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OFF-SHELL "N=1 D=6" AND "CONFORMAL N=2 D=4" SUPERGRAVITY THEORIES

by

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Abstract

The superspace torsions constraints are imposed and a solution of the Bianchi identities in terms of superfields and their covariant derivatives is given. We show the component fields, their supersymmetry transformation and we exhibit a Lagrangian which couples supergravity and a matter multiplet. Finally it is shown that after dimensional reduction the N=1 D=6 off-shell supergravity theory yields the N=2 D=4 off-shell conformal supergravity theory.

Key-words: N=1 D=6 and conformal N=2 D=4 supergravity theories.

1. Introduction

Supersymmetry is a symmetry of space-time [1-3] which can be combined with internal symmetries [4-6] in a non-trivial way, circumventing then well known "no-go" theorems [7].

During the last decade supersymmetry [8] has been a subject that has deserved much attention. When this new symmetry is realized locally we have supergravity [8-9] which puts together supersymmetry and the theory of gravity.

Unification of gravity with other interactions, via supergravity, which has been proved to be renormalizable [10] at the two-loop level, is an important point. It is also a subject of research the renormalizability [11] at the three-loop level. Another interesting aspect is that concerned with quantum supergravity where one has remarkable cancellations of infinities [10]. Quantum properties like these are consequences of supersymmetry and one might expect this good behaviour in extended supergravity [4,12-16].

The N=1 supergravity theory in six dimensions presents, besides the extra simplicity of higher dimensional theories [17], the useful features for studying unification of an ordinaty gauge theory with supergravity, via dimensional reduction [18], giving then a geometrical interpretation for the internal quantum number in the reduced theory [19]. Also as mentioned in reference [20] it may allow a better understanding of the ultraviolet divergence properties than those in four dimensions [19]. This theory is constructed here in the corresponding superspace [21] whose importance has already been pointed out [22], i.e., that of having a well defined mathematical formalism, differential geometry in superspace.

In section 2 a brief review of supergravity in six dimensions is presented. In section 3 the superspace differential geometry is introduced. In section 4 we impose the torsion constraints and present a complete solution of the Bianchi identities. We know that there is some arbitrarines in defining torsion constraints since some choices lead to the same solution [23]. So only the analysis of the Bianchi identities will tell us if the torsion constraints are too restrictive or not, i.e., if they imply or not equations of motion. We will find that our formulation is off-shell. In section 5 we present the independent component fields and their supersymmetry transformation. In section 6 we exhibit a Lagrangian with a coupling between supergravity and matter recovering then the results of ref. [24] with the advantage that we can give a superfield formulation to it. In section 7 the dimensional reduction [25] from D=6 to the four dimensional space time is performed. We then show that the reduced theory fits the N=2 off-shell conformal supergravity [16,26]. In the appendix some useful relations are compiled.

2. Supergravity in six dimensions

We can define [27] six eight-dimensional matrices which are elements of a Clifford Algebra C_6:

$$\Gamma^{\underline{a}} = \begin{pmatrix} 0 & \Sigma^{\underline{a}}_{\underline{\alpha}\underline{\dot{\beta}}} \\ \bar{\Sigma}^{\underline{a}\,\underline{\dot{\alpha}}\underline{\beta}} & 0 \end{pmatrix} \tag{2.1}$$

$$\{\Gamma^{\underline{a}},\Gamma^{\underline{b}}\} = -2\eta^{\underline{ab}}\mathbb{1} \qquad \eta^{\underline{ab}} \sim (-1,1,1,1,1,1)$$

$$\underline{a}\,\varepsilon\{0,1,..,5\} \quad ; \quad \underline{\alpha},\underline{\dot{\beta}}\,\varepsilon\{1,2,3,4\}$$

$$\Sigma^{\underline{a}}_{\underline{\alpha}\underline{\dot{\beta}}} = (\Sigma^{a}_{\underline{\alpha}\underline{\dot{\beta}}},\Sigma^{4}_{\underline{\alpha}\underline{\dot{\beta}}},\Sigma^{5}_{\underline{\alpha}\underline{\dot{\beta}}}) =$$

$$= \left(\begin{pmatrix} \sigma^{a}_{\alpha\dot{\beta}} & 0 \\ 0 & \bar{\sigma}^{a\dot{\alpha}\beta} \end{pmatrix}, \begin{pmatrix} 0 & -i\delta_{\alpha}^{\beta} \\ i\delta^{\dot{\alpha}}_{\dot{\beta}} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\delta_{\alpha}^{\beta} \\ -\delta^{\dot{\alpha}}_{\dot{\beta}} & 0 \end{pmatrix}\right) \tag{2.2}$$

$$\bar{\Sigma}^{\underline{a}\underline{\dot{\alpha}}\underline{\beta}} = (\Sigma^{a\underline{\dot{\alpha}}\underline{\beta}},\Sigma^{4\underline{\dot{\alpha}}\underline{\beta}},\Sigma^{5\underline{\dot{\alpha}}\underline{\beta}}) =$$

$$\left(\begin{pmatrix} \bar{\sigma}^{a\dot{\alpha}\beta} & 0 \\ 0 & \sigma^{a}_{\alpha\dot{\beta}} \end{pmatrix}, \begin{pmatrix} 0 & i\delta_{\alpha}^{\beta} \\ -i\delta^{\dot{\alpha}}_{\dot{\beta}} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \delta_{\alpha}^{\beta} \\ \delta^{\dot{\alpha}}_{\dot{\beta}} & 0 \end{pmatrix}\right) \tag{2.3}$$

$$a\,\varepsilon\{0,1,2,3\} \quad ; \quad \alpha,\dot{\beta}\,\varepsilon\{1,2\}$$

From $\Gamma^{\underline{a}}$ we can construct 64 matrices which constitute a basis in the space of 8x8 matrices.

We can define matrices which are antisymmetric in the Lorentz indices:

$$(\Sigma^{\underline{ab}})_{\underline{\alpha}}{}^{\underline{\beta}} = \Sigma^{\underline{a}}_{\underline{\alpha}\underline{\dot{\lambda}}}\,\bar{\Sigma}^{\underline{b}\underline{\dot{\lambda}}\underline{\beta}} - \Sigma^{\underline{b}}_{\underline{\alpha}\underline{\dot{\lambda}}}\,\bar{\Sigma}^{\underline{a}\underline{\dot{\lambda}}\underline{\beta}}$$

$$(\Sigma^{\underline{ab}})^{\underline{\dot{\alpha}}}{}_{\underline{\dot{\beta}}} = \bar{\Sigma}^{\underline{a}\underline{\dot{\alpha}}\underline{\lambda}}\Sigma^{\underline{b}}_{\underline{\lambda}\underline{\dot{\beta}}} - \bar{\Sigma}^{\underline{b}\underline{\dot{\alpha}}\underline{\lambda}}\Sigma^{\underline{a}}_{\underline{\lambda}\underline{\dot{\beta}}} \tag{2.4}$$

The generators of the Lorentz group, which is the structure group, are:

$$(L_{\underline{ab}})_{\underline{C}}{}^{\underline{D}} = \begin{vmatrix} (L_{\underline{ab}})_{\underline{c}}{}^{\underline{d}} & & \\ & \frac{1}{4}(\Sigma_{\underline{ab}})_{\underline{\alpha}}{}^{\underline{\beta}} & \\ & & \frac{1}{4}(\Sigma_{\underline{ab}})^{\underline{\dot\alpha}}{}_{\underline{\dot\beta}} \end{vmatrix} \tag{2.5}$$

where

$$(L_{\underline{ab}})_{\underline{c}}{}^{\underline{d}} = -(\eta_{\underline{ac}}\delta^{\underline{d}}_{\underline{b}} - \eta_{\underline{bc}}\delta^{\underline{d}}_{\underline{a}}) \tag{2.6}$$

and

$$L_{\underline{ab}} = \frac{1}{4}(\Sigma_{\underline{ab}})_{\underline{\alpha}}{}^{\underline{\beta}} L_{\underline{\beta}}{}^{\underline{\alpha}} = \frac{1}{4}(\bar\Sigma_{\underline{ab}})^{\underline{\dot\beta}}{}_{\underline{\dot\alpha}} L^{\underline{\dot\alpha}}{}_{\underline{\dot\beta}} \tag{2.7a}$$

$$L_{\alpha}{}^{\beta} = -\frac{1}{4}(\Sigma^{\underline{a}}\bar\Sigma^{\underline{b}})_{\alpha}{}^{\beta} L_{ab} \; ; \; (L_{\alpha}{}^{\beta})^{T} = L^{\beta}{}_{\alpha} \tag{2.7b}$$

$$L^{\underline{\dot\alpha}}{}_{\underline{\dot\beta}} = -\frac{1}{4}(\bar\Sigma^{\underline{a}}\Sigma^{\underline{b}})^{\underline{\dot\alpha}}{}_{\underline{\dot\beta}} L_{\underline{ab}} \; ; \; (L^{\underline{\dot\alpha}}{}_{\underline{\dot\beta}})^{T} = L_{\underline{\dot\beta}}{}^{\underline{\dot\alpha}} \tag{2.7c}$$

The matrices $(\bar\Sigma^{\underline{ab}})^{T}$ and $-\Sigma^{\underline{ab}}$ are equivalent representations of the Lorentz group.

$$(\bar\Sigma^{\underline{ab}})^{T} = -X^{-1}\Sigma^{\underline{ab}}X \tag{2.8}$$

where

$$X_{\underline{\alpha}}{}^{\underline{\dot{\beta}}} \sim \begin{pmatrix} 0 & \varepsilon_{\alpha\beta} \\ -\varepsilon^{\dot{\alpha}\dot{\beta}} & 0 \end{pmatrix} \quad ; \quad X^{-1\underline{\beta}}_{\underline{\dot{\alpha}}} \sim \begin{pmatrix} 0 & -\varepsilon_{\dot{\alpha}\dot{\beta}} \\ \varepsilon^{\alpha\beta} & 0 \end{pmatrix}$$

$$\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha} \quad ; \quad \varepsilon_{\dot{\alpha}\dot{\beta}} = -\varepsilon_{\dot{\beta}\dot{\alpha}} \tag{2.9}$$

$$\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha} \quad ; \quad \varepsilon^{\dot{\alpha}\dot{\beta}} = -\varepsilon^{\dot{\beta}\dot{\alpha}}$$

$$\varepsilon_{12} = -\varepsilon^{12} = -1$$

For the Lorentz group (SO(1,5)) the spinor representation is four-dimensional. For this group the dotted and undotted representations are equivalent.

$$\psi_{\underline{\alpha}} = X_{\underline{\alpha}}{}^{\underline{\dot{\beta}}} \bar{\phi}_{\underline{\dot{\beta}}} \quad ; \quad \bar{\phi}_{\underline{\dot{\alpha}}} = X^{-1\underline{\beta}}_{\underline{\dot{\alpha}}} \psi_{\underline{\beta}} \tag{2.10}$$

$$\psi^{\underline{\alpha}} = X^{-1T\underline{\alpha}}{}_{\underline{\dot{\beta}}} \phi^{\underline{\dot{\beta}}} \quad ; \quad \phi^{\underline{\dot{\alpha}}} = X^{T\underline{\dot{\alpha}}}{}_{\underline{\beta}} \psi^{\underline{\beta}}$$

3. Superspace differential geometry

The elements of the superspace are denoted by $z^{\underline{M}} \sim (x^{\underline{m}}, \theta^{\underline{\mu}}, \bar{\theta}^{\underline{\dot{\mu}}})$ where $X^{\underline{m}}$ = ($\underline{m}$=0,1,...5) are the space-time coordinates and $\theta^{\underline{\mu}}$, $\theta^{\underline{\dot{\mu}}}$($\underline{\mu}$,$\underline{\dot{\mu}}$,1,2,3,4) are elements of a Grassmann algebra. The index notation is the usual: Latin letters (m) denote world tensor indices, Greek letters ($\underline{\mu}$,$\underline{\dot{\mu}}$) worlds spinor indices. A capital letter ($\underline{M}$) denotes a superspace index.

The vielbein $E_{\underline{M}}{}^{\underline{A}}(z)$ relates world indices ($\underline{M}$, from the middle of the alphabet) to tangent space indices ($\underline{A}$, from the beginning of the alphabet). The frame $E^{\underline{A}}$ is related to the coordinate one-form $dz^{\underline{M}}$ by

$$E^{\underline{A}} = dz^{\underline{M}} E_{\underline{M}}{}^{\underline{A}} \tag{3.1}$$

The vielbein is invertible and its inverse is given by $E_{\underline{A}}{}^{\underline{M}}$. The summation convention is:

$$E_{\underline{M}}{}^{\underline{A}}E_{\underline{A}}{}^{\underline{N}} = E_{\underline{M}}{}^{\underline{a}}E_{\underline{a}}{}^{\underline{N}} + E_{\underline{M}}{}^{\underline{\alpha}}E_{\underline{\alpha}}{}^{\underline{N}} - E_{\underline{M}}{}^{\underline{\dot\alpha}}E_{\underline{\dot\alpha}}{}^{\underline{N}} = \delta_{\underline{M}}^{\underline{N}} \tag{3.2}$$

The connection $\phi_{\underline{MA}}{}^{\underline{B}}(z)$ is Lie-algebra valued. This means that only the components $\phi_{\underline{ma}}{}^{\underline{b}}$, $\phi_{\underline{m\alpha}}{}^{\underline{\beta}}$, $\phi_{\underline{m\dot\alpha}}{}^{\underline{\dot\beta}}$ are different from zero and have the same properties, with respect to the indices AB, as the generators of the Lorentz group.

