g_{\beta]\gamma} \\ &= R_{\alpha\beta}, \end{aligned} \tag{B.5}$$

$$W^{\gamma\lambda}{}_{\gamma\lambda} = R^{\gamma}{}_{\gamma} = R, \tag{B.6}$$

e, substituindo (B.5) e (B.6) em (B.4), escrevemos:

$$W^{*}_{\alpha\beta}{}^{*}_{\mu\nu} = -W_{\alpha\beta\mu\nu} + g_{\alpha[\mu}\, R_{\nu]\beta} - g_{\beta[\mu}\, R_{\nu]\alpha} - \frac{1}{2}\, R\, g_{\alpha[\mu}\, g_{\nu]\beta},$$

onde os três últimos termos acima são nulos, já que o *background* cujas perturbações estamos estudando é o do modelo de Kasner, onde vale:

$$R_{\alpha\beta} = 0, \quad \forall \quad \alpha, \beta \implies R = 0.$$

Neste caso, escrevemos afinal:

$$W^{*}_{\alpha\beta}{}^{*}_{\mu\nu} = -W_{\alpha\beta\mu\nu}. \tag{B.7}$$

Para finalizarmos a demonstração de (B.2), vamos aplicar a operação dual sobre um dos pares de índices de $\left(W^{*}_{\alpha\beta\mu\nu} - W_{\alpha\beta}{}^{*}_{\mu\nu}\right)$. Calculando sobre o primeiro par, vem:

$$\begin{aligned}\left(W^{**}_{\alpha\beta\mu\nu} - W^{*}_{\alpha\beta}{}^{*}_{\mu\nu}\right) &= -W_{\alpha\beta\mu\nu} - (-W_{\alpha\beta\mu\nu}) \\ &= 0,\end{aligned}$$

e, repetindo a operação para o segundo par de índices,

$$\begin{aligned}\left(W^{*}_{\alpha\beta}{}^{*}_{\mu\nu} - W_{\alpha\beta}{}^{**}_{\mu\nu}\right) &= -W_{\alpha\beta\mu\nu} - (-W_{\alpha\beta\mu\nu}) \\ &= 0,\end{aligned}$$

onde usamos os resultados (B.3) e (B.7) em ambos os cálculos acima. Logo, podemos escrever que o dual da relação (B.2) é nulo, donde a própria relação também o é:

$$\left(W^{*}_{\alpha\beta\mu\nu} - W_{\alpha\beta}{}^{*}_{\mu\nu}\right)^{*} = 0 \Longrightarrow \left(W^{*}_{\alpha\beta\mu\nu} - W_{\alpha\beta}{}^{*}_{\mu\nu}\right) = 0,$$

e fica, finalmente, demosntrada a relação (B.2).

Apêndice C

Cálculos Auxiliares Referentes à Base Tensorial

Apresentaremos aqui os cálculos associados aos resultados (2.96)-(2.103) e (2.107)-(2.120), referentes à base tensorial e seu dual para os casos especiais das soluções de Kasner e Milne, respectivamente. A prova das relações (2.96) e (2.107) fica óbvia se levarmos em conta os resultados (1.45), (1.47), (2.86) e (2.94). As relações (2.96) e (2.114) são também facilmente obtidas. Calculando componente a componente vem:

$$\begin{aligned} \alpha = \beta = 1 \Longrightarrow \left(\sigma^{\alpha}{}_{\mu}\hat{U}^{*\mu}{}_{\beta} - \sigma^{\mu}{}_{\beta}\hat{U}^{*\alpha}{}_{\mu}\right) &= \left(\sigma^{1}{}_{1}\hat{U}^{*1}{}_{1}\right) \\ &= 0, \end{aligned} \tag{C.1}$$

e o mesmo vale para $\alpha = \beta = 2$ e $\alpha = \beta = 3$. Usando novamente as quatro relações citadas acima, obtemos também:

$$\alpha = 1,\ \ \beta = 2 \Longrightarrow \left(\sigma^{\alpha}{}_{\mu}\,\hat{U}^{*\mu}{}_{\beta} - \sigma^{\mu}{}_{\beta}\,\hat{U}^{*\alpha}{}_{\mu}\right) \;=\; \sigma^{1}{}_{1}\,\hat{U}^{*1}{}_{2} - \sigma^{2}{}_{2}\,\hat{U}^{*2}{}_{2}$$

$$= \left(\sigma^{1}{}_{1} - \sigma^{2}{}_{2}\right)\, i\,\alpha = 0,$$

$$\alpha = 1,\ \ \beta = 3 \Longrightarrow \left(\sigma^{\alpha}{}_{\mu}\,\hat{U}^{*\mu}{}_{\beta}\right) \;=\; \sigma^{1}{}_{1}\,\hat{U}^{*1}{}_{3} - \sigma^{3}{}_{3}\,\hat{U}^{*3}{}_{1}$$

$$= 0,$$

$$\alpha = 2,\ \ \beta = 3 \Longrightarrow \left(\sigma^{\alpha}{}_{\mu}\,\hat{U}^{*\mu}{}_{\beta} - \sigma^{\mu}{}_{\beta}\,\hat{U}^{*\alpha}{}_{\mu}\right) \;=\; \sigma^{2}{}_{2}\,\hat{U}^{*2}{}_{3} - \sigma^{3}{}_{3}\,\hat{U}^{*3}{}_{2}$$

$$= 0.$$

As relações (2.97) e (2.115) são calculadas componente a componente de maneira análoga à exibida em (2.89), usando-se (1.45), (1.47), (2.86) e (2.94). Os resultados (2.99) e (2.116) são obtidos escrevendo-se:

$$\left[\eta^{\lambda\gamma\alpha\beta}\, V_{\gamma}\, \sigma^{\mu}{}_{\lambda}\, h_{\varepsilon\nu}\, \hat{U}^{*\varepsilon}{}_{\alpha;\beta} + \eta^{\lambda\gamma\alpha\beta}\, V_{\gamma}\, \sigma_{\nu\lambda}\, h_{\varepsilon}{}^{\mu}\, \hat{U}^{*\varepsilon}{}_{\alpha;\beta}\right] =$$

$$= \eta^{\lambda\gamma\alpha\beta}\, V_{\gamma}\, \left(\sigma^{\mu}{}_{\lambda}\, h_{\varepsilon\nu} + \sigma_{\nu\lambda}\, h_{\varepsilon}{}^{\mu}\right) \left[\hat{U}^{*\varepsilon}{}_{\alpha;\beta} + \Gamma^{\varepsilon}{}_{\tau\beta}\, \hat{U}^{*\tau}{}_{\alpha} - \Gamma^{\tau}_{\alpha\beta}\, \hat{U}^{*\varepsilon}{}_{\tau}\right]. \tag{C.2}$$

O primeiro termo de conexão acima se anula por (1.3), já que todos os seus índices são espaciais. O segundo termo é simétrico em (α,β) e, multiplicado por $\eta^{\lambda\gamma\alpha\beta}$, também se anula. Com isto, vem, usando (2.95):

$$
\begin{aligned}
\hat{U}^{*\mu}{}_{\nu,j} &= -g_{11}\, k_3 \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \hat{U}^{\mu}{}_{\nu,j} \\
&= i\, g_{11}\, k_3\, k_j \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \hat{U}^{\mu}{}_{\nu} \\
&= -i\, k_j\, \hat{U}^{*\mu}{}_{\nu}, \qquad \text{(C.3)}
\end{aligned}
$$

e escrevemos então:

$$
\left[\eta^{\lambda\gamma\alpha\beta}\, V_\gamma\, \sigma^{\mu}{}_{\lambda}\, h_{\varepsilon\nu}\, \hat{U}^{*\varepsilon}{}_{\alpha;\beta} + \eta^{\lambda\gamma\alpha\beta}\, V_\gamma\, \sigma_{\nu\lambda}\, h_{\varepsilon}{}^{\mu}\, \hat{U}^{*\varepsilon}{}_{\alpha;\beta}\right] =
$$

$$
= \quad -i\, \eta^{\lambda\gamma\alpha\beta}\, V_\gamma\, k_\beta\, \left(\sigma^{\mu}{}_{\lambda}\, h_{\varepsilon\nu} + \sigma_{\nu\lambda}\, h_{\varepsilon}{}^{\mu}\right)\, \hat{U}^{*\varepsilon}{}_{\alpha}.
$$

Usando, agora, a expressão (2.76) para o dual, obtemos:

$$
\left[\eta^{\lambda\gamma\alpha\beta}\, V_\gamma\, h_{\varepsilon\nu}\, \hat{U}^{*\varepsilon}{}_{\alpha;\beta} + \eta^{\lambda\gamma\alpha\beta}\, V_\gamma\, \sigma_{\nu\lambda}\, h_{\varepsilon}{}^{\mu}\, \hat{U}^{*\varepsilon}{}_{\alpha;\beta}\right]
$$

$$
\begin{aligned}
= \quad & -\frac{1}{2}\, \Big[\eta^{\lambda\gamma\alpha\beta}\, \eta_{\chi}{}^{\omega\tau\varepsilon}\, V_\gamma\, V_\omega\, k_\beta\, k_\tau\, \left(\sigma^{\mu}{}_{\lambda}\, h_{\varepsilon\nu} + \sigma_{\nu\lambda}\, h_{\varepsilon}{}^{\mu}\right)\, \hat{U}^{\chi}{}_{\alpha} + \\
& + \eta^{\lambda\gamma\alpha\beta}\, \eta^{\chi\omega\tau}{}_{\alpha}\, V_\gamma\, V_\omega\, k_\beta\, k_\tau\, \left(\sigma^{\mu}{}_{\lambda}\, h_{\varepsilon\nu} + \sigma_{\nu\lambda}\, h_{\varepsilon}{}^{\mu}\right)\, \hat{U}^{\varepsilon}{}_{\chi}\Big]
\end{aligned}
$$

$$
= \quad -\frac{1}{2}\, \Big[\eta^{\lambda\gamma\alpha\beta}\, \eta_{\chi}{}^{\omega\tau\nu}\, V_\gamma\, V_\omega\, k_\beta\, k_\tau\, \sigma^{\mu}{}_{\lambda}\, \hat{U}^{\chi}{}_{\alpha} +
$$

$$+\eta^{\lambda\gamma\alpha\beta}\,\eta_{\chi}{}^{\omega\tau\mu}\,V_{\gamma}\,V_{\omega}\,k_{\beta}\,k_{\tau}\,\sigma_{\nu\lambda}\,\hat{U}^{\chi}{}_{\alpha}+$$

$$+\eta^{\lambda\gamma\alpha\beta}\eta_{\chi}^{\omega\tau}{}_{\alpha}\,V_{\gamma}\,V_{\omega}\,k_{\beta}\,k_{\tau}\,\sigma^{\mu}{}_{\lambda}\,\hat{U}^{\chi}{}_{\nu}+$$

