

DILATON QUANTUM COSMOLOGY WITH A SCHRÖDINGER-LIKE EQUATION

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Abstract

A quantum cosmological model with radiation and a dilaton scalar field is analysed. The Wheeler-deWitt equation in the mini-superspace induces a Schrödinger equation, which can be solved. An explicit wavepacket is constructed for a particular choice of the ordering factor. A consistent solution is possible only when the scalar field is a phantom field. Moreover, although the wavepacket is time dependent, a Bohmian analysis allows to extract a bouncing behaviour for the scale factor.

1 Introduction

A scalar field is the simplest fundamental matter field that we may introduce in a cosmological model, since it adds essentially just one degree of freedom and it is naturally invariant under coordinate transformation. Even though, we can quote two different couplings of a scalar field: the minimal coupling, leading to the so-called Einstein's frame; the non-minimal coupling, corresponding to the Jordan's frame. Of course, the minimal coupling is the most

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simple one. But, the non-minimal coupling is one of the predictions of string theory: the scalar field emerging from this theory, called dilaton field, appears non-minimally coupled to the gravity term [1]. Moreover, being an interesting alternative to the General Relativity theory, the Brans-Dicke theory [2], is based on the non-minimal coupling.

Both frames, the Einstein and Jordan ones, may be connected by a conformal transformation. There is an intensive discussion in the literature about the meaning of such connection, in the sense that it may be just a mathematical mapping of one system into another, or may hide a deep physical meaning, see for example [3] and references therein. For some observables, it is clear that the predictions in one frame are not equivalent to the predictions in the other frame. In quantum cosmology this problem has been treated in references [4, 5], showing that it is possible - even at quantum level - to map the equations in one frame into another frame, but with different predictions to the evolution of the universe.

In references [4, 5] a system consisted of gravity and a scalar field, minimally or non-minimally coupled has been considered. In this case, the analysis present a major challenge, typical of quantum cosmology: how to obtain predictions for the evolution of the universe as function of time? In fact, it is well known that, considering the Einstein-Hilbert action, in quantizing it the resulting equation, the Wheeler-de Witt equation, has no time parameter due to the invariance of the gravitational system under time reparametrization. The introduction of a scalar field does not change this situation. In those references, the *WKB* method has been used in order to obtain predictions for the evolution of the universe from the wavefunction determined from the (timeless) Wheeler-de Witt in the minisuperspace. Concerning this question, see reference [6] and references therein.

Another possibility to obtain the time evolution of a quantum cosmological system in the minisuperspace is to introduce matter in the form of a fluid, employing the Schutz description based in terms of potential that conveys the degrees of freedom of the fluid [9, 10]. It has been shown that such description leads, always in the minisuperspace, to a Schrödinger-like equation, with the degrees of freedom of the matter playing the rôle of time. This proposal has been presented in reference [11], and extensively analysed in reference [12].

Our aim in this work is to address the problem of time evolution of the universe in the presence of a dilaton-like field from a quantum cosmological perspective. Since we are interested mainly in the primordial universe,

a radiative fluid will be introduced. This may allow to recover the time variable through the employment of the Schutz formulation. Performing a conformal transformation (which does not affect the radiative fluid, since it is conformal invariant), we re-write the dilaton-gravity system in the Einstein's frame, from which the Wheeler-de Witt equation in the minisuperspace is constructed, resulting in a Schrödinger-like equation. From this Schrödinger-like equation, an explicit solution is obtained by using a specific ordering factor. A wavepacket is determined, but its norm is time-dependent. Hence, it does not fit in the usual many-worlds interpretation of quantum mechanics [7, 6], but it admits a Bohmian analysis [6, 8]. The Bohmian trajectories contain universes that are singularity-free.

This paper is organized as follows. In the next section, we quantize the dilaton-gravity-radiation system in the mini-superspace. In section 3, a wavepacket is constructed and a (formal) many-world analysis is performed. In section 4 the Bohmian trajectories are studied. In section 5 we present our conclusions.

