

Connections among three roads to cosmic acceleration: decaying vacuum, bulk viscosity, and nonlinear fluids

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We discuss the connection among three distinct classes of models often used to explain the late cosmic acceleration: decaying cosmological term, bulk viscous pressure, and nonlinear fluids. We focus on models that are equivalent at zeroth order, in the sense they lead to the same solutions for the evolution of the scale factor. More specifically, we show explicit examples where this equivalence is manifest, which include some well known models belonging to each class, such as a power law Λ -term, a model with constant viscosity, and the Modified Chaplygin Gas. We also obtain new analytic solutions for some of these models, including a new *Ansatz* for the cosmic term.

I. INTRODUCTION

The combination of a large set of astrophysical observations shows that the Universe is currently undergoing a phase of accelerated expansion. A natural explanation for this phenomenon, in the framework of Einstein's theory of gravitation, would be the presence of a cosmological constant. However, if this constant is interpreted as the vacuum energy, a mismatch of about 50 to 100 orders of magnitude occurs between the different contributions to the vacuum energy and the observed value of Λ (this is known as the *cosmological constant problem*). Besides, if the cosmological term is constant, it implies that we live in a very special period of cosmic history where the contribution of Λ to the total energy density is of the same order of magnitude of that of the matter fields (this is the so-called *coincidence problem*).

An interesting alternative, put forward by M. P. Bronstein already in 1933 [1], is the possibility of a *decaying cosmological term*. Since then, several phenomenological models were proposed for this component. Some authors argue that a dynamical Λ is even a requirement of Quantum Field Theory and provide first principles estimations of its dependence on cosmic expansion (see, e.g., refs. [2, 3]). A suitable Λ decaying model could potentially connect the primordial inflation with the present acceleration and also alleviate the coincidence problem.

Another possibility as a driving force for the accelerated expansion is the presence of an isotropic viscous pressure (see,

e.g., refs. [4, 5, 6]), which can arise from dissipation processes or as a consequence of particle creation.

A third road to cosmic acceleration is provided by so-called *exotic fluids*, which have negative relativistic pressure, such as the the widely used linear equation of state (EOS) $p = w\rho$, with $w < -1/3$. However, a fluid with nonlinear EOS may have more suitable properties, such as being stable with respect to pressure perturbations. An example is given by the Chaplygin gas [7, 8]. A particularly interesting property of this type of fluid is the possibility of unifying *dark energy* (DE) and *dark matter* (DM) (for a review of this scenario, see, e.g., ref. [9]).

Many other frameworks which lead to an acceleration of the cosmic expansion have been proposed, most notably those based on scalar fields — generically denoted as quintessence. For recent reviews on DE, with special emphasis on quintessence, see, e.g., refs. [10, 11, 12]. It is worth noting that all these approaches assume that the constant contribution of vacuum energy plus cosmological constant are identically zero due to some compensation mechanism and therefore do not address the cosmological constant problem.

In this work we discuss the connections among the first three possible explanations for the accelerated expansion mentioned above, namely, decaying Λ , bulk viscous pressure, and exotic fluids with nonlinear EOS. We show that these three classes of models modify in the same way the evolution of the expansion rate. Therefore for a given model in one class, it is possible to find equivalent models belonging to the other classes that produce the same expansion history of the Universe.

We discuss some explicit examples of this connection, displaying the equivalent models for two popular sets of Λ decaying cosmologies: one where the cosmic term has a power

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law dependence on the scale factor and another where it has a quadratic form in terms of the expansion rate.

In the process of investigating the equivalent models, we derive analytical solutions for the scale factor in some particular cases. We also introduce and find analytical solutions for a new model of decaying Λ term. These solutions are useful to test numerical codes, they provide some insight for more generic numerical solutions, and are handy for pedagogical purposes.

A corollary of the discussion presented in this work is that one cannot observationally distinguish these three possible explanations for the accelerated expansion using only background results (i.e., the zeroth order homogeneous evolution of the Universe), such as the supernovae distance-redshift relation.

The paper is organized as follows. In section II we present a brief review of the three frameworks to explain the accelerated expansion on which we focus in this work, making apparent the connections among them. Some specific examples are discussed in section III, where equivalent models are explicitly shown and analytic solutions are found. We summarize our results in section IV. Finally, we present our concluding remarks, and discuss avenues of future research in section V.

Throughout the text we use natural units, where $c = G = \hbar = k_B = 1$.

II. THREE POSSIBILITIES FOR COSMIC ACCELERATION

In this section we briefly review the three roads to cosmic acceleration and present some models that will be discussed throughout this work. We introduce the basic equations that govern the evolution of the scale factor in these frameworks, exhibiting the formal analogy between them.

A. Decaying cosmological term

A dynamic cosmological term (i.e., a component with energy-momentum tensor $T_{\mu\nu} = \Lambda(x) g_{\mu\nu}$) emerges naturally in the framework of quantum field theory (see, e.g., ref. [2] and refs. therein). Cosmological models with time-varying Λ were introduced in the 1980's (see, e.g., refs. [13, 14, 15, 16, 17, 18]) either as alternatives to the inflationary paradigm and to a primordial singularity, or to reconcile inflation with observational data. Such Λ decaying cosmologies attracted more interest in the 1990's, also in connection with the age problem, and more models were proposed and tested against the available data (see, e.g., refs. [19, 20, 21, 22, 23, 24, 25]). Finally, this type of model became popular in the turn of the century, impelled by supernovae and cosmic microwave background (CMB) evidence for cosmic acceleration and a smoothly distributed DE.