$$+\eta^{\lambda\gamma\alpha\beta}\,\eta^{\chi\omega\tau}{}_{\alpha}\,V_{\gamma}\,V_{\omega}\,k_{\beta}\,k_{\tau}\,\sigma_{\nu\lambda}\,\hat{U}^{\mu}{}_{\chi}\Big]$$

$$\equiv\ -\frac{1}{2}\sum_{j=1}^{4}T_{j}.$$

Calculando cada um dos termos acima isoladamente, vem:

$$
\begin{aligned}
T_1 &\equiv \eta^{\lambda\gamma\alpha\beta}\,\eta_{\chi}^{\omega\tau}{}_{\nu}\,V_{\gamma}\,V_{\omega}\,k_{\beta}\,k_{\tau}\,\sigma^{\mu}{}_{\lambda}\,\hat{U}^{\chi}{}_{\alpha} \\
&= -\delta^{\lambda\gamma\alpha\beta}_{\chi\omega\tau\nu}\,V_{\gamma}\,V^{\omega}\,k_{\beta}\,k^{\tau}\,\sigma^{\mu}{}_{\lambda}\,\hat{U}^{\chi}{}_{\alpha} \\
&= -k_{\beta}\,k^{\beta}\,\sigma^{\mu}{}_{\lambda}\,\hat{U}^{\lambda}{}_{\nu} \\
&= -k^{2}\,\sigma^{\mu}{}_{\lambda}\,\hat{U}^{\lambda}{}_{\nu},
\end{aligned}
$$

$$
\begin{aligned}
T_2 &\equiv \eta^{\lambda\gamma\alpha\beta}\,\eta_{\chi}{}^{\omega\tau\mu}\,V_{\gamma}\,V_{\omega}\,k_{\beta}\,k_{\tau}\,\sigma_{\nu\lambda}\,\hat{U}^{\chi}{}_{\alpha} \\
&= -\delta^{\chi\omega\tau\mu}_{\lambda\gamma\alpha\beta}\,V^{\gamma}\,V_{\omega}\,k^{\beta}\,k_{\tau}\,\sigma_{\nu}{}^{\lambda}\,\hat{U}^{\alpha}{}_{\chi} \\
&= k^{\beta}\,k_{\beta}\,\sigma_{\nu}{}^{\lambda}\,\hat{U}^{\alpha}{}_{\chi} \\
&= -k^{2}\,\sigma_{\nu}{}^{\lambda}\,\hat{U}^{\mu}{}_{\lambda},
\end{aligned}
$$

$$
\begin{aligned}
T_3 &\equiv \eta^{\lambda\gamma\alpha\beta}\,\eta_{\chi}{}^{\omega\tau}{}_{\alpha}\,V_\gamma\,V_\omega\,k_\beta\,k_\tau\,\sigma^{\mu}{}_{\lambda}\,\hat{U}^{\chi}{}_{\nu}\\
&= -\eta^{\lambda\gamma\alpha\beta}\,\eta_{\chi\omega\alpha\tau}\,V_\gamma\,V^\omega\,k_\beta\,k^\tau\,\sigma^{\mu}{}_{\lambda}\,\hat{U}^{\chi}{}_{\nu}\\
&= \delta^{\lambda\gamma\beta}_{\chi\omega\tau}\,V_\gamma\,V^\omega\,k_\beta\,k^\tau\,\sigma^{\mu}{}_{\lambda}\,\hat{U}^{\chi}{}_{\nu}\\
&= k_\beta\,k^\beta\,\sigma^{\mu}{}_{\lambda}\,\hat{U}^{\lambda}{}_{\nu}\\
&= -k^2\,\sigma^{\mu}{}_{\lambda}\,\hat{U}^{\lambda}{}_{\nu},
\end{aligned}
$$

$$
\begin{aligned}
T_4 &\equiv \eta^{\lambda\gamma\alpha\beta}\,\eta^{\chi\omega\tau}{}_{\alpha}\,V_\gamma\,V_\omega\,k_\beta\,k_\tau\,\sigma_{\nu\lambda}\,\hat{U}^{\mu}{}_{\chi}\\
&= -\eta_{\lambda\gamma\alpha\beta}\,\eta^{\chi\omega\tau\alpha}\,V^\gamma\,V_\omega\,k^\beta\,k_\tau\,\sigma_{\nu}{}^{\lambda}\,\hat{U}^{\mu}{}_{\chi}\\
&= \delta^{\chi\omega\tau}_{\lambda\gamma\beta}\,V^\gamma\,V_\omega\,k^\beta\,k_\tau\,\sigma_{\nu}{}^{\lambda}\,\hat{U}^{\mu}{}_{\chi}\\
&= k^\beta\,k_\beta\,\sigma_{\nu}{}^{\lambda}\,\hat{U}^{\mu}{}_{\lambda}\\
&= -k^2\,\sigma_{\nu}{}^{\lambda}\,\hat{U}^{\mu}{}_{\lambda}.
\end{aligned}
$$

E daí vem então:

$$
\left[\eta^{\lambda\gamma\alpha\beta}\,V_\gamma\,\sigma^{\mu}{}_{\lambda}\,h_{\varepsilon\nu}\,\hat{U}^{*\varepsilon}{}_{\alpha;\beta} + \eta^{\lambda\gamma\alpha\beta}\,V_\gamma\,\sigma_{\nu\lambda}\,h_{\varepsilon}{}^{\mu}\,\hat{U}^{*\varepsilon}{}_{\alpha;\beta}\right] =
$$

$$= -\frac{1}{2}\left(-2\,k^2\right)\left(\sigma^{\mu}{}_{\lambda}\,\hat{U}^{\lambda}{}_{\nu} + \sigma^{\lambda}{}_{\nu}\,\hat{U}^{\mu}{}_{\lambda}\right)$$

$$= k^2\left(\sigma^{\mu}{}_{\lambda}\,\hat{U}^{\lambda}{}_{\nu} + \sigma^{\lambda}{}_{\nu}\,\hat{U}^{\mu}{}_{\lambda}\right),$$

o que nos dá — usando (2.89) e (2.108) para as soluções de Kasner e Milne respectivamente — os seguintes resultados:

$$\left[\eta^{\lambda\gamma\alpha\beta}\,V_{\gamma}\,\sigma^{\mu}{}_{\lambda}\,h_{\varepsilon\nu}\,\hat{U}^{*\varepsilon}{}_{\alpha;\beta} + \eta^{\lambda\gamma\alpha\beta}\,V_{\gamma}\,\sigma_{\nu\lambda}\,h_{\varepsilon}{}^{\mu}\,\hat{U}^{*\varepsilon}{}_{\alpha;\beta}\right] =$$

$$= \begin{cases} \frac{2}{3}\,\theta\,k^2\,\hat{U}^{\mu}{}_{\nu} & \textit{(solução de Kasner)}, \\ -\frac{2}{3}\,\theta\,k^2\,\hat{U}^{\mu}{}_{\nu} & \textit{(solução de Milne)}. \end{cases} \tag{C.4}$$

As relações (2.100) e (2.117) são obtidas de:

$$\left[\eta^{\mu\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta} + \eta_{\nu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h^{\mu}{}_{\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta}\right] =$$

$$= \left(\eta^{\mu\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu\lambda} + \eta_{\nu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h^{\mu}{}_{\lambda}\right)\left[\hat{U}^{*\lambda}{}_{\alpha,\beta} + \Gamma^{\lambda}_{\tau\beta}\,\hat{U}^{*\tau}{}_{\alpha} - \Gamma^{\tau}_{\alpha\beta}\,\hat{U}^{*\lambda}{}_{\tau}\right],$$

onde novamente eliminamos os termos de conexão pelos mesmos argumentos discutidos na demonstração anterior. Usando as relações (2.76) e (2.61) em seguida, vem então que:

$$\left[\eta^{\mu\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta} + \eta_{\nu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h^{\mu}{}_{\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta}\right] =$$

$$= -\frac{1}{2}\,k_{\beta}\,k_{\varepsilon}\left(\eta^{\mu\gamma\alpha\beta}\,V_{\gamma}\,h_{\nu\lambda} + \eta_{\nu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,h^{\mu}{}_{\lambda}\right)\left[\eta^{\tau\varphi\varepsilon}{}_{\alpha}\,V_{\varphi}\,\hat{U}^{\lambda}{}_{\tau} + \eta_{\tau}{}^{\varphi\varepsilon\lambda}\,V_{\varphi}\,\hat{U}^{\tau}{}_{\alpha}\right]$$

$$
\begin{aligned}
&= -\frac{1}{2}\, k_\beta\, k_\varepsilon\, \Big[\eta^{\mu\gamma\alpha\beta}\, \eta_\tau{}^{\varphi\varepsilon}{}_\alpha\, V_\gamma\, V_\varphi\, \hat{U}^\tau{}_\nu + \eta^{\mu\gamma\alpha\beta}\, \eta_\tau{}^{\varphi\varepsilon}{}_\nu\, V_\gamma\, V_\varphi\, \hat{U}^\tau{}_\alpha + \\
&\quad + \eta_\nu{}^{\gamma\alpha\beta}\, \eta^{\tau\varphi\varepsilon}{}_\alpha\, V_\gamma\, V_\varphi\, \hat{U}^\mu{}_\tau + \eta_\nu{}^{\gamma\alpha\beta}\, \eta_\tau{}^{\varphi\varepsilon\mu}\, V_\gamma\, V_\varphi\, \hat{U}^\tau{}_\alpha\Big] \\
&\equiv -\frac{1}{2} \sum_{j=1}^{4} T_j,
\end{aligned}
$$

e, calculando termo a termo, vem (usando-se a equação (2.66)):

$$
\begin{aligned}
T_1 &\equiv \eta^{\mu\gamma\alpha\beta}\, \eta_\tau{}^{\varphi\varepsilon}{}_\alpha\, V_\gamma\, V_\varphi\, k_\beta\, k_\varepsilon\, \hat{U}^\tau{}_\nu \\
&= -\eta^{\mu\gamma\alpha\beta}\, \eta_{\tau\varphi\alpha\varepsilon}\, V_\gamma\, V^\varphi\, k_\beta\, k^\varepsilon\, \hat{U}^\tau{}_\nu \\
&= \delta^{\mu\gamma\beta}_{\tau\varphi\varepsilon}\, V_\gamma\, V^\varphi\, k_\beta\, k^\varepsilon\, \hat{U}^\tau{}_\nu \\
&= -k^2\, \hat{U}^\mu{}_\nu - h^{\alpha\mu}\, k_\alpha\, \left(k_\beta\, \hat{U}^\beta{}_\nu\right) \\
&= -k^2\, \hat{U}^\mu{}_\nu,
\end{aligned}
$$

