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THE POTTS MODEL AND FLOWS.

II. MANY-SPIN CORRELATION FUNCTION

by

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ABSTRACT

A graph theoretic analysis is made of the m-spin correlation functions of the λ-state Potts model. In paper I, the correlation functions for m=2 were expressed in terms of rooted mod-λ flow polynomials. Here we introduce a more general type of polynomial , the partitioned m-rooted flow polynomial , which plays a fundamental role in the calculation of the multi-spin correlation functions. The m-rooted equivalent transmissivities of Tsallis and Levy are interpreted in terms of percolation theory and are expressed as linear combinations of the above correlation functions.

Key-words: Potts model; Many-spin correlation function; Graph theory; Statistical mechanics.

1 INTRODUCTION

Since its formulation (Potts 1952), the λ-state Potts model has been extensively studied (for a review of the subject see Wu 1982). An interesting and widely quoted probabilistic interpretation of this model has been given (Kasteleyn and Fortuin 1969, Wu 1978) which places the Potts model in the context of percolation theory. This relation has advantages in both directions. On the one hand it allows the powerful techniques developed in the theory of thermal critical phenomena to be used in the study of percolation problems. On the other hand it allows the geometric insights gained from the study of percolation theory to become useful in understanding critical phenomena in general.

Kasteleyn and Fortuin (1969) used a variable p_e, directly related to the interaction parameter, which is the probability that the edge e is present in the corresponding percolation model. Recently Essam and Tsallis (1985, paper I of this series which will herein be referred to as PF1) showed that, by using a different but related variable t_e, the connection with percolation theory could be maintained but in addition their interpretation related the partition function and the spin pair correlation function to the expected number of mod-λ flows. Thus the geometric intuition associated with the motion of fluid in a network could be brought to bear on the Potts model. The variable t_e is the "thermal transmissivity" of the edge e used by Tsallis and Levy (1981) in their real space renormalisation group studies (see also Yeomans and Sthinchcombe 1980); t is also the w-variable of Domb (1976). It was also shown in PF1 that for any spin cluster with graph G, the correlation function between spins s_1 and s_2 is proportional to the "equivalent transmissivity" $t_{12}^{eq}(t,G)$ between roots 1 and 2 of G which was so important in the development of real space renormalisation group methods for the Potts model (Tsallis and Levy 1981; for a review see Tsallis 1985).

The "equivalent transmissivity" is the ratio of two multi-linear forms in the t_e variables, a numerator $N_{12}(t,G)$ and a denominator $D(t,G)$ which is proportional to the partition function $Z(t,G)$. For $G' \subseteq G$ the coefficient of $\prod_{\ell \in G'} t_e$ in $D(t,G)$ was shown (PF1) to be equal to the number of proper mod-λ flows on G' which is a polynomial in λ called the flow polynomial $F(\lambda,G')$ (Tutte 1954, Rota 1964, Tutte 1984). Thus $D(t,G)$ is the generating function for $F(\lambda,G')$. Similarly $N_{12}(t,G)$ is the generating function for the polynomials $F_{12}(\lambda,G')$, $G' \subseteq G$, called rooted flow polynomials in PF1. $F_{12}(\lambda,G')$ is the number of proper mod-λ flows on G' in the presence of fixed non-zero "external" flow between the roots. $F(\lambda,G')$ and $F_{12}(\lambda,G')$ are both topological invariants and $F_{12}(\lambda,G')$ satisfies the same "deletion-contraction rule" as was derived for $F(\lambda,G')$ by Tutte (1954). The latter rule was shown in PF1 to lead to the "break-collapse equation" conjectured by Tsallis and Levy (1981) for $t_{12}^{eq}(t,G)$.

In the present paper we extend the results of PF1 for the spin pair correlation function to the correlation function $\Gamma_{12\ldots m}(G)$ among the components of m spins along one of the λ

special directions in which the spins are allowed to point. $\Gamma_{12}(G)$ and $\Gamma_{123}(G)$ appear in the field theoretic formulation of the renormalisation group equations for the Potts model (Amit 1976). An extension of $t_{12}^{eq}(t,G)$ to m-rooted graphs ($m \geq 2$) has been introduced by Tsallis and Levy (1981) and $t_{123}^{eq}(t,G)$ has been used in real space renormalisation group calculations on the triangular and honeycomb lattices (Tsallis and Levy 1981, de Magalhaes et al 1982, Tsallis and dos Santos 1983). Here we prove that $t_{123}^{eq}(G)$ is proportional to $\Gamma_{123}(G)$, however for $m \geq 4$ no such simple relation has been found.

It turns out that $\Gamma_{12...m}(G)$ is a linear combination of the equivalent transmissivities $t_{\mathbb{P}}^{eq}(G)$ corresponding to all possible partitions P of the m roots into blocks. This relation (see eq.3.18a) is one of the main results of the present paper. We call $t_{\mathbb{P}}^{eq}(G)$ a partitioned m-rooted equivalent transmissivity. The equivalent transmissivity $t_{12...m}^{eq}(G)$ is the special case of $t_{\mathbb{P}}^{eq}(G)$ in which $\mathbb{P}$ has only one block. In the limit $\lambda \to 1$, $t_{\mathbb{P}}^{eq}(G)$ becomes the probability $C_{\mathbb{P}}(G)$ that the roots 1,2,...,m of G are connected in blocks according to the partition $\mathbb{P}$ in a bond percolation process. In other words, roots which belong to the same block of $\mathbb{P}$ are connected by an open path (i.e. a path formed by edges which are present) while roots which belong to different blocks are not connected in this way. $C_{12}(G)$ is the usual pair-connectedness function (Essam 1972) and we call $C_{\mathbb{P}}(G)$ the partitioned m-rooted connectedness. A similar partitioning also appears in the transfer matrix formulation of the Potts model (Blöte and Nightingale 1982).

In order to express $t_{\mathbb{P}}^{eq}(G)$ in terms of correlation funtions it is necessary to introduce corresponding partitioned correlation functions $\Gamma_{\mathbb{P}}(G)$ involving spin components along several of the different special directions (one direction for each block). For given m, we show (eqs. (3.41a) and (3.44) together) that $\Gamma_{\mathbb{P}}(G)$ is a linear combination of the $t_{\mathbb{P}'}^{eq}(G)$, $\mathbb{P}' \in \mathcal{P}(M)$ where $\mathcal{P}(M)$ is the set of all partitions of $M=\{1,2,\ldots,m\}$. For m=4 these relations are explicitly inverted (eqs. (3.46) and (3.40a)) to give $t_{\mathbb{P}}^{eq}(G)$ as a linear combination of the $\Gamma_{\mathbb{P}'}(G)$, $P' \in \mathcal{P}(M)$. We expect that the relations for $m > 4$ are also invertible and this provides an interpretation of the m-rooted equivalent transmissivities of Tsallis and Levy (1981) in terms of the more usual spin correlation functions. In papers III and IV (de Magalhães and Essam 1985a, b, which we shall refer to as PF3 and PF4 respectively) we develop powerful techniques which enable the computation of $t_{\mathbb{P}}^{eq}(G)$ for large clusters. The above relations enable the partitioned correlation functions to be found once the equivalent transmissivities have been calculated.

All the partitioned equivalent transmissivities for a given graph G may be expressed as the ratio of two bond percolation averages having a common denominator D(G) and a numerator $N_{\mathbb{P}}(G)$ depending on the partition (see eq.(3.18b)). As previously stated, the denominator when expressed as a multi-linear form in the t_e variables is the generating function for the flow polynomials of its subgraphs. The coefficients in the corresponding multi-linear expansion of $N_{\mathbb{P}}(t,G)$ (eq.(3.23)) are polynomials in λ, $F_{\mathbb{P}}(\lambda,G')$, having all of the properties of flow polynomials (see §4.3) and we shall call them partitioned m-rooted flow polynomials. Similarly to $F_{12}(\lambda,G')$, we show that $F_{123}(\lambda,G')$ is the number of proper mod-λ flows on G' given that there are two fixed non-zero external flows (between the pairs 1, 2 and 1, 3 say). Examples of $F_{12}(\lambda,G)$ and $F_{123}(\lambda,G)$ may be seen in Amit's paper (1976), but the interpretation in terms of flows seems not to have been recognised. For $m \geqslant 4$ it appears that there is no similar combinatorial problem which determines $F_{\mathbb{P}}(\lambda,G')$. However we shall show that, for general m, the partitioned flow polynomials may be expressed as linear combinations of unrooted flow polynomials.

The $F_{\mathbb{P}}(\lambda,G)$ will play a vital role in the development of formulae for the calculation of partitioned equivalent transmissivities (and hence correlation functions) of large networks by decomposition (or construction from) smaller parts (PF3 and PF4). They are, of course, also important in the derivation of series expansions for correlation functions of lattice systems. Finally, $F_{\mathbb{P}}(\lambda,G)$ evaluated at $\lambda=1$ will be called the $d_{\mathbb{P}}$-weight of G since it is a generalisation of the d-weights which arise in the pair-connectedness of bond percolation theory (Essam 1971b).

All our results concerning $\Gamma_{12...m}(G)$, and more generally $\Gamma_{\mathbb{P}}(G)$, are expressed in a general variable $t_e(\mu)$ which contains the p_e and t_e variables as particular cases. The multilinear forms of D(G) and $N_{\mathbb{P}}(G)$ in the p_e variables are the generating functions for the chromatic polynomials of graph theory $P(\lambda,G')$ (see, for example, Tutte 1984) and what we call the partitioned m-rooted chromatic polynomials $P_{\mathbb{P}}(\lambda, G')$ respectively. These polynomials in λ, unlike $F(\lambda, G')$ and $F_{\mathbb{P}}(\lambda, G')$, are not topological invariants and they do not vanish for graphs with "dangling ends". This is one of the disadvantages of the use of the p-variable. In fact, we show that the t-variable is more convenient than the p-variable in many respects.

The plan of this paper is as follows. First (section 2) we define the model, review the known results for the partition function and two-rooted functions previously obtained in the p and t-variables (Kästeleyn and Fortuin 1969 and PF1 respectively), and show that they can be derived simultaneously using the general $t(\mu)$-variable. In section 3, we derive two alternative expressions for $\Gamma_{12\ldots m}(G)$ (and $\Gamma_{\mathbb{P}}(G)$) which are then expanded in the $t_e(\mu)$ variables. In section 4, we study other quantities related to $F_{\mathbb{P}}(\lambda, G)$ and show that a knowledge of the $F_{\mathbb{P}}$'s and F's allows the calculation of $Z(G)$, $\Gamma_{12\ldots m}(G)$ (and $\Gamma_{\mathbb{P}}(G)$) in both the t and p-variables. The properties of $F_{\mathbb{P}}(\lambda, G)$ are also derived in this section. The resulting properties of $N_{\mathbb{P}}(t, G)$ and hence $t_{\mathbb{P}}^{eq}(t, G)$ are derived in section 5 and the corresponding properties in the p-variable are also given.The advantages of the t over the p-variable are given in section 6. Finally the limiting case of bond percolation ($\lambda \to 1$) is studied in section 7.

2 MODEL AND REVIEW OF KNOWN RESULTS

2.1 Partition Function expressed in the t and p variables

Let us consider a graph G with vertex set V and edge set E. We shall denote by $|V|$ and $|E|$ the numbers of vertices and edges of G respectively; in general we shall denote by $|\Omega|$ the number of elements in a given set Ω.

Let us associate with each vertex i of V a spin vector s_i of length s which can take on one of λ values e_α ($\alpha = 1,2,..,\lambda$) which are the position vectors of the corners of a $(\lambda-1)$ dimensional hypertetrahedron relative to its centre.

The hamiltonian of the λ-state Potts model associated with a graph G can be written in terms of these spin vectors as:

$$\mathcal{H}(G) = - \sum_{e \in E} J_e \, \vec{s}_i \cdot \vec{s}_j \tag{2.1}$$

where J_e is the coupling constant between the spins $\mathbf{s}_i$ and $\mathbf{s}_j$ associated with the edge e. The sum in eq (2.1) includes all the interacting spin pairs on G.

The partition function Z(G) associated with the graph G is defined by

$$Z(G) \equiv \mathrm{tr}_G \left\{ \exp \left[\sum_{e \in E} K_e \, \vec{s}_i \cdot \vec{s}_j \right] \right\} \tag{2.2a}$$

with

$$K_e \equiv J_e \beta \qquad\qquad (\beta \equiv 1/k_B T) \tag{2.2b}$$

where tr_G means sum over all positions of the spin vectors $\mathbf{s}_i$, $i \in V$.

