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2011 J. Phys.: Conf. Ser. 296 012024

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The dynamics of beltramized flows and its relation with the Kelvin waves.

Rafael González¹ and E Sergio Santini²

¹Instituto de Desarrollo Humano, Universidad Nacional de General Sarmiento
Pcia. de Buenos Aires, Argentina and Depto. de Física FCEyN, Universidad de Buenos Aires,
Buenos Aires, Argentina

²Instituto de Ciencias, Universidad Nacional de General Sarmiento
Pcia. de Buenos Aires, Argentina and ICRA-BR, Centro Brasileiro de Pesquisas Físicas, Rio
de Janeiro, Brasil

E-mail: rgonzale@ungs.edu.ar

Abstract. We define the beltramized flow as the sum of an uniform translation and an uniform rotation with a Beltrami flow. Some of their features are studied by solving the Euler equations, for different geometries, taking into account the boundary conditions, and for different symmetries. We show that the Kelvin waves are beltramized flows. Finally, we show that the variational principle found in a previous work, remains valid for the beltramized flow.

1. Introduction

The Beltrami flow defined as $\boldsymbol{\omega}_B = \nabla \times \mathbf{v}_B = \gamma \mathbf{v}_B$,. has been treated in several contexts [1], [2], [3]. Its importance lies on that it has nonlinear solutions of Euler and Navier-Stokes equations, and it has relevance for the lagrangian turbulence. And some other properties as being eigenfunctions of the curl operator and then they can be superposed linearly.

In previous works [4], [5] we have considered meanly two aspects: the formation of Kelvin waves ¹ with, what we called, Beltrami structure, in an expansion, and the existence of a variational principle similar to that introduced by Woltjer in Magnetohydrodynamics concerning the force-free magnetic fields that have a *topological analogy* with the vorticity field. But in the first case [4], the whole flow is not a pure Beltrami flow but the sum of a Rankine flow plus a Beltrami flow what we will denominate *beltramized flow*. And it is related to the Kelvin waves and is also important because the *breakdown* phenomena can develop from it [8]. Nevertheless when considering Kelvin waves [9] or steady axisymmetric swirling [10] flow in an expansion the Beltrami structure of the flow is not generally considered as it is done, for example, by Dritschel [2].

The aim of this paper is to consider the two aspects just pointed out. On the one hand we establish the dynamics of the beltramized flow and get solutions by considering different geometries and symmetries, showing that it results to be Kelvin Waves. We see that the eigenvalue of the beltramized flow is closely related to the boundary conditions. And that the steady state swirling flow is obtained as a particular case. On the other hand, we extend the variational principle introduced for pure Beltrami flows [5] to beltramized flows.

¹ Here the Kelvin waves are given in the sens introduced by Kelvin [6]

In section (2) write the dynamic equation for the beltramized flow. In section (3) we get the solutions for different geometries and symmetries and taking into account the boundary conditions. In section (4) we extend the variational principle for Beltrami flows to a beltramized ones. Finally in section (5) we write the conclusions.

2. The dynamic equation for the beltramized flow.

The Beltrami flow is defined as a field \mathbf{v}_B that satisfies $\boldsymbol{\omega}_B = \nabla \times \mathbf{v}_B = \gamma \mathbf{v}_B$, with $\gamma = \text{constant}$.

On the other hand, we define a *beltramized flow* as

$$\mathbf{v} = U\mathbf{e}_z + \Omega r\mathbf{e}_\theta + \mathbf{v}_B, \quad (1)$$

being U and Ω constants, i.e a beltramized flow is the superposition of a Rankine flow with a Beltrami one. As the dynamical equations are satisfied in any inertial frame of reference, we do not consider here the uniform translation, or, in other words, we work in a frame which is translating with velocity $U\mathbf{e}_z$.

Furthermore, to take into account the uniform rotation, we will consider our analysis in a non-inertial frame rotating with the angular velocity $\boldsymbol{\Omega} = \Omega\mathbf{e}_z$, and recover the uniform translation when it corresponds through the boundary conditions. Therefore the inertial terms, i.e. Centrifugal and Coriolis ones, are added to the RHS of Euler's equation (see for example [9] paragraph 14):

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \times \boldsymbol{\omega} - 2\boldsymbol{\Omega} \times \mathbf{v} - \frac{1}{\rho} \nabla H, \quad (2)$$

where $H \equiv p + \frac{v^2}{2} - \frac{1}{2}\rho(\boldsymbol{\Omega} \times \mathbf{r})^2$.