In most Λ decaying models, the density associated to the cosmological term is either determined by the dynamics of a scalar field or is given as an explicit expression in terms of the scale factor a and/or the expansion factor H . Here we shall

consider only the later case. For example, Özer and Taha [14] introduced a model in which $\Lambda \propto a^{-2}$. This same dependence was proposed by Chen and Wu [19], based on dimensional arguments in the context of quantum cosmology (see also [20, 26]). Gasperini, using the thermal interpretation of Λ [27], introduced the more general power law form [17]

$$\Lambda \propto a^{-m}. \quad (1)$$

Several observational consequences of this model were investigated by Waga and collaborators [21] and it was found to be consistent with the existing data for $m \gtrsim 1.6$. More recently, models in which the energy density associated to the cosmic term has the form

$$\rho_\Lambda = \rho_{\Lambda 0} + \frac{\varepsilon \rho_{m0}}{3 - \varepsilon} a^{-3+\varepsilon}$$

where also proposed [3, 28] and compared with observational data [29, 30]. In particular, a comparison of Large Scale Structure (LSS) data with CMB observations leads to the constraint $\varepsilon \lesssim 0.7 \times 10^{-3}$ [29].

Models in which Λ has a scaling in terms of the expansion factor were also proposed. For example, investigations using the renormalization group approach lead to a running of the cosmic term of the form¹ [3, 31, 32]

$$\Lambda = \Lambda_0 + \nu H^2. \quad (2)$$

This model is consistent with a number of observational data, but again a comparison with LSS and CMB data implies $\nu \lesssim 2.3 \times 10^{-3}$ [29] (see also [30]). On the other hand, an estimate of the cosmological term from the trace anomaly of quantum chromodynamics (QCD) yields [33]

$$\Lambda = \sigma H, \quad (3)$$

where the proportionality constant would be related to the QCD cutoff energy scale ($\sigma = \Lambda_{QCD}^3$).

Several other models, with different phenomenological scalings in H , a , or a combination of the two, have also been introduced (see, for instance, refs.² [22, 23, 24, 25, 35]). However, in what refers to specific models, in the remaining of the paper we shall restrict to the simple scalings given by eqs. (1), (2), and (3).

Now, let us state the basic equations that govern the dynamics of models with a cosmic fluid and a cosmological term (in a Friedmann–Lemaître–Robertson–Walker Universe). This will allow to make the equivalence with the other two frameworks explicit. Since a component with $p_\Lambda = -\rho_\Lambda$ can only be dynamic if it interacts with another matter-energy component, we shall impose only that the total energy-momentum is conserved, leading to the energy conservation equation

$$\dot{\rho} + 3H(p + \rho) = -\dot{\rho}_\Lambda, \quad (4)$$

¹ A quadratic scaling was also introduced previously, motivated by dimensional arguments [22].

² See also ref. [34], for a collection of several phenomenological Λ -decay laws and some historical remarks.

where $H = \dot{a}/a$ and the dot denotes derivative with respect to the cosmic time t . Differentiating the Friedmann equation

$$H^2 = \frac{8\pi}{3}(\rho + \rho_\Lambda) - \frac{k}{a^2} \quad (5)$$

and using equation (4) leads to

$$\dot{H} - \frac{k}{a^2} = -4\pi(p + \rho). \quad (6)$$

From now on, we shall consider only the case in which the cosmic fluid has a linear barotropic equation of state:

$$p = (\gamma - 1)\rho. \quad (7)$$

Using eq. (5) and inserting the expression above into eq. (6) one gets

$$\dot{H} + \frac{3\gamma}{2}H^2 + \frac{k}{a^2} \left(\frac{3\gamma}{2} - 1 \right) = 4\pi\gamma\rho_\Lambda. \quad (8)$$

Once an expression for ρ_Λ in terms of H and/or a is given, this equation can be solved to obtain the time behavior of the scale factor.

B. Viscous pressure

The investigation of possible roles of dissipative processes in the Universe has accompanied several developments of cosmology in the past decades. The consideration of models with dissipation started to raise considerable interest in the 1970's, both as a candidate to explain the high entropy per baryon ratio inferred from the CMB (see, e.g., ref. [36]), as well as a mechanism for isotropization and homogenization of the Universe (see, e.g., [37]). Later, viscosity was invoked as a way to avoid the primordial singularity (see, e.g., ref. [38]) and to drive an inflationary expansion (see, e.g., refs. [39, 40, 41, 42]). Its role in the transition from a de Sitter epoch to the Friedmann epoch (deflationary phase) was also investigated (see e.g. [43, 44]). More recently, models with bulk viscous pressure were investigated as possible sources for the current phase of accelerated expansion of the Universe (see e.g. refs. [4, 5]).

In the context of a homogeneous and isotropic Universe, dissipation can only be present through a bulk viscous pressure (also known as second viscosity), whose effect in the energy-momentum tensor (and therefore, on the equations of motion) is to add a new term to the isotropic dynamic pressure

$$p = P - \Pi, \quad (9)$$

where P is the equilibrium (thermodynamic) pressure and Π is a correction term present in dissipative situations. This term can appear either due to a real bulk viscosity — to be derived from first principles from kinetic theory (see, e.g., ref. [36, 45]) — or as an effective pressure associated to the phenomenon of particle creation, which arises naturally in the context of quantum processes (see e.g. [46]).