2 Dilaton-gravity system with radiative fluid

Let us consider the non-minimal coupling of gravity and a scalar field, represented by the Brans-Dicke theory:

$$\mathcal{L} = \sqrt{-\tilde{g}}\phi \left\{ \tilde{R} - \tilde{\omega} \frac{\phi_{;\rho}\phi^{;\rho}}{\phi^2} \right\} + \mathcal{L}_m, \quad (1)$$

where L_m is the matter Lagrangian supposed to be conformal invariant (a radiative fluid). In the strict string dilatonic case we have $\tilde{\omega} = -1$. This defines the theory in the Jordan's frame. Performing a conformal transformation such that $g_{\mu\nu} = \phi^{-1}\tilde{g}_{\mu\nu}$, we transpose the action (1) to the corresponding expression written in the Einstein's frame:

$$\mathcal{L} = \sqrt{-g} \left\{ R - \omega \frac{\phi_{;\rho}\phi^{;\rho}}{\phi^2} \right\} + \mathcal{L}_m, \quad (2)$$

where $\omega = \tilde{\omega} + 3/2$.

Restricting ourselves to the FLRW metric, in an appropriate coordinate system, the line element can be written as

$$ds^2 = N(t)^2 dt^2 - a(t)^2 \gamma_{ij} dx^i dx^j \quad (3)$$

where $N(t)$ is the lapse function, $a(t)$ is the scale factor and γ_{ij} is the induced metric of the homogeneous and isotropic spatial hypersurfaces with curvature $k = 0, \pm 1$. From now on, we will fix $k = 0$. With this metric, the gravitational Lagrangian becomes,

$$\mathcal{L}_G = \frac{V_0 a^3}{N} \left\{ -6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 - \frac{\dot{a} \dot{N}}{a N} \right] - \omega \frac{\dot{\phi}^2}{\phi^2} \right\} \quad , \quad (4)$$

where V_0 is a constant and can be interpreted as the physical volume of the compact universe divided by a^3 . Since we shall have an identical multiplicative constant in front of the matter lagrangian we can drop it from our analysis (this can also be understood as a normalization of the fields). Hence, discarding a surface term, the gravitational Lagrangian can be written as,

$$\mathcal{L}_G = \frac{1}{N} \left\{ 6a\dot{a}^2 - \omega a^3 \frac{\dot{\phi}^2}{\phi^2} \right\}. \quad (5)$$

Defining,

$$\sigma = \sqrt{|\omega|} \ln \phi, \quad (6)$$

we obtain,

$$\mathcal{L}_G = \frac{1}{N} \left\{ 6a\dot{a}^2 - \epsilon a^3 \dot{\sigma}^2 \right\}, \quad (7)$$

where $\epsilon = \pm 1$ according ω is positive (upper sign) or negative (lower sign). The canonical momenta associated with the scale factor and the scalar field are respectively:

$$p_a = 12 \frac{a\dot{a}}{N} \quad , \quad p_\sigma = -2\epsilon \frac{a^3 \dot{\sigma}}{N} \quad . \quad (8)$$

This leads to the following expression in terms of the conjugate momentum:

$$\mathcal{L}_G = p_a \dot{a} + p_\sigma \dot{\sigma} - N \left\{ \frac{1}{24} \frac{p_a^2}{a} - \epsilon \frac{p_\sigma^2}{4a^3} \right\} \quad . \quad (9)$$

Considering a radiative matter component (for the computation of the conjugate momentum associated with the fluid, see references [11, 12]), the total Hamiltonian is:

$$H = N \left\{ \frac{1}{24} \frac{p_a^2}{a} - \epsilon \frac{p_\sigma^2}{4a^3} - \frac{p_T}{a} \right\}. \quad (10)$$

The resulting Schrödinger equation is,

$$-\frac{\partial^2 \Psi}{\partial a^2} + \frac{\epsilon}{a^2} \frac{\partial^2 \Psi}{\partial \sigma^2} = i \frac{\partial \Psi}{\partial T}, \quad (11)$$

where we made the redefinition $\frac{\sigma}{\sqrt{6}} \rightarrow \sigma$ and $\frac{T}{24} \rightarrow T$.