If now we set $\mathbf{v} = \mathbf{v}_B \Rightarrow \mathbf{v}_B \times \boldsymbol{\omega}_B = 0$. Then

$$\frac{\partial \mathbf{v}_B}{\partial t} = 2\mathbf{v}_B \times \boldsymbol{\Omega} - \frac{1}{\rho} \nabla H. \quad (3)$$

Finally, applying $\nabla \times$ to both members of the last equation and using the definition of Beltrami flow we have

$$\gamma \frac{\partial \mathbf{v}_B}{\partial t} = -2\boldsymbol{\Omega} \nabla \cdot \mathbf{v}_B + 2(\boldsymbol{\Omega} \cdot \nabla) \mathbf{v}_B, \quad (4)$$

which can be condensed in the following system of equations:

$$\frac{\partial \mathbf{v}_B}{\partial t} = \frac{2\boldsymbol{\Omega}}{\gamma} \frac{\partial \mathbf{v}_B}{\partial z}, \quad (5)$$

$$\nabla \cdot \mathbf{v}_B = 0, \quad (6)$$

$$\nabla \times \mathbf{v}_B = \gamma \mathbf{v}_B, \quad (7)$$

where the eigenvalue γ will be determined together with their solutions. This system of equations in fact verifies that a Rankine flow plus a Beltrami flow gives a solution of Euler's equations, as was demonstrated in [5].

3. Solutions of the Euler equations by considering the geometries and the symmetries

Here we consider different geometries and symmetries in order to solve the dynamical system given by (5), (6) and (7) which, from now on, will be called DS.

3.1. *Rectangular geometry (plane waves).*

We seek the solution of DS as a plane wave in an infinite fluid

$$\mathbf{v}_B = \mathbf{A}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \quad (8)$$

where $\mathbf{A} = \text{constant}$.

Substituting (8) in DS we have

$$\omega = -\frac{2\Omega}{\gamma}k_z, \quad (9)$$

$$\mathbf{k} \cdot \mathbf{A} = 0, \quad (10)$$

$$\frac{\mathbf{k}}{\gamma} \times \mathbf{v}_B = -i\mathbf{v}_B. \quad (11)$$

From where, according to (10) and (11) we obtain $\gamma = k$, and then Eq. (9) becomes

$$\omega = -\frac{2\Omega k_z}{k}. \quad (12)$$

which coincides with Eq. (14.8) of [9].

Now, we follow an analog approach as that given in [9] section 14. Defining $\mathbf{n} \equiv \frac{\mathbf{k}}{k}$, if we use a complex wave amplitude in the form $\mathbf{A} = \mathbf{a} + i\mathbf{b}$ with real vectors \mathbf{a} and \mathbf{b} , it follows that $\mathbf{n} \times \mathbf{b} = -\mathbf{a}$ and $\mathbf{n} \times \mathbf{a} = \mathbf{b}$: the vectors \mathbf{a} and \mathbf{b} (both lying in the plane perpendicular to \mathbf{k}) are orthogonal and equal in magnitude. By taking their directions as the x and y axes, and separating real and imaginary parts in Eq. (8), we obtain

$$v_{Bx} = a \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) ; v_{By} = -a \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) \quad (13)$$

or

$$v_{Bx} = a \cos(kz + 2\Omega t) ; v_{By} = -a \sin(kz + 2\Omega t) \quad (14)$$

where a is a constant of finite amplitude.

3.2. *Tubes with cylindrical and non-cylindrical geometries*

Here we consider two possible geometries:

a) Cylindrical geometry: we consider a cylinder of radius R .

b) Non-cylindrical geometry: tubes with variable section, as for example in fig. 1. In this case the conditions of continuities between regions should be considered besides the boundary conditions, in order to find the solution for the whole tube region.