For first order deviations from equilibrium it may be shown that the generic form of the extra term is given by [36, 47]

$$\Pi = \zeta\theta, \quad (10)$$

where θ is the divergence of the four-velocity field u^μ , which in a homogeneous Universe, for comoving observers, is simply given by $\theta = 3H$. The viscosity coefficient ζ can be a function of the dynamic variables (ρ , p , etc.), but not of its (space-time) derivatives [36, 47]. The second law of thermodynamics requires that $\zeta > 0$ [36]. On phenomenological grounds, it is usually assumed that ζ is a function of the energy density ρ only. For example, a generic power law form for the viscosity coefficient

$$\zeta = \alpha\rho^n \quad (11)$$

was introduced and investigated in [48] and several analytic solutions were discussed in [40] and in subsequent papers.

As mentioned above, a bulk viscous pressure can also appear as a consequence of quantum effects, such as particle production and trace anomaly (for a calculation from first principles in the case of a scalar field, see ref. [46]). An *Ansatz* to represent the effects of particle creation, which includes non-linear terms in the expansion factor, was given by Novello and Araújo [49]. In their work, the viscous term takes the form,

$$\Pi = \sum_{k=1}^N \alpha_k \theta^k, \quad (12)$$

where the coefficients can be functions of ρ .

Prigogini and collaborators [50] applied the thermodynamics of open systems to cosmology and derived the bulk viscosity from particle creation. Assuming that the created particles are in thermal equilibrium with the existing ones and that the creation process occurs at constant specific entropy, they found³

$$\Pi = \frac{\rho + P}{n\theta} \Psi, \quad (13)$$

where n is the particle number density and Ψ is the particle source (if $\Psi > 0$) or sink (if $\Psi < 0$), which is given by $\Psi = N^\mu{}_{;\mu}$, where $N^\mu = nu^\mu$ is the particle flux vector. Calvão et al. [51] give a thermodynamical description of particle creation processes in the Universe, still restricting to adiabatic transformations, but relaxing the assumption of constant specific entropy. They propose the *Ansatz*

$$\Pi = \frac{\beta\Psi}{\theta}, \quad (14)$$

which generalizes expression (13).

As in the previous sections, we shall consider only a fluid with the equation of state⁴ $P = (\gamma - 1)\rho$. Inserting expres-

³ This form of Prigogini et al.'s result [50] is taken from ref. [51].

⁴ In models with $\gamma > 2/3$, the bulk viscous pressure can drive the accelerated expansion if $\Pi > (\gamma - 2/3)\rho$. However, in this case the correction term in (9) dominates over the thermodynamic pressure and is not a small correction to the energy-momentum tensor. Therefore this term is more naturally interpreted as arising from particle creation processes.

sion (9) in equation (6) and using the Friedmann equation (5) with no cosmological term we obtain

$$\dot{H} + \frac{3\gamma}{2}H^2 + \frac{k}{a^2} \left(\frac{3\gamma}{2} - 1 \right) = 4\pi\Pi. \quad (15)$$

A comparison with equation (8) makes evident that the viscous pressure can be equivalent to a cosmological term. If both Π and ρ_Λ are expressed as functions of the scale factor or the Hubble parameter alone, the explicit equivalence of the two models is straightforward. For example, a (linear) bulk viscosity of the form (10) with constant viscosity coefficient is equivalent to a cosmological term proportional to the expansion factor: $\rho_\Lambda = (3\zeta/\gamma)H$. Therefore, Schützhold's model (eq. 3) for Λ is equivalent to a linear viscosity with constant coefficient. However, in general, if Π and ρ_Λ have different explicit expressions in terms of ρ , H , or a , the equivalent models can only be found after the solution of the equation of motion is obtained.

C. Nonlinear equations of state

Models with nonlinear equations of state have raised a substantial interest recently, mainly due to the possibility of DE and DM unification [52, 53]. For instance, if p is a nonlinear function of the energy density ρ , it could be vanishing in high density regions, behaving as DM, and be close to $-\rho$ in low density ones, acting as DE. This behavior would lead naturally to a transition from matter domination to a de Sitter state in the cosmic evolution and could have a minor impact in processes taking place in the early Universe. Moreover, such equation of state can lead to a positive (adiabatic) sound velocity ($c_s^2 = dp/d\rho$), even for negative pressures.

An example of an equation of state exhibiting these properties is given by an inverse power law [7, 52, 53]

$$p = \frac{M^{4(\alpha+1)}}{\rho^\alpha}, \quad (16)$$

where $M > 0$ has dimension of mass and α is a real dimensionless constant. The case $\alpha = 1$ can be derived from the Born-Infeld action for a scalar field [54] and arises from the embedding of theories in higher dimensional space-times (in particular from $(3+1)$ -branes immersed in a $(4+1)$ -bulk), being connected with string theory [7, 55, 56]. Because of its similarity with an EOS proposed long ago in the context of aerodynamics [8], this fluid became known as Chaplygin gas, and the generic power law (16) as Generalized Chaplygin Gas (GCG [7, 53]). This type of matter component was dubbed as quartessence [57], since in this framework, only a fourth component, besides radiation, baryons, and neutrinos, would be needed to describe the cosmic evolution.

The GCG model is consistent with a number of observational data involving the background evolution of the Universe for $-1/2 \lesssim \alpha \lesssim 1/2$ (see, e.g., refs. [57, 58]). On the other hand, equation (16) fails to reproduce LSS and CMB data if it is applied to the fluctuations [59]. However, if the sound velocity vanishes (for example, due to the introduction of a cer-

tain type of intrinsic entropy perturbations), then the model is again in agreement with the data [60, 61].