$$
\begin{aligned}
T_2 &\equiv \eta^{\mu\gamma\alpha\beta}\, \eta_\tau{}^{\varphi\varepsilon}{}_\nu\, V_\gamma\, V_\varphi\, k_\beta\, k_\varepsilon\, \hat{U}^\tau{}_\alpha \\
&= \eta^{\mu\gamma\alpha\beta}\, \eta_{\tau\varphi\varepsilon\nu}\, V_\gamma\, V^\varphi\, k_\beta\, k^\varepsilon\, \hat{U}^\tau{}_\alpha \\
&= -\delta^{\mu\gamma\alpha\beta}_{\tau\varphi\varepsilon\nu}\, V_\gamma\, V^\varphi\, k_\beta\, k^\varepsilon\, \hat{U}^\tau{}_\alpha \\
&= -k_\nu\, \left(k^\beta\, \hat{U}^\mu{}_\beta\right) - k^2\, \hat{U}^\mu{}_\nu + h^{\alpha\gamma}\, k_\alpha\, h^\mu{}_\nu\, \left(k_\beta\, \hat{U}^\beta{}_\gamma\right) - h^{\alpha\mu}\, k_\alpha\, \left(k_\beta\, \hat{U}^\beta{}_\nu\right)
\end{aligned}
$$

$$= \quad -k^2\, \hat{U}^{\mu}{}_{\nu},$$

$$\begin{aligned} T_3 &\equiv \eta_{\nu}{}^{\gamma\alpha\beta}\, \eta^{\tau\varphi\varepsilon}{}_{\alpha}\, V_{\gamma}\, V_{\varphi}\, k_{\beta}\, k_{\varepsilon}\, \hat{U}^{\mu}{}_{\tau} \\ &= -\eta_{\nu\gamma\alpha\beta}\, \eta^{\tau\varphi\alpha\varepsilon}\, V^{\gamma}\, V_{\varphi}\, k^{\beta}\, k_{\varepsilon}\, \hat{U}^{\mu}{}_{\tau} \\ &= \delta^{\tau\varphi\varepsilon}_{\nu\gamma\beta}\, V^{\gamma}\, V_{\varphi}\, k^{\beta}\, k_{\varepsilon}\, \hat{U}^{\mu}{}_{\tau} \\ &= -k^2\, \hat{U}^{\mu}{}_{\nu} - k_{\nu}\, \left(k^{\beta}\, \hat{U}^{\mu}{}_{\beta}\right) \\ &= -k^2\, \hat{U}^{\mu}{}_{\nu}, \end{aligned}$$

$$\begin{aligned} T_4 &\equiv \eta_{\nu}{}^{\gamma\alpha\beta}\, \eta_{\tau}{}^{\varphi\varepsilon\mu}\, V_{\gamma}\, V_{\varphi}\, k_{\beta}\, k_{\varepsilon}\, \hat{U}^{\tau}{}_{\alpha} \\ &= \eta_{\nu\gamma\alpha\beta}\, \eta^{\tau\varphi\varepsilon\mu}\, V^{\gamma}\, V_{\varphi}\, k^{\beta}\, k_{\varepsilon}\, \hat{U}^{\alpha}{}_{\tau} \\ &= -\delta^{\tau\varphi\varepsilon\mu}_{\nu\gamma\alpha\beta}\, V^{\gamma}\, V_{\varphi}\, k^{\beta}\, k_{\varepsilon}\, \hat{U}^{\alpha}{}_{\tau} \\ &= -h^{\alpha\mu}\, k_{\alpha}\, \left(k_{\beta}\, \hat{U}^{\beta}{}_{\nu}\right) - k^2\, \hat{U}^{\mu}{}_{\nu} + \delta^{\mu}{}_{\nu}\, h^{\alpha\gamma}\, k_{\alpha}\, \left(k_{\beta}\, \hat{U}^{\beta}{}_{\gamma}\right) - k_{\nu}\, \left(k^{\beta}\, \hat{U}^{\mu}{}_{\beta}\right), \end{aligned}$$

donde vem afinal:

$$\left[\eta^{\mu\gamma\alpha\beta}\, V_{\gamma}\, h_{\nu\lambda}\, \hat{U}^{*\lambda}{}_{\alpha;\beta} + \eta_{\nu}{}^{\gamma\alpha\beta}\, V_{\gamma}\, h^{\mu}{}_{\lambda}\, \hat{U}^{*\lambda}{}_{\alpha;\beta}\right] = 2\, k^2\, \hat{U}^{\mu}{}_{\nu}. \tag{C.5}$$

A prova para os resultados (2.101) e (2.118) é dada a seguir:

$$\left[\eta^{\mu\gamma\alpha\beta}\, V_\gamma\, \sigma_{\nu\lambda}\, \hat{U}^{*\lambda}{}_{\alpha;\beta} + \eta_\nu{}^{\gamma\alpha\beta}\, V_\gamma\, \sigma^\mu{}_\lambda\, \hat{U}^{*\lambda}{}_{\alpha;\beta}\right] =$$

$$\begin{aligned}
&= \left(\eta^{\mu\gamma\alpha\beta}\, V_\gamma\, \sigma_{\nu\lambda} + \eta_\nu{}^{\gamma\alpha\beta}\, V_\gamma\, \sigma^\mu{}_\lambda\right) \left[\hat{U}^{*\lambda}{}_{\alpha,\beta} + \Gamma^\lambda_{\tau\beta}\, \hat{U}^{*\tau}{}_\alpha - \Gamma^\tau_{\alpha\beta}\, \hat{U}^{*\lambda}{}_\tau\right] \\
&= \left(\eta^{\mu\gamma\alpha\beta}\, V_\gamma\, \sigma_{\nu\lambda} + \eta_\nu{}^{\gamma\alpha\beta}\, V_\gamma\, \sigma^\mu{}_\lambda\right) \hat{U}^{*\lambda}{}_{\alpha,\beta} \\
&= -\frac{i}{2}\, k_\varepsilon \left(\eta^{\mu\gamma\alpha\beta}\, V_\gamma\, \sigma_{\nu\lambda} + \eta_\nu{}^{\gamma\alpha\beta}\, V_\gamma\, \sigma^\mu{}_\lambda\right) \left[\eta_\tau{}^{\varphi\varepsilon\lambda}\, V_\varphi\, \hat{U}^\tau{}_{\alpha,\beta} + \eta^{\tau\varphi\varepsilon}{}_\alpha\, V_\varphi\, \hat{U}^\lambda{}_{\tau,\beta}\right] \\
&= -\frac{1}{2}\, k_\beta\, k_\varepsilon \left[\eta^{\mu\gamma\alpha\beta}\, \eta_\tau{}^{\varphi\varepsilon\lambda}\, V_\gamma\, V_\varphi\, \sigma_{\nu\lambda}\, \hat{U}^\tau{}_\alpha + \eta^{\mu\gamma\alpha\beta}\, \eta^{\tau\varphi\varepsilon}{}_\alpha\, V_\gamma\, V_\varphi\, \sigma_{\nu\lambda}\, \hat{U}^\lambda{}_\tau + \right. \\
&\quad \left. + \eta_\nu{}^{\gamma\alpha\beta}\, \eta_\tau{}^{\varphi\varepsilon\lambda}\, V_\gamma\, V_\varphi\, \sigma^\mu{}_\lambda\, \hat{U}^\tau{}_\alpha + \eta_\nu{}^{\gamma\alpha\beta}\, \eta^{\tau\varphi\varepsilon}{}_\alpha\, V_\gamma\, V_\varphi\, \sigma^\mu{}_\lambda\, \hat{U}^\lambda{}_\tau\right] \\
&\equiv -\frac{1}{2} \sum_{j=1}^{4} T_j,
\end{aligned}$$

onde, calculando cada um dos termos acima em separado, obtemos:

$$\begin{aligned}
T_1 &\equiv \eta^{\mu\gamma\alpha\beta}\, \eta_\tau{}^{\varphi\varepsilon\lambda}\, V_\gamma\, V_\varphi\, k_\beta\, k_\varepsilon\, \sigma_{\nu\lambda}\, \hat{U}^\tau{}_\alpha \\
&= \eta^{\mu\gamma\alpha\beta}\, \eta_{\tau\varphi\varepsilon\lambda}\, V_\gamma\, V^\varphi\, k_\beta\, k^\varepsilon\, \sigma_\nu{}^\lambda\, \hat{U}^\tau{}_\alpha \\
&= -\delta^{\mu\gamma\alpha\beta}_{\tau\varphi\varepsilon\lambda}\, V_\gamma\, V^\varphi\, k_\beta\, k^\varepsilon\, \sigma_\nu{}^\lambda\, \hat{U}^\tau{}_\alpha \\
&= k^2\, \sigma_\nu{}^\alpha\, \hat{U}^\mu{}_\alpha - k_\beta\, \sigma_\nu{}^\beta\, h^{\mu\gamma} \left(k_\lambda\, \hat{U}^\lambda{}_\gamma\right)
\end{aligned}$$

$$= \; -k^2\, \sigma_\nu{}^\alpha\, \hat{U}^\mu{}_\alpha,$$

$$T_2 \;\equiv\; \eta^{\mu\gamma\alpha\beta}\, \eta^{\tau\varphi\varepsilon}{}_\alpha\, V_\gamma\, V_\varphi\, k_\beta\, k_\varepsilon\, \sigma_{\nu\lambda}\, \hat{U}^\lambda{}_\tau$$

$$= \; -\eta^{\mu\gamma\alpha\beta}\, \eta_{\tau\varphi\alpha\varepsilon}\, V_\gamma\, V^\varphi\, k_\beta\, k^\varepsilon\, \sigma_\nu{}^\lambda\, \hat{U}^\tau{}_\lambda$$

$$= \; \delta^{\mu\gamma\beta}_{\tau\varphi\varepsilon}\, V_\gamma\, V^\varphi\, k_\beta\, k^\varepsilon\, \sigma_\nu{}^\lambda\, \hat{U}^\tau{}_\lambda$$

$$= \; -k^2\, \sigma^\mu{}_\alpha\, \hat{U}^\alpha{}_\nu,$$

$$T_3 \;=\; \eta_\nu{}^{\gamma\alpha\beta}\, \eta_\tau{}^{\varphi\varepsilon\lambda}\, V_\gamma\, V_\varphi\, k_\beta\, k_\varepsilon\, \sigma^\mu{}_\lambda\, \hat{U}^\tau{}_\alpha$$

$$= \; \eta^{\chi\gamma\alpha\beta}\, \eta_{\tau\varphi\varepsilon}\, V_\gamma\, V^\varphi\, k_\beta\, k^\varepsilon\, h_{\nu\chi}\, \sigma^{\mu\lambda}\, \hat{U}^\tau{}_\alpha$$

$$= \; -\delta^{\chi\gamma\alpha\beta}_{\tau\varphi\varepsilon\lambda}\, V_\gamma\, V^\varphi\, k_\beta\, k^\varepsilon\, h_{\nu\chi}\, \sigma^{\mu\lambda}\, \hat{U}^\tau{}_\alpha$$