The boundary conditions for the beltramized flow are:

Case a)

given that $\mathbf{n} = \mathbf{e}_r$ it is $\mathbf{n} \cdot \mathbf{e}_z = \mathbf{n} \cdot \mathbf{e}_\theta = 0$ and then from (1)

$$\mathbf{v} \cdot \mathbf{n} = 0 \text{ and } \boldsymbol{\omega} \cdot \mathbf{n} = 0 \Rightarrow \mathbf{v}_B \cdot \mathbf{n} = 0 .$$

Case b)

$$\mathbf{v} \cdot \mathbf{n} = 0 = U\mathbf{e}_z \cdot \mathbf{n} + \Omega r(\mathbf{n} \cdot \mathbf{e}_\theta) + \mathbf{v}_B \cdot \mathbf{n}, \text{ and}$$

$\boldsymbol{\omega} \cdot \mathbf{n} = 0 = 2\Omega\mathbf{e}_z \cdot \mathbf{n} + \gamma\mathbf{v}_B \cdot \mathbf{n}$, but, because of the circular geometry of the tube, $\mathbf{n} \cdot \mathbf{e}_\theta = 0$ and from last equations we obtain that $(2\Omega - \gamma U)\mathbf{e}_z \cdot \mathbf{n} = 0$. Now, for tubes with regions for which $\mathbf{e}_z \cdot \mathbf{n} \neq 0$ it must be $2\Omega - \gamma U = 0$, or

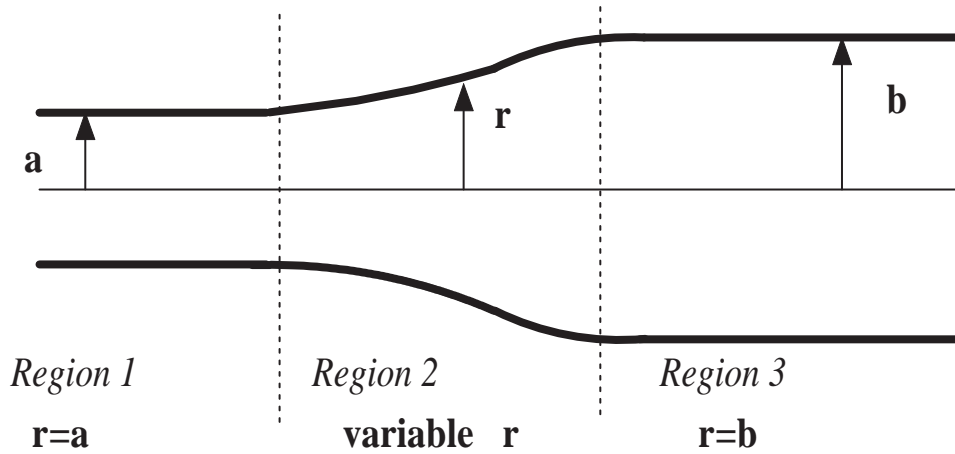


Figure 1. Non cylindrical tube geometry: it may have regions of constant and variable sections

$$\gamma = \frac{2\Omega}{U}. \quad (15)$$

From now on we consider different symmetries of the flow in a cylinder (case a) or in different regions of the non-cylindrical geometry (case b).

3.2.1. Flow with axial symmetry

For this symmetry, the general solution for the DS is:

$$\mathbf{v}_B = \mathbf{v}(r)e^{i(kz-\omega t+\Delta)}, \quad (16)$$

with k, ω and Δ constants quantities. From Eq. (5) we obtain

$$\omega = -\frac{2\Omega}{\gamma}k, \quad (17)$$

and from Eq(6) it follows

$$\mathbf{v}_B = \left(-\frac{1}{r}\frac{\partial\psi}{\partial z}, v_\theta, \frac{1}{r}\frac{\partial\psi}{\partial r}\right), \quad (18)$$

where $v_\theta = v_\theta(r, z)$. Then from (7) we have

$$\nabla \times \mathbf{v}_B = -\frac{1}{r}\frac{\partial}{\partial z}(rv_\theta)\mathbf{e}_r + \left[-\frac{1}{r}\frac{\partial^2\psi}{\partial z^2} - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right]\mathbf{e}_\theta + \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\mathbf{e}_z = \gamma\mathbf{v}_B, \quad (19)$$

and from (18) and (19) it follows that:

$$v_\theta = \gamma\frac{\psi}{r}, \quad (20)$$

$$\frac{\partial^2\psi}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \gamma^2\psi = 0 \quad (21)$$

from where, defining $\psi(r, z) = F(r)\sin(kz - \omega t + \Delta)$ we arrives to a solution which is well behaved at $r = 0$, i.e. it does not diverges at that point:

$$\psi = ArJ_1[(\gamma^2 - k^2)^{1/2}r] \sin(kz - \omega t + \Delta). \quad (22)$$

We consider now the boundary conditions so we have by taking into account of Eq. (17).