Let G' be a partial graph of G, i.e., a subgraph of G with vertex set V and edge set $E' \subseteq E$ and let $Q(G')$ be a function defined on the set of such subgraphs. For example, $Q(G')$ could be the number of components, $\omega(G')$, or the number of independent cycles, $c(G')$, in G'. Suppose also that $u_e (e \in E)$ is a function defined on the edge set E of G, and denote the edge set of the complement of G' with respect to G by $E \backslash E'$, then we call

$$\langle Q \rangle_{G,u} = \sum_{G' \subseteq G} Q(G') \prod_{e \in E'} u_e \prod_{e \in E \backslash E'} (1 - u_e) \tag{2.3}$$

the percolation average of Q relative to the u-variable. For $0 \leq u_e \leq 1$ ($\forall e \in E$), this quantity is indeed the average over all configurations in a bond percolation model in which the edge e has probability u_e of being present independently of all other edges (see, for example, Wu 1978).

If $s^2 = \lambda - 1$ and

$$p_e = 1 - e^{-\lambda K_e} \tag{2.4}$$

then $Z(G)$ can be expressed as (Kasteleyn and Fortuin 1969):

$$Z(p, G) = \left\{ \prod_{e \in E} A_e^{(p)} \right\} \langle \lambda^{\omega} \rangle_{G,p} \tag{2.5a}$$

where the symbol p represents the vector $(p_1, p_2, \cdots, p_{|E|})$ and

$$A_e^{(p)} \equiv e^{(\lambda - 1) K_e} \tag{2.5b}$$

In PF1 the variable

$$t_e = \frac{1 - e^{-\lambda K_e}}{1 + (\lambda - 1) e^{-\lambda K_e}} \tag{2.6}$$

was used instead of p_e and it was found, for $s^2=\lambda-1$, that

$$Z(t,G) = \lambda^{|V|} \left\{ \prod_{\ell\in E} A_\ell^{(t)} \right\} \langle \lambda^c \rangle_{G,t} \tag{2.7a}$$

where the symbol t represents $(t_1,t_2,\ldots,t_{|E|})$ and

$$A_\ell^{(t)} = \frac{1}{\lambda} \left\{ e^{(\lambda-1)K_\ell} + (\lambda-1)\, e^{-K_\ell} \right\}. \tag{2.7b}$$

The variable t_e was also used by Domb(1974) in a series expansion analysis of the partition function. t_e appeared also in many real space renormalisation group calculations (Yeomans and Stinchcombe 1980, Tsallis and Levy 1981, etc) and was called by Tsallis and Levy (1981) the "thermal transmissivity" associated with edge e.

The averages $\langle \lambda^\omega \rangle_{G,p}$ and $\langle \lambda^c \rangle_{G,t}$ may be interpreted(PF1) in terms of expected numbers of colourings and flows respectively. The fact that the number of flows is a topological invariant was shown in PF1 to lead to considerable simplification in the coefficients of the multi-linear form of $\langle \lambda^c \rangle_{G,t}$.

We shall find in our analysis of the correlation functions that there is a parallelism between formulae using the t-variable and the corresponding formulae using the p-variable. This can be seen already in expressions (2.5a) and (2.7a) for Z(G) and we now show that these expressions have a common origin.

Using (3.2) of PF1 it is easily seen that for arbitrary μ :

$$e^{K_\ell\, \vec{e}_{\alpha_i}\cdot \vec{e}_{\alpha_j}} = A_\ell(\mu)\left[\mu\, t_\ell(\mu)\, \delta(\alpha_i,\alpha_j) + (1-t_\ell(\mu))\right] \tag{2.8a}$$

where

$$t_\ell(\mu) = \frac{1 - \exp\left[-\lambda K_\ell s^2/(\lambda-1)\right]}{1 + (\mu-1)\exp\left[-\lambda K_\ell s^2/(\lambda-1)\right]} \tag{2.8b}$$

and

$$A_\ell(\mu) = \frac{1}{\mu}\left\{ e^{K_\ell s^2} + (\mu-1)\, e^{-K_\ell s^2/(\lambda-1)} \right\} . \tag{2.8c}$$

Now

$$Z(G) = \sum_{\alpha_1=1}^{\lambda} \cdots \sum_{\alpha_{|V|}=1}^{\lambda} \prod_{\ell\in E} e^{K_\ell \vec{e}_{\alpha_i}\cdot \vec{e}_{\alpha_j}} \tag{2.9}$$

and if we substitute eq.(2.8a) into eq.(2.9) and expand the product we obtain:

$$Z(t(\mu),G) = \left(\prod_{\ell\in E} A_\ell(\mu)\right) \sum_{\alpha_1=1}^{\lambda} \cdots \sum_{\alpha_{|V|}=1}^{\lambda} \left\{ \sum_{G'\subseteq G} \prod_{\ell\in E'} \left[\mu t_\ell(\mu) \times \delta(\alpha_i,\alpha_j)\right] \prod_{\ell\in E\setminus E'} \left[1 - t_\ell(\mu)\right] \right\} . \tag{2.10}$$

Since the δ functions impose the condition that two spins linked by an edge have the same state variable α, we get:

$$Z(t(\mu),G) = \left(\prod_{\ell\in E} A_\ell(\mu)\right) \sum_{G'\subseteq G} \lambda^{w(G')} \mu^{|E'|} \prod_{\ell\in E'} t_\ell(\mu) \prod_{\ell\in E\setminus E'} \left[1 - t_\ell(\mu)\right] . \tag{2.11}$$

Now by Euler's law (Harary 1969)

$$c(G') = |E'| - |V| + w(G') \tag{2.12}$$

and hence

$$Z(t(\mu),G) = \mu^{|V|}\left(\prod_{\ell\in E} A_\ell(\mu)\right) \left\langle \left(\frac{\lambda}{\mu}\right)^{\omega} \mu^{c} \right\rangle_{G,t(\mu)} . \qquad (2.13)$$

Notice that if $s^2 = \lambda - 1$, then, from eq.(2.8b), it follows that $t_e(1) = p_e$ and $t_e(\lambda) = t_e$,and, from eq.(2.8c), that $A_e(1) = A_e^{(p)}$ and $A_e(\lambda) = A_e^{(t)}$. Therefore eqs (2.5a) and (2.7a) correspond to the respective cases $\mu = 1$ and $\mu = \lambda$ of eq. (2.13).

Expansion of the product over $(1-t_e(\mu))$ in the average $\langle(\lambda/\mu)^{\omega}\mu^{c}\rangle_{G,t(\mu)}$ leads to:

$$D(t(\mu),G) \equiv \langle(\lambda/\mu)^{\omega}\mu^{c}\rangle_{G,t(\mu)} = \sum_{G'\subseteq G} F(\mu,\lambda,G') \prod_{\ell\in E'} t_\ell(\mu) \qquad (2.14a)$$

with

$$F(\mu,\lambda,G') = \sum_{G''\subseteq G'} (-1)^{|E'\setminus E''|} \left(\frac{\lambda}{\mu}\right)^{\omega(G'')} \mu^{c(G'')} \qquad (2.14b)$$

where G" is a partial graph of G′. Throughout this paper we shall assume that the superscript " ′ " on any graph refers to any of its partial graphs.

Eq. (2.14a) reduces for $\mu = \lambda$ and $s^2 = \lambda - 1$ to the multilinear form of D(t,G) obtained in PF1, namely

$$D(t,G) \equiv \langle \lambda^{c} \rangle_{G,t} = \sum_{G'\subseteq G} F(\lambda,G') \prod_{\ell\in E'} t_\ell \qquad (2.15)$$

where $F(\lambda,G)$ is the flow polynomial of G(see PF1 and references therein) which is given by:

$$F(\lambda,G) = \sum_{G'\subseteq G} (-1)^{|E\setminus E'|} \lambda^{c(G')} \quad . \tag{2.16}$$

In the case of $\mu=1$ and $s^2=\lambda-1$, eq. (2.14a) becomes:

$$D(p,G) \equiv \langle \lambda^{\omega} \rangle_{G,p} = \sum_{G'\subseteq G} (-1)^{|E'|} P(\lambda,G') \prod_{\ell\in E'} p_\ell \tag{2.17}$$

where $P(\lambda,G)$ is given by:

$$P(\lambda,G) = \sum_{G'\subseteq G} (-1)^{|E'|} \lambda^{\omega(G')} \quad . \tag{2.18}$$

Eq. (2.18) is a well known formula (Birkhoff 1912) for the chromatic polynomial $P(\lambda,G)$ of the graph G with λ colours. It represents the number of ways in which we can colour the vertices of G with λ colours in such a way that no two adjacent vertices are coloured alike.

2.2) Pair Correlation Function in the t-variable

It has been proved (PF1) that the correlation function between the spins s_1 and s_2 can be expressed in the t-variable as:

$$\langle \vec{s}_1 \cdot \vec{s}_2 \rangle^T_{G,T} = (\lambda-1) \; t^{eq}_{12}(t,G) \tag{2.19}$$

where $\langle \; \rangle^T_G$ means a thermal average (this should not be confused with the average defined in eq.(2.3)) and

$$t^{eq}_{12}(t,G) = \frac{N_{12}(t,G)}{D(t,G)} \quad . \tag{2.20}$$

$D(t,G)$ is defined in eq.(2.15) and $N_{12}(t,G)$ is:

$$N_{12}(t,G) = \langle \lambda^{c}\, \gamma_{12} \rangle_{G,t} \tag{2.21a}$$

where, in the defining equation (2.3),

$$\gamma_{12}(G') = \begin{cases} 1 & \text{if 1 is connected to 2 on } G' \\ 0 & \text{otherwise} \end{cases} \tag{2.21b}$$

1 and 2 are the vertices where the spins s_1 and s_2 are respectively located and they are called the roots of the graph G.

$t_{12}^{eq}(t,G)$ is what Tsallis and Levy (1981) called the "equivalent transmissivity" between 1 and 2 in a graph G. It represents the thermal transmissivity of a single equivalent edge linking the spins s_1 and s_2 which interact through an equivalent (or effective) coupling constant J_{eq} (see PF1).

In the same way as $D(t,G)$, $N_{12}(t,G)$ also has an interpretation in terms of flows (PF1). The multi-linear expansion of $N_{12}(t,G)$ in the t-variable is (cf PF1):

$$N_{12}(t,G) = \sum_{G'\subseteq G} F_{12}(\lambda,G') \prod_{\ell\in E'} t_\ell \tag{2.22}$$

where the two - rooted flow polynomial $F_{12}(\lambda,G)$ is a topological invariant given by:

$$F_{12}(\lambda,G) = \sum_{G'\subseteq G} (-1)^{|E\setminus E'|}\, \lambda^{c(G')}\, \gamma_{12}(G') . \tag{2.23}$$

$F_{12}(\lambda,G)$ is related to the unrooted flow polynomial through (PF1) :

$$F_{12}(\lambda,G) = \frac{F(\lambda, G \cup g)}{(\lambda - 1)} \tag{2.24}$$

where g is an extra edge which links the roots 1 and 2. Hence $F_{12}(\lambda,G)$ represents (see PF1) the number of proper mod-λ flows in $(G \cup g)$ with a fixed non-zero value on g. In other words, $F_{12}(\lambda,G)$ is the number of proper mod-λ flows in G in the presence of a fixed non-null external flow.

Similar results were obtained for $\langle s_1 . s_2 \rangle_G^T$ in the p-variable by Kasteleyn and Fortuin (1969). The results for the pair correlation function in both t and p-variables could have been obtained simultaneously by following a similar procedure to that of §(2.1). In the next section we shall do this for the m-spin correlation functions.

3 m-SPIN CORRELATION FUNCTIONS

The correlation function $\Gamma_{12\ldots m}(G)$ among the components $s_{11}, s_{21},\ldots, s_{m1}$ of the m spins $\mathbf{s}_1, \mathbf{s}_2,\ldots, \mathbf{s}_m$ along the direction of $\mathbf{e}_1$, is defined as

$$\Gamma_{12\ldots m}(G) \equiv \langle s_{11}\, s_{21} \cdots s_{m1} \rangle_G^T \equiv$$

$$\equiv \left[Z(G)\right]^{-1} \mathrm{tr}_G \left\{ s_{11} s_{21} \cdots s_{m1} \exp\left[\sum_{\ell \in E} K_\ell\, \vec{s}_i \cdot \vec{s}_j \right] \right\} \tag{3.1a}$$

where in general the spin component along the direction of $\mathbf{e}_\alpha$ is given by:

$$s_{i\alpha} = \frac{\vec{s}_i \cdot \vec{e}_\alpha}{s} \qquad (i=1,2,\ldots,m) \tag{3.1b}$$

and the trace is over all positions of the $|V|$ spin vectors $\mathbf{s}_j$ in G. The special vertices 1,2,...,m on which $\mathbf{s}_1, \mathbf{s}_2,\ldots, \mathbf{s}_m$ are respectively located constitute the roots of G.