$$= \; -k^2\, \sigma^\mu{}_\alpha\, \hat{U}^\alpha{}_\nu,$$

$$T_4 \;\equiv\; \eta_\nu{}^{\gamma\alpha\beta}\, \eta^{\tau\varphi\varepsilon}{}_\alpha\, V_\gamma\, V_\varphi\, k_\beta\, k_\varepsilon\, \sigma^\mu{}_\lambda\, \hat{U}^\lambda{}_\tau$$

$$= \; -\eta_{\nu\gamma\alpha\beta}\, \eta^{\tau\varphi\varepsilon\alpha}\, V^\gamma\, V_\varphi\, k^\beta\, k_\varepsilon\, \sigma^\mu{}_\lambda\, \hat{U}^\lambda{}_\tau$$

$$= \; \delta^{\tau\varphi\varepsilon}_{\nu\gamma\beta}\, V^\gamma\, V_\varphi\, k^\beta\, k_\varepsilon\, \sigma^\mu{}_\lambda\, \hat{U}^\lambda{}_\tau$$

$$= \quad -k^2\,\sigma^{\mu}{}_{\alpha}\,\hat{U}^{\alpha}{}_{\nu},$$

e temos então que:

$$\left[\eta^{\mu\gamma\alpha\beta}\,V_{\gamma}\,\sigma_{\nu\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta} + \eta_{\nu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,\sigma^{\mu}{}_{\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta}\right] =$$

$$= \quad -\frac{1}{2}\left[-2\,k^2\left(\sigma^{\mu}{}_{\alpha}\,\hat{U}^{\alpha}{}_{\nu} + \sigma^{\alpha}{}_{\nu}\,\hat{U}^{\mu}{}_{\alpha}\right)\right]$$

$$= \quad k^2\left(\sigma^{\mu}{}_{\alpha}\,\hat{U}^{\alpha}{}_{\nu} + \sigma^{\alpha}{}_{\nu}\,\hat{U}^{\mu}{}_{\alpha}\right),$$

onde — de (2.89) e (2.108), para as soluções de Kasner e Milne, respectivamente — temos afinal:

$$\left[\eta^{\mu\gamma\alpha\beta}\,V_{\gamma}\,\sigma_{\nu\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta} + \eta_{\nu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,\sigma^{\mu}{}_{\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta}\right] =$$

$$= \quad \begin{cases} \frac{2}{3}\,\theta\,k^2\,\hat{U}^{\mu}{}_{\nu} & \textit{(solução de Kasner)}, \\ -\frac{2}{3}\,\theta\,k^2\,\hat{U}^{\mu}{}_{\nu} & \textit{(solução de Milne)}. \end{cases} \tag{C.6}$$

As relações (2.102) e (2.119) podem ser demonstradas da seguinte maneira:

$$\left[\eta^{\gamma\tau\alpha\beta}\,V_{\tau}\,\sigma_{\gamma\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta}\right] \quad = \quad \eta^{\gamma\tau\alpha\beta}\,V_{\tau}\,\sigma_{\gamma\lambda}\left[\hat{U}^{*\lambda}{}_{\alpha,\beta} + \Gamma^{\lambda}_{\tau\beta}\,\hat{U}^{*\tau}{}_{\alpha} - \Gamma^{\tau}_{\alpha\beta}\,\hat{U}^{*\lambda}{}_{\tau}\right]$$

$$= \quad \eta^{\gamma\tau\alpha\beta}\,V_{\tau}\,\sigma_{\gamma\lambda}\,\hat{U}^{*\lambda}{}_{\alpha,\beta}$$

$$= \quad -\frac{i}{2}\,\eta^{\gamma\tau\alpha\beta}\,V_{\tau}\,\sigma_{\gamma\lambda}\,k_{\varepsilon}\left[\eta_{\omega}{}^{\varphi\varepsilon\lambda}\,V_{\varphi}\,\hat{U}^{\omega}{}_{\alpha,\beta} + \eta^{\omega\varphi\varepsilon}{}_{\alpha}\,V_{\varphi}\,\hat{U}^{\lambda}{}_{\omega,\beta}\right]$$

$$
\begin{aligned}
&= -\frac{1}{2}\,\eta^{\gamma\tau\alpha\beta}\,V_\tau\,k_\beta\,k_\varepsilon\,\sigma_{\gamma\lambda}\,\left(\eta_\omega{}^{\varphi\varepsilon\lambda}\,V_\varphi\,\hat{U}^\omega{}_\alpha + \eta^{\omega\varphi\varepsilon}{}_\alpha\,V_\varphi\,\hat{U}^\lambda{}_\omega\right)\\
&= -\frac{1}{2}\left[\eta^{\gamma\tau\alpha\beta}\,\eta_\omega{}^{\varphi\varepsilon\lambda}\,V_\tau\,V_\varphi\,k_\beta\,k_\varepsilon\,\sigma_{\gamma\lambda}\,\hat{U}^\omega{}_\alpha + \eta^{\gamma\tau\alpha\beta}\,\eta^{\omega\varphi\varepsilon}{}_\alpha\,V_\tau\,V_\varphi\,k_\beta\,k_\varepsilon\,\sigma_{\gamma\lambda}\,\hat{U}^\lambda{}_\omega\right]\\
&\equiv -\frac{1}{2}\sum_{j=1}^{2} T_j,
\end{aligned}
$$

e, calculando termo a termo, vem:

$$
\begin{aligned}
T_1 &\equiv \eta^{\gamma\tau\alpha\beta}\,\eta_\omega{}^{\varphi\varepsilon\lambda}\,V_\tau\,V_\varphi\,k_\beta\,k_\varepsilon\,\sigma_{\gamma\lambda}\,\hat{U}^\omega{}_\alpha\\
&= \eta^{\gamma\tau\alpha\beta}\,\eta_{\omega\varphi\varepsilon\lambda}\,V_\gamma\,V^\varphi\,k_\beta\,k^\varepsilon\,\sigma_\gamma{}^\lambda\,\hat{U}^\omega{}_\alpha\\
&= -\delta^{\gamma\tau\alpha\beta}_{\omega\varphi\varepsilon\lambda}\,V_\gamma\,V^\varphi\,k_\beta\,k^\varepsilon\,\sigma_\gamma{}^\lambda\,\hat{U}^\omega{}_\alpha\\
&= -k_\beta\,\sigma^{\beta\lambda}\,\left(k_\alpha\,\hat{U}^\alpha{}_\lambda\right)\\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
T_2 &\equiv \eta^{\gamma\tau\alpha\beta}\,\eta^{\omega\varphi\varepsilon}{}_\alpha\,V_\gamma\,V_\varphi\,k_\beta\,k_\varepsilon\,\sigma_{\gamma\lambda}\,\hat{U}^\lambda{}_\omega\\
&= -\eta_{\gamma\tau\beta\alpha}\,\eta^{\omega\varphi\varepsilon\alpha}\,V^\tau\,V_\varphi\,k^\beta\,k_\varepsilon\,\sigma^\gamma{}_\lambda\,\hat{U}^\lambda{}_\omega\\
&= \delta^{\omega\varphi\varepsilon}_{\gamma\tau\beta}\,V^\tau\,V_\varphi\,k^\beta\,k_\varepsilon\,\sigma^\gamma{}_\lambda\,\hat{U}^\lambda{}_\omega
\end{aligned}
$$

$$= \quad -k_\beta\,\sigma^{\beta\lambda}\,\left(k_\alpha\,\hat{U}^\alpha{}_\lambda\right)$$

$$= \quad 0,$$

e vem finalmente que:

$$\left[\eta^{\gamma\tau\alpha\beta}\,V_\gamma\,\sigma_{\gamma\lambda}\,\hat{U}^{*\lambda}{}_{\alpha;\beta}\right] = 0, \tag{C.7}$$

um resultado que é geral para qualquer solução específica do modelo de Kasner.

Para finalizarmos, exibiremos a demonstração das relações (2.103) e (2.120), onde usamos o resultado (2.76):

$$\eta^{\alpha\beta\mu\nu}\,V_\beta\,\sigma_{\mu\gamma}\,\hat{U}^{*\gamma}{}_\nu = -\frac{i}{2}\,k_\varepsilon\,\eta^{\alpha\beta\mu\nu}\,V_\beta\,\sigma_{\mu\gamma}\,\left(\eta_\lambda{}^{\varphi\varepsilon\gamma}\,V_\varphi\,\hat{U}^\lambda{}_\nu + \eta^{\lambda\varphi\varepsilon}{}_\nu\,V_\varphi\,\hat{U}^\gamma{}_\lambda\right)$$

$$\begin{aligned} = \quad & -\frac{i}{2}\,\Big[\eta^{\alpha\beta\mu\nu}\,\eta_\lambda{}^{\varphi\varepsilon\gamma}\,V_\beta\,V_\varphi\,k_\varepsilon\,\sigma_{\mu\gamma}\,\hat{U}^\lambda{}_\nu + \\ & +\eta^{\alpha\beta\mu\nu}\,\eta^{\lambda\varphi\varepsilon}{}_\nu\,V_\beta\,V_\varphi\,k_\varepsilon\,\sigma_{\mu\gamma}\,\hat{U}^\gamma{}_\lambda\Big] \end{aligned}$$

$$\begin{aligned} = \quad & -\frac{i}{2}\,\Big[\eta^{\alpha\beta\mu\nu}\,\eta_{\lambda\varphi\varepsilon\gamma}\,V_\beta\,V^\varphi\,k^\varepsilon\,\sigma_\mu{}^\gamma\,\hat{U}^\lambda{}_\nu + \\ & +\eta^{\alpha\beta\mu\nu}\,\eta_{\lambda\varphi\varepsilon\nu}\,V_\beta\,V^\varphi\,k^\varepsilon\,\sigma_\mu{}^\gamma\,\hat{U}^\lambda{}_\gamma\Big] \end{aligned}$$

$$= \quad \frac{i}{2}\,\left[\delta^{\alpha\beta\mu\nu}_{\lambda\varphi\varepsilon\gamma}\,V_\beta\,V^\varphi\,k^\varepsilon\,\sigma_\mu{}^\gamma\,\hat{U}^\lambda{}_\nu + \delta^{\alpha\beta\mu}_{\lambda\varphi\varepsilon}\,V_\beta\,V^\varphi\,k^\varepsilon\,\sigma_\mu{}^\gamma\,\hat{U}^\lambda{}_\gamma\right]$$

$$\begin{aligned} = \quad & \frac{i}{2}\,\Big[k^\alpha\,\left(\sigma^\beta{}_\gamma\,\hat{U}^\gamma{}_\beta\right) - \sigma^{\alpha\gamma}\,\left(k_\beta\,\hat{U}^\beta{}_\gamma\right) - k_\beta\,\sigma^{\beta\gamma}\,\hat{U}^\alpha{}_\gamma + \\ & +k_\beta\,\sigma^{\beta\gamma}\,\hat{U}^\alpha{}_\gamma - k^\alpha\,\left(\sigma^\beta{}_\gamma\,\hat{U}^\gamma{}_\beta\right)\Big] \end{aligned}$$

$$= \quad -\frac{i}{2}\,\sigma^{\alpha\gamma}\,\left(k_\beta\,\hat{U}^\beta{}_\gamma\right)$$

$$= \quad 0, \tag{C.8}$$

o qual é também válido para qualquer solução específica do modelo de Kasner.