Case a:

$$\psi = ArJ_1[k(\frac{4\Omega^2}{\omega^2} - 1)^{1/2}r] \sin(kz - \omega t + \Delta), \quad (23)$$

with $k(\frac{4\Omega^2}{\omega^2} - 1)^{1/2}R = x_n$ as the eigenvalue equation for which $v_r(R) = 0$, where x_n are the zeros of $J_1[x]$.

Case b:

$$\psi = ArJ_1[(\frac{4\Omega^2}{U^2} - k^2)^{1/2}r] \sin[k(z + Ut) + \Delta], \quad (24)$$

from which we can get an steady state solution by moving in a reference frame with velocity $U\mathbf{e}_z$. Here A is a constant of finite amplitude.

3.2.2. Flow with cylindrical symmetry

In this situation the flow do not depend on θ nor on z so from Eq.(5) we have an steady solution, and it reads as:

case a:

$$\psi = ArJ_1[\gamma r], \quad (25)$$

case b:

$$\psi = ArJ_1[\frac{2\Omega}{U}r], \quad (26)$$

where A is a constant of finite amplitude.

3.2.3. Flow with helical symmetry

In this case defining $\phi = \theta - \kappa z$ we have $(\frac{\partial \mathbf{v}_B}{\partial z})_{\theta,r} = -\kappa (\frac{\partial \mathbf{v}_B}{\partial \phi})_{r,z}$, so that we can put (5) in the form:

$$\frac{\partial \mathbf{v}_B}{\partial t} = -\kappa \frac{2\Omega}{\gamma} \frac{\partial \mathbf{v}_B}{\partial \phi}. \quad (27)$$

Then we consider the solution:

$$\mathbf{v}_B = \mathbf{v}(r)e^{i(n\phi - \omega t)}, \quad (28)$$

with n integer, and then we have:

$$\omega = -\frac{2\Omega}{\gamma} \kappa n. \quad (29)$$

The Eqs. (6) and (7) are satisfied if the flow \mathbf{v}_B takes the form [5]

$$v_{Br} = -C[\frac{\mu_n}{\kappa} J'_n(\mu_n r) - \frac{\gamma}{\kappa^2 r} J_n(\mu_n r)] \cos[\Lambda(n, t)], \quad (30)$$

$$v_{B\theta} = C[\frac{n}{\kappa r} J_n(\mu_n r) - \frac{\mu_n \gamma}{\kappa^2 n} J'_n(\mu_n r)] \sin[\Lambda(n, t)], \quad (31)$$

$$v_{Bz} = C[\frac{\mu_n^2}{\kappa^2 n} J_n(\mu_n r)] \sin[\Lambda(n, t)], \quad (32)$$

where C is a constant of finite amplitude, and where :

case a:

$$\mu^2 = \gamma^2 - n^2 \kappa^2, \quad \Lambda(n, t) = n(\phi + \frac{2\Omega}{\gamma} \kappa t), \quad (33)$$

case b:

$$\mu^2 = n^2(\frac{4\Omega^2}{U^2} - \kappa^2), \quad \Lambda(n, t) = n(\phi + U \kappa t). \quad (34)$$

4. A variational property

In previous works [4], [5] we have considered the topological analogy between the Hydrodynamics and the Magnetohydrodynamics, and showed that the enstrophy plays the same role than the magnetic energy in Woljter's theorem [7], in the sense that the Beltrami flow equilibrium with constant eigenvalue is obtained when the enstrophy is extremized with the constraint that the helicity is conserved.

We show that the same principle is valid for beltramized flows as defined by Eq. 1. For doing that we take the flow

$$\mathbf{v} = U\mathbf{e}_z + \Omega r\mathbf{e}_\theta + \tilde{\mathbf{v}}, \quad (35)$$

$$\boldsymbol{\omega} = 2\Omega\mathbf{e}_z + \tilde{\boldsymbol{\omega}}, \quad (36)$$

and demonstrate that when $\tilde{\mathbf{v}} = \tilde{\mathbf{v}}_B$, then $\delta[\Phi - \gamma H_{\boldsymbol{\omega}}] = 0$, where $\Phi = \frac{1}{2} \int \omega^2 dV$ and $H_{\boldsymbol{\omega}} = \int \mathbf{v} \cdot \boldsymbol{\omega} dV$.

In this case $\tilde{\mathbf{v}}$, and $\tilde{\boldsymbol{\omega}} = \nabla \times \tilde{\mathbf{v}}$ are the velocity and the vorticity with regards to the uniform translation and rotation.