Several other nonlinear EOS have been discussed in the literature (see, for instance, refs. [9, 52, 62, 63, 64, 65, 66]). Effective equations of state appear generically in scalar field models with noncanonical purely kinetic Lagrangians [67]. For example, a generalization of the Born-Infeld action [68] leads to equation (16), while other generalizations yield new classes of quartessence models (see, e.g., ref. [69]).

A natural generalization of equation (16), in the context of the linear EOS (7) considered in sections (II A) and (II B) is given by [70, 71, 72, 73, 74]

$$p = (\gamma - 1)\rho - M^{4(\alpha+1)}\rho^{-\alpha}, \quad (17)$$

which became known as the Modified Chaplygin Gas (MCG) EOS.

This type of EOS appears in several settings with different physical motivations. For example, the consideration of Schwarzschild–Anti de Sitter black holes in 5D [75] leads to the EOS

$$p = -\frac{2}{3}\rho - \frac{12}{\rho l^2}, \quad (18)$$

where l is associated to the curvature scale of the asymptotic anti-de Sitter space.

Another particular case of equation (17) that is useful in the cosmological setting, is given by

$$p = \frac{\rho}{3} \left(1 - \frac{\rho_a^2}{\rho^2} \right), \quad (19)$$

which provides an excellent approximation for a relativistic gas of massive particles [76].

For the purpose of illustrating the equivalence of the three frameworks, it is convenient to consider more generic EOS with a linear and a nonlinear part

$$p = (\gamma - 1)\rho - f(\rho). \quad (20)$$

Inserting this expression into equation (6) and using the Friedmann equation (5) with no cosmological term we obtain

$$\dot{H} + \frac{3\gamma}{2}H^2 + \frac{k}{a^2} \left(\frac{3\gamma}{2} - 1 \right) = 4\pi f(\rho). \quad (21)$$

Clearly, the similarity of the equation above with equations (8) and (15) is manifest, showing that a variable cosmic term, a bulk viscous pressure, and a nonlinear term in the EOS can play an equivalent role in the determination of the background cosmological dynamics. Although the aforementioned equivalence between the three frameworks holds for generic equations of state, we have restricted to fluids with a linear term to simplify the discussion of some specific examples that shall be developed in the next section.

It is worth noticing how the basic equations are modified in these three classes of models. The introduction of a new matter-energy component — the dynamical Λ term — alters

both the energy conservation and the Friedmann equations (eqs. 4 and 5 respectively) with respect to a single fluid. On the other hand, the inclusion of a viscous pressure modifies only the energy conservation, through the inclusion of Π . Finally, a nonlinear term in the EOS does not change any equation formally (although it appears in both equations).

III. EXAMPLES AND ANALYTIC SOLUTIONS

In this section we shall illustrate the connection among the three roads to cosmic acceleration discussed above through some examples, showing the explicit equivalence among specific models in each framework. In particular, we shall consider two common types of dynamical Λ models: a power law cosmic term of the form (1) and Λ -decaying models with a quadratic dependence on the expansion rate, including models (2) and (3) as particular cases. We explicitly find their analogues in terms of a bulk viscous pressure and a nonlinear EOS in some particular cases. In this process we derive novel analytical solutions to these models.

A. Cosmological term depending on the scale factor

For models in which the Λ term is given as a function of the scale factor, the equivalence with a model with nonlinear EOS of the form (20) can be easily tested if the energy conservation equation allows an analytic solution, providing an explicit expression for $\rho(a)$. In this case, $\rho_\Lambda(a)$ is directly linked to the non-linear term $f(\rho(a))$.

As an example, let us consider the following EOS:

$$\begin{aligned} p &= (\gamma - 1)\rho - \frac{B\rho}{1 + \sqrt{1 + C\rho}} \\ &= (\gamma - 1)\rho + \frac{B}{C} \left(1 - \sqrt{1 + C\rho}\right), \end{aligned} \quad (22)$$

where the parameter B is dimensionless, and $C > 0$ has dimensions of inverse energy density. The solution of the energy conservation equation (4), for $\gamma \neq 0$,⁵ is

$$\frac{\left(\frac{\sqrt{1+C\rho+1-B/\gamma}}{\sqrt{1+C\rho_0+1-B/\gamma}}\right)^{2\left(\frac{B-\gamma}{B-2\gamma}\right)}}{\left(\frac{\sqrt{1+C\rho-1}}{\sqrt{1+C\rho_0-1}}\right)^{\frac{2\gamma}{B-2\gamma}}} = \left(\frac{a}{a_0}\right)^{-3\gamma}. \quad (23)$$

Choosing $B = \gamma$ this expression can be easily inverted, leading to

$$\rho = c_1 a^{-3\gamma} + c_2 a^{-3\gamma/2}, \quad (24)$$

where $c_1 = C^{-1} (\sqrt{1 + C\rho_0} - 1)^2 a_0^{3\gamma}$ and $c_2 = 2\sqrt{c_1/C}$. Therefore, the nonlinear term is given by

$$f(\rho) = \frac{\gamma}{C} \left(\sqrt{1 + C\rho} - 1\right) \left(\frac{a}{a_0}\right)^{-3\gamma/2} = \gamma \frac{c_2}{2} a^{-3\gamma/2}. \quad (25)$$

We thus see that this nonlinear term plays the same role as a dynamical cosmic term of the form (1), with $m = 3\gamma/2$. Hence, in this case, the power law Λ decaying model with a γ -fluid (eq. 7) is equivalent to a single fluid with EOS (22), with $B = \gamma$. From equation (25), it is clear that these models are also analogous to a fluid with linear EOS and viscous pressure of the form $\Pi = \Pi_0 a^{-3\gamma/2}$ (see eq. 15).