In §3.1 and §3.2 we derive, by two different procedures, alternative expressions, as well as their corresponding multilinear expansions, for $\Gamma_{12\ldots m}(t(\mu),G)$ in the general $t(\mu)$-variable (see def. 2.8b). Explicit expressions are given for $\Gamma_{12\ldots m}(t,G)$ and $\Gamma_{12\ldots m}(p,G)$. In the first approach, by introducing a "ghost spin" which couples with $\mathbf{s}_1, \mathbf{s}_2,\ldots, \mathbf{s}_m$, we obtain an expression for $\Gamma_{12\ldots m}(t(\mu),G)$ which involves partial derivatives of $D(t(\mu),G^+)$ (defined in 2.14a), where G^+ is the graph obtained from G by adding an extra vertex which links to the m roots. The multi-linear expansion of the numerator of $\Gamma_{12\ldots m}(G)$ in t(or p) involves flow polynomials(or chromatic polynomials) of subgraphs of G^+. The second procedure is an extension of the one used in §2.1 for the partition function, and $\Gamma_{12\ldots m}$ (G) is expressed in terms of m-rooted partitioned equivalent transmissivities $t_{\mathbb{P}}^{eq}$ (G). In §3.3 similar expressions are found for the partitioned correlation functions $\Gamma_{\mathbb{P}}(G)$.The multi-linear form in the t-variable of the numerator $N_{\mathbb{P}}(t,G)$ of $t_{\mathbb{P}}^{eq}(t,G)$ involves partitioned m-rooted flow polynomials $F_{\mathbb{P}}(\lambda,G')$ whose importance has already been stressed in the introduction.

3.1 Expression of $\Gamma_{12\ldots m}(G)$ in terms of unrooted functions

One way of calculating $\Gamma_{12\ldots m}(G)$ is by introducing a "ghost spin" s_g which interacts with the m spins s_1, s_2,..., s_m through the respective coupling constants J_1, J_2,..., and J_m. We shall denote by G^+ the graph $G\cup K_{1,m}$ where $K_{1,m}$ is the star graph formed by $g_1\cup g_2\ldots\cup g_m$, g_i being the edge which links s_g to s_i . Differentiating $Z(G^+)$ m times we get

$$\frac{\partial}{\partial K_m}\cdots\frac{\partial}{\partial K_2}\frac{\partial}{\partial K_1} Z(G^+) = \mathrm{tr}_{G^+}\left\{(\vec{s}_1\cdot\vec{s}_g)(\vec{s}_2\cdot\vec{s}_g)\cdots(\vec{s}_m\cdot\vec{s}_g)\, e^{-\beta\mathcal{H}(G^+)}\right\}. \qquad (3.2)$$

Tracing over the positions of s_g we easily derive that, at $K_1=K_2=\ldots=K_m=0$:

$$\Gamma_{12\ldots m}(G) = s^{-m}\left.\frac{\frac{\partial}{\partial K_m}\cdots\frac{\partial}{\partial K_2}\frac{\partial}{\partial K_1} Z(G^+)}{Z(G^+)}\right|_{K_1=K_2=\ldots=K_m=0} . \qquad (3.3)$$

Combining eqs. (2.13), (2.8b), (2.8c) and (3.3) we get that:

$$\Gamma_{12\ldots m}(t(\mu),G) = \left(\frac{s}{\lambda-1}\right)^m \left[D(t(\mu),G^+)\right]^{-1} \times$$

$$\times\left.\left\{\sum_{j=0}^{m}\ \sum_{\{i_1,i_2,\ldots,i_j\}\in\mathcal{C}_j(M)}\left(\frac{\lambda}{\mu}-1\right)^{m-j}\left(\frac{\lambda}{\mu}\right)^j\frac{\partial}{\partial t_{i_1}(\mu)}\frac{\partial}{\partial t_{i_2}(\mu)}\cdots\frac{\partial D(t(\mu),G^+)}{\partial t_{i_j}(\mu)}\right\}\right|_{t_i(\mu)=0} \qquad (3.4)$$

where $\mathcal{C}_j(M)$ is the set of all possible combinations of j indices $\{i_1,i_2,\ldots,i_j\}$ chosen from $M=\{1,2,\ldots m\}$; for $j=0$ there is only one term, namely $((\lambda/\mu)-1)^m\, D(G^+)$, in the second sum.

Using the multi-linear form of $D(t(\mu),G)$ (eq.2.14a) and (eq.3.4) we finally get that:

$$\Gamma_{12\ldots m}(t(\mu), G) = \left(\frac{s}{\lambda-1}\right)^m \left(\frac{\mu}{\lambda}\right) \times$$

$$\times \frac{\displaystyle\sum_{G'\subseteq G} \sum_{j=0}^{m} \sum_{\{i_1,\ldots,i_j\}\in \mathcal{E}_j(M)} \left(\frac{\lambda}{\mu}-1\right)^{m-j}\left(\frac{\lambda}{\mu}\right)^j F(\mu,\lambda, G'\cup g_{i_1}\cup\ldots\cup g_{i_j}) \prod_{\ell\in E'} t_\ell(\mu)}{\displaystyle\sum_{G'\subseteq G} F(\mu,\lambda,G') \prod_{\ell\in E'} t_\ell(\mu)} \qquad (3.5a)$$

with the convention that for j=0, instead of ghost edges we have a ghost site:

$$F(\mu,\lambda, G'\cup g_{i_1}\cup\ldots\cup g_{i_j})\Big|_{j=0} = F(\mu,\lambda, G'\cup\bullet) \qquad (3.5b)$$

Notice that for $\mu=\lambda$, only the term j=m contributes to $\Gamma_{12\ldots m}(G)$ and we obtain for $s^2=\lambda-1$ that:

$$\Gamma_{12\ldots m}(t, G) = (\lambda-1)^{-m/2} \frac{\displaystyle\sum_{G'\subseteq G} F(\lambda, G'\cup K_{1,m}) \prod_{\ell\in E'} t_\ell}{\displaystyle\sum_{G'\subseteq G} F(\lambda,G') \prod_{\ell\in E'} t_\ell} \qquad (3.6)$$

For $\mu=1$ and $s^2=\lambda-1$ eq. (3.5a) reduces to:

$$\Gamma_{12\ldots m}(p, G) = (\lambda-1)^{-m/2}\, \lambda^{-1} \times$$

$$\times \frac{\displaystyle\sum_{G'\subseteq G} \sum_{j=0}^{m} \sum_{\{i_1,\ldots,i_j\}\in \mathcal{E}_j(M)} (\lambda-1)^{m-j}(-\lambda)^j(-1)^{|E'|} P(\lambda, G'\cup g_{i_1}\cup\ldots\cup g_{i_j}) \prod_{\ell\in E'} p_\ell}{\displaystyle\sum_{G'\subseteq G} (-1)^{|E'|} P(\lambda,G') \prod_{\ell\in E'} p_\ell} \qquad (3.7a)$$

where for the j=0 term :

$$P(\lambda, G' \cup g_{i_1} \cup \dots \cup g_{i_j})\Big|_{j=0} = P(\lambda, G' \cup \bullet) = \lambda P(\lambda, G') \quad . \tag{3.7b}$$

From eqs.(3.6) and (3.7a) , we see that the form of $\Gamma_{12\dots m}(G)$ is much simpler when expressed in the t-variable than in the p-variable.

3.2 Expression of $\Gamma_{12\dots m}(G)$ in terms of partitioned equivalent transmissivities

An alternative procedure for deriving $\Gamma_{12\dots m}(G)$ which leads to results that generalise the ones quoted in §2.2 follows along the same lines as the procedure used in §2.1. The combination of eq.(2.8a) with eq(3.1a) leads to the following expression similar to eq.(2.10):

$$\Gamma_{12\dots m}(t(\mu), G) = \left[\langle \lambda^{\omega} \mu^{|E|} \rangle_{G, t(\mu)}\right]^{-1} \sum_{G' \subseteq G} \sum_{\alpha_1 = 1}^{\lambda} \dots \sum_{\alpha_{|V|}=1}^{\lambda} \Big\{ \Delta_{11} \Delta_{21} \dots \Delta_{m1} \times \prod_{\ell \in E'} \left[\mu t_\ell(\mu)\, \delta(\alpha_i, \alpha_j)\right] \prod_{\ell \in E \setminus E'} \left[1 - t_\ell(\mu)\right] \Big\} \quad . \tag{3.8}$$

Each G' defines a partition $\mathbb{P}'$ of the set $M=\{1,2,\dots,m\}$ of roots into b' blocks B_1, B_2, ..., $B_{b'}$ where the roots in each block are connected in G' and roots belonging to different blocks are not connected. In the example shown in Fig.1b, the partition $\mathbb{P}'$ has two blocks $B_1=\{1,2\}$ and $B_2=\{3,4\}$.

Carrying out the α-summations for the spins which do not belong to the rooted components of G′ we obtain:

$$\Gamma_{12\ldots m}(t(\mu),G) = \frac{\langle (\lambda/\mu)^{w}\,\mu^{c}\,T_{12\ldots m}\rangle_{G,t(\mu)}}{\langle (\lambda/\mu)^{w}\,\mu^{c}\rangle_{G,t(\mu)}} \tag{3.9a}$$

where

$$T_{12\ldots m}(G') = \prod_{B\in\mathbb{P}'}\left[\frac{1}{\lambda}\sum_{\alpha=1}^{\lambda}\left(\frac{\vec{e}_{\alpha}\cdot\vec{e}_{1}}{\Lambda}\right)^{\ell_B}\right] . \tag{3.9b}$$

In definition (3.9b), ℓ_B is the number of roots in the block B of the partition $\mathbb{P}'$.

Since by symmetry $(e_{\alpha}\cdot e_{\beta})^{\ell_B}$ when summed on α is independent of β, it follows that:

$$\frac{1}{\lambda}\sum_{\alpha=1}^{\lambda}\left(\frac{\vec{e}_{\alpha}\cdot\vec{e}_{1}}{\Lambda}\right)^{\ell_B} = \frac{1}{\lambda^2}\sum_{\alpha=1}^{\lambda}\sum_{\beta=1}^{\lambda}\left(\frac{\vec{e}_{\alpha}\cdot\vec{e}_{\beta}}{\Lambda}\right)^{\ell_B} . \tag{3.10}$$

But from a straightforward generalisation, to an arbitrary value of s, of eq.(5.4) of PF1, we see that the flow polynomial for the graph R_{ℓ_B} consisting of two vertices u and v with ℓ_B edges in parallel is given by:

$$F(\lambda,R_{\ell_B}) = \frac{1}{\lambda^2}\,\mathrm{tr}_{R_{\ell_B}}\left\{\left[\vec{\Lambda}_{\mu}\cdot\vec{\Lambda}_{\nu}\left(\frac{\lambda-1}{\Lambda^2}\right)\right]^{\ell_B}\right\}$$

$$= \frac{1}{\lambda^2}\left(\frac{\lambda-1}{\Lambda}\right)^{\ell_B}\sum_{\alpha=1}^{\lambda}\sum_{\beta=1}^{\lambda}\left(\frac{\vec{e}_{\alpha}\cdot\vec{e}_{\beta}}{\Lambda}\right)^{\ell_B} . \tag{3.11}$$

Combining eqs.(3.9b), (3.10) and (3.11) we obtain that:

$$T_{12\ldots m}(G') = \left(\frac{\Delta}{\lambda-1}\right)^m F(\lambda, I_{\mathbb{P}'}) \tag{3.12a}$$

where

$$F(\lambda, I_{\mathbb{P}'}) = \prod_{B\in\mathbb{P}'} F(\lambda, R_{\ell_B}) \tag{3.12b}$$

is the flow polynomial for the "interface graph" $I_{\mathbb{P}'}$ which has a vertex for each block of $\mathbb{P}'$ and the vertex corresponding to block B is connected to an "external" vertex u by an edge of multiplicity ℓ_B.

Notice that the general formula for $F(\lambda, R_{\ell_B})$ can be derived from eq.(2.16). It is given by:

$$F(\lambda, R_{\ell_B}) = \frac{(\lambda-1)}{\lambda}\left\{(\lambda-1)^{\ell_B-1} + (-1)^{\ell_B}\right\} \qquad (\ell_B=1,2,\ldots). \tag{3.12c}$$

Now in order to get an expression for $\Gamma_{12..m}(G)$ in terms of equivalent transmissivities which extends eq.(2.19), we need to define a generalisation of the "connectedness indicator" (eq.2.21b), namely :

$$\gamma_{\mathbb{P}'}(G') = \begin{cases} 1 & \text{if any pair of roots in the same block of } \mathbb{P}' \text{ is connected in } G' \text{ and there is no connection among roots in different blocks.} \\ 0 & \text{otherwise} \end{cases} \tag{3.13}$$

where $\mathbb{P}'=\{B_1, B_2, \ldots, B_{b'}\}$. For a specific $\mathbb{P}'$ we shall use as subscript of γ the actual roots separated by commas whenever they belong to different blocks. For example in fig.(1b): $\gamma_{12,34}(G')=1$ because 1 is connected to 2 and 3 is connected to 4 on G', but $\gamma_{123,4}(G')=0$ because 1,2 and 3 are not connected among themselves and 4 is not unconnected to the other roots.