Two questions related to (11) appear immediately. First, what is the sign of ϵ ? If it is positive, we have a hyperbolic Schrödinger equation. This implies that the "energy" E is not positive defined, as it can be seen by multiplying the Schrödinger equation by Ψ^* and integrating in σ and a . Moreover, in this case, the argument of the Bessel function can become imaginary, what may not be very serious, but may pose some problems to construct the wavepacket, since the integration in E must be done along all real axis, and for $E < 0$ the Bessel function becomes modified Bessel function. On the other hand, if the sign is -1 the positivity of E seems to be assured, and the construction of the wavepacket seems not problematic, since the integration on E is done only in the positive semi-axis. For all these reasons, for the moment, we consider the case $\epsilon = -1$. It corresponds to a phantom scalar behaviour induced to the dilaton field.

Second, we have also not taken into account a possible factor ordering. If such factor ordering is introduced, with $\epsilon = -1$, we have the following Schrödinger equation:

$$\frac{\partial^2 \Psi}{\partial a^2} + \frac{p}{a} \frac{\partial \Psi}{\partial a} + \frac{1}{a^2} \frac{\partial^2 \Psi}{\partial \sigma^2} = -i \frac{\partial \Psi}{\partial T} \quad . \quad (12)$$

Even though there is no unique way to chose the factor ordering, a possible consistent and adequate choice is the covariant ordering which is invariant through fields re-definitions. The minisuperspace for our model is two dimensional. Thus, using the covariant factor ordering, we can re-write the Wheeler-DeWitt equation in a Klein-Gordon like equation by defining the minisuperspace metric such that eq. (12) reads

$$\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu) \Psi = -i \frac{\partial \Psi}{\partial T} \quad , \quad (13)$$

where $\mu, \nu = 0, 1$ and the coordinates represents the fields, i.e. $x^0 = a$ and $x^1 = \sigma$. The only possible value for p in eq. (12) which allows the covariant factor ordering is $p = 1$. In this case, we have

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix} \Rightarrow \sqrt{g} = a \quad (14)$$

and the Wheeler-DeWitt equation reads

$$\left[\frac{1}{a} \frac{\partial}{\partial a} \left(a \frac{\partial}{\partial a} \right) + \frac{1}{a^2} \frac{\partial^2}{\partial \sigma^2} \right] \Psi = -i \frac{\partial \Psi}{\partial T} . \quad (15)$$

This equation can be solved using the technique of separation of variables. Hence, through the ansatz

$$\Psi(a, \sigma, T) = \phi(a) e^{ik\sigma} e^{-iET}, \quad (16)$$

we obtain,

$$\phi'' + \frac{\phi'}{a} + \left\{ E - \frac{k^2}{a^2} \right\} \phi = 0, \quad (17)$$

where the prime means derivative with respect to a . Noticing that this is just a Bessel's equation, its solution reads

$$\Psi = A J_\nu(\sqrt{E}a) e^{ik\sigma} e^{-iET}, \quad \nu = k, \quad (18)$$

with A being a normalization constant.

3 Wavepacket and expectation values

Let us construct a wavepacket by choosing conveniently the function $A = A(E, k)$. We have, for (18),

$$\Psi(a, \sigma, T) = \int_0^\infty \int_0^\infty r^{\nu+1} e^{-(\gamma+iT)r^2} e^{-(\alpha-i\sigma)k} J_k(ra) dk dr, \quad (19)$$

where $r = \sqrt{E}$ and α, γ are positive constants. Using the formula (6.631-4) from [13], the final result is:

$$\Psi(a, \sigma, T) = C \frac{e^{-\frac{a^2}{4B(T)}}}{B(T) g_\alpha(a, B, \sigma)}, \quad (20)$$

where

$$B(T) = (\gamma + iT), \quad g_\alpha(a, \sigma, T) = -\alpha + \ln \left[\frac{a}{2B(T)} \right] \pm i\sigma, \quad (21)$$

and C is a normalization constant. Remark that, in order to give physical meaning for this wave packet, the condition $\Re g_\alpha(a, \sigma, T) < 0$ must be satisfied, assuring that the integral in the separation parameter k is convergent.