Apêndice D

Cálculos Auxiliares Referentes à Base Vetorial

Apresentaremos aqui os cálculos relativos a termos que surgem nas equações Quase-Maxwellianas para perturbações vetoriais. As demonstrações são listadas na ordem em que surgem em cada equação.

$$
\begin{aligned}
\eta_{(\mu}{}^{\gamma\alpha\beta}\, V_\gamma\, h_{\nu)}{}^\lambda\, \hat{P}^*_{\alpha;\beta} &= \eta_{(\mu}{}^{\gamma\mu\nu}\, V_\gamma\, h_{\nu)}{}^\lambda \left[\hat{P}^*_{\lambda\alpha,\beta} - \Gamma^\tau_{\lambda\beta}\,\hat{P}^*_{\tau\alpha} - \Gamma^\tau_{\alpha\beta}\,\hat{P}^*_{\lambda\tau}\right] \\
&= \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_\gamma\, h_{\nu)}{}^\lambda\, \hat{P}^*_{\lambda\alpha,\beta} \\
&= \eta_{(\mu}{}^{\gamma\alpha\beta}\, h_{\nu)}{}^\lambda\, \eta_{(\lambda}{}^{\varepsilon\tau\chi}\, V_\gamma\, V_\varepsilon\, m_{\alpha)}\, m_\tau\, \hat{P}_{\chi,\beta} \\
&= -i\,\eta_{(\mu}{}^{\gamma\alpha\beta}\left[\eta_{\nu)}{}^{\varepsilon\tau\chi}\, m_\alpha + \eta_\alpha{}^{\varepsilon\tau\chi}\, m_{\nu)}\right] V_\gamma\, V_\varepsilon\, m_\tau\, m_\beta\, \hat{P}_\chi \\
&= -i\,\eta_{(\mu\gamma\alpha\beta}\,\eta^{\alpha\varepsilon\tau\chi}\, V^\gamma\, V_\varepsilon\, m_{\nu)}\, m^\beta\, m_\tau\, \hat{P}_\chi
\end{aligned}
$$

$$
\begin{aligned}
&= -i\,\eta_{(\mu\gamma\beta\alpha}\,\eta^{\varepsilon\tau\lambda\alpha}\,V^{\gamma}\,V_{\varepsilon}\,m_{\nu)}\,m^{\beta}\,m_{\tau}\,\hat{P}_{\lambda} \\
&= i\,\delta^{\varepsilon\tau\lambda}_{(\mu\gamma\beta}\,V^{\gamma}\,V_{\varepsilon}\,m_{\nu)}\,m^{\beta}\,m_{\tau}\,\hat{P}_{\lambda} \\
&= i\,\Big(\delta^{\varepsilon}_{(\mu}\,\delta^{\tau}_{\gamma}\,\delta^{\lambda}_{\beta} + \delta^{\varepsilon}_{\gamma}\,\delta^{\tau}_{\beta}\,\delta^{\lambda}_{\gamma} + \delta^{\varepsilon}_{\beta}\,\delta^{\tau}_{(\mu}\,\delta^{\lambda}_{\gamma} - \\
&\quad -\delta^{\varepsilon}_{(\mu}\,\delta^{\tau}_{\beta}\,\delta^{\lambda}_{\gamma} - \delta^{\varepsilon}_{\gamma}\,\delta^{\tau}_{(\mu}\,\delta^{\lambda}_{\beta} - \delta^{\varepsilon}_{\beta}\,\delta^{tau}_{\gamma}\,\delta^{\lambda}_{(\mu}\Big)\,V^{\gamma}\,V_{\varepsilon}\,m_{\nu)}\,m^{\beta}\,m_{\tau}\,\hat{P}_{\lambda} \\
&= i\,\Big[-m^{2}\,m_{(\mu}\,\hat{P}_{\nu)} - m_{(\mu}\,m_{\nu)}\,m^{\beta}\,\hat{P}_{\beta}\Big] \\
&= m^{2}\,\Big(-i\,m_{(\mu}\,\hat{P}_{\nu)}\Big) \\
&= m^{2}\,\hat{P}_{\mu\nu}, \qquad \text{(D.1)}
\end{aligned}
$$

onde utilizamos a relação (2.27). Esta mesma equação também é usada na relação abaixo:

$$
\begin{aligned}
h^{\alpha\beta}\,\hat{P}_{\alpha;\beta} &= h^{\alpha\beta}\,\Big[\hat{P}_{\alpha,\beta} - \Gamma^{\alpha\beta}\,\hat{P}_{\gamma}\Big] \\
&= h^{\alpha\beta}\,\hat{P}_{\alpha,\beta} \\
&= -i\,h^{\alpha\beta}\,m_{\beta}\,\hat{P}_{\alpha} \\
&= 0, \qquad \text{(D.2)}
\end{aligned}
$$

e ainda:

$$
\begin{aligned}
\hat{P}^{\alpha}{}_{;\alpha} &= \left(h^{\alpha\beta}\,\hat{P}_{\beta}\right)_{;\alpha} \\
&= -\left(V^{\alpha}{}_{;\alpha}\,V^{\beta} + V^{\alpha}\,V^{\beta}{}_{;\alpha}\right)\,\hat{P}_{\beta} + h^{\alpha\beta}\,\hat{P}_{\beta;\alpha} \\
&= h^{\alpha\beta}\,\left[\hat{P}_{\beta,\alpha} - \Gamma^{\gamma}_{\beta\alpha}\,\hat{P}_{\gamma}\right] \\
&= -i\,h^{\alpha\beta}\,m_{\alpha}\,\hat{P}_{\beta} \\
&= 0.
\end{aligned}
\tag{D.3}
$$

Usando (2.33), prova-se facilmente que:

$$
\begin{aligned}
h^{\alpha}{}_{(\mu}\,h^{\beta}{}_{\nu)}\,\hat{P}_{\alpha;\beta} &= -i\,h^{\alpha}{}_{(\mu}\,h^{\beta}{}_{\nu)}\,m_{\beta}\,\hat{P}_{\alpha} \\
&= -i\,m_{(\mu}\,\hat{P}_{\nu)} \\
&= \hat{P}_{\mu\nu},
\end{aligned}
\tag{D.4}
$$

$$
\begin{aligned}
\sigma^{\alpha\beta}\,\hat{P}_{\alpha;\beta} &= \sigma^{\alpha\beta}\,\left[\hat{P}_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta}\,\hat{P}_{\gamma}\right] \\
&= \sigma^{\alpha\beta}\,\hat{P}_{\alpha,\beta} \\
&= -i\,\sigma^{\alpha\beta}\,m_{\beta}\,\hat{P}_{\alpha}
\end{aligned}
$$

$$= \quad 0. \tag{D.5}$$

$$\begin{aligned} h^{(\alpha}{}_{\mu}\, h^{\beta)}{}_{\nu}\, \hat{P}_{\alpha;\beta} &= -i\, m_{\beta}\, \hat{P}_{\alpha}\, h^{(\alpha}{}_{\mu}\, h^{\beta)}{}_{\nu} \\ &= \left(h^{\alpha}{}_{\mu}\, h^{\beta}{}_{\nu} + h^{\beta}{}_{\mu}\, h^{\alpha}{}_{\nu}\right)\, \hat{P}_{\alpha\beta} \\ &= 2\, \hat{P}_{\mu\nu}, \end{aligned} \tag{D.6}$$

$$\begin{aligned} h^{\alpha}{}_{(\mu}\, \sigma^{\beta}{}_{\nu)}\, \hat{P}_{(\alpha;\beta)} &= -i\, h^{\alpha}{}_{(\mu}\, \sigma^{\beta}{}_{\nu)}\, m_{(\alpha}\, \hat{P}_{\beta)} \\ &= h^{\alpha}{}_{(\mu}\, \sigma^{\beta}{}_{\nu)}\, \hat{P}_{\alpha\beta} \\ &= \sigma^{\alpha}{}_{(\mu}\, \hat{P}_{\nu)\alpha} \end{aligned} \tag{D.7}$$

A próxima relação a ser demonstrada é consequência direta das equações (1.45)-(1.46), válidas para a solução de Kasner ($p_1 = p_2 = 2/3$, $p_3 = -1/3$). Delas, é facilmente obtido que:

$$E^{\alpha}{}_{\beta} = \frac{2}{3}\, \theta\, \sigma^{\alpha}{}_{\beta}, \tag{D.8}$$

donde vem então que

$$E^{\alpha}{}_{(\mu}\, \hat{P}_{\nu)\alpha} = \frac{2}{3}\, \theta\, \sigma^{\alpha}{}_{(\mu}\, \hat{P}_{\nu)\alpha}, \tag{D.9}$$

$$h^{\alpha}{}_{(\mu}\, E^{\beta}{}_{\nu)}\, \hat{P}_{(\alpha;\beta)} = \frac{2}{3}\, \theta\, h^{\alpha}{}_{(\mu}\, \sigma^{\beta}{}_{\nu)}\, \hat{P}_{(\alpha;\beta)}$$

$$= \frac{2}{3}\,\theta\,\sigma^{\alpha}{}_{(\mu}\,\hat{P}_{\nu)\alpha}, \tag{D.10}$$

onde usamos a relação (D.7) acima, e:

$$\begin{aligned}
E^{\alpha\beta}\,\hat{P}_{\alpha;\beta} &= -i\,m_{\beta}\,E^{\alpha\beta}\,\hat{P}_{\alpha} \\
&= -i\,m_{\beta}\left(h^{\alpha\gamma}\,E^{\beta}{}_{\gamma}\right)\hat{P}_{\alpha} \\
&= -i\,h^{\alpha\gamma}\,m_{\beta}\left(\frac{2}{3}\,\theta\,\sigma^{\beta}{}_{\gamma}\right)\hat{P}_{\alpha} \\
&= -\frac{2}{3}\,\theta\,i\,\sigma^{\alpha\beta}\,m_{\beta}\,\hat{P}_{\alpha} \\
&= 0,
\end{aligned} \tag{D.11}$$

onde fizemos uso de (2.36).