Let us define $\mathcal{P}(M)$ to be the set of all possible partitions of the set $M=\{1,2,\ldots,m\}$. For example, if $M=\{1,2,3\}$ then

$$\mathcal{P}(1,2,3)=\{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \mathbb{P}_4, \mathbb{P}_5\}$$

where

$\mathbb{P}_1=\{\{1\};\{2\};\{3\}\}$

$\mathbb{P}_2=\{\{1,2\};\{3\}\}$

$\mathbb{P}_3=\{\{1,3\};\{2\}\}$

$\mathbb{P}_4=\{\{2,3\};\{1\}\}$

$\mathbb{P}_5=\{\{1,2,3\}\}$.

Observe that, in this case, the following equality holds for an arbitrary three-rooted graph G:

$$\gamma_{1,2,3}(G)+\gamma_{12,3}(G)+\gamma_{13,2}(G)+\gamma_{23,1}(G)+\gamma_{123}(G)=1 \quad . \tag{3.14}$$

In general for an m-rooted graph G, the following identity holds

$$\sum_{\mathbb{P}\in\mathcal{P}(M)} \gamma_{\mathbb{P}}(G) = 1 \quad . \tag{3.15}$$

Using identity (3.15) we can rewrite $T_{12..m}(G')$ (eq.3.12a) as:

$$T_{12\ldots m}(G') = \left(\frac{\Delta}{\gamma-1}\right)^m \sum_{\mathbb{P}\in\mathcal{P}(M)} \gamma_{\mathbb{P}}(G')\ F(\gamma, I_{\mathbb{P}}) \,. \tag{3.16}$$

Substituting eq.(3.16) into eq.(3.9a) and interchanging the sum over G´ implied by the angular brackets and the sum over $\mathbb{P}$ we obtain:

$$\Gamma_{12\ldots m}(t(\mu),G) = \left(\frac{\Delta}{\gamma-1}\right)^m \sum_{\mathbb{P}\in\mathcal{P}(M)} \frac{\langle(\gamma/\mu)^w \mu^c \gamma_{\mathbb{P}}\rangle_{G,t(\mu)}}{\langle(\gamma/\mu)^w \mu^c\rangle_{G,t(\mu)}}\ F(\gamma, I_{\mathbb{P}}) \,. \tag{3.17}$$

In fact we do not need to consider all the partitions $\mathbb{P}$ of $\mathcal{P}(M)$ since $F(\gamma,R_1)=0$ (cf. eq.3.12c) and hence $F(\gamma,I_{\mathbb{P}})=0$ for all $\mathbb{P}$ which contain blocks with isolated roots (i.e. with $\ell_B=1$). If we define $\tilde{\mathcal{P}}(M)$ as the set of all partitions of M into blocks each of which has at least two roots, then we can rewrite eq.(3.17) as:

$$\Gamma_{12\ldots m}(t(\mu),G) = \left(\frac{\Delta}{\gamma-1}\right)^m \sum_{\mathbb{P}\in\tilde{\mathcal{P}}(M)} t^{eq}_{\mathbb{P}}(t(\mu),G)\ F(\gamma, I_{\mathbb{P}}) \tag{3.18a}$$

where

$$t^{eq}_{\mathbb{P}}(t(\mu),G) \equiv \frac{N_{\mathbb{P}}(t(\mu),G)}{D(t(\mu),G)} \tag{3.18b}$$

with $D(t(\mu),G)$ defined in eq.(2.14a) and

$$N_{\mathbb{P}}(t(\mu),G) \equiv \langle(\gamma/\mu)^w \mu^c \gamma_{\mathbb{P}}\rangle_{G,t(\mu)} \,. \tag{3.18c}$$

Notice that for $\mu=\gamma$ and $s^2=\gamma-1$ (and hence $t_e(\mu)=t_e$), $t^{eq}_{\mathbb{P}}(t(\mu),G)$ becomes:

$$t_{\mathbb{P}}^{eq}(t,G) = \frac{N_{\mathbb{P}}(t,G)}{D(t,G)} \tag{3.19a}$$

where $D(t,G)$ is given by eq.(2.15) and $N_{\mathbb{P}}(t,G)$ is a generalisation of $N_{12}(t,G)$ (eq. 2.21a) given by:

$$N_{\mathbb{P}}(t,G) = \langle \lambda^{c} \gamma_{\mathbb{P}} \rangle_{G,t} \quad . \tag{3.19b}$$

In the case when $\mu=1$ and $s^2=\lambda-1$ (where $t_e(\mu)=p_e$), $t_{\mathbb{P}}^{eq}(t(\mu),G)$ reduces to:

$$t_{\mathbb{P}}^{eq}(p,G) = \frac{N_{\mathbb{P}}(p,G)}{D(p,G)} \tag{3.20a}$$

where $D(p,G)$ is defined in eq.(2.17) and $N_{\mathbb{P}}(p,G)$ is:

$$N_{\mathbb{P}}(p,G) = \langle \lambda^{\omega} \gamma_{\mathbb{P}} \rangle_{G,p} \quad . \tag{3.20b}$$

We shall call $t_{\mathbb{P}}^{eq}(t(\mu),G)$ the **partitioned m-rooted equivalent transmissivity** expressed in the $t(\mu)$-variable since it reduces , for $\mathbb{P}$ containing only one block (i.e., for $b=1$; we shall, throughout this paper, denote by b the number of blocks of a partition $\mathbb{P}$) and $t_e(\mu)=t_e$, to the equivalent transmissivity $t_{12\ldots m}^{eq}(t,G)$ among the roots $1,2,\ldots m$ which appears in Tsallis and Levy (1981) as $G\{t_i\}$.

If we set $s^2 = \lambda -1$ then eq.(3.18a) particularised for $m=2,3$ and 4 becomes respectively:

$$\Gamma_{12}(t(\mu),G) = t_{12}^{eq}(t(\mu),G) \tag{3.21a}$$

$$\Gamma_{123}(t(\mu),G) = (\gamma-2)(\gamma-1)^{-1/2}\, t_{123}^{eq}(t(\mu),G) \tag{3.21b}$$

$$\Gamma_{1234}(t(\mu),G) = (\gamma^2-3\gamma+3)\,(\gamma-1)^{-1}\, t_{1234}^{eq}(t(\mu),G) + t_{12,34}^{eq}(t(\mu),G) + t_{13,24}^{eq}(t(\mu),G) + t_{14,23}^{eq}(t(\mu),G)\,. \tag{3.21c}$$

It may be shown that the $(\gamma-1)$ cartesian components $\langle s_{1x_i}\, s_{2x_i}\rangle_G^T$ $(i=1,2,\ldots,\gamma-1)$ of $\langle \mathbf{s}_1\cdot\mathbf{s}_2\rangle_G^T$ are equal, and we see that eq.(3.21a) for $t_e(\mu)=t_e$ agrees with eq.(2.19) obtained previously (PF1). It is also clear from eqs(3.21), and more generally from eq.(3.18a), that $\Gamma_{12\ldots m}(G)$ is proportional to $t^{eq}_{12\ldots m}(G)$ only for m=2 and 3; for $m\geqslant 4$ the m-spin correlation function necessarily involves several partitioned m-rooted equivalent transmissivities.

Following the same procedure as was used for $D(t(\mu),G)$ in §2.1 we obtain that:

$$N_{\mathbb{P}}(t(\mu),G) = \sum_{G'\subseteq G} F_{\mathbb{P}}(\mu,\gamma,G') \prod_{e\in E'} t_e(\mu) \tag{3.22a}$$

where

$$F_{\mathbb{P}}(\mu,\gamma,G') = \sum_{G''\subseteq G'} (-1)^{|E'\setminus E''|} \left(\frac{\gamma}{\mu}\right)^{\omega(G'')} \mu^{c(G'')}\, \gamma_{\mathbb{P}}(G'')\,. \tag{3.22b}$$

Eq.(3.22a) for $\mu=\gamma$ and $s^2=\gamma-1$ generalizes eq.(2.22):

$$N_{\mathbb{P}}(t,G) \equiv \langle \gamma^c \gamma_{\mathbb{P}}\rangle_{G,t} = \sum_{G'\subseteq G} F_{\mathbb{P}}(\gamma,G') \prod_{e\in E'} t_e \tag{3.23}$$

where what we shall call the **partitioned m-rooted flow polynomial** $F_{\mathbb{P}}(\lambda,G)$ generalizes $F_{12}(\lambda,G)$ (eq.2.23) and is defined by:

$$F_{\mathbb{P}}(\lambda,G) = \sum_{G'\subseteq G} (-1)^{|E\setminus E'|} \lambda^{c(G')} \gamma_{\mathbb{P}}(G') \quad . \qquad (3.24)$$

For $\mu=1$ and $s^2=\lambda-1$, eq. (3.22a) becomes:

$$N_{\mathbb{P}}(p,G) \equiv \langle \lambda^{\omega} \gamma_{\mathbb{P}} \rangle_{G,p} = \sum_{G'\subseteq G} (-1)^{|E'|} P_{\mathbb{P}}(\lambda,G') \prod_{\ell\in E'} p_{\ell} \qquad (3.25)$$

where what we call the **partitioned m-rooted chromatic polynomial** $P_{\mathbb{P}}(\lambda,G)$ is defined by:

$$P_{\mathbb{P}}(\lambda,G) = \sum_{G'\subseteq G} (-1)^{|E'|} \lambda^{\omega(G')} \gamma_{\mathbb{P}}(G') \quad . \qquad (3.26)$$

Combination of eqs.(3.18a), (3.19a), (3.23) and (2.15) leads to the following form for $\Gamma_{12...m}(t,G)$:

$$\Gamma_{12...m}(t,G) = (\lambda-1)^{-m/2} \frac{\sum_{G'\subseteq G} \left\{ \sum_{\mathbb{P}\in \tilde{P}(M)} F(\lambda,I_{\mathbb{P}})\, F_{\mathbb{P}}(\lambda,G') \prod_{\ell\in E'} t_{\ell} \right\}}{\sum_{G'\subseteq G} \left\{ F(\lambda,G') \prod_{\ell\in E'} t_{\ell} \right\}} \qquad (3.27)$$

where we inverted the order of the sums over $\mathbb{P}$ and G' since $F(\lambda,I_{\mathbb{P}})$ does not depend on G'.

From eqs.(3.18a), (3.20), (3.22) and (2.17) we get in the p-variable:

$$\Gamma_{12\ldots m}(p,G) = (\lambda-1)^{-m/2} \times$$

$$\times \frac{\displaystyle\sum_{G'\subseteq G}\left\{\sum_{\mathbb{P}\in\tilde{\mathcal{P}}(M)} F(\lambda,I_{\mathbb{P}})\,(-1)^{|E'|}\,P_{\mathbb{P}}(\lambda,G')\prod_{e\in E'} p_e\right\}}{\displaystyle\sum_{G'\subseteq G}\left\{(-1)^{|E'|}\,P(\lambda,G')\prod_{e\in E'} p_e\right\}} \qquad (3.28)$$

3.3 Partitioned Correlation Functions

In the previous subsection (eq.(3.21)) we saw that the correlation functions $\Gamma_{12}(G)$ and $\Gamma_{123}(G)$ were proportional to the equivalent transmissivities $t_{12}^{eq}(G)$ and $t_{123}^{eq}(G)$ respectively. However $\Gamma_{1234}(G)$ involves the partitioned m-rooted equivalent transmissivities $t_{12,34}^{eq}(G)$, $t_{13,24}^{eq}(G)$ and $t_{14,23}^{eq}(G)$ in addition to $t_{1234}^{eq}(G)$. Here we introduce the partitioned correlation functions $\Gamma_{\mathbb{P}}(G)$ and show that a knowledge of the partitioned equivalent transmissivities also determines the $\Gamma_{\mathbb{P}}(G)$ and vice-versa.

