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# DYNAMICS OF THE INTERACTION BETWEEN CLASSICAL AND QUANTUM SYSTEMS AND THE REDUCTION OF THE WAVE PACKET

by

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ABSTRACT

A variational form for the dynamics of a classical system interacting with a quantum one is proposed as a nonlinear generalization of the nonrelativistic quantum theory. In addition to the relevance of this kind of problem for the calculation of nuclear models in which some degrees of freedom are treated quantum-mechanically and others classically, the equations derived in this paper are designed to describe the reduction of the wave packet.

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The interaction between quantum and classical systems, fundamental in order to understand the quantum measurement problem[1,2], has been widely studied in the literature[3,4,5,6]. On the other hand, there is a variety of physical problems that claims for a combined quantum-classical treatment: quantizations of solitons, theories of nuclear fission, heavy ion scattering, quantum corrections to classical radiation, semiclassical theory of gravitation, and others. Partially motivated by some of these problems and studying the transitions from one mechanics to the other, Shirokov[4] developed a combined algebra for quantum and classical mechanics and showed that an unified formulation based on it is possible. Interested in the semi-classical theory of gravity, Kibble and collaborators[6] (see also ref. (7)) derived a time-dependent Schrödinger equation coupled to the classical Einstein's equation from an action principle. In this letter we will obtain from a variational principle, the dynamic equations for the interaction between classical and quantum sub-systems, irrespectively of any particular model. The obtained equations describe a kind of classical-quantum dynamics (cq-dynamics). An alternative derivation of these equations was already obtained by us in a previous work[8]. Here our main purpose is to explore these coupled equations, describing the interaction between system (quantum) and apparatus (classical), in order to avoid the well know "collapse of the state vector" or "reduction of the wavepacket" arising from the Von-Neumann[1] theory of measurement.

First one defines a whole system, which will be called classical-quantum system (cq-system) throughout this paper, constituted by two sub-systems in interaction, one being classical with R degrees of freedom and the other quantum with N degrees of freedom. The classical sub-system is described by R classical variables (c-numbers) and the quantum one is given by a complete set of basis states, which in the absence of the c-number variables are the eigenstates of the time-independent Hamiltonian operator

$$\hat{H}_q = \hat{H}_q(\hat{x}^j, \hat{p}_j) \quad , \tag{1}$$

where $\hat{x}^j\,(j=1,2,\ldots,N)$ represents the coordinate operators of the N quantum degrees of freedom (q-numbers) and $\hat{p}_j = -i\hbar \frac{\partial}{\partial x^j}$.

It is well-known that the fundamental equation of a non-relativistic quantum dynamics may be established from the action integral

$$A[\phi] = \int_{t_1}^{t_2} dt \langle \phi(t) | i\hbar \frac{\partial}{\partial t} - \hat{H}_q(\hat{x}^j, \hat{p}_j) | \phi(t) \rangle \tag{2}$$

by means of the variational principle $\delta A = 0$, in which the variation is assumed to be zero at the limits, and provided that the normalization of state vectors $|\phi(t)\rangle$ is maintained.

Second, in order to characterize the dynamic of the cq-system, we define an extended Hamiltonian operator

$$\hat{H} = \hat{H}(\hat{x}^j, \hat{p}_j; \alpha^J, \dot{\alpha}^J) \tag{3}$$

where the c-number coordinates, as well as their respective velocities $\dot{\alpha}^J = \frac{d\alpha^J}{dt}$ take the role of continuous parameters. From

(3) we construct the following extended action integral

$$A\left[\psi,\alpha^J\right] = \int_{t_1}^{t_2} dt \langle\psi| i\hbar \frac{\partial}{\partial t} - \hat{H}(\hat{x}^j,\hat{p}_j;\alpha^J,\dot{\alpha}_J)|\psi\rangle \qquad (4)$$

where $|\psi\rangle$ is the state vector of the cq-system and Dirac's bracket symbol means integration over the configurational space $\{x^j\}$ only. Requiring as before that $\delta A = 0$, but splitting the variation symbol into two independent parts