Let us calculate this normalization constant. Unitarity requires,

$$N = \int \Psi^* \Psi da d\sigma = 1. \quad (22)$$

But,

$$\Psi^* \Psi = C^2 \frac{e^{-\frac{\gamma a^2}{2B^*B}}}{2B^*B} \frac{1}{g^*g}. \quad (23)$$

Remark that,

$$B = \gamma + iT = D e^{i\theta}, \quad (24)$$

where

$$D = \sqrt{\gamma^2 + T^2} = \sqrt{B^*B}, \quad (25)$$

$$\theta = \arctan\left(\frac{T}{\gamma}\right). \quad (26)$$

Hence,

$$g^*g = h^2 + (\sigma - \theta)^2, \quad h = -\alpha + \ln\left(\frac{a}{2(B^*B)^{\frac{1}{2}}}\right). \quad (27)$$

For the scalar field σ and for a we must compute a double integral. In doing so, we must fix the range of σ . In principle, it is from $-\infty < \sigma < +\infty$, implying $0 \leq \phi < +\infty$, that is, a positive gravitational coupling. The norm of the wave function becomes

$$N = \frac{C^2}{(2B B^*)^{1/2}} \int_0^\infty \int_{-\infty}^{+\infty} \frac{e^{-\gamma \frac{a^2}{2B B^*}}}{(2B^*B)^{\frac{1}{2}} [h^2 + (\sigma - \theta)^2]} d\sigma da. \quad (28)$$

Under the substitutions,

$$u = \frac{a}{(2B^*B)^{\frac{1}{2}}}, \quad v = -\theta + \sigma, \quad (29)$$

the integral above becomes,

$$N = \frac{C^2}{(B B^*)^{1/2}} \int_0^\infty \int_{-\infty}^{+\infty} \frac{e^{-\gamma u^2}}{h^2 + v^2} du dv. \quad (30)$$

But,

$$\int_{-\infty}^{+\infty} \frac{dv}{h^2 + v^2} = \frac{\pi}{h}. \quad (31)$$

Hence,

$$N = \frac{C^2}{(B B^*)^{1/2}} \pi \int_0^\infty \frac{e^{-\gamma u^2}}{\alpha + \ln\left(\frac{u}{2}\right)} du = \frac{C^2}{(B B^*)^{1/2}} \pi I_1, \quad (32)$$

where I_1 is the definite integral,

$$I_1 = \int_0^\infty \frac{e^{-\gamma u^2}}{\alpha + \ln\left(\frac{u}{2}\right)} du. \quad (33)$$

The wavepacket determined above is time dependent and does not convey unitarity.

As a formal exercise, even in absence of unitarity we evaluate the expectation value of the scalar field and of the scale factor for this wavepacket:

$$\langle \sigma \rangle_T = \frac{\int_0^\infty \int_{-\infty}^{+\infty} \Psi^* \sigma \Psi da d\sigma}{\int_0^\infty \int_{-\infty}^{+\infty} \Psi^* \Psi da d\sigma}, \quad (34)$$

$$\langle a \rangle_T = \frac{\int_0^\infty \int_{-\infty}^{+\infty} \Psi^* a \Psi da d\sigma}{\int_0^\infty \int_{-\infty}^{+\infty} \Psi^* \Psi da d\sigma}. \quad (35)$$

Let us first evaluate the expectation value for the scalar field. In the numerator of (34), we have:

$$\int_{-\infty}^{+\infty} \frac{\sigma d\sigma}{h^2 + (\sigma - \theta)^2} = \int_{-\infty}^{+\infty} \frac{(v + \theta) dv}{h^2 + v^2}. \quad (36)$$

The first integral is zero, and the second one leads to,

$$\int_{-\infty}^{+\infty} \frac{\sigma d\sigma}{h^2 + (\sigma - \theta)^2} = \theta \int_{-\infty}^{+\infty} \frac{dv}{h^2 + v^2} = \theta \frac{\pi}{h}. \quad (37)$$

Re-inserting in the expression for the scale factor, we find:

$$\langle \sigma \rangle_T = \arctan\left(\frac{T}{\gamma}\right). \quad (38)$$

For the scale factor, integrating in σ we have:

$$\langle a \rangle_T = \pi \int_0^\infty \frac{a}{h} \frac{e^{-2\gamma \frac{a^2}{B^*B}}}{B^*B} da. \quad (39)$$

Performing, as before, the substitution

$$\frac{a}{\sqrt{B^*B}} = \frac{a}{\sqrt{\gamma^2 + T^2}} = u, \quad (40)$$

we obtain:

$$\langle a \rangle_T = \sqrt{\gamma^2 + T^2} \int_0^\infty \frac{u}{h} e^{-2\gamma u^2} du = \frac{I_2}{I_1} \sqrt{\gamma^2 + T^2}, \quad (41)$$

where

$$I_2 = \int_0^\infty \frac{u}{h} e^{-2\gamma u^2} du. \quad (42)$$

Hence,

$$\langle a \rangle_T = a_0(\gamma^2 + T^2)^{1/2}, \quad (43)$$

which is formally the same solution as when there is no scalar field [12]. Remark that the norm of the wavepacket is time dependent and a unitary framework can not be invoked.