Let us define $\Gamma_{\mathbb{P}}(G)$ in terms of the spin components of eq.(3.1b):

$$\Gamma_{\mathbb{P}}(G) \equiv \left\langle \prod_{B\in\mathbb{P}} \prod_{i\in B} s_{i\alpha_B} \right\rangle_G^T \qquad (3.29)$$

where the component index α_B is the same for all spins s_i in a given block B but different for spins in different blocks (i.e. $\alpha_{B_1} \neq \alpha_{B_2} \neq \ldots \neq \alpha_{B_b}$). By symmetry of the Potts model the value of this correlation function is independent of which components are chosen for each block, for example:

$$\Gamma_{1,2}(G) = \langle s_{11} \, s_{22} \rangle_{G,T} = \langle s_{12} \, s_{23} \rangle_G^T \tag{3.30}$$

where the comma indicates merely that different components of the spins s_1 and s_2 are involved. $\Gamma_{12\ldots m}(G)$ referred to previously is the case when $\mathbb{P}$ has a single block.

In order to relate $\Gamma_{\mathbb{P}}(G)$ to the partitioned equivalent transmissivities, it is convenient to introduce a "ghost" spin for each block and define:

$$\Gamma_0(G_{\mathbb{P}}^+) = \left\langle \prod_{B\in\mathbb{P}} \prod_{i\in B} \left(\frac{\vec{s}_B \cdot \vec{s}_i}{s} \right) \right\rangle_{G_{\mathbb{P}}^+}^T \tag{3.31}$$

where s_B is the "ghost" spin for block B and the thermal average is relative to the Hamiltonian of G but includes averaging over states of the b "ghost" spins. The graph $G_{\mathbb{P}}^+$ is the graph G augmented by an extra vertex for each block and a "ghost" edge connecting each vertex in block B to the extra vertex for that block. In the case of a single block (b=1) $G_{\mathbb{P}}^+$ reduces to the graph G^+ introduced in §3.1. The subscript 0 indicates that no interaction is associated with the "ghost" edges.

The method of §3.1 can be easily extended to give the generalisation of eq.(3.5a) by noting that $\Gamma_o(G^+_{\mathbb{P}})$ may be obtained by introducing an interaction parameter for each "ghost" edge and then using eq.(3.3) with G^+ replaced by $G^+_{\mathbb{P}}$. The multi-linear form of $\Gamma_o(t(\mu),G^+_{\mathbb{P}})$ is similar to the one of $\Gamma_{12\ldots m}(t(\mu),G)$, the only difference being the proportionality constant (μ/λ) which should be replaced by $(\mu/\lambda)^b$. In particular, $\Gamma_o(G^+_{\mathbb{P}})$ is expressed in terms of flow polynomials and the t_e variables by eq.(3.6) with $G' \cup K_{1,m}$ replaced by the subgraph of $G^+_{\mathbb{P}}$ having the edges of G' together with all of the "ghost" edges, namely

$$\Gamma_o(t,G^+_{\mathbb{P}}) = (\lambda-1)^{-m/2} \frac{\sum\limits_{G'\subseteq G} F(\lambda, G' \cup K_{1,\ell_{B_1}} \cup K_{1,\ell_{B_2}} \cup \ldots \cup K_{1,\ell_{B_b}}) \prod\limits_{\ell\in E'} t_\ell}{\sum\limits_{G'\subseteq G} F(\lambda, G') \prod\limits_{\ell\in E'} t_\ell} \,. \tag{3.32}$$

This result shows immediately that:

$$\Gamma_o(G^+_{\mathbb{P}}) = 0 \quad \text{if P has a block with a single root} \tag{3.33}$$

since in this case all the required flow polynomials are for graphs with a "dangling" edge and are, therefore, zero.

For $m \leq 3$ the only partitions having blocks which contain at least two roots are the ones with a single block (previously discussed) and we now use eq.(3.33) to determine $\Gamma_{1,2}(G)$, $\Gamma_{12,3}(G)$ and $\Gamma_{1,2,3}(G)$ in terms of $\Gamma_{12}(G)$ and $\Gamma_{123}(G)$. In general

$$\Gamma_o(G^+_{\mathbb{P}}) = \lambda^{-b} \sum_{\alpha_{B_1}=1}^{\lambda} \cdots \sum_{\alpha_{B_b}=1}^{\lambda} \left\langle \prod_{B\in\mathbb{P}} \prod_{i\in B} \left(\frac{\vec{e}_{\alpha_B}\cdot \vec{s}_i}{s} \right) \right\rangle_G^T$$

$$= \lambda^{-b} \sum_{\alpha_{B_1}=1}^{\lambda} \cdots \sum_{\alpha_{B_b}=1}^{\lambda} \left\langle \prod_{B\in\mathbb{P}} \prod_{i\in B} s_{i\alpha_B} \right\rangle_G^T \tag{3.34}$$

which is a linear combination of partitioned correlations (eq.(3.29)), but terms in which some of the α_{B_i} are equal correspond to coarser partitions than $\mathbb{P}$. For example:

$$\Gamma_0(G^+_{1,2}) = \gamma^{-2}\left[\gamma\,\Gamma_{12}(G) + \gamma(\gamma-1)\,\Gamma_{1,2}(G)\right] \tag{3.35}$$

and using eq.(3.33) gives

$$\Gamma_{1,2}(G) = -\Gamma_{12}(G)\,/\,(\gamma-1)\quad . \tag{3.36}$$

Similarly

$$\Gamma_{12,3}(G) = -\,\Gamma_{123}(G)/(\gamma-1) \tag{3.37}$$

and using

$$\Gamma_0(G^+_{1,2,3}) = \gamma^{-3}\Big\{\gamma\,\Gamma_{123}(G) + \gamma(\gamma-1)\left[\Gamma_{12,3}(G) + \Gamma_{13,2}(G) + \Gamma_{1,23}(G)\right] + \gamma(\gamma-1)(\gamma-2)\,\Gamma_{1,2,3}(G)\Big\} \tag{3.38}$$

together with eqs.(3.33) and (3.37) gives

$$\Gamma_{1,2,3}(G) = 2\,\Gamma_{123}(G)\,/\,(\gamma-1)(\gamma-2)\quad . \tag{3.39}$$

The $\Gamma_{\mathbb{P}}(G)$ with $m \leq 3$ are therefore expressible in terms of m-rooted equivalent transmissivities using eqs.(3.21) together with the above results. The generalisation of eqs.(3.35) and (3.38) is

$$\Gamma_0(G^+_{\mathbb{P}}) = \lambda^{-b} \sum_{\substack{\mathbb{P}'\in P(M)\\ \mathbb{P}'\geq \mathbb{P}}} (\lambda)_{b'}\ \Gamma_{\mathbb{P}'}(G) \tag{3.40a}$$

where

$$(\lambda)_{b'} = \lambda(\lambda-1)\ \dots\ (\lambda-b'+1) \tag{3.40b}$$

and $P'\geq P$ whenever every block of $\mathbb{P}$ is contained in a block of $\mathbb{P}'$. The relation (3.40a) may be inverted (see, for example, Rota 1964 and references therein) to give

$$\Gamma_{\mathbb{P}'}(G) = \frac{1}{(\lambda)_{b'}} \sum_{\mathbb{P}\in P(M)} \mu(\mathbb{P}',\mathbb{P})\ \lambda^{b}\ \Gamma_0(G^+_{\mathbb{P}}) \tag{3.41a}$$

where $\mu(\mathbb{P}',\mathbb{P})$ is the Mobius function of the lattice of partitions of m elements. Explicitly

$$\mu(\mathbb{P}',\mathbb{P}) = \begin{cases} (-1)^{b'-b}(2!)^{a_3}(3!)^{a_4}\dots((m-1)!)^{a_m} & \text{if } \mathbb{P}'\leq \mathbb{P} \\ 0 & \text{otherwise.} \end{cases} \tag{3.41b}$$

In the case that $\mathbb{P}'$ is the partition $\{1,2,\dots,m\}$ then a_i is the number of blocks in $\mathbb{P}$ with i elements. Otherwise a_i is determined in the same way but identifying elements in $\mathbb{P}$ which are in the same block of $\mathbb{P}'$. For example, if $\mathbb{P}'=\{16,2,35,4,7,8,9\}$ and $\mathbb{P}=\{1269,34578\}$ then $\mu(\mathbb{P}',\mathbb{P}) = -\,2!3!$.

We now consider the generalisation of eq.(3.18a) to partitioned correlation functions. Equation (3.9a) is the same except that if $\Gamma_{12\ldots m}(G)$ is replaced by $\Gamma_0(G^+_{\mathbb{P}})$ then $T_{12\ldots m}$ must be replaced by $T^0_{\mathbb{P}}$ where

$$T^0_{\mathbb{P}}(G') = \frac{1}{\lambda^{b+b'}}\, tr\left\{\prod_{B\in\mathbb{P}}\ \prod_{i\in B}\left(\frac{\vec{s}_B\cdot\vec{s}_{B'_i}}{s}\right)\right\} \qquad (3.42)$$

and $s_{B'_i}$ is a representative spin of the block B'_i of the partition $\mathbb{P}'$ of the roots of G induced by G′ which contains root i, and the trace is over the b+b′ spins which occur in the product. Using the extension of eq.(5.4) of PF1 to arbitrary s shows that

$$T^0_{\mathbb{P}}(G') = \left(\frac{s}{\lambda-1}\right)^m F(\lambda, I_{\mathbb{P},\mathbb{P}'}) \qquad (3.43)$$

where $I_{\mathbb{P},\mathbb{P}'}$ is the bipartite "interface graph" having a vertex for each block in $\mathbb{P}$ and $\mathbb{P}'$, and for each B of $\mathbb{P}$ an edge for each root $i\in B$ linking it to the block B′ of $\mathbb{P}'$ which contains i. For example, if m=4, $\mathbb{P}=\{12,34\}$ and $\mathbb{P}'=\{13,24\}$ then $I_{\mathbb{P},\mathbb{P}'}$ is a square with $F(\lambda, I_{\mathbb{P},\mathbb{P}'}) = \lambda-1$. Continuing to follow the derivation of eq. (3.18a) we arrive at

$$\Gamma_0(t(\mu), G^+_{\mathbb{P}}) = \left(\frac{s}{\lambda-1}\right)^m \sum_{\mathbb{Q}\in\tilde{P}(M)} t^{eq}_{\mathbb{Q}}(t(\mu), G)\; F(\lambda, I_{\mathbb{P},\mathbb{Q}})\,. \qquad (3.44)$$

For example, if m=4 then

$$\begin{pmatrix}\Gamma_0(G^+_{1234})\\ \Gamma_0(G^+_{12,34})\\ \Gamma_0(G^+_{13,24})\\ \Gamma_0(G^+_{14,23})\end{pmatrix} = \left(\frac{s}{\lambda-1}\right)^4 \begin{pmatrix}(\lambda-1)(\lambda^2-3\lambda+3) & (\lambda-1)^2 & (\lambda-1)^2 & (\lambda-1)^2\\ (\lambda-1)^2 & (\lambda-1)^2 & (\lambda-1) & (\lambda-1)\\ (\lambda-1)^2 & (\lambda-1) & (\lambda-1)^2 & (\lambda-1)\\ (\lambda-1)^2 & (\lambda-1) & (\lambda-1) & (\lambda-1)^2\end{pmatrix} \begin{pmatrix}t^{eq}_{1234}(G)\\ t^{eq}_{12,34}(G)\\ t^{eq}_{13,24}(G)\\ t^{eq}_{14,23}(G)\end{pmatrix}. \qquad (3.45)$$

Inverting the above matrix we get

$$\begin{pmatrix} t^{eq}_{1234}(G) \\ t^{eq}_{12,34}(G) \\ t^{eq}_{13,24}(G) \\ t^{eq}_{14,23}(G) \end{pmatrix} = \left(\frac{\lambda-1}{\lambda}\right)^4 \frac{1}{(\lambda)_4} \begin{pmatrix} \lambda+1 & -(\lambda-1) & -(\lambda-1) & -(\lambda-1) \\ -(\lambda-1) & (\lambda^2-3\lambda+1) & 1 & 1 \\ -(\lambda-1) & 1 & (\lambda^2-3\lambda+1) & 1 \\ -(\lambda-1) & 1 & 1 & (\lambda^2-3\lambda+1) \end{pmatrix} \begin{pmatrix} \Gamma_o(G^+_{1234}) \\ \Gamma_o(G^+_{12,34}) \\ \Gamma_o(G^+_{13,24}) \\ \Gamma_o(G^+_{14,23}) \end{pmatrix} .$$

(3.46)

Combining eq.(3.46) for $s^2 = \lambda - 1$ and $\mu = \lambda$ together with eqs.(3.19a), (2.15), (3.23) and (3.32) we arrive at

$$F_{1234}(\lambda, G) = \frac{1}{(\lambda)_4} \Big\{ (\lambda+1)\, F(\lambda, G \cup K_{1,4}) - (\lambda-1) \Big[F(\lambda, G \cup e_{12} \cup e_{34}) + F(\lambda, G \cup e_{13} \cup e_{24}) + F(\lambda, G \cup e_{14} \cup e_{23}) \Big] \Big\} \quad (3.47)$$

and

$$F_{\alpha\beta,\gamma\delta}(\lambda, G) = \frac{1}{(\lambda)_4} \Big\{ -(\lambda-1)\, F(\lambda, G \cup K_{1,4}) + (\lambda^2-3\lambda+1)\, F(\lambda, G \cup e_{\alpha\beta} \cup e_{\gamma\delta}) + F(\lambda, G \cup e_{\alpha\gamma} \cup e_{\beta\delta}) + F(\lambda, G \cup e_{\alpha\delta} \cup e_{\beta\gamma}) \Big\}$$

$$(\alpha, \beta, \gamma, \delta = 1,2,3,4)$$

(3.48)

where $e_{\alpha\beta}$ ($\alpha, \beta = 1,2,3,4$) is an extra edge linking α and β . We see therefore that $F_{1234}(\lambda, G)$ and $F_{\alpha\beta,\gamma\delta}(\lambda, G)$ are linear combinations of unrooted flow polynomials.