If on the one hand the evolution of the scale factor will be the same in the three models, on the other hand analytical solutions may be easier to find with one specific choice of the equivalent descriptions. In particular, while it is not possible to find a simple analytic solution for the energy conservation equation (4) with a power law decaying Λ term and a linear fluid, such solution is easily found for the nonlinear EOS (22). The advantage of one description over the others in deriving the evolution of the system will become even more apparent in the discussion below.

Analytic solutions for $a(t)$ for the equivalent models discussed above can be easily found for $\gamma = 2n/3$ (where n is an integer number). For example, let us consider the ‘‘radiation era’’ ($\gamma = 4/3$), such that in the decaying Λ model we have $\rho_\Lambda \propto a^{-2}$ (or, in the case of viscous models, $\Pi \propto a^{-2}$). In this case, equations (8), (15), and (21) allow the simple analytic solution⁶

$$a = \left[\left(\frac{16\pi\sigma}{3} - k \right) t^2 + At \right]^{1/2}, \quad (26)$$

where σ is the constant of proportionality present in each model. The constant of integration A is left free, since equations (8), (15), and (21) are second order differential equations (and we have already used $a(0) = 0$).

The solution above is obtained in a simpler way for the nonlinear fluid model, since in this case we have an explicit expression for the energy density as a function of the scale factor (eq. 24), which can be inserted in the Friedmann equation (5), without a cosmological term, yielding

$$H^2 = \frac{8\pi}{3} \left(c_1 a^{-3\gamma} + c_2 a^{-3\gamma/2} \right) - \frac{k}{a^2}. \quad (27)$$

For $\gamma = 4/3$ this equation becomes

$$H^2 = \frac{8\pi c_1}{3} a^{-4} - \left(k - \frac{8\pi c_2}{3} \right) a^{-2}, \quad (28)$$

which is analogous to the one obtained for a Universe with only radiation and curvature, with the quantity $k - 8\pi c_2/3$ in

⁵ The particular case $\gamma = 0$ leads to the relation

$$\frac{(\sqrt{1+C\rho}-1) \exp \sqrt{1+C\rho}}{(\sqrt{1+C\rho_0}-1) \exp \sqrt{1+C\rho_0}} = \left(\frac{a}{a_0}\right)^{3B/2}.$$

⁶ Here and throughout we shall impose the condition $a(0) = 0$. On the other hand, to simplify the expressions, the ‘‘normalization’’ of $a_0 = a(t_0)$ will be left arbitrary.

the place of the curvature parameter k . Now we have a first order differential equation which can be easily solved to get

$$a = \left[\left(\frac{8\pi c_2}{3} - k \right) t^2 + 2t \sqrt{\frac{8\pi c_1}{3}} \right]^{1/2}. \quad (29)$$

Now the constant A has a clear interpretation in terms of the EOS parameters and the “normalization” $\rho(a_0) = \rho_0$.

A simple analytic solution can also be found from eq. (27) in the case $\gamma = 2/3$:

$$a = \frac{2\pi c_2}{3} t^2 + t \sqrt{\frac{8\pi c_1}{3} - k}. \quad (30)$$

Of course, this solution is also valid for a cosmological term decaying as a^{-1} in a Universe dominated by a fluid with EOS $p = -\rho/3$ (e.g., a gas of nonrelativistic cosmic strings) for any curvature k .

These simple examples illustrate how the use of equivalent models is handy for obtaining analytic solutions. In particular, for decaying Λ models where ρ_Λ is a function of the scale factor a , it may be more advantageous to use an equivalent model in terms of a nonlinear fluid, since it may allow to obtain analytic solutions for $\rho(a)$.

B. Cosmological term depending on the expansion rate

We shall now consider the case where the cosmic term is expressed as a function of H . More specifically, we introduce the *Ansatz*

$$\Lambda = \Lambda_0 + \sigma H + \nu H^2, \quad (31)$$

which can be regarded as a series expansion of a generic $\Lambda(H)$.

Clearly, from equations (8) and (15), it is easy to see that the dynamics of a Universe with a cosmological term of the form above and filled by a perfect fluid is equivalent to the one of a Universe with no cosmic term, but with and imperfect fluid with viscous pressure of the form (12) with constant coefficients (and including a constant term).

On the other hand, to make the comparison with nonlinear fluid models — for generic curvature — the combination of the Friedmann and energy conservation equations has to be used. Some examples that admit analytic solutions for $a(t)$, are discussed below.

1. Trace anomaly cosmological term

Let us first consider Schützhold’s model [33] of decaying Λ (eq. 3), which has the form (31) with $\Lambda_0 = 0$ and $\nu = 0$. As mentioned in section II B, in the case of a linear fluid (eq. 7), this model is equivalent to a viscous fluid with constant viscosity coefficient.