We can define covariant derivatives:*

$$\mathcal{D}_{\underline{M}}v^{\underline{A}} = \partial_{\underline{M}}v^{\underline{A}} + (-1)^{\underline{mb}}\, v^{\underline{B}}\,\phi_{\underline{MB}}{}^{\underline{A}} \tag{3.3}$$

$$\mathcal{D}_{\underline{M}}u_{\underline{A}} = \partial_{\underline{M}}u_{\underline{A}} - \phi_{\underline{MA}}{}^{\underline{B}}\, u_{\underline{B}} \tag{3.4}$$

$$\mathcal{D}_{\underline{B}}v^{\underline{A}} = E_{\underline{B}}{}^{\underline{M}}\mathcal{D}_{\underline{M}}v^{\underline{A}} \;;\; \mathcal{D}_{\underline{B}}u_{\underline{A}} = E_{\underline{B}}{}^{\underline{M}}\mathcal{D}_{\underline{M}}u_{\underline{A}} \tag{3.5}$$

The torsion and curvature tensors are defined by:

$$T^{\underline{A}} = \frac{1}{2}\, E^{\underline{C}}{}_{\wedge}E^{\underline{B}}\, T_{\underline{BC}}{}^{\underline{A}} = dE^{\underline{A}} + E^{\underline{B}}{}_{\wedge}\phi_{\underline{B}}{}^{\underline{A}} \tag{3.6}$$

$$R_{\underline{A}}{}^{\underline{B}} = \frac{1}{2}\, E^{\underline{C}}{}_{\wedge}E^{\underline{D}}\, R_{\underline{DCA}}{}^{\underline{B}} = d\phi_{\underline{A}}{}^{\underline{B}} + \phi_{\underline{A}}{}^{\underline{C}}{}_{\wedge}\phi_{\underline{C}}{}^{\underline{B}} \tag{3.7}$$

where $E^{\underline{C}}{}_{\wedge}E^{\underline{B}}$ means the exterior product [28] of $E^{\underline{C}}$ and $E^{\underline{B}}$. It follows from the definition that $R_{\underline{A}}{}^{\underline{B}}$ is Lie-algebra valued.

From the Poincaré lemma (dd=0) we obtain the Bianchi identities:

* $(-1)^{mb}$ is defined as: m(b)=0 if m(b) is vectorial, m(b)=1 if m(b) is spinorial.

$$dT^{\underline{A}} + T^{\underline{B}} \wedge \phi_{\underline{B}}{}^{\underline{A}} - E^{\underline{B}} \wedge R_{\underline{B}}{}^{\underline{A}} = 0 \tag{3.8}$$

$$dR_{\underline{A}}{}^{\underline{B}} + R_{\underline{A}}{}^{\underline{C}} \wedge \phi_{\underline{C}}{}^{\underline{B}} - \phi_{\underline{A}}{}^{\underline{C}} \wedge \phi_{\underline{C}}{}^{\underline{B}} = 0 \tag{3.9}$$

or

$$E^{\underline{C}} \wedge E^{\underline{D}} \wedge E^{\underline{E}} (\mathcal{D}_{\underline{E}} T_{\underline{D}\underline{C}}{}^{\underline{A}} - R_{\underline{E}\underline{D}\underline{C}}{}^{\underline{A}} + T_{\underline{E}\underline{D}}{}^{\underline{F}} T_{\underline{F}\underline{C}}{}^{\underline{A}}) \equiv E^{\underline{C}} \wedge E^{\underline{D}} \wedge E^{\underline{E}} I_{\underline{E}\underline{D}\underline{C}}{}^{\underline{A}} = 0 \tag{3.10}$$

$$E^{\underline{C}} \wedge E^{\underline{D}} \wedge E^{\underline{E}} (\mathcal{D}_{\underline{E}} R_{\underline{D}\underline{C}\underline{A}}{}^{\underline{B}} + T_{\underline{E}\underline{D}}{}^{\underline{F}} R_{\underline{F}\underline{C}\underline{A}}{}^{\underline{B}}) = 0 \tag{3.11}$$

The problem of choosing constraints is very intricate and so far only the technic of field redefinitions [29] has been generally used. The linearized version of a supergravity theory can also help in analyzing if a given set of constraints is too restrictive or not, i.e., if they imply or not equations of motion. It has also been shown [23] that there is some arbitrariness in defining a set of constraints since different choices can lead to the same solution of the Bianchi identities. So we have imposed the following constraints, in analogy with those for N=2 D=4 supergravity theory of reference [16]:

$$T^{\underline{a}}_{\underline{\alpha}\dot{\underline{\beta}}} = 2i\,\Sigma^{\underline{a}}_{\underline{\alpha}\dot{\underline{\beta}}} \;,\; T_{\underline{a}\underline{b}}{}^{\underline{c}} = 0$$

$$T_{\underline{\alpha}\underline{a}}{}^{\underline{b}} = T_{\dot{\underline{\alpha}}\underline{a}}{}^{\underline{b}} = T_{\underline{\alpha}\dot{\underline{\beta}}}{}^{\underline{\gamma}} = T_{\dot{\underline{\alpha}}\underline{\beta}}{}^{\dot{\underline{\gamma}}} = 0$$

$$T_{\underline{\alpha}\underline{\beta}}{}^{\underline{\gamma}} = T_{\dot{\underline{\alpha}}\dot{\underline{\beta}}}{}^{\dot{\underline{\gamma}}} = T_{\underline{\alpha}\underline{\beta}}{}^{\dot{\underline{\gamma}}} = T_{\dot{\underline{\alpha}}\dot{\underline{\beta}}}{}^{\underline{\gamma}} = T_{\underline{\alpha}\underline{\beta}}{}^{\underline{a}} = T_{\dot{\underline{\alpha}}\dot{\underline{\beta}}}{}^{\underline{a}} = 0 \tag{3.12}$$

The solution of the Bianchi identities showed us that these constraints did not imply any equation of motion i.e., we did not find any differential equation in the x-space for the physical components of the superfields of our theory. Then our formulation is an off-shell one.

Equivalently we can define $R_{\underline{A}}{}^{\underline{B}}$ and $T^{\underline{A}}$ by the Lie-algebra:*

$$[\mathcal{D}_{\underline{A}},\mathcal{D}_{\underline{B}}] = -R_{\underline{A}\underline{B}} - T_{\underline{A}\underline{B}}{}^{\underline{C}}\mathcal{D}_{\underline{C}} \tag{3.13}$$

The Jacoby identities are equivalent to the Bianchi identities.

4. Solution of the Bianchi identities

Due to the choice of the constraints the number of equations that constitute the first set of the Bianchi identities is reduced to twenty four equations but some of them are related by complex conjugation operation.

It has already been pointed out [30] that all components of the curvature are functions of the torsion components and that the second set of the Bianchi identities does not contain new information. So we can concentrate ourselves in the first set of the Bianchi identities.

Due to the choice of the structure group and constraints the equations corresponding to the Bianchi identities $I_{\underline{\alpha}\underline{\beta}\underline{\gamma}}{}^{\underline{\dot{\delta}}}$, $I_{\underline{\dot{\alpha}}\underline{\dot{\beta}}\underline{\dot{\gamma}}}{}^{\underline{\delta}}$, $I_{\underline{\alpha}\underline{\beta}\underline{\gamma}}{}^{\underline{a}}$, $I_{\underline{\dot{\alpha}}\underline{\dot{\beta}}\underline{\dot{\gamma}}}{}^{\underline{a}}$, $I_{\underline{\alpha}\underline{\beta}\underline{\dot{\gamma}}}{}^{\underline{a}}$ and $I_{\underline{\dot{\alpha}}\underline{\dot{\beta}}\underline{\gamma}}{}^{\underline{a}}$ vanish identically.

* $[\mathcal{D}_{\underline{A}},\mathcal{D}_{\underline{B}}]_+ = \mathcal{D}_{\underline{A}}\mathcal{D}_{\underline{B}} - (-1)^{ab}\mathcal{D}_{\underline{B}}\mathcal{D}_{\underline{A}}$

Using the technique of Young-tableaux [31] the solution of the identities $I_{\varepsilon\dot{\delta}c}{}^{a}$ and $I_{\varepsilon\dot{\delta}\gamma}{}^{\alpha}$ read:

$$T_{\underline{\beta}\underline{\gamma}\underline{\lambda}}{}^{\underline{\tau}} = 2\delta_{\underline{\beta}}{}^{\underline{\tau}} X_{\underline{\gamma}}{}^{\underline{\dot{\rho}}} G_{\underline{\dot{\rho}}\underline{\lambda}} - \delta_{\underline{\gamma}}{}^{\underline{\tau}} X_{\underline{\beta}}{}^{\underline{\dot{\rho}}} G_{\underline{\dot{\rho}}\underline{\lambda}} + \delta_{\underline{\lambda}}{}^{\underline{\tau}} X_{\underline{\beta}}{}^{\underline{\dot{\rho}}} G_{\underline{\dot{\rho}}\underline{\gamma}} \tag{4.1}$$

$$R_{\underline{\varepsilon}\underline{\dot{\delta}}\underline{c}\underline{a}} = -\frac{1}{2}(\Sigma_{\underline{ca}})_{\underline{\alpha}}{}^{\underline{\gamma}} R_{\underline{\varepsilon}\underline{\dot{\delta}}\underline{\gamma}}{}^{\underline{\alpha}} \tag{4.2}$$

$$R_{\underline{\varepsilon}\underline{\dot{\delta}}\underline{\gamma}}{}^{\underline{\alpha}} = -8i\delta_{\underline{\varepsilon}}{}^{\underline{\alpha}} G_{\underline{\dot{\delta}}\underline{\gamma}} - 8iX^{-1}_{\underline{\dot{\delta}}}{}^{\alpha} X_{\underline{\varepsilon}}{}^{\dot{\lambda}} G_{\underline{\dot{\lambda}}\underline{\gamma}} \tag{4.3}$$

$$R_{\underline{\dot{\varepsilon}}\underline{\delta}\underline{\dot{\gamma}}}{}^{\underline{\dot{\alpha}}} = 8i\delta_{\underline{\dot{\varepsilon}}}{}^{\underline{\dot{\alpha}}} G_{\underline{\delta}\underline{\dot{\gamma}}} + 8iX_{\underline{\delta}}{}^{\underline{\dot{\alpha}}} X^{-1}_{\underline{\dot{\varepsilon}}}{}^{\underline{\lambda}} G_{\underline{\lambda}\underline{\dot{\gamma}}} \tag{4.4}$$

Where

$$G_{\underline{\dot{\alpha}}\underline{\gamma}} \equiv X^{-1}_{\underline{\dot{\alpha}}}{}^{\underline{\varepsilon}} T_{\underline{\varepsilon}\underline{\gamma}\underline{\beta}}{}^{\underline{\beta}}$$

$$G_{\underline{\gamma}\underline{\dot{\alpha}}} \equiv X_{\underline{\gamma}}{}^{\underline{\dot{\varepsilon}}} T_{\underline{\dot{\varepsilon}}\underline{\dot{\alpha}}\underline{\dot{\beta}}}{}^{\underline{\dot{\beta}}}$$

$$G_{\underline{\dot{\lambda}}\underline{\beta}} = -G_{\underline{\beta}\underline{\dot{\lambda}}}$$

$$X_{\underline{\varepsilon}}{}^{\underline{\dot{\beta}}} G_{\underline{\dot{\beta}}\underline{\gamma}} = -X_{\underline{\gamma}}{}^{\underline{\dot{\beta}}} G_{\underline{\dot{\beta}}\underline{\varepsilon}} \tag{4.5}$$

Analogously the solutions of the identities $I_{\dot{\varepsilon}\dot{\delta}c}{}^{a}$ and $I_{\dot{\varepsilon}\dot{\delta}\dot{\gamma}}{}^{\dot{\alpha}}$ are:

$$T_{\underline{\dot{\varepsilon}}\underline{\alpha}\underline{\beta}}{}^{\underline{\gamma}} = \delta_{\underline{\alpha}}{}^{\underline{\gamma}} F_{\underline{\dot{\varepsilon}}\underline{\beta}} - \delta_{\underline{\beta}}{}^{\underline{\gamma}} F_{\underline{\dot{\varepsilon}}\underline{\alpha}} + 2X^{-1}_{\underline{\dot{\varepsilon}}}{}^{\underline{\gamma}} X_{\underline{\beta}}{}^{\underline{\dot{\lambda}}} F_{\underline{\dot{\lambda}}\underline{\alpha}} \tag{4.6}$$

$$R_{\underline{\dot{\varepsilon}}\underline{\dot{\delta}}\underline{c}\underline{a}} = -\frac{1}{2}(\Sigma_{\underline{ca}})_{\underline{\alpha}}{}^{\underline{\gamma}} R_{\underline{\dot{\varepsilon}}\underline{\dot{\delta}}\underline{\gamma}}{}^{\underline{\alpha}} \tag{4.7}$$

$$R_{\underline{\dot{\varepsilon}}\underline{\dot{\delta}}\underline{\gamma}}{}^{\underline{\alpha}} = 8i(X^{-1}_{\underline{\dot{\delta}}}{}^{\underline{\alpha}} F_{\underline{\dot{\varepsilon}}\underline{\gamma}} + X^{-1}_{\underline{\dot{\varepsilon}}}{}^{\underline{\alpha}} F_{\underline{\dot{\delta}}\underline{\gamma}}) \tag{4.8}$$

$$R_{\dot{\varepsilon}\dot{\delta}\dot{\gamma}}{}^{\underline{\dot{\alpha}}} = -8i(\delta_{\dot{\delta}}{}^{\underline{\dot{\alpha}}} X^{-1}_{\dot{\gamma}}{}^{\underline{\beta}} F_{\dot{\varepsilon}\beta} + \delta_{\dot{\varepsilon}}{}^{\underline{\dot{\alpha}}} X^{-1}_{\dot{\gamma}}{}^{\underline{\beta}} F_{\dot{\delta}\beta}) \tag{4.9}$$