4 THE PARTITIONED m-ROOTED FLOW POLYNOMIALS $F_{\mathbb{P}}(\lambda,G)$

We next turn to the study of the coefficients $F_{\mathbb{P}}(\lambda,G')$ which appear in the expansions of $\Gamma_{12...m}(G)$ and $\Gamma_{\mathbb{P}}(G)$ in the t_e variables. We derive their relationship with the unrooted flow polynomials and we give an interpretation for $F_{123}(\lambda,G)$ in terms of the number of proper mod-λ flows under fixed external flows (section 4.1). We show that the unrooted and the partitioned m-rooted flow polynomials are very important since from them we can calculate the partition function and m-spin correlation functions in both t and p-variables (section 4.2). We also give the graph theoretic properties of $F_{\mathbb{P}}(\lambda,G')$ (section 4.3) which will be used in forthcoming papers (PF3 and PF4).

4.1 Relationship between $F_{\mathbb{P}}(\lambda,G)$ and $F(\lambda,G)$

Comparison between eqs.(3.6) and (3.27) leads to:

$$F(\lambda, G\cup K_{1,m}) = \sum_{\mathbb{P}\in\tilde{P}(M)} F(\lambda, I_{\mathbb{P}})\; F_{\mathbb{P}}(\lambda, G) \qquad (4.1)$$

which we shall derive in a forthcoming paper (PF4) by an independent procedure (namely, by the extended subgraph break-collapse method).

Observe that eq.(4.1) particularised for m=2 reduces to eq.(2.24) since homeomorphic graphs have the same value of $F(\lambda,G)$ (see property (iii) of $F(\lambda,G)$ in PF1).

In the particular case of m=3 eq.(4.1) leads to:

$$F_{123}(\lambda, G) = F(\lambda, G\cup K_{1,3}) \,/\, (\lambda-1)(\lambda-2) \qquad (4.2)$$

From the above relation, we can interpret $F_{123}(\lambda,G)$ as being the number of proper mod-λ flows in $G \cup g_1 \cup g_2 \cup g_3$ with fixed non-zero and different values Φ_1 and Φ_2 on two of these "ghost" edges, say, on g_1 and g_2 (observe that the value of the flow on g_3 is then automatically given by $-(\Phi_1+\Phi_2)$). Eq.(4.1) particularised for m=4 gives:

$$F(\lambda,G\cup K_{1,4}) = (\lambda^2-3\lambda+3)(\lambda-1)\,F_{1234}(\lambda,G) + (\lambda-1)^2\left[F_{12,34}(G) + F_{13,24}(\lambda,G) + F_{14,23}(\lambda,G)\right] \tag{4.3}$$

which combined with eq.(3.48) would lead to eq.(3.47).

We see thus that $F_{1234}(\lambda,G)$, unlike $F_{12}(\lambda,G)$ and $F_{123}(\lambda,G)$, is not proportional to $F(\lambda,G\cup K_{1,m})$ but involves flow polynomials of other graphs as well (cf. eq.(3.47)), and we were unable to find an interpretation for $F_{1234}(\lambda,G)$ in terms of the number of proper mod-λ flows under fixed external flows. For general m,we expect that eq.(3.44) may be inverted and, by following the same procedure used in the derivation of eqs.(3.47) and (3.48), it would lead to a linear relation between partitioned m-rootedflow polynomials and unrooted flow polynomials.

4.2 Relationships between $F_{\mathbb{P}}(\lambda,G)$ and other quantities

In this subsection we shall prove that we can calculate $P(\lambda,G)$, $P_{\mathbb{P}}(\lambda,G)$, $D(p,G)$, $N_{\mathbb{P}}(p,G)$, $D(t,G)$, $N_{\mathbb{P}}(t,G)$, the Whitney rank function $W^G(x,y)$ and its partitioned extension $W^G_{\mathbb{P}}(x,y)$ in terms of unrooted and partitioned rooted flow polynomials.

From eqs.(3.26) and (2.12) we can write $P_{\mathbb{P}}(\lambda,G)$ as:

$$P_{\mathbb{P}}(\lambda,G) = \lambda^{|V|-|E|}(\lambda-1)^{|E|}\sum_{G'\subseteq G}\left\{\lambda^{c(G')}\,\gamma_{\mathbb{P}}(G')\left(\frac{1}{1-\lambda}\right)^{|E'|}\times\right.$$
$$\left.\times\left(1-\frac{1}{1-\lambda}\right)^{E\setminus E'}\right\}. \tag{4.4}$$

Notice that the sum over G' on the right hand side of eq.(4.4) is exactly $\langle\lambda^c\gamma_{\mathbb{P}}\rangle_{G,t}$ calculated at $t_e=t=(1-\lambda)^{-1}$ ($\forall e\in E$). Therefore, taking into account its polynomial form (eq.(3.23)), we finally get the following relation between $P_{\mathbb{P}}(\lambda,G)$ and $F_{\mathbb{P}}(\lambda,G')$:

$$P_{\mathbb{P}}(\lambda,G) = \lambda^{|V|-|E|}\sum_{G'\subseteq G}(-1)^{|E'|}(\lambda-1)^{|E\setminus E'|}F_{\mathbb{P}}(\lambda,G'). \tag{4.5}$$

Combining eqs.(4.5) and (3.25) we get:

$$N_{\mathbb{P}}(p,G) = \sum_{G'\subseteq G}\left\{(-1)^{|E'|}\lambda^{|V|-|E'|}\times\right.$$
$$\left.\times\left[\sum_{G''\subseteq G'}(-1)^{|E''|}(\lambda-1)^{|E'\setminus E''|}F_{\mathbb{P}}(\lambda,G'')\right]\prod_{e\in E'}p_e\right\}. \tag{4.6}$$

It has been pointed out(Essam 1971a) that the Whitney rank function $W^G(x,y)$ contains , as particular cases, $P(\lambda,G)$ and $D(p,G)$. It can be shown that $F(\lambda,G)$ and $D(t,G)$ may also be obtained from $W^G(x,y)$ as follows:

$$F(\lambda, G) = (-1)^{|E|} \; W^G(-1, -\lambda) \tag{4.7}$$

$$D(t, G) = (1-t)^{|E|} \; W^G\left(\frac{t}{1-t}, \frac{\lambda t}{1-t}\right) . \tag{4.8}$$

We shall, therefore, introduce a partitioned m-rooted rank function $W_{\mathbb{P}}^{G}(x,y)$ which, similarly to $W^G(x,y)$, is related to $P_{\mathbb{P}}(\lambda,G)$, $N_{\mathbb{P}}(p,G)$, $F_{\mathbb{P}}(\lambda,G)$ and $N_{\mathbb{P}}(t,G)$. We shall define $W_{\mathbb{P}}^{G}(x,y)$ by:

$$W_{\mathbb{P}}^{G}(x,y) \equiv \sum_{G' \subseteq G} x^{r(G')} \; y^{c(G')} \; \gamma_{\mathbb{P}}(G') \tag{4.9}$$

where $r(G')$ is the cocycle rank of G' given by:

$$r(G') = |V| - \omega(G') . \tag{4.10}$$

Notice that for $\gamma_{\mathbb{P}}(G')=1$ ($\forall G' \subseteq G$), $W_{\mathbb{P}}^{G}(x,y)$ reduces to the Whitney rank function.

Using eqs. (2.12), (4.9) and (4.10) we get that:

$$W_{\mathbb{P}}^{G}(x, \lambda x) = \sum_{G' \subseteq G} x^{|E'|} \; \lambda^{c(G')} \; \gamma_{\mathbb{P}}(G') \tag{4.11}$$

Now, let us derive the relation between $W_{\mathbb{P}}^{G}(x,\lambda x)$ and $F_{\mathbb{P}}(\lambda,G')$. In order to do this, we multiply and divide the right hand side of eq.(4.11) by $(1+x)^{|E|}$. We obtain, thus:

$$W_{\mathbb{P}}^{G}(x,\lambda x) = (1+x)^{|E|} \sum_{G' \subseteq G} \lambda^{c(G')} \; \gamma_{\mathbb{P}}(G') \left(\frac{x}{1+x}\right)^{|E'|} \left(1 - \frac{x}{1+x}\right)^{|E \setminus E'|} \tag{4.12}$$

The sum over G' in eq.(4.12) is just $N_{\mathbb{P}}(t=x/(1+x), G)$. Using eq.(3.23) we finally get that:

$$W_{\mathbb{P}}^{G}(x, \gamma x) = (1+x)^{|E|} \sum_{G' \subseteq G} F_{\mathbb{P}}(\gamma, G') \left(\frac{x}{1+x}\right)^{|E'|} . \quad (4.13)$$

Eqs.(3.23), (4.5), (4.6) and (4.13) show that the knowledge of $F_{\mathbb{P}}(\gamma,G')$ for all partial graphs G' of G allows the calculation of $N_{\mathbb{P}}(t,G)$, $P_{\mathbb{P}}(\gamma,G)$, $N_{\mathbb{P}}(p,G)$ and $W_{\mathbb{P}}^{G}(x,y=\gamma x)$. If we make $\gamma_{\mathbb{P}}(G')=1$ for all $G' \subseteq G$ we get similar formulae (by dropping the subscript $\mathbb{P}$) relating $F(\gamma,G')$ to $D(t,G)$, $P(\gamma,G)$, $D(p,G)$ and $W^{G}(x,y=\gamma x)$. We conclude therefore that **from unrooted and rooted flow polynomials we can ultimately calculate the partition function and the m-spin correlation functions for any given graph G in the t or in the p-variable.**

4.3 Properties of $F_{\mathbb{P}}(\gamma,G)$

Similarly to the two-rooted flow polynomial(see PF1), $F_{\mathbb{P}}(\gamma,G)$ is a topological invariant which has the following properties:

(i) Deletion-contraction rule. If the edge e of G is not a loop then:

$$F_{\mathbb{P}}(\gamma, G) = F_{\mathbb{P}}(\gamma, G_e^{\gamma}) - F_{\mathbb{P}}(\gamma, G_e^{\delta}) \quad (4.14)$$

where G_e^{γ} and G_e^{δ} are obtained from G by contracting and deleting the edge e respectively.

We can prove eq.(4.14) by considering the definition of $F_{\mathbb{P}}(\gamma,G)$ (eq.3.24) and splitting the sum over G' into two parts according to the presence or absence of the edge e, namely:

$$F_{\mathbb{P}}(\lambda, G) = \sum_{\substack{G' \subseteq G \\ e \in E'}} (-1)^{|E \setminus E'|} \lambda^{c(G')} \gamma_{\mathbb{P}}(G') + \sum_{\substack{G' \subseteq G \\ e \notin E'}} (-1)^{|E \setminus E'|} \lambda^{c(G')} \gamma_{\mathbb{P}}(G') . \tag{4.15}$$

Noticing that in the first sum $c(G')=c(G'^{\gamma}_{e})$, $|E'|=|E(G'^{\gamma}_{e})|+1$, $\gamma_{\mathbb{P}}(G')=\gamma_{\mathbb{P}}(G'^{\gamma}_{e})$ and that in the second sum all the graphs G' are equal to G'^{δ}_{e}, we finally get eq.(4.14) (remembering that $|E|=|E(G^{\gamma}_{e})|+1=|E(G^{\delta}_{e})|+1$).

(ii) If at least one root of a block of the partition $\mathbb{P}$ is not connected in G to the other roots in the same block then $F_{\mathbb{P}}(\lambda,G) = 0$.