In a flat Universe, this model leads to the analytic solution [41, 77, 78]

$$a = [\exp(\sigma\gamma t/2) - 1]^{2/3\gamma}. \quad (32)$$

As pointed out by S. Carneiro [79], this same solution is found in the case of the MCG (eq. 17) with $\alpha = -1/2$. In fact the energy conservation equation allows a simple analytic solution for this type of fluid⁷:

$$\rho = \rho_0 \left[\left(1 - \frac{A}{\gamma} \right) a^{-3\gamma(1+\alpha)} + \frac{A}{\gamma} \right]^{1/(1+\alpha)}, \quad (33)$$

where ρ_0 is a constant of integration and $A = (M^4/\rho_0)^{1+\alpha}$. Thus, in the flat case and for $\alpha = -1/2$, the nonlinear term is given by

$$f(\rho) = M^2 \rho^{1/2} = H M^2 \sqrt{\frac{8\pi}{3}}, \quad (34)$$

showing the equivalence of the two models.

For some values of γ , this nonlinear fluid model with $\alpha = -1/2$ allows analytic solutions for any value of the curvature k . For example, if $\gamma = 4/3$ (which can be regarded as a “generalized radiation fluid”) one has,

$$a = \left[e^{\sqrt{6\pi} M^2 t} - 1 - \frac{k}{\frac{4\sqrt{\rho_0}}{3M^2} - 1} \left(\frac{\sinh \sqrt{\frac{3\pi}{2}} M^2 t}{\sqrt{\frac{3\pi}{2}} M^2} \right)^2 \right]^{1/2}, \quad (35)$$

while for $\gamma = 2/3$ (“generalized cosmic string gas”) one has

$$a = 2\pi\rho_0^{1/2} M^2 \left(1 - \frac{3M^2}{2\rho_0^{1/2}} \right) \left(\frac{\sinh \sqrt{\frac{3\pi}{2}} M^2 t}{\sqrt{\frac{3\pi}{2}} M^2} \right)^2 + \left[\frac{8\pi\rho_0}{3} \left(1 - \frac{3M^2}{2\rho_0^{1/2}} \right)^2 - k \right]^{1/2} \frac{\sinh \sqrt{6\pi} M^2 t}{\sqrt{6\pi} M^2}. \quad (36)$$

These new solutions generalize equation (32), for these two choices of γ , for arbitrary curvature. Although these solutions are not directly related to a constant bulk viscosity or cosmic term proportional to H , they exemplify how the use of this nonlinear fluid model, by analytically solving the energy conservation equation, simplifies the obtention of analytic solutions for $a(t)$.

Analytic solutions for $\alpha = 1$ and $\gamma = 4/3$, in the flat case (i.e., for EOS 19), were obtained in reference [76]. It is worth mentioning that this model is equivalent (again in the flat case) to a linear fluid with particle creation, leading to a bulk viscous term of the form (14) with constant source and $\beta \propto \rho^{-1/2}$. On the other hand, if $\beta = \text{const.}$, this particle creation model is equivalent to a MCG with $\alpha = 1/2$.

⁷ For $\gamma = 0$ the solution is

$$\rho = \rho_0 \left[1 + 3(1+\alpha) \left(\frac{M^4}{\rho_0} \right)^{(1+\alpha)} \ln \frac{a}{a_0} \right]^{1/(1+\alpha)}.$$

2. Renormalization group cosmological term

Another interesting case is found for $\sigma = 0$ in equation (31), which gives the form (2) representing the effect of quantum corrections to the vacuum energy in the renormalization group approach [3, 31, 32]. In the flat case, the effect of this cosmic term is clearly analogous to having a term $f(\rho) = (\Lambda_0/8\pi) + (\gamma\nu/3)\rho$ in equation (21). This analogy is still valid for generic curvature, as can be seen by inserting expression (2) into equation (8):

$$\dot{H} + \frac{\gamma}{2}(3-\nu)H^2 + \frac{k}{a^2}\left(\frac{3\gamma}{2} - 1\right) = \frac{\gamma\Lambda_0}{2}. \quad (37)$$

This equation has the same solution as for a model with constant Λ and linear EOS of the form (7), but with $\gamma' = \gamma(1-\nu/3)$, $k' = k(3\gamma/2 - 1)/(\gamma(3-\nu)/2 - 1)$, and $\Lambda'_0 = \Lambda_0/(1-\nu/3)$.

The case $\gamma = 1/(3-\nu)$ (i.e., $\gamma' = 1/3$), allows an analytic solution given by

$$a(t) = \sinh^2 \left[\frac{t}{2} \sqrt{\frac{\Lambda_0}{3-\nu}} \right] + \left[\frac{k}{\Lambda_0} (2\nu - 3) \right]^{\frac{1}{2}} \left[e^{t\sqrt{\Lambda_0/(3-\nu)}} - 1 \right]. \quad (38)$$

An explicit solution for $\gamma = 4/(3-\nu)$ can also be obtained as a particular case of equation (44), presented in the next section.

3. Quadratic expansion rate cosmological term

As a final example, let us consider the full quadratic form (eq. 31). Inserting this expression in equation (8) and introducing the new variable u , such that $a = u^n$, with

$$n = \frac{2}{\gamma(3-\nu)}, \quad (39)$$

one has

$$\ddot{u} - \frac{\gamma\sigma}{2}\dot{u} - \frac{\gamma\Lambda_0}{2n}u = -\frac{k}{n}\left(\frac{3\gamma}{2} - 1\right)u^{1-2n}. \quad (40)$$

For $k = 0$ (or $\gamma = 2/3$), the solution is

$$u = u_H := e^{\gamma\sigma t/4} \left\{ c_1 \exp \left[t \left(\frac{\gamma^2\sigma^2}{16} + \frac{\gamma\Lambda_0}{2n} \right)^{\frac{1}{2}} \right] + c_2 \exp \left[-t \left(\frac{\gamma^2\sigma^2}{16} + \frac{\gamma\Lambda_0}{2n} \right)^{\frac{1}{2}} \right] \right\}, \quad (41)$$

where c_1 and c_2 are arbitrary constants.