With the equation (4.9) the identity $I_{\dot{\varepsilon}\dot{\delta}\dot{\gamma}}{}^{\dot{\alpha}}$ is identically fulfilled and the identity $I_{\dot{\gamma}\dot{\varepsilon}a}{}^{\tau}$ implies:

$$\mathcal{D}_{\dot{\gamma}} F_{\dot{\beta}\alpha} = 0 \tag{4.10}$$

Where

$$F_{\dot{\varepsilon}\gamma} \equiv -T_{\dot{\varepsilon}\gamma\lambda}{}^{\lambda}$$

$$X_{\delta}{}^{\dot{\varepsilon}} F_{\dot{\varepsilon}\gamma} = -X_{\gamma}{}^{\dot{\varepsilon}} F_{\dot{\varepsilon}\delta} \tag{4.11}$$

Now let us define:

$$B^{\underline{a}}_{\underline{\beta\lambda}} D^{\underline{b}\,\underline{\lambda\rho}} T_{\underline{ab}}{}^{\underline{\alpha}} \equiv M_{\beta}{}^{\underline{\rho\alpha}} \tag{4.12}$$

therefore

$$T_{\underline{\gamma\delta}}{}^{\underline{\lambda\eta\alpha}} = \frac{1}{16} B^{\underline{a}}_{\underline{\gamma\delta}} D^{\underline{b\lambda\eta}} T_{\underline{ab}}{}^{\underline{\alpha}} =$$

$$= \frac{1}{32} \left(\delta^{\underline{\eta}}_{\underline{\gamma}} M_{\underline{\delta}}{}^{\underline{\lambda\alpha}} - \delta^{\underline{\lambda}}_{\underline{\gamma}} M_{\underline{\delta}}{}^{\underline{\eta\alpha}} - \delta^{\underline{\eta}}_{\underline{\delta}} M_{\underline{\gamma}}{}^{\underline{\lambda\alpha}} + \delta^{\underline{\lambda}}_{\underline{\delta}} M_{\underline{\gamma}}{}^{\underline{\eta\alpha}}\right) \tag{4.13}$$

The identity $I_{\dot{\varepsilon}dc}{}^{a}$ allows to express $R_{\underline{\dot{\varepsilon}adc}}$ in terms of $T_{\underline{ad}}{}^{\underline{\alpha}}$.

$$R_{\underline{\dot{\varepsilon}adc}} = i\textstyle\sum_{\underline{d\alpha\dot{\varepsilon}}} T_{\underline{ca}}{}^{\underline{\alpha}} + i\sum_{\underline{c\alpha\dot{\varepsilon}}} T_{\underline{ad}}{}^{\underline{\alpha}} - i\sum_{\underline{a\alpha\dot{\varepsilon}}} T_{\underline{dc}}{}^{\underline{\alpha}} \tag{4.14}$$

Equation (4.14) in spinor notation (see appendix) becomes:

$$R_{\underline{\dot{\varepsilon}}\underline{a}\underline{\gamma}}{}^{\underline{\alpha}} = -\frac{i}{4} B^{\underline{d}}_{\underline{\gamma\lambda}} D^{\underline{c}\lambda\underline{\alpha}} \sum_{\underline{d}\underline{\beta}\underline{\dot{\varepsilon}}} T_{\underline{ca}}{}^{\underline{\beta}} +$$

$$-\frac{i}{4} B^{\underline{d}}_{\underline{\gamma\lambda}} D^{\underline{c}\lambda\underline{\alpha}} \sum_{\underline{c}\underline{\beta}\underline{\dot{\varepsilon}}} T_{\underline{ad}}{}^{\underline{\beta}} + \frac{i}{4} B^{\underline{d}}_{\underline{\gamma\lambda}} D^{\underline{c}\lambda\underline{\alpha}} \sum_{\underline{a}\underline{\beta}\underline{\dot{\varepsilon}}} T_{\underline{dc}}{}^{\underline{\beta}} \quad (4.15)$$

But

$$T_{\underline{ab}}{}^{\underline{\alpha}} = -\frac{1}{16} (\textstyle\sum_{\underline{ab}})_{\underline{\rho}}{}^{\underline{\beta}} M_{\underline{\beta}}{}^{\underline{\rho\alpha}} \quad (4.16)$$

Then eq. (4.15) becomes:

$$R_{\underline{\dot{\varepsilon}}\underline{a}\underline{\gamma}}{}^{\underline{\alpha}} = \frac{i}{64} [B^{\underline{d}}_{\underline{\gamma\lambda}} D^{\underline{c}\lambda\underline{\alpha}} \sum_{\underline{d}\underline{\beta}\underline{\dot{\varepsilon}}} (\textstyle\sum_{\underline{ca}})_{\underline{\rho}}{}^{\underline{\delta}} +$$

$$+ B^{\underline{d}}_{\underline{\gamma\lambda}} D^{\underline{c}\lambda\underline{\alpha}} \sum_{\underline{c}\underline{\beta}\underline{\dot{\varepsilon}}} (\textstyle\sum_{\underline{ad}})_{\underline{\rho}}{}^{\underline{\delta}} - B^{\underline{d}}_{\underline{\gamma\lambda}} D^{\underline{c}\lambda\underline{\alpha}} \sum_{\underline{a}\underline{\beta}\underline{\dot{\varepsilon}}} (\textstyle\sum_{\underline{dc}})_{\underline{\rho}}{}^{\underline{\delta}}] M_{\underline{\delta}}{}^{\underline{\rho\beta}} \quad (4.17)$$

Analogously we find

$$R_{\underline{\dot{\varepsilon}}\underline{a}\underline{\dot{\gamma}}}{}^{\underline{\dot{\alpha}}} = -\frac{i}{64} C^{\underline{d}}_{\underline{\dot{\gamma}\dot{\lambda}}} A^{\underline{c}\dot{\lambda}\underline{\dot{\alpha}}} [(\textstyle\sum_{\underline{ca}})_{\underline{\rho}}{}^{\underline{\beta}} \sum_{\underline{d}\underline{\alpha}\underline{\dot{\varepsilon}}} +$$

$$+ (\textstyle\sum_{\underline{ad}})_{\underline{\rho}}{}^{\underline{\beta}} \sum_{\underline{c}\underline{\alpha}\underline{\dot{\varepsilon}}} - (\textstyle\sum_{\underline{dc}})_{\underline{\rho}}{}^{\underline{\beta}} \sum_{\underline{a}\underline{\alpha}\underline{\dot{\varepsilon}}}] M_{\underline{\beta}}{}^{\underline{\rho\alpha}} \quad (4.18)$$

Now we insert eq. (4.17) into the identity $I_{\underline{\dot{\varepsilon}}\underline{d}\underline{\gamma}}{}^{\underline{\alpha}}$ and eq. (4.18) into the identity $I_{\underline{\dot{\varepsilon}}\underline{d}\underline{\dot{\gamma}}}{}^{\underline{\dot{\alpha}}}$ after symmetrisation in $\dot{\varepsilon}\dot{\gamma}$. Then the resulting equations yield after some algebra:

$$M_{\underline{\gamma}}{}^{\underline{\alpha\delta}} = 4i\delta_{\underline{\gamma}}{}^{\underline{\delta}} X^{-1}_{\underline{\dot{\rho}}}{}^{\lambda} \mathcal{D}_{\underline{\lambda}} F^{\underline{\dot{\rho}\alpha}} + 4i X^{-1}_{\underline{\dot{\rho}}}{}^{\delta} \mathcal{D}_{\underline{\gamma}} F^{\underline{\dot{\rho}\alpha}} \quad (4.19)$$

and

$$2X_{\underline{\gamma}}{}^{\dot{\gamma}}\mathcal{D}_{\underline{\gamma}}G^{\dot{\lambda}\underline{\alpha}} - \mathcal{D}_{\underline{\gamma}}F^{\dot{\lambda}\underline{\alpha}} + X_{\underline{\gamma}}{}^{\dot{\lambda}}X^{-1}_{\dot{\underline{\rho}}}{}^{\lambda}\mathcal{D}_{\underline{\lambda}}F^{\dot{\rho}\underline{\alpha}} + \delta^{\underline{\alpha}}_{\underline{\gamma}}\mathcal{D}_{\underline{\lambda}}F^{\dot{\lambda}\lambda} = 0 \quad (4.20)$$

where

$$F^{\dot{\underline{\rho}}\underline{\beta}} = \frac{1}{2}\,\varepsilon^{\underline{\alpha\beta\gamma\delta}}X_{\underline{\alpha}}{}^{\dot{\rho}}X_{\underline{\gamma}}{}^{\dot{\lambda}}F_{\dot{\underline{\lambda}}\underline{\delta}} \quad (4.21)$$

$$G^{\dot{\underline{\rho}}\underline{\beta}} = \frac{1}{2}\,\varepsilon^{\underline{\alpha\beta\gamma\delta}}X_{\underline{\alpha}}{}^{\dot{\rho}}X_{\underline{\gamma}}{}^{\dot{\lambda}}G_{\dot{\underline{\lambda}}\underline{\delta}} \quad (4.22)$$

With the results obtained until now and using eq. (3.13) the non-linear identities $I_{\underline{ed}\dot{\underline{\gamma}}}{}^{\underline{\alpha}}$ and $I_{\underline{edc}}{}^{\underline{\alpha}}$ and their complex conjugate are identically fulfilled. Identity $I_{\underline{dc\varepsilon}}{}^{\underline{\alpha}}$ in spinor notation reads:

$$B^{\underline{c}}_{\underline{\varepsilon\lambda}}D^{\underline{d\lambda\alpha}}R_{\underline{abcd}} = -4R_{\underline{ab\varepsilon}}{}^{\underline{\alpha}} \therefore R_{\underline{abcd}} = \frac{1}{4}(\Sigma_{\underline{cd}})_{\underline{\alpha}}{}^{\underline{\varepsilon}}R_{\underline{ab\varepsilon}}{}^{\underline{\alpha}} \quad (4.23)$$

Again let us define

$$\frac{1}{2}(\Sigma^{\underline{cd}})_{\underline{\gamma}}{}^{\underline{\delta}}R_{\underline{cd\varepsilon}}{}^{\underline{\alpha}} \equiv \phi_{\underline{\gamma}}{}^{\underline{\delta}}{}_{\underline{\varepsilon}}{}^{\underline{\alpha}} \quad (4.24)$$

Then

$$R_{\underline{ab\varepsilon}}{}^{\underline{\alpha}} = -\frac{1}{16}(\Sigma_{\underline{ab}})_{\underline{\delta}}{}^{\underline{\gamma}}\phi_{\underline{\gamma}}{}^{\underline{\delta}}{}_{\underline{\varepsilon}}{}^{\underline{\alpha}} \quad (4.25)$$

and

$$R_{\underline{abcd}} = -\frac{1}{64}(\Sigma_{\underline{ab}})_{\underline{\delta}}{}^{\underline{\gamma}}(\Sigma_{\underline{cd}})_{\underline{\alpha}}{}^{\underline{\varepsilon}}\phi_{\underline{\gamma}}{}^{\underline{\delta}}{}_{\underline{\varepsilon}}{}^{\underline{\alpha}} \quad (4.26)$$

From identity $I_{\underline{dc\varepsilon}}{}^{\underline{\alpha}}$ one obtains directly $R_{\underline{dc\varepsilon}}{}^{\underline{\alpha}}$. Then using eq. (4.24) the expression for $\phi_{\underline{\gamma}}{}^{\underline{\delta}}{}_{\underline{\varepsilon}}{}^{\underline{\alpha}}$ reads:

$$\phi_{\underline{\gamma}}{}^{\underline{\delta}}{}_{\underline{\varepsilon}}{}^{\underline{\alpha}} = \mathcal{D}_{\underline{\varepsilon}} M_{\underline{\gamma}}{}^{\underline{\delta}\underline{\alpha}} + (\Sigma^{\underline{d}\underline{c}})_{\underline{\gamma}}{}^{\underline{\delta}} [\frac{i}{4} \bar{\Sigma}_{\underline{d}}^{\dot{\underline{\beta}}\underline{\eta}} \times$$

$$\times \{\mathcal{D}_{\underline{\eta}}, \mathcal{D}_{\dot{\underline{\beta}}}\} (D_{\underline{c}}^{\underline{\lambda}\underline{\rho}} \delta_{\underline{\varepsilon}}^{\underline{\alpha}} X_{\underline{\rho}}{}^{\dot{\underline{\lambda}}} G_{\dot{\underline{\lambda}}\underline{\lambda}} - D_{\underline{c}}^{\underline{\lambda}\underline{\alpha}} X_{\underline{\varepsilon}}{}^{\dot{\underline{\lambda}}} G_{\dot{\underline{\lambda}}\underline{\lambda}}) +$$

$$+4D_{\underline{d}}^{\underline{\lambda}\underline{\beta}} D_{\underline{c}}^{\underline{\eta}\underline{\alpha}} X_{\underline{\varepsilon}}{}^{\dot{\underline{\rho}}} X_{\underline{\beta}}{}^{\dot{\underline{\lambda}}} G_{\dot{\underline{\rho}}\underline{\lambda}} G_{\dot{\underline{\lambda}}\underline{\eta}} +$$

$$+4A_{\underline{d}}^{\dot{\underline{\lambda}}\dot{\underline{\rho}}} D_{\underline{c}}^{\underline{\rho}\underline{\alpha}} X_{\underline{\varepsilon}}{}^{\dot{\underline{\beta}}} X^{-1}_{\dot{\underline{\lambda}}}{}^{\underline{\lambda}} F_{\dot{\underline{\beta}}\underline{\rho}} F_{\underline{\lambda}\dot{\underline{\rho}}} +$$

$$+4A_{\underline{d}}^{\dot{\underline{\lambda}}\dot{\underline{\beta}}} D_{\underline{c}}^{\underline{\rho}\underline{\alpha}} F_{\underline{\varepsilon}\dot{\underline{\lambda}}} F_{\dot{\underline{\beta}}\underline{\rho}} +$$

$$-4A_{\underline{d}}^{\dot{\underline{\lambda}}\dot{\underline{\beta}}} D_{\underline{c}}^{\underline{\rho}\underline{\lambda}} F_{\underline{\varepsilon}\dot{\underline{\lambda}}} F_{\dot{\underline{\gamma}}\underline{\rho}} X^{-1}_{\dot{\underline{\beta}}}{}^{\underline{\alpha}} X_{\underline{\lambda}}{}^{\dot{\underline{\gamma}}} +$$

$$-4A_{\underline{d}}^{\dot{\underline{\lambda}}\dot{\underline{\rho}}} D_{\underline{c}}^{\underline{\rho}\underline{\lambda}} \delta_{\underline{\varepsilon}}^{\underline{\alpha}} X^{-1}_{\dot{\underline{\lambda}}}{}^{\underline{\delta}} X_{\underline{\lambda}}{}^{\dot{\underline{\gamma}}} F_{\underline{\delta}\dot{\underline{\rho}}} F_{\dot{\underline{\gamma}}\underline{\rho}}] \qquad (4.27)$$