This follows from the fact that if the roots in each block are not connected in G then, by deleting any number of edges in G (thus forming G'), this fact continues to be true. Consequently $\gamma_{\mathbb{P}}(G')$ vanishes for all partial graphs G' of G, and hence $F_{\mathbb{P}}(\lambda,G)$ vanishes also (cf. eq. 3.24).

(iii) If G_1 and G_2 differ only by some number of isolated non-rooted vertices then $F_{\mathbb{P}}(\lambda,G_1) = F_{\mathbb{P}}(\lambda,G_2)$.

This follows trivially from the definition of $F_{\mathbb{P}}(\lambda,G)$ (eq.3.24).

(iv) If G has a non-rooted vertex j of degree one (i.e., if G has a "dangling end") then $F_{\mathbb{P}}(\lambda,G)=0$.

This follows from the application of properties (i) (where e is the incident edge at j) and (iii) (where $G_1=G_e^{\gamma}$ and $G_2=G_e^{\delta}$).

(v) If H and L are disjoint graphs or have at most one vertex in common and if all the roots belong to H then:

$$F_{\mathbb{P}}(\lambda, H\cup L) = F(\lambda, L)\, F_{\mathbb{P}}(\lambda, H).$$

This follows immediately from the definitions (2.16) and (3.24).

(vi) If e is a loop then $F_{\mathbb{P}}(\lambda,G)=(\lambda-1)\, F_{\mathbb{P}}(\lambda,G_e^{\delta})$.

The proof follows along the same lines as the one for property (i),except for the fact that the first sum in eq.(4.15) is rewritten in terms of G'^{δ}_e .Taking into account that,if the loop belongs to G', $c(G')= c(G'^{\delta}_e)+1$, $\gamma_{\mathbb{P}}(G')= \gamma_{\mathbb{P}}(G'^{\delta}_e)$ and $|E'|=|E(G'^{\delta}_e)|+1$, we easily arrive at property(vi).

(vii) Edge doubling. If G_{ef} is the graph obtained from G by replacing the edge e by the double edge ef then $F_{\mathbb{P}}(\lambda,G_{ef})$ satisfies the following relation:

$$F_{\mathbb{P}}(\lambda, G_{\ell f}) = (\lambda-2)\, F_{\mathbb{P}}(\lambda, G) + (\lambda-1)\, F_{\mathbb{P}}(\lambda, G_{\ell}^{\delta}). \tag{4.16}$$

This property extends property (v) for F_{ij}(see PF1).

Proof: If we apply eq.(4.14) to the edge f of G_{ef} and then property (vi) to $(G_{ef})_f^{\gamma}$ we get

$$F_{\mathbb{P}}(\lambda, G_{\ell f}) = (\lambda-1)\, F_{\mathbb{P}}(\lambda, G_{\ell}^{\gamma}) - F_{\mathbb{P}}(\lambda, G).$$

Substituting $F_{\mathbb{P}}(\lambda, G_e^{\gamma})$ of eq.(4.14) into the above equation we finally arrive at eq.(4.16).

(viii) If G_1 and G_2 are homeomorphic (see PF1) then

$$F_{\mathbb{P}}(\lambda, G_1) = F_{\mathbb{P}}(\lambda, G_2).$$

Proof: Let us suppose for the moment that G_1 differs from G_2 by only one non-rooted vertex j ($j \in V_1$) of degree two. If we apply property (i) (where e is any of the two incident edges at j) then $(G_1)_{\ell}^{\gamma} = G_2$ and $(G_1)_{\ell}^{\delta}$ will contain a non-rooted vertex of degree one. Applying property (iv) to $(G_1)_{\ell}^{\delta}$ we finally get that $F_{\mathbb{P}}(\lambda, G_1) = F_{\mathbb{P}}(\lambda, G_2)$. If G_1 differs from G_2 by any number ℓ of vertices of degree two then we just have to repeat the above procedure successive times.

All these properties, except (ii), continue to be valid for $F(\lambda, G)$ (see PF1). Another interesting property of $F(\lambda, G)$ is that it vanishes for a graph which contains an articulation edge (see property (i) in PF1).

5 PROPERTIES OF RELATED QUANTITIES

In this section we derive the properties of the configurational averaged quantities and of the partitioned rooted chromatic polynomials which appeared in our expressions for m-spin correlation functions. These properties will be used in extensions of the break-collapse method (Tsallis and Levy 1981) which will appear in forthcoming papers (PF3 and PF4).

5.1 Properties of $N_{\mathbb{P}}(t,G)$

The following properties of $N_{\mathbb{P}}(t,G)$ can be deduced easily from the corresponding ones of $F_{\mathbb{P}}(\lambda,G)$ using eq.(3.23):

(i) The break-collapse equation (BCE) is given by:

$$N_{\mathbb{P}}(t,G) = (1-t_\ell)\, N_{\mathbb{P}}(t,G_\ell^s) + t_\ell\, N_{\mathbb{P}}(t,G_\ell^\gamma) \tag{5.1}$$

which extends the BCE for $N_{ij}(t,G)$ (see PF1) and is a particular case ($Q=\lambda^c \gamma_{\mathbb{P}}$; $p_\ell = t_e$) of eq.(2) of Kasteleyn and Fortuin (1969).

(ii) If at least one root of a block of the partition $\mathbb{P}$ is not connected to the other roots in the same block on G then $N_{\mathbb{P}}(t,G)=0$.

(iii) If G_1 and G_2 differ by any number of non-rooted vertices of degree zero then $N_{\mathbb{P}}(t,G_1)=N_{\mathbb{P}}(t,G_2)$.

(iv) If the edge e is incident at a non-rooted vertex of degree one then $N_{\mathbb{P}}(t,G)=N_{\mathbb{P}}(t,G_e^{\gamma})$.

(v) If H and L are disjoint graphs or have at most one vertex in common and if all the roots belong to H then:

$$N_{\mathbb{P}}(t, H\cup L) = D(t,L)\; N_{\mathbb{P}}(t,H).$$

(vi) If e is a loop then $N_{\mathbb{P}}(t,G)=[1+(\lambda-1)t_e]\; N_{\mathbb{P}}(t,G_e^{\delta})$.

(vii) Edge doubling.If G_{ef} is the graph obtained by replacing the edge e by the edges e and f then:

$$N_{\mathbb{P}}(t,G_{ef}) = \left[t_e + t_f + (\lambda-2)\, t_e t_f\right]\left[N_{\mathbb{P}}(t,G_e^{\gamma}) - N_{\mathbb{P}}(t,G_e^{\delta})\right] + \left[1+(\lambda-1)\, t_e t_f\right] N_{\mathbb{P}}(t,G_e^{\delta})$$

which extends property (v) for $N_{ij}(t,G_{ef})$ in PF1.

5.2 Properties of $t_{\mathbb{P}}^{eq}(t,G)$

D(t,G) does not depend on the roots of G by definition (cf. eq.(2.15)) and its properties, which were given in PF1, are similar to those of $N_{\mathbb{P}}(t,G)$. The properties of $t_{\mathbb{P}}^{eq}(t,G)$ can be easily derived from the ones of $N_{\mathbb{P}}(t,G)$ and D(t,G) (cf. eq.3.19a), namely:

(i) the BCE for $t_{\mathbb{P}}^{eq}(t,G)$ is:

$$t_{\mathbb{P}}^{eq}(t,G) = \frac{(1-t_e)\, N_{\mathbb{P}}(t,G_e^{\delta}) + t_e\, N_{\mathbb{P}}(t,G_e^{\gamma})}{(1-t_e)\, D(t,G_e^{\delta}) + t_e\, D(t,G_e^{\gamma})} \tag{5.2}$$

which extends the BCE for $t^{eq}_{12\ldots m}(G;t)$ stated by Tsallis and Levy (1981).

(ii), (iii) and (iv) $N_{\mathbb{P}}$ may be replaced by $t^{eq}_{\mathbb{P}}$ in the corresponding properties for $N_{\mathbb{P}}(t,G)$.

(v) If H and L are disjoint graphs or have at most one vertex in common and if all the roots belong to H then:

$$t^{eq}_{\mathbb{P}}(t, H\cup L) = t^{eq}_{\mathbb{P}}(t, H).$$

(vi) If e is a loop then $t^{eq}_{\mathbb{P}}(t,G)=t^{eq}_{\mathbb{P}}(t,G^{s}_{e})$.

5.3 Properties of $P_{\mathbb{P}}(\lambda,G)$ and $P(\lambda,G)$

$P_{\mathbb{P}}(\lambda,G)$ has the following properties:

(i) Deletion-contraction rule. If the edge e of G is not a loop then:

$$P_{\mathbb{P}}(\lambda, G) = P_{\mathbb{P}}(\lambda, G^{s}_{e}) - P_{\mathbb{P}}(\lambda, G^{\gamma}_{e}). \tag{5.3}$$

Proof: If we split the sum in eq.(3.26) into two parts according to the presence or absence of e we get:

$$P_{\mathbb{P}}(\lambda,G) = \sum_{\substack{G'\subseteq G\\ e\in E'}} (-1)^{|E'|}\lambda^{\omega(G')}\gamma_{\mathbb{P}}(G') + \sum_{\substack{G'\subseteq G\\ e\notin E'}} (-1)^{|E'|}\lambda^{\omega(G')}\gamma_{\mathbb{P}}(G'). \tag{5.4}$$

Following along the same lines as in property (i) of $F_{\mathbb{P}}(\lambda,G)$ and noticing that $\omega(G')=\omega(G'^{\gamma}_{e})$ if $e\in E'$ we easily arrive at eq.(5.3)

(ii) If at least one root of a block of the partition $\mathbb{P}$ is not connected to the other roots in the same block on G then $P_{\mathbb{P}}(\lambda,G)=0$.

The proof is similar to the one given in property(ii) of $F_{\mathbb{P}}(\lambda,G)$.

(iii) If G_1 and G_2 differ by $|V_0|$ non-rooted isolated vertices then $P_{\mathbb{P}}(\lambda,G_1)=\lambda^{|V_0|}P_{\mathbb{P}}(\lambda,G_2)$.

This follows trivially from the definition (3.26).

(iv) If e is an edge incident at a non-rooted vertex of degree one (i.e., e is a "dangling end") then $P_{\mathbb{P}}(\lambda,G)=(\lambda-1)P_{\mathbb{P}}(\lambda,G_e^{\gamma})$.

The proof follows from the application of eq.(5.3) to the edge e and subsequent application of property (iii) to G_e^{δ}.

(v) If H and L are disjoint graphs and if all the roots belong to H then:

$$P_{\mathbb{P}}(\lambda, H\cup L) = P(\lambda, L)\, P_{\mathbb{P}}(\lambda, H).$$

This follows trivially from the definitions(2.18) and (3.26).

(vi) If G contains a loop then $P_{\mathbb{P}}(\lambda,G)=0$.

The proof follows immediately from eq.(5.4) where now e is a loop. In this case the first sum becomes simply $-P_{\mathbb{P}}(\lambda,G_e^{\delta})$ while the second sum continues to be $P_{\mathbb{P}}(\lambda,G_e^{\delta})$ and hence $P_{\mathbb{P}}(\lambda,G)$ vanishes.

(vii) The replacement of a multiple edge by a single edge does not change $P_{\mathbb{P}}(\lambda,G)$.

Proof: Suppose that G contains, among other edges, two edges e_1 and e_2 in parallel between the vertices i and j. If we apply eq.(5.3) to, let us say, the edge e_1 then it follows that $P_{\mathbb{P}}(\lambda,G) = P_{\mathbb{P}}(\lambda,G^{\delta}_{e_1})$ since $G^{\gamma}_{e_1}$ contains a loop (which comes from the edge e_2 collapsed) and consequently $P_{\mathbb{P}}(\lambda,G^{\gamma}_{e})=0$ (cf.(vi)).

(viii) Insertion of a vertex of degree two. If G_{fg} is the graph obtained from G by replacing the edge e by two edges f and g in series, then $P_{\mathbb{P}}(\lambda,G_{fg})$ satisfies the following relation:

$$P_{\mathbb{P}}(\lambda, G_{fg}) = (\lambda-1)\, P_{\mathbb{P}}(\lambda, G^{\delta}_{e}) - P_{\mathbb{P}}(\lambda, G). \qquad (5.5)$$

The proof follows from the application of eq.(5.3) to the edge f of G_{fg} and subsequent application of property (iv) to $(G_{fg})^{\delta}_{f}$. It is clear from eq.(5.5) that, unlike $F_{\mathbb{P}}(\lambda,G)$ (see property (viii) of §4.3), $P_{\mathbb{P}}(\lambda,G)$ is not a topological invariant.

All these properties, except (ii), remain valid for $P(\lambda,G)$ if the subscript $\mathbb{P}$ is ignored.