To show that, it is necessary to prove that

$$\delta[\Phi - \gamma H_{\boldsymbol{\omega}}] = \delta[\tilde{\Phi} - \gamma H_{\tilde{\boldsymbol{\omega}}}], \quad (37)$$

where $\tilde{\Phi} = \frac{1}{2} \int \tilde{\omega}^2 dV$ and $H_{\tilde{\boldsymbol{\omega}}} = \int \tilde{\mathbf{v}} \cdot \tilde{\boldsymbol{\omega}} dV$, and then to follow the steps given in [4] for the RHS of Eq.(37) assuming that the variations $\delta\tilde{\mathbf{v}} = \delta\tilde{\boldsymbol{\omega}} = 0$ on the boundary.

Really, taking into account Eqs. (35),(36) we have

$$\delta\Phi = \int \Omega\mathbf{e}_z \cdot \delta\tilde{\boldsymbol{\omega}} dV + \delta\tilde{\Phi}. \quad (38)$$

But

$$\int \Omega\mathbf{e}_z \cdot \delta\tilde{\boldsymbol{\omega}} dV = \int \Omega\mathbf{e}_z \cdot \nabla \times \delta\tilde{\mathbf{v}} dV = \int \nabla \cdot (\delta\tilde{\mathbf{v}} \times \Omega\mathbf{e}_z) dV = \int \hat{\mathbf{n}} \cdot (\delta\tilde{\mathbf{v}} \times \Omega\mathbf{e}_z) dS = 0. \quad (39)$$

In the same way

$$\delta H_{\boldsymbol{\omega}} = \int U\mathbf{e}_z \cdot \delta\tilde{\boldsymbol{\omega}} dV + \int \Omega r\mathbf{e}_\theta \cdot \delta\tilde{\boldsymbol{\omega}} dV + \int 2\Omega\mathbf{e}_z \cdot \delta\tilde{\mathbf{v}} dV + \delta H_{\tilde{\boldsymbol{\omega}}}. \quad (40)$$

And as in Eq (39), we can see that

$$\int U\mathbf{e}_z \cdot \delta\tilde{\boldsymbol{\omega}} dV = \int \Omega r\mathbf{e}_\theta \cdot \delta\tilde{\boldsymbol{\omega}} dV = \int 2\Omega\mathbf{e}_z \cdot \delta\tilde{\mathbf{v}} dV = 0, \quad (41)$$

where we have used the $\Omega r \mathbf{e}_\theta \cdot \nabla \times \delta \tilde{\mathbf{v}} = \nabla \cdot (\delta \tilde{\mathbf{v}} \times \Omega r \mathbf{e}_\theta) + 2\Omega \mathbf{e}_z \cdot \delta \tilde{\mathbf{v}}$, and that as $\nabla \cdot \tilde{\mathbf{v}} = 0$ is $\tilde{\mathbf{v}} = \nabla \times \mathbf{A}$.

Therefore Eq (37) is accomplished, and then from $\delta[\tilde{\Phi} - \gamma H_{\tilde{\omega}}] = 0$ we get $\tilde{\omega} = \gamma \tilde{\mathbf{v}}$ that is to say that $\tilde{\mathbf{v}} = \tilde{\mathbf{v}}_B$.

5. Conclusions

We can resume our conclusions with the following points

- The Beltrami flow obeys to the dynamic equations SD Eqs.(5-7).
- Their general solutions are represented by traveling waves of finite amplitude that are the so called Kelvin waves showing that these waves have a Beltrami structure.
- The γ eigenvalue of the Beltrami flow depends on the geometry considered by mean of the boundary conditions.
 - i) In a rectangular geometry in an infinite fluid the eigenvalue is $\gamma = k$ the modulus of the vector wave.
 - ii) In a cylindrical geometry of radius R , γ results from the boundary condition $v_{(r=R)} = 0$
 - iii) In a non-cylindrical geometry of tubes of variable section, the eigenvalue is given by $\gamma = \frac{2\Omega}{U}$. In this case we could have different regions (see Figure 1), some of which have cylindrical geometry, but in order to accomplish the boundary conditions of the whole flow, γ must have this unique value. On the other hand, the whole solution is obtained by considering the continuity conditions among different regions, which means to determine the constant coefficients of the solutions in each region.
- The Beltrami flow is the result of extremizing the enstrophy subject to the helicity conservation, extending in this way the result [4] for a pure Beltrami flow.

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