The fact that the result for the scale factor is the same, essentially, as when the scalar field is absent may be seen as contradictory. But, we remark that in the Friedmann equation it appears the logarithmic derivative of the scale factor and the derivative of the scalar field. Hence, the presence of the scale field just change the value of a_0 with respect to the case it is absent.

4 Bohmian trajectories

In order to extract some physical insight on the model developed until here, given the nature of the wavepacket employed in the previous section, we will use a Bohmian analysis [6]. Note that the computation of the Bohmian trajectories makes sense even in absence of unitarity, this being one of the main features of the ontological interpretation of quantum mechanics.

The Bohm-de Broglie interpretation is most easily understood through the polar form of the wave function. Indeed, decomposing the wave function

as $\Psi = Re^{iS}$, the Wheeler-DeWitt equation (15) split in two real non-linear equations for the two real function $R(a, \sigma, T)$ and $S(a, \sigma, T)$

$$\frac{\partial R^2}{\partial T} + \frac{1}{a} \frac{\partial}{\partial a} \left(R^2 2a \frac{\partial S}{\partial a} \right) + \frac{1}{a} \frac{\partial}{\partial \sigma} \left(R^2 \frac{2}{a} \frac{\partial S}{\partial \sigma} \right) = 0 \quad (44)$$

$$\frac{\partial S}{\partial T} + \left(\frac{\partial S}{\partial a} \right)^2 + \left(\frac{1}{a} \frac{\partial S}{\partial \sigma} \right)^2 + Q = 0 \quad (45)$$

with

$$Q \equiv -\frac{1}{R} \left[\frac{1}{a} \frac{\partial}{\partial a} \left(a \frac{\partial R}{\partial a} \right) + \frac{1}{a^2} \frac{\partial^2 R}{\partial \sigma^2} \right] \quad (46)$$

Equation (45) is a modified Hamilton-Jacobi equation with the presence of the quantum potential Q . Through this equation, we can identify the momenta as

$$P_a = \frac{\partial S}{\partial a} \quad , \quad P_\sigma = \frac{\partial S}{\partial \sigma} \quad . \quad (47)$$

The metric in the minisuperspace (14) allows to define covariant derivatives, see reference [14]. Therefore, equation (44) represents a continuity equation written in that minisuperspace, with R^2 playing the rôle of a probability density and we can directly read the associated velocity within each parenthesis, i.e.

$$v^\mu = g^{\mu\nu} \frac{\partial S}{\partial x^\nu} \quad \Rightarrow \quad \dot{a} = 2 \frac{\partial S}{\partial a} \quad , \quad \dot{\sigma} = \frac{2}{a^2} \frac{\partial S}{\partial \sigma}$$

The wavefunction can be written as,

$$\begin{aligned} \Psi(a, \sigma, T) &= A \frac{e^{-\frac{a^2}{4B}}}{g B} = A \frac{e^{-\frac{a^2 B^*}{4BB^*}}}{g^* g B^* B} g^* B^* , \\ &= f(a, \sigma, T) e^{iS(a, \sigma, T)} , \end{aligned} \quad (48)$$

with,

$$f(a, \sigma, T) = A \frac{e^{-\frac{a^2 \gamma}{4BB^*}}}{\sqrt{g^* g B^* B}} , \quad (49)$$

$$S(a, \sigma, T) = \frac{a^2 T}{4(\gamma^2 + T^2)} - \arctan\left(\frac{T}{\gamma}\right) + \arctan\left(\frac{-\theta + \sigma}{h}\right) . \quad (50)$$

Remembering that,

$$h = -\alpha + \ln\left(\frac{a}{2\sqrt{\gamma^2 + T^2}}\right) = \Re g_T(a, \sigma). \quad (51)$$

and that, for a radiative fluid $N = a$ in (8) [12], taking into account the time and field redefinitions, we have the following expressions for the Bohmian trajectories:

$$\dot{a} = \frac{aT}{(\gamma^2 + T^2)} + \frac{2}{a} \frac{\theta - \sigma}{h^2 + (\theta - \sigma)^2}, \quad (52)$$

$$\dot{\sigma} = \frac{2}{a^2} \frac{h}{h^2 + (\theta - \sigma)^2}, \quad (53)$$

where θ and h are given by (26,51). These equations must be integrated numerically⁶.