The consequences of identity $I_{\underline{abc}}{}^{d}$ are:

$$\phi_{\underline{\alpha}}{}^{\underline{\beta}}{}_{\underline{\gamma}}{}^{\underline{\delta}} = \phi_{\underline{\gamma}}{}^{\underline{\delta}}{}_{\underline{\alpha}}{}^{\underline{\beta}} \qquad (4.28)$$

$$\phi_{\underline{\alpha}}{}^{\underline{\beta}}{}_{\underline{\beta}}{}^{\underline{\gamma}} = \frac{1}{4} \delta_{\underline{\alpha}}{}^{\underline{\gamma}} \phi \qquad (4.29)$$

where

$$\phi \equiv \phi_{\underline{\alpha}}{}^{\underline{\beta}}{}_{\underline{\beta}}{}^{\underline{\alpha}} \qquad (4.30)$$

Let us compile all formulas which can be obtained from the constraints we have imposed and from the Bianchi identities. The components of the torsion which are different from zero are:

$$T_{\underline{\alpha\dot\beta}}{}^{\underline{c}} = T_{\underline{\dot\beta\alpha}}{}^{c} = 2i\,\Sigma^{\underline{c}}_{\underline{\alpha\dot\beta}} \qquad \text{(I)}$$

$$T_{\underline{\beta a}}{}^{\underline{\tau}} = D^{\underline{\gamma\lambda}}_{\underline{a}}\, T_{\underline{\beta\gamma\lambda}}{}^{\underline{\tau}}$$

$$T_{\underline{\beta\gamma\lambda}}{}^{\underline{\tau}} = 2\delta^{\underline{\tau}}_{\underline{\beta}}\, X_{\underline{\gamma}}{}^{\underline{\dot\rho}}\, G_{\underline{\dot\rho\lambda}} - \delta^{\underline{\tau}}_{\underline{\gamma}}\, X_{\underline{\beta}}{}^{\underline{\dot\rho}}\, G_{\underline{\dot\rho\lambda}} + \delta^{\underline{\tau}}_{\underline{\lambda}}\, X_{\underline{\beta}}{}^{\underline{\dot\rho}}\, G_{\underline{\dot\rho\gamma}}\,.$$

$$T_{\underline{\dot\beta a}}{}^{\underline{\dot\tau}} = A^{\underline{\dot\gamma\dot\lambda}}_{\underline{a}}\, T_{\underline{\dot\beta\dot\gamma\dot\lambda}}{}^{\underline{\dot\tau}}$$

$$T_{\underline{\dot\beta\dot\gamma\dot\lambda}}{}^{\underline{\dot\tau}} = -2\delta^{\underline{\dot\tau}}_{\underline{\dot\beta}}\, X^{-1}_{\underline{\dot\gamma}}{}^{\underline{\rho}}\, G_{\underline{\rho\dot\lambda}} + \delta^{\underline{\dot\tau}}_{\underline{\dot\gamma}}\, X^{-1}_{\underline{\dot\beta}}{}^{\underline{\rho}}\, G_{\underline{\rho\dot\lambda}} - \delta^{\underline{\dot\tau}}_{\underline{\dot\lambda}}\, X^{-1}_{\underline{\dot\beta}}{}^{\underline{\rho}}\, G_{\underline{\rho\dot\gamma}}$$

$$T_{\underline{\dot\varepsilon a}}{}^{\underline{\gamma}} = D^{\underline{\alpha\beta}}_{\underline{a}}\, T_{\underline{\dot\varepsilon\alpha\beta}}{}^{\underline{\gamma}}$$

$$T_{\underline{\dot\varepsilon\alpha\beta}}{}^{\underline{\gamma}} = \delta^{\underline{\gamma}}_{\underline{\alpha}}\, F_{\underline{\dot\varepsilon\beta}} - \delta^{\underline{\gamma}}_{\underline{\beta}}\, F_{\underline{\dot\varepsilon\alpha}} + 2X^{-1}_{\underline{\dot\varepsilon}}{}^{\underline{\gamma}}\, X_{\underline{\beta}}{}^{\underline{\dot\lambda}}\, F_{\underline{\dot\lambda\alpha}}$$

$$T_{\underline{\varepsilon a}}{}^{\underline{\dot\gamma}} = A^{\underline{\dot\alpha\dot\beta}}_{\underline{a}}\, T_{\underline{\varepsilon\dot\alpha\dot\beta}}{}^{\underline{\dot\gamma}}$$

$$T_{\underline{\varepsilon\dot\alpha\dot\beta}}{}^{\underline{\dot\gamma}} = \delta^{\underline{\dot\gamma}}_{\underline{\dot\alpha}}\, F_{\underline{\varepsilon\dot\beta}} - \delta^{\underline{\dot\gamma}}_{\underline{\dot\beta}}\, F_{\underline{\varepsilon\dot\alpha}} + 2X_{\underline{\varepsilon}}{}^{\underline{\dot\gamma}}\, X^{-1}_{\underline{\dot\beta}}{}^{\underline{\lambda}}\, F_{\underline{\lambda\dot\alpha}}$$

$$T_{\underline{ab}}{}^{\underline{\alpha}} = D^{\underline{\gamma\delta}}_{\underline{a}}\, B_{\underline{b\lambda\eta}}\, T_{\underline{\gamma\delta}}{}^{\underline{\lambda\eta\alpha}}$$

$$B^{\underline{a}}_{\underline{\beta\lambda}}\, D^{\underline{b\lambda\rho}}\, T_{\underline{ab}}{}^{\underline{\alpha}} \equiv M_{\underline{\beta}}{}^{\underline{\rho\alpha}}$$

$$T_{\underline{\gamma\delta}}{}^{\underline{\lambda\eta\alpha}} = \frac{1}{32}\left(\delta^{\underline{\eta}}_{\underline{\gamma}}\, M_{\underline{\delta}}{}^{\underline{\lambda\alpha}} - \delta^{\underline{\lambda}}_{\underline{\gamma}}\, M_{\underline{\delta}}{}^{\underline{\eta\alpha}} - \delta^{\underline{\eta}}_{\underline{\delta}}\, M_{\underline{\gamma}}{}^{\underline{\lambda\alpha}} + \delta^{\underline{\lambda}}_{\underline{\delta}}\, M_{\underline{\gamma}}{}^{\underline{\eta\alpha}}\right)$$

$$M_{\underline{\gamma}}{}^{\underline{\alpha\delta}} = 4i\,\delta^{\underline{\delta}}_{\underline{\gamma}}\, X^{-1}_{\underline{\dot\rho}}{}^{\underline{\lambda}}\, \mathcal{D}_{\underline{\lambda}} F^{\underline{\dot\rho\alpha}} + 4i\,X^{-1}_{\underline{\dot\rho}}{}^{\underline{\delta}}\, \mathcal{D}_{\underline{\gamma}} F^{\underline{\dot\rho\alpha}}$$

$$T_{\underline{ab}}{}^{\underline{\dot\alpha}} = A^{\underline{\dot\gamma\dot\delta}}_{\underline{a}}\, C_{\underline{b\dot\lambda\dot\eta}}\, T_{\underline{\dot\gamma\dot\delta}}{}^{\underline{\dot\lambda\dot\eta\dot\alpha}}$$

$$C^{\underline{a}}_{\underline{\dot\beta\dot\lambda}}\, A^{\underline{b\dot\lambda\dot\rho}}\, T_{\underline{ab}}{}^{\underline{\dot\alpha}} \equiv M_{\underline{\dot\beta}}{}^{\underline{\dot\rho\dot\alpha}}$$

$$T_{\underline{\dot\gamma}\underline{\dot\delta}}{}^{\dot\lambda\dot\eta\dot\alpha} = \frac{1}{32}\left(\delta^{\dot\eta}_{\dot\gamma} M_{\underline{\dot\delta}}{}^{\dot\lambda\dot\alpha} - \delta^{\dot\lambda}_{\dot\gamma} M_{\underline{\dot\delta}}{}^{\dot\eta\dot\alpha} - \delta^{\dot\eta}_{\dot\delta} M_{\underline{\dot\gamma}}{}^{\dot\lambda\dot\alpha} + \delta^{\dot\lambda}_{\dot\delta} M_{\underline{\dot\gamma}}{}^{\dot\eta\dot\alpha}\right)$$

$$M_{\underline{\dot\gamma}}{}^{\dot\alpha\dot\delta} = 4i\delta^{\dot\delta}_{\dot\gamma} X_{\underline{\rho}}{}^{\dot\lambda} \mathcal{D}_{\underline{\dot\lambda}} F^{\rho\dot\alpha} + 4i X_{\underline{\rho}}{}^{\dot\delta} \mathcal{D}_{\underline{\dot\gamma}} F^{\rho\dot\alpha}$$

The components of the curvature read:

$$R_{\underline{\varepsilon}\underline{\dot\delta}\underline{c}\underline{a}} = -\frac{1}{2}(\Sigma_{\underline{ca}})_{\underline{\alpha}}{}^{\gamma} R_{\underline{\varepsilon}\underline{\dot\delta}\underline{\gamma}}{}^{\alpha} \qquad \text{(II)}$$

$$R_{\underline{\varepsilon}\underline{\dot\delta}\underline{\gamma}}{}^{\alpha} = -8i\delta^{\alpha}_{\varepsilon} G_{\underline{\dot\delta}\underline{\gamma}} - 8i X^{-1}_{\underline{\dot\delta}}{}^{\alpha} X_{\underline{\varepsilon}}{}^{\dot\lambda} G_{\underline{\dot\lambda}\underline{\gamma}}$$

$$R_{\underline{\dot\varepsilon}\underline{\dot\delta}\underline{\gamma}}{}^{\dot\alpha} = 8i\delta^{\dot\alpha}_{\dot\varepsilon} G_{\underline{\delta}\underline{\dot\gamma}} + 8i X_{\underline{\delta}}{}^{\dot\alpha} X^{-1}_{\underline{\dot\varepsilon}}{}^{\lambda} G_{\underline{\lambda}\underline{\dot\gamma}}$$

$$R_{\underline{\dot\varepsilon}\underline{\dot\delta}\underline{c}\underline{a}} = -\frac{1}{2}(\Sigma_{\underline{ca}})_{\underline{\alpha}}{}^{\gamma} R_{\underline{\dot\varepsilon}\underline{\dot\delta}\underline{\gamma}}{}^{\alpha}$$

$$R_{\underline{\dot\varepsilon}\underline{\dot\delta}\underline{\gamma}}{}^{\alpha} = 8i\left(X^{-1}_{\underline{\dot\delta}}{}^{\alpha} F_{\underline{\dot\varepsilon}\underline{\gamma}} + X^{-1}_{\underline{\dot\varepsilon}}{}^{\alpha} F_{\underline{\dot\delta}\underline{\gamma}}\right)$$

$$R_{\underline{\dot\varepsilon}\underline{\dot\delta}\underline{\dot\gamma}}{}^{\dot\alpha} = -8i\left(\delta^{\dot\alpha}_{\dot\delta} X^{-1}_{\underline{\dot\gamma}}{}^{\beta} F_{\underline{\dot\varepsilon}\underline{\beta}} + \delta^{\dot\alpha}_{\dot\varepsilon} X^{-1}_{\underline{\dot\gamma}}{}^{\beta} F_{\underline{\dot\varepsilon}\underline{\beta}}\right)$$

$$R_{\underline{\dot\varepsilon}\underline{a}\underline{d}\underline{c}} = \frac{1}{4}(\Sigma_{\underline{dc}})_{\underline{\alpha}}{}^{\gamma} R_{\underline{\dot\varepsilon}\underline{a}\underline{\gamma}}{}^{\alpha}$$

$$R_{\underline{\dot\varepsilon}\underline{a}\underline{\gamma}}{}^{\alpha} = \frac{i}{64} B^{\underline{d}}_{\underline{\gamma\lambda}} D^{\underline{c}\lambda\alpha}\Big[\Sigma_{\underline{d}\underline{\beta}\underline{\dot\varepsilon}}(\Sigma_{\underline{ca}})_{\underline{\rho}}{}^{\delta} +$$

$$+ \Sigma_{\underline{c}\underline{\beta}\underline{\dot\varepsilon}}(\Sigma_{\underline{ad}})_{\underline{\rho}}{}^{\delta} - \Sigma_{\underline{a}\underline{\beta}\underline{\dot\varepsilon}}(\Sigma_{\underline{dc}})_{\underline{\rho}}{}^{\delta}\Big] M_{\underline{\delta}}{}^{\rho\beta}$$

$$R_{\underline{\dot\varepsilon}\underline{a}\underline{\dot\gamma}}{}^{\dot\alpha} = -\frac{i}{64} C^{\underline{d}}_{\underline{\dot\gamma}\underline{\dot\lambda}} A^{\underline{c}\dot\lambda\dot\alpha}\Big[\Sigma_{\underline{d}\underline{\alpha}\underline{\dot\varepsilon}}(\Sigma_{\underline{ca}})_{\underline{\rho}}{}^{\beta} +$$