Imposing $a(0) = 0$ as in the previous cases, we have

$$a(t) = \left\{ e^{\gamma\sigma t/4} \sinh \left[t \left(\frac{\gamma^2\sigma^2}{16} + \frac{\gamma\Lambda_0}{2n} \right)^{\frac{1}{2}} \right] \right\}^n. \quad (42)$$

Notice that for $\nu = \Lambda_0 = 0$ we recover the solution (32), as it should be. Also, for $\sigma = 0$ and $\gamma = 1/(3-\nu)$, we recover expression (38) for $k = 0$.

Solutions for $k \neq 0$ can be found in some particular cases. For example, for $n = 1/2$, such that $\gamma = 4/(3-\nu)$, the solution is

$$u = u_H + k \left(\frac{3\gamma}{2} - 1 \right) \frac{2}{\gamma\Lambda_0}. \quad (43)$$

Again, imposing $a(0) = 0$, we have

$$a(t) = \left\{ e^{\gamma\sigma t/4} \sinh \left[t \left(\frac{\gamma^2\sigma^2}{16} + \gamma\Lambda_0 \right)^{\frac{1}{2}} \right] + \frac{2k}{\gamma\Lambda_0} \left(\frac{3\gamma}{2} - 1 \right) \left(1 - e^{\gamma\sigma t/4} \cosh \left[t \left(\frac{\gamma^2\sigma^2}{16} + \gamma\Lambda_0 \right)^{\frac{1}{2}} \right] \right) \right\}^{\frac{1}{2}} \quad (44)$$

A nonlinear fluid model that leads to the same solutions as the quadratic $\Lambda(H)$ model, in some cases, is given by the following EOS

$$p = (\gamma - 1)\rho - M^2\rho^{1/2} - A, \quad (45)$$

with A being a constant. This model allows an analytic solution for the energy conservation equation, which, for $\gamma \neq 0$, is given by⁸

$$\left(\frac{a}{a_0} \right)^{-3\gamma} = \frac{\left(\frac{\rho^{1/2} - \frac{M^2}{2\gamma} - \sqrt{\frac{M^4}{4\gamma^2} + \frac{A}{\gamma}}}{\rho_0^{1/2} - \frac{M^2}{2\gamma} - \sqrt{\frac{M^4}{4\gamma^2} + \frac{A}{\gamma}}} \right)^{\frac{M^2}{\sqrt{M^4+4A\gamma}}+1}}{\left(\frac{\rho^{1/2} - \frac{M^2}{2\gamma} + \sqrt{\frac{M^4}{4\gamma^2} + \frac{A}{\gamma}}}{\rho_0^{1/2} - \frac{M^2}{2\gamma} + \sqrt{\frac{M^4}{4\gamma^2} + \frac{A}{\gamma}}} \right)^{\frac{M^2}{\sqrt{M^4+4A\gamma}}-1}}. \quad (46)$$

Clearly, for $k = 0$ the model above (eq. 45) is equivalent to the quadratic Λ model (eq. 31), and has a solution of the form (42). However, this solution would hardly be found from inverting relation (46) and inserting the result in the Friedmann equation. This is an example where the use of a nonlinear EOS makes it harder to obtain analytic solutions, contrary to the examples of sections III A and III B 1.

IV. SUMMARY

In this work it was shown that three distinct classes of cosmological models (decaying Λ , bulk viscous pressure and fluids with exotic nonlinear equations of state) are equivalent, in the sense that in each of the classes there are models that reproduce exactly the same expansion history of the Universe.

⁸ For $\gamma = 0$ the relation is

$$\rho^{1/2} - \rho_0^{1/2} - \frac{A}{M^2} \ln \left(\frac{M^2\rho^{1/2} + A}{M^2\rho_0^{1/2} + A} \right) = \frac{3M^2}{2} \ln \frac{a}{a_0}.$$

This equivalence is apparent from the formal analogy among equations (8), (15), and (21).⁹

As explicit examples where the equivalence is manifest, we have considered two popular classes of “decaying vacuum” cosmologies: a power law Λ term (sec. III A) and a cosmic term depending on the expansion rate (sec. III B).

We have analyzed the case of a generic quadratic cosmological term of the form $\Lambda = \Lambda_0 + \sigma H + \nu H^2$, which is equivalent to a viscous term of the form (12) with $N = 2$ and constant coefficients, plus a constant term. In the flat case (and for a linear EOS $p = (\gamma - 1)\rho$) this model is also equivalent to the MCG Gas with $\alpha = -1/2$ plus a constant term (eq. 45).

We obtained a complete analytical solution for these models in the flat case (eq. 41). For arbitrary curvature, we have shown a solution for $\gamma = 4/(3 - \nu)$ (eq. 44). As far as we know, it is the first time that such type of model is discussed in the literature.

For $\Lambda_0 = 0$, and in the flat case, this model is equivalent to a linear viscosity of the form (10) with coefficient $\zeta = \zeta_0 + \zeta_1 \rho^{1/2}$ and, for a linear EOS, to the MCG (eq. 17) with $\alpha = -1/2$.