As próximas relações a serem demonstradas são:

$$\begin{aligned}
\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,\sigma_{\nu)}{}^{\lambda}\,\hat{P}^{*}_{\lambda\alpha;\beta} &= -i\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,V_{\gamma}\,\sigma_{\nu)}{}^{\lambda}\left[\eta_{\lambda}{}^{\varepsilon\tau\chi}\,m_{\alpha}+\eta_{\alpha}{}^{\varepsilon\tau\chi}\,m_{\lambda}\right]V_{\varepsilon}\,m_{\tau}\,m_{\beta}\,\hat{P}_{\chi} \\
&= -i\,\eta_{(\mu}{}^{\gamma\alpha\beta}\,\eta_{\alpha}{}^{\varepsilon\tau\chi}\,V_{\gamma}\,V_{\varepsilon}\,\sigma_{\nu)}{}^{\lambda}\,m_{\lambda}\,m_{\tau}\,m_{\beta}\,\hat{P}_{\chi} \\
&= -i\,\eta_{(\mu\gamma\beta\alpha}\,\eta^{\varepsilon\tau\chi\alpha}\,V^{\gamma}\,V_{\varepsilon}\,\sigma_{\nu)}{}^{\lambda}\,m_{\lambda}\,m_{\tau}\,m^{\beta}\,\hat{P}_{\chi} \\
&= i\,\delta^{\varepsilon\tau\chi}_{(\mu\gamma\beta}\,V^{\gamma}\,V_{\varepsilon}\,\sigma_{\nu)}{}^{\lambda}\,m_{\lambda}\,m_{\tau}\,m^{\beta}\,\hat{P}_{\chi}
\end{aligned}$$

$$
\begin{aligned}
&= i\,\Big(\delta^{\varepsilon}_{(\mu}\,\delta^{\tau}_{\gamma}\,\delta^{\chi}_{\beta} + \delta^{\varepsilon}_{\gamma}\,\delta^{\tau}_{\beta}\,\delta^{\chi}_{(\mu} + \delta^{\varepsilon}_{\beta}\,\delta^{\tau}_{(\mu}\,\delta^{\chi}_{\gamma} - \delta^{\varepsilon}_{(\mu}\,\delta^{\tau}_{\beta}\,\delta^{\chi}_{\gamma} - \\
&\quad -\delta^{\varepsilon}_{\gamma}\,\delta^{\tau}_{(\mu}\,\delta^{\chi}_{\beta} - \delta^{\varepsilon}_{\beta}\,\delta^{\tau}_{\gamma}\,\delta^{\chi}_{(\mu}\Big)\; V^{\gamma}\,V_{\varepsilon}\,\sigma_{\nu)}{}^{\lambda}\,m_{\lambda}\,m_{\tau}\,m^{\beta}\,\hat{P}_{\chi} \\
&= -i\,m^2\,m_{\lambda}\,\sigma^{\lambda}{}_{\mu}\,\hat{P}_{\nu)} - i\,m_{\lambda}\,\sigma^{\lambda}{}_{(\mu}\,m_{\nu)}\,m^{\beta}\,\hat{P}_{\beta} \\
&= m^2\,\sigma^{\alpha}{}_{(\mu}\,\hat{P}_{\nu)\alpha}, \qquad \text{(D.12)}
\end{aligned}
$$

$$
\begin{aligned}
\eta^{\gamma\lambda\alpha\beta}\,V_{\lambda}\,\sigma_{\gamma(\mu}\,h_{\nu)}{}^{\tau}\,\hat{P}^{*}_{\tau\alpha;\beta} &= \eta^{\gamma\lambda\alpha\beta}\,V_{\lambda}\,\sigma_{\gamma(\mu}\,h_{\nu)}{}^{\tau}\,\left[\hat{P}^{*}_{\tau\alpha,\beta} - \Gamma^{\varepsilon}_{\tau\beta}\,\hat{P}^{*}_{\varepsilon\alpha} - \Gamma^{\varepsilon}_{\alpha\beta}\,\hat{P}^{*}_{\tau\varepsilon}\right] \\
&= \eta^{\gamma\lambda\alpha\beta}\,V_{\lambda}\,\sigma_{\gamma(\mu}\,h_{\nu)}{}^{\tau}\,\left(\eta_{(\tau}{}^{\varepsilon\chi\zeta}\,V_{\varepsilon}\,m_{\alpha)}\,m_{\chi}\,\hat{P}_{\zeta}\right)_{,\beta} \\
&= -i\,\eta^{\gamma\lambda\alpha\beta}\,V_{\lambda}\,\sigma_{\gamma(\mu}\,h_{\nu)}{}^{\tau}\,\left[\eta_{\tau}{}^{\varepsilon\chi\zeta}\,m_{\alpha} + \eta_{\alpha}{}^{\varepsilon\chi\zeta}\,m_{\tau}\right]\,V_{\varepsilon}\,m_{\chi}\,m_{\beta}\,\hat{P}_{\zeta} \\
&= -i\,\eta^{\gamma\lambda\alpha\beta}\,\eta_{\alpha\varepsilon\chi\tau}\,V_{\lambda}\,V^{\varepsilon}\,\sigma_{\gamma(\mu}\,m_{\nu)}\,m_{\beta}\,m^{\chi}\,\hat{P}^{\tau} \\
&= i\,\delta^{\gamma\lambda\beta}_{\varepsilon\chi\tau}\,V_{\lambda}\,V^{\varepsilon}\,\sigma_{\gamma(\mu}\,m_{\nu)}\,m_{\beta}\,m^{\chi}\,\hat{P}^{\tau} \\
&= i\begin{vmatrix} \delta^{\gamma}_{\varepsilon} & \delta^{\gamma}_{\chi} & \delta^{\gamma}_{\tau} \\ \delta^{\lambda}_{\varepsilon} & \delta^{\lambda}_{\chi} & \delta^{\lambda}_{\tau} \\ \delta^{\beta}_{\varepsilon} & \delta^{\beta}_{\chi} & \delta^{\beta}_{\tau} \end{vmatrix}\,V_{\lambda}\,V^{\varepsilon}\,\sigma_{\gamma(\mu}\,m_{\nu)}\,m_{\beta}\,m^{\chi}\,\hat{P}^{\tau} \\
&= i\,\left(-m^2\,\sigma^{\alpha}_{(\mu}\,m_{\nu)}\,\hat{P}_{\alpha} - \sigma^{\alpha}{}_{(\mu}\,m_{\nu)}\,m_{\alpha}\,m^{\beta}\,\hat{P}_{\beta}\right) \\
&= m^2\,\sigma^{\alpha}{}_{(\mu}\,\left(-i\,m_{\nu)}\,\hat{P}_{\alpha}\right) \\
&= m^2\,\sigma^{\alpha}{}_{(\mu}\,\hat{P}_{\nu)\alpha}. \qquad \text{(D.13)}
\end{aligned}
$$

$$
\begin{aligned}
\eta^{\gamma\tau\alpha\beta}\, V_\tau\, \sigma_\gamma{}^\lambda\, \hat{P}^*_{\lambda\alpha;\beta} &= \eta^{\gamma\tau\alpha\beta}\, V_\tau\, \sigma_\gamma{}^\lambda\, \left[\hat{P}^*_{\lambda\alpha,\beta} - \Gamma^\varepsilon_{\lambda\beta}\, \hat{P}^*_{\varepsilon\alpha} - \Gamma^\varepsilon_{\alpha\beta}\, \hat{P}^*_{\lambda\varepsilon}\right] \\
&= \eta^{\gamma\tau\alpha\beta}\, V_\tau\, \sigma_\gamma{}^\lambda\, \left(\eta_{(\lambda}{}^{\varepsilon\chi\zeta}\, V_\varepsilon\, m_{\alpha)}\, m_\chi\, \hat{P}_\zeta\right)_{,\beta} \\
&= -i\, \eta^{\gamma\tau\alpha\beta}\, \left[\eta_\lambda{}^{\varepsilon\chi\zeta}\, m_\alpha + \eta_\alpha{}^{\varepsilon\chi\zeta}\, m_\lambda\right]\, V_\tau\, V_\varepsilon\, \sigma_\gamma{}^\lambda\, m_\chi\, m_\beta\, \hat{P}_\zeta \\
&= -i\, \eta^{\gamma\tau\alpha\beta}\, \eta_{\alpha\varepsilon\chi\zeta}\, V_\tau\, V^\varepsilon\, \sigma_\gamma{}^\lambda\, m_\lambda\, m^\chi\, m_\beta\, \hat{P}^\zeta \\
&= i\, \delta^{\gamma\tau\beta}_{\varepsilon\chi\zeta}\, V_\tau\, V^\varepsilon\, \sigma_\gamma{}^\lambda\, m_\lambda\, m^\chi\, m_\beta\, \hat{P}^\zeta \\
&= i\, \left(-m^2\, \sigma^{\alpha\beta}\, m_\alpha\, \hat{P}_\beta - \sigma^{\alpha\beta}\, m_\alpha\, m_\beta\, m^\gamma\, \hat{P}_\gamma\right) \\
&= m^2\, \sigma^{\alpha\beta}\, \left(-i\, m_\alpha\, \hat{P}_\beta\right) \\
&= m^2\, \sigma^{\alpha\beta}\, \hat{P}_{\alpha\beta} \\
&= 0. \qquad \text{(D.14)}
\end{aligned}
$$

Temos, ainda, que:

$$
\begin{aligned}
\eta_{(\mu}{}^{\gamma\alpha\beta}\, V_\gamma\, h_{\nu)}{}^\lambda\, \hat{P}_{\lambda\alpha;\beta} &= \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_\gamma\, h_{\nu)}{}^\lambda\, \left[\hat{P}_{\lambda\alpha,\beta} - \Gamma^\tau_{\lambda\beta}\, \hat{P}_{\tau\alpha} - \Gamma^\tau_{\alpha\beta}\, \hat{P}_{\lambda\tau}\right] \\
&= -i\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_\gamma\, h_{\nu)}{}^\lambda\, \left(m_\lambda\, \hat{P}_\alpha + m_\alpha\, \hat{P}_\lambda\right)_{,\beta} \\
&= -i\, \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_\gamma\, h_{\nu)}{}^\lambda\, \left[m_\lambda\, \hat{P}_{\alpha,\beta} + m_\alpha\, \hat{P}_{\lambda,\beta}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= -\eta_{(\mu}{}^{\gamma\alpha\beta}\, V_\gamma\, h_{\nu)}{}^{\lambda}\, \left[m_\lambda\, m_\beta\, \hat{P}_\alpha + m_\alpha\, m_\beta\, \hat{P}_\lambda\right] \\
&= -\eta_{(\mu}{}^{\gamma\alpha\beta}\, V_\gamma\, m_{\nu)}\, m_\beta\, \hat{P}_\alpha \\
&= \eta_{(\mu}{}^{\gamma\alpha\beta}\, V_\gamma\, m_{\nu)}\, m_\beta\, \hat{P}_\alpha \\
&= \hat{P}^*_{\mu\nu}. \qquad \text{(D.15)}
\end{aligned}
$$