$$+ \Sigma_{\underline{c}\underline{\alpha}\underline{\dot\varepsilon}}(\Sigma_{\underline{ad}})_{\underline{\rho}}{}^{\beta} - \Sigma_{\underline{a}\underline{\alpha}\underline{\dot\varepsilon}}(\Sigma_{\underline{dc}})_{\underline{\rho}}{}^{\beta}\Big] M_{\underline{\beta}}{}^{\rho\alpha}$$

$$R_{\underline{\varepsilon}\underline{\delta}\underline{c}\underline{a}} = \frac{1}{4}(\Sigma_{\underline{ca}})_{\underline{\alpha}}{}^{\gamma} R_{\underline{\varepsilon}\underline{\delta}\underline{\gamma}}{}^{\alpha}$$

$$R_{\underline{\varepsilon}\underline{\delta}\underline{\gamma}}{}^{\underline{\alpha}} = 8i(\delta_{\underline{\delta}}^{\underline{\alpha}} X_{\underline{\gamma}}{}^{\dot{\underline{\beta}}} F_{\underline{\varepsilon}\dot{\underline{\beta}}} + \delta_{\underline{\varepsilon}}^{\underline{\alpha}} X_{\underline{\gamma}}{}^{\dot{\underline{\beta}}} F_{\underline{\delta}\dot{\underline{\beta}}})$$

$$R_{\underline{\varepsilon}\underline{\delta}\dot{\underline{\gamma}}}{}^{\dot{\underline{\alpha}}} = -8i(X_{\underline{\delta}}{}^{\dot{\underline{\alpha}}} F_{\underline{\varepsilon}\dot{\underline{\gamma}}} + X_{\underline{\varepsilon}}{}^{\dot{\underline{\alpha}}} F_{\underline{\delta}\dot{\underline{\gamma}}})$$

$$R_{\underline{\varepsilon}\underline{a}\dot{\underline{\gamma}}}{}^{\dot{\underline{\alpha}}} = -\frac{i}{64} C^{\underline{d}}_{\dot{\underline{\gamma}}\dot{\underline{\lambda}}} A^{\underline{c}\dot{\underline{\lambda}}\dot{\underline{\alpha}}} \Big[\textstyle\sum_{\underline{d}\underline{\varepsilon}\dot{\underline{\beta}}} (\bar{\sum}_{\underline{c}\underline{a}})^{\dot{\underline{\delta}}}{}_{\dot{\underline{\rho}}} +$$

$$+ \textstyle\sum_{\underline{c}\underline{\varepsilon}\dot{\underline{\beta}}} (\bar{\sum}_{\underline{a}\underline{d}})^{\dot{\underline{\delta}}}{}_{\dot{\underline{\rho}}} - \sum_{\underline{a}\underline{\varepsilon}\dot{\underline{\beta}}} (\bar{\sum}_{\underline{d}\underline{c}})^{\dot{\underline{\delta}}}{}_{\dot{\underline{\rho}}} \Big] M_{\dot{\underline{\delta}}}{}^{\dot{\underline{\rho}}\dot{\underline{\beta}}}$$

$$R_{\varepsilon a\gamma}{}^{\alpha} = \frac{i}{64} B^{d}_{\gamma\lambda} D^{c\lambda\alpha} \Big[\textstyle\sum_{d\varepsilon\dot{\alpha}} (\bar{\sum}_{ca})^{\dot{\beta}}{}_{\dot{\rho}} +$$

$$+ \textstyle\sum_{c\varepsilon\dot{\alpha}} (\bar{\sum}_{ad})^{\dot{\beta}}{}_{\dot{\rho}} - \sum_{a\varepsilon\dot{\alpha}} (\bar{\sum}_{dc})^{\dot{\beta}}{}_{\dot{\rho}} \Big] M_{\dot{\beta}}{}^{\dot{\underline{\rho}}\dot{\underline{\alpha}}}$$

$$R_{\underline{a}\underline{b}\underline{c}\underline{d}} = \frac{1}{4} (\textstyle\sum_{\underline{c}\underline{d}})_{\underline{\alpha}}{}^{\underline{\varepsilon}} R_{\underline{a}\underline{b}\underline{\varepsilon}}{}^{\underline{\alpha}}$$

$$\frac{1}{2} (\textstyle\sum^{\underline{c}\underline{d}})_{\underline{\gamma}}{}^{\underline{\delta}} R_{\underline{c}\underline{d}\underline{\varepsilon}}{}^{\underline{\alpha}} \equiv \phi_{\underline{\gamma}}{}^{\underline{\delta}}{}_{\underline{\varepsilon}}{}^{\underline{\alpha}}$$

So

$$R_{\underline{a}\underline{b}\underline{\varepsilon}}{}^{\underline{\alpha}} = -\frac{1}{16} (\textstyle\sum_{\underline{a}\underline{b}})_{\underline{\delta}}{}^{\underline{\gamma}} \phi_{\underline{\gamma}}{}^{\underline{\delta}}{}_{\underline{\varepsilon}}{}^{\underline{\alpha}}$$

and

$$R_{\underline{a}\underline{b}\dot{\underline{\varepsilon}}}{}^{\dot{\underline{\alpha}}} = \frac{1}{256} (\textstyle\sum^{\underline{c}\underline{d}})^{\dot{\underline{\alpha}}}{}_{\dot{\underline{\varepsilon}}} (\sum_{\underline{c}\underline{d}})_{\underline{\alpha}}{}^{\underline{\varepsilon}} (\sum_{\underline{a}\underline{b}})_{\underline{\delta}}{}^{\underline{\gamma}} \phi_{\underline{\gamma}}{}^{\underline{\delta}}{}_{\underline{\varepsilon}}{}^{\underline{\alpha}}$$

$$\phi_{\underline{\gamma}}{}^{\underline{\delta}}{}_{\underline{\varepsilon}}{}^{\underline{\alpha}} = \mathcal{D}_{\underline{\varepsilon}} M_{\underline{\gamma}}{}^{\underline{\delta}\underline{\alpha}} + (\textstyle\sum^{\underline{d}\underline{c}})_{\underline{\gamma}}{}^{\underline{\delta}} \Big[\frac{i}{4} \bar{\sum}_{\underline{d}}{}^{\dot{\underline{\beta}}\underline{\eta}} \times$$

$$\times \{\mathcal{D}_{\underline{\eta}}, \mathcal{D}_{\dot{\underline{\beta}}}\} (D_{\underline{c}}{}^{\underline{\lambda}\underline{\rho}} \delta_{\underline{\varepsilon}}^{\underline{\alpha}} X_{\underline{\rho}}{}^{\dot{\underline{\lambda}}} G_{\dot{\underline{\lambda}}\underline{\lambda}} - D_{\underline{c}}{}^{\underline{\lambda}\underline{\alpha}} X_{\underline{\varepsilon}}{}^{\dot{\underline{\lambda}}} G_{\dot{\underline{\lambda}}\underline{\lambda}}) +$$

$$+ 4 D_{\underline{d}}{}^{\underline{\lambda}\underline{\beta}} D_{\underline{c}}{}^{\underline{\eta}\underline{\alpha}} X_{\underline{\varepsilon}}{}^{\dot{\underline{\rho}}} X_{\underline{\beta}}{}^{\dot{\underline{\lambda}}} G_{\dot{\underline{\rho}}\underline{\lambda}} G_{\dot{\underline{\lambda}}\underline{\eta}} +$$

$$+4A_{\underline{d}}^{\underline{\dot\lambda}\underline{\dot\rho}}\, D_{\underline{c}}^{\underline{\rho}\underline{\alpha}} X_{\underline{\varepsilon}}{}^{\underline{\dot\beta}}\, X^{-1}_{\underline{\dot\lambda}}{}^{\underline{\lambda}} F_{\underline{\dot\beta}\underline{\rho}} F_{\underline{\lambda}\underline{\dot\rho}} \quad +$$

$$+4A_{\underline{d}}^{\underline{\dot\lambda}\underline{\dot\beta}}\, D_{\underline{c}}^{\underline{\rho}\underline{\alpha}} F_{\underline{\varepsilon}\underline{\dot\lambda}} F_{\underline{\dot\beta}\underline{\rho}} \quad +$$

$$-4A_{\underline{d}}^{\underline{\dot\lambda}\underline{\dot\beta}}\, D_{\underline{c}}^{\underline{\rho}\underline{\lambda}} F_{\underline{\varepsilon}\underline{\dot\lambda}} F_{\underline{\dot\gamma}\underline{\rho}} X^{-1}_{\underline{\dot\beta}}{}^{\underline{\alpha}} X_{\underline{\lambda}}{}^{\underline{\dot\gamma}} \quad +$$

$$-4A_{\underline{d}}^{\underline{\dot\lambda}\underline{\dot\rho}}\, D_{\underline{c}}^{\underline{\rho}\underline{\lambda}}\, \delta_{\underline{\varepsilon}}^{\underline{\alpha}} X^{-1}_{\underline{\dot\lambda}}{}^{\underline{\delta}} X_{\underline{\lambda}}{}^{\underline{\dot\gamma}} F_{\underline{\delta}\underline{\dot\rho}} F_{\underline{\dot\gamma}\underline{\rho}}$$

The superfields F and G are subject to the following conditions:

$$G_{\underline{\dot\lambda}\underline{\beta}} = -G_{\underline{\beta}\underline{\dot\lambda}} \qquad \text{(III)}$$

$$X_{\underline{\varepsilon}}{}^{\underline{\dot\beta}} G_{\underline{\dot\beta}\underline{\gamma}} = -X_{\underline{\gamma}}{}^{\underline{\dot\beta}} G_{\underline{\dot\beta}\underline{\varepsilon}}$$

$$X^{-1}_{\underline{\dot\varepsilon}}{}^{\underline{\beta}} G_{\underline{\beta}\underline{\dot\gamma}} = -X^{-1}_{\underline{\dot\gamma}}{}^{\underline{\beta}} G_{\underline{\beta}\underline{\dot\varepsilon}}$$

$$X_{\underline{\delta}}{}^{\underline{\dot\varepsilon}} F_{\underline{\dot\varepsilon}\underline{\gamma}} = -X_{\underline{\gamma}}{}^{\underline{\dot\varepsilon}} F_{\underline{\dot\varepsilon}\underline{\delta}}$$

$$X^{-1}_{\dot\delta}{}^{\varepsilon} F_{\varepsilon\dot\gamma} = -X^{-1}_{\dot\gamma}{}^{\varepsilon} F_{\varepsilon\dot\delta}$$

$$\mathcal{D}_{\dot\gamma} F_{\dot\beta\alpha} = 0 \qquad \mathcal{D}_{\gamma} F_{\beta\dot\alpha} = 0$$

$$2X_{\underline{\gamma}}{}^{\underline{\dot\gamma}} \mathcal{D}_{\underline{\dot\gamma}} G^{\underline{\dot\lambda}\underline{\alpha}} - \mathcal{D}_{\underline{\gamma}} F^{\underline{\dot\lambda}\underline{\alpha}} + X_{\underline{\gamma}}{}^{\underline{\dot\lambda}} X^{-1}_{\underline{\dot\rho}}{}^{\underline{\lambda}} \mathcal{D}_{\underline{\lambda}} F^{\underline{\dot\rho}\underline{\lambda}} + \delta_{\underline{\gamma}}^{\underline{\alpha}} \mathcal{D}_{\underline{\lambda}} F^{\underline{\dot\lambda}\underline{\lambda}} = 0$$

$$2X^{-1}_{\underline{\dot\gamma}}{}^{\underline{\gamma}} \mathcal{D}_{\underline{\gamma}} G^{\underline{\lambda}\underline{\dot\alpha}} - \mathcal{D}_{\underline{\dot\gamma}} F^{\underline{\lambda}\underline{\dot\alpha}} + X^{-1}_{\underline{\dot\gamma}}{}^{\underline{\lambda}} X_{\underline{\rho}}{}^{\underline{\dot\lambda}} \mathcal{D}_{\underline{\dot\lambda}} F^{\underline{\rho}\underline{\dot\alpha}} + \delta_{\underline{\dot\gamma}}^{\underline{\dot\alpha}} \mathcal{D}_{\underline{\dot\lambda}} F^{\underline{\lambda}\underline{\dot\lambda}} = 0$$

$$F^{\underline{\dot\rho}\underline{\beta}} = \frac{1}{2}\, \varepsilon^{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}}\, X_{\alpha}{}^{\dot\rho} X_{\gamma}{}^{\dot\lambda} F_{\dot\lambda\delta}$$

$$G^{\dot{\underline{\rho}}\underline{\beta}} = \frac{1}{2}\,\varepsilon^{\underline{\alpha\beta\gamma\delta}}\,X_{\alpha}{}^{\dot{\rho}}\,X_{\gamma}{}^{\dot{\lambda}}\,F_{\dot{\lambda}\delta}$$

$$F^{\underline{\rho}\dot{\underline{\beta}}} = \frac{1}{2}\,\varepsilon^{\underline{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}}\,X^{-1}_{\underline{\dot{\alpha}}}{}^{\underline{\rho}}\,X^{-1}_{\underline{\dot{\gamma}}}{}^{\underline{\lambda}}\,F_{\underline{\lambda\dot{\delta}}}$$

$$G^{\rho\dot{\beta}} = \frac{1}{2}\,\varepsilon^{\underline{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}}\,X^{-1}_{\underline{\dot{\alpha}}}{}^{\underline{\rho}}\,X^{-1}_{\underline{\dot{\gamma}}}{}^{\underline{\lambda}}\,F_{\underline{\lambda\dot{\delta}}}$$