The numerical solutions of (52,53) provide an interesting perspective depending on the initial conditions and on the free parameters of the model, namely γ and α . We can show that there is at least a class of non-singular solutions, satisfying the wavepacket conditions of consistency. More precisely, in figure 1, the a non-singular solution is displayed. The function h is also plotted, showing that the condition for the convergence of the wavepacket is satisfied. In the same figure we plot the results obtained use the expectation value of the previous section, and the numerical solutions for the Bohmian trajectories. The Bohmian trajectories predict an asymmetric bounce, while the expectation value display a perfect symmetric bounce.

5 Conclusions

The possibility to make predictions and subsequently extract a family of solutions (trajectories in the minisuperspace) is a recurrent problem in quantum cosmology. Although the well-known *WKB* method, including decoherence features, has provided a vast range of particular results, there is an alternative

⁶It can already be stated that the solutions are not equivalent to the expectation values found before, a consequence of the absence of unitarity: when the wavepacket is unitary, the Bohmian trajectories reproduces the expectation values.

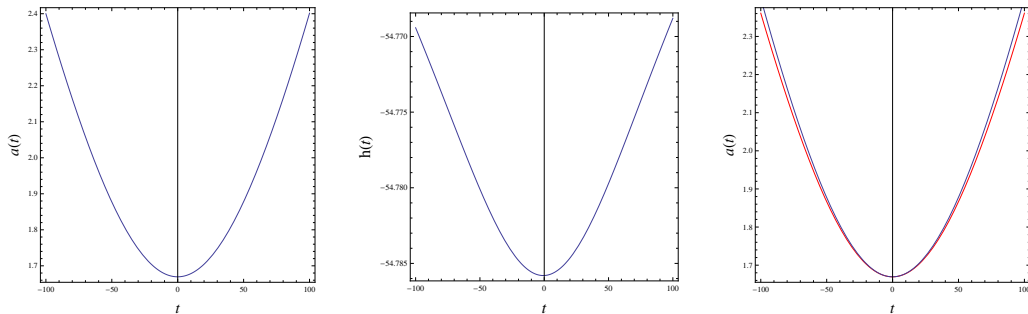


Figure 1: In the left panel, the behaviour of the scale factor for $\gamma = 100$ and $\alpha = 50$ is shown. In the central panel, the function h is displayed. In the right panel, the solution obtained using the Bohmian trajectories (blue) is compared with the solution using the expectation value (red).

framework by means of transforming the WdW equation into a Schrödinger equation, with the time variable induced by a matter component [9, 10, 11]. Quite interesting results have appeared in the literature [12] using this procedure.

In this paper, we investigated how we could obtain some predictions with that alternative framework, using a system characterized by a dilaton field and radiation expressed using the Schutz variables. The dynamics of the radiative fluid implies a time variable.

We found that, in order the Schrödinger equation to be elliptic, leading to a positive energy spectrum, the dilaton field must have a phantom behaviour. On the other hand, the construction of a quasi gaussian superposition, conjugated with some convergence criteria, lead to a wavepacket with a time dependent norm and hence unitarity could not be invoked. Nevertheless, this still allows us to investigate this wavepacket under a Bohmian perspective. Interestingly, we found a bouncing behaviour for the scale factor, i.e., the singularity is avoided. In this sense we extend the results of reference [15], where the imposition of unitarity led to a conclusion that the dilaton field should be constant. At same time, such approach open new perspective concerning anisotropic quantum models [16].

There are still many open directions to explore in this program. In particular, we must look for, (i) other wavepackets, presumably by using numerical methods, (ii) study the dynamics in the Jordan frame. We hope to address these problems in a future work.

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