$$
\begin{aligned}
h^\alpha{}_{(\mu}\, h^\beta{}_{\nu)}\, \hat{P}^*_{\alpha;\beta} &= h^\alpha{}_{(\mu}\, h^\beta{}_{\nu)}\, \left[\hat{P}^*_{\alpha,\beta} - \Gamma^\gamma_{\alpha\beta}\, \hat{P}^*_\gamma\right] \\
&= h^\alpha{}_{(\mu}\, h^\beta{}_{\nu)}\, \left[\eta_\alpha{}^{\tau\gamma\lambda}\, V_\tau\, \left(\hat{\nabla}_\gamma\, \hat{P}_\lambda\right)\right]_{,\beta} \\
&= \eta_\alpha{}^{\tau\gamma\lambda}\, V_\tau\, h^\alpha{}_{(\mu}\, h^\beta{}_{\nu)}\, \left(\hat{\nabla}_\gamma\, \hat{P}_\lambda\right)_{,\beta} \\
&= -i\, \eta_{(\mu}{}^{\tau\gamma\lambda}\, V_\tau\, h_{\nu)}{}^{\beta}\, \left(m_\gamma\, \hat{P}_\lambda\right)_{,\beta} \\
&= -\eta_{(\mu}{}^{\tau\gamma\lambda}\, V_\tau\, h_{\nu)}{}^{\beta}\, m_\gamma\, m_\beta\, \hat{P}_\lambda \\
&= -\eta_{(\mu}{}^{\tau\gamma\lambda}\, V_\tau\, m_{\nu)}\, m_\gamma\, \hat{P}_\lambda \\
&= -\hat{P}^*_{\mu\nu}. \qquad \text{(D.16)}
\end{aligned}
$$

$$
\hat{P}^{*\alpha}{}_{;\alpha} = \hat{P}^{*\alpha}{}_{,\alpha} + \Gamma^\alpha_{\beta\alpha}\, \hat{P}^{*\beta}
$$

$$
\begin{aligned}
&= \left(h^{\alpha\beta}\,\hat{P}^*_\beta\right)_{,\alpha} \\
&= h^{\alpha\beta}\,\hat{P}^*_{\beta,\alpha} \\
&= -i\,h^{\alpha\beta}\,m_\alpha\,\hat{P}^*_\beta \\
&= 0, \qquad \text{(D.17)}
\end{aligned}
$$

$$
\begin{aligned}
h^\alpha{}_{(\mu}\,\sigma^\beta{}_{\nu)}\,\hat{P}^*_{[\alpha;\beta]} &= h^\alpha{}_{(\mu}\,\sigma^\beta{}_{\nu)}\,\left(\hat{P}^*_{\alpha;\beta} - \hat{P}^*_{\beta;\alpha}\right) \\
&= h^\alpha{}_{(\mu}\,\sigma^\beta{}_{\nu)}\,\left(\hat{P}^*_{\alpha,\beta} - \hat{P}^*_{\beta,\alpha} - \Gamma^\gamma_{\alpha\beta}\,\hat{P}^*_\gamma + \Gamma^\gamma_{\alpha\beta}\,\hat{P}^*_\gamma\right) \\
&= h^\alpha{}_{(\mu}\,\sigma^\beta{}_{\nu)}\,\left[\eta_\alpha{}^{\tau\gamma\lambda}\,V_{tau}\,\left(\hat{\nabla}_\gamma\,\hat{P}_\lambda\right)_{,\beta} - \eta_\beta{}^{\tau\gamma\lambda}\,V_\tau\,\left(\hat{\nabla}_\gamma\,\hat{P}_\lambda\right)_{,\alpha}\right] \\
&= h^\alpha{}_{(\mu}\,\sigma^\beta{}_{\nu)}\,\left[-\eta_{[\alpha}{}^{\tau\gamma\lambda}\,V_\tau\,m_{\beta]}\,m_\gamma\,\hat{P}_\lambda\right] \\
&= 0, \qquad \text{(D.18)}
\end{aligned}
$$

já que estamos multiplicando um termo antissimétrico nos índices (α,β) — entre colchetes — por outro que está simetrizado em (μ,ν) e, por extensão, nos mesmos índices (α,β). Prosseguindo com as demonstrações, temos:

$$
\eta_{\alpha\beta}{}^{\mu\nu}\,V_\mu\,\hat{P}^*_\nu = \eta_{\alpha\beta}{}^{\mu\nu}\,V_\mu\,\eta_\nu{}^{\gamma\lambda\tau}\,V_\gamma\,\hat{\nabla}_\lambda\,\hat{P}_\tau
$$

$$
\begin{aligned}
&= \eta_{\alpha\beta\mu\nu}\, \eta^{\nu\gamma\lambda\tau}\, V^{\mu}\, V_{\gamma}\, \hat{\nabla}_{\lambda}\, \hat{P}_{\tau} \\
&= \delta^{\alpha\beta\mu}_{\gamma\lambda\tau}\, V^{\mu}\, V_{\gamma}\, \hat{\nabla}_{\lambda}\, \hat{P}_{\tau} \\
&= \hat{\nabla}_{[\alpha}\, \hat{P}_{\beta]} \\
&= h^{\mu}{}_{[\alpha}\, h^{\gamma}{}_{\beta]}\, \left(h^{\nu}{}_{\gamma}\, \hat{P}_{\nu}\right)_{;\mu} \\
&= h^{\mu}{}_{[\alpha}\, h^{\nu}{}_{\beta]}\, \left[-\left(V^{\nu}{}_{;\mu}\, V_{\gamma} + V^{\nu}\, V_{\gamma;\mu}\right) \hat{P}_{\nu} + h^{\nu}{}_{\gamma}\, \hat{P}_{\nu;\mu}\right] \\
&= h^{\mu}{}_{[\alpha}\, h^{\nu}{}_{\beta]}\, \hat{P}_{[\mu;\nu]} \qquad \text{(D.19)}
\end{aligned}
$$

$$
\begin{aligned}
h_{\alpha}{}^{\beta}\, \hat{P}_{\beta}{}^{\mu}{}_{;\mu} &= h_{\alpha}{}^{\beta}\, \left(\hat{P}_{\beta}{}^{\mu}{}_{;\mu} - \Gamma^{\gamma}_{\beta\mu}\, \hat{P}_{\gamma}{}^{\mu} + \Gamma^{\mu}_{\gamma\mu}\, \hat{P}_{\beta}{}^{\gamma}\right) \\
&= -i\, h_{\alpha}{}^{\beta}\, \left(m_{\beta}\, \hat{P}^{\mu} + m^{\mu}\, \hat{P}_{\beta}\right)_{,\mu} \\
&= -h_{\alpha}{}^{\beta}\, \left(m_{\beta}\, m_{\mu}\, \hat{P}^{\mu} + m_{\mu}\, m^{\mu}\, \hat{P}_{\beta}\right) \\
&= m^{2}\, \hat{P}_{\alpha}. \qquad \text{(D.20)}
\end{aligned}
$$

$$
\begin{aligned}
\eta_{\alpha\beta}{}^{\mu\nu}\, V_{\mu}\, \sigma_{\nu}{}^{\gamma}\, \hat{P}^{*}_{\gamma} &= -\eta_{\alpha\beta}{}^{\mu\nu}\, \eta_{\gamma}{}^{\lambda\tau\varepsilon}\, V_{\mu}\, V_{\lambda}\, \sigma_{\nu}{}^{\gamma}\, \hat{\nabla}_{\tau}\, \hat{P}_{\varepsilon} \\
&= -\eta_{\alpha\beta\mu\nu}\, \eta^{\gamma\lambda\tau\varepsilon}\, V^{\mu}\, V_{\lambda}\, \sigma^{\gamma}{}_{\nu}\, \hat{\nabla}_{\tau}\, \hat{P}_{\varepsilon}
\end{aligned}
$$

$$= \delta^{\gamma\lambda\tau\varepsilon}_{\alpha\beta\mu\nu} V^{\mu} V_{\lambda} \sigma^{\gamma}{}_{\nu} \hat{\nabla}_{\tau} \hat{P}_{\varepsilon}$$

$$= \sigma^{\gamma}{}_{[\alpha} \hat{\nabla}_{\gamma} \hat{P}_{\beta]} - \sigma^{\gamma}{}_{[\alpha} \hat{\nabla}_{\beta]} \hat{P}_{\gamma}, \tag{D.21}$$

$$- \eta_{[\alpha}{}^{\mu\gamma\lambda} V_{\gamma} \sigma_{\beta]\mu} \hat{P}^{*}_{\lambda} = -\eta_{[\alpha}{}^{\mu\gamma\lambda} \eta_{\lambda}{}^{\tau\varepsilon\nu} V_{\gamma} V_{\tau} \sigma_{\beta]\mu} \hat{\nabla}_{\varepsilon} \hat{P}_{\nu}$$

$$= -\eta_{[\alpha\mu\gamma\lambda} \eta^{\lambda\tau\varepsilon\nu} V^{\gamma} V_{\tau} \sigma_{\beta]}{}^{\mu} \hat{\nabla}_{\varepsilon} \hat{P}_{\nu}$$

$$= -\delta^{\tau\varepsilon\nu}_{[\alpha\mu\gamma} V^{\gamma} V_{\tau} \sigma_{\beta]}{}^{\mu} \hat{\nabla}_{\varepsilon} \hat{P}_{\nu}$$

$$= -\sigma^{\gamma}{}_{[\alpha} \hat{\nabla}_{\gamma} \hat{P}_{\beta]} + \sigma\gamma_{[\alpha} \hat{\nabla}_{\beta]} \hat{P}_{\gamma} \tag{D.22}$$

Os termos que aparecem na primeira e segunda projeções, equações (6.19) e (6.21) respectivamente, são escritos como:

$$h_{\alpha}{}^{\beta} h^{\mu\nu} \hat{P}_{\beta\mu;\nu} = h_{\alpha}{}^{\beta} h^{\mu\nu} \left[\hat{P}_{\beta\mu,\nu} - \Gamma^{\gamma}_{\beta\nu} \hat{P}_{\gamma\mu} - \Gamma^{\gamma}_{\mu\nu} \hat{P}_{\beta\gamma} \right]$$

$$= h_{\alpha}{}^{\beta} h^{\mu\nu} \hat{P}_{\beta\mu,\nu}$$

$$= h_{\alpha}{}^{\beta} h^{\mu\nu} \left(\hat{\nabla}_{(\beta} \hat{P}_{\mu)} \right)_{,\nu}$$