5. Component fields and supersymmetry transformation

In the last section the superfields $F_{\underline{\dot{\alpha}\beta}}$ and $G_{\underline{\dot{\alpha}\beta}}$ and their complex conjugates were shown to satisfy the equation (4.20) and its complex conjugate one. Using them plus equation (3.13) we can prove that the only independent component fields [32] of $F_{\underline{\dot{\alpha}\beta}}$ and $G_{\underline{\dot{\alpha}\beta}}$ are:

$$F_{\underline{\dot{\alpha}\beta}}\Big|_{\theta=\bar{\theta}=0} \equiv A_{\underline{\dot{\alpha}\beta}} \quad ; \quad G_{\underline{\dot{\alpha}\beta}}\Big|_{\theta=\bar{\theta}=0} \equiv B_{\underline{\dot{\alpha}\beta}} \tag{5.1}$$

$$\mathcal{D}_{\underline{\alpha}}F_{\underline{\dot{\beta}\gamma}}\Big|_{\theta=\bar{\theta}=0} \equiv C_{\underline{\alpha\dot{\beta}\gamma}} \quad ; \quad \mathcal{D}_{\underline{\delta}}\mathcal{D}_{\underline{\alpha}}F_{\underline{\dot{\beta}\gamma}}\Big|_{\theta=\bar{\theta}=0} \equiv D_{\underline{\delta\alpha\dot{\beta}\gamma}}$$

The higher components of F and G contain only derivatives on and multilinear combinations of x-space fields already appearing in the superfields. The component fields of the vielbein and connection superfields, in a suitable gauge are:

$$E_{\underline{M}}{}^{\underline{A}}(x,0,0) \sim \begin{pmatrix} e_{\underline{m}}{}^{\underline{a}} & \psi_{\underline{m}}{}^{\underline{\alpha}} & \psi_{\underline{m}}{}^{\underline{\dot{\alpha}}} \\ 0 & \delta^{\underline{\alpha}}_{\underline{\mu}} & 0 \\ 0 & 0 & \delta^{\underline{\dot{\alpha}}}_{\underline{\mu}} \end{pmatrix} \tag{5.2}$$

$$\phi_{\underline{B}}{}^{\underline{A}}(x,0,0) \sim dx^{\underline{m}}\,\phi_{\underline{mB}}{}^{\underline{A}} \qquad (5.3)$$

The ($SU^{+}(4)$) irreducible content of the independent component fields of $F_{\dot{\alpha}\beta}$ and $G_{\dot{\alpha}\beta}$ given above can be obtained using Young-tableau. We find

$$A_{\dot{\alpha}\beta} \; : 1+15$$

$$B_{\dot{\alpha}\beta} \; : 1+15$$

$$C_{\alpha\dot{\beta}\gamma} \; : 4+4+20+36$$

$$D_{\delta\alpha\dot{\beta}\gamma} : 6+6+6+10+10+10+\overline{10}+64+64+70 \qquad (5.4)$$

For any superfield M we define the supersymmetry transformation by:

$$\delta_{\xi}M = -(\xi^{\underline{\alpha}}\mathcal{D}_{\underline{\alpha}} - \bar{\xi}^{\underline{\dot{\alpha}}}\mathcal{D}_{\underline{\dot{\alpha}}})M \qquad (5.5)$$

Then we have:

$$\delta e_{\underline{m}}{}^{\underline{a}} = -2i(\psi_{\underline{m}}{}^{\underline{\alpha}}\,\Sigma^{\underline{a}}_{\underline{\alpha\dot{\beta}}}\,\bar{\xi}^{\underline{\dot{\beta}}} - \xi^{\underline{\beta}}\,\Sigma^{\underline{a}}_{\underline{\beta\dot{\alpha}}}\,\psi_{\underline{m}}{}^{\underline{\dot{\alpha}}}) \qquad (5.6)$$

$$\delta\psi_{\underline{m}}{}^{\underline{\alpha}} = 2\xi^{\underline{\alpha}}\,\bar{\Sigma}_{\underline{c}}^{\underline{\dot{\delta}\rho}}\,e_{\underline{m}}{}^{\underline{c}}\,G_{\underline{\dot{\delta}\rho}} - 2\xi^{\underline{\beta}}\,e_{\underline{m}}{}^{\underline{c}}\,\Sigma_{\underline{c}}^{\underline{\dot{\delta}\alpha}}\,G_{\underline{\dot{\delta}\beta}} +$$

$$+2\bar{\xi}^{\underline{\dot{\beta}}}\,e_{\underline{m}}{}^{\underline{c}}\,D_{\underline{c}}^{\underline{\rho\alpha}}\,F_{\underline{\dot{\beta}\rho}} + 2\bar{\xi}^{\underline{\dot{\beta}}}\,e_{\underline{m}}{}^{\underline{c}}\,\bar{\Sigma}_{\underline{d}}^{\underline{\dot{\delta}\rho}}\,X_{\underline{\dot{\beta}}}^{-1}{}^{\underline{\alpha}}F_{\underline{\dot{\delta}\rho}} - \mathcal{D}_{\underline{m}}\xi^{\underline{\alpha}} \qquad (5.7)$$

$$\delta\phi_{\underline{mA}}{}^{\underline{B}} = -\,\xi^{\underline{\alpha}}e_{\underline{m}}{}^{\underline{a}}R_{\underline{\alpha a A}}{}^{\underline{B}} - \xi^{\underline{\alpha}}\psi_{\underline{m}}{}^{\underline{\beta}}R_{\underline{\alpha\beta A}}{}^{\underline{B}} +$$

$$+ \xi^{\underline{\alpha}} \bar{\psi}_{\underline{m}}{}^{\dot{\underline{\beta}}} R_{\underline{\alpha}\dot{\underline{\beta}}\underline{A}}{}^{\underline{B}} + \bar{\xi}^{\dot{\underline{\alpha}}} e_{\underline{m}}{}^{\underline{a}} R_{\dot{\underline{\alpha}}\underline{a}\underline{A}}{}^{\underline{B}} +$$

$$+ \bar{\xi}^{\dot{\underline{\alpha}}} \psi_{\underline{m}}{}^{\underline{\beta}} R_{\dot{\underline{\alpha}}\underline{\beta}\underline{A}}{}^{\underline{B}} - \bar{\xi}^{\dot{\underline{\alpha}}} \psi_{\underline{m}}{}^{\dot{\underline{\beta}}} R_{\dot{\underline{\alpha}}\dot{\underline{\beta}}\underline{A}}{}^{\underline{B}} \tag{5.8}$$

where all the expressions for $R_{\underline{ABC}}{}^{D}$ have already been obtained.

$$\delta A_{\dot{\underline{\alpha}}\underline{\beta}} = -\xi^{\underline{\rho}} C_{\underline{\rho}\dot{\underline{\alpha}}\underline{\beta}} \tag{5.9}$$

$$\delta B_{\dot{\underline{\alpha}}\underline{\beta}} = \xi^{\underline{\rho}} (X_{\underline{\rho}}{}^{\dot{\underline{\lambda}}} C_{\dot{\underline{\lambda}}\underline{\beta}\dot{\underline{\alpha}}} - \frac{1}{2} X_{\dot{\underline{\alpha}}}^{-1\,\underline{\lambda}} X_{\dot{\underline{\beta}}}^{-1\,\underline{\gamma}} X_{\dot{\underline{\eta}}}^{-1\,\underline{\sigma}} \varepsilon_{\underline{\beta\lambda\rho\sigma}} \times$$

$$\varepsilon^{\dot{\underline{\eta}}\dot{\underline{\lambda}}\dot{\underline{\beta}}\dot{\underline{\rho}}} C_{\dot{\underline{\lambda}}\underline{\gamma}\dot{\underline{\rho}}}) - \xi^{\dot{\underline{\rho}}} (X_{\dot{\underline{\rho}}}^{-1\,\underline{\gamma}} C_{\underline{\gamma}\dot{\underline{\alpha}}\underline{\beta}} - \frac{1}{2} X_{\underline{\beta}}{}^{\dot{\underline{\delta}}} X_{\underline{\rho}}{}^{\dot{\underline{\eta}}}$$

$$X_{\underline{\sigma}}{}^{\dot{\underline{\lambda}}} \varepsilon_{\dot{\underline{\alpha}}\dot{\underline{\delta}}\dot{\underline{\rho}}\dot{\underline{\lambda}}} \varepsilon^{\underline{\sigma\beta\rho\delta}} C_{\underline{\beta}\dot{\underline{\eta}}\underline{\delta}}) \tag{5.10}$$

$$\delta C_{\underline{\gamma}\dot{\underline{\alpha}}\underline{\beta}} = -\xi^{\underline{\rho}} D_{\underline{\rho\gamma}\dot{\underline{\alpha}}\underline{\beta}} + \bar{\xi}^{\dot{\underline{\rho}}} 8i (\delta_{\dot{\underline{\rho}}}^{\dot{\underline{\lambda}}} B_{\underline{\gamma}\dot{\underline{\alpha}}} +$$

$$+ X_{\underline{\gamma}}{}^{\dot{\underline{\lambda}}} X_{\dot{\underline{\rho}}}^{-1\,\underline{\lambda}} B_{\underline{\lambda}\dot{\underline{\alpha}}}) A_{\dot{\underline{\lambda}}\underline{\beta}} + \xi^{\dot{\underline{\rho}}} 8i (\delta_{\underline{\gamma}}^{\underline{\lambda}} B_{\dot{\underline{\rho}}\underline{\beta}} +$$

$$+ X_{\dot{\underline{\rho}}}^{-1\,\underline{\lambda}} X_{\underline{\gamma}}{}^{\dot{\underline{\lambda}}} B_{\dot{\underline{\lambda}}\underline{\beta}}) A_{\dot{\underline{\alpha}}\underline{\lambda}} - 2i\bar{\xi}^{\dot{\underline{\rho}}} \sum_{\underline{\gamma}\dot{\underline{\rho}}}^{\underline{f}} \mathcal{D}_{\underline{f}} A_{\dot{\underline{\alpha}}\underline{\beta}} \tag{5.11}$$

$$\delta D_{\underline{\delta\gamma}\dot{\underline{\alpha}}\underline{\beta}} = -8i\xi^{\dot{\underline{\rho}}} (\delta_{\underline{\delta}}^{\underline{\lambda}} B_{\dot{\underline{\rho}}\underline{\gamma}} + X_{\dot{\underline{\rho}}}^{-1\,\underline{\lambda}} X_{\underline{\delta}}{}^{\dot{\underline{\lambda}}} B_{\dot{\underline{\lambda}}\underline{\gamma}}) C_{\underline{\lambda}\dot{\underline{\alpha}}\underline{\beta}} +$$

$$-8i\bar{\xi}^{\dot{\underline{\rho}}} (\delta_{\dot{\underline{\rho}}}^{\dot{\underline{\lambda}}} B_{\underline{\delta}\dot{\underline{\alpha}}} + X_{\underline{\delta}}{}^{\dot{\underline{\lambda}}} X_{\dot{\underline{\rho}}}^{-1\,\underline{\lambda}} B_{\underline{\lambda}\dot{\underline{\alpha}}}) C_{\underline{\gamma}\dot{\underline{\lambda}}\underline{\beta}}$$

$$+8i\bar{\xi}^{\dot{\underline{\rho}}} (\delta_{\underline{\delta}}^{\underline{\lambda}} B_{\dot{\underline{\rho}}\underline{\beta}} + X_{\dot{\underline{\rho}}}^{-1\,\underline{\lambda}} X_{\underline{\delta}}{}^{\dot{\underline{\lambda}}} B_{\dot{\underline{\lambda}}\underline{\beta}}) C_{\underline{\gamma}\dot{\underline{\alpha}}\underline{\lambda}} +$$

$$\text{terms envolving } (C_{\dot{\underline{\alpha}}\underline{\beta}\dot{\underline{\gamma}}} A_{\dot{\underline{\lambda}}\underline{\sigma}} + B_{\dot{\underline{\alpha}}\underline{\beta}} A_{\dot{\underline{\lambda}}\underline{\sigma}} + A_{\underline{\alpha}\dot{\underline{\beta}}} A_{\dot{\underline{\gamma}}\underline{\sigma}} + \ldots) \tag{5.12}$$

6. Supergravity and matter coupling

It has been pointed out in reference [24] that the action of supergravity in six dimensions cannot be written in a manifestly Lorentz-invariant form.

Here we want to show that what we have in fact, is supergravity coupled to a matter multiplet.

For this purpose let us define

$$\chi^{\underline{\alpha}} \equiv \frac{1}{\sqrt{k}} \mathcal{D}_{\underline{\dot{\alpha}}} F^{\underline{\alpha}\underline{\dot{\alpha}}} \tag{6.1}$$

$$C_{\underline{\alpha}\underline{\beta}[\underline{\gamma}\underline{\delta}]} \equiv \mathcal{D}_{\underline{\alpha}} \mathcal{D}_{\underline{\beta}} (X_{\underline{\gamma}}{}^{\dot{\lambda}} F_{\underline{\dot{\lambda}}\underline{\delta}}) \tag{6.2}$$

$$D \equiv \mathcal{D}_{\underline{\alpha}} \mathcal{D}_{\underline{\beta}} (X^{-1}_{\underline{\dot{\lambda}}}{}^{\underline{\alpha}} F^{\underline{\dot{\lambda}}\underline{\beta}}) + c.c. \tag{6.3}$$

where k is the Newton's constant.

Performing the Clebsh-Gordon decomposition of the tensor product $4 \otimes 4 \otimes 4 \otimes 4$ we can see that the fifteen dimensional representation $A_{[\underline{a}\underline{b}]}$ appears as an irreducible piece of the tensor $C_{\underline{\alpha}\underline{\beta}[\underline{\gamma}\underline{\delta}]}$. This representation can be identified with $A_{\underline{m}\underline{n}}$ $(\equiv e_{\underline{m}}{}^{\underline{a}} e_{\underline{n}}{}^{\underline{b}} A_{\underline{a}\underline{b}})$ of ref. [24].