For $\Lambda_0 = 0 = \nu$ we recover the “trace anomaly cosmological term”, which is equivalent to a linear viscous term with constant viscosity coefficient. Again, in the flat case and for a linear EOS, this model is equivalent to the MCG with $\alpha = -1/2$.

We have found two analytic solutions for the MCG with $\alpha = -1/2$ and arbitrary curvature: one for the “generalized radiation fluid” (eq. 35) and another for the “generalized cosmic string gas” (eq. 36). This generalizes the solution for the flat case discussed in [78].

For $\sigma = 0$ we recover the “renormalization group cosmic term”. We have show that this Λ -decaying model is equivalent to a model with constant Λ and a linear fluid, with redefined values for the cosmological constant, the curvature, and the equation of state parameter (see sec. III B 2). We have obtained an analytic solution for the case $\gamma = 1/(3 - \nu)$ and arbitrary curvature.

Regarding the power law Λ term, we found that models with a linear equation of state $p = (\gamma - 1)\rho$ and a power law cosmic term of the form $\Lambda \propto a^{-3\gamma/2}$ (or equivalently a viscous pressure of the form $\Pi = \Pi_0 a^{-3\gamma/2}$) are equivalent to a model with EOS given by $p = (\gamma - 1)\rho + \gamma C^{-1} (1 - \sqrt{1 + C\rho})$ and no Λ . We have displayed the analytical solutions for $a(t)$ for $\gamma = 4/3$ and $\gamma = 2/3$, for generic curvature.

The use of equivalent models can be handy in finding analytical solutions. In fact, in some cases one particular model excels the other ones in the task of deriving explicit solutions.

Here, we have illustrated this property with particular examples where the solutions to a particular model are easily found using the equivalent model belonging to another class.

V. DISCUSSION AND CONCLUDING REMARKS

As mentioned above, a practical application of the mathematical equivalence pointed out in this work is in the search for analytical solutions, which can be useful for pedagogical purposes and to test numerical codes. But, besides this operational use, does the equivalence among the three classes of models have a deeper physical connection? Can we observationally distinguish these models?

Certainly they cannot be distinguished with observables that are mostly dependent on the background cosmology, such as the redshift-distance diagram of type Ia supernovae. On the other hand, although their dynamic role is equivalent, these models may have different thermodynamical behavior. For example, it is know that bulk viscosity and matter creation processes are thermodynamically distinct (see, e.g. [82])¹⁰. Thus, one may try to use thermodynamic arguments to distinguish among the three frameworks.

The existence of equivalent models in the three classes discussed in this work is not very surprising given the limited degrees of freedom of a homogeneous Universe. In fact a similar equivalence was pointed out in refs. [84] connecting scalar field models with nonlinear fluids in the homogeneous case.

A way to distinguish these models is to study their behavior when more degrees of freedom are present. Conversely, if the equivalence among the three frameworks remains in more generic configurations, this could point to a deeper connection among them. Thus, the natural next step in this investigation is to study the behavior of linear perturbations in the three classes of models.

Naturally, if Λ decaying models are considered homogeneous by construction, the perturbations of the inhomogeneous component will be clearly different from viscous models or models with nonlinear equations of state, since the viscous or nonlinear terms are intrinsically inhomogeneous. On the other hand, the vacuum contribution can be regarded being as space-time dependent. For example, in models where the cosmic term depends on the expansion rate, H can be replaced by a third of the local value of the velocity divergence (see, e.g., [85]).

Models with bulk viscous pressure are readily applicable to inhomogeneous situations by using the local values of the density and velocity divergence in the expression of the viscous term. Of course, models which are identical in the flat case for homogeneous configurations will have a different behavior in the perturbations.

⁹ It is worth mentioning that the equivalence among these frameworks has been discussed in previous works for some particular cases. For example, the equivalence between variable- Λ cosmological models and models with bulk viscosity was pointed out in refs. [80] (for one specific model in the flat case) and [81] (for two models). Also, the similarity of the trace anomaly cosmic term with the MCG with $\alpha = -1/2$, in the flat case, was shown in ref. [79].

¹⁰ The thermodynamics of a vacuum decaying model is discussed by Alcaniz & Lima [30], whereas the thermodynamic behavior of the GCG was investigated by Santos, Bedran, & Soares [83].

It is also straightforward to consider adiabatic perturbations in models with nonlinear equations of state. However, for quartessence models such as the GCG, this type of perturbations generate large scale structures that are inconsistent with observational data (see, e.g. [59, 63]). On the other hand, a specific type of entropy perturbation brings these models back in agreement with the data [60, 61, 63]. Several other ways to generalize to the inhomogeneous case equations of state that correspond to the GCG for the background have also been considered (see, e.g., refs. [86, 87, 88]).

The study of perturbations in the three frameworks is rather involving and presents several subtleties. Therefore, this analysis is left for subsequent works (see, e.g., [89]).

It is also worth investigating the microphysical motivation of the three frameworks. For example, it is known that models with nonlinear equations of state may arise from scalar fields with noncanonical kinetic terms [67, 68, 69].

Another avenue of research is to investigate whether the equivalence holds with respect to other frameworks for cosmic acceleration, such as self-interacting gases [4, 5] and other dark energy models.

Finally, several cosmological observables (such as large-

scale structure, cosmic microwave background fluctuations, and distance-redshift relations) have to be obtained in order to check the viability of these models with respect to the diverse set of observational data available.

Once these steps are undertaken, the study of equivalent models may prove useful for finding physical insights and motivations to phenomenological and first principle models that provide a good description of the available astro-cosmological data.

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