$$= -i\, h_{\alpha}{}^{\beta} h^{\mu\nu} m_{(\beta} \hat{P}_{\mu),\nu}$$

$$= -h_{\alpha}{}^{\beta} h^{\mu\nu} m_{\nu} m_{(\beta} \hat{P}_{\mu)}$$

$$= \quad 0, \tag{D.23}$$

$$\begin{aligned}
\eta_\alpha{}^{\beta\mu\nu} V_\beta \sigma_\mu{}^\gamma \hat{P}^*_{\nu\gamma} &= \eta_\alpha{}^{\beta\mu\nu} \eta_{(\nu}{}^{\varepsilon\lambda\tau} V_\beta V_\varepsilon \sigma_\mu{}^\gamma m_{\gamma)} m_\lambda \hat{P}_\tau \\
&= \eta_\alpha{}^{\beta\mu\nu} \eta_\nu{}^{\varepsilon\lambda\tau} V_\beta V_\varepsilon \sigma_\mu{}^\gamma m_\gamma m_\lambda \hat{P}_\tau + \\
&+ \eta_\alpha{}^{\beta\mu\nu} \eta_\gamma{}^{\varepsilon\lambda\tau} V_\beta V_\varepsilon \sigma_\mu{}^\gamma m_\nu m_\lambda \hat{P}_\tau \\
&= \eta_{\alpha\beta\mu\nu} \eta^{\nu\varepsilon\lambda\tau} V^\beta V_\varepsilon \sigma^{\mu\gamma} m_\gamma m_\lambda \hat{P}_\tau + \\
&+ \eta_{\alpha\beta\mu\nu} \eta^{\gamma\varepsilon} V^\beta V^\varepsilon \sigma^\mu{}_\gamma m^\nu m_\lambda \hat{P}_\tau \\
&= \delta^{\varepsilon\lambda\tau}_{\alpha\beta\mu} V^\beta V_\varepsilon \sigma^{\mu\gamma} m_\gamma m_\lambda \hat{P}_\tau - \delta^{\gamma\varepsilon\lambda\tau}_{\alpha\beta\mu\nu} V^\beta V_\varepsilon \sigma^\mu{}_\gamma m^\nu m_\lambda \hat{P}_\tau \\
&= \sigma^{\mu\gamma} m_\gamma m_\mu \hat{P}_\alpha - \sigma^{\mu\gamma} m_\gamma m_\alpha \hat{P}_\mu - \sigma^\mu{}_\alpha m^\nu m_\mu \hat{P}_\nu + \\
&+ \sigma^\mu{}_\alpha m^\lambda m_\lambda \hat{P}_\mu + \sigma^\mu{}_\gamma m^\gamma m_\mu \hat{P}_\alpha - \sigma^\mu{}_\gamma m^\gamma m_\alpha \hat{P}_\mu \\
&= 2\,\sigma^{\beta\gamma} m_\beta m_\gamma \hat{P}_\alpha - 2\,\sigma^{\beta\gamma} m_\gamma m_\alpha \hat{P}_\beta - \sigma_\alpha{}^\beta m_\beta h^{\gamma\lambda} m_\gamma \hat{P}_\lambda - m^2 \sigma_\alpha{}^\beta \hat{P}_\beta \\
&= -m^2 \sigma_\alpha{}^\beta \hat{P}_\beta + 2\left(\sigma^{\beta\gamma} m_\beta m_\gamma\right) \hat{P}_\alpha \\
&= -m^2 \sigma_\alpha{}^\beta \hat{P}_\beta + 2\,i\,\sigma^{\beta\gamma} m_\beta \hat{P}_{\alpha\gamma} - 2\,i\,\sigma^{\beta\gamma} m_\beta \hat{\nabla}_\alpha \hat{P}_\gamma \\
&= -m^2 \sigma_\alpha{}^\beta \hat{P}_\beta - 2\,\sigma^{\beta\gamma} m_\beta m_\alpha \hat{P}_\gamma \\
&= -m^2 \sigma_\alpha{}^\beta \hat{P}_\beta,
\end{aligned} \tag{D.24}$$

$$
\begin{aligned}
\eta^{\mu\beta\gamma\lambda} V_\gamma \left(\sigma_{\alpha\beta} + \frac{4}{3}\theta h_{\alpha\beta}\right) \hat{P}^*_{\lambda;\mu} &= \eta^{\mu\beta\gamma\lambda} V_\gamma \left(\sigma_{\alpha\beta} + \frac{4}{3}\theta h_{\alpha\beta}\right) \left[\hat{P}^*_{\lambda,\mu} - \Gamma^\tau_{\lambda\mu} \hat{P}^*_\tau\right] \\
&= \eta^{\mu\beta\gamma\lambda} V_\gamma \left(\sigma_{\alpha\beta} + \frac{4}{3}\theta h_{\alpha\beta}\right) \left(\eta_\lambda{}^{\varepsilon\tau\nu} V_\varepsilon \hat{\nabla}_\nu \hat{P}_\tau\right)_{,\mu} \\
&= \eta^{\mu\beta\gamma\lambda} \eta_{\lambda\varepsilon\tau\nu} V_\gamma V^\varepsilon \left(\sigma_{\alpha\beta} + \frac{4}{3}\theta h_{\alpha\beta}\right) \left(-i\, m^\nu \hat{P}^\tau\right)_{,\mu} \\
&= -i\, \delta^{\mu\beta\gamma}_{\varepsilon\tau\nu} V_\gamma V^\varepsilon \left(\sigma_{\alpha\beta} + \frac{4}{3}\theta h_{\alpha\beta}\right) m^\nu \left(-i\, m_\mu \hat{P}^\tau\right) \\
&= -\left(\sigma_\alpha{}^\beta + \frac{4}{3}\theta h_\alpha{}^\beta\right) m^\mu m_\beta \hat{P}_\mu + \left(\sigma_\alpha{}^\beta + \frac{4}{3}\theta h_\alpha{}^\beta\right) m^\mu m_\mu \hat{P}_\beta \\
&= -m^2 \sigma_\alpha{}^\beta \hat{P}_\beta - \frac{4}{3}\theta \hat{P}_\alpha, \qquad \text{(D.25)}
\end{aligned}
$$

$$
\begin{aligned}
h_\alpha{}^\beta h^{\mu\nu} \hat{P}^*_{\beta\mu;\nu} &= h_\alpha{}^\beta h^{\mu\nu} \left[\hat{P}^*_{\beta\mu,\nu} - \Gamma^\gamma_{\beta\nu} \hat{P}^*_{\gamma\mu} - \Gamma^\gamma_{\mu\nu} \hat{P}^*_{\beta\gamma}\right] \\
&= h_\alpha{}^\beta h^{\mu\nu} \hat{P}^*_{\beta\mu,\nu} \\
&= h_\alpha{}^\beta h^{\mu\nu} \left(\eta_{(\beta}{}^{\gamma\lambda\tau} V_\gamma m_{\mu)} m_\lambda \hat{P}_\tau\right)_{,\nu} \\
&= h_\alpha{}^\beta h^{\mu\nu} \left(\eta_\beta{}^{\gamma\lambda\tau} V_\gamma m_\mu m_\lambda + \eta_\mu{}^{\gamma\lambda\tau} V_\gamma m_\beta m_\lambda\right) \hat{P}_{\tau,\nu} \\
&= -i\, h_\alpha{}^\beta h^{\mu\nu} \left(\eta_\beta{}^{\gamma\lambda\tau} V_\gamma m_\mu m_\lambda m_\nu \hat{P}_\tau + \eta_\mu{}^{\gamma\lambda\tau} V_\gamma m_\beta m_\lambda m_\nu \hat{P}_\tau\right) \\
&= -i \left(-\eta_\alpha{}^{\gamma\lambda\tau} V_\gamma m^2 m_\lambda \hat{P}_\tau + \eta^{\nu\gamma\lambda\tau} V_\gamma m_\alpha m_\lambda m_\nu \hat{P}_\tau\right) \\
&= i\, m^2 \eta_\alpha{}^{\gamma\lambda\tau} V_\gamma m_\lambda \hat{P}_\tau
\end{aligned}
$$

$$= -m^2\, \eta_\alpha{}^{\gamma\lambda\tau}\, V_\gamma\, \hat{\nabla}_\lambda\, \hat{P}_\tau$$

$$= m^2\, \eta_\alpha{}^{\gamma\tau\lambda}\, V_\gamma\, \hat{\nabla}_\lambda\, \hat{P}_\tau$$

$$= m^2\, \hat{P}^*_\alpha, \tag{D.26}$$

$$\eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, \hat{P}_{\mu;\nu} = \eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, \left(\hat{P}_{\mu,\nu} - \Gamma^\gamma_{\mu\nu}\, \hat{P}_\gamma\right)$$

$$= -i\, \eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, m_\nu\, \hat{P}_\mu$$

$$= \eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, \hat{\nabla}_\nu\, \hat{P}_\mu$$

$$= \hat{P}^*_\alpha, \tag{D.27}$$

$$\eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, \sigma_\mu{}^\gamma\, \hat{P}_{\nu\gamma} = \eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, \sigma_\mu{}^\gamma\, \hat{\nabla}_{(\nu}\, \hat{P}_{\gamma)}$$

$$= \eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, \sigma_\mu{}^\gamma\, \left(\hat{\nabla}_\nu\, \hat{P}_\gamma + \hat{\nabla}_\gamma\, \hat{P}_\nu\right)$$

$$= \eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, \sigma_\mu{}^\gamma\, \hat{\nabla}_\nu\, \hat{P}_\gamma + \eta_\alpha{}^{\beta\mu\nu}\, V_\beta\, \sigma_\mu{}^\gamma\, \hat{\nabla}_\gamma\, \hat{P}_\nu$$

$$
= \begin{cases} -3\,\sigma_1{}^1\,\hat{P}^*_x & \text{para } \alpha = 1 \\ \\ -3\,\sigma_2{}^2\,\hat{P}^*_y & \text{para } \alpha = 2 \\ \\ 0 \quad \text{para } \alpha = 3 \end{cases}
$$

$$
= -3\,\sigma_\alpha{}^\beta\,\hat{P}^*_\beta. \tag{D.28}
$$

$$
\begin{aligned}
h_{\mu(\alpha}\,h_{\beta)}{}^\nu\,\hat{P}^{*\mu}_{;\nu} &= h_{\mu(\alpha}\,h_{\beta)}{}^\nu\,\left[\hat{P}^{*\mu}_{,\nu} + \Gamma^\mu_{\gamma\nu}\,\hat{P}^{*\gamma}\right] \\
&= h_{\mu(\alpha}\,h_{\beta)}{}^\nu\,\hat{P}^{*\mu}_{,\nu} \\
&= h_{\mu(\alpha}\,h_{\beta)}{}^\nu\,\eta^{\mu\gamma\tau\lambda}\,V_\gamma\,\left(\hat{\nabla}_\tau\,\hat{P}_\lambda\right)_{,\nu} \\
&= h_{\mu(\alpha}\,h_{\beta)}{}^\nu\,\eta^{\mu\gamma\tau\lambda}\,V_\gamma\,\left[-i\,m_\tau\,\left(-i\,m_\nu\,\hat{P}_\lambda\right)\right] \\
&= h_{\mu(\alpha}\,h_{\beta)}{}^\nu\,\eta^{\mu\gamma\tau\lambda}\,V_\gamma\,m_\tau\,m_\nu\,\hat{P}_\lambda \\
&= \eta_{(\alpha}{}^{\gamma\tau\lambda}\,V_\gamma\,m_{\beta)}\,m_\tau\,\hat{P}_\lambda \\
&= \hat{P}^*_{\alpha\beta}.
\end{aligned} \tag{D.29}
$$

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"TEORIA DE PERTURBAÇÕES EM UNIVERSOS ANISOTRÓPICOS: MODELO KASNER"

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Rio de Janeiro, 03 de abril de 1998