Then we have the gauge multiplet

$$\psi_{\underline{m}}{}^{\mu} = \psi_{\underline{m}}{}^{\alpha} e_{\underline{\alpha}}{}^{\mu} \quad ; \quad A^{+}_{\underline{m}\underline{n}} \quad ; \; e_{\underline{m}}{}^{\underline{a}} e_{\underline{a}\underline{n}} = g_{\underline{m}\underline{n}} \tag{6.4}$$

plus the matter multiplet

$$D \quad ; \quad \chi^{\mu} = \chi^{\alpha} e_{\underline{\alpha}}{}^{\mu} \quad ; \quad A^{-}_{\underline{m}\underline{n}} \tag{6.5}$$

where $A^{+}_{\underline{mn}}$ gives origin to the self-dual part of $G_{\underline{mnr}}$ i.e.,

$$G_{\underline{mnr}} + \tilde{G}_{\underline{mnr}} \tag{6.6}$$

with

$$\tilde{G}_{\underline{mnr}} = \varepsilon_{\underline{mnrpqs}} G^{\underline{pqs}} \tag{6.7}$$

and $A^{-}_{\underline{mn}}$ gives origin to the anti-self dual part of $G_{\underline{mnr}}$ i.e.,

$$G_{\underline{mnr}} - \tilde{G}_{\underline{mnr}} \tag{6.8}$$

We can observe through the supersymmetry transformations that the matter multiplet cannot be consistently removed or, alternatively we cannot have an action for supergravity theory in six dimensions written in a manifestly Lorentz-invariant form in agreement with reference [24].

Then we fit bellow the Lagrangian of ref. [24].

$$\mathcal{L} = \frac{e^2}{4k^2} R - \frac{e}{2}(\partial_{\underline{m}} D)^2 - \frac{e}{12} G^2_{\underline{mnr}} +$$

$$-e(\chi \Sigma^{\underline{m}} \mathcal{D}_{\underline{m}} \bar{\chi}) - e\psi_{\underline{m}} \Sigma^{\underline{mnp}} \mathcal{D}_{\underline{n}} \bar{\psi}_{\underline{p}} +$$

$$+\frac{e}{\sqrt{2}} \partial_{\underline{n}} D\{\chi \Sigma^{\underline{m}} \Sigma^{\underline{n}} \bar{\psi}_{\underline{m}} + \psi_{\underline{m}} \Sigma^{\underline{n}} \Sigma^{\underline{m}} \bar{\chi}\} k +$$

$$+\frac{e}{4}(G + \tilde{G})^{\underline{mnr}}\{\psi_{\underline{m}} \Sigma_{\underline{nr}} \bar{\chi} - \chi \Sigma_{\underline{nr}} \bar{\psi}_{\underline{m}}\} k +$$

$$- \frac{e}{12} G_{\underline{mnr}} \chi \Sigma^{\underline{mnr}} \bar{\chi} k \quad +$$

$$+ \frac{e}{2} (G - \tilde{G})^{\underline{mnr}} \psi_{\underline{m}} \Sigma_{\underline{n}} \bar{\psi}_{\underline{r}} k \quad + \qquad (6.9)$$

$$+ \text{ quartic fermion terms}$$

where

$$G_{\underline{mnr}} = e^{k\sqrt{2} D} 3 \partial_{[\underline{m}} A_{\underline{nr}]} \qquad (6.10)$$

7. Dimensional reduction

We are going to show that the N=1 D=6 off-shell supergravity theory yields the N=2 D=4 off-shell conformal supergravity. That is an useful result once one remembers that the conformal supergravity is very important if one wants to gain insight into the structure of off-shell Poincaré and de Sitter supergravity theories [33,34].

It will be seen that after dimensional reduction we will have more component fields than those of off-shell conformal supergravity. But the "extra" component fields [16] can constitute an entire real scalar supermultiplet whose components can be identified with those of the scale parameter superfield.

We are going to postulate the following dimensional reduction when going from D=6 to D=4 space-time:

a) for spinors

$$A_{\underline{\alpha}} \sim \begin{pmatrix} A^1_{\alpha} \\ \bar{A}^{\dot{\alpha}}_2 \end{pmatrix} \qquad \bar{A}_{\dot{\underline{\alpha}}} \sim (A_{\dot{\alpha}1}, A^{\alpha 2}) \qquad (7.1)$$

$$A^{\underline{\alpha}} \sim (A_1^{\alpha}, \bar{A}_{\dot{\alpha}}^{2}) \qquad \bar{A}^{\dot{\alpha}} \sim \begin{pmatrix} \bar{A}^{\dot{\alpha}1} \\ A_{\alpha 2} \end{pmatrix} \tag{7.2}$$

$$\alpha, \dot{\beta} \in [1,2]$$

b) for matrices

b.1) $M_{\underline{\alpha}\underline{\beta}}$

$$M_{\underline{\alpha}\underline{\beta}} \sim \begin{pmatrix} M^{1}_{\alpha\beta 1} & M_{\alpha}^{1\ \dot{\beta}2} \\ M^{\dot{\alpha}}_{2\beta 1} & M_{2}^{\dot{\alpha}\beta 2} \end{pmatrix} \tag{7.3}$$

b.2) $M_{\dot{\underline{\alpha}}\dot{\underline{\beta}}}$

$$M_{\dot{\underline{\alpha}}\underline{\beta}} \sim \begin{pmatrix} M_{\dot{\alpha}1\ \dot{\beta}}^{\ \ \ 1} & M_{\dot{\alpha}1\ 2}^{\ \ \ \beta} \\ M^{\alpha 2\ 1}_{\ \ \ \ \dot{\beta}} & M^{\alpha 2\ \beta}_{\ \ \ \ 2} \end{pmatrix} \tag{7.4}$$

b.3) $M_{\dot{\underline{\alpha}}\beta}$

$$M_{\dot{\underline{\alpha}}\underline{\beta}} \sim \begin{pmatrix} M_{\dot{\alpha}1\ \beta}^{\ \ \ 1} & M_{\dot{\alpha}1\ 2}^{\ \ \ \dot{\beta}} \\ M^{\alpha 2\ 1}_{\ \ \ \ \beta} & M^{\alpha 2\ \dot{\beta}}_{\ \ \ \ 2} \end{pmatrix} \tag{7.5}$$

If $M_{\dot{\underline{\alpha}}\underline{\beta}}$ satisfies the relation $X_{\underline{\gamma}}{}^{\dot{\underline{\alpha}}} M_{\dot{\underline{\alpha}}\underline{\beta}} = -X_{\underline{\beta}}{}^{\dot{\underline{\alpha}}} M_{\dot{\underline{\alpha}}\underline{\gamma}}$ then we obtain that

$$M_{\lambda\ 2}^{2\ \dot{\alpha}} = M_{1\ \alpha}^{\dot{\lambda}\ 1} \tag{7.6}$$

$$M_{\alpha\ \beta}^{2\ 1} = M^{[21]}_{(\alpha\beta)} + M^{(21)}_{[\alpha\beta]} \tag{7.7}$$

$$M_{\dot{\alpha}\ \dot{\beta}}^{2\ 1} = M^{[21]}_{(\dot{\alpha}\dot{\beta})} + M^{(21)}_{[\dot{\alpha}\dot{\beta}]} \tag{7.8}$$

Analogously

$$M_{\underline{\alpha}\underline{\dot{\beta}}} \sim \begin{pmatrix} M^{1}_{\alpha\ \dot{\beta}1} & M^{1\ \beta 2}_{\alpha} \\ M^{\dot{\alpha}}_{2\ \dot{\beta}1} & M^{\dot{\alpha}\ \beta 2}_{2} \end{pmatrix} \tag{7.9}$$

Here also

$$X^{-1\ \underline{\beta}}_{\underline{\dot{\gamma}}} M_{\underline{\beta}\underline{\dot{\alpha}}} = -X^{-1\ \underline{\beta}}_{\underline{\dot{\alpha}}} M_{\underline{\beta}\underline{\dot{\gamma}}} \tag{7.10}$$

implies

$$M^{1\gamma}{}_{\dot{\alpha}1} = M_{\dot{\gamma}2}{}^{\alpha 2} \tag{7.11}$$

$$M^{12}_{\gamma\alpha} = M^{[12]}_{(\gamma\alpha)} + M^{(12)}_{[\gamma\alpha]} \tag{7.12}$$

$$M^{\dot{1}\dot{2}}_{\dot{\gamma}\dot{\alpha}} = M^{[\dot{1}\dot{2}]}_{(\dot{\gamma}\dot{\alpha})} + M^{(\dot{1}\dot{2})}_{[\dot{\gamma}\dot{\alpha}]} \tag{7.13}$$

c) for tensors like

c.1)

$$C_{\underline{\gamma}\underline{\dot{\alpha}}\underline{\beta}}$$

we postulate

$$C_{\underline{\gamma}\underline{\dot{\alpha}}\underline{\beta}} = \psi_{\underline{\gamma}} \otimes \phi_{\underline{\dot{\alpha}}\underline{\beta}} \tag{7.14}$$

But we already know how to reduce the two terms on the right-hand side of (7.14). Then $C_{\underline{\gamma}\underline{\dot{\alpha}}\underline{\beta}}$ when reduced give us as independent component fields (considering also the complex conjugate of $C_{\gamma\dot{\alpha}\beta}$).

$$M^{k}_{\gamma\dot{\alpha}i}{}^{\beta j}\ ,\ N^{k}_{\gamma}{}^{ij}_{\alpha\beta}\ ,\ P^{k}_{\dot{\gamma}}{}^{ij}_{\alpha\beta}\ ,\ldots\ (i\neq j) \tag{7.15}$$

c.2)

$$D_{\underline{\gamma\delta\dot{\alpha}\beta}}$$

we postulate

$$D_{\underline{\gamma\delta\dot{\alpha}\beta}} = M_{\underline{\gamma\delta}} \otimes N_{\underline{\dot{\alpha}\beta}} \tag{7.16}$$

Then using (7.3) and (7.5) one obtains the component fields in the reduced theory (also considering the complex conjugate of $D_{\underline{\gamma\delta\dot{\alpha}\beta}}$).

$$R^{ij}_{\gamma\delta}{}_{\dot{\alpha}k\beta}^{\ \ \ell}\ ,\ S^{ij}_{\gamma\delta}{}^{\alpha\beta}_{k\ell}\ ,\ldots\ \begin{pmatrix} i\neq j\\ k\neq\ell\end{pmatrix} \tag{7.17}$$

We can find in the literature [16] that the conformal supergravity theory can be written in a superspace with SL(2,C) x U(N) as the tangent space group.

In the case of N=2 this theory is described in terms of a conformaly covariant chiral superfield $P^{ij}_{\alpha\beta}$ together with some additional non-Weyl covariant superfields $R^{ij}_{\alpha\beta}$, T^{ij} and $J^{i}_{\alpha\dot{\beta}j}$ where P and R are symmetric on their spinor indices and antisymmetric on their internal indices. T is symmetric on ij and the field J is hermitian.

The component field content of N=2 off-shell conformal supergravity [35] is given by:

spin 0: $U^{ij}{}_{k\ell}$ (mass dimension 2) spin $\frac{1}{2}$: $N_{\alpha ij}{}^{k}$ (dimension $\frac{3}{2}$)

spin 1: $M^{ij}_{\alpha\beta}$ (dimension 1); $X^{ij}_{\alpha\beta}$ (dimension 2)

spin $\frac{3}{2}$: $\psi_{\alpha\beta\gamma i}$ (dimension $\frac{3}{2}$)

spin 2: $S_{\alpha\beta\gamma\delta}$ (dimension 2) (7.18)

where M, ψ, X and S are totally antisymmetric on their U(2) indices and totally symmetric on their spinor indices. U is skew on both pairs of indices and N is skew on ij.

We see that we have more component fields than those given above. But the additional ones can fit an entire real scalar superfield. The components of this supermultiplet can be identified [16] with those of the scale parameter superfield of the conformal theory.

Let us see how the component fields (7.18) fit in the reduced theory.

Using (7.5) we have

$$A_{\underset{\sim}{\dot\alpha}\underset{\sim}{\beta}} \sim \begin{pmatrix} A_{1\dot\alpha}{}^{1}{}_{\beta} & A_{1\dot\alpha}{}^{\dot\beta}{}_{2} \\ A^{2\alpha}{}^{1}{}_{\beta} & A^{2\alpha\,\dot\beta}{}_{2} \end{pmatrix} \quad \text{(dimension 1)} \qquad (7.19)$$

$$A_{\underset{\sim}{\alpha}\underset{\sim}{\dot\beta}} \sim \begin{pmatrix} A^{1}_{\alpha\dot\beta 1} & A^{1\ \beta 2}_{\alpha} \\ A^{\dot\alpha}_{2\dot\beta 1} & A^{\dot\alpha\ \beta 2}_{2} \end{pmatrix} \quad \text{(dimension 1)} \qquad (7.20)$$

where

$$A^{ij}_{\alpha\beta} = A^{[ij]}_{(\alpha\beta)} + A^{(ij)}_{[\alpha\beta]} \quad (i\neq j) \qquad (7.21)$$

So we can identify $A^{[ij]}_{(\alpha\beta)}$ with $M^{[ij]}_{(\alpha\beta)}$ (7.22)

Now

$$\mathcal{D}_{\gamma}F_{\underline{\dot{\alpha}}\underline{\beta}}\Big|_{\theta=\bar{\theta}=0} = C_{\underline{\gamma}\underline{\dot{\alpha}}\underline{\beta}} \qquad \text{(dimension 2)} \tag{7.23}$$

So using (7.14) we can identify

$$C^{j}{}_{(\gamma\alpha j\beta i)} \quad \text{with} \quad \psi_{(\alpha\beta\gamma)i} \tag{7.24}$$

and

$$H^{\gamma k}{}_{\gamma[ij]\alpha} \quad \text{with} \quad N_{\alpha[ij]}{}^{k} \tag{7.25}$$

Finally

$$\mathcal{D}_{\delta}\mathcal{D}_{\gamma}F_{\dot{\alpha}\underline{\beta}}\Big|_{\theta=\bar{\theta}=0} = D_{\delta\gamma\dot{\alpha}\beta} \qquad \text{(dimension 2)} \tag{7.26}$$

Then using (7.16) we can identify

$$D^{[ij]}_{(\delta\gamma)}{}^{(\delta\gamma)}_{[k\ell]} \quad \text{with} \quad U^{[ij]}{}_{[k\ell]} \tag{7.27}$$

if one imposes the reality condition on $D^{[ij]}_{(\delta\gamma)}{}^{(\delta\gamma)}_{[k\ell]}$ and

$$L^{[ij]}_{(\alpha\beta\gamma\delta)[ij]} \quad \text{with} \quad S_{(\alpha\beta\gamma\delta)} \tag{7.28}$$

and $\quad V^{[ij]\,\gamma}_{(\alpha\,\gamma\,k\,\beta\ell)}\,g^{ke}$ with $X^{[ij]}_{(\alpha\beta)}$; $g^{ij} = -g^{ji}$ $g^{12} = 1$ (7.29)

Thus we have proved that the reduced theory fits the N=2 off--shell conformal supergravity theory.