$$\delta = \delta_q + \delta_c \qquad (5)$$

where $\delta_q$ represents a variational operator due to the transformation of the state vectors with c-number coordinates being held fixed, and $\delta_c$ represents a variation due to the transformation of the c-number coordinates freezing the functional space of the state vectors, we obtain from (4) and (5)

$$\delta A = \int_{t_1}^{t_2} dt \left[\delta_q \langle\psi| i\hbar \frac{\partial}{\partial t} - \hat{H}|\psi\rangle + \delta_c \langle\psi| i\hbar \frac{\partial}{\partial t} - \hat{H}|\psi\rangle\right] = 0 \qquad (6)$$

and therefrom

$$\int_{t_1}^{t_2} dt \left[\langle\delta_q\psi| i\hbar \frac{\partial}{\partial t} - \hat{H}|\psi\rangle + \langle\psi| i\hbar \frac{\partial}{\partial t} - \hat{H}|\delta_q\psi\rangle - \delta_c\langle\psi|\hat{H}|\psi\rangle\right] = 0 \quad (7)$$

In order to satisfy this variational condition for all c-number variables $\alpha^J$, state vectors $\psi$ and their corresponding complex conjugates, the above condition must be satisfied by each of the three terms separately. The formal rules for computing variations provide that the first two terms of Eq. (7) go over to the time-dependent Schrődinger equation and its complex conjugate with the extended Hamiltonian (3). The last term in Eq. (7) leads to

$$\int_{t_1}^{t_2} dt \sum_J \left[ \frac{\partial H'}{\partial \alpha^J} - \frac{\partial}{\partial t}\left(\frac{\partial H'}{\partial \dot{\alpha}^J}\right) \right] \delta_c \alpha^J - \sum_J \frac{\partial H'}{\partial \dot{\alpha}^J} \delta_c \alpha^J \Big|_{t_1}^{t_2} = 0 \qquad (8)$$

where $H' \equiv H'(\alpha^J, \dot{\alpha}^J) = \langle\psi|\hat{H}|\psi\rangle$. The fixed endpoint conditions, $\alpha^J(t_1) = \alpha^J(t_2) = 0$, make null the last term of Eq. (8). As usual, the coefficients of $\delta_c \alpha^J$ in this last term will be called generalized mementum $p_J$,

$$p_J = \frac{\partial H'}{\partial \dot{\alpha}^J} = \frac{\partial}{\partial \dot{\alpha}^J} \langle\psi|H|\psi\rangle \qquad (9)$$

Therefore, the variational principle $\delta A = 0$, with Eq. (4) and Eq. (5), is equivalent to the following system of coupled equations:

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle \qquad (10)$$

and

$$\left[ \frac{\partial}{\partial \alpha^J} - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\alpha}^J}\right) \right] \langle\psi|H|\psi\rangle = 0 \qquad (11)$$

with $\langle x^j|\psi\rangle = \psi(x^j; \alpha^J, \dot{\alpha}^J, t)$.

The self-consistent coupled equations (10) and (11) are the variational forms searched for, which are a time-dependent Schrődinger equation coupled, by means of the extended Hamiltonian operator $\hat{H}$, to an Euler-Lagrange equation where the quantity

$<\psi|\hat{H}|\psi>$ plays the role of the classical Lagrangian.

It should be emphasized that, although the Schrödinger equation (10) alone is linear, it appears here as intrinsically nonlinear. This is so because, in equation (10), the state vectors $|\psi>$ depend on the c-number parameters which in turn depend on them, according to equation (11). In consequence, the superposition principle is violated. Extra explicit nonlinearities can be obtained by adding terms to the action integral (4), cf. Kibble[6,7]. Specific solutions of these equations will be examined in a paper to come.

In order to study the consistence of these equations, it is convenient to introduce a model Hamiltonian

$$\hat{H}(\hat{x}^j,\hat{p}_j;\alpha^J,\dot{\alpha}^J) = H_c(\alpha^J,\dot{\alpha}^J) + H_q(\hat{x}^j,\hat{p}_j) + \hat{V}(\hat{x}^J;\alpha^J,\dot{\alpha}^J) \qquad (12)$$

that separates the extended Hamiltonian (3) into a c-number part $H_c(\alpha^J,\dot{\alpha}^J)$ and q-number part $H_q(\hat{x}^j,\hat{p}_j)$ whith the coupling between them described by the interaction operator $\hat{V}=\hat{V}(\hat{x}^j;\alpha^J,\dot{\alpha}^J)$.