5.4 Properties of $N_{\mathbb{P}}(p,G)$, $D(p,G)$ and $t^{eq}_{\mathbb{P}}(p,G)$

Using eq.(3.20b) and the properties of $P_{\mathbb{P}}(\lambda,G)$ we can easily derive the following properties of $N_{\mathbb{P}}(p,G)$:

(i) The break-collapse equation (BCE) is given by:

$$N_{\mathbb{P}}(p,G) = (1-p_e)\,N_{\mathbb{P}}(p,G_e^{\delta}) + p_e\,N_{\mathbb{P}}(p,G_e^{\gamma}) \qquad (5.6)$$

which is a particular case ($Q=\lambda^{\omega}\gamma_{\mathbb{P}};\, p_{\ell}=p_e$) of eq.(2) of Kasteleyn and Fortuin (1969).

(ii) If at least one root of a block of the partition $\mathbb{P}$ is not connected to the other roots in the same block on G then $N_{\mathbb{P}}(p,G)=0$.

(iii) If G_1 and G_2 differ by $|V_0|$ non-rooted vertices of degree zero then $N_{\mathbb{P}}(p,G_1)=\lambda^{|V_0|}\,N_{\mathbb{P}}(p,G_2)$.

(iv) If e is an edge incident at a non-rooted vertex of degree one then $N_{\mathbb{P}}(p,G)=[\lambda-(\lambda-1)p_e]\,N_{\mathbb{P}}(p,G_e^{\gamma})$.

(v) If H and L are disjoint graphs and if all the roots belong to H then:

$$N_{\mathbb{P}}(p,H\cup L) = D(p,L)\,N_{\mathbb{P}}(p,H)$$

(vi) If e is a loop then $N_{\mathbb{P}}(p,G)=N_{\mathbb{P}}(p,G_e^{\delta})$.

(vii) Edge doubling. If G_{ef} is the graph obtained by replacing the edge e by the edges e and f then:

$$N_{\mathbb{P}}(p,G_{ef}) = (p_e+p_f-p_ep_f)\,[N_{\mathbb{P}}(p,G_e^{\gamma})-N_{\mathbb{P}}(p,G_e^{\delta})] + N_{\mathbb{P}}(p,G_e^{\delta})\ .$$

(viii) Insertion of a vertex of degree two. If G_{fg} is the graph obtained from G by replacing the edge e by two edges f and g in series, then:

$$N_{\mathbb{P}}(p,G_{fg}) = \left[\lambda - (\lambda-1)(p_f+p_g) + (\lambda-2)p_f p_g\right] N_{\mathbb{P}}(p,G_e^{\delta}) + p_f p_g\, N_{\mathbb{P}}(p,G_e^{\sigma}) .$$

Notice that D(p,G) has properties similar to the ones of $N_{\mathbb{P}}(p,G)$ (just replace $N_{\mathbb{P}}$ by D) except property (ii). Consequently (cf eq.3.20a) it is easy to prove that the properties of $t_{\mathbb{P}}^{eq}(t,G)$, which were listed in §5.2, remain valid for $t_{\mathbb{P}}^{eq}(p,G)$.

6 ADVANTAGES OF t OVER p

We now compare the variables t_e and p_e. First of all, the variable t_e, unlike the p-variable, has the important physical interpretation of being the correlation function $\Gamma_{ij}(t,e)$ between the components s_{i1} and s_{j1} of the spins s_i and s_j of the graph consisting of a single edge e.

Second, concerning high-temperature multi-linear expansions of the partition function and m-spin correlation functions, t is much more convenient than the p-variable. In fact, if we compare properties (iv) and (viii) of $F_{\mathbb{P}}(\lambda,G)$ and $P_{\mathbb{P}}(\lambda,G)$ as well as the corresponding properties of $F(\lambda,G)$ and $P(\lambda,G)$, we see that:

a) graphs with "dangling ends" do not contribute to the multi-linear forms of $\Gamma_{12\ldots m}(t,G)$ (eq.3.27), $\Gamma_{\mathbb{P}}(t,G)$ (see eqs. (3.32) and (3.41a)) and of $Z(t,G)$ (eqs.(2.7a) and (2.15)), while they do contribute to the multi-linear forms of $\Gamma_{12..m}(p,G)$ (eq.(3.28)), $\Gamma_{\mathbb{P}}(p,G)$ and $Z(p,G)$(eqs.(2.5a) and (2.17)).

b) because $F(\lambda,G)$ vanishes also for graphs with articulation edges, only the graphs in which every edge belongs to a cycle contribute to the multi-linear form of $Z(t,G)$, unlike the case of the p-variable.

c) the multi-linear forms of $\Gamma_{12\ldots m}(t,G)$, $\Gamma_{\mathbb{P}}(t,G)$ and $Z(t,G)$, unlike those of $\Gamma_{12\ldots m}(p,G)$, $\Gamma_{\mathbb{P}}(p,G)$ and $Z(p,G)$, are determined essentially by the topology of G, the insertion of unrooted vertices of degree two giving rise to a trivial change(i.e. replace t_e by $t_g t_f$).

Notice also that the alternative multi-linear forms of $\Gamma_{12\ldots m}(p,G)$ (eq.(3.7a)) and $\Gamma_{\mathbb{P}}(p,G)$ have more terms to be calculated than the corresponding ones of $\Gamma_{12\ldots m}(t,G)$ (eq.(3.6)) and $\Gamma_{\mathbb{P}}(t,G)$. Furthermore, a comparison between the two expressions for $\Gamma_{12\ldots m}(p,G)$ (eqs.(3.7a)) and (3.28)) leads to a relation between unrooted and rooted partitioned chromatic polynomials which is rather complicated. Consequently there is no simple interpretation of $P_{\mathbb{P}}(\lambda,G)$ even for the simple cases of $P_{12}(\lambda,G)$ and $P_{123}(\lambda,G)$, contrary to what happens for rooted flow polynomials.

As we shall see in a forthcoming paper (PF3), there is a powerful procedure for calculating $\Gamma_{12\ldots m}(t,G)$ and $\Gamma_{\mathbb{P}}(t,G)$ which does not apply to $\Gamma_{12\ldots m}(p,G)$ and $\Gamma_{\mathbb{P}}(p,G)$.

7 THE PARTITIONED m-ROOTED CONNECTEDNESS IN BOND PERCOLATION ($\lambda\to 1$)

Let us consider in this section the $\lambda=1$ limit which corresponds to the bond percolation problem (see Kasteleyn and Fortuin 1969). In this limit, both the flow polynomial and the chromatic polynomial, by their definitions, vanish except for the null graph $N_{|V|}$ with $|V|$ isolated vertices, i.e:

$$F(1,G) = P(1,G) = \delta(G,N_{|V|}) \tag{7.1a}$$

where

$$\delta(G,N_{|V|}) = \begin{cases} 1 & \text{if } G=N_{|V|} \\ 0 & \text{otherwise} \end{cases} \quad . \tag{7.1b}$$

From eqs.(2.15) and (2.17) it follows immediately that:

$$D(t,G)\Big|_{\lambda=1} = D(p,G)\Big|_{\lambda=1} = 1 \qquad \forall G \tag{7.2}$$

and from definitions (3.19b) and (3.20b):

$$N_{\mathbb{P}}(t,G)\Big|_{\lambda=1} = N_{\mathbb{P}}(p,G)\Big|_{\lambda=1} = \langle\gamma_{\mathbb{P}}\rangle_G \equiv C_{\mathbb{P}}(G) \tag{7.3}$$

where the variables t_e and p_e become identical and equal to (cf defs (2.6) and (2.4)):

$$t_e\Big|_{\lambda=1} = p_e\Big|_{\lambda=1} = 1-e^{-K_e} \quad . \tag{7.4}$$

(7.4)

Combining eqs.(7.2), (7.3), (3.19a) and (3.20a) we get that:

$$t_{\mathbb{P}}^{eq}(t,G)\Big|_{\lambda=1} = t_{\mathbb{P}}^{eq}(p,G)\Big|_{\lambda=1} = C_{\mathbb{P}}(G) \quad . \tag{7.5}$$

$C_{\mathbb{P}}(G)$ generalises the pair connectedness $C_{12}(G) = \langle \gamma_{12} \rangle_G$, a function which is well known in bond percolation theory. $C_{\mathbb{P}}(G)$ represents the probability of the roots of G being connected according to the partition $\mathbb{P}$ (i.e., roots in the same block of $\mathbb{P}$ are connected and roots in different blocks are not connected). We shall call $C_{\mathbb{P}}(G)$ the **partitioned m-rooted connectedness**. The coefficients of the series expansion of $C_{12}(G)$ in the variable defined in eq.(7.4) are known as weak pair connectedness weights or "d-weights" for bond percolation (see, for example, Essam 1971b). These d-weights are equal to the two-rooted flow polynomials evaluated at $\lambda=1$. Therefore, we shall call the coefficients of the series expansion of $C_{\mathbb{P}}(G)$ the **$d_{\mathbb{P}}$-weights** which are given by (cf. eq. (3.24)):

$$d_{\mathbb{P}}(G) = F_{\mathbb{P}}(1,G) = \sum_{G' \subseteq G} (-1)^{|E \setminus E'|} \gamma_{\mathbb{P}}(G') \quad . \tag{7.6}$$

Observe that for $\gamma_{\mathbb{P}}(G) = \gamma_{12}(G)$, eq.(7.6) reduces to eq. (3.19) of Essam (1972).

The properties of $d_{\mathbb{P}}(G)$ can be easily deduced from the ones of $F_{\mathbb{P}}(\lambda,G)$ by making $\lambda=1$ and using eq.(7.1a). Notice that

$d_{\mathbb{P}}(G)$ vanishes for graphs having: (a) components without roots or just one root (cf(v)); (b) loops (cf(vi)); (c) "dangling ends" (cf(iv)). The graphs which have non-zero $d_{\mathbb{P}}$-weights are the ones in which each component G_i has no loops and is a one-irreducible multi-rooted graph, i.e., the deletion of any vertex of G_i leaves it with at least one root in each of the components of G_i which result from the deletion of this vertex. Observe also that, for $\gamma_{\mathbb{P}}(G)= \gamma_{12}(G)$, the above properties are in agreement with well known properties of the d-weights (Essam 1971b).

The properties of $C_{\mathbb{P}}(G)$ follow easily from the ones of $N_{\mathbb{P}}(t,G)$ or $N_{\mathbb{P}}(p,G)$ by making $\lambda=1$.

It is worth emphasizing that when the partition $\mathbb{P}$ contains only one block, then $C_{12..m}(G)$ gives the probability that all pairs of points $\{i,j\}$ $(i,j=1,2,...m; i\neq j)$ are connected. Arrowsmith (1979) proposed, in the context of directed bond percolation, one way of calculating $d_{12...m}(G)$ which involves a sum of directed weights over all possible orientations of G. The evaluation of $F_{12...m}(\lambda,G)$ at $\lambda=1$ provides an alternative procedure for calculating $d_{12...m}(G)$.

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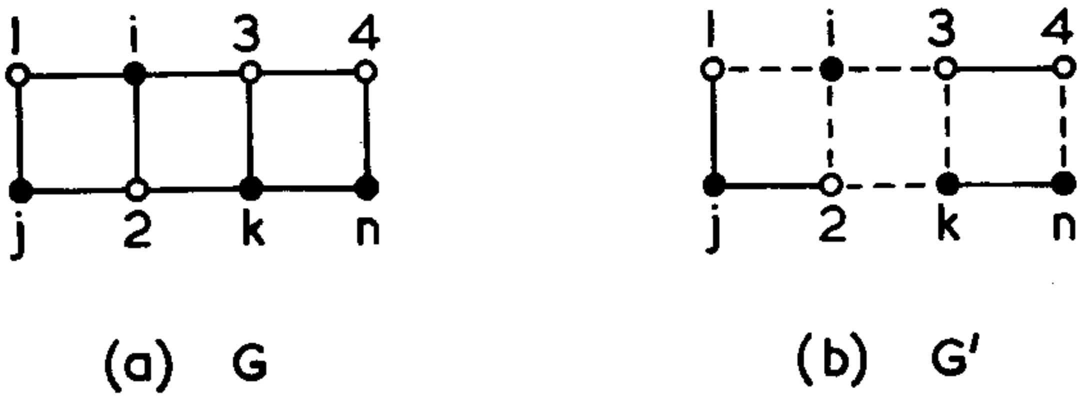

Fig. 1

Figure Caption

Figure 1. Example of a partial graph G′(fig.1b) of a graph G (fig.1a) with roots 1,2,3,4 (indicated by open circles) partitioned into $b'=2$ blocks: $B_1=\{1,2\}$ and $B_2=\{3,4\}$. The missing edges are represented by dashed lines and the non-rooted vertices by full circles.

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