Appendix

Generalities of the six dimensional supergravity theory.

$$\varepsilon^{1234} = 1$$

$$\varepsilon^{\underline{\alpha\beta\gamma\delta}}\,\varepsilon_{\underline{\rho\lambda\gamma\delta}} = 2(\delta^{\underline{\alpha}}_{\underline{\rho}}\,\delta^{\underline{\beta}}_{\underline{\lambda}} - \delta^{\underline{\alpha}}_{\underline{\lambda}}\,\delta^{\underline{\beta}}_{\underline{\rho}})$$

$$(\Sigma^{\underline{ab}})_{\underline{\alpha}}{}^{\underline{\beta}} = (\Sigma^{\underline{a}}\bar\Sigma^{\underline{b}} - \Sigma^{\underline{b}}\bar\Sigma^{\underline{a}})_{\underline{\alpha}}{}^{\underline{\beta}}$$

$$(\bar\Sigma^{\underline{ab}})^{\underline{\dot\alpha}}{}_{\underline{\dot\beta}} = (\bar\Sigma^{\underline{a}}\Sigma^{\underline{b}} - \Sigma^{\underline{b}}\bar\Sigma^{\underline{a}})^{\underline{\dot\alpha}}{}_{\underline{\dot\beta}}$$

$$(\bar\Sigma^{\underline{ab}\underline{\dot\alpha}}{}_{\underline{\dot\beta}})^{T} = -X^{-1}_{\underline{\dot\beta}}{}^{\underline{\lambda}}\,\Sigma^{\underline{ab}}_{\underline{\lambda}}{}^{\underline{\rho}}\,X_{\underline{\rho}}{}^{\underline{\dot\alpha}}$$

$$X_{\alpha}{}^{\underline{\dot\beta}} \sim \begin{pmatrix} 0 & \varepsilon_{\alpha\beta} \\ -\varepsilon^{\dot\alpha\dot\beta} & 0 \end{pmatrix}; \quad X^{-1}_{\dot\alpha}{}^{\underline{\beta}} \sim \begin{pmatrix} 0 & -\varepsilon_{\dot\alpha\dot\beta} \\ \varepsilon^{\alpha\beta} & 0 \end{pmatrix}$$

$$(\Sigma^{\underline{a}}\bar\Sigma^{\underline{b}} + \Sigma^{\underline{b}}\bar\Sigma^{\underline{a}})_{\underline{\alpha}}{}^{\underline{\beta}} = -2\delta^{\underline{\beta}}_{\underline{\alpha}}\,\eta^{\underline{ab}}$$

$$(\bar\Sigma^{\underline{a}}\Sigma^{\underline{b}} + \Sigma^{\underline{b}}\bar\Sigma^{\underline{a}})^{\underline{\dot\alpha}}{}_{\underline{\dot\beta}} = -2\delta^{\underline{\dot\alpha}}_{\underline{\dot\beta}}\,\eta^{\underline{ab}}$$

$$\Sigma^{\underline{a}}_{\underline{\alpha\dot\beta}}\,\bar\Sigma^{\underline{b}\underline{\dot\beta}\underline{\alpha}} = -4\eta^{\underline{ab}}$$

$$\Sigma^{\underline{a}}_{\underline{\alpha\dot\alpha}}\,\bar\Sigma^{\underline{\dot\beta\beta}}_{\underline{a}} = 2(X_{\underline{\alpha}}{}^{\underline{\dot\beta}}\,X^{-1}_{\underline{\dot\alpha}}{}^{\underline{\beta}} - \delta^{\underline{\dot\beta}}_{\underline{\dot\alpha}}\,\delta^{\underline{\beta}}_{\underline{\alpha}})$$

$$\Sigma^{\underline{a}}_{\underline{\alpha\dot\alpha}}\,\Sigma_{\underline{a\beta\dot\beta}} = 2\varepsilon_{\underline{\alpha\gamma\beta\delta}}\,X^{-1}_{\underline{\dot\alpha}}{}^{\underline{\gamma}}\,X^{-1}_{\underline{\dot\beta}}{}^{\underline{\delta}} = 2\varepsilon_{\underline{\dot\alpha\dot\gamma\dot\beta\dot\delta}}\,X_{\underline{\alpha}}{}^{\underline{\dot\gamma}}\,X_{\underline{\beta}}{}^{\underline{\dot\delta}}$$

$$\bar\Sigma^{\underline{a\dot\alpha\alpha}}\,\bar\Sigma^{\underline{\dot\beta\beta}}_{\underline{a}} = 2\varepsilon^{\underline{\dot\alpha\dot\gamma\dot\beta\dot\delta}}\,X^{-1}_{\underline{\dot\gamma}}{}^{\underline{\alpha}}\,X^{-1}_{\underline{\dot\delta}}{}^{\underline{\beta}} = 2\varepsilon^{\underline{\alpha\gamma\beta\delta}}\,X_{\underline{\gamma}}{}^{\underline{\dot\alpha}}\,X_{\underline{\delta}}{}^{\underline{\dot\beta}}$$

$$\Sigma^{\underline{bc}}\Sigma^{\underline{a}} = 2(\eta^{\underline{ba}}\Sigma^{\underline{c}} - \eta^{\underline{ca}}\Sigma^{\underline{b}}) - \frac{1}{3}\,\varepsilon^{\underline{bcadef}}\,\Sigma_{d}\bar\Sigma_{e}\Sigma_{f}$$

$$\Sigma^{a}\bar{\Sigma}^{bc} = -2(\eta^{ab}\Sigma^{c} - \eta^{ac}\Sigma^{b}) - \frac{1}{2}\,\varepsilon^{abcdef}\Sigma_{d}\bar{\Sigma}_{e}\Sigma_{f}$$

$$\mathrm{tr}\;\Sigma^{bc} = \mathrm{tr}\;\bar{\Sigma}^{bc} = 0$$

$$\mathrm{tr}(\Sigma^{ab}\Sigma^{cd}) = -16(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc})$$

$$A^{a\dot{\alpha}\dot{\beta}} = \bar{\Sigma}^{a\dot{\alpha}\lambda}\,X_{\lambda}{}^{\dot{\beta}}$$

$$B^{a}_{\alpha\beta} = \Sigma^{a}_{\alpha\dot{\lambda}}\,X_{\beta}{}^{\dot{\lambda}}$$

$$C^{a}_{\dot{\alpha}\dot{\beta}} = X^{-1}_{\dot{\alpha}}{}^{\lambda}\,\Sigma^{a}_{\lambda\dot{\beta}}$$

$$D^{a\alpha\beta} = X^{-1}_{\dot{\lambda}}{}^{\alpha}\,\bar{\Sigma}^{a\dot{\lambda}\beta}$$

$$A^{a\dot{\alpha}\dot{\beta}}\,A_{a}{}^{\dot{\gamma}\dot{\delta}} = 2\varepsilon^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}$$

$$B^{a}_{\alpha\beta}\,B_{a\gamma\delta} = 2\varepsilon_{\alpha\beta\gamma\delta}$$

$$C^{a}_{\dot{\alpha}\dot{\beta}}\,C_{a\dot{\gamma}\dot{\delta}} = 2\varepsilon_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}$$

$$D^{a\alpha\beta}\,D^{\gamma\delta} = 2\varepsilon^{\alpha\beta\gamma\delta}$$

$$A^{a\dot{\alpha}\dot{\beta}}\,C_{a\dot{\gamma}\dot{\delta}} = 2(\delta^{\dot{\alpha}}_{\dot{\gamma}}\,\delta^{\dot{\beta}}_{\dot{\delta}} - \delta^{\dot{\alpha}}_{\dot{\delta}}\,\delta^{\dot{\beta}}_{\dot{\gamma}})$$

$$A^{a\dot{\alpha}\dot{\beta}}\,C^{b}_{\dot{\alpha}\dot{\beta}} = 4\eta^{ab}$$

$$A^{a\dot{\alpha}\dot{\beta}}\,C^{b}_{\dot{\beta}\dot{\gamma}} + A^{b\dot{\alpha}\dot{\beta}}\,C^{a}_{\dot{\beta}\dot{\gamma}} = -2\eta^{ab}\delta^{\dot{\alpha}}_{\dot{\gamma}}$$

$$D^{a\alpha\beta}\,B^{b}_{\beta\gamma} + D^{b\alpha\beta}\,B^{a}_{\beta\gamma} = -2\eta^{ab}\delta^{\alpha}_{\gamma}$$

$$A^{\underline{a}\dot{\underline{\alpha}}\dot{\underline{\beta}}}\, C^{\underline{b}}_{\dot{\underline{\beta}}\dot{\underline{\gamma}}} = (\bar{\Sigma}^{\underline{a}}\Sigma^{\underline{b}})^{\dot{\underline{\alpha}}}{}_{\dot{\underline{\gamma}}} \quad ; \quad D^{\underline{a}\underline{\alpha}\underline{\beta}}\, B^{\underline{b}}_{\underline{\beta}\underline{\gamma}} = (\Sigma^{\underline{b}}\bar{\Sigma}^{\underline{a}})_{\underline{\gamma}}{}^{\underline{\alpha}}$$

Spinor notation

$$v^{\dot{\underline{\alpha}}\dot{\underline{\beta}}} = -\frac{1}{4} A^{\underline{a}\dot{\underline{\alpha}}\dot{\underline{\beta}}}\, v_{\underline{a}} \therefore v_{\underline{a}} = C_{\underline{a}\dot{\underline{\beta}}\dot{\underline{\alpha}}}\, v^{\dot{\underline{\alpha}}\dot{\underline{\beta}}}$$

$$v_{\underline{\alpha}\underline{\beta}} = -\frac{1}{4} B^{\underline{a}}_{\underline{\alpha}\underline{\beta}}\, v_{\underline{a}} \therefore v_{\underline{a}} = D_{\underline{a}}{}^{\beta\underline{\alpha}}\, v_{\underline{\alpha}\underline{\beta}}$$

$$v_{\dot{\underline{\alpha}}\dot{\underline{\beta}}} = -\frac{1}{4} C^{\underline{a}}_{\dot{\underline{\alpha}}\dot{\underline{\beta}}}\, v_{\underline{a}} \therefore v_{\underline{a}} = A_{\underline{a}}{}^{\dot{\beta}\dot{\underline{\alpha}}}\, v_{\dot{\underline{\alpha}}\dot{\underline{\beta}}}$$

$$v^{\underline{\alpha}\underline{\beta}} = -\frac{1}{4} D^{\underline{a}\underline{\alpha}\underline{\beta}}\, v_{\underline{a}} \therefore v_{\underline{a}} = B_{\underline{a}\underline{\beta}\underline{\alpha}}\, v^{\underline{\alpha}\underline{\beta}}$$

$v_{\dot{\underline{\alpha}}\dot{\underline{\beta}}}$ and $v^{\dot{\underline{\alpha}}\dot{\underline{\beta}}}$ are equivalent tensor representations

$$v_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}\, \varepsilon_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}\, v^{\dot{\gamma}\dot{\delta}}$$

the same is true for $v_{\underline{\alpha}\underline{\beta}}$ and $v^{\underline{\alpha}\underline{\beta}}$

$$B^{\underline{a}}_{\underline{\alpha}\underline{\beta}}\, D^{\underline{b}\underline{\gamma}\underline{\delta}}\, L_{\underline{a}\underline{b}} = 16 L_{\underline{\alpha}\underline{\beta}}{}^{\underline{\gamma}\underline{\delta}} = -2\delta^{\underline{\delta}}_{\underline{\alpha}}\, L_{\underline{\beta}}{}^{\underline{\gamma}} +$$

$$+ 2\delta^{\underline{\gamma}}_{\underline{\alpha}}\, L_{\underline{\beta}}{}^{\underline{\delta}} + 2\delta^{\underline{\delta}}_{\underline{\beta}}\, L_{\underline{\alpha}}{}^{\underline{\gamma}} - 2\delta^{\underline{\gamma}}_{\underline{\beta}}\, L_{\underline{\alpha}}{}^{\underline{\delta}}$$

$$A^{\underline{a}\dot{\underline{\gamma}}\dot{\underline{\delta}}}\, C^{\underline{b}}_{\dot{\underline{\alpha}}\dot{\underline{\beta}}}\, L_{\underline{a}\underline{b}} = 16 L^{\dot{\underline{\gamma}}\dot{\underline{\delta}}}{}_{\dot{\underline{\alpha}}\dot{\underline{\beta}}} = -2\delta^{\dot{\underline{\delta}}}_{\dot{\underline{\alpha}}}\, L^{\dot{\underline{\gamma}}}{}_{\dot{\underline{\beta}}} +$$

$$+ 2\varepsilon^{\dot{\gamma}}_{\dot{\alpha}}\, L^{\dot{\delta}}{}_{\dot{\beta}} + 2\delta^{\dot{\delta}}_{\dot{\beta}}\, L^{\dot{\gamma}}{}_{\dot{\alpha}} - 2\delta^{\dot{\gamma}}_{\dot{\beta}}\, L^{\dot{\delta}}{}_{\dot{\alpha}}$$

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