Usually in many nuclear problems, specially in heavy ion collision as was already noticed by Liran et al.[9], it is presupposed, in disagreement with equations (10) and (11), that there is no reaction of the quantum on the classical dynamics and all the influence of the classical subsystem on the quantum subsystem is produced directly, via the interaction potential $\hat{V}(\hat{x}^j;\alpha^J,\dot{\alpha}^J)$, imposed a priori. Such procedure is equivalent to considering only the influence of the c-number dynamic over the q-number one without taking into account the corresponding reaction in a self-consistent way. In that case, the classical time dependent variables $\alpha^J(t)$ are actually prescribed in time.

In the limit case, when $\hat{V}(\hat{x}^j;\alpha^J,\dot{\alpha}^J) = 0$, separating the c and q variables

$$\psi(x^j;\alpha^J,\dot{\alpha}^J,t) = f(\alpha^J,\dot{\alpha}^J)\phi(x^j,t) \quad , \tag{13}$$

we get from (10), (11) and (12) two uncoupled equations,

$$i\hbar \frac{\partial\phi}{\partial t}(x^j,t) = H_q(\hat{x}^j,\hat{p}_j)\phi(x^j,t) + \lambda\phi(x^j,t) \tag{14}$$

and

$$\left[\frac{\partial}{\partial\alpha^J} - \frac{d}{dt}\left(\frac{\partial}{\partial\dot{\alpha}^J}\right)\right] |f(\alpha^J,\dot{\alpha}^J)|^2 = 0 \quad , \tag{15}$$

one describing the quantum susbsystem and the other the classical one, as expected. The $\lambda$ in equation (14) is a constant, arising from the method of separation of variables, that must be defined by the boundary conditions.

This $\lambda$ parameter, a natural consequence of the classical-quantum dynamics in the limit $\hat{V} = 0$, has a very simple physical interpretation, it describes the stationary ($\lambda = 0$) or the decay (imaginary $\lambda$) characters of the quantum subsystem.

The information that we can obtain from the microworld, regarded as subject to quantum laws, are always given by means of macroscopic instruments, which are described by classical laws. The description of possibles correlations between the state of the observed system and the instruments is the main feature of the measurement problem of the quantum theory. Thus we expect that the kind of formulation developed in this letter, in which the quantum and classical systems are considered in the same footing, would be one way to avoid the measurement problem. For this it is enough to consider, in disagreement with the Copenhagen interpretation, where both system and apparatus are regarded as subject to quantum laws, that the micro-system is described by the q-number variables and the apparatus by the c-number variables with the dynamical interaction between them described by the coupled equations (10) and (11). The purely quantum or the purely classical description are to be considered as the two limit of the cq-dynamics given by equations (14) and (15). The question of how select the degrees of freedom which must be considered as classical or quantum is to be solved solely by the experience. One may continue to interpret the absolute square of the wave function as the probability of findind the localized particle at a given point, but now this probability distribution has naturally become observer dependent, that is, a function of $\alpha^J$. Frederick[10] has shown that this dependence would eliminate the acausal behaviour of the collapse of the wave function.

All this is consistent with the fact, already being claimed by others[11], that the description of the dynamical reduction of the state vector could be obtained through a non linear generalization of the wave equation.

In summary from a variational principle a system of coupled equations is obtained which treats the classical and quantum systems in interaction in a self-consistent manner. This system of coupled equations goes over to the usual Schrödinger and Euler-Lagrange equations when no interactions is present between them. It seems that these coupled equations are interesting for the calculation of nuclear models which can be characterized by dynamic interaction between collective, classically treated degrees of freedom and intrinsic, quantically treated ones. Finally, we expect that this kind of formulation clarifies the dynamics inherent in the quantum measurement problem.

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